SECTION III.

DEVELOPEMENT OF THE SOLUTION.

The two Curves described by the Instantaneous Pole of Rotation.

THE above illustration of the rotatory motion of a body leads us at once, and as it were by the hand, to the calculations necessary to measure all the different affections of this motion.

And first this succession of points, at which the central ellipsoid comes into contact with the fixed plane of the impressed couple, traces on the surface of the ellipsoid the path of the instantaneous pole in the interior of the body, and the corresponding succession of points on the fixed plane traces its path in absolute space. We can therefore determine immediately these two curved lines, and consider them as the bases of two conical surfaces having the same vertex, one of which, moving with the body, would by rolling on the other, which is fixed in absolute space, cause in the body the precise motion with which it is endued.

To find the first curve we have only to determine the succession of points in which the ellipsoid is touched by a plane which is always at the same distance from its centre; or what is the same thing, which touches a concentric sphere whose radius is equal to the given distance.

While this plane traces on the ellipsoid the path of the instantaneous pole, we may remark that it traces on the sphere the path that the pole of the couple, which is fixed in space, would appear to describe in the interior of the moveable body; a curve of the same nature which we shall have also occasion to consider.

But to speak of the first only: we see that it is *a re-entering curve of double curvature*, having like the ellipse *four principal vertices*, at which it is divided into four equal and symmetrical parts; a species of elliptical *wheel*, whose *axle* is always either the *greatest* or *least radius* of the central ellipsoid, according as the radius of the sphere is given greater or less than the mean radius of the ellipsoid. This curve of double curvature is projected in *a complete ellipse* on the plane perpendicular to the axis which forms its *axle*, in *an elliptic arc* on the other plane, and always in *an hyperbolic arc* on the plane perpendicular to *the mean radius*. (19)

(19) Let Pg (fig. 14.) be a perpendicular section of the tangent plane touching the ellipsoid in P and a concentric sphere whose radius $Gg = r$ in g.

Let x, y, z, be the co-ordinates of P,
x', y', z', ... of g,
x'', y'', z'',...................... of any point in the tangent plane.

Then we have two equations to this plane; viz:

$$\frac{xx''}{a^2} + \frac{yy''}{b^2} + \frac{zz''}{c^2} = 1,$$

$$\text{and } \frac{x'x''}{r^2} + \frac{y'y''}{r^2} + \frac{z'z''}{r^2} = 1,$$

which must coincide; therefore $\dfrac{x'}{r^2} = \dfrac{x}{a^2}$,

$$\text{and } \frac{x'^2}{r^4} = \frac{x^2}{a^4},$$

$$\frac{y'^2}{r^4} = \frac{y^2}{b^4},$$

$$\frac{z'^2}{r^4} = \frac{z^2}{c^4};$$

$$\text{Therefore } \frac{1}{r^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4};$$

$$\text{Also } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

which are the equations to the curve of double curvature. Let a be the greatest and c the least semi-axis. Then if $r > b$, the curve can never meet the plane perpendicular to the major axis;

$$\text{for if } x = 0, \quad z^2 = c^2 . \frac{\dfrac{1}{r^2} - \dfrac{1}{b^2}}{\dfrac{1}{c^2} - \dfrac{1}{b^2}},$$

a negative quantity. The curve therefore lies in this case wholly about the major axis. And *vice versâ* if $r < b$. Also the equation to the projection on the principal plane perpendicular to c is

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + c^2\left(\frac{1}{r^2} - \frac{x^2}{a^4} - \frac{y^2}{b^4}\right),$$

$$\text{or } \frac{x^2}{a^2} \cdot \frac{1 - \frac{c^2}{a^2}}{1 - \frac{c^2}{r^2}} + \frac{y^2}{b^2} \cdot \frac{1 - \frac{c^2}{b^2}}{1 - \frac{c^2}{r^2}} = 1,$$

which is evidently the equation to an ellipse whose semi-axes are

$$a\sqrt{\frac{1 - \frac{c^2}{r^2}}{1 - \frac{c^2}{a^2}}}, \quad \text{and } b\sqrt{\frac{1 - \frac{c^2}{r^2}}{1 - \frac{c^2}{b^2}}},$$

the first of which is always $< a$; and if $r < b$, the second is $< b$, or the projection is a complete ellipse, concentric to the principal section of the ellipsoid; but if $r > b$ the minor semi-axis is $> b$, and the projection is only part of an ellipse.

