SECTION I.

ON THE ACTUAL MOTION OF BODIES, AND ON THE FORCES CAPABLE OF PRODUCING ANY GIVEN MOTION.

Signification of the terms, Simple Rotatory Motion, and Angular Velocity.

THE only rotatory motion of which we have a clear idea, is that of a body which turns on an immoveable axis. For we see plainly all the circumferences of the circles which the different points of the body describe about this axis, and which they can really describe at the same time, without changing in any respect their relative position, or what we may denominate the form of the body.

We have an equally clear idea of the quantity or measure of this rotatory motion; for since all the points in it describe similar arcs in the same time, the ratio of the velocity of a point to the radius of the circle which it describes is the same for all points, and it is this constant ratio which forms the measure, or, as it is called, the *angular velocity* of rotation. (1)

(1) Let OPp (fig. 1) be the axis of rotation, R, r, any two points describing the circles RR', rr', which must lie in planes perpendicular to Op, and therefore parallel to each other.

Let RR', rr', be arcs described uniformly in the same time (t).

Draw PS parallel to pr, and when pr comes into the position pr', let PS' be the corresponding position of PS.

Then $\angle SPS' = \angle rpr'$. (Euc. XI. 10.)

But since the body is rigid,

$$SR = S'R',$$

and therefore, $\angle RPR' = \angle SPS' = \angle rpr'$.

Hence every point in the body describes round Op in the time (t) an angle $= RPR'$.

Let (ω) be the angle described in $1''$ by every point,

then $\angle RPR' = t\omega$,

and the actual velocity (v) of $R = \dfrac{RR'}{t}$

$$= \frac{PR \, . \, \angle RPR'}{t} = PR \, . \, \omega,$$

and $\dfrac{v}{PR} = \omega$, which is the same for every point, is called the *Angular Velocity* of the body, or the *Velocity of Rotation*.

Composition of Rotatory Motion.

From these simple notions, and from the primary elements of Geometry, we may conclude that if, from the influence of any two separate causes, a body tended to turn at the same time round the two sides of a parallelogram, with two angular velocities respectively proportional to the lengths of these sides, the body would turn round the diagonal, with an angular velocity proportional to the length of this diagonal. (2)

(2) Let the straight lines Oa, Ob, (fig. 2.) lying in the plane of the paper intersect each other in O, and suppose two impulses to act simultaneously upon a body, one of which would cause it to turn about Oa with an angular velocity $= e \cdot OA$, and the other to turn about Ob with an angular velocity $= e \cdot OB$. Draw AP, BP, parallel to OB, OA, and PM, PN perpendicular to OA, OB; and let QPQ' be the circle which the point P would describe, if the first impulse were communicated singly, about Oa,

RPR' ....t....t.t...t....t.t.t.
........................second ....
.... Ob.

Then the planes in which these circles lie will both of them be perpendicular to the plane bOa passing through the axes, and therefore their intersection pPp' will be perpendicular to this plane, and therefore to each of the lines PM, PN, which are the radii of the circles, and will therefore be a tangent to both circles at P.

The first impulse therefore, would make P tend to move in the direction Pp with a velocity $= MP \times$ (angular velocity round Oa)

$$= e \cdot OA \cdot MP = e \cdot OA \cdot OB \sin BOA,$$

and the second in the direction Pp' with a velocity $= NP \times$ (angular velocity round Ob)

$$= e \cdot OB \cdot NP = e \cdot OB \cdot OA \sin BOA,$$

that is, in a direction exactly opposite, with the same velocity. If therefore these impulses were communicated together, the point P would remain at rest; and the same may evidently be proved of every other point in the line OP, which would therefore remain entirely at rest in consequence of the two impulses.

Draw AK, AL perpendicular to OB, OP. Then it is evident that the first impulse would not cause any tendency to motion in the point A, and the second would cause it to move in the direction of the tangent to a circle whose radius is AK, and whose plane is perpendicular to Ob, that is, in a direction perpendicular to the plane bOa, with a velocity $= e \,.\, OB \,.\, AK$. But in consequence of the two impulses, the line OP remains fixed in space, and since the body is rigid, the distance AL is invariable, and the direction of A's motion coincides with the tangent to a circle whose radius is AL, and whose plane is perpendicular to OP. The point A will therefore describe about L the circle SAS', with an angular velocity $= \dfrac{e \,.\, OB \,.\, AK}{AL} = e \,.\, OP$, (by similar triangles OPN, PAL.) And in the same manner, it may be shewn that every point in OA will describe a circle, in a plane perpendicular to OP, with an angular velocity $= e \,.\, OP$. Hence the whole line OA, and consequently the rigid body in which it lies, will turn about OP with an angular velocity $= e \,.\, OP$.

The same reasoning clearly holds if an interval occur between the communication of the impulses.

The velocities of the highest points Q' and R of the circles described by a point lying within the $\angle\, bOa$, are here supposed to be in the same direction, when resolved perpendicularly to a plane bisecting $\angle\, bOa$; and the rotatory motions are consequently said to be in the same direction.

If they are in opposite directions, we must take $OB' = OB$ (fig. 3.) on the other side of OA and complete the parallelogram, when OP' may he shewn as before to be the new axis, and $e.OP'$ the angular velocity of the body about it.

