CHAPTER V.

RELATIVE VELOCITIES IN MECHANISMS.

§ 14. RELATIVE LINEAR VELOCITIES

WE have so far considered motion only as change of position, entirely without reference to the time occupied by the change, that is, to the velocity of the different points of the body while moving; and we have seen that there are many kinematic problems which can be treated entirely without consideration of velocity. Connected with velocity, however, there are two distinct sets of problems which we have to examine, and one of these we can now take up. The absolute velocity of any point in a machine, as well as the changes in that velocity, depend, as we shall see presently, upon the forces acting upon the different parts of the machine. With these we have not at present anything to do. But the *relative velocities* of different points in the machine at any given instant can be determined by purely geometric considerations, so that we have already the means of dealing with them. We have seen that at each instant every body[1] in a machine or mechanism is virtually turning about some particular point, and have seen, further, how to find that point. **Every link of the machine, therefore, is simply in the condition of a wheel turning**

[1] Limiting ourselves to *plane motion*; see end of § 2.

about its axis, or a lever vibrating on its fulcrum, and this no matter how complex in appearance, or even in reality, the connection between the different parts of the machine may be. But in such a case it is obvious that the velocities of the different points must be simply proportional to their distance from the centre of rotation, that is proportional to their real, or virtual, radii or "leverage." The velocity of any one point being then known, the determination of the velocities of the others becomes a mere matter of finding the virtual centre and the distances of the various points from it. And even without knowing the *absolute* velocity of any point the same method gives us the *proportionate* velocities of all the points, quite independently of their absolute velocities. We must now look at this somewhat more in detail, especially in reference to *angular* as well as *linear* velocity.

When a body is turning about any fixed axis its motion is characterised by two conditions: (i.) the angular velocity of every point in it is equal, and (ii.) the linear velocities of its different points are proportional to their radial distances from the fixed axis, the linear velocities of points at equal distances from this axis being therefore equal. These conditions being characteristic of rotation simply, without reference to whether it occur for a short or a long time, are as entirely applicable to rotation about a virtual as about a permanent axis or centre. The difference is merely that in the former case the results obtained apply merely to one position of the body, while in the latter they apply equally to all its positions. We have seen that the motion of every link in a mechanism relatively to every other may at any instant be considered as a simple rotation about some point in that other. Hence it follows that at any instant every point in a link has the same angular velocity—that it describes, that is, equal angles in equal times.[1] It follows also that the linear velocities of different points in any link vary in direct proportion to the virtual radii of those points. Take Fig. 35 as an example, supposing d to be fixed, and

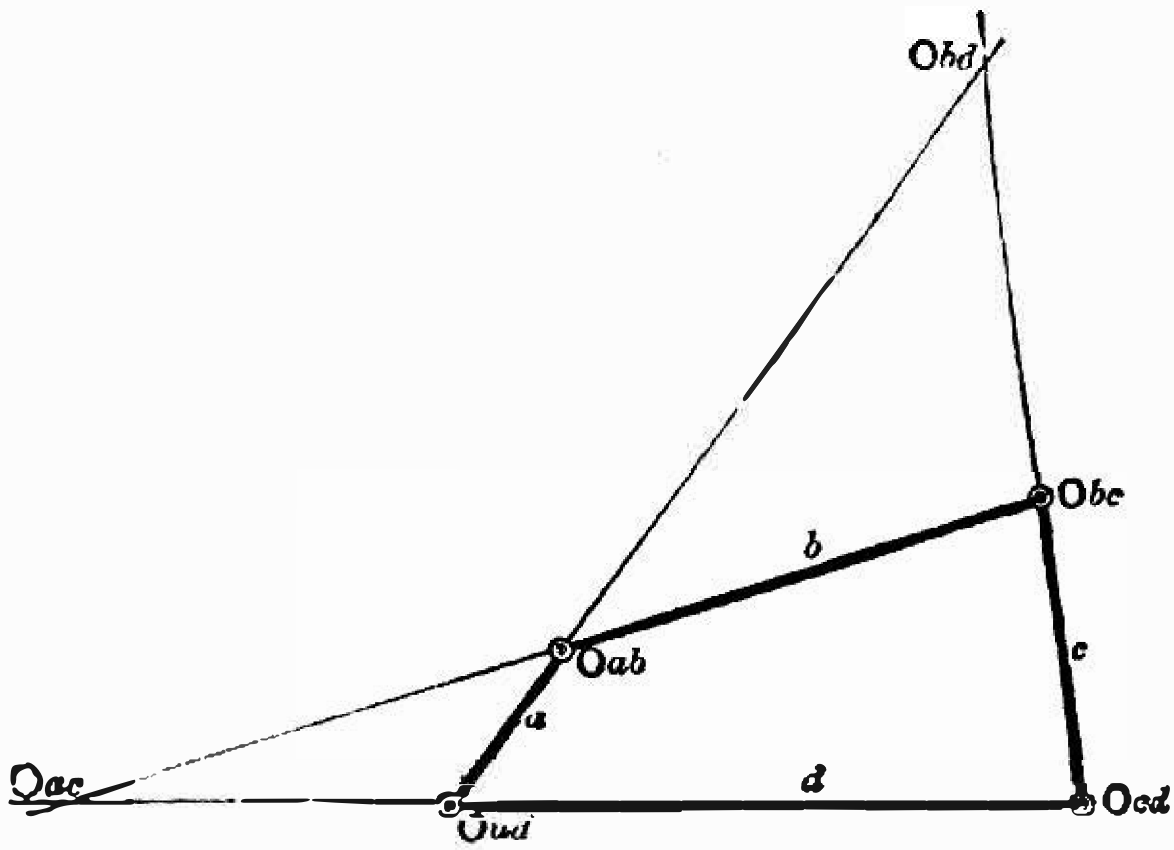

FIG. 35.

the motions of the other three links observed relatively to it. Every point in a is, at the instant, turning with the same angular velocity about O_{ad}, every point in b with the same angular velocity about O_{bd}, and every point in c with the same angular velocity about O_{cd}. Further, a point in a at any given distance from O_{ad} moves with just half the linear velocity of a point in a twice as far from O_{ad}, and with double the linear velocity of a point half its distance from O_{ad}, and similarly with the other links, whether the centres about which they are turning be permanent or virtual.

As we have seen, this makes it an extremely simple matter to find the velocity of all the points in any link if

[1] More fully that all the points *would* describe equal angles in equal times if they continued to move with the velocities which they have at the instant of observation.

only that of one point be known. Suppose, for instance, that the velocity of the point A_1 (Fig. 36) be given, to find that of A_2, both points belonging to the link *a*. Arithmetically it might be found by measuring $\overline{O_{ad}A_2}$ and $\overline{O_{ad}A_1}$, to any scale, and multiplying the given velocity by the ratio between them, *i.e.* by $\dfrac{\text{virt. rad. } A_2}{\text{virt. rad. } A_1}$. We shall

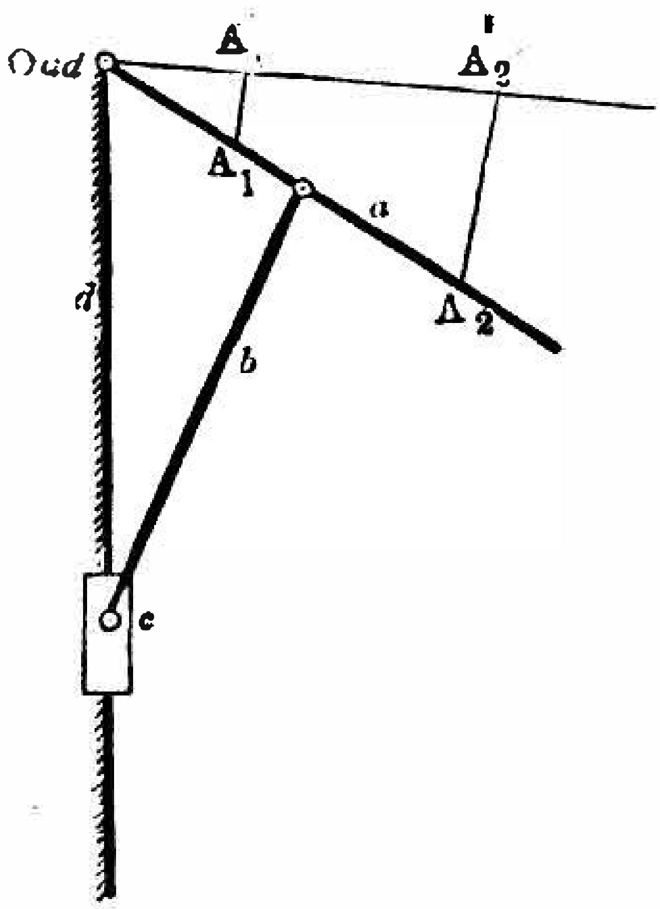

FIG. 36.

find it often more convenient, however, and it involves less measurement and no arithmetical multiplication, to solve the problem by a construction, as follows: Set off $A_1A'_1$ through the point A_1 in any convenient direction, to represent the given velocity of that point on any scale. Through O_{ad} draw a line through A'_1, and through A_2 a line parallel to $A_1A'_1$, calling the join of these lines A'_2;—the segment $A_2A'_2$ represents the velocity of A_2 on the same scale as that on which $A_1A'_1$ represents that of A_1. For the ratio

$$\frac{A_2A'_2}{A_1A'_1} = \frac{O_{ad}A_2}{O_{ad}A_1} = \frac{\text{virtual radius of } A_2}{\text{virtual radius of } A_1}.$$

Fig. 37 shows another construction, and one often more convenient than the foregoing, for solving a similar problem. Let $B_1 B_2$ be two points of a link b, and let $B_1 B'_1$ be the known velocity of B_1, to find that of B_2. Join both points to the virtual centre of b relatively to the fixed link, viz. O_{bd}.

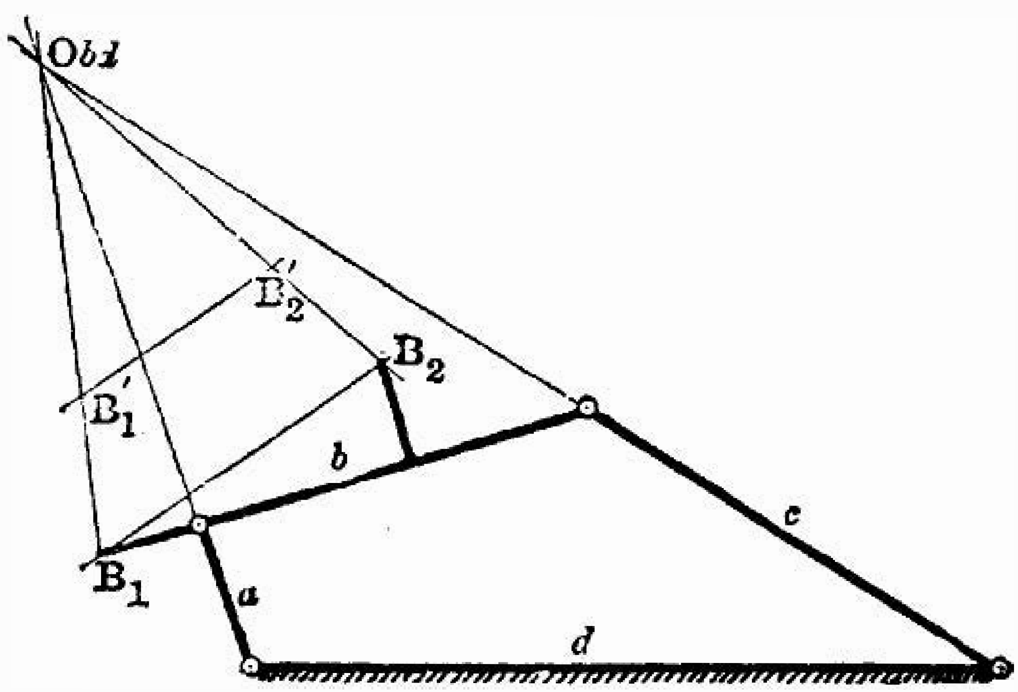

FIG. 37.