The equation to the projection perpendicular to the mean semi-axis is

$$\frac{x^2}{a^2} \cdot \frac{1 - \frac{b^2}{a^2}}{1 - \frac{b^2}{r^2}} + \frac{z^2}{c^2} \cdot \frac{1 - \frac{b^2}{c^2}}{1 - \frac{b^2}{r^2}} = 1,$$

which is manifestly the equation to an hyperbola, the coefficients of x^2 and z^2 having necessarily different signs.

The four vertices of this curve are the points where the radius vector, and consequently the velocity of rotation, attain their *maximum* and *minimum* values; and we may remark that the *maximum* always occurs when the instantaneous pole passes through the two vertices which lie in the *mean* principal plane of the ellipsoid, and the *minimum* when it passes through the other two vertices. (20)

(20) Since $GP = \sqrt{x^2 + y^2 + z^2}$,

$$0 = d_x GP = GP \,.\, d_x GP$$

$$= x + y d_x y + z d_x z,$$

also $0 = \frac{x}{a^2} + \frac{y}{b^2} d_x y + \frac{z}{b^2} d_x z,$

$$0 = \frac{x}{a^4} + \frac{y}{b^4} d_x y + \frac{z}{b^4} d_x z,$$

whence we derive separately $y = 0$, $x = 0$, which give possible values for GP when $r > b$, the former satisfying the conditions for a maximum, and the latter for a minimum; the value of the maximum radius (P) being

$$\sqrt{x^2 + z^2} = \frac{a^2\sqrt{r^2 - c^2} + cr\sqrt{a^2 - r^2}}{r\sqrt{a^2 - c^2}},$$

and of the minimum (ρ),

$$\sqrt{x^2 + y^2} = \frac{a^2\sqrt{r^2 - b^2} + br\sqrt{a^2 - r^2}}{r\sqrt{a^2 - b^2}}.$$

The second curve, being traced by that which rolls about the centre on the fixed plane of the couple, is therefore a plane curve which encircles the projection of the centre, forming equal and regular undulations corresponding to the equal and symmetrical arcs of the rolling orbit which produces it: it is a species of circular curve whose radius varies periodically, and which winds for ever between two concentric circles whose circumferences it touches alternately. (21)

(21) The projection (*gp*) of the radius vector *GP* on the fixed plane of the couple will evidently arrive at its maximum and minimum values cotemporaneously with *GP*; for when *GP* is greatest its inclination to this plane is least.

With centre *g*, (fig. 15.) the projection of *G*, and radii ρ and P, describe two circles; then the curve being perpendicular to the radius vector *gp* at the points where it attains its maximum and minimum values, will manifestly touch these circles at those points; that is, will touch them alternately, since the curve passes alternately the mean and major principal planes.

The consequences deduced in this and the last note for $r > b$ are easily adapted to the case when $r < b$.

If the angle at the centre which corresponds to two consecutive vertices of these equidistant undulations is commensurable with four right angles, the curve re-enters itself after a certain number of revolutions; and the instantaneous pole which describes it returns at once to the same position, both in the body and in space. But in the contrary case the curve never re-enters itself, and the pole, which always returns periodically to the same place in the body, can never return at the same time to the same point in space.