Whence it follows that the rotatory motions about different axes which intersect in any point are compounded in precisely the same manner as simple forces applied at the point.

And this similarity of composition is not confined to rotatory motions about axes which intersect; but what is very remarkable, it extends to rotatory motions about axes situated any how in space.

Thus rotatory motions about two parallel axes are compounded into a single one equal to their sum, about an axis parallel to them, which divides their distance in the inverse ratio of the component rotatory motions. (3)

(3) Let the axes aa', bb' (fig. 4.) be as before in the plane of the paper. Draw AB perpendicular to them, and let

$$\text{angular velocity round } aa' = e.BP,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad bb' = e.AP,$$

then as before,

Velocity of P upwards, in a direction perpendicular to the plane of the paper, due to the rotatory motion round aa', $= AP \times$ (angular velocity round aa')

$$= e.BP.AP.$$

Velocity downwards, due to the rotatory motion round bb',

$$= e.AP.BP.$$

And P, and similarly every point in pp', remains at rest, and therefore pp' becomes the new axis of rotation.

And the velocity of A perpendicular to AP, which is that due to the rotatory motion round bb', $= e \,.\, AP \,.\, AB$; therefore velocity of A round P, or velocity of the body's rotation round pp',

$$= \frac{e \,.\, AP \,.\, AB}{AP} = e \,.\, AB = e \,.\, (BP + AP),$$

$$= (\text{velocity of rotation round } aa')$$
$$+ (\text{velocity of rotation round } bb').$$

What is meant by saying that the rotatory motions are in the same direction is evident from the last note: that is, that the motions of Q' and R are in the same direction.

If these are in opposite directions, the resultant rotatory motion is equal to their difference, and the position of the axes is determined by the same laws as that of the resultant of two parallel forces acting in opposite directions. (4)

(4) If the rotatory motions are in opposite directions, the motion of r', (fig. 5.) the highest point of the circle described by a point P' round bb', will be in an opposite direction to that of Q', which we may suppose to be the same as in the last note.

Suppose the angular velocities to be unequal, and let that round aa' be the greater. In BA produced take a point P, such that

angular vel. round aa', which we may call ω_a, $= e \,.\, PB$,
.... bb', ... ω_b, $= e \,.\, PA$,

then it is evident that the motion of R' will be parallel to that of r', and the motion of P round bb' will be in the direction RPR', while the motion round aa' is in the direction $Q'PQ$; and therefore as before,

velocity of P downwards $= \omega_a \,.\, PA = e \,.\, PB \,.\, PA$,
.... upwards $= \omega_b \,.\, PB = e \,.\, PA \,.\, PB$;

therefore pp' is the new axis of rotation, and the velocity of rotation round pp', or ω_p,

$$= \frac{e\,.\,PB\,.\,AB}{PB} = e\,.\,AB = e\,PB - e\,PA = \omega_a - \omega_b.$$

We may therefore conclude that the resultant of two rotatory motions ω_a, ω_b, about parallel axes aa', bb' is a rotatory motion about an axis parallel to them in the same plane, which $= \omega_a \pm \omega_b$, according as the component rotatory motions are in the same or opposite directions; the distance of the axis from bb' being equal to BP

$$= BA\,.\,\frac{e\,.\,BP}{e\,BP \pm e\,AP} = BA\,.\,\frac{\omega_a}{\omega_a \pm \omega_b},$$

BA being measured in a positive direction.

If the motions are in opposite directions and $\omega_b > \omega_a$, BP will be negative, and P will lie on the other side of B.

If $\omega_a = \omega_b$, BP will be infinite, and every point will describe a circle of infinite radius. This case is considered in the next note. (*See Pritchard's Couples, Appendix*, Prop. C.)

If these two parallel and opposite rotatory motions are equal, they can never be reduced to a single one. They form in that case what may be called a *Couple of Rotatory Motions*: a rotatory motion *sui generis*, and which can never be reduced to a simple rotatory motion about any axis whatever. And in fact it is easy to see that the result of such a couple would be to give the body a simple motion of translation in space, in a direction perpendicular to the plane of the couple, and measured by its *moment*, that is, by the product of one of the rotatory motions and the distance between the parallel axes. (5)

(5) Let the plane of the paper be *perpendicular* to the parallel axes aa', bb', (fig. 6.) which meet it in the points A and B; and suppose two impulses to be communicated to the body which, separately, would generate angular velocities, each $= \omega_a$, round them in opposite directions. Take $AB = a$. Then it is clear that if for 1″ the motion round aa' were alone communicated, B would describe the arc $BB'' = a\omega_a$, and if the motion round aa' were then to cease and that round bb' to commence, A would describe an arc AA'' also $= a \,.\, \omega_a$. At this time therefore the position of the distance between the axes $(A''B'')$ is parallel to the initial position (AB) and the points of intersection (A) and (B) of the axes with the paper have moved in the direction BA through $A'A'' = a\,.\,\text{versin}\,\omega_a$. If we suppose the motions to be communicated in reversed order we shall have the new position of the distance $(A'''B''')$ still parallel to AB, while the points of intersection will have moved in the direction AB through $A'A''' = a\,.\,\text{versin}\,\omega_a$. And it is clear that the effects will be similar however small the intervals may be taken. If therefore the impulses be communicated together, the direction of the distance at any instant will be parallel to AB, and consequently the motion round bb', which tends to make A move in the direction AB, will tend (since AB may be considered as an incompressible rod) to make B move in the same direction, while an equal and contrary effect will be produced directly on B by the motion round aa'. Hence every point in AB and consequently every point in the body will move in a direction perpendicular to AB.

