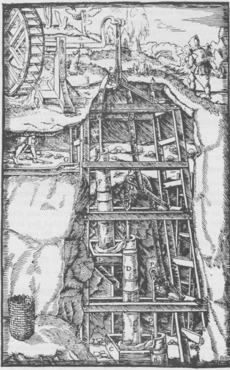
6

FOUR-BAR COUPLER-POINT CURVES

6-1 THE FOUR-BAR LINKAGE

Linkwork, in its early applications, consisted mainly of revolute-connected members and was widely used for converting the continuous rotation of a water wheel into a reciprocating motion suited to piston pumps (Fig. 6-1). The piston-cylinder combination at the end of the line represents a prismatic pair, of course, but ahead of this there are only the revolute connections generally associated with linkwork. Agricola's arrangements show wheel and pump—power source and point of work—fairly close together. Such compactness did not always prevail; linkworks of magnificent proportions were also part of the past. A linkwork is a means of power transmission as well as being a motion transformer. Before the introduction of rope transmissions and the now universal electric wire, linkwork was employed for long-distance transmission of power. Gigantic linkages, principally for mine pumping operations, connected water wheels at the riverbank to pumps high up on the hillside. One such installation (1713) in Germany was 3 km long.

Such linkages consisted in the main of what we call four-bar linkages, i.e., planar four-revolute mechanisms, and terminated in a slider-crank mechanism with a prismatic pair.



A—SHAFT. B—BOTTOM PUMP. C—FIRST TANK. D—SECOND PUMP. E—SECOND TANK. F—THIRD PUMP. G—TROUGH. H—THE IRON SET IN THE AXLE. I—FIRST PUMP ROD. K—SECOND PUMP ROD. L—THIRD PUMP ROD. M—FIRST PISTON ROD. N—SECOND PISTON ROD. O—THIRD PISTON ROD. P—LITTLE AXLES. Q—"CLAWS."

FIGURE 6-1 Linkwork between water wheel and pumps, sixteenth century. [*From Agricola's "De re metallica," Hoover translation* (1912).]

With Watt's invention of the "straight-line motion" (1784) the four-bar linkage was used in a new way, for the significant motion output was not that of the follower but that of the coupler: Watt had found a coupler point describing a curve of special usefulness.

The first analytical investigation of a coupler curve, the curve of the Watt mechanism, was undertaken by Prony,[1] who examined Watt's "straight-line motion" for deviations (1796). Samuel Roberts showed (1876) that the "three-bar curve"[2]—today we call it the coupler curve of the four-bar—is an algebraic curve of the sixth order; i.e., a straight line will cut it in not more than six points. Cayley and others showed further properties of the curve. Their interest lay in exploring linkages hypothetically able to generate specific algebraic curves of any order: the applications to mechanisms were to be made later.

6-2 EQUATION OF COUPLER CURVES[3]

Derivation

The equation of the coupler-point curve for a four-bar linkage may be obtained by analytic geometry. The derivation presented follows that of Samuel Roberts, with only slight changes in notation. The equation will be written in cartesian coordinates, with the x axis along the line of centers O_AO_B and the y axis perpendicular to that line at O_A (Fig. 6-2). Let (x', y'), (x'', y''), and (x, y) be, respectively, the coordinates of points A, B, and coupler point M; then

$$x' = x - b \cos \theta \qquad y' = y - b \sin \theta$$

and

$$x'' = x - a \cos (\theta + \gamma) \qquad y'' = y - a \sin (\theta + \gamma)$$

Since A and B describe circles (or arcs of circles) about centers O_A and O_B, respectively,

$$x'^2 + y'^2 = r^2 \qquad \text{and} \qquad (x'' - p)^2 + y''^2 = s^2$$

[1] Gaspard François Prony (1755–1839), engineer, was an associate of the famous bridge builder Perronet and became his successor as director of the École des Ponts et Chaussées. Also professor of mathematics at the École Polytechnique, Prony wrote textbooks on mechanics and hydraulics but is perhaps best remembered for the friction brake, or absorption dynamometer.

[2] Only the moving links were counted and called bars. In recent years, only Svoboda ("Computing Mechanisms and Linkages," Massachusetts Institute of Technology Radiation Laboratory Series, vol. 27, McGraw-Hill Book Company, New York, 1948) has used the term "three-bar."

[3] This section may be omitted at first reading.

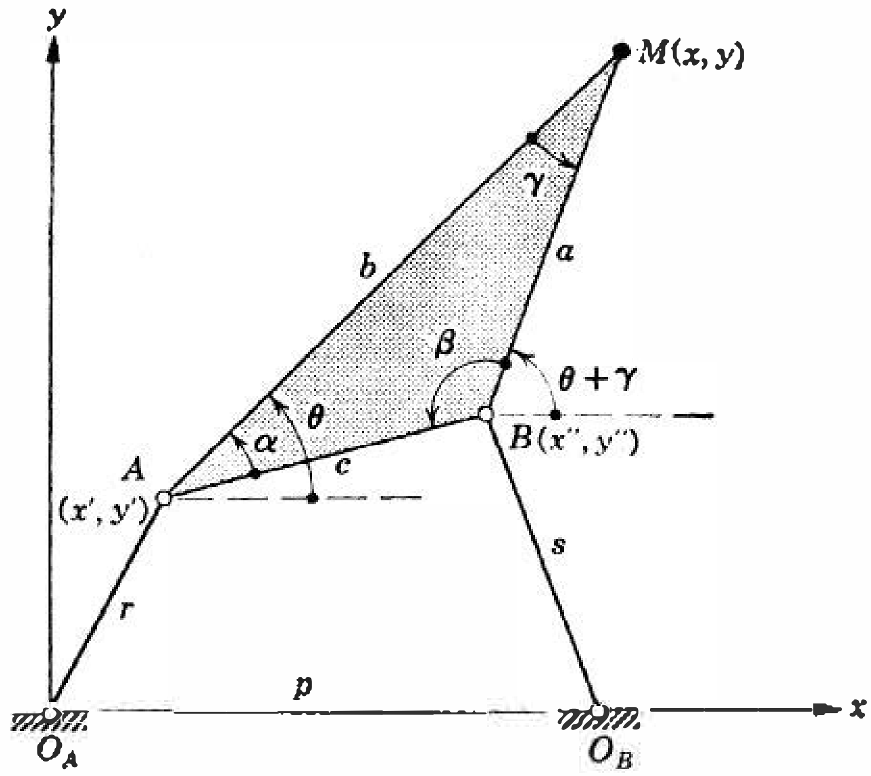


FIGURE 6-2 Coordinate system and notations used to derive equation of coupler curve.

Substituting the values of x', y' and x'', y'' into the last two equations yields

$$(x - b\cos\theta)^2 + (y - b\sin\theta)^2 = r^2$$

and $$[x - a\cos(\theta + \gamma) - p]^2 + [y - a\sin(\theta + \gamma)]^2 = s^2$$

which, by application of trigonometric identities and ordering of terms, become

$$x\cos\theta + y\sin\theta = \frac{x^2 + y^2 + b^2 - r^2}{2b}$$

and

$$[(x - p)\cos\gamma + y\sin\gamma]\cos\theta - [(x - p)\sin\gamma - y\cos\gamma]\sin\theta = \frac{(x - p)^2 + y^2 + a^2 - s^2}{2a}$$

The equation of the coupler-point curve may now be obtained by elimination of θ between the last two equations. Solving these equations for $\cos\theta$ and $\sin\theta$ and substituting the values obtained into the identity $\cos^2\theta + \sin^2\theta = 1$ yields the general four-bar coupler-curve equation

$$\begin{aligned}&\{\sin\alpha[(x - p)\sin\gamma - y\cos\gamma](x^2 + y^2 + b^2 - r^2)\\ &\quad + y\sin\beta[(x - p)^2 + y^2 + a^2 - s^2]\}^2\\ &\quad + \{\sin\alpha[(x - p)\cos\gamma + y\sin\gamma](x^2 + y^2 + b^2 - r^2)\\ &\quad - x\sin\beta[(x - p)^2 + y^2 + a^2 - s^2]\}^2\\ &\qquad = 4k^2\sin^2\alpha\sin^2\beta\sin^2\gamma[x(x - p) - y - py\cot\gamma]^2 \qquad (6\text{-}1)\end{aligned}$$

In this, k is the constant of the sine law applied to the triangle ABM,

$$k = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

This equation is of the sixth degree and because of its properties also bears the formidable title of tricircular sextic. One of its properties has been mentioned: a straight line will intersect it in no more than six points. Its further features may be studied either geometrically or algebraically. The geometric examination begins with Sec. 6-3. An introduction to the algebraic study of coupler curves will be given by considering some properties deduced by Roberts and others. As a guide to an understanding of the methods of analytic geometry, second-order curves (conic sections) will be examined en route.

Circle of Foci

On setting

$$\begin{aligned} L &= \sin\alpha[(x-p)\sin\gamma - y\cos\gamma] & M &= y\sin\beta \\ N &= \sin\alpha[(x-p)\cos\gamma + y\sin\gamma] & P &= -x\sin\beta \\ \phi &= x^2+y^2+b^2-r^2 & \psi &= (x-p)^2+y^2+a^2-s^2 \end{aligned}$$

Eq. (6-1) takes the form

$$(L\phi + M\psi)^2 + (N\phi + P\psi)^2 - 4k^2(LP - NM)^2 = 0 \qquad \text{(6-1}a\text{)}$$

Note that the equation

$$LP - NM = 0$$

or

$$x(x-p) + y^2 - py\cot\gamma = 0$$

represents a circle[1] passing through O_A and O_B (Fig. 6-3). For reasons that will appear later, this circle is called the *circle of singular foci*.

Multiple Points

A multiple point of a curve, as, for example, a cusp or a crunode (see Sec. 6-3), is a point where the curve has several tangents. We propose to show that the coupler curve has multiple points at each of its intersections with the circle of foci. When a curve is defined by an equation of the form

$$F(x, y) = 0$$

its tangent may be found by equating to zero the differential of the function $F(x, y)$,

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

[1] In rectangular coordinates every equation of the form

$$x^2 + y^2 + Dx + Ey + F = 0$$

represents a circle.

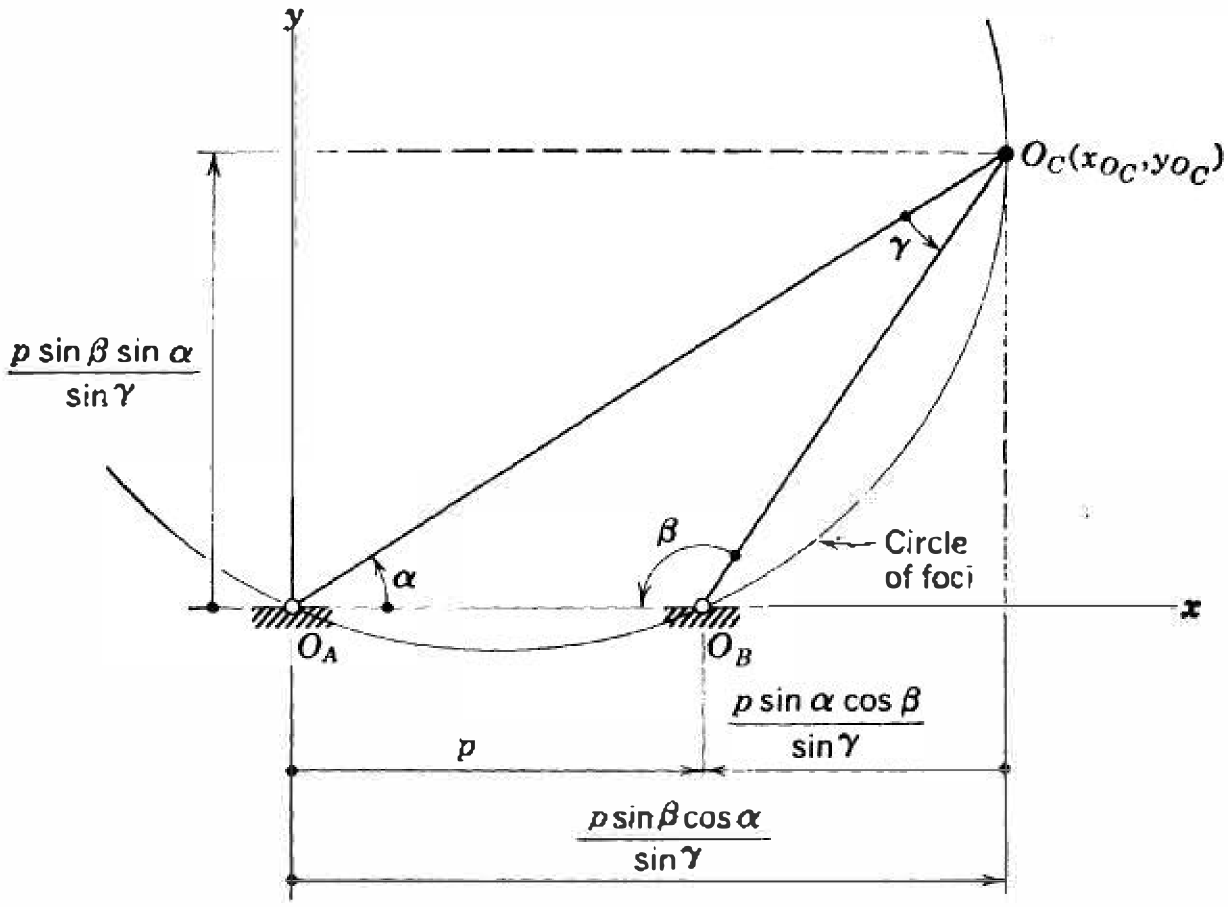


FIGURE 6-3 Triangle of the singular foci $O_AO_BO_C$ similar to coupler triangle ABM.

Since dx and dy are infinitesimal changes in the coordinates x and y along the curve, they also define the slope of the tangent as

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

At a multiple point, where the curve has several tangents, the above expression must be indeterminate, which means that both $\partial F/\partial x$ and $\partial F/\partial y$ must be zero to satisfy the indeterminancy.

The left member of Eq. (6-1a) is the function $F(x, y)$ corresponding to the coupler curve, whence

$$\frac{\partial F}{\partial x} = 2(L\phi + M\psi)\frac{\partial}{\partial x}(L\phi + M\psi) + 2(N\phi + P\psi)\frac{\partial}{\partial x}(N\phi + P\psi) - 8k^2(LP - NM)\frac{\partial}{\partial x}(LP - NM)$$

A similar expression is formed for $\partial F/\partial y$ on replacing x by y. Since the points of intersection of two curves are found by considering their equations as simultaneous, the intersection of the coupler curve and the circle of foci is given by the pair of equations

$$(L\phi + M\psi)^2 + (N\phi + P\psi)^2 - 4k^2(LP - NM) = 0 \qquad LP - NM = 0$$

Since the left member of the second equation is zero, each term of the first equation is likewise zero, and

$$L\phi + M\psi = 0 \qquad N\phi + P\psi = 0$$

The last two quantities serve to make $\partial F/\partial x$ and $\partial F/\partial y$ zero also, thus satisfying the requirement of an indeterminate slope at the point of intersection of the two curves and establishing the presence of a multiple point.

Imaginary Points

Further properties of the curve may be deduced from its equation by considering in addition to real points of the plane, whose coordinates x and y are real numbers, imaginary points having complex numbers as coordinates. Complex numbers of the form $z = x + iy$ are used in this text to represent real points of coordinates x and y, in which both x and y are real. The situation in this section is different, because x and y are themselves complex: such points are called imaginary. As remarked, these points have no geometric or physical meaning, for they cannot, with their four coordinates, represent real points in a plane. They are useful, however, because of their analytic resemblance to real points. Although no imaginary point exists in a material plane (the plane is already completely "filled" with real points), consideration of such points is sometimes helpful in the study of curves by means of their equations. As in the consideration of n-dimensional spaces with n larger than 3, it often turns out to be convenient and suggestive to think in geometric language about quantities having only analytic meaning.

Thus, although the coupler curve is a closed curve which does not extend to infinity, we shall be able to speak of its imaginary points at infinity and determine its asymptotes at those points. The asymptotes, as may be expected, turn out to be imaginary lines, but three of their intersections are real and very significant points. It is from a consideration of these points that Roberts deduced for the first time what we call the Roberts-Chebyshev theorem, that the same coupler curve may be generated by three different four-bar linkages.

Second-order Curves

An asymptote of a curve is a straight line such that a point, tracing a curve and receding to infinity, approaches indefinitely near to the straight line. An asymptote may also be considered as a tangent to a curve at a point an infinite distance from the origin.

Before considering the sixth-order curve of primary interest, the matter of points at infinity and asymptotes of a curve will be reviewed in terms of the more familiar second-order curves whose general equation is

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0 \tag{6-2}$$

As may be recalled, the above equation—depending on the values of the coefficients A, B, and C—represents an ellipse, a parabola, or a hyperbola. Such curves intersect any straight line in two points which may be real or imaginary, at finite distances or at infinity. For points at infinity on such curves, the coordinates x and y are infinite, whence the first three terms of highest power are so large that the last three terms may be neglected. Points at infinity then lie on the curve defined by the equation

$$Ax^2 + By^2 + Cxy = 0 \tag{6-3}$$

The directions t of the points at infinity on a curve are defined from the equation of a straight line passing through the origin, $y = x/t$ or $t = x/y$, in which $t = \cot \phi$ (Fig. 6-4). Since the coordinates of the intersection between line and curve must satisfy the equations of both, we have the pair of equations

$$Ax^2 + By^2 + Cxy = 0 \qquad x = ty$$

Elimination of x and y produces

$$At^2 + Ct + B = 0 \tag{6-4}$$

from which

$$t' = \frac{-C + \sqrt{C^2 - 4AB}}{2A} \qquad t'' = \frac{-C - \sqrt{C^2 - 4AB}}{2A}$$

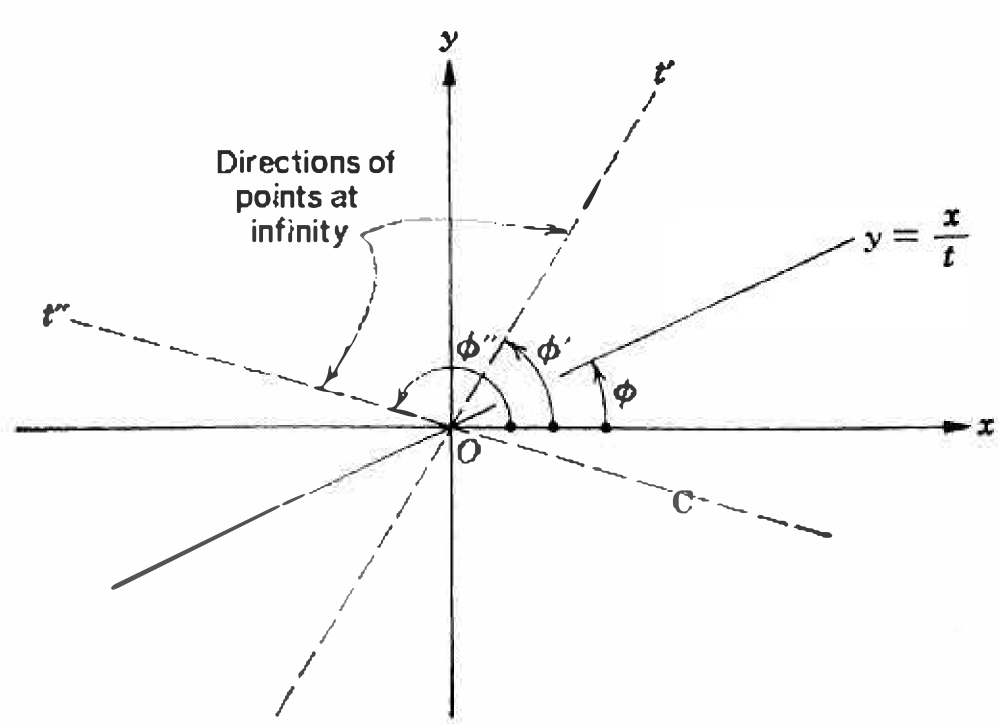


FIGURE 6-4 Directions of points at infinity of a second-order curve.

Each value of t indicates the direction of a point at infinity, i.e., the angles ϕ' and ϕ'' (Fig. 6-4).