Join also B_1 and B_2, set off $B_1 B'_1$ along the radius of B_1 and draw $B'_1 B'_2$ parallel to $B_1 B_2$. Then $B_2 B'_2$ represents the linear velocity of the point B_2 on the same scale as that used in setting off $B_1 B'_1$. The proof is the same as before, simply that (by similarity of triangles)

$$\frac{B_1 B'_1}{B_2 B'_2} = \frac{O_{bd} B_1}{O_{bd} B_2} = \frac{\text{virtual radius } B_1}{\text{virtual radius } B_2}, \text{ as was required.}$$

It should be always most distinctly remembered that the bodies which are represented in our figures by straight links may be of any form whatever (see p. 67). We shall find that we have very often to do with points like B_2, Fig. 37, not lying at all on the axes of the bodies to which they belong. It should be noticed also that the line $A_1 A'_1$, &c., Fig. 36, were not set off in the direction of motion of A_1, &c., but in any direction that happened to be convenient.

We have compared the linear velocities of points of one and the same link,—but we can in just the same way compare the velocities of points in different links, or find the velocities of such points, if that of any one point be given. We do this by help of the theorem which we have already so often utilised, that the virtual centre of any link relatively to any other is a point common to both,—a point which has the same motion to whichever of the links we suppose it to belong. Let the velocity of a point A_1

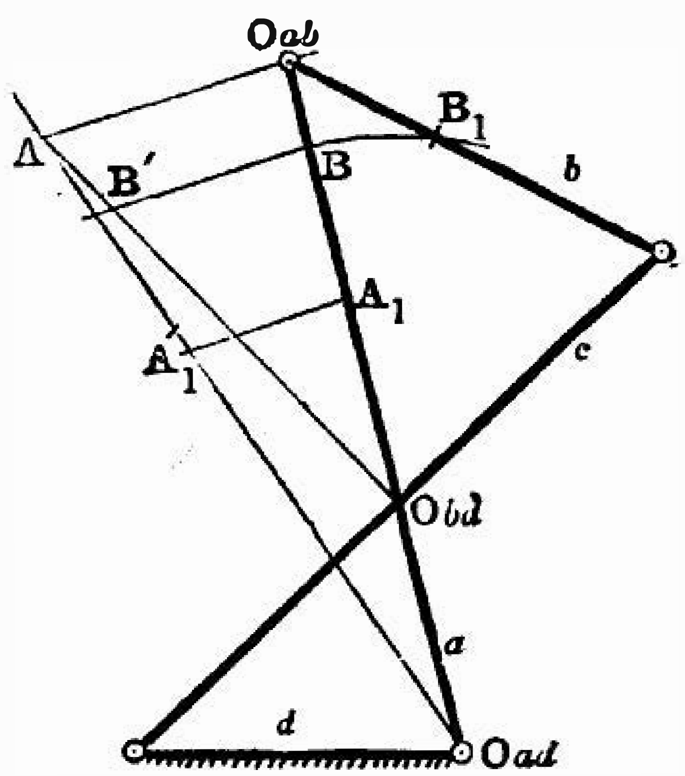

FIG. 38.

on the link a, for instance, be given;—to find from it the velocity of a point B_1 on the link b. The process is simply to find first the velocity of the common point of a and b as a point of a, and then treating it as a point in b to find from it the velocity of B_1. The necessary construction is shown in Fig. 38. $A_1A'_1$ is drawn to scale in any convenient direction for the velocity of A_1; by the former construction $O_{ab}A$ represents on the same scale the velocity of O_{ab} considered as a point of a. But this point has the same velocity as a point of b, so that by joining A to O_{bd} and carrying the radius of B_1 round to B, as in the figure, we get BB' for the velocity of B_1, to be measured on the same scale as before.

The construction applies equally to opposite as to adjacent links. To find, for instance, the velocity of the point C_1 in c, having given the velocity of A_1 in a as before, we should proceed as in Fig. 39, finding the velocity of

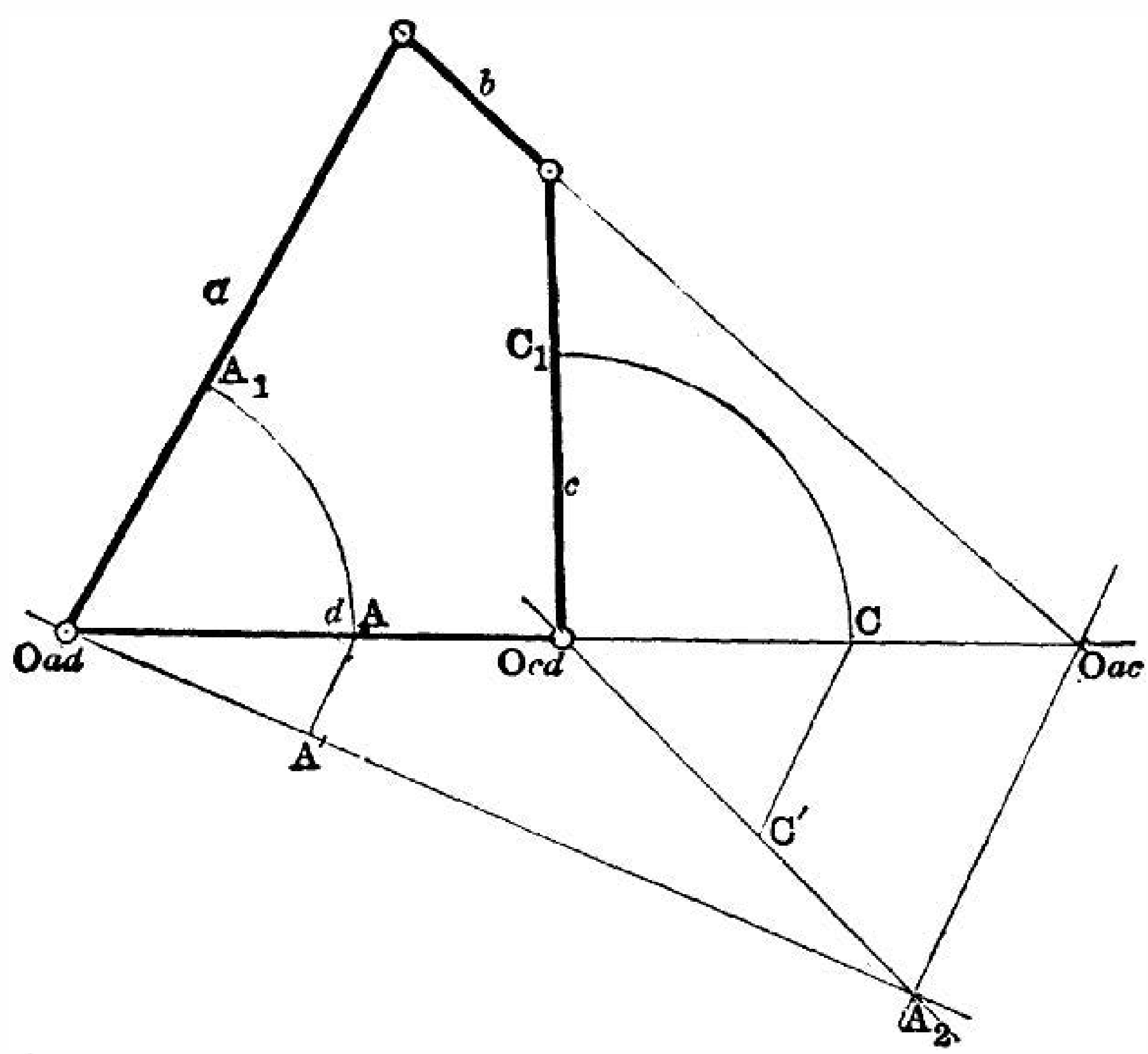

FIG. 39.

the point O_{ac} as a point in a, and then, treating it as a point of c, obtaining by its help the velocity of C_1. For convenience' sake we carry A_1 round to the line which is the axis of d, then setting off AA' as before, we obtain the line $O_{ac}A_2$ as the velocity of the point O_{ac}. Joining A_2 to O_{cd}, and carrying C_1 over to the axis of d (as we had previously done with A_1), we can at once draw CC' parallel to AA', and representing on the same scale the velocity of C_1.

Although the constructions of Figs. 37, 38, and 39 are very easy, both to work and to prove, they are not always the most convenient for practical purposes, mainly on account of the fact that it often happens that such points as O_{bd}, Fig. 37, or O_{ac}, Fig. 39, are at inconveniently great distances, often entirely beyond the limits of a drawing-board. There is never, however, any difficulty in dispensing with the actual construction of a line to such a point. Fig. 40, a very useful construction, shows how easily this

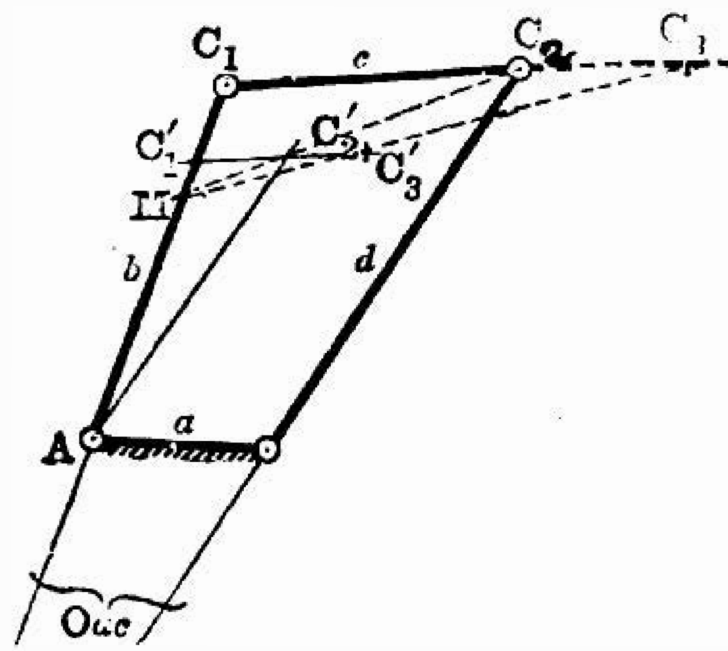

FIG. 40.

can be done. Let it be here required to find the velocity of the point C_2, that of C_1 being given. The point O_{ac} is inaccessible, but we have seen that it is sufficient for our purposes to have *any* two lines parallel to the virtual radii of C_1 and C_2, and not necessarily those radii themselves, with their (here) inaccessible join. If, therefore, we draw through O_{ab}, here the point A, a line parallel to d, we have at once such lines as we want in the most direct fashion. Setting off AC'_1 along b to represent the given velocity of C_1, and drawing $C'_1C'_2$ parallel to C_1C_2, then AC'_2 is the required velocity of C_2. For the triangles $O_{ac}C_1C_2$ and $AC'_1C'_2$ are similar, and hence $\dfrac{AC_1}{AC'_2} = \dfrac{\text{virtual radius } C_1}{\text{virtual radius } C_2}$, which is all that is required.