And if P be any point in AB,

P's velocity $= AP \,.\, \omega_a + BP \,.\, \omega_a$

$= a \,.\, \omega_a$; which is the moment of the couple.

A *Couple of Rotatory Motions* is therefore equivalent to a single force, applied at the centre of gravity of the body, in a direction perpendicular to the plane of the couple, and equal to the product of its moment by the mass of the body.

These couples may be transformed into others equivalent to them, may be turned about or removed at pleasure in their own planes or into planes parallel to their own, without any change being produced in the motion of the body. Their composition and decomposition follow exactly the same law with those of ordinary couples, and we may apply to them without exception the corresponding theorems. (6)

(6) If aa', bb', *in* the plane of the paper be the axes of a couple whose moment is $a \cdot \omega_a$, it is evident that the same motion will be produced in the point P, perpendicular to the plane, by a couple of rotatory motions round cc', dd', whose distance $CD = b$, with angular velocities each $= \omega_b$, if $a \cdot \omega_a = b \cdot \omega_b$. (*See Pritchard*, Prop. IV.)

Also if the axes be turned in the same plane into the positions ee', ff', parallel to each other the effect on P is the same. (v. *Pritchard*, Prop. III.)

And what is here true of P is true of every point in the body.

The propositions corresponding to (*Pritchard*, Prop. II. and Cor.) are self-evident.

The effect of any number of couples in the same or parallel planes is clearly equal to their sum, the angular velocities being taken with their proper signs.

If two couples lie in planes inclined to each other they may be transformed and removed, until their distances are equal and coincide with the intersection of the planes and with each other, (v. *Pritchard*, Prop. VI.) when the motions of translation which they produce in the point P, taken as before, may be compounded as in Elementary Dynamics. And the same will apply to the parallel motions of every other point in the body.

From the parallelogram of *Simple Rotatory Motions*, and the parallelogram of *Couples of Rotatory Motions* results the composition of any number whatever of rotatory motions, situated any how in space, and this general composition perfectly resembles the general composition ofe forces. (7)

(7) If $\angle\, bOa = 90^0$ (fig. 8.), we have from note (2) the rotatory motion round Op with an angular velocity $(\omega_p) = e\,.\,OP$, equivalent to

rotatory motion round Oa with angular vel. $(\omega_a) = e\,.\,Oa$,
.... Ob .... $(\omega_b) = e\,.\,Ob$;

$$\therefore\ (\omega_p)^2 = (\omega_a)^2 + (\omega_b)^2,$$

and $\omega_a = \omega_p \cos\alpha$, $\omega_b = \omega_p \sin\alpha$, where $\alpha = pOa = \tan^{-1}\dfrac{\omega_b}{\omega_a}$,

and similarly if Op be the diagonal of a parallelopiped, and α, β, γ, the angles which it makes with the three edges passing through O, the rotatory motion about it may be resolved into three rotatory motions about these axes with angular velocities,

$$\omega_p \cos\alpha, \quad \omega_p \cos\beta, \quad \omega_p \cos\gamma, \text{ respectively.}$$

We may therefore resolve any number of rotatory motions about axes passing through a point, into others about three rectangular axes passing through that point, which we may call the origin, and taking the sums about each axis, compound them again for a general resultant.

If any one of the axes do not pass through the origin we must suppose two opposite rotatory motions, each equal to the motion about it, to be impressed upon the body, about an axis parallel to it through the origine we shall then have a motion about the latter axis, and also a couple of rotatory motions to consider, in addition to any supposed to exist previously in the body. And any number of couples of rotatory motions impressed on the body may be resolved into equivalents in the three co-ordinate planes, and a resultant obtained as for the simple rotatory motions.

Thus, in the same way that any number of forces whatever may always be reduced to a single one passing through a given point and a single couple, any number of rotatory motions about different axes situated any how in space, may always be reduced to a single rotatory motion about an axis passing through a point, selected at pleasure, and a single couple of two equal and opposite rotatory motions about axes parallel to one another. And if, as in the case of forces, we wish to reduce them so as to leave nothing arbitrary, we may always assign a position for the point above-mentioned, such that the plane of the *couple* shall be perpendicular to the axis of the resultant rotatory motion, which may in this case be denominated the *Central Axis* of the couplesoof rotatory motions*.

Since a couple of rotatory motions is equivalent to a simple motion of translation of the body in

* See *Pritchard*, Prop. VII.

the direction of the *axis* of the couple, the above analysis ultimately reduces the whole motion to a rotatory motion about a determinate axis, and a simultaneous motion of translation in the direction of this axis. Which is, as we shall see farther on, the most general motion that a body can have in absolute space.

Rotatory Motion about a Point.

We now proceed to give an idea of the motion of a body about a point on which it appears to turn in every direction.

The motion of a body which turns on an immoveable axis being the only one of which we have a clear idea, it is to our notions thereof that we must endeavour to reduce the motion of a body turning any how about a fixed point.