The species of conics may be distinguished by the number of real directions in which lines passing through the origin meet the curve at infinity. The discriminant of the expression for t serves for the testing:

1. If $C^2 - 4AB < 0$, the directions t' and t'' are imaginary and the curve is an ellipse. Points at infinity on the curve are imaginary, since they lie on lines having imaginary directions. As an example, consider the circle (a special case of the ellipse) given by the equation

$$x^2 + y^2 = R^2$$

Following the procedure produces

$$t^2 + 1 = 0 \qquad \text{whence } t' = +i,\ t'' = -i$$

which is to say that the circle has no real points at infinity (they would have to lie on real lines, i.e., be lines having a real direction). For analytical purposes, the presence of a direction is tantamount to the existence of some kind of point at infinity. If the direction is found to be real, then a real point exists at infinity on the curve; if the direction is found to be either $+i$ or $-i$, then the imaginary point at infinity is called a *cyclic* point.[1] The circle is then said to have two points at infinity, one in each of the directions i and $-i$, that is, the two cyclic points.

2. If $C^2 - 4AB = 0$, the directions are real and equal, $t' = t''$, and the curve is a parabola. The two real points are coincident at infinity and are a double point.

3. If $C^2 - 4AB > 0$, the directions are real and distinct, $t' \neq t''$, and the curve is a hyperbola. There are two real points at infinity because of the two branches.

Asymptotes

An asymptote to a curve is its tangent at a point at infinity; since second-order curves have two points at infinity, there will be an asymptote for each direction. To find these asymptotes, it is convenient to transform the x, y coordinates of a point P on the curve into coordinates related to the directions of the points at infinity, i.e., into coordinates related to the oblique axes defined by t' and t''. These new coordinates (Fig. 6-5) will be called X and Y and will be expressed in terms of ratios of perpendicular distances measured from P to the new axes t', t'' to perpendicular distances measured from an invariant point $Q(1, 0)$ of the old, or x, y, system, viz.,

[1] A cyclic point is also known as a circular point at infinity.

$$X = \frac{PM}{QH} = \frac{PM}{1 \sin \phi'} \qquad \text{and} \qquad Y = \frac{PN}{QK} = \frac{PN}{1 \sin \phi''}$$

The change in coordinates between the systems is then

$$X = \frac{x \sin \phi' - y \cos \phi'}{\sin \phi'} = x - t'y$$

and

$$Y = \frac{x \sin \phi'' - y \cos \phi''}{\sin \phi''} = x - t''y$$

The equation of the second-order curves [Eq. (6-2)] becomes

$$-XY[2At't'' + 2B + C(t' + t'')] - X(t' - t'')(Dt'' + E) \\ + Y(t' - t'')(Dt' + E) + F(t' - t'') = 0 \quad (6\text{-}5)$$

where t' and t'' are the familiar solutions of Eq. (6-4).

Two conditions apply to the asymptotes:

1. They must be parallel to the directions t' and t'' of the points at infinity.

2. They must intersect the curve at two points at infinity. (Recall that an asymptote is a tangent at infinity and that furthermore a tangent is the limit of a secant whose two points of intersection with the curve have become coincident.)

Now, a line parallel to the t'' axis has, in terms of the new coordinates, an equation of the form $Y =$ const, where the constant determines the distance from the axis t''. Such a line will intersect the curve at infinity as well as at a point P (Fig. 6-5). The X coordinate of that

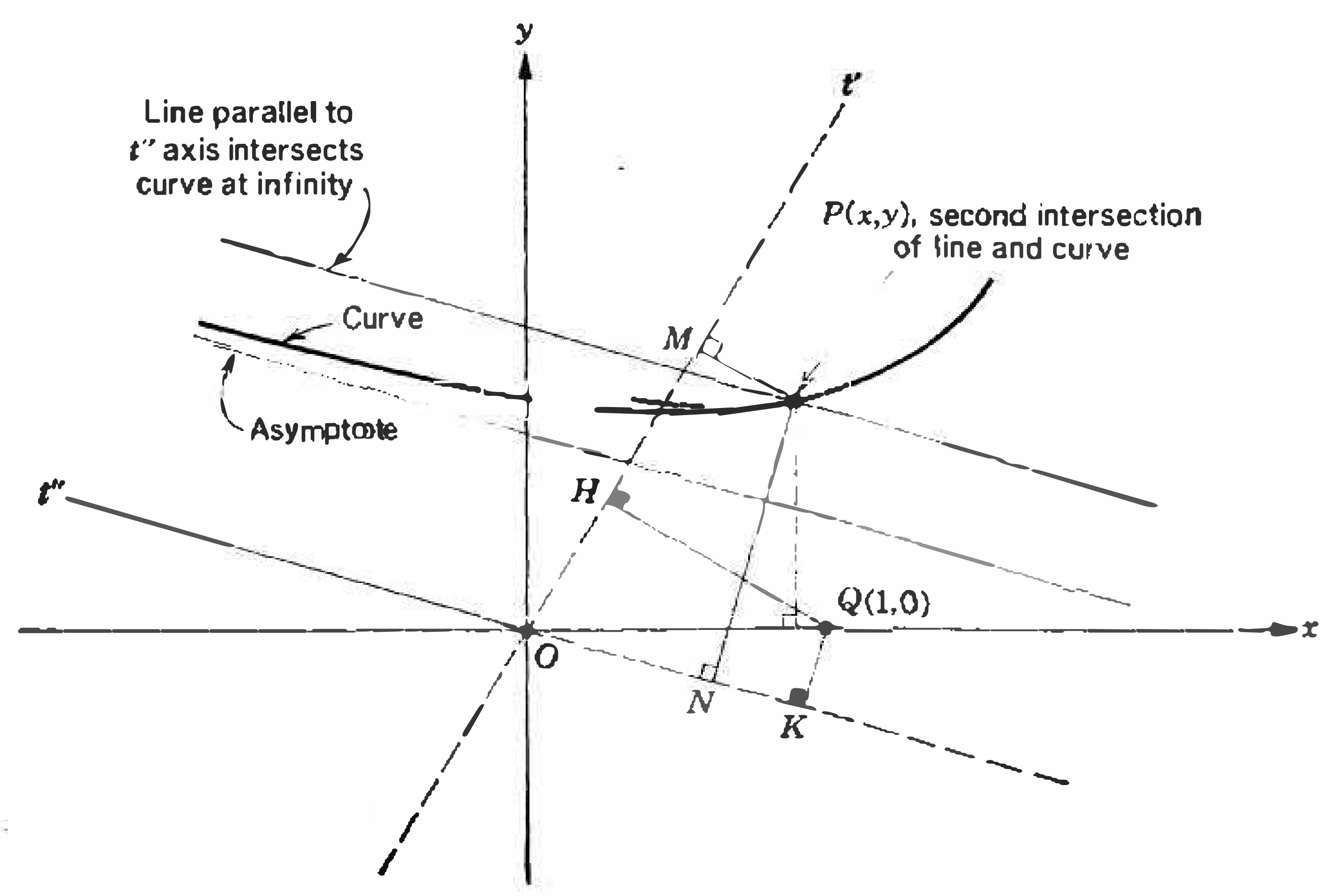


FIGURE 6-5 Relations of oblique axes t' and t'', curve and asymptote.

second point is given by Eq. (6-5) when Y is replaced by the constant of the line. Since property 2 above requires that the asymptote intersect the curve at two points at infinity, P must also be at infinity and its coordinate X must be infinite. With X infinite, the terms independent of X in Eq. (6-5) are trivial and the coefficient of X must be zero, giving

$$-Y[2At't'' + 2B + C(t' + t'')] - (t' - t'')(Dt'' + E) = 0$$

On returning to the rectangular coordinates x and y

$$x - t''y = \frac{Dt'' + E}{\sqrt{C^2 - 4AB}}$$

This is the equation of the asymptote in the t'' direction.

The equation of the asymptote parallel to the t' direction is found in a similar manner by equating to zero the coefficient of Y in Eq. (6-5); this gives

$$x - t'y = -\frac{Dt' + E}{\sqrt{C^2 - 4AB}}$$

As an example, consider the hyperbola

$$x^2 - 2y^2 + 2x + 2 = 0$$

The directions of its points at infinity are found from $t^2 - 2 = 0$, that is, $t' = \sqrt{2}$ and $t'' = -\sqrt{2}$. The asymptotes are then given by

$$x - y\sqrt{2} = -1$$

and $x + y\sqrt{2} = 1$, that is, by the real lines

$$y = \frac{x}{\sqrt{2}} + \frac{1}{\sqrt{2}} \qquad \text{and} \qquad y = -\frac{x}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

As a further example, consider the circle $x^2 + y^2 - R^2 = 0$, for which we already know that its points at infinity, the cyclic points, lie in the directions $t' = i$ and $t'' = -i$. Upon rewriting the equation of this circle in terms of $X = x - iy$ and $Y = x + iy$, it becomes $XY - R^2 = 0$. The coefficient of X is Y, which must be zero to satisfy the equation with X infinite, whence the asymptote parallel to the direction $t'' = -i$ is

$$Y = 0 \qquad \text{or} \qquad x + iy = 0$$

Similarly, the asymptote parallel to the direction $t' = i$ is found to be

$$X = 0 \qquad \text{or} \qquad x - iy = 0$$

Both these asymptotes are imaginary and possess one real point, the origin of coordinates at which they intersect.

Asymptotes of the Four-bar Coupler Curve

Examination of the curve equation (6-1) shows its terms of highest degree to be $x^6 + y^6$. In consequence the significant equation for points at infinity is $x^6 + y^6 = 0$. As with the second-order curves, this equation is taken with $y = x/t$, giving

$$t^6 + 1 = 0 \qquad \text{or} \qquad (t^3 - i)(t^3 + i) = 0 \tag{6-6}$$

as the equation the directions of the points at infinity must satisfy.

This equation of the sixth degree has six solutions—all imaginary—and the coupler curve therefore has six imaginary points at infinity in the directions

$$t' = i \qquad \text{and} \qquad t'' = -i$$

each being a triple solution of Eq. (6-6). The cyclic points are therefore triple points of the coupler curve. The asymptotes, also imaginary, must be parallel to the directions i and $-i$, and since each cyclic point is triple, there will be a total of six asymptotes forming two sets of three parallel imaginary lines. The determination of the asymptotes follows the method described for second-order curves.

In order to carry out the computations more conveniently, Eq. (6-1) is first rewritten, use being made of the identity

$$U^2 + V^2 = (U + iV)(U - iV)$$

with U and V as the parentheses appearing in the left-hand member of Eq. (6-1). Thus, after some algebraic manipulations,

$$\begin{aligned}\{\sin\alpha(x^2 + y^2 + b^2 - r^2)(ix - y - ip)e^{-i\gamma} \\ + \sin\beta[(x-p)^2 + y^2 + a^2 - s^2](y - ix)\} \\ \times \{\sin\alpha(x^2 + y^2 + b^2 - r^2)(-ix - y + ip)e^{i\gamma} \\ + \sin\beta[(x-p)^2 + y^2 + a^2 - s^2](y + ix)\} \\ = 4k^2 \sin^2\alpha \sin^2\beta \sin^2\gamma[x(x-p) + y^2 - py\cot\gamma]^2\end{aligned}$$

The substitution of X for $x + iy$ and Y for $x - iy$ in this equation yields

$$\begin{aligned}\{-\sin\alpha(XY + b^2 - r^2)(X - p)e^{-i\gamma} \\ + \sin\beta[(X-p)(Y-p) + a^2 - s^2]X\} \\ \times \{\sin\alpha(XY + b^2 - r^2)(Y - p)e^{i\gamma} \\ - \sin\beta[(X-p)(Y-p) + a^2 - s^2]Y\} \\ = 4ik^2 \sin^2\alpha \sin^2\beta \sin^2\gamma \\ \times \left\{XY - \frac{p}{2}[X(1 - i\cot\gamma) + Y(1 + i\cot\gamma)]\right\}^2\end{aligned} \tag{6-7}$$

The highest powers of X and Y in this equation are X^3 and Y^3; the asymptotes are therefore obtained by equating their coefficients to zero.

The coefficient of X^3 is

$$[-\sin \alpha Y e^{-i\gamma} + \sin \beta (Y - p)][\sin \alpha Y (Y - p) e^{i\gamma} - \sin \beta Y (Y - p)]$$

The requirement that this coefficient be zero will be met if any one of the three equations

$$Y = 0 \qquad Y = p \qquad \sin \alpha Y e^{-i\gamma} = \sin \beta (Y - p)$$

is satisfied. When written in terms of cartesian coordinates, these equations become

$$x - iy = 0 \qquad x - p - iy = 0$$
$$x - p\frac{\sin \beta \cos \alpha}{\sin \gamma} - i\left(y - p\frac{\sin \beta \sin \alpha}{\sin \gamma}\right) = 0 \tag{6-8}$$

They represent a set of three parallel asymptotes corresponding to the triple point at infinity in the direction $t' = i$. (Parallel lines intersect at infinity.) Another set of three parallel asymptotes in the direction $t'' = -i$ is obtained by equating to zero the coefficient of Y^3 in Eq. (6-7), i.e.,

$$[-\sin \alpha Y (X - p) e^{-i\gamma} + \sin \beta (X - p) X][\sin \alpha X e^{i\gamma} - \sin \beta (X - p)] = 0$$

This requirement will be satisfied if one of the following equations holds:

$$X = 0 \qquad X = p \qquad \sin \alpha X e^{i\gamma} = \sin \beta (X - p)$$

These transformed into cartesian coordinates are

$$x + iy = 0 \qquad x - p + iy = 0$$
$$x - p\frac{\sin \beta \cos \alpha}{\sin \gamma} + i\left(y - p\frac{\sin \beta \sin \alpha}{\sin \gamma}\right) = 0 \tag{6-9}$$

Singular Foci

For a curve passing through the cyclic points—as the four-bar coupler curve does—the points of intersection of asymptotes of the curve in the direction of the cyclic points are called singular foci.[1] Since the coupler curve has two sets of three parallel asymptotes of this type, it has a total of nine singular foci. Examination of Eqs. (6-8) and (6-9) shows that three of these intersections are real; i.e., there are three real singular foci. They are the origin O_A, the point O_B ($x = b$, $y = 0$), and a third point O_C of coordinates

$$x_{O_C} = p\frac{\sin \beta \cos \alpha}{\sin \gamma} \qquad y_{O_C} = p\frac{\sin \beta \sin \alpha}{\sin \gamma}$$

[1] E. N. Laguerre, Sur les courbes planes algébriques (1865), from "Oeuvres de Laguerre," vol. II, Gauthier-Villars, Paris, 1905.

which also lies on the circle of foci defined earlier in this section. It may be further observed (Fig. 6-3) that the angles at O_A, O_B, and O_C of the triangle $O_AO_BO_C$ are respectively equal to the angles at A, B, and M of the triangle ABM: the triangles $O_AO_BO_C$ and ABM are therefore similar.

In summary: Starting from the coupler-curve equation, a series of manipulations identified the points O_A, O_B, and O_C with unique properties of a sixth-degree equation, viz., that they constitute what are called the three real singular foci of the curve. Repeating almost exactly the words of Roberts, we are led to remark that, since the singular foci are similarly related to the coupler curve, we might have taken as fixed centers the focus O_A and the third focus O_C, and by means of links of suitable lengths we should obtain the same coupler curve. In like manner we might have taken as centers O_B and O_C. We conclude, then, that the coupler curve can be described in three different ways by four-bar linkages having fixed centers at any two of the singular foci and couplers forming triangles that are similar to the triangle of the three foci.

This problem of the triple generation of coupler curves will be reconsidered in Sec. 6-4 by geometric means, and a complete determination of the three four-bar linkages will be given there.

6-3 DOUBLE POINTS AND SYMMETRY

Coupler curves for 10 four-bar linkages are shown in Figs. 6-31 to 6-40 of the appendix of this chapter.[1] The four-bar linkages generating these curves fulfill the Grashof condition and are of the crank-rocker type. The crank at the left has unit length in each figure. The lengths of the coupler A, the rocker B, and the frame C are given in multiples of the unit crank length. Coupler points are indicated by small circles; they are spaced at unit intervals on a rectangular grid carried by the coupler, giving the coupler points convenient coordinates with respect to the coupler. The length of each dash corresponds to 10° of crank rotation (in the original there were twice as many dashes, each worth 5°). From the length of the dashes a fair idea of the linear velocity of a coupler point may be gained; the estimate of the linear acceleration will be rougher.

Coupler curves have a variety of shapes, as inspection of the figures will show. These figures, selected for their possible utilization in problems of this text, do not show all possible features because of the arbitrary disposition of the points on the coupler grid and the ratios of link lengths. In general, coupler curves may possess double points

[1] Redrawn from "Analysis of the Four Bar Linkage" by Hrones and Nelson by permission of the Massachusetts Institute of Technology Press.

(cusps and crunodes) as well as symmetry about an axis. The properties of the curves, difficult to study algebraically because of the unwieldy equation, will be examined geometrically. This chapter devotes itself to geometric features other than curvature; this very important topic is the subject of Chap. 7.

A double point is a point on a curve at which the curve has two tangents. A double point may be of two types: a *crunode*, at which the tangents are distinct, the curve crossing itself; and a *cusp*, at which the tangents are coincident, the curve being tangent to itself.

Cusps

The most familiar example of the cusp is derived from the curve traced by a point on the periphery of a rolling wheel (Fig. 6-6a). The curve is the common cycloid, one of the special cases of the trochoid. We should recognize, before going further, that P is a point on the moving centrode π_m and that I, a point on the fixed centrode π_f, was the instantaneous center of velocity for the moment that P and I were coincident. It is quite evident that P came down to I, stopped, and moved off in a direction opposite to that of approach. The velocity of P, although zero at an instant, experienced no discontinuity. We note that a cusp is a curve property associated with a point on a moving centrode and with the relative motion of centrodes. Thus, if a coupler point happens to lie on the moving centrode of the coupler, a cusp will develop at the position where point and instant center are coincident; furthermore, the tangent