Such are the two curves described by the instantaneous pole, the one in the interior of the body, and the other in absolute space. And although these curves are of such different forms, yet since it is one and the same point which describes them both, the equations to them, between the radius vector and the arc, exactly coincide. (22)

(22) To find this equation to the two curves, we have

$$p^2 = x^2 + y^2 + z^2,$$

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2},$$

$$\frac{1}{r^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}.$$

Substituting in the second and third the values of z^2 derived from the first, multiplying the former by $\frac{1}{b^2} + \frac{1}{c^2}$, and subtracting, we have

$$x^2\left(\frac{1}{a^2} - \frac{1}{c^2}\right)\left(\frac{1}{b^2} - \frac{1}{a^2}\right) + \frac{p^2}{b^2 c^2} = \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{r^2};$$

$$\therefore\ x^2 = a^4 . \frac{\frac{b^2 c^2}{r^2} - \left(\frac{1}{b^2} + \frac{1}{c^2}\right) + p^2}{(a^2 - b^2)(a^2 - c^2)},$$

or $x = a' . \sqrt{\alpha^2 + p^2}$, suppose;

$$\therefore \; d_p x = \frac{a' p}{\sqrt{a^2 + p^2}}.$$

Similarly, $d_p y = \dfrac{b' p}{\sqrt{\beta^2 + p^2}}$, $\quad d_p z = \dfrac{c' p}{\sqrt{\gamma^2 + p^2}}$;

$$\therefore \; d_p s, \text{ which} = \sqrt{(d_p x)^2 + (d_p y)^2 + (d_p z)^2}$$

$$= p \sqrt{\frac{a'^2}{\alpha^2 + p^2} + \frac{b'^2}{\beta^2 + p^2} + \frac{c'^2}{\gamma^2 + p^2}}.$$

The *rolling cone* of which the first curve forms the base is simply *a right cone of the second degree;* but the *fixed cone* on which it rolls is a *transcendent cone*, whose surface undulates for ever about the fixed axis of the couple: it is a species of right circular cone, whose surface however is *fluted* according to the regular undulations of the curve which forms its base.

Proposed Names for the two Curves.

We know that a heavy body projected any how in space turns on its centre of gravity, exactly as if it were free from the action of gravity. The two remarkable curves therefore above described are presented constantly to our notice in the motion of projectiles, and merit names as much as the path of the centre of gravity which is called a parabola.

I propose therefore to give them the names of relative and absolute *Poloids;* or rather, in order to distinguish them by their respective forms, to call the first simply the *poloid*, and the second the *serpoloid*.

It is evident that the forms of the curves will depend entirely on four given quantities, viz. the three semi-axes of the central ellipsoid which are always given by the nature of the body, and the height of the centre above the tangent plane of the couple, which is given by the direction of the impressed couple.

Particular Case in which the Poloid becomes an Ellipse, and the Serpoloid a Spiral.

In the particular case in which the height above the plane of the couple is exactly equal to the *mean radius* of the ellipsoid, the poloid becomes an *ellipse*, whose plane passes through the mean radius, and the serpoloid a spiral, which when examined throughout its whole extent appears to be a sort of *double spiral.* I mean that it throws out on opposite sides of a vertex two equal branches whose generating points revolve in opposite directions about a fixed centre, which they continually approach, but which is a species of *asymptotic point* that they never attain. In this case therefore the instantaneous pole is always a new point both in the body and in absolute space, although the length of the spiral described is finite, and moreover equal to the semi-circumference of the rolling ellipse which produces it. (23)

(23) When $r = b$, we have

$$\frac{x^2}{a^2} \cdot \frac{b^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \cdot \frac{b^2}{c^2} = 1,$$

$$\text{and } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$\text{whence } x = \frac{a^2}{c^2} \cdot \sqrt{\frac{b^2 - c^2}{a^2 - b^2}} \cdot z,$$

and the curve lies wholly in a plane passing through the axis of y. It is therefore an ellipse whose axes are

$$\sqrt{\frac{a^2 b^2 + b^2 c^2 - a^2 c^2}{a^2 - b^2}} = a' \text{ and } b.$$

Now the latter is evidently the value of Gg (fig. 14.) in this case. And as the value of GP diminishes from the former, the angle PGg, which is the angle (θ) between the instantaneous axis and the fixed axis of the couple, perpetually diminishes, and the accelerating couple arising from the centrifugal forces ($\omega_p . M \sin \theta$) becomes less than any assignable quantity; that is, no finite couple M can make the pole coincide with g (fig. 16.) though it continually approaches it in moving either way from K, which lies in the circumference of the circle described with radius $GK = a'$ on the plane of the couple.