Now it is shewn that this motion, whatever it may be, if considered only for a single instant, is none other than a simple rotatory motion about a certain axis passing through the fixed point, whose direction remains immoveable during this instant*.

Hence it is concluded that in the following instanteit is likewise a simple motion of rotation, but about another axis; and so on, from one instant to another, in such wise that the motion of the body may be considered as a succession of simple rotatory motions, of each of which the

* See Appendix.

mind has a distinct idea. It is in the same way that, in order to form a notion of the motion of a point in a curved line, we represent this point as describing successively the sides of an infinitesimal polygon inscribed in this curve. And the same is true of the instantaneous axis of rotation in the motion of a body, as of the tangent to a curve in the motion of a point which describes it.

Sensible Illustration of this Rotatory Motion.

Though the preceding analysis is exact, and I have laboured to render it clear, it seems to me that our idea of a body, turning about an axis which is perpetually changing, is still rather obscure. It is therefore desirable to render it clearer, and to present to the mind a distinct and sensible image of it.

Now I demonstrate in the most simple manner that, *the rotatory motion of a body about an axis which incessantly varies its position round a fixed point, is identical with the motion of a certain cone, whose vertex coincides with this point, and which rolls, without sliding, on the surface of a fixed cone having the same vertex.*

I mean to say that the moveable cone, supposed to have a rigid connection with the body, which it carries along with it, if made to roll on the other cone, which is fixed in absolute space, will cause the body to perform the precise motion with which it is supposed to be endued.

The line of contact of the two cones will be at every instant the axis about which the body turns for this instant, or as it is called, the *instantaneous axis*; whence we perceive that this axis is moveable at the same time in the body and in absolute space, describing in space the surface of the fixed cone, and in the interior of the body the surface of the moveable cone of which we have just spoken. (8)

(8) The most general motion of which a rigid body is capable about a fixed point O (fig. 9.) is manifestly such, that any point P in it, taken at pleasure, may be made to describe any given curve of double curvature lying on the surface of a sphere, of which O is the centre, and OP the radius, with a velocity varying as any function of its position.

The normal planes at successive points of such a curve of double curvature intersect in a conical surface*, whose vertex is O. Suppose this surface to be disconnected from the body; and let the body be rigidly connected with a plane which rolls upon the cone, both of them being in the interior of the body and perfectly rough. Let OR be the line of contact and $\angle POR = \alpha$, which will be the inclination of the osculating plane† at P to the rolling plane.

If now an angular velocity (ω) be communicated to the body about OR it may be resolved into two, one about an axis perpendicular to OP in the plane POR, which will cause P to move in the osculating plane, with an angular velocity $= \omega \sin \alpha$, and the other about OP, which being resolved again, part of it about an axis OQ, perpendicular to the rolling plane and therefore in the normal plane POR, will be destroyed

* Hymers' *Analytical Geometry*, Art. 64. † Ibid. Art. 63.

by friction, leaving finally an angular velocity about $OR = \omega \cos \alpha \sin \alpha$.

Let t be the small time which elapses before the plane comes into contact with the next axis of rotation, which will manifestly depend on the form of the cone; and suppose the *whole system* to turn about OR in an opposite direction with an angular velocity[†] β. Then $t.(\omega \sin \alpha \cos \alpha - \beta)$ will be the angle described by the rolling plane in space, and $t\omega \sin \alpha \cos \alpha$ in the system.

Now the inclination of the new position of the osculating plane to the rolling plane, that is, the new value of α, depends on the latter only[‡] while the new position of the osculating plane in space depends on the former. Hence the *relation* between the alteration of P's velocity in the osculating plane and the change in the position of this plane[‡] may be varied arbitrarily by varying β. But we may evidently give what value we please to β at any point, by introducing a moveable cone[‡] instead of a moveable plane, to which the body is rigidly attached, and supposing the system absolutely at rest; for the rolling plane revolving backwards in space would constantly touch such a cone. So that, conversely, we may assign such forms to the cones, and such a function of the position for the velocity of the moveable cone, as to make P describe any curve whatever; with a velocity depending in any manner on the position[†]

And this, I believe, is the greatest degree of clearness of which an idea so complicated and obscure, as that of the motion of a body which turns any how about a fixed centre, is susceptible. There is no motion of this nature which cannot be produced exactly, by making a certain cone roll on a fixed cone having the same vertex; so that if we figure to ourselves all the possible cones which can be made to roll in this manner upon one another, we have a faithful illustration of all the possible motions of a body round a point, on which it is at liberty to turn in every direction.

Also, if the rotatory motion to be considered were discontinuous, that is, if the axis of rotation, instead of changing its position by insensible degrees, were to leap abruptly from one position to the next through a finite angle, we could imitate equally well the motion of the body, by taking, instead of two cones, two pyramids having the same vertex, and causing one to roll on the other, so that the moveable pyramid turning on their common edge should bring into contact all its different faces, one after the other, with the faces respectively equal to them of the fixed pyramid.

If the motion of the body is given, it is clear that the two cones or pyramids are also given, as well as the velocity of rotation round the line of contact, and consequently the velocity with which the instantaneous axis traces at the same time the two surfaces. And reciprocally, if of these different quantities, which come under our consideration in discussing the motion, any three are given, we may say that the fourth is also, and that the motion of the body is thoroughly determined.