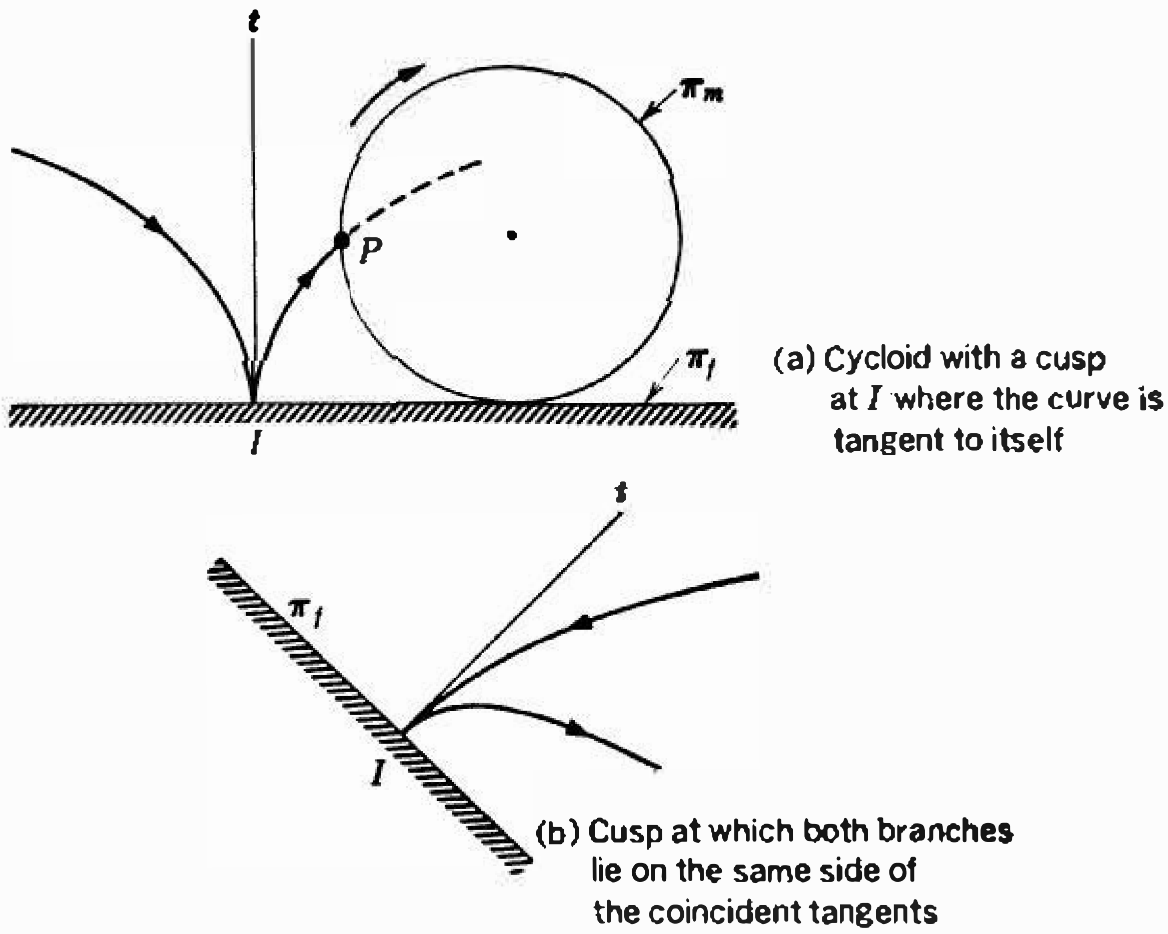
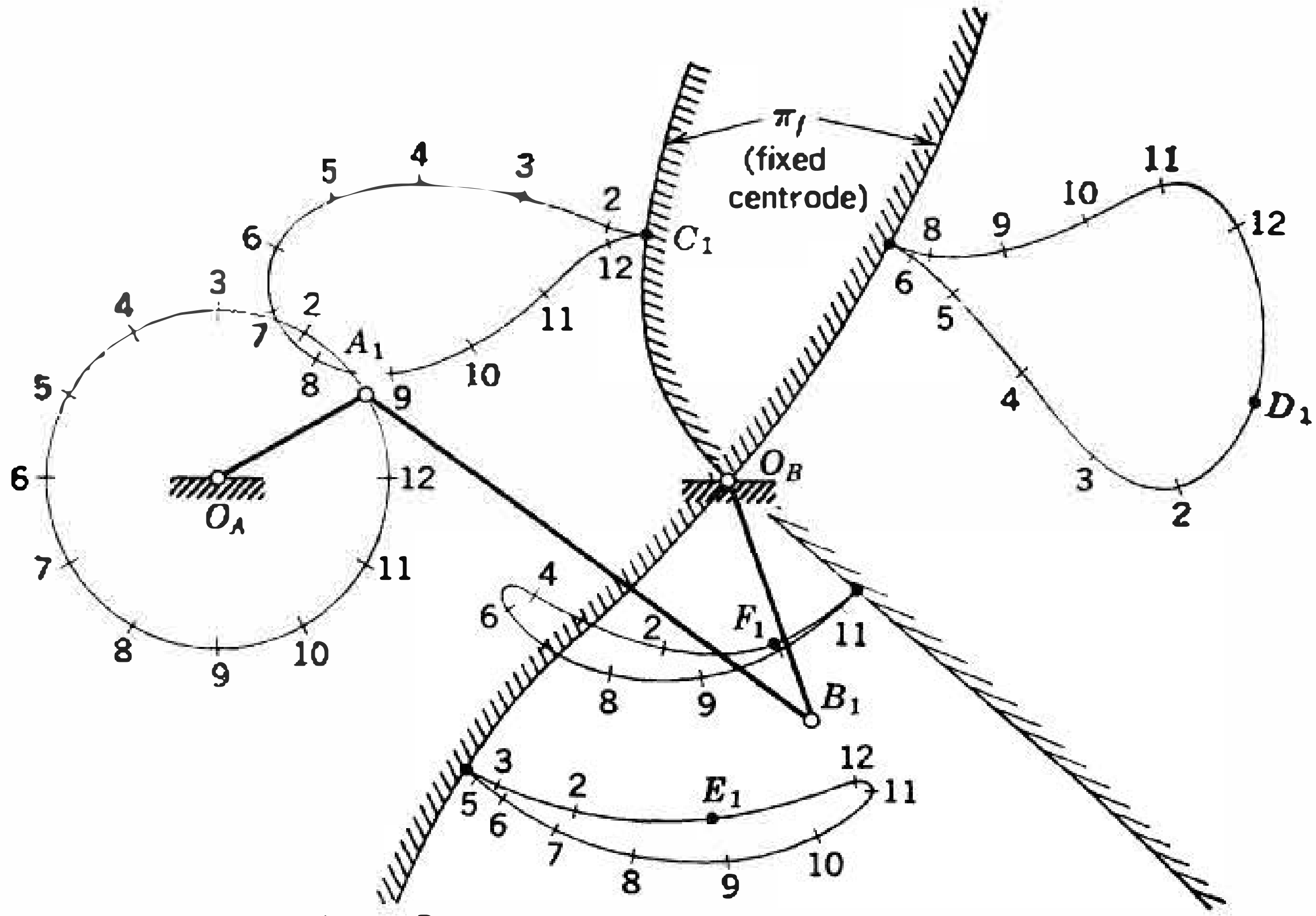
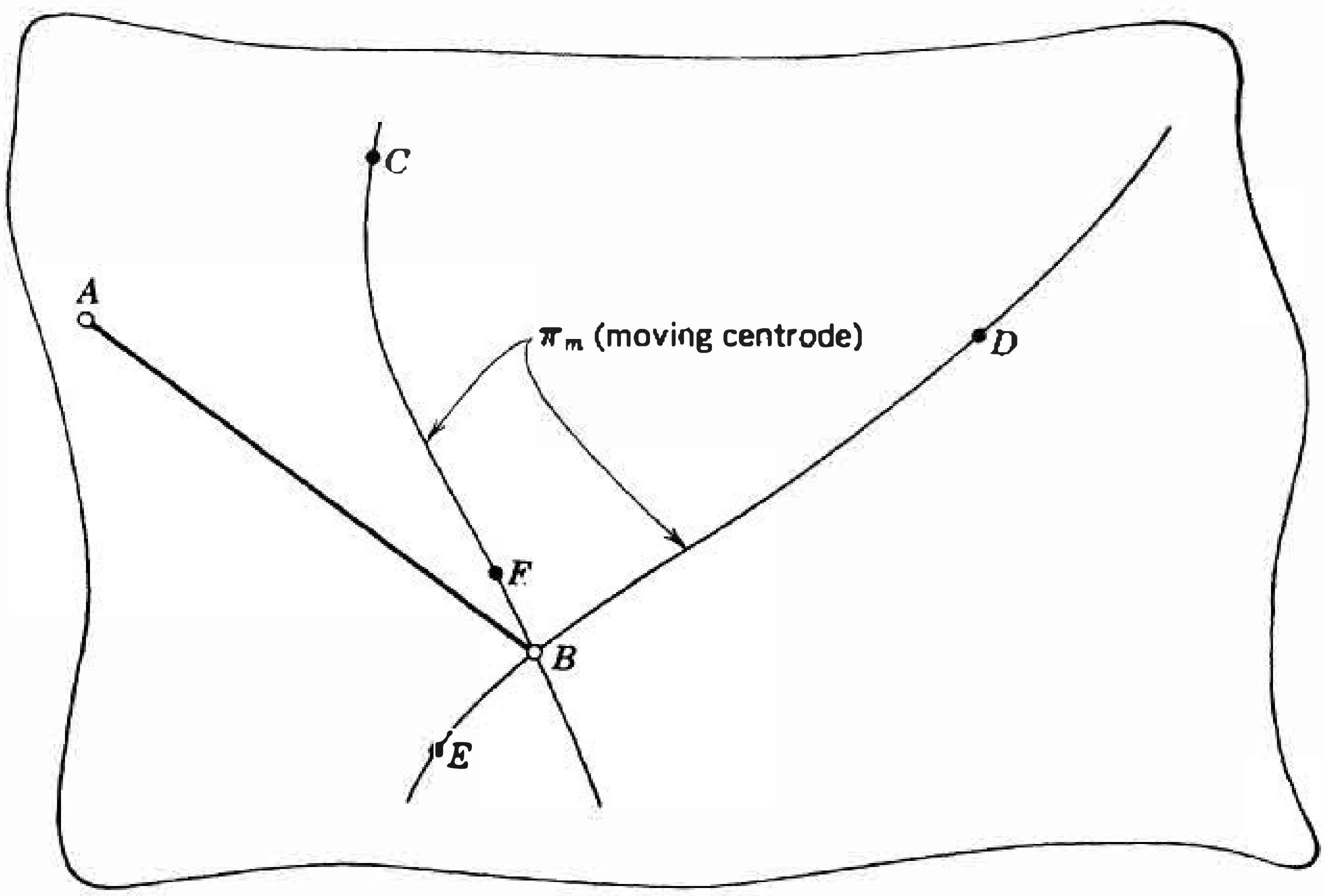


FIGURE 6-6 The cusp, a double point with coincident tangents.



(a) Four-bar linkage $O_A ABO_B$ (in position 1) and fixed centrode π_f



(b) Plane of coupler showing the moving centrode π_m

FIGURE 6-7 Cusps of coupler curves.

of the cusp will be normal to the fixed centrode. The cusp may also take the form shown in Fig. 6-6*b*, in which both branches of the curve lie on the same side of the common tangent; this tangent is also normal to the fixed centrode.

We may see the action in a four-bar from Fig. 6-7. The linkage is shown in its entirety in Fig. 6-7*a*; the coupler link is *AB*, located in

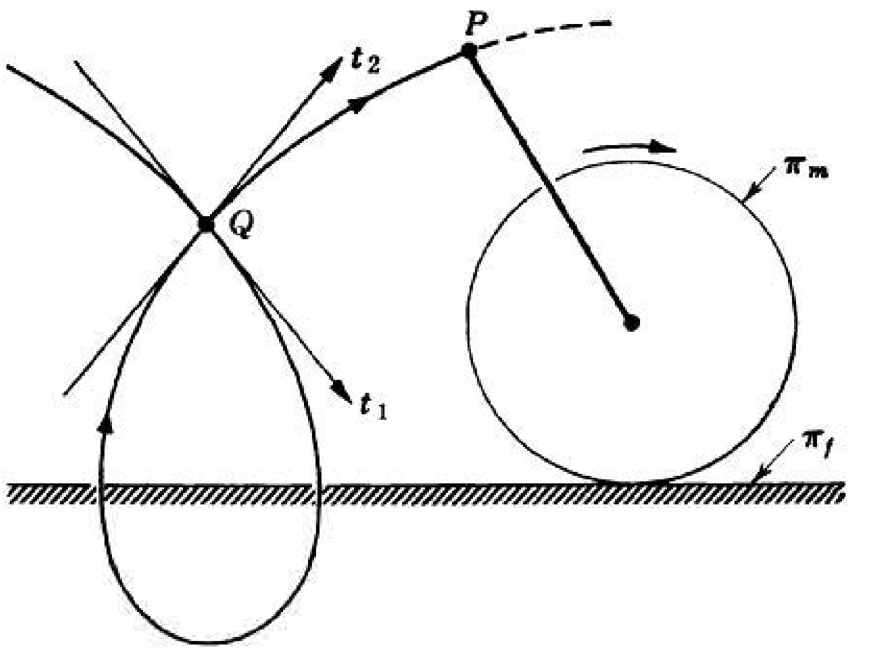


FIGURE 6-8 Prolate cycloid with a (symmetrical) crunode at Q.

position 1. Figure 6-7*b* shows the isolated coupler, a portion of its plane with the moving centrode sketched in, and four coupler points—C, D, E, and F—located on the moving centrode. The curves that these points trace on the fixed plane are shown in Fig. 6-7*a*; each coupler curve shows a cusp for the instant at which the point on π_m touches π_f. For the instant depicted, the moving and fixed centrodes are in contact at C_1, whence the cusp there. With rotation of the crank, the centrodes roll, and cusps are formed by the other coupler points at appropriate positions.

Crunode

The crunode is a more obvious form of double point than the cusp; as noted earlier, the curve crosses itself and therefore has two distinct tangents. A simple example again derives from a special case of the trochoid, specifically the prolate cycloid (Fig. 6-8). With regard to the four-bar, it will be seen that a crunode is related to the circle of singular foci corresponding to the coupler point.

A coupler curve with two crunodes is shown in Fig. 6-9. For

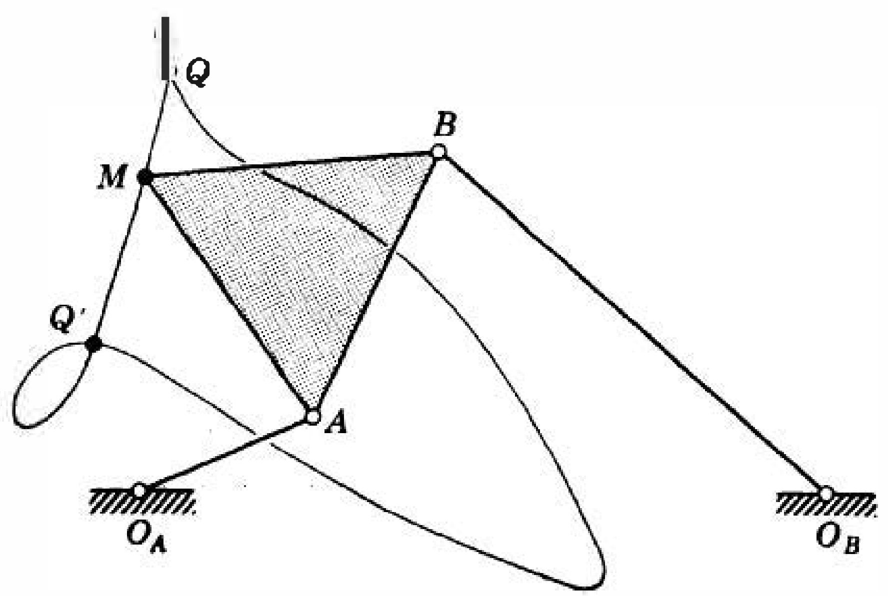


FIGURE 6-9 Coupler curve with double points Q and Q'.

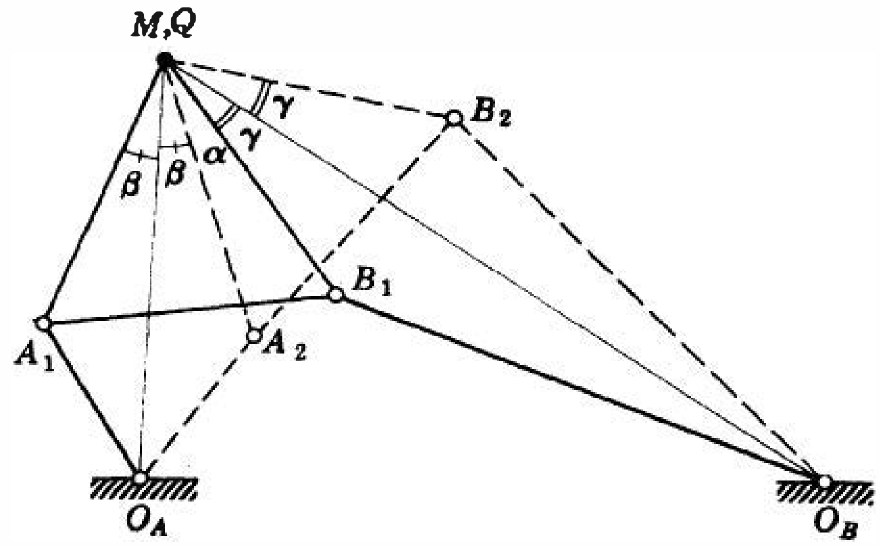


FIGURE 6-10 Two positions of a four-bar corresponding to a double point at Q.

the crunode Q (Fig. 6-10), there must be two positions of the coupler AB such as A_QB_1 and A_QB_2 for which the coupler point M assumes the same position[1] Q on the fixed plane. Considering the quadrilateral $O_AA_1QA_2$, in which $O_AA_1 = O_AA_2$ and $A_1Q = A_2Q$, it is clear that O_AQ bisects the angle $A_1QA_2 = 2\beta$. Similarly, O_BQ is the bisector of the angle $B_1QB_2 = 2\gamma$. However, since the coupler is rigid, $2\beta = 2\gamma$, or $\beta = \gamma$; the vertex angle of A_1QB_1 is $2\beta + \alpha$; for A_2QB_2 it is $\alpha + 2\gamma$. Since $\beta = \gamma$, the angle $O_AQO_B = \beta + \alpha + \gamma$ is therefore the same as the vertex angle of the coupler,

$$O_AQO_B = A_1QB_1 = A_2QB_2$$

Consider now a triangle $O_AO_BO_C$ similar to ABM constructed with O_AO_B as base (Fig. 6-11); the last equalities then yield

$$O_AQO_B = O_AO_CO_B$$

which implies that the point Q must necessarily belong to the circle passing through points O_A, O_B, O_C. It may be recalled that this point O_C

[1] This assumption is valid provided that the IC be uniquely defined and be distinct from Q. See Bricard, vol. II, p. 308.

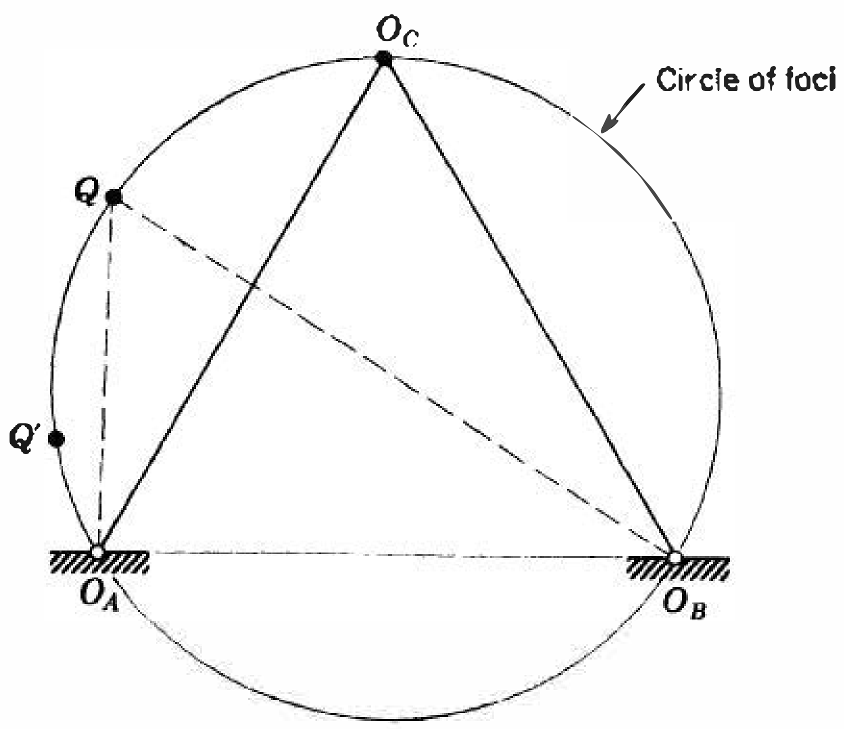


FIGURE 6-11 The double points of a coupler curve, if any, lie on the circle of foci.

is the third singular focus of the coupler curve (Sec. 6-2) and the circle $O_AO_BO_C$ the circle of foci whence a coupler curve has crunodes at each of its intersections with the corresponding circle of foci. If the curve does not intersect the circle of foci, then it has no double points.

Symmetry

Coupler curves which are symmetrical about an axis may be generated by a four-bar linkage with a coupler base AB and follower of equal length, $AB = O_BB$ (Fig. 6-12). The coupler point generating a symmetrical curve must then lie anywhere on the circle centered at B and passing through A.