Particular Cases in which they are reduced to a Point.

If the distance of the centre from the tangent plane is given equal to one of the *extreme radii* of the ellipsoid, then since this can only happen at a single point in the surface, the poloid and serpoloid are both reduced to a single point, and the instantaneous pole remains immoveable both in the body and in space during the whole motion; and the same thing will happen in a single variety of the particular case before considered, namely, when the plane of the couple touches the central ellipsoid at its mean pole. (24)

(24) In all these cases the axis of the impressed couple coincides with a principal axis, and therefore the theorem of page 29 is applicable, and the axis of the couple being the instantaneous axis, $\theta = 0$ and the centrifugal couple vanishes, that is, the elementary centrifugal couples mutually destroy each other.

Particular Case, arising from the Constitution of the Body, in which the two Curves become Circles.

Lastly, if the body is one of those which have two of their principal moments of inertia equal to one another, in which case the ellipsoid becomes a solid of revolution, the poloid becomes a circle about the axis of this spheroid, and the serpoloid another circle about the fixed axis of the couple. In all bodies of this kind the movement is that of a right cone whose base is a circle rolling uniformly on a fixed cone of the same kind. It is one of the simplest cases of rotatory motion, but we must remark that if, as is usually the case, the circumferences of these circles are incommensurable, the instantaneous pole can never return at the same time to the same point in the body and in absolute space. (25)

(25) If $a=b$, the equations to the poloid become

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1,$$

$$\text{and}\quad \frac{x^2+y^2}{a^4}+\frac{z^2}{c^4}=\frac{1}{r^2},$$

whence we have

$$\frac{z^2}{c^2}\cdot\left(\frac{1}{c^2}-\frac{1}{a^2}\right)=\frac{1}{r^2}-\frac{1}{a^2},$$

$$\text{and}\quad \frac{x^2+y^2}{a^2}\cdot\left(\frac{1}{c^2}-\frac{1}{a^2}\right)=\frac{1}{c^2}-\frac{1}{r^2},$$

the equations to a circle in a plane perpendicular to the axis of z.

Hence GP, and therefore the angular velocity, is constant.

And the serpoloid being traced out by P (fig. 14.), which always remains at the same distance $\sqrt{GP^2-r^2}$ from g, will also be a circle.

The uniformity of the motions of the pole in the body and in space will be shewn hereafter.

We need not examine the simplest case of all, in which the central ellipsoid is a perfect sphere: for in whatever manner the couple is applied, the axis of rotation and the axis of the couple coincide, and the instantaneous pole remains immoveable in the body and in absolute space.

Velocities, with which the Pole describes the two Curves, with which it approaches and recedes from the Centre, with which it revolves about the fixed Axis of the Couple, &c.

After having examined the nature of the two curves described by the pole, which have the same differential equation between the radius vector drawn from the centre of the ellipsoid and the arc, we may consider the velocity with which the pole describes them both; that with which it recedes from or approaches the centre, which is the increment or *fluxion* of the radius vector and consequently of the angular velocity; and the angular motion of the pole about the fixed axis of the impressed couple. It is easy to determine the remarkable points where these different velocities have their *maximum* and *minimum* values, which occur at the alternate vertices of the waves of the serpoloid. But we may make a curious remark on the velocity of the pole along this curve; viz. that it may have, in certain cases, a *minimum* value at the *superior* vertex of the wave, and likewise a *minimum* at the *inferior* vertex, and consequently a *maximum* at a third point intermediate to them; which will not hold for the other velocities, whose maximum and minimum values occur only at the vertices of the curve described by the pole. (26)

(26) The velocity in the poloid $(d_{t}s) = PR$ (fig. 13.)