Thus the Earth turns in one day on its axis, while this axis describes in an opposite direction a right cone* about the axis of the ecliptic, with

* See Appendix.

a velocity measured by the retrograde motion (*Precession*) of the Equinoxes, and which amounts to about 50″ in a year; we may therefore determine in the case of the Earth the cone which, rolling on the former in the interior of the globe, would generate in the Earth a motion corresponding precisely to that which we observe in it. And it is easy to see that on the Earth the circumference of the circle which is the base of this little moveable cone, is to that of the base of the fixed cone, as a day to the period of a complete revolution of the equinoxes: which gives scarcely six feet for the little circumference which the instantaneous pole of rotation of the Earth describes every day on its surface. (This is on the hypothesis of an uniform diurnal Precession.) (9)

(9) Let P (fig. 10.) be the pole of the Earth, K the point where a line perpendicular to the plane of the ecliptic at the Earth's centre meets the surface. Then we may consider the centre of gravity as a fixed centre of rotation; and consequently PQR, the fixed circle in space which the pole describes, will have its circumference equal to $2\pi \times$ radius of Earth $\times$ sin (PK = Earth's obliquity), and the little circle which rolls upon it, and which is the path of the instantaneous pole on the Earth's surface, will have its circumference equal to the portion of PQR which is described in one day, that is

$$= \frac{\frac{1}{365}}{\frac{360 \times 60 \times 60}{50}} \times \text{radius of } \oplus \times \sin 23^{\circ}.28'$$

$= 5\frac{1}{2}$ feet nearly.

And every point in this little circle becomes successively in the course of a day the instantaneous pole of rotation. Hence we may find at any instant the actual position, both in space and on the Earth itself, of the instantaneous pole.

The most general Motion that a Body can have in Absolute Space considered.

From the simple idea of a mere motion of translation, which carries forward at every instant all the equal molecules of the body through small equal and parallel lines in space, and from the simple idea of the rotation of the body about an axis, which remains immoveable during this instant, results the complex idea of the most general motion of which a body is capable in absolute space. Nothing is more clear than this resolution of any kind of motion into two others which we can conceive perfectly, and which we may consider separately, since they are such that, if at every instant they were executed one after the other, every point in the body would be brought to the same place at which it arrives, by its natural motion, at the end of the instant of which we are speaking.

But we may have a curiosity to form for ourselves a notion of the real and single motion with which the body is endued, for the purpose of seeing, in some degree, the nature of the simultaneous curves which the different points describe, and which they can really describe at the same time without causing any change in the form of the body.

Now since a motion of translation may always be considered as a *Couple* of equal and opposite rotatory motions, it follows that the motion of a body, whatever it may be, can always be reduced to a simple rotatory motion about an axis passing through a point selected arbitrarily in space, and a certain couple of rotatory motions, whose plane will be in general inclined to this axis. But instead of taking a point at pleasure, we may always assign for it such a position, that the plane of the couple shall be perpendicular to the axis of simple rotation; and then the whole motion is reduced to a rotatory motion about a determinate axis, and a motion of translation in the direction of this axis. Whence it results that the motion is identical with that of an external screw which turns in the corresponding internal screw. All the points in the body therefore describe on concentric cylinders small arcs of *helices* which have all ~~the same~~ *furrow**. In the next instant it is a different screw, with another axis and a different *furrow*: and so on, whence we see the way in which are formed the simultaneous curves which all the points describe in space, and in which they move as in curvilinear canals, wherein we may suppose them to be inclosed.

Sometimes the *furrow* of this screw vanishes, and then the whole motion is reduced to a simple

* "*Pas*," the distance between two contiguous threads.

rotatory motion about the axis of the screw, which becomes what is called the *spontaneous axis* of rotation.

But in general the furrow of the screw does not vanish, and there is no spontaneous axis properly so called: that is, there is no straight line in the body, all the points in which remain immoveable for an instant. But there is always what we may call *a sliding spontaneous axis*, that is, a succession of points, forming a straight line, which have no motion other than a simple one of translation, in the direction of this line.

Such are the simplest notions and the clearest illustrations that we can form for ourselves of the motion of bodies. The mere motion of translation and the mere rotatory motion require no explanation to enable us to conceive them. Any motion whatever may always be reduced, and that in an infinite number of ways, to two such motions. And among this infinite number of reductions there is always one which furnishes the axis of rotation in the actual direction of the translation: so that, *the most general motion of which a body is capable is, as we before observed, that of a certain external screw which turns in the corresponding internal screw.*

After considering the actual motion of bodies in a point of view purely geometrical, I proceed to enquire into the forces capable of producing it, in order to determine conversely the motion due to any given forces; which is the natural object of the science of Dynamics.

Forces capable of producing a given Motion.

Whatever be the motion of a body there always exist forces capable of producing it. For at any instant during the motion of the body we may consider every molecule as if it were at rest, but acted on by a force capable of giving it the actual velocity which we suppose it to possess. Therefore an infinity of similar forces, applied individually to all the equal molecules of the body, are capable of producing in it the actual motion which we observe; and this too spontaneously, I mean to say even if these molecules have no mutual connexion, and consequently without causing any violent strain, which would tend to destroy their actual connection.