Since $BO_B = BA = BM$, the above circle also passes through O_B and the inscribed angle AO_BM satisfies the relation

$$AO_BM = \frac{ABM}{2} = \frac{\hat{B}}{2} = \text{const}$$

Consider now the linkage in two positions $O_AA_1B_1O_B$ and $O_AA_2B_2O_B$ for which points A_1 and A_2 are symmetrical with respect to the line of fixed centers O_AO_B (Fig. 6-13). For these positions, triangles $O_BA_1B_1$ and $O_BA_2B_2$ are equal, since corresponding sides are equal, whence $\beta_1 = \beta_2$. Now, the isosceles triangles $O_BB_1M_1$ and $O_BB_2M_2$ are equal,

$$O_BM_1 = O_BM_2$$

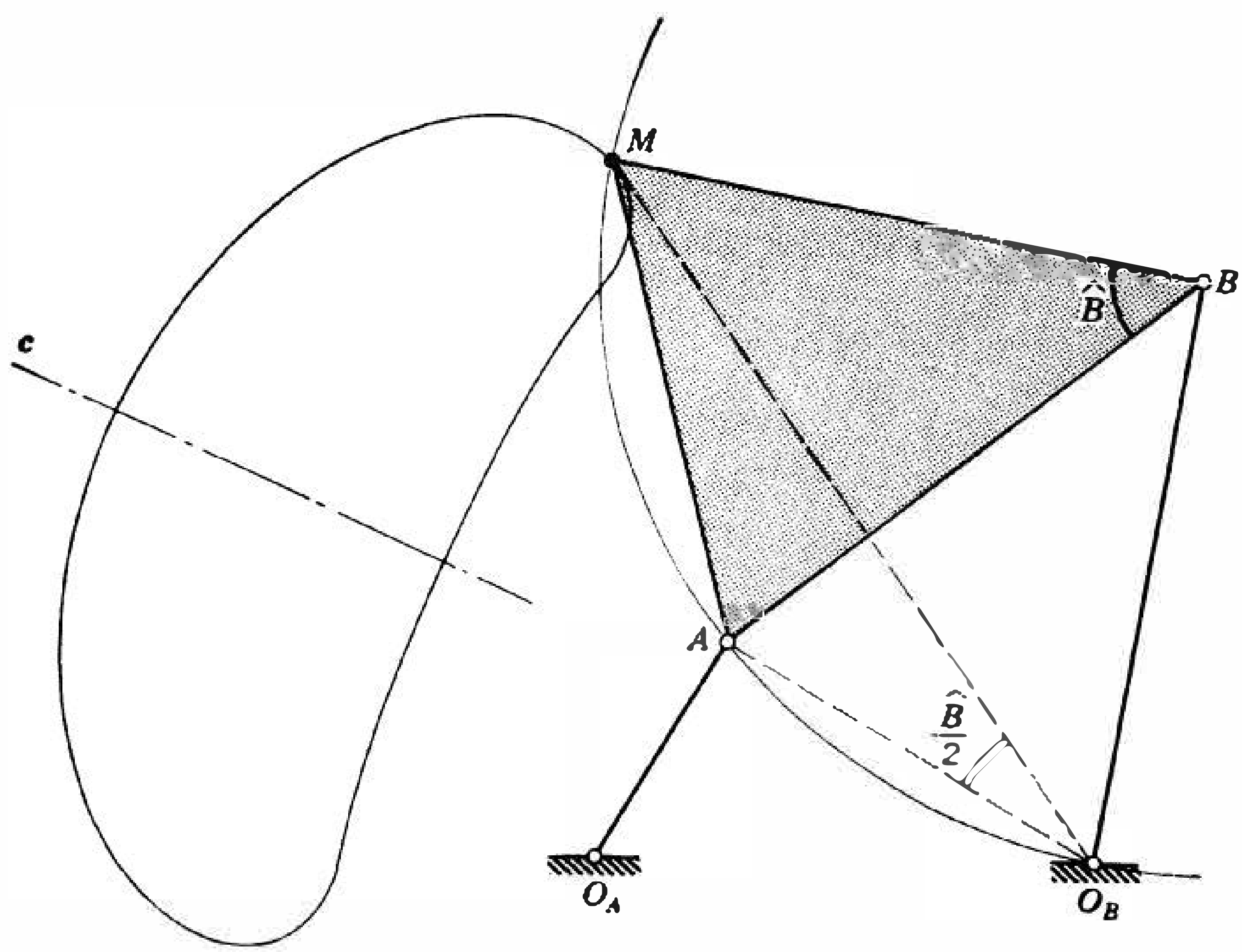


FIGURE 6-12 Symmetrical coupler curve.

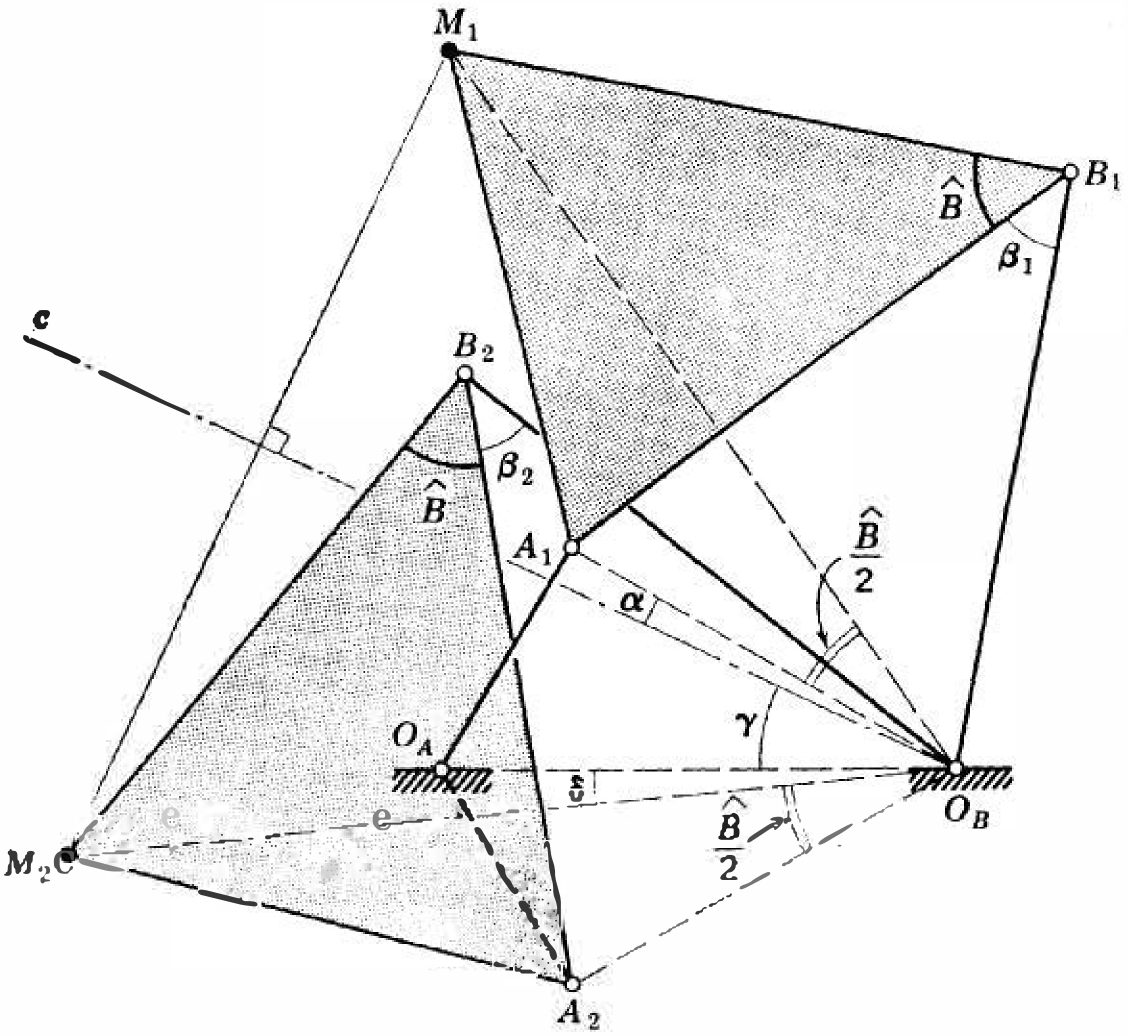


FIGURE 6-13 Two positions of a four-bar corresponding to symmetrical points M_1 and M_2 on coupler curve.

and the midnormal c to M_1M_2 passes through O_B and bisects the angle $M_1O_BM_2$, whence

$$\delta + \gamma = \alpha + \frac{\hat{B}}{2}$$

Since the angles $O_AO_BA_1$ and $O_AO_BA_2$ are also equal (A_1 symmetric to A_2 with respect to O_AO_B),

$$\gamma + \alpha = \frac{\hat{B}}{2} + \delta$$

Adding the last two equations yields

$$\gamma = \frac{\hat{B}}{2}$$

The midnormal c to M_1M_2 therefore makes a constant angle with the line of fixed centers O_AO_B, whence it is an axis of symmetry for the coupler curve generated by point M. It may further be noted that symmetric points on the coupler curve correspond to symmetric positions of the crank with respect to the line of centers O_AO_B.

The foregoing situation was a sufficient condition for symmetry, but not a necessary one. For example, the Watt straight-line mechanism, in which the coupler point lies on the coupler line AB, traces a coupler curve symmetrical with respect to the line of centers O_AO_B.* Note that the line of centers is also the circle of foci and that the curve has a crunode at its intersection with the line of centers.

6-4 THE ROBERTS-CHEBYSHEV THEOREM[1]

A remarkable property of the planar four-bar linkage is found in the Roberts-Chebyshev theorem: *Three different planar four-bar linkages will trace identical coupler curves.* The development of this theorem is presented in modern form, and extensions related to six-bar and slider-crank mechanisms are given.

Coupler Curves and Cognate Linkages

Kinematically equivalent four-bar linkages—usually called equivalent linkages—are commonly used for velocity and acceleration analyses of planar direct-contact mechanisms such as cams and noncircular gears. The equivalent linkage is really an analog of the direct-contact mechanism, and it will be remembered that the dimensions of the linkage change with time, i.e., with the position of the mechanism: the equivalent linkage might well be called an instantaneous linkage.

When the designer's concern is only with the curve traced by a coupler point of a planar four-bar, then other planar linkages tracing an identical coupler-point curve may be found by the application of the Roberts-Chebyshev theorem. For want of a name, these linkages—related through their common coupler curve—will be called *cognate* linkages. It is emphasized here that these linkages do not look alike: their relation stems only from the identical coupler curves they trace. However, in contrast to the equivalent linkages, the dimensions of the cognate linkages do *not* change with time, and a cognate linkage may therefore be substituted for the entire cycle of the motion of its related linkage: this provides a linkage whose space requirements may be more favorable than that of the (original) linkage being replaced. The velocity and acceleration characteristics of the cognate linkages will not in general be identical link for link.

* See Prob. 6-4.

[1] Much of the material of this and the following section appeared in *Machine Design*, Apr. 16, 1959, and is reprinted by courtesy of the Penton Publishing Company, Cleveland.

The Roberts-Chebyshev Theorem

Roberts[1] and Chebyshev were mathematicians of considerable stature during the latter part of the nineteenth century. Both were members of the Royal Society, and both studied, among many other things, coupler curves of four-bar linkages.

The French and German literature speak of the "Roberts' theorem," while the dual name appears in the Russian. It seems proper to use both names: Roberts announced his discovery in 1875, Chebyshev in 1878. Except for the final result, there is no resemblance between the two developments, for the approaches are as different as they can be.

To explore the theorem, *three different planar four-bar linkages will trace identical coupler curves,* we shall consider a four-bar linkage and see what can happen to it (Fig. 6-14). The given four-bar is O_AABO_B, shown in Fig. 6-14*a* (drawn in solid lines) and carrying the coupler point M, which traces a planar curve (not illustrated). On the left, the dash-line parallelogram involving O_AA_1 and MA_1 is added, and the triangle A_1MC_1 is constructed similar to triangle ABM—note where the angles are. On the right, a similar construction of a parallelogram brings us to the point C_2. A third parallelogram is then constructed to bring us to the point O_C, which we shall assume for the time being to be a frame point.

We can distinguish a total of three four-bar linkages mutually connected at their coupler points M:

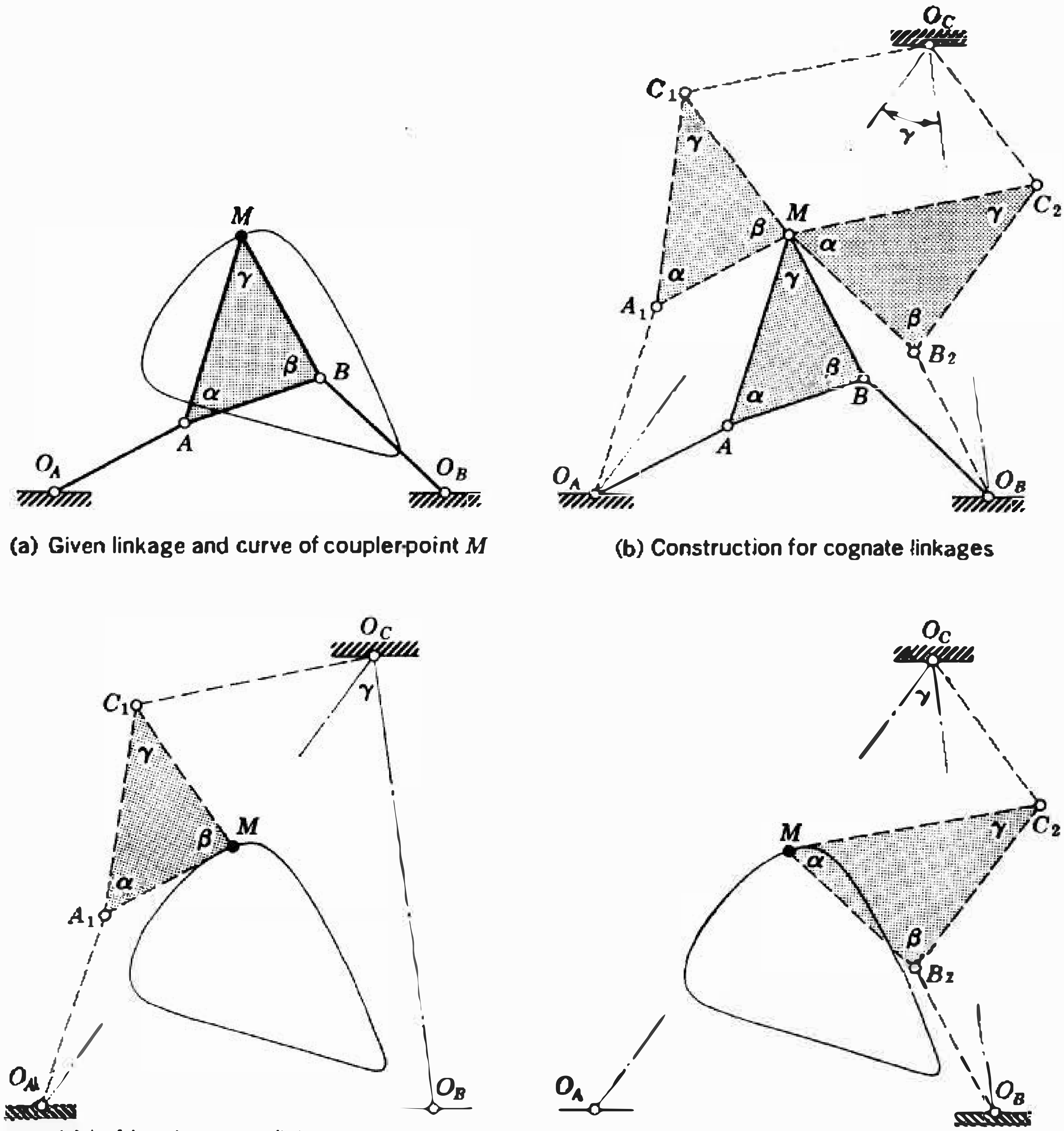
O_AABO_B (solid)	the given linkage
$O_AA_1C_1O_C$	link O_AO_C not shown
$O_BB_2C_2O_C$	link O_BO_C not shown

We also recognize, intuitively because of the parallelograms, that the entire complex of 10 links is movable. (The 10-bar linkage is, in fact, overclosed—there are 2 more links than necessary; thus, 2 links such as O_AA_1 and O_BB_2 could be removed, to leave the 8 remaining links as a constrained mechanism.)

It is thus clear that each of the three four-bar linkages traces, through the identical coupler point M, identical coupler curves. This allows the replacement of the given linkage by either of the other two, whose space requirements are different because of link lengths and one frame-joint location, O_C. These are then cognate linkages, the relation stemming from the common coupler-point curve.

For the theorem to be true, it is necessary that the frame joint O_C be truly fixed, as are O_A and O_B: we must justify our earlier assumption.

[1] Samuel Roberts (1827–1913). Another Roberts, Richard (1789–1864), has given his name to the Roberts "straight-line" motion, which he described prior to 1841.



(a) Given linkage and curve of coupler-point M

(b) Construction for cognate linkages

(c) Lefthand cognate linkage and curve of coupler-point M

(d) Righthand cognate linkage and curve of coupler-point M

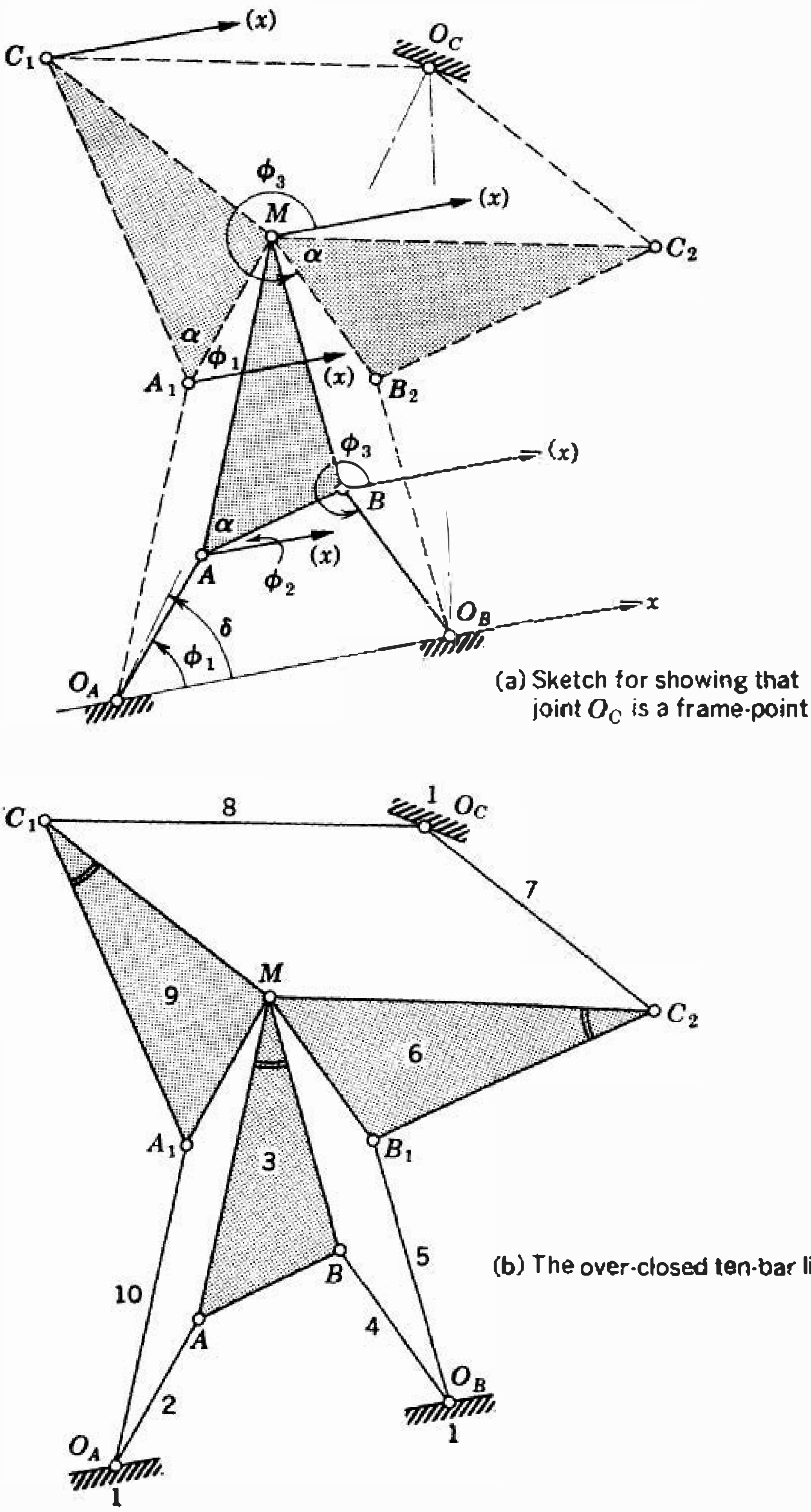
FIGURE 6-14 Three cognate four-bars and their identical coupler curves—the Roberts-Chebyshev theorem.