If the points C_1 and C_2 were both points the directions of whose virtual radii were known, as in the figure, it would be still simpler to proceed as in Fig. 41, where $C_1C'_1$ again represents the velocity of C_1, and $C'_1C'_2$ is drawn parallel to C_1C_2. Here $C_2C'_2$ obviously gives the velocity of C_2,—it is unnecessary to go through the proof. But we

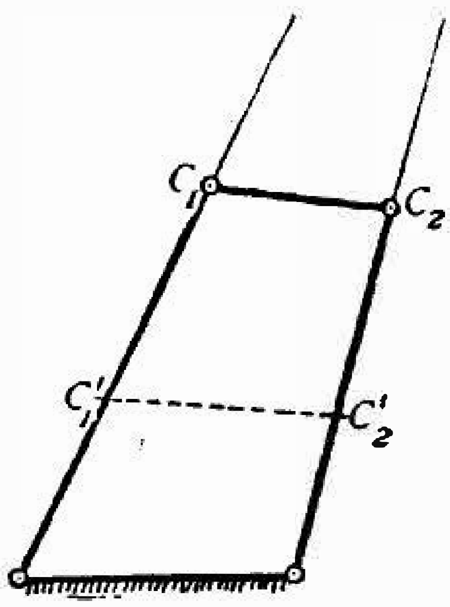

FIG. 41.

often have to deal with a point such as C_3 in Fig. 40, whose virtual radius is not a line in the mechanism, and is therefore not directly known, unless we can draw a line to the virtual centre, which is here supposed impossible. In this case we may conveniently proceed, as in Fig. 40, thus :—Find the join of the line $C_2C'_2$ with the link b, say M, and draw MC_3, which is cut by $C'_1C'_2$ in C'_3. The distance AC'_3 represents the velocity of C_3. The details of the proof may be left to the student.[1]

If in a case such as Fig. 39, the point O_{ac} be inaccessible, it can easily be dispensed with by the construction shown in Fig. 42. Here O_{cd} is joined to S, and a line drawn parallel

[1] It may be noticed that this construction gives one very easy way of drawing a line from a given point (as C_3), through an inaccessible point given as the join of two lines, b and d, on the paper. For of course a line through C_3 parallel to C'_3A would lie in the required direction.

to d in any convenient position, cutting the three lines which meet in S in the points 1, 2, and 3. The ratios of the distances between these points are exactly the same as between the three points O_{ad}, O_{cd}, and O_{ac}, which lie upon d, and it is only these ratios, not the points themselves, which we essentially require. The construction given in Fig. 39 may therefore be worked equally well by substituting 1 for O_{ad}, 2 for O_{cd}, and 3 for O_{ac}, and at the same time taking A' for A and C' for C,—in other respects working

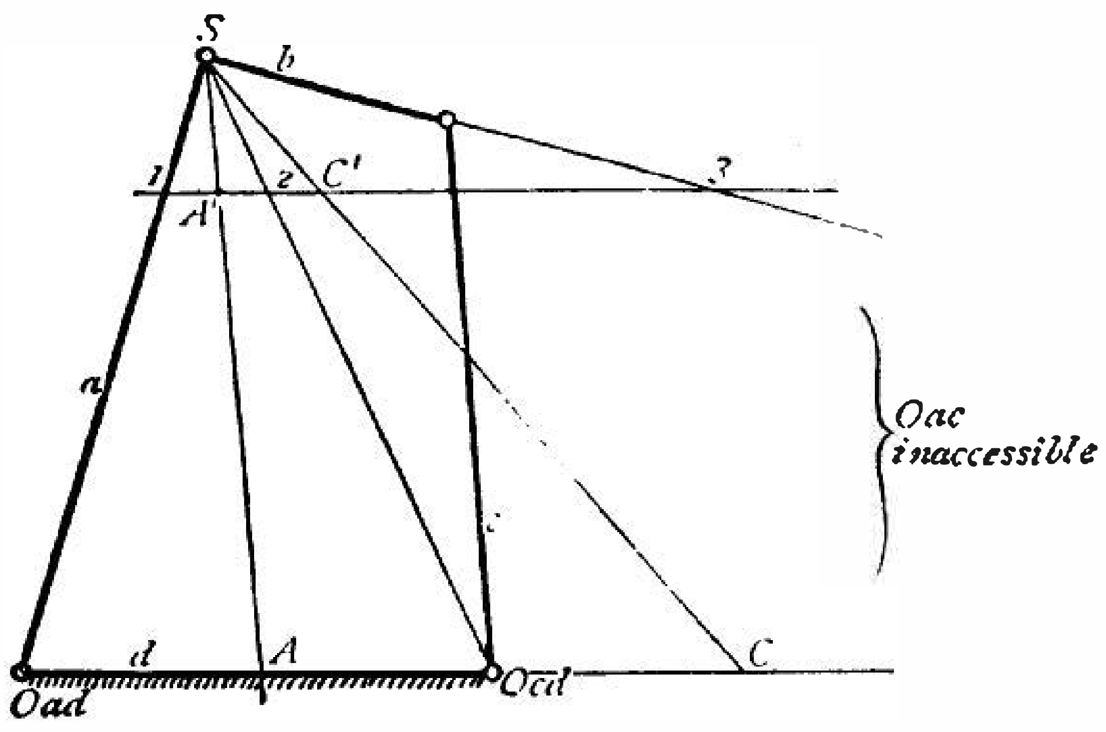

FIG. 42.

exactly as before. It can easily be seen that in effect this is nothing more than taking a different and more convenient *scale* for part of the construction. These constructions for getting over difficulties caused by inaccessible points are only examples of many that can be used, but are themselves quite generally useful, and are besides sufficient to indicate to the student the sort of procedure to be adopted in any particular case. Examples of other, more or less similar, constructions will be found further on, in connection with problems where they are required.

The whole matter which we have gone over in this section may now be summed up. Our problem has been: Given the linear velocity v_1 of any point A of a link a in a mechanism having plane motion, to find the simultaneous linear velocity v_2 of any point C of any other link c of the same mechanism, the fixed link being (say) d. Finding first the three virtual centres O_{ac} (which we may call O), O_{ad}, and O_{cd}, we have found that

$$\frac{\text{vel } C}{\text{vel } A} = \frac{v_2}{v_1} = \frac{OO_{ad}}{AO_{ad}} \times \frac{CO_{cd}}{OO_{cd}} = \frac{OO_{ad}}{OO_{cd}} \times \frac{CO_{cd}}{AO_{ad}}.$$

Put into words this is equivalent to saying that **the velocity of C is to that of A directly as the virtual radii of those two points relatively to the fixed link, and inversely as the virtual radius in c and in a of the common point (O_{ac}) of those bodies.**

If the two points belong to the same link, the ratio $\dfrac{OO_{ad}}{OO_{cd}}$ goes out, and we have simply that the velocities of the two points are proportional directly to their virtual radii. Here, however, one special case requires looking at. If both the points belonged to such a body as the link c in Fig. 36, their virtual radii,—no matter what their position in the body,—would always be equal. For the virtual centre of c relatively to d is a point at infinity, the distance of which from all points in our paper must be taken to be the same. Hence if the virtual centre of a body be at infinity, *i.e.* if it have only a motion of translation, all its points are moving with equal velocities. Exactly the same thing is true in reference to the link b in Fig. 43. In this mechanism, the **parallelogram** or **double-crank**,[1] opposite links are made equal, *i.e.* $b = d$, and $a = c$. Opposite links are therefore

[1] As to important properties of this mechanism see further, §§ 54 and 55.

always parallel, and their join is always at an infinite distance;—the points O_{bd} and O_{ac} are at infinity for all possible positions of the mechanism. Whichever link, therefore, is fixed, all the points of the opposite link are moving at any instant in the same direction and with the same velocity. The difference between the case of the link *c* in Fig. 36 and that of *b* in Fig. 43 is that the virtual centre of the former

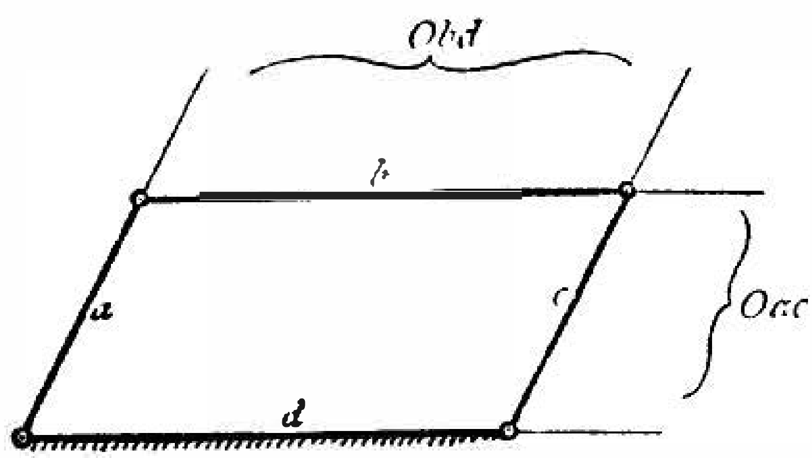

FIG. 43.

is a *permanent* centre, while that of the latter is only an *instantaneous* one. In the former case not only are all points moving in the same direction at any one instant, but this direction remains unchanged from instant to instant, whereas in Fig. 42 the direction of motion of *b* changes with every change in its position, although in any one position all its points are moving in the same direction. The difference is in essence precisely the same as that between the rotation of such links as *a* and *b* in Figs. 36 or 37. The motion of each link is at each instant a rotation about some one point. But in the case of *a* the rotation is always about the same point, in the case of *b* about a point which changes with every change in the position of the link.

§ 15. RELATIVE ANGULAR VELOCITIES.

Just as linear velocity may be expressed in different units,—as a velocity of a foot, a metre, or a mile per unit of time,—so angular velocity is a quantity measured by more than one standard.[1] The unit most commonly occurring in connection with engineering questions is *one revolution per unit of time*, the latter being generally a minute. Thus a shaft is said to have an angular velocity of 30 if it be turning at the instant at such a rate as would, if uniformly continued for one minute, cause it to make 30 complete turns in that time. To find the linear from the given angular velocity of the point in this case it is necessary simply to multiply the latter by the radius of the point and by 2π, that is, by the length of the circumference of the circle in which the point is moving. This assumes, of course, that the units of distance and of time are the same for both linear and angular velocities. For mathematical purposes the unit of angular velocity is generally taken as motion through an arc equal in length to its own radius in a unit of time. This arc subtends an angle of $\left(\frac{360}{2\pi}\right)$, or 57·3 degrees nearly, so that an angular velocity of 30 would represent, on this scale, a motion through $(30 \times 57{\cdot}3)$ degrees, or about 4·77 complete turns per unit of time. To convert angular into linear velocities on this scale the former have only to be multiplied by the radius. To convert angular velocities expressed in the former standard, therefore, to the latter, they must be divided by 2π, and *vice*

[1] The principal questions relating to linear and angular velocities are discussed in Chapter VII. What is said in the present section is not intended to do more than make clear the numerical relations of the units used as far as is necessary for the constructions given.

versâ, the time-unit being supposed the same in both cases. For general scientific purposes the second is the most convenient unit of time, but for many engineering problems the minute is preferable. For angular velocities expressed as number of revolutions, for instance, the minute is almost invariably made the time-unit.