Such are the elementary forces capable of producing a given motion in a system of molecules, either free, or arbitrarily connected with each other.

But if these molecules constitute, as I here suppose, a system of invariable form, we may compound together all these elementary forces, and so replace them by a single force and a single couple, which will be equally capable of producing a given motion in the *solid body* under consideration. (10)

(10) This follows immediately from the theory of couples,* and includes d'Alembert's principle, which is, that "if any number of forces act upon a rigid system to produce motion, the elementary forces which indi-

* *Pritchard*, Prop. VII.

vidually propel every molecule of the system, and which are called the *effective* forces, are statically equivalent to these *impressed* forces." Since the system is rigid, we may reduce each set of forces to a single force applied at a given point, and a couple whose plane passes through this point. Suppose a set of forces, respectively equal and opposite to each of the *impressed* forces, to be applied at the same points; then these may be reduced to a single force, applied at the same point as before, and a couple whose plane also passes through this point. Now the force will clearly be equal to the resultant of the *impressed* forces, and in an opposite direction, and the couple will be equivalent to and in the same plane with the resultant couple of the impressed forces, and tend in an opposite direction. But such a set of forces would entirely prevent motion; therefore their resultant force and resultant couple must exactly balance the resultant force and resultant couple of the *effective* forces. Hence the impressed and effective forces are statically equivalent.

The impressed force at any one of the points of application may beemore than adequate to produce the actual motion of the molecule on which it immediately acts, and hence we might imagine a part of it to be *lost*. On the other hand at some points the impressed forces may be inadequate to produce the actual motion, and at others there may be no impressed forces at all; whence arises an idea of forces being *gained* at these points. It follows immediately from what has been said above that "the forces lost and gained are statically equivalent," and this is the original form in which the principle was stated.

Let us enquire therefore what are respectively the force and the couple which together correspond to a given motion.

And first, if the body has only a *mere motion of translation* in space, so that all the equal molecules have velocities equal, parallel and in the same direction, it is manifest that all the elementary forces capable of producing these velocities are also equal, parallel, and in the same direction, and consequently reducible to a single force, parallel and in the same direction, and equal to their sum, applied at the centre of gravity of the body. Hence we see conversely, that *the effect of any force, applied arbitrarily at the centre of gravity of a body, is to transport all the particles thereof in its own direction, with a velocity measured by the magnitude of the force divided by the mass of the body;* which is, so to speak, self-evident.

In the second place, if the body has only a *mere rotatory motion* about any axis, it is evident that the velocities, and consequently the forces by which the individual molecules are impelled, are all proportional to the distances of the molecules from this axis, and in directions always perpendicular to these distances, and to the axis of which we speak. Now these elementary forces are always reducible to a single force and a couple. But if the axis passes through the centre of gravity of the body the force vanishes, and the whole is reduced to a couple, whose plane is in general inclined to this axis. If in addition the axis is one of the three rectangular ones which exist in all bodies, and which are called the three *principal axes*, we find that the couple is perpendicular to this axis, and that it is measured by the product of the angular velocity by the *moment of inertia* of the body about this principal axis. Whence we conclude conversely, that *the effect of a couple acting on a body in a plane perpendicular to one of its three principal axes is to maheo the body turn about this axis, with an angular velocity equal to the moment of the couple divided by the moment of inertia of the body about this axis.* (11)

(11) Let a rigid system turn about an axis OG (fig. 11.) with an angular velocity ω, and let any point P in it, whose co-ordinates are $ON = x$, $NP = y$, $OG = z$, he the position of a molecule whose mass is m, PT being a tangent at P to the circle which it describes about OG.

Then m is impelled by a moving force equal to the product of its mass and velocity $= m \,.\, \omega \,.\, OP$, applied at the point P in the direction TP; which we may resolve into two others at the same point, one

$$= m\omega o OP \,.\cos TPN = m\omega \,.\, OP \,.\sin OPN = m\omega x$$

in the direction NP or Gy, and another $= m\omega y$ in the direction NO, or $= - m\omega y$ in the direction Gx.

Now we may suppose the former force to act on the body at any point N in the line of its direction. Let two opposite forces, each equal and parallel to it be applied at O; then we shall have a force at O in the direction $Gy = \omega m x$, and a *couple* whose moment is $\omega m x^2$ in the plane NOP; and the former will give us in the same manner a force at G in the direction $Gy = \omega m x$, and a couple in the plane zy whose moment is $\omega m x z$. Proceeding in the same way with the other force, we obtain, A force at G in the direction $Gx = -\omega m y$, a couple in the plane NOP which tends in the same direction with the former and whose moment is $= \omega m y^2$,

And a couple in the plane zx which tends in an opposite direction * to that in the plane zy, and whose moment $= -\omega m y z$ when estimated in the same direction. And the couples in the plane NOP may be removed into the plane xy. Hence the resolved parts of all the elementary forces may be summed up, and reduced to,

(i) Two forces applied at G; viz.

$\Sigma(\omega m . x) = \omega\Sigma(mx)$ in the direction Gy,

and $-\omega\Sigma(my)$ .... Gx.

(ii) Two couples; viz.