To show this by purely geometric construction is tedious, but it engaged the attention of mathematicians such as Cayley, Clifford, and Kleiber, among others. The simplest demonstration seems to be that of Ja. B. Schor (1941) making use of complex numbers: it is given by Bloch and is somewhat as follows.[1] In Fig. 6-15a the point O_C will be fixed if

$$z = O_A O_C e^{i\delta} = \text{const} \tag{6-10}$$

[1] The complex-number proof goes back to Hart in the "Messenger of Mathematics," p. 32, 1883 (quaternion proof of the triple generation of three-bar motion).

This means that O_AO_C and the angle δ will have to be expressed in terms of the invariant dimensions of the first linkage O_AABO_B and must be independent of the angular displacements ϕ_1, ϕ_2, and ϕ_3 of this linkage. These angles are measured counterclockwise from the x axis, for convenience laid through the line O_AO_B; ϕ_1 and ϕ_3 also appear in the cognate linkages.



(a) Sketch for showing that joint O_C is a frame-point

(b) The over-closed ten-bar linkage

FIGURE 6-15 Cognate four-bar linkages.

Starting from O_A, and taking advantage of the parallelograms, we may write

$$z = O_AA_1e^{i(\phi_2+\alpha)} + A_1C_1e^{i(\phi_1+\alpha)} + C_1O_Ce^{i(\phi_3+\alpha)} \tag{6-11}$$

From the parallelograms and similar triangles, the first factors of the above equation may be found in terms of the link lengths of the given linkage,

$$O_AA_1 = AM \qquad \frac{A_1C_1}{A_1M} = \frac{AM}{AB} \tag{a}$$

or since $A_1M = O_AA$,

$$A_1C_1 = \frac{AM}{AB}O_AA \tag{b}$$

$$\frac{MC_2}{MB_2} = \frac{AM}{AB}$$

or since $MC_2 = C_1O_C$ and $MB_2 = BO_B$,

$$C_1O_C = \frac{AM}{AB}BO_B \tag{c}$$

With a hint from Eqs. (*b*) and (*c*), Eq. (*a*) may be put into a more convenient form,

$$O_AA_1 = \frac{AM}{AB}AB \tag{a'}$$

Substituting Eqs. (*a'*), (*b*), and (*c*) in Eq. (6-11), ordering the terms, and factoring, we get

$$z = \frac{AM}{AB}e^{i\alpha}(O_AAe^{i\phi_1} + ABe^{i\phi_2} + BO_Be^{i\phi_3}) \tag{6-12}$$

The terms in the parentheses are recognized as a vector sum equivalent to O_AO_B, or

$$z = \frac{AM}{AB}O_AO_Be^{i\alpha} \tag{6-13}$$

On comparing Eq. (6-10) with Eq. (6-13), it is seen that (1) $O_AO_C = (AM/AB)O_AO_B =$ const and (2) $\delta = \alpha =$ const.

This shows not only that O_C is a fixed point but also that the triangle $O_AO_CO_B$ is similar to the original triangle of the coupler, AMB. From this last information, we may immediately locate the point O_C: it is merely necessary to construct on the frame link O_AO_B a triangle similar to AMB; the upper vertex is O_C.

The location of O_C was part of Roberts' demonstration. Cayley suggested the plan of Fig. 6-16; it is a simple way of determining the link lengths of the cognate mechanisms. In this construction the given linkage is pulled out straight, as it were, and the parallel lines are drawn to define the cognate linkages.

The four-bar linkage invoked so far for demonstration purposes had the coupler point M lying to one side of the line AB. The theorem still applies when the coupler point lies on the line AB, either between A and B or beyond A or B. The determination of the cognate linkages requires a bit more care now, for all the links of Fig. 6-16 will lie on top of each other, the cognate couplers having become "lines."

As M moves closer to the line AB, it is apparent that the general geometry is preserved as the links approach collinearity with each other and the line O_AO_B; O_AO_B will be divided in the same ratio as M divides AB. Note that C_1 and C_2 will divide A_1M and MB_2 similarly.

As an example of the case where M lies between A and B, consider the linkage of Fig. 6-17, which would be a Watt linkage if the coupler point M lay at the midpoint of AB. Referring to Fig. 6-17b, we then note the following:

1. The frame point O_C lies on the line O_AO_B and divides it in the same ratio as M divides the line AB.

2. O_AA_1 is parallel to AM, and MA_1 is parallel to O_AA, thus defining A_1.

3. O_BB_2 is parallel to MB, and MB_2 is parallel to O_BB, thus defining B_2.

4. C_1 will divide the line A_1M in the same ratio as M divided AB, allowing the link C_1O_C to be drawn to complete the left-hand linkage.

5. C_2 will divide the line MB_2 in the same ratio as M divided AB, allowing the link C_2O_C to be drawn to complete the right-hand linkage.

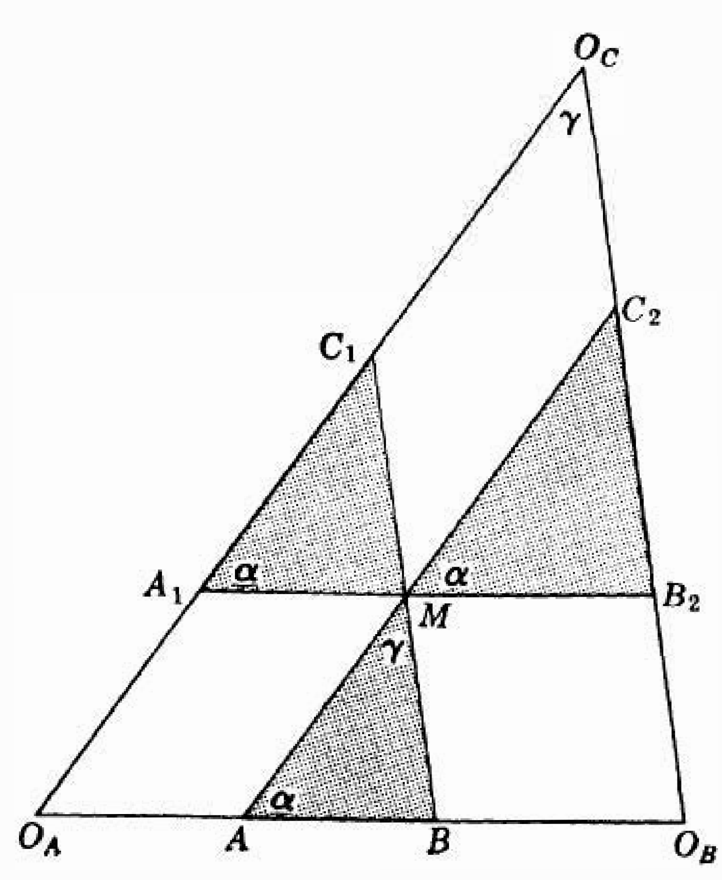


FIGURE 6-16 Plan for determining the lengths of cognate links.

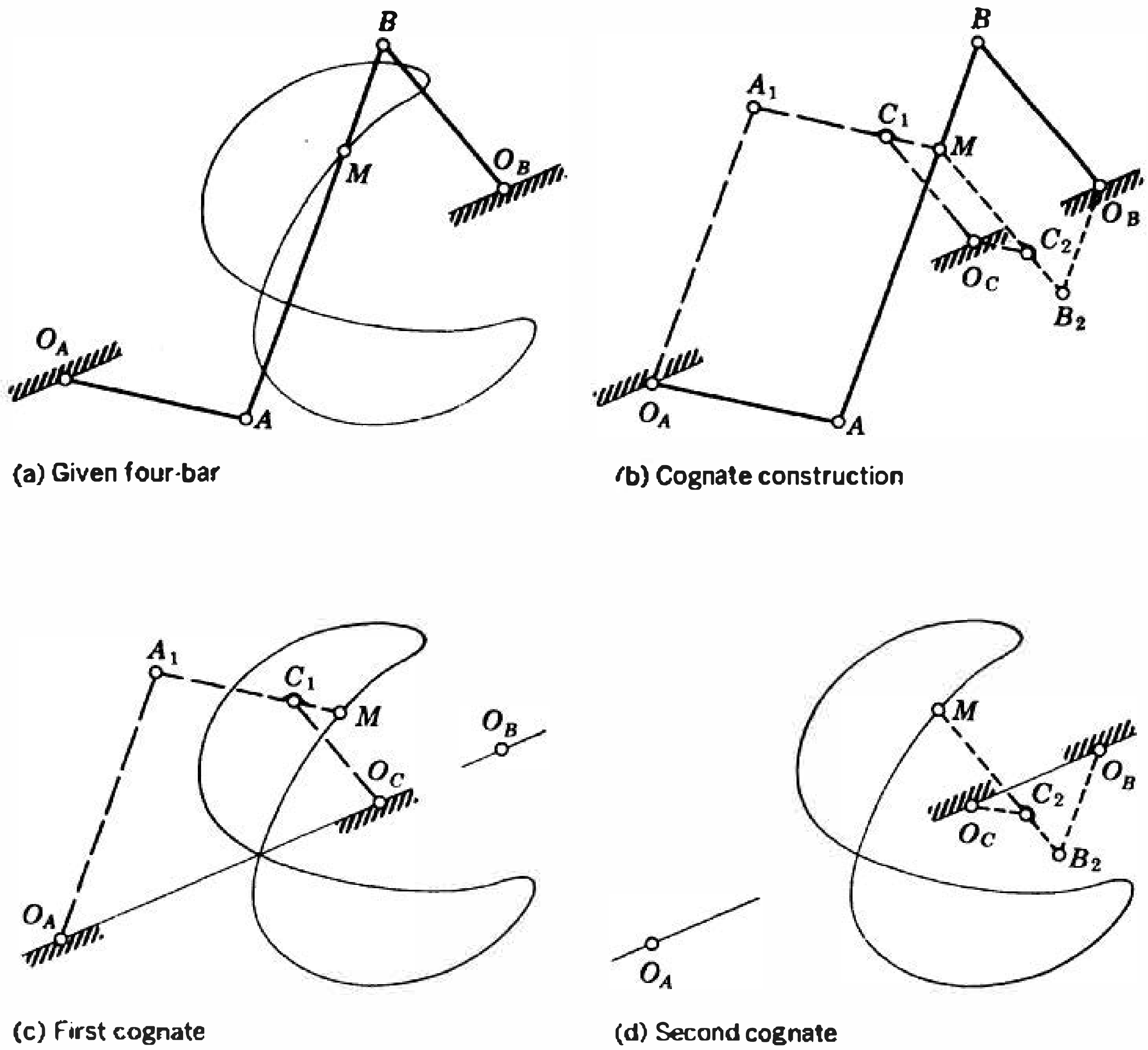


FIGURE 6-17 Cognate four-bars, M between A and B.

For the case in which M lies on an extension of AB, the method is the same as before, except that the line division is now external, following the position of M.

Cayley showed that the coupler point M and the instantaneous centers of coupler and frame of each of the three linkages are collinear at all times and that this line is the normal to the coupler curve (see Fig. 6-18).

Velocity Relations in Cognate Linkages

Returning to Fig. 6-15b, assume that the angular velocity of link O_AA of the given linkage O_AABO_B is ω_2. The angular velocities of the other moving links AB and O_BB, determined in some convenient way, would be ω_3 and ω_4. Since O_AA is parallel to A_1M, O_CC_2 parallel to C_1M, and the angle A_1MC_1 fixed, the angular velocity of link O_CC_2 of the right-hand cognate linkage is also ω_2. Similar considerations are applicable to other links of the cognate linkages, and the following table of velocity equivalences results:

LEFT COGNATE		GIVEN		RIGHT COGNATE
ω_9	=	ω_2	=	ω_7
ω_{10}	=	ω_3	=	ω_5
ω_8	=	ω_4	=	ω_6

These velocity relations must be taken into account when the point M is to be driven along the coupler curve C with prescribed velocities. For example, if the desired motion M is obtained from the linkage O_AABO_B, with O_AA driven at a constant angular velocity ω_2, the same motion of M (curve and velocities) will be obtained by using the right-hand cognate linkage $O_BB_1C_2O_C$ and driving the link O_CC_2 at the same constant angular velocity ω_2. If the left-hand cognate linkage $O_AA_1C_1O_C$ must be used, then it will have to be driven at a variable angular velocity corresponding to the angular velocity of either AB or O_BB when O_AA of the given linkage is driven at the constant angular velocity ω_2.

Historical Note

There is a salient contrast between the development and statement of the theorem as given here and the original works of Roberts and Chebyshev. Parametric equations for the coordinates of the coupler point of the Watt linkage had been derived by the French engineer Prony (1755–

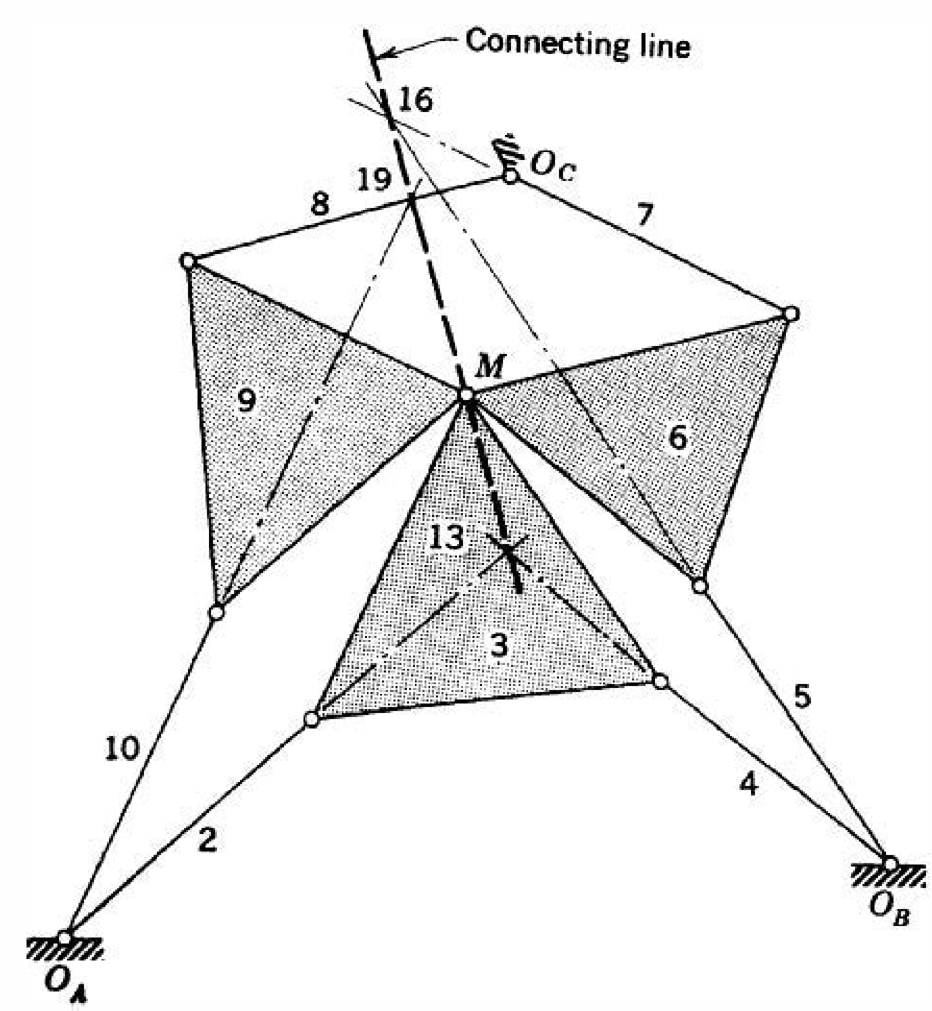


FIGURE 6-18 Collinearity of the three coupler-and-frame instantaneous centers and point M.

1839). Eighty years later, Roberts' analytical investigation showed the curve to be of the sixth order and demonstrated the presence of the third singular focus of the coupler curve, i.e., the point O_C which had with respect to the coupler curve all the properties of the fixed centers O_A and O_B. He deduced that the same curve could therefore be generated by three different four-bars.

Chebyshev had studied four-bar linkages to generate approximate straight-line segments, arriving at the particular form known as the Chebyshev straight-line motion. By geometric perception, and reasoning from similar triangles, he derived the theorem for this simple case, before generalizing it to all coupler-point situations.

6-5 EXTENSIONS OF THE ROBERTS-CHEBYSHEV THEOREM

a. Cognate Slider-crank Mechanisms

A modification of the Roberts-Chebyshev theorem applies to the slider-crank mechanism: *two different planar slider-crank mechanisms will trace identical coupler curves.*

If a sliding pair replaces one of the turning pairs of a four-bar linkage, a slider-crank mechanism results (Fig. 6-19*a*). The center O_B is at infinity, with the link BO_B now infinitely long. On "straightening" this linkage (Fig. 6-19*b*), we can draw only a portion of the schematic, for BO_B now extends to infinity to the right (arguing by analogy). Under these circumstances the linkage "above" O_B vanishes, for practical purposes, and O_C finds itself at infinity. Beyond what might have been C_2 lies O_C—at infinity—whence C_1 must also be a slider. However, the cognate linkage, the slider-crank $O_AA_1C_1$, sharing the coupler point M with the given linkage, has been defined.

The construction of the cognate slider crank may be followed from Fig. 6-19*c*; a parallelogram and a similar triangle are added as shown. The point 13 is the instantaneous center of the given coupler and frame, 35 is the transfer center, and 15 applies to the cognate coupler. The course of the new slider is then along the perpendicular to 15-57 drawn through 57.

It is still necessary to justify that O_AC_1 is a straight line of constant inclination. Considering Fig. 6-19*c*, we note that the triangles $C_1O_AA_1$ and O_ABA are similar, whence $\alpha_1 = \alpha$. Now $\beta_1 = \delta + \alpha_1$, and

$$\beta = \theta + \alpha$$

but $\beta_1 = \beta$ or $\delta + \alpha_1 = \theta + \alpha$, which with $\alpha_1 = \alpha$ means that

$$\delta = \theta = \text{const}$$

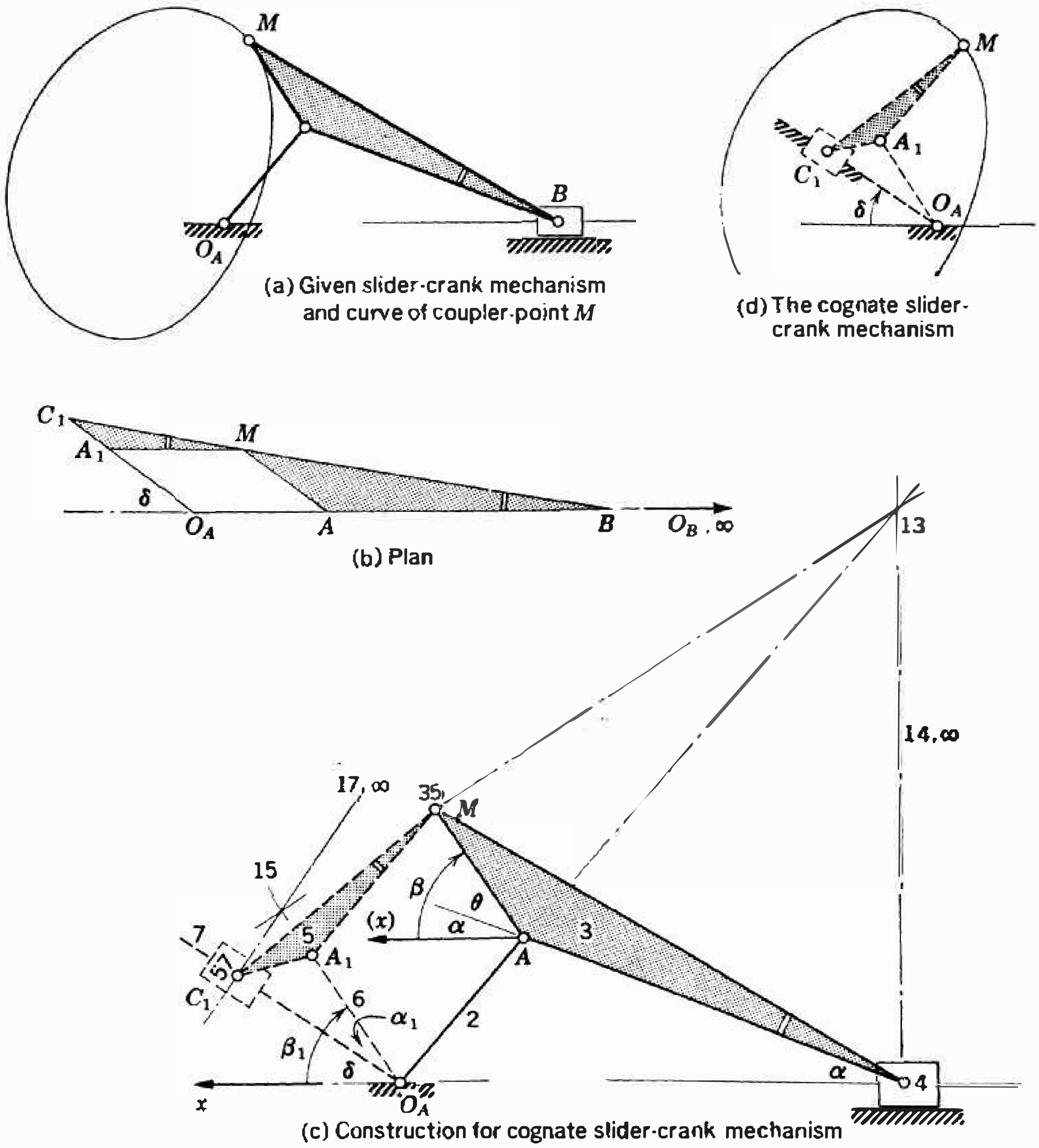


FIGURE 6-19 Cognate slider-crank mechanisms.