There is obviously no more difficulty in solving problems connected with relative angular velocities than we have found in connection with relative linear velocities. It has only to be remembered, in addition to the characteristics of pure rotation already mentioned, that if two points of different bodies have the same radius, and have equal linear velocities, their angular velocities are also equal; and that otherwise (*i.e.* if the points have *unequal* linear velocities), their angular velocities are directly proportional to their linear velocities. If the two points have the same linear velocities but different radii, their angular velocities are inversely to their radii. In general, therefore, the angular velocities of two points in different bodies are proportional directly to their linear velocities and inversely to their radii. But as all points in a body must have, at each instant, the same angular velocity, we may say, even more generally, that **the angular velocities of any two bodies having plane motion are proportional directly to the linear velocities of any two of their points having the same radius, and inversely to the radii of any two of their points having the same linear velocity**, and in the general case to the ratio $\frac{\text{linear velocity}}{\text{radius}}$ for any two of their points whatever. We may put this down in symbols as follows:—calling a the linear velocity of any point A of a body α, and b that of any point B of another body β, the radii of the points (instantaneous or permanent) being r_a and r_b respectively. Expressing angular velocities according to the second standard given on p. 95, we have—

$$\text{Ang. vel. } \alpha = \frac{a}{r_a}$$

$$\text{Ang. vel. } \beta = \frac{b}{r_b}$$

$$\text{If } r_a = r_b, \text{ therefore, } \frac{\angle \text{ vel. } \alpha}{\angle \text{ vel. } \beta} = \frac{a}{b};$$

$$\text{If } a = b, \ \frac{\angle \text{ vel. } \alpha}{\angle \text{ vel. } \beta} = \frac{r_b}{r_a};$$

$$\text{Or generally, } \frac{\angle \text{ vel. } \alpha}{\angle \text{ vel. } \beta} = \frac{a . r_b}{b . r_a}:$$

these three equations expressing the three conditions supposed above.

It remains only to show how, by the aid of these relations, we can find the angular velocity of any link in a mechanism having given that of any other link, or, in other words, the proportionate angular velocities of any two links. This we can always do by the method of virtual centres, generally in several ways. Taking the mechanism Fig. 44, let us compare the angular velocities of the links *a*, *b*, and *c* relatively to *d*, the angular velocity of *a* relatively to *d* being given. Comparing first *a* and *b* relatively to *d* we can proceed as follows:—*a* and *b* have a common point, the point O_{ab}; this point is therefore a point in each link which has the same linear velocity relatively to *d*. The angular velocities of *b* and *a* ares therefore inversely proportional to the virtual radius in each of them of the point O_{ab}. To solve the problem by construction draw *any line* through O_{ab} and make the segment $O_{ab}A$ equal on any scale to the given angular velocity of the link a. Then through O_{ad} draw a line parallel to the join of O_{bd} and A (which need not itself be drawn, but which is drawn for distinctness' sake in the

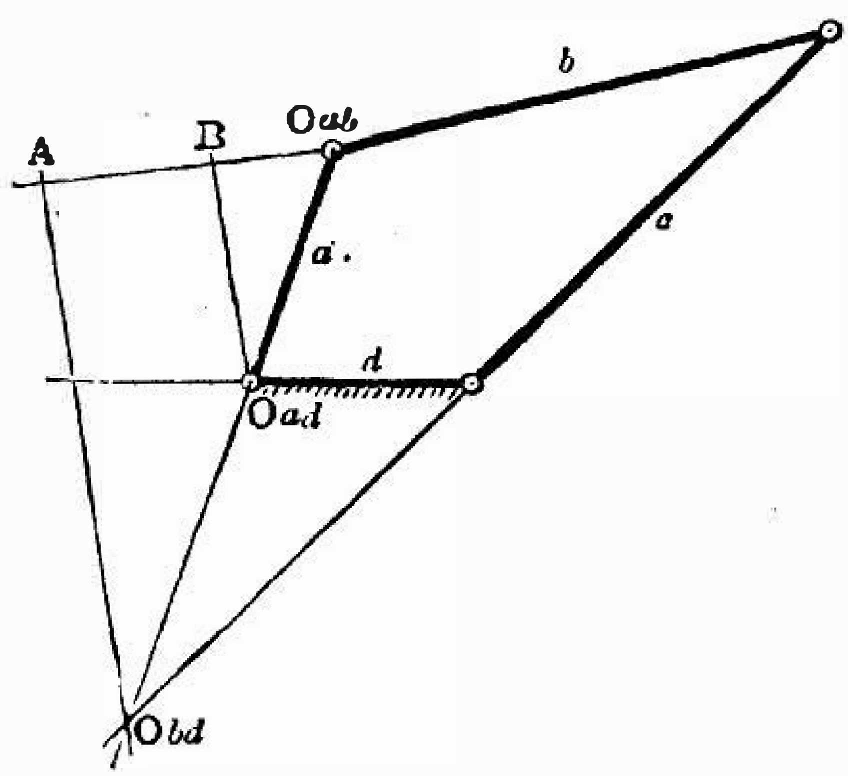

FIG. 44.

figure) and cutting the first line drawn in B. $O_{ab}B$ represents the angular velocity of the link b on the same scale as that on which $O_{ab}A$ represents that of a. The proof is simply that the triangles $O_{ab}A\ O_{bd}$ and $O_{ab}B\ O_{ad}$ are similar, and that therefore the ratio

$$\frac{O_{ab}A}{O_{ab}B} = \frac{O_{ab}O_{bd}}{O_{ab}O_{ad}} = \frac{\text{virtual radius of } O_{ab} \text{ as a point of } b}{\text{virtual radius of } O_{ab} \text{ as a point of } a}$$

exactly as required.

The construction must always have the same simplicity as in this case, for the problem always concerns the comparison of the motion of two bodies relatively to a third, and the three essential points used in the construction are the three virtual centres of these three bodies taken in pairs (here O_{ab}, O_{bd} and O_{ad}), and these points invariably, as we have seen, lie in one line.

To avoid confusion we may illustrate the other part of the problem, viz., the finding of the relative angular velocities of c and a, by another figure (Fig. 45). This case is one which has more often direct application in practice than Fig. 44. Proceeding precisely as before we take their common point O_{ac}, draw through it any line whatever on which

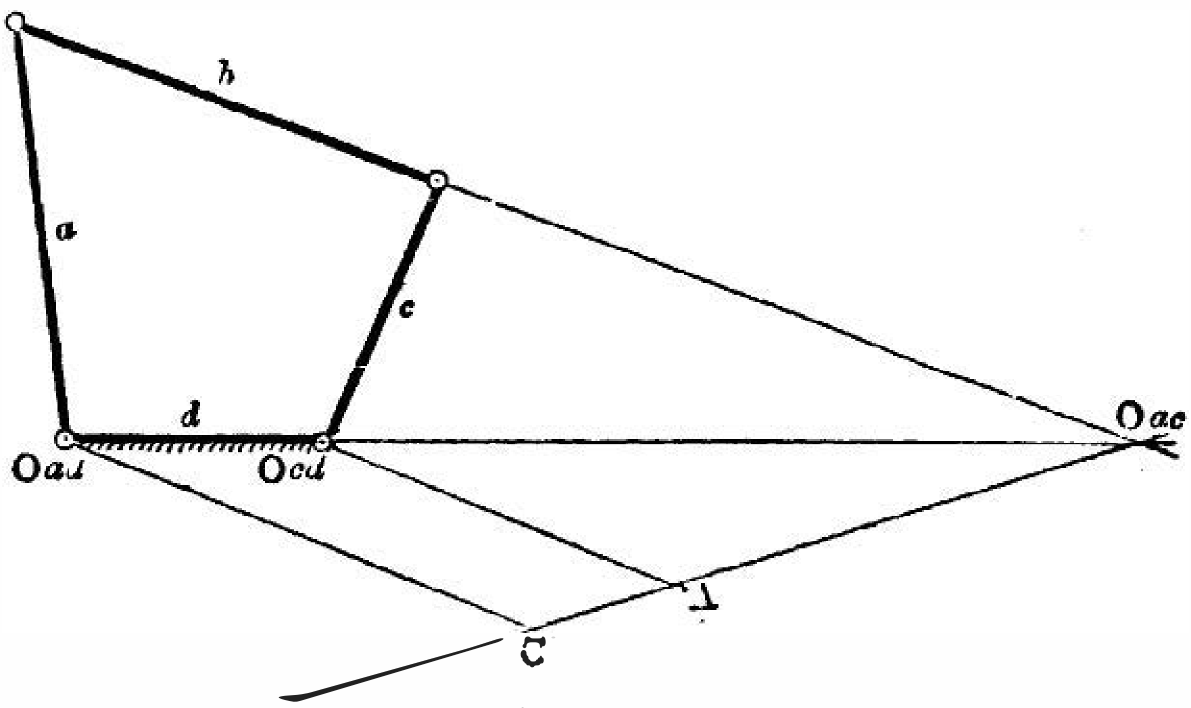

FIG. 45.

to set off a segment $O_{ac}A$ for the given angular velocity of a, and then draw $O_{ad}C$ parallel to the join of O_{cd} and A. Then $O_{ac}C$ is the angular velocity required, and

$$\frac{O_{ac}\,C}{O_{ac}\,A} = \frac{\text{angular vel. } c}{\text{angular vel. } a}.$$

Conveniently A and C may be taken on b. Alternatively we may proceed by setting off parallels through O_{ad} and O_{cd}, making $O_{cd}A$ equal to the velocity of a, and drawing $O_{ac}A$ to cut off on the other parallel the distance $O_{ad}C$, which is equal to the required angular velocity of c.