$\omega\Sigma(mxz)$ in the plane zy

$-\omega\Sigma(myz)$ .... zx,

tending to make the body turn about some axis in the plane of xy.

(iii) Two couples; viz:

$\omega(\Sigma m x^2)$ and $\omega(\Sigma m y^2)$ in the plane xy.

If G be the centre of gravity

$$\Sigma m x = 0, \quad \Sigma m y = 0,$$

therefore the two forces vanish.

$$\text{If } \Sigma m x z = 0, \text{ and } \Sigma m y z = 0,$$

the resultant couple lies wholly in the plane of xy, and is equal to

$$\omega\{(\Sigma m x^2) + (\Sigma m y^2)\} = \omega\{\Sigma m(x^2 + y^2)\}$$
$$= \omega . \Sigma(m . OP^2),$$

* See Note (2).

which is called the ***moment of inertia*** of the body about the axis GO.

For an investigation of the number and directions of the axes for which, when they are severally taken as the axis of z, the quantities Σmxz and Σmyz separately vanish, and fornthe method of determining the moments of inertia of bodies, see Appendix.

Now this one theorem gives us immediately the effect of a couple acting on a body in any plane whatsoever. For whatever be the couple, we can always decompose it into three others, respectively perpendicular to the three principal axes of the bodye and dividing each of them by the moment of inertia relative to its axis, we shall have the three angular velocities with which these three couples would tend to make the body turn about their respective axes. If then from the centre as origin, we take three lines in these directions to represent at once the axes and velocities of rotation, and complete the parallelopiped, we shall haveein the diagonal the axis and velocity of the rotatory motion to which the proposed couple gives rise at the first instant*. The simplicity of this operation is evident.

Motion of a Body arising from any given Forces.

Whatever be these forces they may always be reduced to a single one, passing through the centre of gravity of the body, and a single couple.

* See Note (7).

Now the effect of the force is a mere motion of translation in the direction of this force, and the effect of the couple is to impress on the body a rotatory motion about a certain axis passing through the centre of gravity, whose direction is determined by what we have just observed.

But it will be seen from what immediately follows that we can express much more clearly the direction of the instantaneous axis, relative to the plane of the couple which gives birth to it.

The Central Ellipsoid of a Solid Body.

Since the three components of the proposed couple give three rotatory motions which are respectively proportional to these couples, and inversely as the moments of inertia about the three axes, we perceive from geometrical considerations that the axis of rotation is in the direction of the diameter conjugate to the plane of the couple, in an ellipsoid whose axes are reciprocally proportional to the square roots of the moments of inertia about these axes. (12)

(12) Let A, B, C, be the moments of inertia about the principal axes, and let the centre of gravity G (fig. 11.) be taken as origin, and these three rectangular axes as the axes of x, y, and z respectively. Take in them the lines

$$a = \frac{1}{\sqrt{n \, . \, A}}, \qquad b = \frac{1}{\sqrt{n \, . \, B}}, \qquad c = \frac{1}{\sqrt{n \, . \, C}},$$

and about these three lines as axes describe an ellipsoid.

Let α, β, γ, be the angles which the axis of a couple, whose plane passes through the origin, makes with the co-ordinate axes; and let M be its moment. Then the equation to the plane of the couple is*

$$x \cos \alpha + y \cos \beta + z \cos \gamma = 0 \ldots\ldots (\text{i})†$$

and the resolved parts of it respectively perpendicular to the axes are

$$M \cos \alpha, \quad M \cos \beta, \quad M \cos \gamma;$$

and by the theorem laid down in page 29, if ω_a be the velocity generated by the couple $M \cos \alpha$ about the principal axis Ox,

$$\omega_a = \frac{M \cos \alpha}{A} = M n a^2 \cos \alpha;$$

and similarly, $\omega_b = M n b^2 \cos \beta$, $\omega_c = M n c^2 \cos \gamma$;

and if $GP = p$ be the resultant axis of rotation, meeting the surface of the ellipsoid in the point P, whose co-ordinates are x', y', z', and inclined at angles α', β', γ' to the axes,

$$x' = p \cos \alpha' = p \frac{\omega_a}{\omega_p}†, \quad y' = p \frac{\omega_b}{\omega_p}, \quad z' = p \frac{\omega_c}{\omega_p},$$

and the equation to the tangent plane at P, which is

$$\frac{x x'}{a^2} + \frac{y y'}{b^2} + \frac{z z'}{c^2} = 1,$$

becomes $x \cos \alpha + y \cos \beta + z \cos \gamma = \dfrac{\omega_p}{M n p} \ldots\ldots (\text{ii.})$,

wherefore the planes (i.) and (ii.) are parallel; that

* Hymers, Art. 3. Cor. 1.
† Note (7).

is, GP is the diameter conjugate to the plane of the couple*.