Hence the point C_1 (the cognate slider) follows a straight line of constant slope. The point C_1 could thus be used to trace a straight line—no cognate slider is needed for this.

As with the four-bar linkage, the instantaneous centers of the coupler and frame of each of the slider-crank mechanisms are collinear with coupler point M at all times; this line is also the normal to the coupler curve.

The velocity relations are the following:

$$\omega_2 = \omega_5 \qquad \omega_3 = \omega_6$$

$$\frac{v_{C_1}}{v_B} = \frac{O_A A_1}{AB} = \frac{AM}{AB}$$

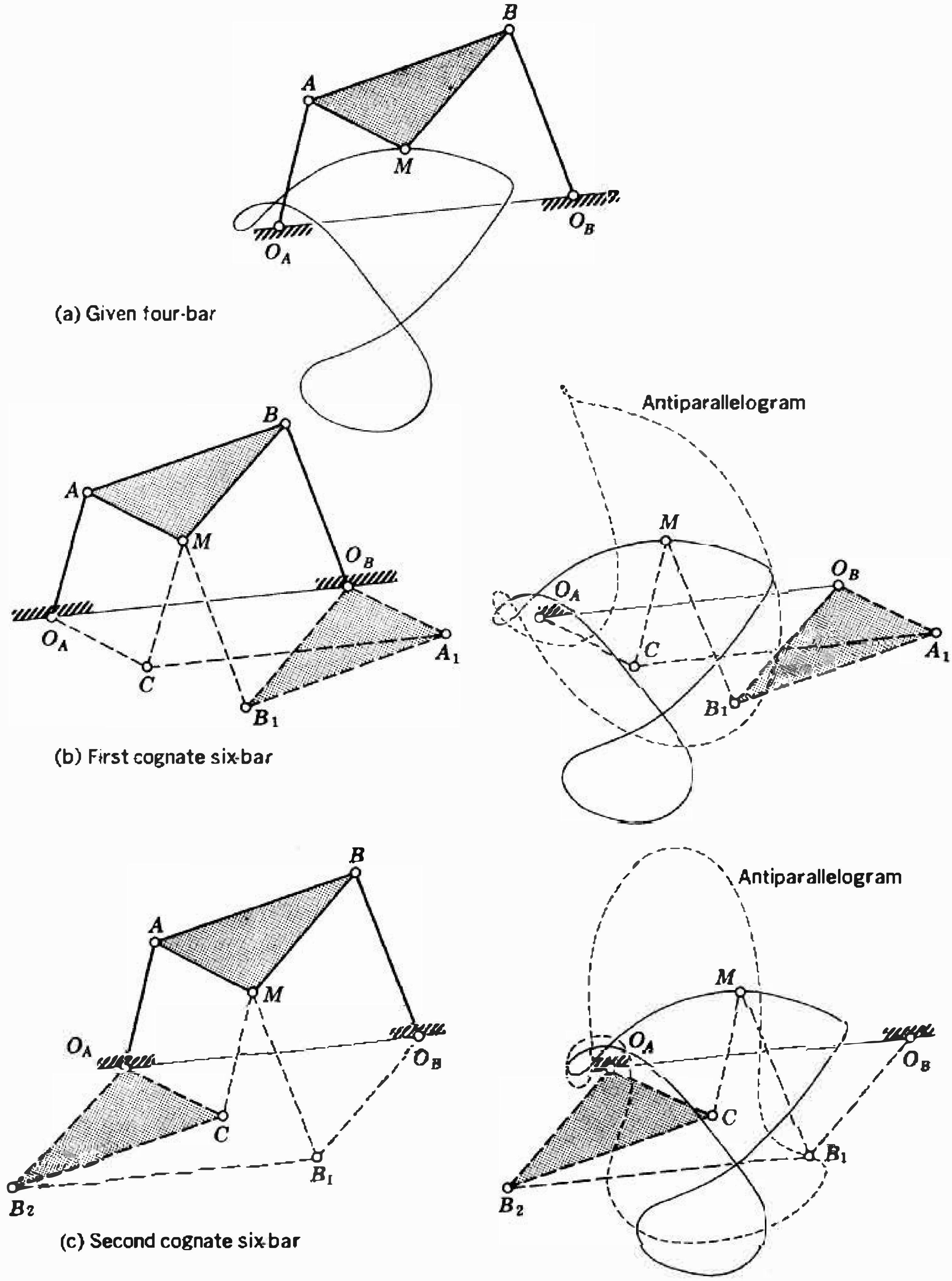


FIGURE 6-20 Four- and six-bar cognate linkages, M off the line AB.

b. Six-bar Linkage Cognate to a Four-bar Linkage

An extension of the Roberts-Chebyshev theorem is as follows: *The coupler-point curve of a planar four-bar linkage is also described by the joint of the dyad of a proper six-bar linkage.*

The course of this extension may be seen from Fig. 6-20*b*. Here O_AABO_B is the given four-bar linkage carrying a coupler point M. The procedure for finding the six-bar linkage may be seen to be as follows:

1. From O_A and M construct a parallelogram locating C.
2. From O_B and M construct a parallelogram locating B_1.
3. The links MC and MB_1 are the dyad whose joint is at M.
4. Construct the triangle $A_1O_BB_1$ similar to AMB: it is reversed and upside down with respect to AMB.
5. Connect C and A_1, forming a third parallelogram $O_AO_BA_1C$. An identical dyad is formed when the other frame point O_B is used (Fig. 6-20*c*).

The situation in which the coupler point M lies on the line AB, either between A and B or beyond A or B, follows the same pattern.

If the frame parallelograms go into the antiparallelogram configuration at the change points, then completely different coupler-point curves will be traced.

It is apparent that the Roberts-Chebyshev theorem is useful in those problems of synthesis involving the coupler-point curve, for the linkages that are related through the curve are identified. The choice of the cognate mechanism is dependent on space requirements (link dimensions and frame-point locations), velocity and acceleration considerations, and the value of the transmission angle during certain phases of the motion.

6-6 STRAIGHT-LINE MECHANISMS—APPROXIMATE AND EXACT

The Watt linkage and "great beam" gave engines of the early nineteenth century vast bulk, and other, more compact "parallel motions" of only pin-connected members were sought. Of course, the Watt linkage was studied for optimum proportions, but different four-bar linkages were devised, among them the Evans (United States), or grasshopper (Fig. 6-21*a*); the Roberts[1] (Fig. 6-21*b*), and the Chebyshev (Fig. 6-21*c*).

Numerous simple planar linkages possessing the geometric relations necessary for generating true straight lines followed. We shall discuss two, the Peaucellier and the Hart.

[1] This Richard Roberts, engineer, is not to be confused with the mathematician Samuel Roberts; see Sec. 6-4.

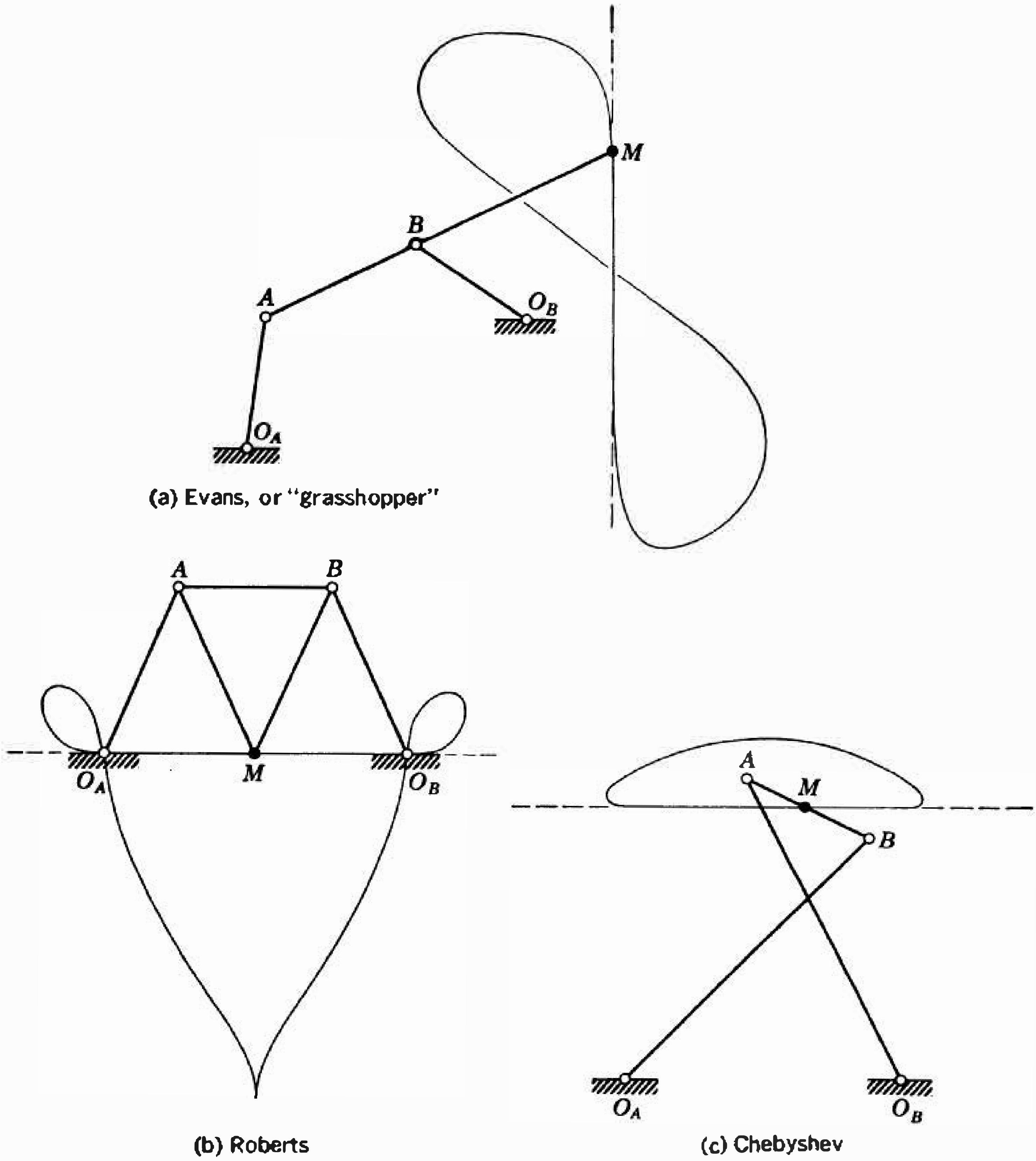


FIGURE 6-21 The best-known "straight-line" motions other than Watt's.

The true straight-line pin-connected linkages involve a geometric relation known as *inversion* (not the kinematic inversion, i.e., the successive fixing of links of a chain to create a variety of mechanisms). If two points P and Q, restrained to move along a straight line passing through a fixed point O, maintain the relation

$$OP \times OQ = k = \text{const of inversion}$$

then:

1. The points P and Q are said to be inversely related.
2. When the straight line rotates about O, the curves traced by P and Q are said to be the inverse of each other. In particular:

a. If one point, either P or Q, traces a circle *not* passing through O, the other point will also describe a circle; i.e., the inverse of a circle is a circle.

b. If one point traces a circle passing through O, the other point's circle will have an infinitely great radius; i.e., it will describe a straight line.

Mechanical devices realizing these conditions are known as inversors, of which the earliest and best known, although not quite the simplest, is that of Peaucellier (1864).

Let us first establish point 2b. In Fig. 6-22, O and O_P are fixed points, with $O_PP = OO_P$, so that P describes a circle passing through O. Let Q be a point on OP satisfying the relation

$$OP \times OQ = k$$

and consider the perpendicular QT to OO_P passing through the point Q. We shall show that the distance OT depends only on the constant k and the diameter of the circle so that, as P describes the circle, Q will describe the fixed straight line TQ. On drawing the line PS we realize that the triangles OPS and OTQ are similar, whence

$$\frac{OP}{OS} = \frac{OT}{OQ} \qquad \text{or} \qquad OP \times OQ = OS \times OT = k$$

and

$$OT = \frac{k}{OS}$$

Consider now the Peaucellier cell, the six-bar chain of Fig. 6-23a. It is formed by connecting a rhombus of equal sides s with two bars of equal length l; the figure is symmetrical about the median m. Three points, seen to be always collinear regardless of the configuration, have been labeled ●, P, and Q. In Fig. 6-23b we now add two more links (the frame and O_PP, with $O_PP = OO_P$), displace m, and draw the line BDC,

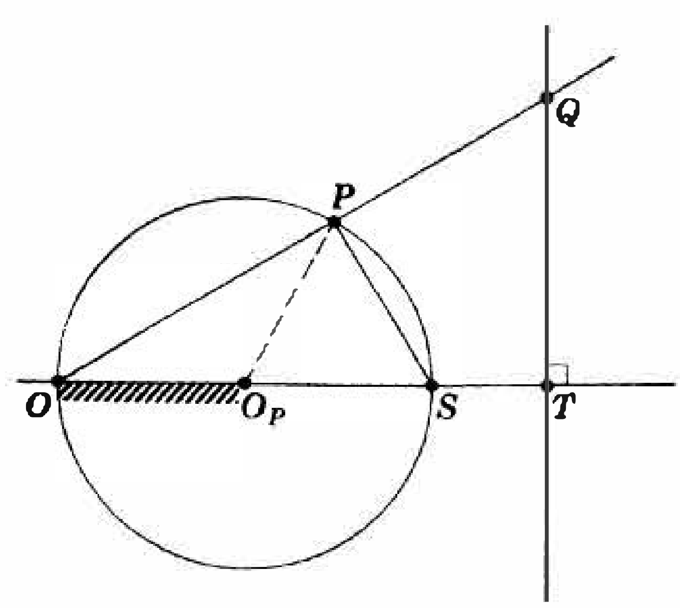


FIGURE 6-22 Proof of proposition 2b.

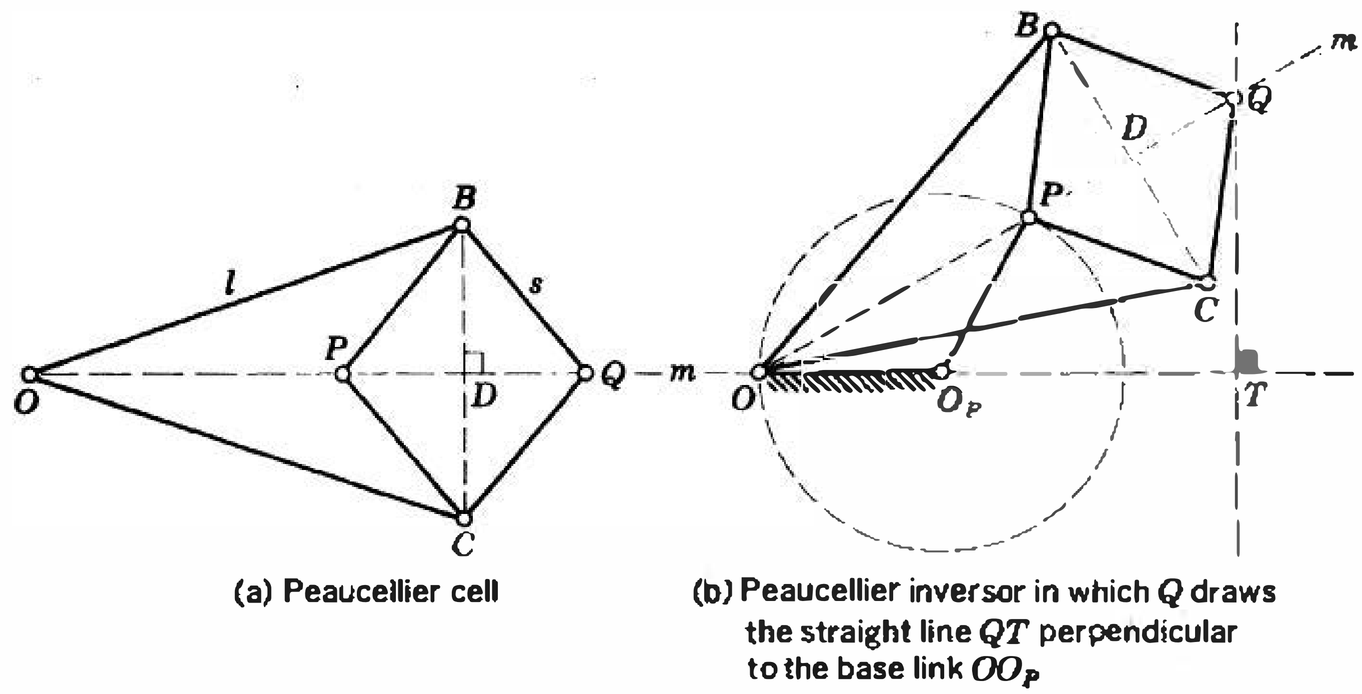


FIGURE 6-23 Peaucellier straight-line mechanism.

which will always be perpendicular to m. Then

$$\begin{aligned} OP \times OQ &= (OD - PD)(OD + PD) \\ &= OD^2 - PD^2 = (OB^2 - BD^2) - (PB^2 - BD^2) \\ &= OB^2 - PB^2 = l^2 - s^2 = \text{const} \end{aligned}$$

The Peaucellier mechanism, since it satisfies the condition

$$OP \times OQ = \text{const}$$

is thus an inversor.

Next we shall consider point 2a. In Fig. 6-24 O and O_P are again fixed points, with $OO_P = d$ and $O_PP = r$, so that P describes a circle (which this time does *not* pass through O, since $r \neq d$). Let P' be the other intersection of OP with the circle and Q be a point on OP satisfying the relation of inversion

$$OP \times OQ = k$$

Draw PS and $P'S'$, and consider the triangles OPS and $OS'P'$. These triangles are similar: their angles at O are equal; their angles at P and S' are also equal, since they are inscribed in the same circle and subtend the same chord $P'S$; the same holds for their angles at P' and S. Then

$$\frac{OP}{OS'} = \frac{OS}{OP'}$$

or $\quad OP \times OP' = OS \times OS' = (d + r)(d - r) = d^2 - r^2 = \text{const}$

On dividing the relation of inversion by the above, we find the ratio

$$\frac{OQ}{OP'} = \frac{k}{d^2 - r^2}$$

When P describes the circle centered at O_P, P' describes the same circle and, because of the last relation, Q must describe a curve similar to this circle, with center of similitude at O: this curve is also a circle, and its radius is

$$R = r\left|\frac{k}{d^2 - r^2}\right|$$

In the Peaucellier cell, $k = l^2 - s^2$, whence

$$R = r\left|\frac{l^2 - s^2}{d^2 - r^2}\right|$$

The center O_Q of the circle traced by Q corresponds to the center O_P in the same similitude, whence

$$\frac{OO_Q}{OO_P} = \frac{l^2 - s^2}{d^2 - r^2}$$

With the proportions shown in Fig. 6-25, $d < r$ and $l > s$, theeratio of similitude is negative, and O_Q is therefore to the left of 0, that is, in the opposite direction from O_P.