$$= \frac{p}{\omega_p} \text{ (velocity round } GQ);$$

and since the plane of the centrifugal couple passes through the axis of the impressed couple, the axis GN of the former lies in the plane of the latter:

$$\text{and } \omega_q = \frac{(\text{moment of centrifugal couple}) \cos NGQ}{\text{moment of inertia round } GQ}$$

$$= \frac{\omega_p \,.\, M \sin\theta \,.\cos NGQ}{Q};$$

$$\therefore\ d_t s = Mnpq^2 \sin\theta \cos NGQ.$$

But by Napier's rules,

$$\cos PGQ = \cos PGU \,.\cos QGU,$$

$$\text{and } \cos PGU = \sin\theta = \frac{\sqrt{p^2 - r^2}}{p},$$

$$\cos PGQ = -\, d_s p\,^*,$$

$$\text{whence } \cos QGN = \left\{1 - \frac{(pd_s p)^2}{p^2 - r^2}\right\}^{\frac{1}{2}},$$

$$\text{and } (pq)^2 \sin^2 PGQ + (qu)^2 \sin^2 UGQ + (pu)^2 \sin^2 PGU$$

$$= (ac)^2 + (ab)^2 + (bc)^2,$$

$$\text{also } p^2 + q^2 + u^2 = a^2 + b^2 + c^2,$$

whence we may obtain q in terms of p and $d_s p$; and substituting, we have $d_t s$ in terms of the same quantities. But from note (22) we have $d_s p = \dfrac{1}{pF(p)}$, and we can therefore determine $d_t s$ in terms of p alone.

And since $\omega_p = Mnp^2 \cos\theta = Mnpr$,

$$d_t \omega_p = Mnr \,.\, d_t p = Mnr\, d_t s \,.\, d_s p,$$

* Miller's *Differential Calculus*, Art. 94.

which is reducible to an elliptic function of the kind treated by Legendre, Chap. XXXIII.

If ρ be the radius vector Gg of the serpoloid,

$$p = \sqrt{r^2 + \rho^2},$$

$$\text{and } d_\rho s = d_p s \,.\, d_\rho p = \frac{\rho}{\sqrt{r^2 + \rho^2}} \,.\, \sqrt{r^2 + \rho^2} \,.\, F(r^2 + \rho^2)$$

$$= \rho F(r^2 + \rho^2),$$

$$d_t s = \rho F(r^2 + \rho^2)\, d_t \rho,$$

and at the vertices $d_t \rho = 0$, $d_\rho s = \infty$.

We may afterwards simplify the equation to the serpoloid by referring it to the radius vector drawn from its own centre in the plane in which it lies, and the angle described by this radius about the centre. And any such expression may be obtained without difficulty. (27)

(27) If θ be the angular distance of the radius vector from a given position,

$$(d_\rho \theta)^2 = \frac{(d_\rho s)^2 - 1}{\rho^2} = \{F(r^2 + \rho^2)\}^2 - \frac{1}{\rho^2},$$

which is reducible as before to an elliptic function.

Determination of the Position of the Body at the end of a given Time.

Lastly, in order to find the formulæ which will enable us to calculate the position of the body at the end of a given time, we must begin by determining the velocity of rotation in terms of the time, which will be done by a single integration, and will give the place of the instantaneous pole on the surface of the central ellipsoid. Afterwards we must integrate the above equation to the serpoloid, which will give the place of the pole on the fixed plane of the couple. And by these two quadratures, which naturally belong to the class of *elliptic transcendents,* we may say that the proposed problem is entirely solved; I mean that we are enabled by means of them to determine the actual position in space in which a body is found at the end of a given time. For we have only to suppose the ellipsoid placed in contact with the fixed plane, so that they touch at the points which we have just determined in the ellipsoid and plane respectively; when the central ellipsoid, and consequently the body, will have the exact position in space at which it arrives by its natural motion at the end of the given time.

We may vary these determinations in different ways, by taking other unknown quantities relative to the position of the body; but whatever be the co-ordinates which we employ, the expression of these quantities in terms of the time will always require two integrations, which necessarily belong to the class of elliptic transcendents.

In the particular case in which the height of the centre above the fixed plane is equal to the mean radius of the ellipsoid, when the poloid becomes a simple ellipse and the serpoloid a spiral, the difficulty is diminished, and the integrals become *logarithmic* or *exponential*. (28)

(28) In this case $d_p s$ is reducible at once to an elliptic function; and the value of q^2 deduced from note (26) is greatly simplified; and thus the values of $d_\iota \omega_p$ and $d_\theta \rho$ are reduced as here stated.