I suppose therefore an ellipsoid to be constructed about the centre of gravity of the body, having its three principal axes in the directions of the principal axes of the body, the squares of their lengths being reciprocally proportional to the moments of inertia of the body about them: and I may here remark that this ellipsoid will possess the remarkable property, that the moment of inertia of the body about any one of its diameters will be inversely as the square of the length of this diameter. (13)

(13) The equation to a plane perpendicular to GP (v. last note) is

$$\delta = x \cos \alpha' + y \cos \beta' + z \cos \gamma';$$

and if Q be any point in this plane, the distance of Q from $GP = \sqrt{GQ^2 - \delta^2}$; therefore moment ($P$) of the body round GP, which $= \Sigma$ mass of a particle $\times$ (distance from $GP)^2$,

$$= \Sigma m (x^2 + y^2 + z^2 - \delta^2).$$

And since the co-ordinate axes are by hypothesis principal axes,

$$\Sigma myz = 0, \qquad \Sigma mxz = 0, \qquad \Sigma mxy = 0;$$

$$\therefore\ P = (\Sigma mx^2) \sin^2 \alpha' + (\Sigma my^2) \sin^2 \beta' + (\Sigma mz^2) \sin^2 \gamma',$$

or since $\sin^2 \alpha' = 1 - \cos^2 \alpha' = \cos^2 \beta' + \cos^2 \gamma'$,

$$P = \cos^2 \alpha' . \Sigma m (y^2 + z^2) + \cos^2 \beta' . \Sigma m (x^2 + z^2)$$
$$+ \cos^2 \gamma' \, \Sigma m (x^2 + y^2)$$

* Hymers, Art. 24. Cor. 1.

$$= A \cos^2 \alpha' + B \cos^2 \beta' + C \cos^2 \gamma'$$

$$= \frac{1}{n a^2} \cos^2 \alpha' + \frac{1}{n b^2} \cos^2 \beta' + \frac{1}{n c^2} \cos^2 \gamma'$$

$$= \frac{1}{n a^2} \cdot \frac{x'^2}{p^2} + \frac{1}{n b^2} \cdot \frac{y'^2}{p^2} + \frac{1}{n c^2} \cdot \frac{z'^2}{p^2}$$

$$= \frac{1}{n p^2}, \quad \text{since } \frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1.$$

Now whatever be the form and constitution of a body, it has always a centre of gravity and three principal axes, as distinctly determinate as those of any homogeneous regular solid *. We may therefore always conceive such an ellipsoid as the above to be constructed in it, which has the triple advantage, of placing before our eyes the centre of gravity and principal axes, of giving us all the moments of inertia which we may have to take into consideration (since the moment about any diameter is expressed in a simple function of its length), and of offering us an easy method of determining the position of the axis of instantaneous rotation, relative to the plane of the impressed couple. Now it is this ellipsoid, the consideration of which will throw great light on the theory of rotatory motion, that I shall for the future denominate the *central Ellipsoid*.

* See Appendix.

The Motion of a Body arising from any given Couple clearly expressed.

Suppose that a body at rest is struck by a couple in any plane drawn through the centre of gravity, which may therefore be considered a diametral plane of the central ellipsoid: we have just seen that the instantaneous axis of rotation to which the couple gives rise is the diameter *conjugate* to the plane of the couple: and the angular velocity will evidently be measured by the moment of the couple, resolved perpendicularly to this diameter, and divided by the moment of inertia of the body about the same diameter. (14)

(14) This may be proved strictly from the preceding notes. For if θ be the angle between the instantaneous axis and the axis of the couple,

$$\cos\theta = \cos\alpha \,.\, \cos\alpha' + \cos\beta \,.\, \cos\beta' + \cos\gamma\cos\gamma'$$

$$= \frac{\omega_a}{Mna^2}\cdot\frac{x'}{p} + \frac{\omega_b}{Mnb^2}\cdot\frac{y'}{p} + \frac{\omega_c}{Mnc^2}\cdot\frac{z'}{p}$$

$$= \frac{\omega_p}{Mnp^2}\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2}\right),$$

$$\therefore\ \omega_p = Mnp^2\cos\theta = \frac{M\cos\theta}{p}.$$

It is manifest also from note (10) that the *impressed* couple M is equal to, and in the same plane with, the resultant couple of the *effective* forces; that is, the resultant of the two sets of couples (ii) and (iii) in note (11).

And resolving M parallel and perpendicular to the plane of the couples (iii), we shall have

$$M \cos\theta = \text{resultant of (iii)} = P \,.\, \omega_p,$$

the same equation as before.

Likewise, $M \sin\theta =$ resultant of the couples (ii).

Hence the axis of an impressed couple can never be the corresponding axis of instantaneous rotation, unless it coincides with a principal axis of the body. For if $\theta = 0$, no part of M can be employed to produce the couples (ii), which form a necessary part of the *effective* forces, requisite about any axis not a principal one to give to each molecule its proper motion. The theorem (page 29) expressed by $M = P \,.\, \omega_p$ only holds therefore for a principal axis.

Since a couple may always be removed into any plane parallel to its own, without causing any change in its effect on the body, we may always suppose the plane of the impressed couple instead of being drawn through the centre, to touch the surface of the ellipsoid: and then we may say, that

If a body is struck by a couple situated in any plane which touches the central ellipsoid of the body, the instantaneous pole of the rotatory motion to which the couple gives birth is the point of contact. And conversely that

If a body turns on any diameter of its central ellipsoid, the couple actually impressed on it is in the tangent plane at the pole.

Which appears to be one of the simplest theorems in Dynamical Science, on the difficult and obscure theory of rotatory motion.

We proceed to enquire how the rotatory motion changes from one instant to another, and to trace throughout its whole course the motion of the body.