Peaucellier, captain of engineers in the French Army at the time of his invention (he rose to general), was motivated in his search by the approximate straight-line motions of his time and their use in engines. However, the day of the big-beamed engines was drawing to a close, the compact direct-connected or slider-crank engines supplanting their bulky ancestors. In consequence, engine applications dwindled. The Peaucellier straight-line mechanism is, nevertheless, of more than passing interest, for it was the first device able to generate—create—a straight line in the

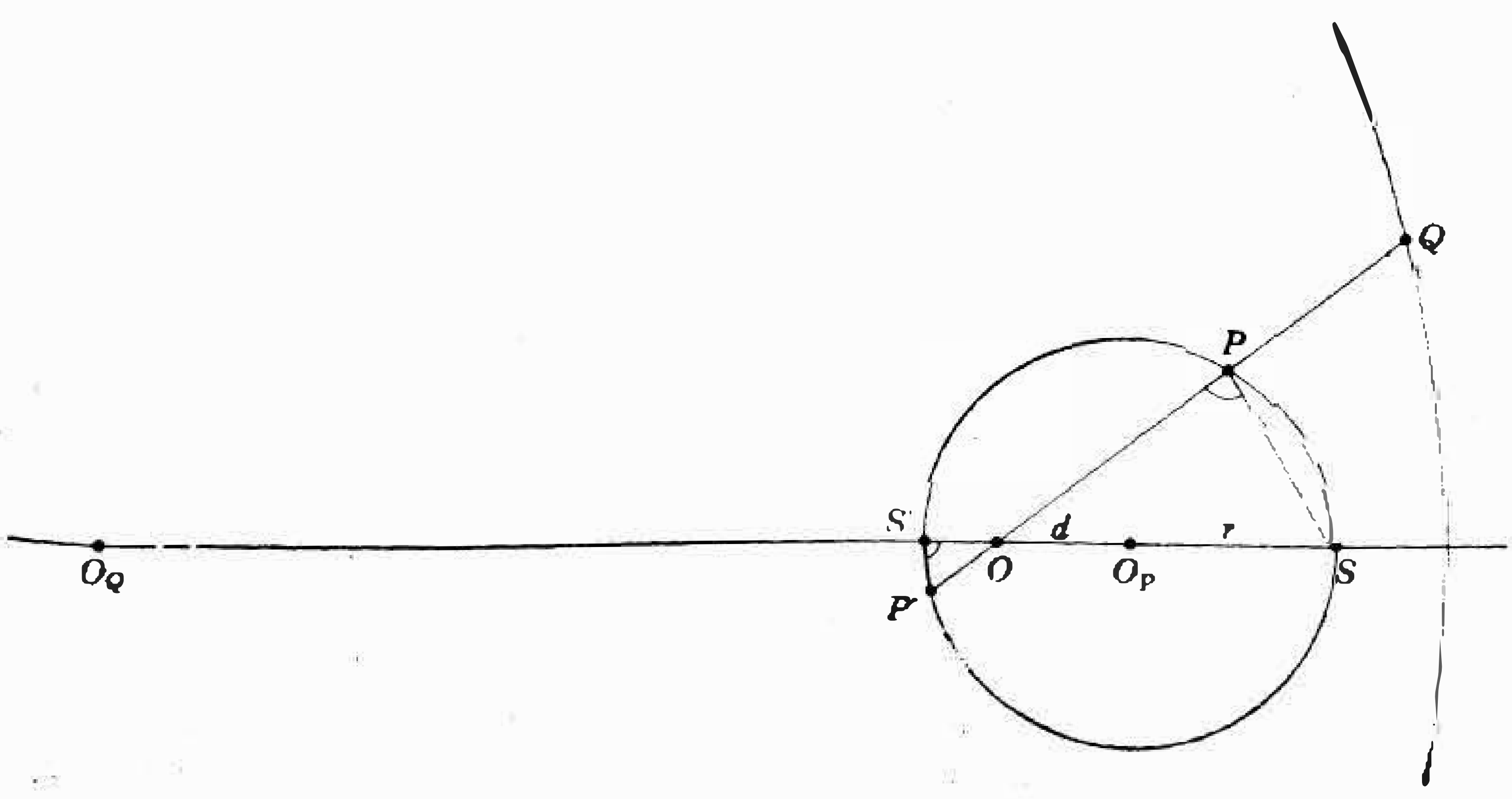


FIGURE 6-24 Proof of proposition 2a.

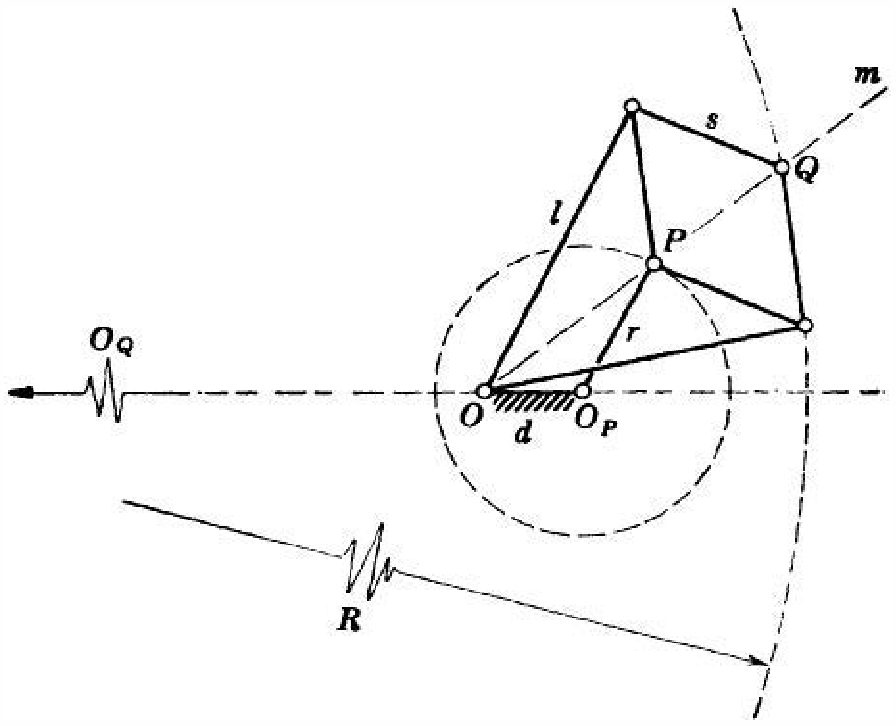


FIGURE 6-25 Peaucellier inversor for drawing circular arcs of large radii.

same sense that a compass generates a circle; using a straight-edge, one merely copies an existing line.

The instrument's ability to generate circular arcs, especially flat arcs beyond the practical range of a trammel bar, gave it some application in manufacturing machinery and in the drafting room. As a compass, the pivot point O_P is made adjustable, allowing the distance d to be selected for a desired radius.

A modern application of the Peaucellier linkage is the autofocusing mechanism of a photographic enlarger. The optical problem involves the proper spacing of negative, lens, and paper to maintain a continuously sharp image over the range of magnification; the problem is summarized by the equation $xx' = f^2$ (x and x' are the distances, f the focal length of the lens). This equation can be mechanized by an inversor, since it is of the form $OP \times OQ = k$.

Figure 6-26*a*, showing the optical system, assumes that both Gauss points are at the center of the lens, justifiable with a normal photographic objective (but not a telephoto lens). Focal points are indicated by F, with f the focal length. The film must be moved a distance x' while the paper moves the distance x, both relative to the lens. Magnification is given by $m = h/h' = u/v = x/f$. The Newton equation for the conjugate distances x and x' has already been noted as $x'x = f^2$.

The rearrangement of Fig. 6-26*b* groups the variables x and x' about the lens and leads directly to the hardware schematic (Fig. 6-26*c*). Here the familiar points of a modified Peaucellier cell are indicated; P, Q, and O' are constrained to move along the same straight line by sliders. We may now write $OP \times OQ = k = l^2 - s^2$. Physically, P is at the wrong spot, but $OP = QO' = x'$, and $OQ = x$. Consequently

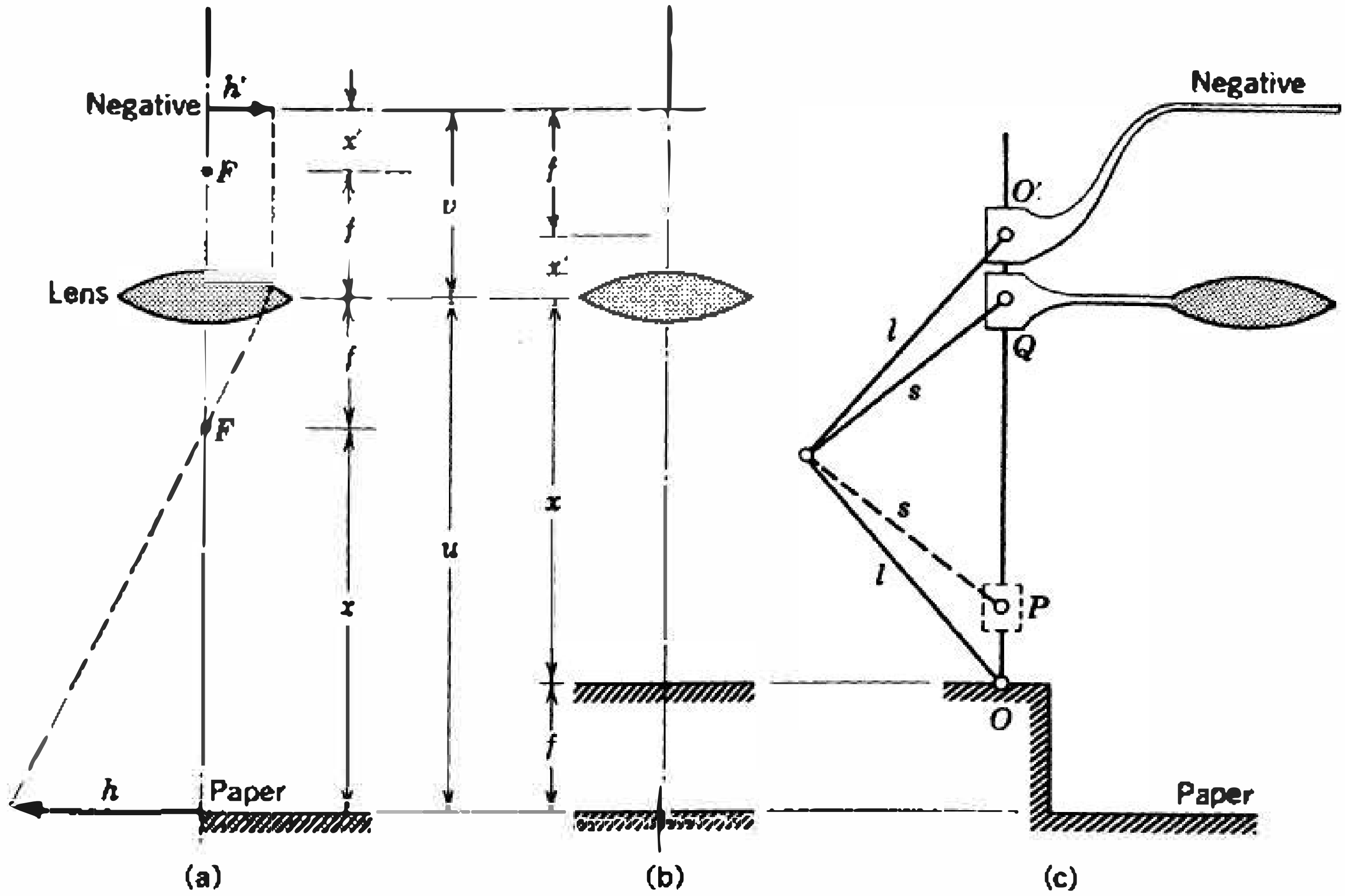


FIGURE 6-26 Autofocus enlarger employing Peaucellier mechanism.

the mechanization of the inversion relation is a direct analog of the optical requirement, and $l^2 - s^2 = f^2$.

Another mathematically correct straight-line mechanism, but of only six links, may be derived from the contraparallelogram chain of four pin-connected links (Fig. 6-27), in which $EB = CD$ and $BD = EC$.

Four points such as O, P, Q, and O', lying on a line m parallel to BC (and hence also to ED), will divide the distances between the pin connections in the same proportion. Furthermore, the points will continue to remain in line when the chain is deformed. These points are also related by inversion; thus, $OP \times OQ = \text{const} = O'Q \times O'P$.

The inversion relation may be established after constructing CC

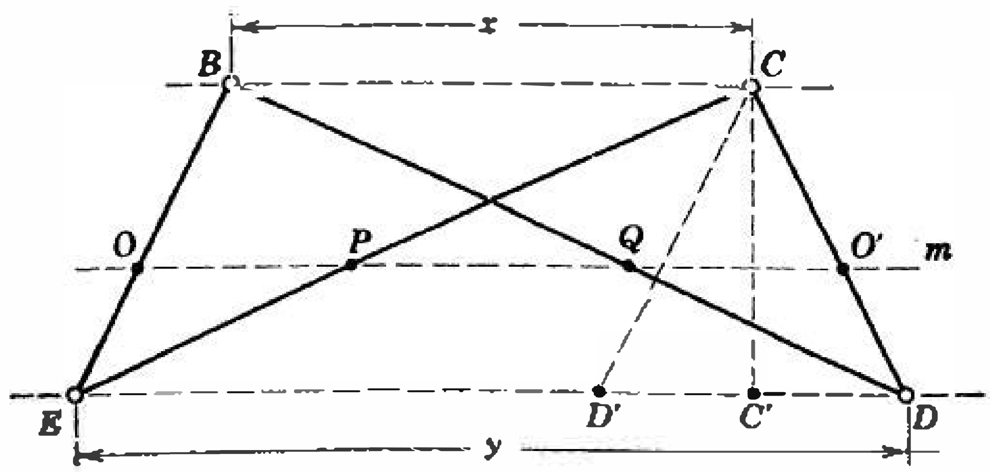


FIGURE 6-27 Contraparallelogram chain of Hart.

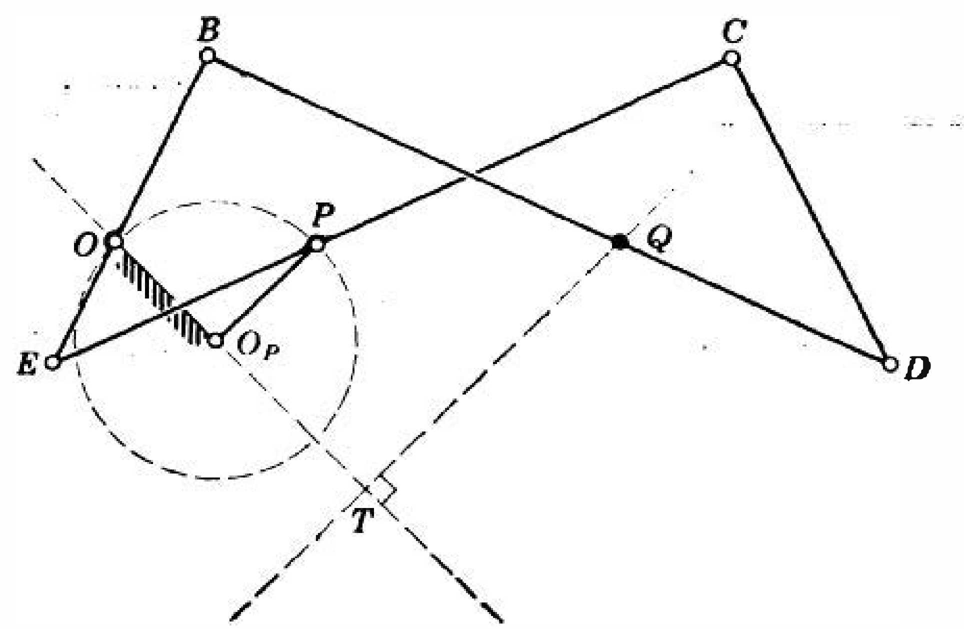


FIGURE 6-28 Hart's inversor.

perpendicular to ED and CD' parallel to BE. From similar triangles

$$\frac{OP}{x} = \frac{OE}{BE} \quad \text{and} \quad \frac{OQ}{y} = \frac{OB}{BE}$$

Then $$OP \times OQ = \frac{OE}{BE} \times \frac{OB}{BE}\, xy = xy \times \text{const}$$

Now $x = EC' - C'D'$, and $y = EC' + C'D'$, whence

$$\begin{aligned} xy &= EC'^2 - C'D'^2 \\ &= (EC^2 - CC'^2) - (CD^2 - CC'^2) \\ &= EC^2 - CD^2 = l^2 - s^2 = \text{const} \end{aligned}$$

and therefore $OP \times OQ = $ const for all configurations.

We recognize that, when O is made a fixed point and P is guided along a circle passing through O, Q will trace a straight line (Fig. 6-28). Point O_P may be chosen at will, subject to the above restriction. The path of Q will be perpendicular to the base line OO_P. This is the six-bar mechanism of H. Hart (1875).

APPENDIX: ATLAS OF FOUR-BAR COUPLER CURVES

The curves shown in this appendix are a selection from the atlas of Hrones and Nelson, as mentioned in Sec. 6-3. This atlas contains

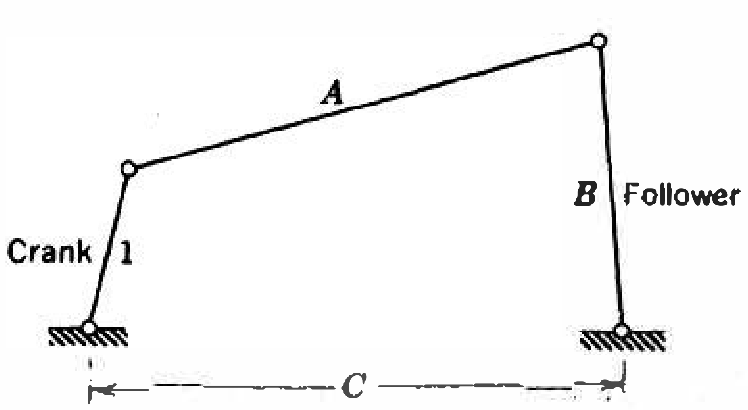


FIGURE 6-29 Notations used in Hrones and Nelson's four-bar coupler curve atlas.

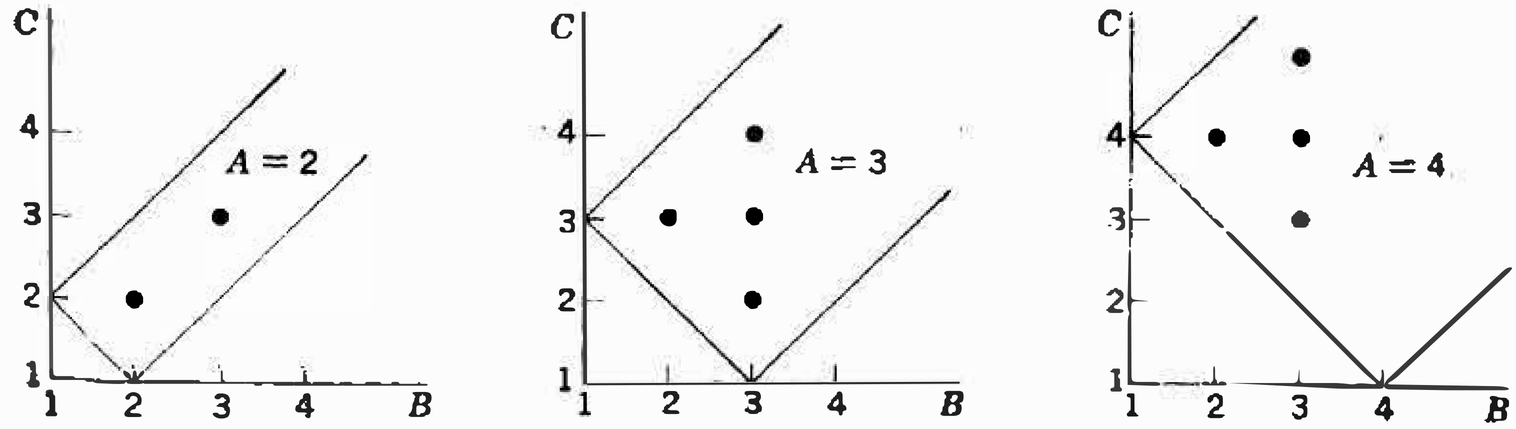


FIGURE 6-30 Determination of the linkages to be considered in this appendix.

approximately 7,300 curves drawn to large scale (730 pages, 11 by 17 in.) and constitutes a very practical tool for the designer, who, by paging through, may find a shape and configuration suitable for a given application.