Lastly for bodies whose central ellipsoid is one of revolution, when the motion becomes wholly uniform and circular, no integration is required to determine the position of the body at the end of a given time. (29)

(29) In this case all the axes in the equatoreal plane AGB (fig. 17.) of the spheroid are principal axes. Let AGC be the plane passing through the axis of the spheroid and the axis of the couple, GB the intersection of the equatoreal plane with the plane of the couple, which will be manifestly perpendicular to AGC. Then the resolved part of M perpendicular to the principal axis $GB = 0$, and therefore the axis of instantaneous rotation lies in the plane AGC.

Therefore AGC is the plane of the centrifugal couple, and GB, which is the diameter conjugate to it, the axis about which this couple tends to make the body turn.

And from note (25) GP $(= p)$ is constant, as indeed appears also from the consideration that GB and there-the small arc PR is perpendicular to GP; substituting therefore in the equation obtained in note (26) and observing that $NGQ = 0$, we have

$$d_\iota s = Mnpa^2 \frac{\sqrt{p^2 - r^2}}{p}, \text{ a constant quantity.}$$

For the application to the theory of Precession, see Appendix.

Properties of the three principal Axes of the Body, relative to the Stability of the Rotatory Motion.

When the plane of the impressed couple is so situated that the instantaneous pole falls on one of the principal poles of the central ellipsoid, it always remains there; so that the instantaneous axis is the immoveable straight line in which the axes of the body and the couple coincide throughout the whole motion. Each of the three principal axes is therefore a permanent axis of rotation. But there exists between them, as is known, a remarkable difference with respect to the stability of the rotatory motion about each.

If the instantaneous pole falls on the *greater* or *lesser* pole of the ellipsoid, and happens by the impulsion of any small disturbing couple to be drawn aside to a little distance therefrom, it will not recede farther, but will describe its poloid about this particular pole of the ellipsoid. But this is not the case when the instantaneous pole falls on the *mean* pole of the ellipsoid; for on any the slightest displacement it will recede farther and farther, and proceed to describe its poloid about the greater or lesser pole, according as the accidental disturbance causes the distance of the tangent plane of the couple from the centre of the ellipsoid to increase or diminish. And if the disturbance is such that this distance is not altered, which happens in the directions of two particular ellipses which cross each other at the mean pole, the instantaneous pole will proceed to describe the particular ellipse along which it has been started, or rather the half of this ellipse, till it arrives at the opposite mean pole; which is the greatest disturbance that the body can suffer: whilst, if it had been started along the other half of the ellipse it would have returned immediately to the same mean pole; which is the least possible disturbance. There is therefore a single case in which the instantaneous axis being drawn aside from the mean axis, with which it coincided at first, not only does not recede farther from it, but even returns towards it immediately, until its distance is less than any assignable quantity. But in all other cases it proceeds to describe an elliptical cone about the major or minor axis, or to trace the plane of one or other of the ellipses which I have mentioned: and we may say that the rotatory motion about the mean axis has no stability. (30)

(30) The rotatory motions about GP, GQ (fig. 13.) are supposed to be in the same direction; and in that case P moves in the *direction* MPU. Suppose the direction of the motion which the impressed couple tends to produce about its own axis to be reversed; then the motion about the instantaneous axis will be reversed, the motion which the centrifugal couple tends to produce about GQ being unchanged. The motions about GP, GQ will therefore be in opposite directions, and in order to find R the new pole we must take q on the other side of G. The pole will therefore move in the direction UPM.

Hence the direction of the motion of the pole in the poloid depends on the direction of the motion which the impressed couple tends to produce about its axis. If therefore, for a given direction of this motion, the instantaneous pole, on being started in one direction from the mean pole along the above-mentioned ellipse, tends to describe the whole semi-ellipse, it will on being started along the other half of this ellipse still tend to move in the same direction as before, i. e. will return towards the mean pole; which from note (24) it will approach nearer than by any assignable quantity.