The constructions given in Figs. 44 and 45 have the advantage that they are easily proved and understood, and that in themselves they are quite simple. They have, however, some drawbacks similar to those of Figs. 37 and 39, especially that there are many cases in which, although it is perfectly easy to find the position of such points as O_{bd} Fig. 44, or O_{ac} Fig. 45, it is practically very difficult to get at them in drawing, as they often enough lie altogether off the drawing-board. This difficulty is fortunately very easily met. All that we require is to know the *ratio* between the lengths of the virtual radii of a certain point which is common to two bodies, the ratio $\frac{O_{ab}\,O_{ad}}{O_{ab}\,O_{bd}}$ in Fig. 44, and $\frac{O_{ac}\,O_{ad}}{O_{ac}\,O_{cd}}$ in Fig. 45. This ratio we must be able to set off upon some other line than the actual line of the three centres (see also p. 92, *ante*), in such a way as to bring all

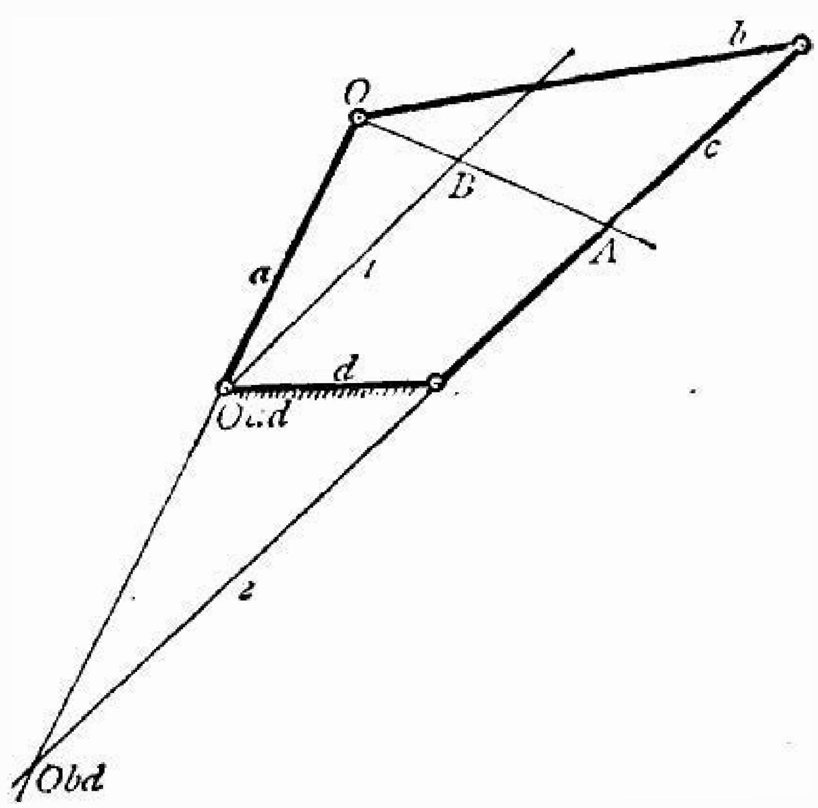

FIG. 46

the three points defining the ratio within easy reach. This can be done in an unlimited number of different ways, of which we shall point out two, one (Fig. 46) as another construction for Fig. 44, and the other (Fig. 47) for Fig. 45.

Let us call the common point, or virtual centre, of the two bodies whose relative angular velocity has to be measured, simply O, for shortness' sake, and to distinguish it better from the virtual centres about which this point is moving in the two bodies, respectively, to which it belongs. Then we proceed thus :—(Fig. 46, corresponding to Fig. 44), —through O_{ad} and O_{bd} draw a pair of parallel lines, 1 and 2, then any line through the point O will be cut by these parallels proportionally to the virtual radii OO_{ad} and OO_{bd}. With a radius representing on any scale the known angular velocity of the link a, cut 2 in a point A, then by drawing OA we have at once OB as the required angular velocity of the link b. It will be seen at a glance that this construction does not require the direct use of the point O_{bd}, because we know that one of the links *must* pass through that point, and it is only necessary to draw the line 1 parallel to the axis of that link. To use this construction it is necessary to take the distance OA on a scale sufficiently large to allow the point A to reach the axis of c in any of its positions.

When the point O itself, and not either of the other two centres, is the inaccessible point (as in Fig. 45), a somewhat different but quite as simple method can be employed. Through the three points O, O_{ad}, and O_{cd} (Fig. 47, corresponding to Fig. 45), draw three lines meeting at one point. By taking this point at one of the opposite vertices of the figure, as S, Fig. 47, two of the lines, including that through the inaccessible point O, will be the axes of links, and only the third (the line $O_{cd}S$ in the figure) will have to be actually drawn. Then any line parallel to $O_{ad}O$ will be divided by the three lines radiating from S in the same ratio as the line of virtual centres is divided. To solve the same problem as that of Fig. 45 it is only necessary to set off $O_{cd}A$ along the line of centres to represent the known angular velocity of a, construct the parallelogram $AA'C'C$ as shown, and then measure $A'C'$ or AC for the required angular velocity of *c*.

Had the angular velocity of *c* been known instead of that of *a*, it would of course have been necessary to set off

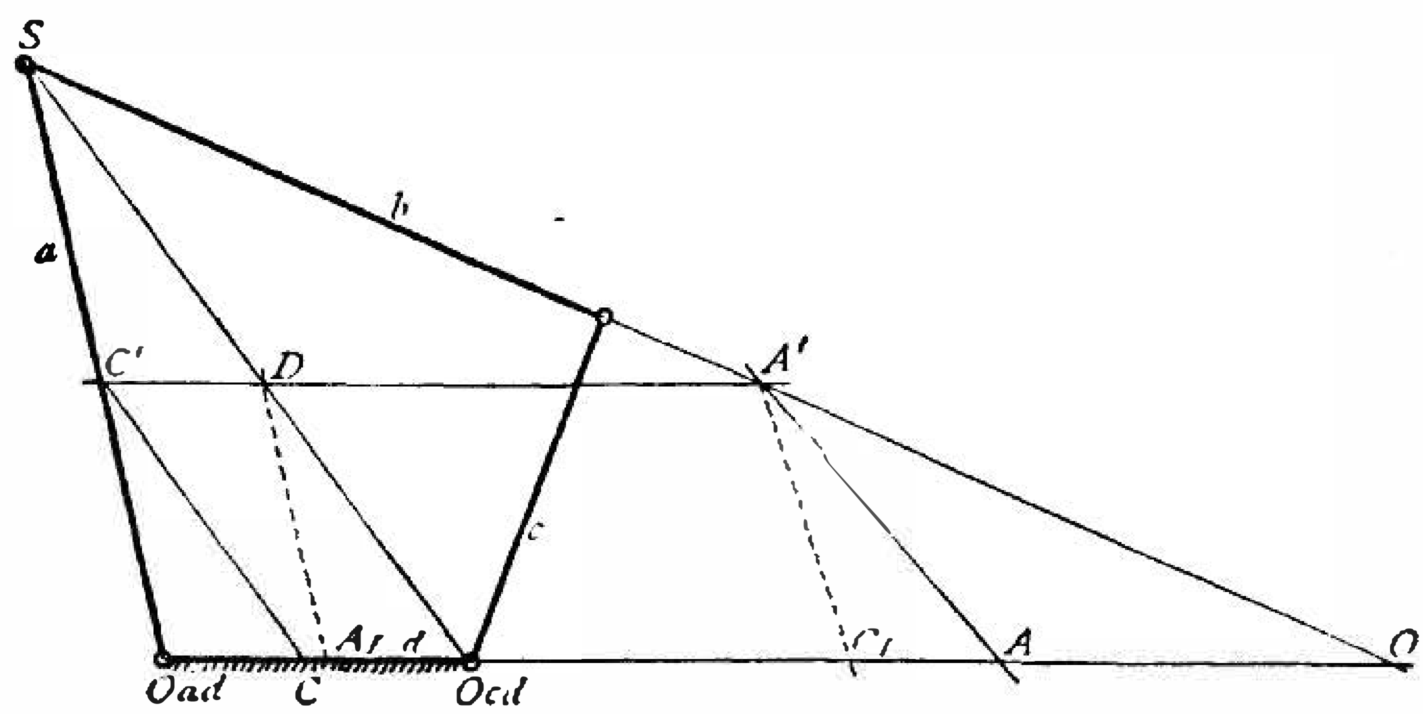

FIG. 47.

a distance $O_{ad}C_1$ in the first place equal to that velocity and then construct for A_1C_1, the velocity of *a*, as shown in dotted lines.

By using the second method of p. 99 (making $O_{cd}A$ = vel. *a*), we are of course already independent of O_{ac}, for the line *b*, on which we may take A and C, is always given and must always pass through O_{ac} (see Fig. 57).

§ 16. DIAGRAMS OF RELATIVE VELOCITIES.

In the last two sections we found how we could, by a simple construction, determine the linear velocity of any point in a mechanism when the linear velocity of any other point was given, or the angular velocity of any link in a mechanism when the angular velocity of any other link was given. In practice it frequently happens that it is of interest to solve these problems for a number of different positions of the mechanism. If this has been done arithmetically the results can be represented in the form of a table; where, however, graphic methods have been used in the solution of the problem the results are most conveniently represented in a diagram, a form which on many grounds is much the more useful of the two. We propose now to work out in some detail, with the aid of illustrations drawn to scale, the methods of making diagrams of velocities as applied to several cases having considerable practical interest.

We take first a very familiar case and one of frequent occurrence. Let it be required to find the velocity with which the piston of an ordinary direct-acting engine is moving comparatively to the velocity of the crank-pin. The mechanism of the engine, the slider-crank, we have already repeatedly had before us,—it is that shown in Fig. 48.

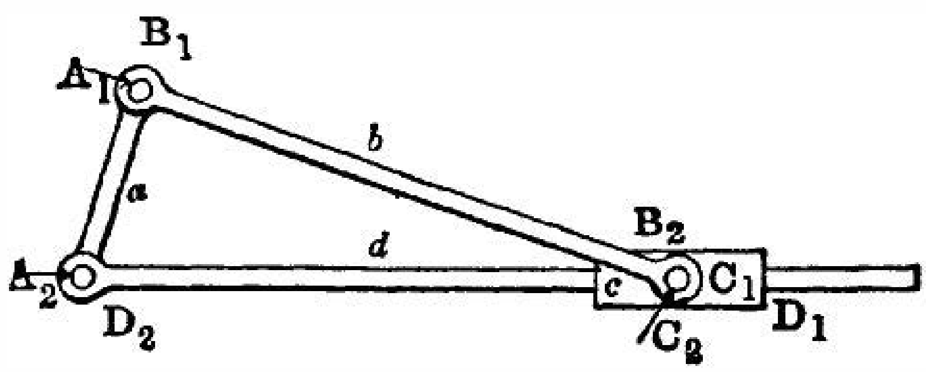

FIG. 48.

The fixed link of the mechanism becomes, in the case of the engine, the link d (see p. 67). The piston forms a part of the link c, and as the virtual centre of that link relatively to d (the point O_{ca}) is a point at infinity (p. 75), all points in it are at the same (infinitely great) distance from the virtual centre, and therefore move with the same velocity. We may therefore take *any* point in it to represent the piston, so far as its velocity goes, and for convenience' sake we take the centre of the pair connecting the links *c* and *b*, that is the point O_{bc} Fig. 49. The crank-pin itself, considered as a solid body revolving about the point O_{ad} along with all the rest of the link *a*, has of course different linear velocities in its different points. What is always meant by the velocity

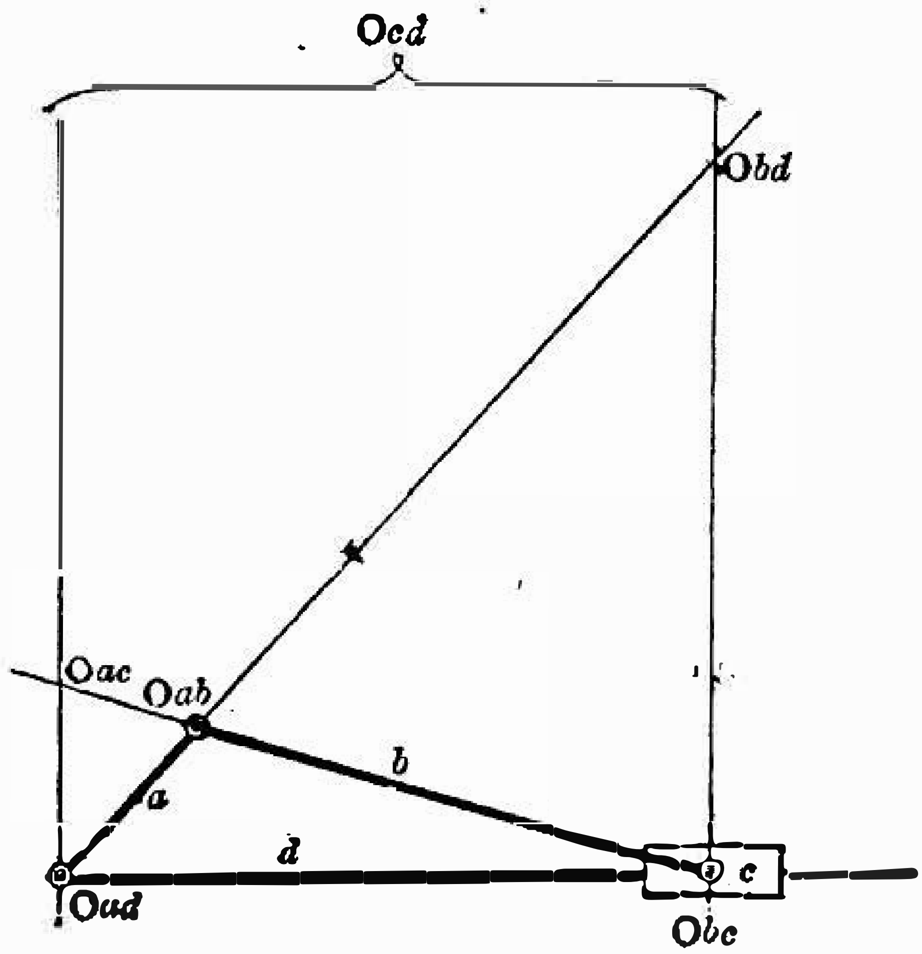

FIG. 49.