The only axes of stable rotatory motion therefore are those about which the moments of inertia of the body are respectively the greatest and least; but we must not conclude that it is equally stable about these two axes: for if one of them differs little from the mean axis, the stability of the motion about it will not be greater than of that about the mean axis, as we shall presently see.

Measure of the Stability about each of the two extreme Axes.

In order to form a distinct idea of this stability, and of that which forms in some degree the measure of it for each of these two axes, imagine the surface of the ellipsoid to be cut into four elliptical sections by the two ellipses above-mentioned, whose planes intersect in the mean axis. The mean pole is therefore at the intersection of these two ellipses; the greater pole is in the centre of one of a pair of sections, and the lesser pole in the centre of the supplementary section.

Now, in the first place, if the instantaneous pole of rotation falls on the mean pole of the ellipsoid, it is clear that if disturbed ever so little it will be thrown into one or other of these two sections, and describe its poloid about one or other of the principal poles of the ellipsoid: or else if the disturbance is in the direction of one of the two ellipses it will either describe the half of this ellipse, or return immediately to its former position; this being the only case of stability about the mean axis.

Again, if the instantaneous pole falls on the *greater* pole of the ellipsoid it may be removed at pleasure to any point in the surrounding section without ceasing to describe its poloid about the same pole: and if it is in this that we make the stability about the major axis to consist, we may say that the magnitude of the section is in some degree the measure of it. Similarly we perceive that the supplementary section is the measure of the stability of the rotatory motion about the minor axis. Now if one of these two axes differ little from the mean axis, the corresponding section is very small, and the supplementary section very great. The axis which differs little from the mean axis affords therefore very little stability, and the other axis very much. It is not therefore correct to say, as people usually do, that if the instantaneous axis is drawn a little aside from the principal axis which corresponds to the greatest or least moment of inertia of the body, it recedes very little from it, and makes only small oscillations during the whole period of the motion: for if the moment of inertia relative to this axis differs little from the mean moment, the instantaneous pole may be thrown by a small disturbance out of the little section in which it lies into the neighbouring section, and proceed to describe therein its poloid about the other axis; or else, if it is only removed to another point in this little section, it may describe therein a narrow and elongated poloid, and consequently make considerable oscillations about the principal pole from which it has been drawn aside. (31)

(31) If $r > b$, the semi-axes of the ellipse which is the projection of the poloid on the plane perpendicular to the axis of x are

$$\frac{c^2}{r}\sqrt{\frac{a^2 - r^2}{a^2 - c^2}}, \quad \text{and} \quad \frac{b^2}{r}\sqrt{\frac{a^2 - r^2}{a^2 - b^2}},^*$$

the latter of which, when b and consequently r very nearly equals a, becomes

$$\frac{b^2}{r}\sqrt{1 - \frac{r^2 - b^2}{a^2}} \text{ very nearly,}$$

which may be made to approach b as nearly as we please by diminishing the difference between r and b, that is by throwing the pole very near to the boundary of the elliptic section mentioned in the text.

The other semi-axis will be increased by this means; but very slightly, as $a^2 - r^2$ is supposed to be very small in comparison with $a^2 - c^2$.

* Note (19). The equation to the projection on yz is obtained from that on xy by writing z for x, a for c, and c for a.

In bodies where one of the extreme moments of inertia differs little from the mean moment, and where consequently the central ellipsoid is nearly a solid of revolution about the other axis, the stability of the rotatory motion is only absolute for this axis. This is the case in the Earth, whose motion is stable about its present axis, but would be very much otherwise about the third axis, which differs, as we know, very little from the mean axis.

Motion of the principal Axes of the Body in absolute Space.

We have considered the motion of the instantaneous pole of rotation both in the body and in space: but it may be asked what are the motions of the poles of the central ellipsoid themselves; the velocities with which they revolve about the fixed axis of the impressed couple, and with which they approach or recede from the plane perpendicular to this axis which gives us their motions of *Precession* and *Nutation:* we may examine into the nature of the three curves or serpoloids which the projections of these three principal poles trace at the same time on a fixed plane, &c. and we shall find in the easiest manner many curious properties of the motion of a body. For example,

The sum of the areas swept out, during the motion of the body, by the projections on the plane of the couple of equal portions of each of the three *principal axes, measured from the centre, is proportional to the time.*

If these three lines, instead of being equal, are proportional to the square roots of the moments of inertia, or to what I call the arms of inertia of the body about the same axes, the sum of the areas is also proportional to the time.

These are simple, and in some degree geometrical theorems, which must however be distinguished from the dynamical theorem relative to the areas traced by all the radii vectores drawn to the several molecules of the body, though it is easy to reduce these expressions, drawn fromethe same principle, to one and the same.

Analogous theorems may be proved relative to the *nutations* of the three principal axes of the body towards theefixed plane of the couple of impulsion. For the sum of the squares of the distances of the three principal poles of the central ellipsoid from the axis of the couple is a *constant quantity:* and,

The sum of the squares respectively multiplied by the moments of inertia of the body is constant throughout the whole motion.

Lastly, if we consider the curves described by the projections of these poles on the same fixed plane, we shall find that they are of the same nature with the serpoloid described by the instantaneous pole of rotation.

In general the pole, whether it be the *greater* or *lesser*, which forms the centre of the poloid, describes a curve having like the serpoloid equal and regular undulations about the same centre; the superior vertices of the one corresponding to the superior vertices of the other, and the inferior to the inferior. During the same time the other two poles describe also curves undulating regularly: but when one of them passes the superior vertices of its path the other is passing the inferior vertices of its similar path; the poles being at an angular distance of 90° from one another.

In the particular cases in which the *poloid* is an *ellipse* and the *serpoloid* a spiral, the mean pole ofe the body describes also a spiral which approaches the centre continually, and nearer than by any assignable limit, without ever reaching it: the two other poles also describe spirals, of a species in some degree resembling the former; for each of them recedes continually from a certain *minimum* distance from the centre to a certain *maximum* which it never reaches; so as to approach continually the circumference of an asymptotic circle. We may make another curious remark on this particular case of the motion of bodies; viz. that there exists in the mean plane of the central ellipsoid a certain diameter which has the remarkable property of remaining always perpendicular to the fixed axis of the impressed couple, and therefore of describing the plane of this couple, and that too with an uniform motion. So that *the whole motion of the body consists in turning on this particular diameter with a variable velocity, while this diameter uniformly describes a circle in space.*

When the central ellipsoid is one of revolution the pole of the figure describes a circle as well as the instantaneous pole. In this case there is no other pole properly speaking than the extremity of the axis of the spheroid: but if we chose to fix arbitrarily on two other points in the equator at an angular distance of 90° from each other and to examine the two curves which their projections describe, we should have two perfectly equal but not circular curves: which would be two equal serpoloids about the same centre, the superior vertices of the one being always at an angular distance of 90° from the corresponding inferior vertices of the other.

We shall have yet more new properties and new illustrations of rotatory motion to present. For instance, it is easy to see that *the section of the central ellipsoid made by the fixed plane of the couple is an ellipse whose area is constant throughout the motion.*

So that if we consider the central ellipsoid as plunged into a non-resisting fluid, of which the fixed plane of the couple forms the level, we may say that the area of the plane of floatation is constant.

We may pass from hence to a new illustration of the motion and represent it by that of an elliptical cone which rolls on the plane of the couple with a variable velocity, and slides with an uniform velocity. All which properties will be developed in the Memoir.

We see how much these illustrations enlighten and correct our ideas of even the most elementary portions of the theory of rotatory motion. Those who cultivate the geometrical properties of surfaces of the second order will draw from them without difficulty a great number of curious theorems relative to this kind of motion: for each proposition in Geometry gives a corresponding one in Dynamics. But the great advantage of our mode of treating the subject consists in the easy demonstrations which it affords of the motions of precession and nutation of the equators of the heavenly bodies, and of the nodes of their orbits: of thus simplifying and sometimes correcting these difficult theories, of which we have already seen an example in the determination of the *single* and *invariable* plane of areas, which I have denominated the *Equator of the system of the Universe.*