of the crank-pin is, however, the velocity of its axis, and this coincides with the centre of the pair connecting the links *a* and *b*, that is the point O_{ab}. The problem then presents itself in this form:—Given the velocity of the point O_{ab} in the plane of the fixed link *d*, to find for a sufficient number of positions of O_{ab} the velocity of the point O_{bc} in the same plane. One of these points is common to the links *a* and *b*, and the other to the links *c* and *b*, both of them therefore are, in all positions of the mechanism, points of the link *b*. Hence the problem is really nothing more than the finding of the relative velocities of two points in the same body, as we have already done on p. 86 (Figs. 36 and 37).

Let the data in our case be as follows :—

Radius of crank (a) . . . = 1·5 feet.
Length of connecting-rod (b) = 6·0 feet.
Speed of crank 56 revolutions per minute.

This gives for the linear velocity of the crank-pin ($2\pi \times 1·5 \times 56$) = 528 feet per minute, or 8·8 feet per second. Divide the crank-circle, as in Fig. 50, into any convenient

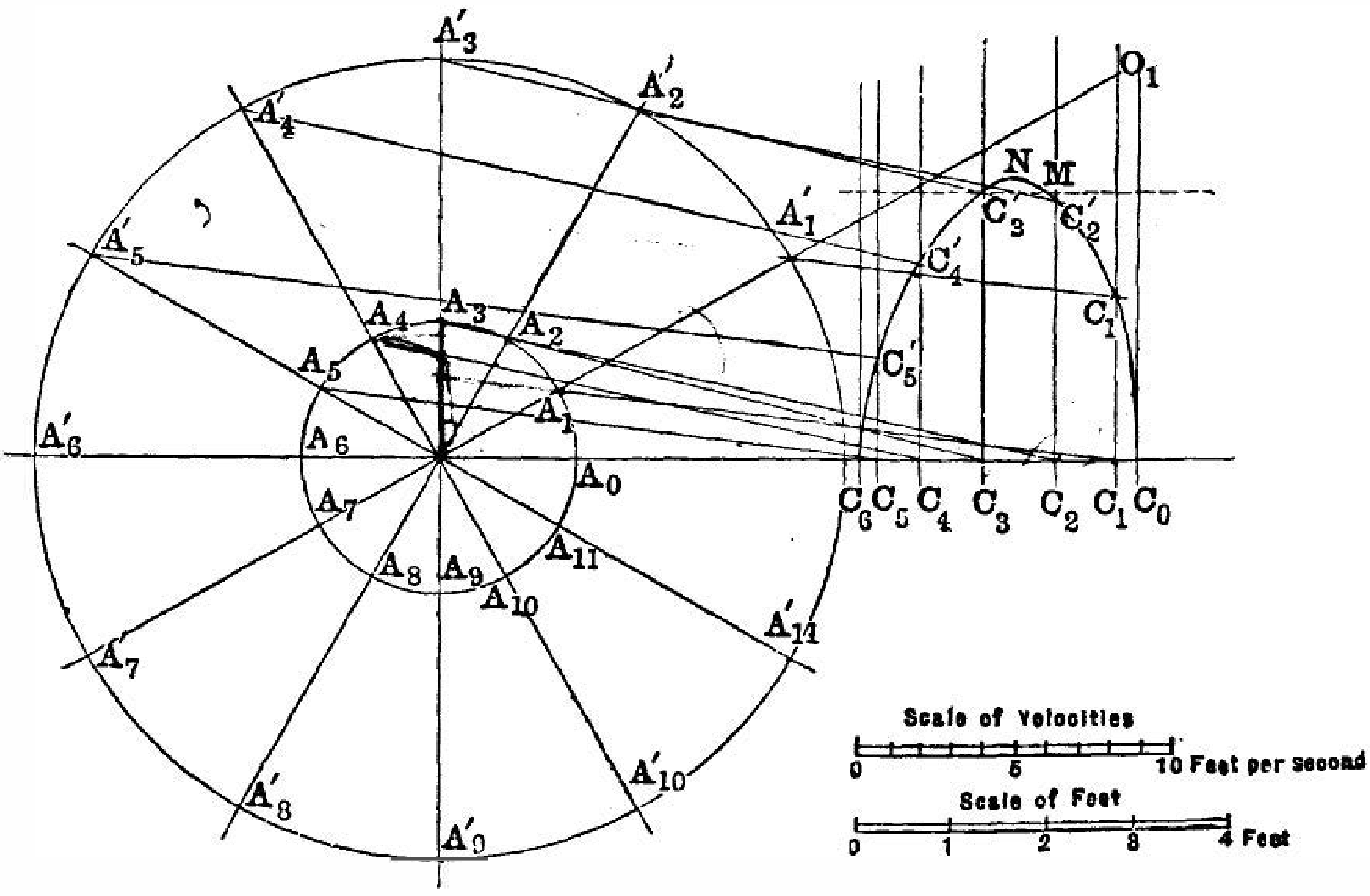

FIG. 50.

number of parts, and for each position of the crank-pin, A_1, A_2, &c., find the corresponding position of the piston C_1, C_2, &c., and the virtual centre of b relatively to d, O_1, O_2, &c. Next set off $A_1A'_1$ = 8·8 on any scale, and draw a circle about the point O_{ad} with the radius $O_{ad}A'_1$. Then lines drawn through the points A'_1, A'_2, &c., parallel to each corresponding position of *b* will cut the corresponding virtual radii of C_1, C_2 (here parallel lines at right angles to the axis of *d*), &c., in points C'_1, C'_2, &c., such that $C_1C'_1$, $C_2C'_2$, &c., are the required velocities of the piston on the same scale as that used for setting off $A_1A'_1$. This construction has already been proved in connection with Fig. 37. A curve joining all the points C'_1, C'_2 &c., gives by its ordinates the velocity of the piston at any required point of its travel.

If it be required rather to represent the velocity of the piston at each position of the crank instead of at each of its own positions, then what is called a *polar* diagram, like that shown in Fig. 51, can be used. Here a circle is drawn

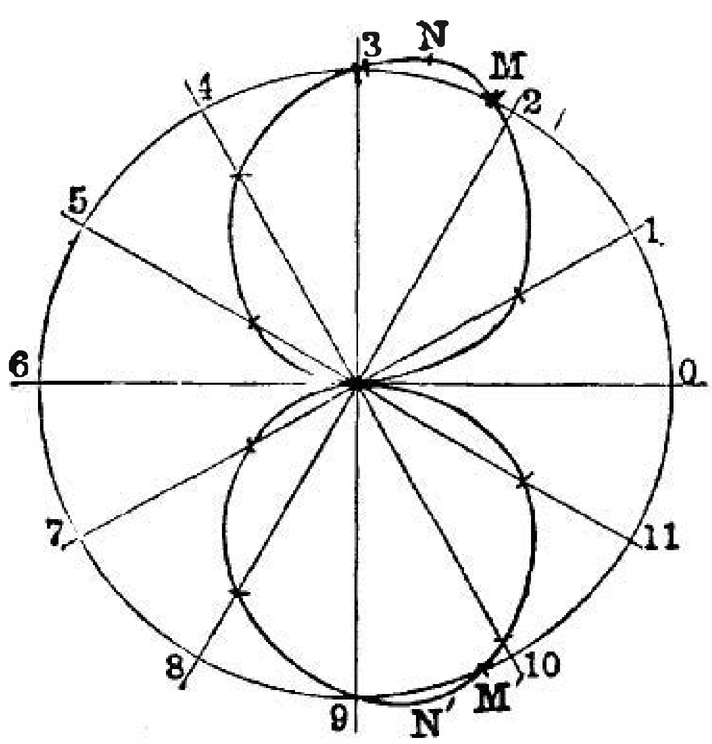

FIG. 51.

with radius = 8·8 on the same scale as before, the different positions of the crank are marked on it, and upon each the corresponding distance $C_1C'_1$, $C_2C'_2$, &c., is set off.

The actual changes of velocity in this case are worth noticing. At the beginning of the stroke the velocity of the piston is of course nothing. This comes out in the construction by the coincidence of the virtual centre with the point C_0. As the piston moves on its velocity increases. At first the angle at C (as $A_1C_1O_1$) is greater than that at A (as $C_1A_1O_1$) so that OA is greater than OC, and therefore the velocity of the piston *less* than that of the crank-pin. At some point, however, the angle at A must become a right-angle (when the axis of the connecting-rod b is tangential to the crank-circle), and then OC must be the hypotenuse of a right-angled triangle, and therefore *greater* than OA, so that the velocity of the piston must be then *greater* than that of the crank-pin. Before this point is reached there must, therefore, be some position in which the velocities of the piston and crank-pin are equal, and it is obvious that this position will be that for which the triangle, AOC, is isosceles, and $OA = OC$. When the crank-pin is in its central position, A_3, and at right-angles to the direction of the piston's motion, OA and OC are again equal, the two virtual radii being parallel, and the point O at infinity. Here again, therefore, the velocity of the piston is equal to that of the crank-pin. After this, until the end of the stroke, OA is always greater than OC, and the velocity of the crank-pin therefore greater than that of the piston, and at A_6 the latter again becomes $= 0$, *i.e.* the piston is for the instant stationary. The same changes of relative velocity occur, in reversed order, as the crank makes its second half revolution from A_6 back to A_0—the lines for these are not shown in the diagram.

The positions of the important points noticed are readily seen in Fig. 51; and also in Fig. 50 if a line be drawn parallel to the axis of d and at a distance from it $= A_1A'_1$, or 8·8. At O and at 6 (or C_0 and C_6) the ordinate of the piston speed-curve is $= 0$. At M and at 3 (and also at 9 and at M') it is equal to the distance which represents the crank-pin velocity, while at N it exceeds that distance.

The actual maximum velocity of the piston would in this case be about 9·3 feet per second, or 5·7 per cent. greater than that of the crank-pin.

It will be found an interesting exercise to draw diagrams of this kind for different lengths of connecting-rod, and note how considerably the shortening of this rod increases the maximum velocity of the piston. It will easily be seen also that while for all lengths of the rod the point 3 (the second point at which crank-pin and piston velocities are equal) remains in the same position, the points *M* and *N* move nearer and nearer to it as the rod is lengthened. If the rod could be made infinitely long these three points would coincide, the maximum velocity of the piston would be equal to the crank-pin velocity, and the position of maximum velocity would occur when the crank is in its mid-position. We have already pointed out that the mechanism of Fig. 52, some properties of which were examined in

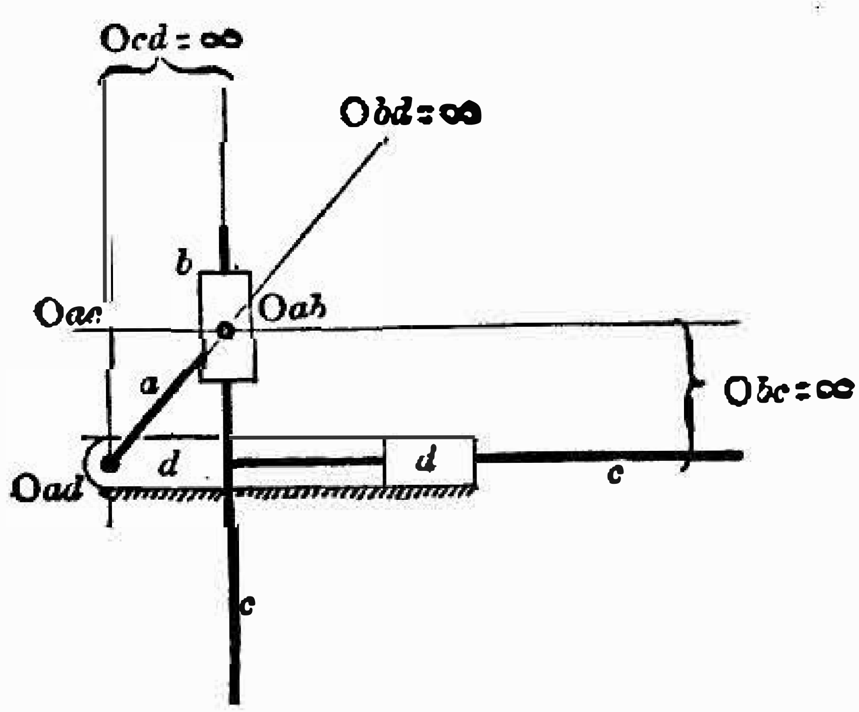

FIG. 52.

§ 12, is constructively equivalent to the one with an infinitely long connecting-rod. It will be worth while to work out for this mechanism a diagram similar to that just worked out for the slider-crank. This is done in Fig. 53, which is drawn to the same scales as Fig. 50, in which also the same data as to speed are assumed, and the same length of crank a. It is more convenient in this case, for reasons

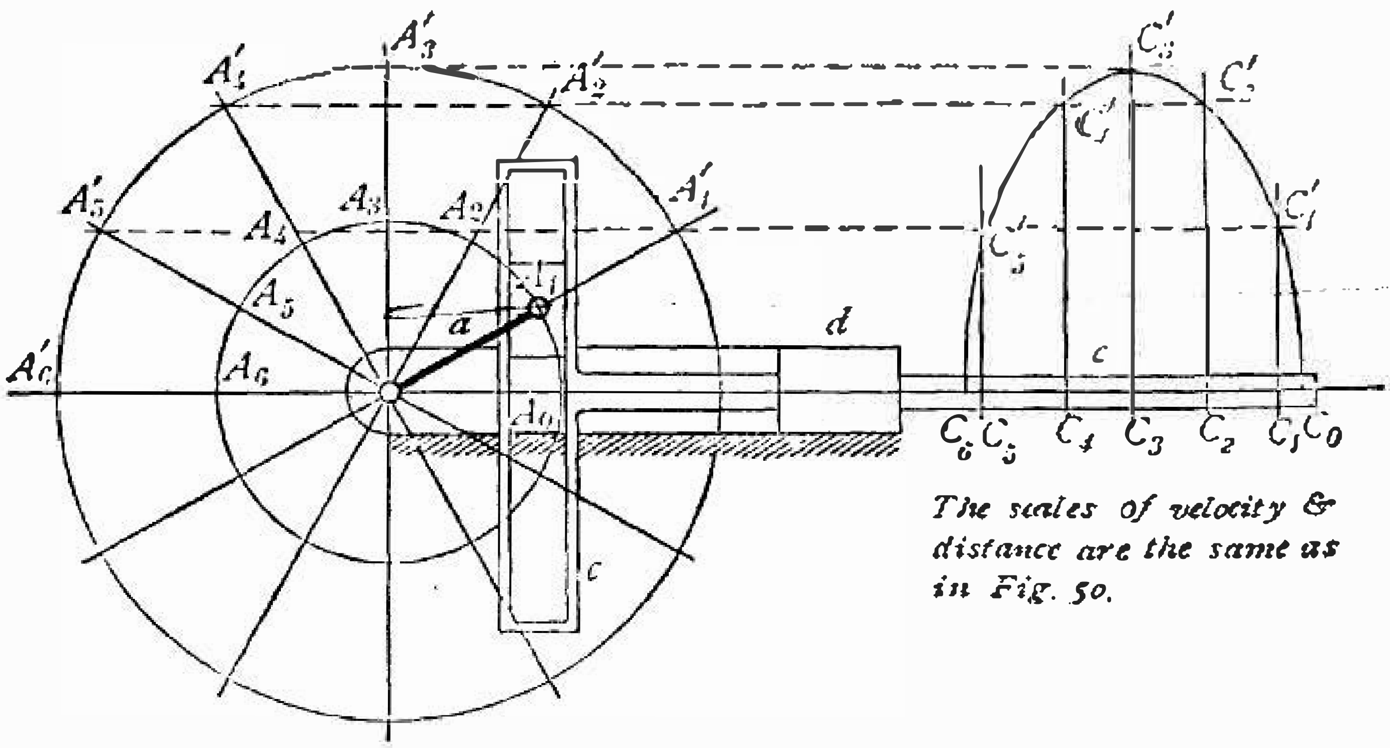

FIG. 53.

which can easily be seen, to set off the crank-pin velocity (8·8 feet per second) from the point O_{ad} instead of, as in Fig. 50, from the point O_{ab}. Then marking the positions A_1, A_2, A_3, &c., as before, and taking any convenient point upon c to represent the piston, the piston velocities, $C_1C'_1$, $C_2C'_2$, &c., are at once found by drawing the lines $A'_1C'_1$, $A'_2C'_2$, &c., parallel to the connecting-rod, *i.e.* parallel always to the direction of motion of the piston, for the infinitely long connecting-rod moves (as we saw on p. 77) always parallel to itself. It will be seen at once that the curve $C'_1C'_2C'_3$ . . . . . is a semi-ellipse of a height equal to the radius of the velocity circle,—that the maximum velocity of the piston is *equal to* the maximum velocity of the crank, and that the maximum velocity is reached by the piston just when the crank is at mid-stroke—these being precisely the characteristics which we pointed out a few lines back as theoretically belonging to a mechanism with an infinitely long connecting-rod. In Fig. 54 is shown a polar diagram for this case corresponding to Fig. 51. Here again there is a great simplification, the two curves are simply circles having

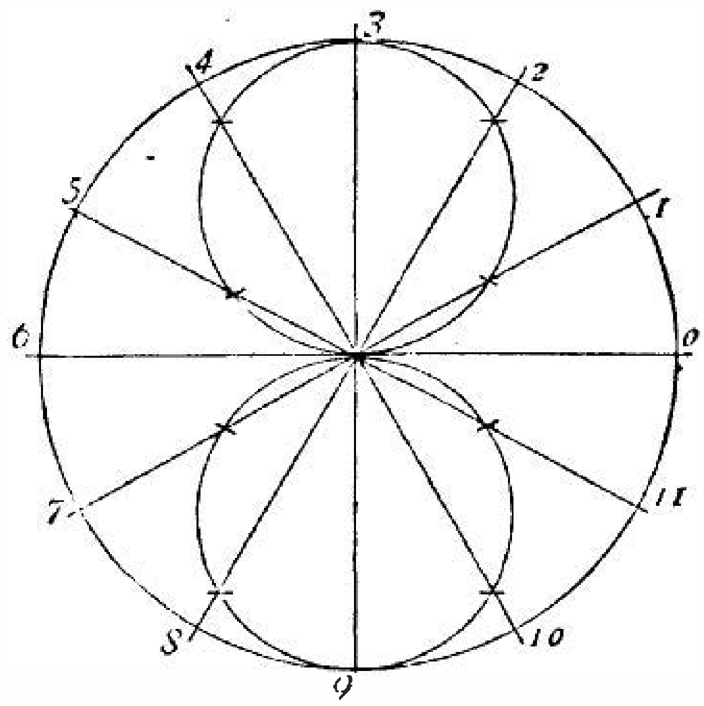

FIG. 54.

diameters equal to the length which stands for the crank-pin velocity.

It may be noted here that sometimes one wishes simply to find out the relative velocities of the piston at different periods of its stroke without reference to any particular crank-pin velocity. In this case it is convenient either to assume the crank-pin velocity = (say) 10 on any convenient scale, or (sometimes) to let the radius of the crank itself stand for its velocity. In the last case the diagram of Fig. 54 becomes very similar in appearance to the well-known Zeuner valve diagram, although its interpretation is very different.

As a last example of the construction of diagrams of velocities we shall take a chain similar to that already shown in Fig. 25 and dealt with in Figs. 40 and 44, &c., but with different proportions. Take the length of the four links as follows—

$$a = 14 \text{ inches}$$
$$b = 32 \quad ,,$$
$$c = 19 \quad ,,$$
$$d = 34 \quad ,,$$

and make the link a the fixed link. Suppose b to turn with a uniform angular velocity of 48 revolutions per

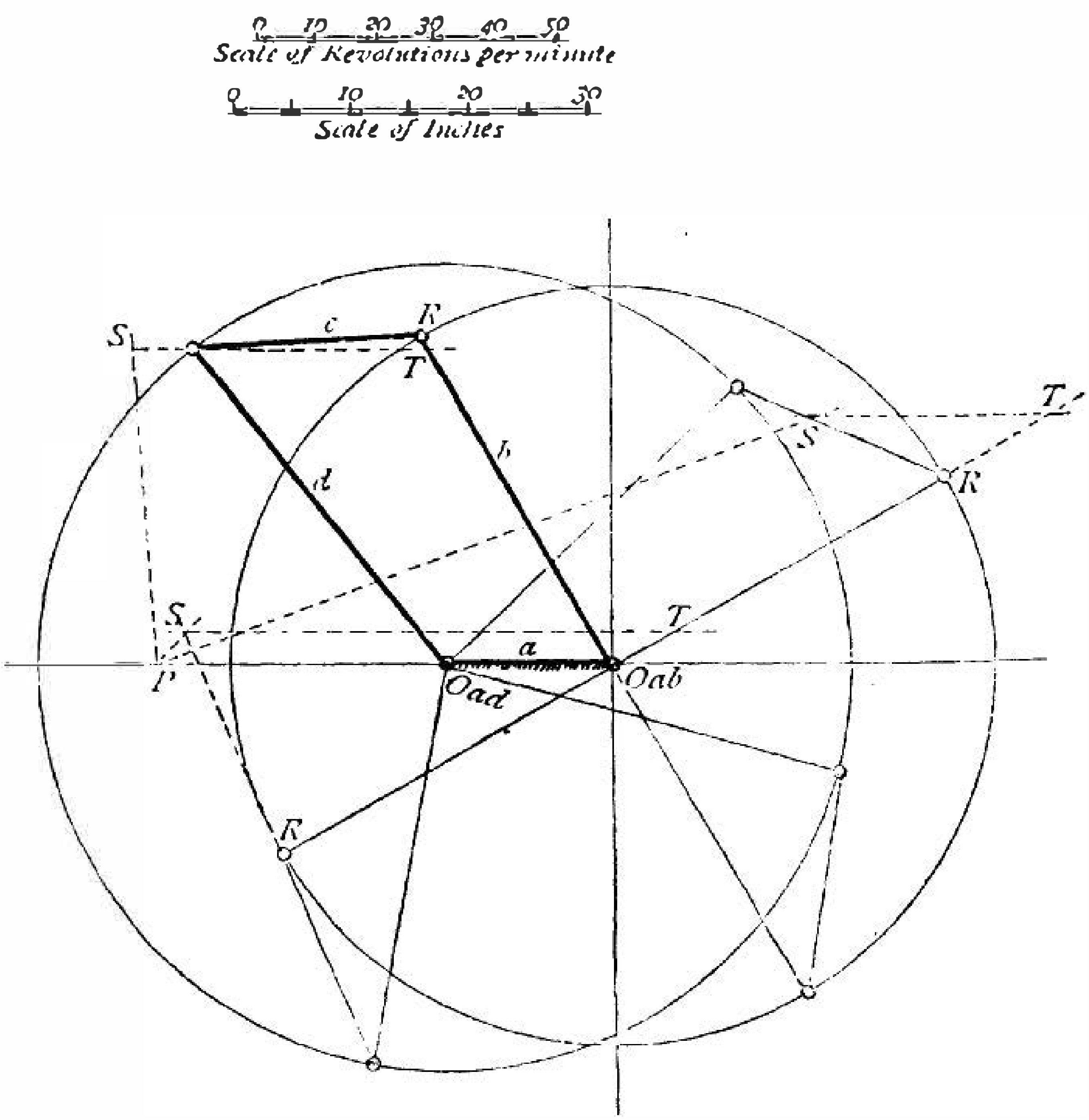

FIG. 55.

minute, and let our problem be to find the angular velocity of d at any number of different positions of b (Fig. 55). In a case of this kind it is most convenient to use some such abbreviated construction as was given in Fig. 47. First set off any required number of positions of the link *b*, and for each construct the corresponding positions of *d* (a few of these positions are shown in the diagram). From the point O_{ad} (*i.e.* the virtual centre relatively to the fixed link of the link *d*, whose velocity has to be found) set off upon the axis of *a* a distance $O_{ad}P$, standing for 48, the angular

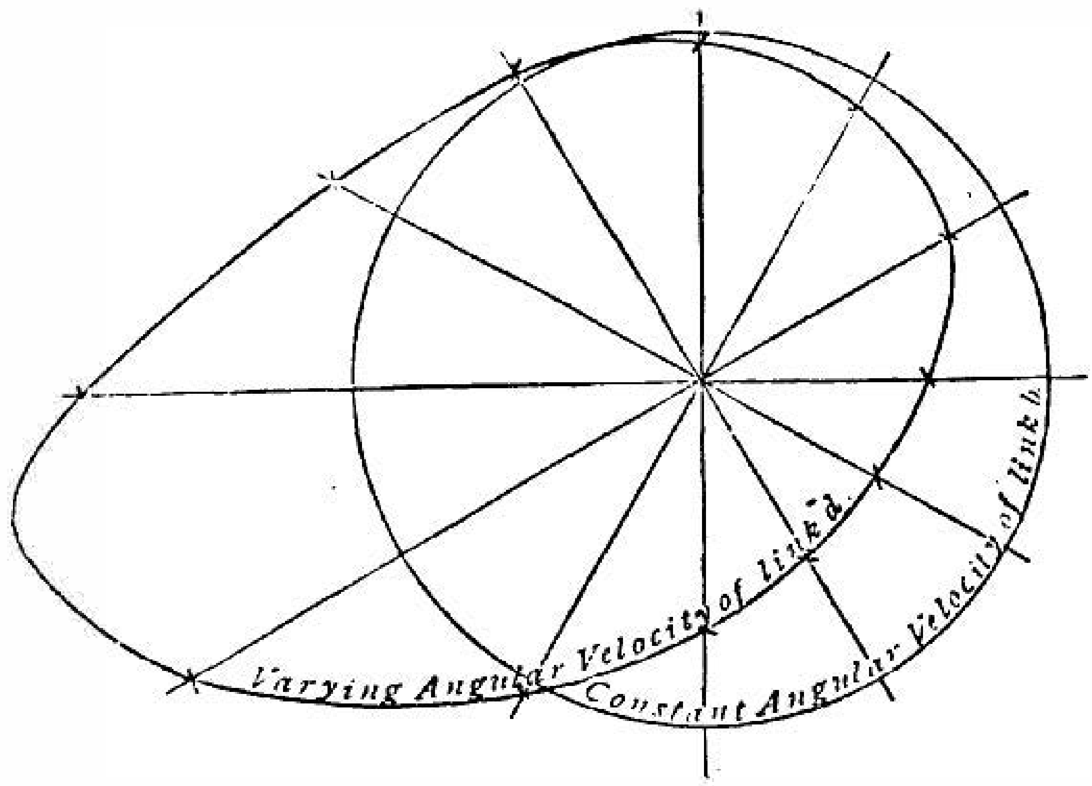

FIG. 56.

velocity of *b*, upon any scale. Then for each position of the mechanism draw a line through *P*, parallel to the line (which need not be itself drawn) joining O_{ad} and *R*, until it cuts the axis of *c* in a point *S*. The distance *ST*, from *S* to the axis of *b* (measured parallel to the line of the three centres, here the axis of *a*) is the required angular velocity of *d*. A diagram may be conveniently made, as in Fig. 56, by drawing a circle with a radius = 48 to stand for the constant angular velocity of *b*, marking on it the positions of *b* used in Fig. 55, and then setting off on each radius the corresponding value of *ST*, the angular velocity of *d* for the given position of *b*.

Such constructions as those given in Figs. 47 or 55 are very general, allowing the velocities to be represented by any distances whatever, *i.e.* drawn on any scale whatever. But by using particular distances and scales the construction may often be greatly simplified. Thus if we take the length of the link d in Fig. 55 to represent the angular velocity of b, we have only to draw through O_{ab} a line parallel to d, and cutting c, to obtain at once the angular velocity of d on the same scale. By comparing this construction (Fig. 57) with

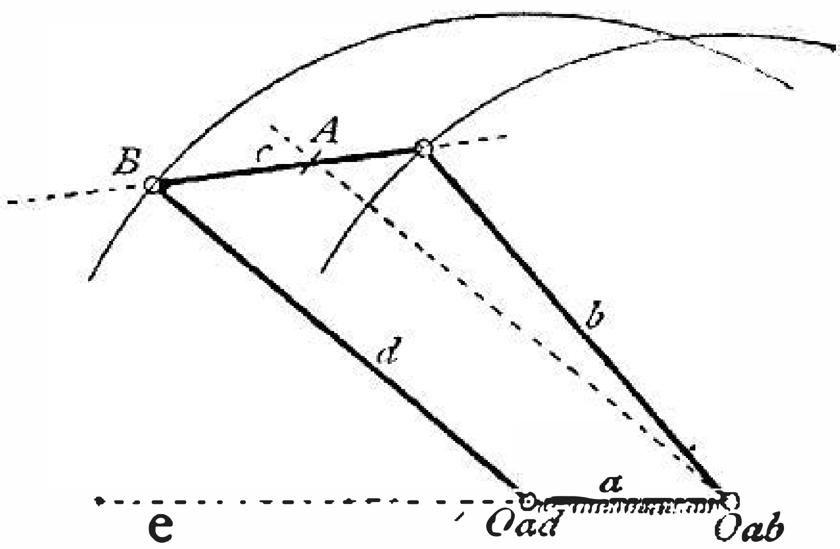

FIG. 57.

that of Fig. 45 it will be seen that BA is a line passing through the (inaccessible) centre O_{bd}, and is in exactly the same position as the axis of b or the line CA in that figure. If this construction, which is the simplest of all possible constructions for this case, be used, the circle in Fig. 56 has, of course, a radius equal to that of the link d.

We should have obtained the same result in Fig. 45 if we had not only taken A and C on the axis of b, as there mentioned,[1] but had set off $O_{cd}A$ on such a scale as to be equal to c. Whether the greater practical convenience lies in the saving of construction lines, or in the use of an even scale of velocities, may be left to the draughtsman to decide in each particular case.

[1] See also end of § 15.

It must be noted that the *mean* angular velocity of *b* and *d* must be equal, for each one takes the same time to make one whole revolution, but if *b* have a *uniform* velocity within the revolution, then *d* has the varying velocity which has been drawn in Fig. 56. In the case supposed, the velocity of *d* would vary within each revolution from the rate of $27\frac{1}{2}$ revolutions per minute to that of 97 revolutions per minute, the actual mean rate being 48 revolutions per minute.

If we had taken the chain of Figs. 32 or 53, and fixed the link *a*, we should have obtained a mechanism in which, as in that of Fig. 55, the links *b* and *d* would both revolve, the one driving the other through the link *c*. This mechanism is that generally known as an "Oldham coupling." It is interesting to try with it the constructions of Figs. 45 or 55 for finding the relative angular velocities of the links *b* and *d*. It will be found that the latter construction gives no result, too many of the points required being at an infinite distance, but the former shows very clearly the leading characteristic of the mechanism, that the angular velocity ratio transmitted between the shafts (the links *b* and *d*) is constant, and is equal to *unity*.

An important analogue of this mechanism, the "universal joint," will be examined in detail in § 64.

A possible misunderstanding may be guarded against before we leave this part of our subject. Let *b* and *c* be any links of a mechanism of which *a* is a third link. We have seen how to find the ratio $\frac{\angle \text{r. vel. } b}{\angle \text{r. vel. } c}$ when both these velocities were measured relatively to *a*. This ratio must not be confused with the angular velocity of *b* relatively to *c*, which is of course an entirely different quantity. If the angular velocities of *b* and *c* relatively to *a* be β and γ respectively, then in order to find the angular velocities of these two links relatively to each other we have only to proceed as in § 3—namely, give to both a velocity equal and opposite to that of one of them. Thus if we give to both an angular velocity of $-\gamma$, we bring *c* to rest and find the angular velocity of *b* relatively to it to be $\beta - \gamma$. Or if we give both an angular velocity of $-\beta$, we bring *b* to rest, and find the angular velocity of *c* relatively to it to be $\gamma - \beta$, which is the same magnitude as before, reversed only in sense.[1] In general, if v_1 and v_2 be the angular velocities of any two bodies relatively to a third, the velocity of each relatively to the other is $v_1 \pm v_2$, where the *positive* sign is to be used if v_1 and v_2 have opposite senses, and the *negative* sign if they have the same sense. Thus in Fig. 56 distances measured radially between the two curves represent the angular velocity of *b* relatively to *d* on the same scale as that on which the radii of the curves represent the angular velocities of *b* and *d* respectively relatively to *a*.

[1] § 3, p. 27.