The four-bar linkages considered here (as well as in the original) are of the crank-rocker type, i.e., having continuous rotation of the crank, with oscillation of the follower. With the notation shown in Fig. 6-29 the link lengths must therefore satisfy the conditions

$$C < A + B - 1$$
$$C > |A - B| + 1$$
$$A, B, C > 1$$

In order to determine what combinations of values of A, B, and C are compatible with these conditions, consider the diagrams of Fig. 6-30,

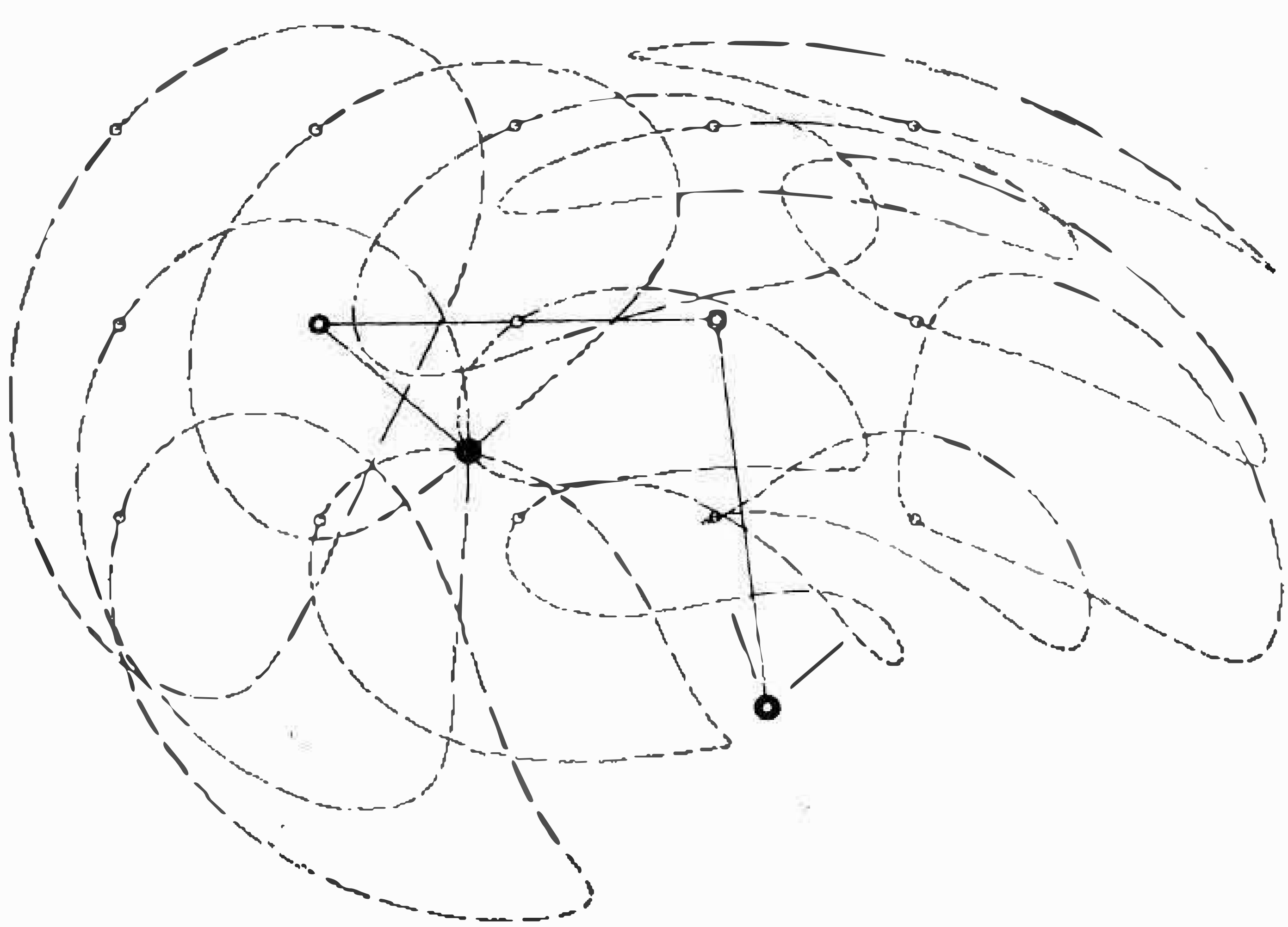


FIGURE 6-31 $A = 2$, $B = 2$, $C = 2$.

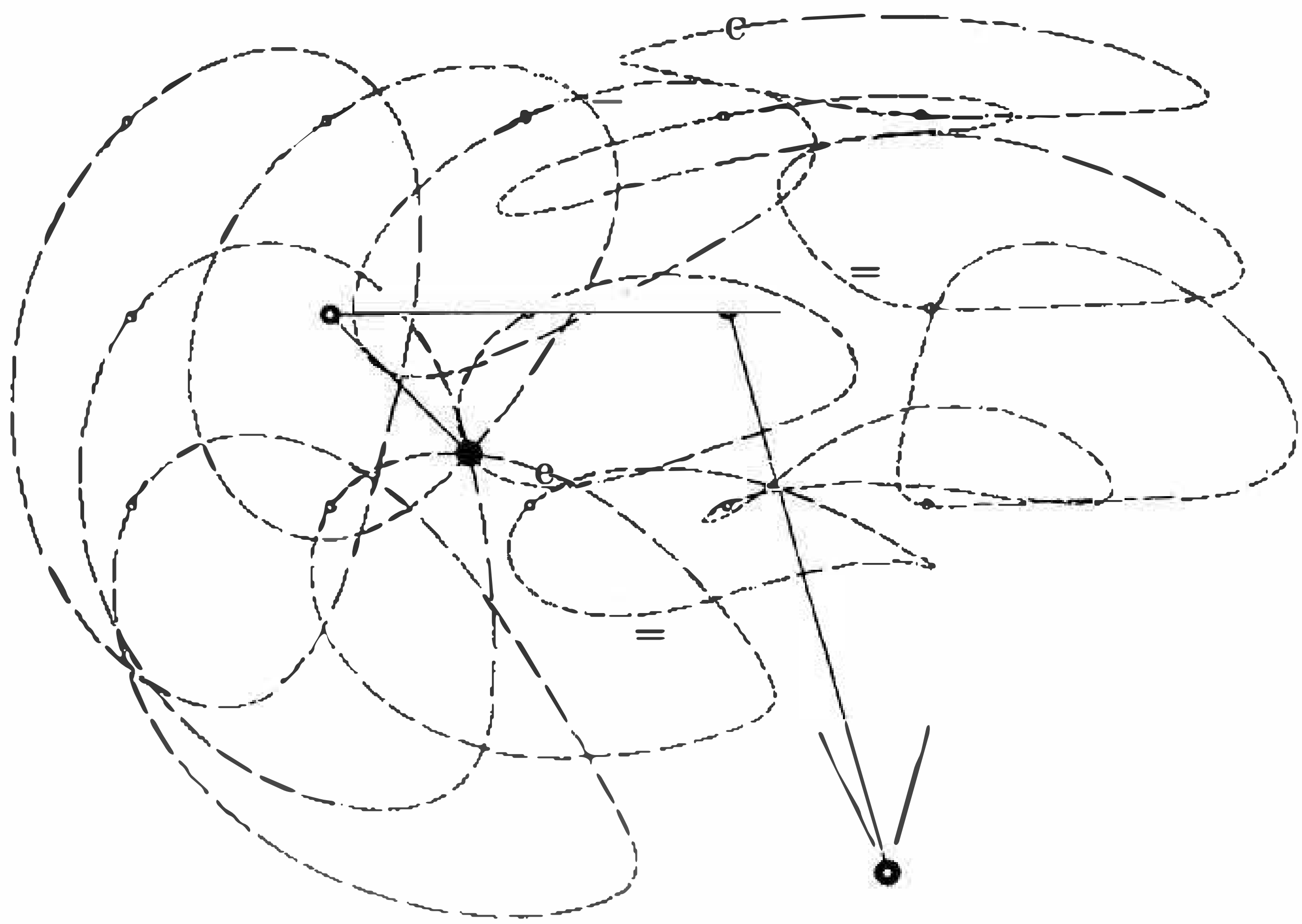


FIGURE 6-32 $A = 2$, $B = 3$, $C = 3$.

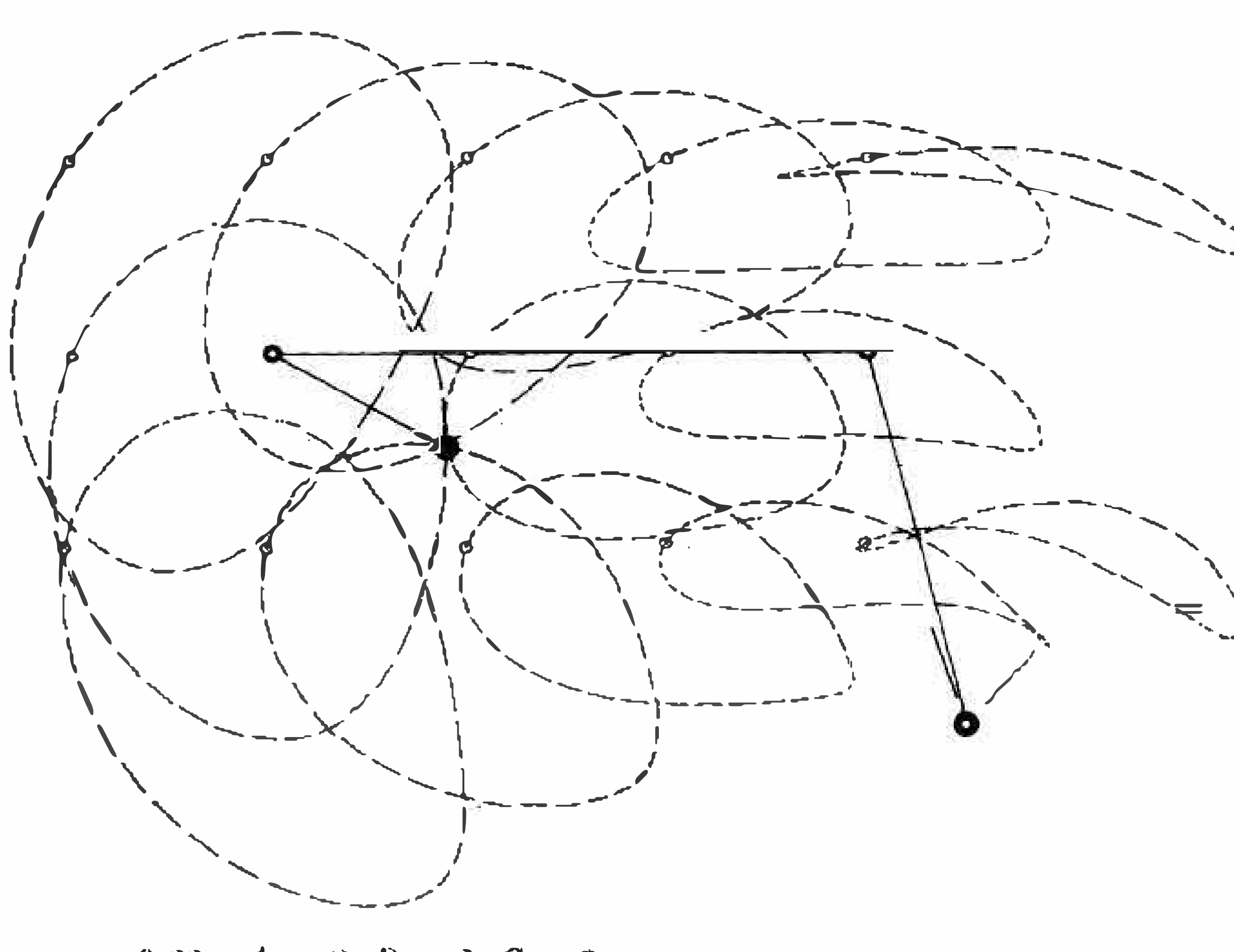


FIGURE 6-33 $A = 3$, $B = 2$, $C = 3$.

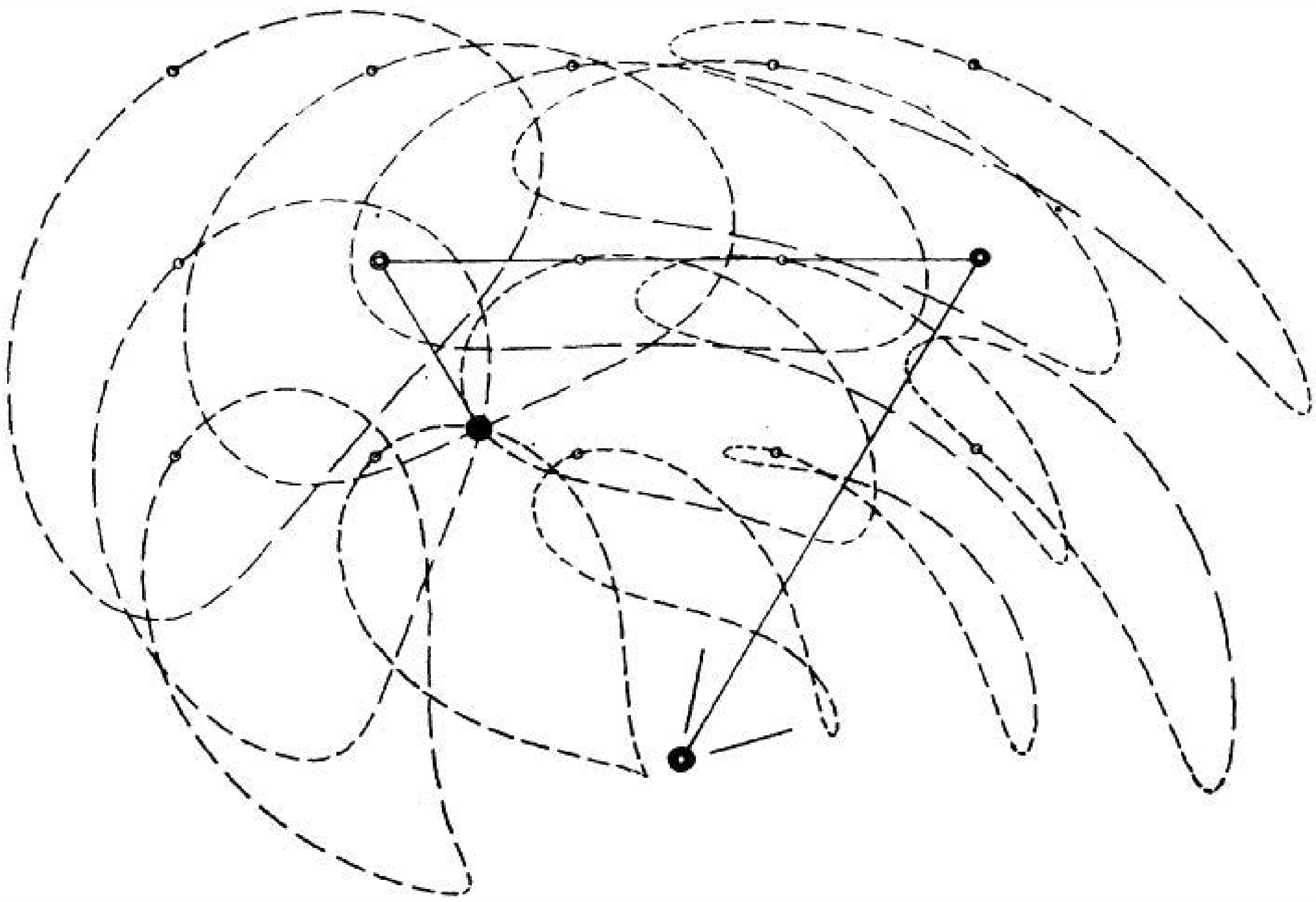


FIGURE 6-34 $A = 3$, $B = 3$, $C = 2$.

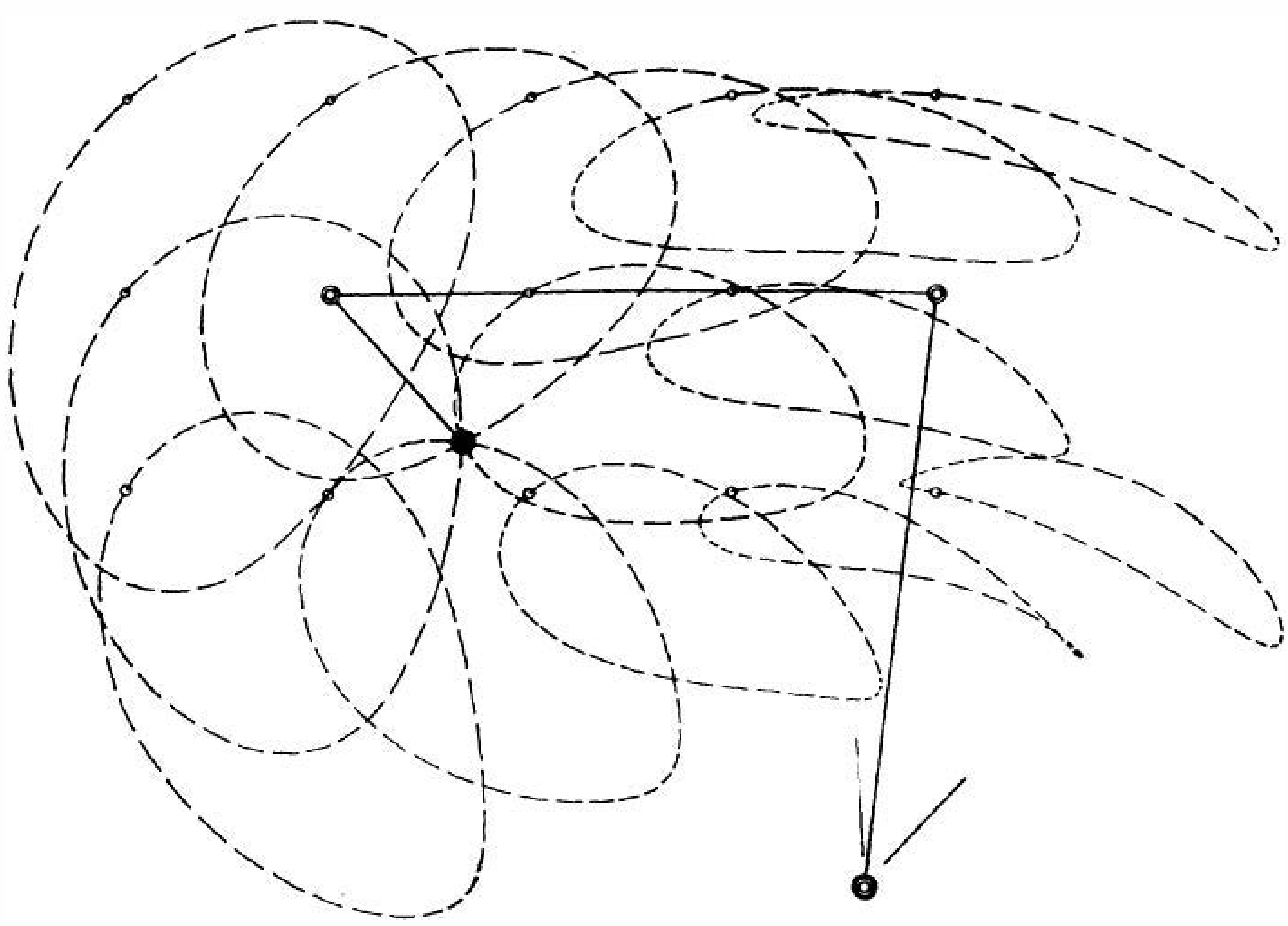


FIGURE 6-35 $A = 3$, $B = 3$, $C = 3$.

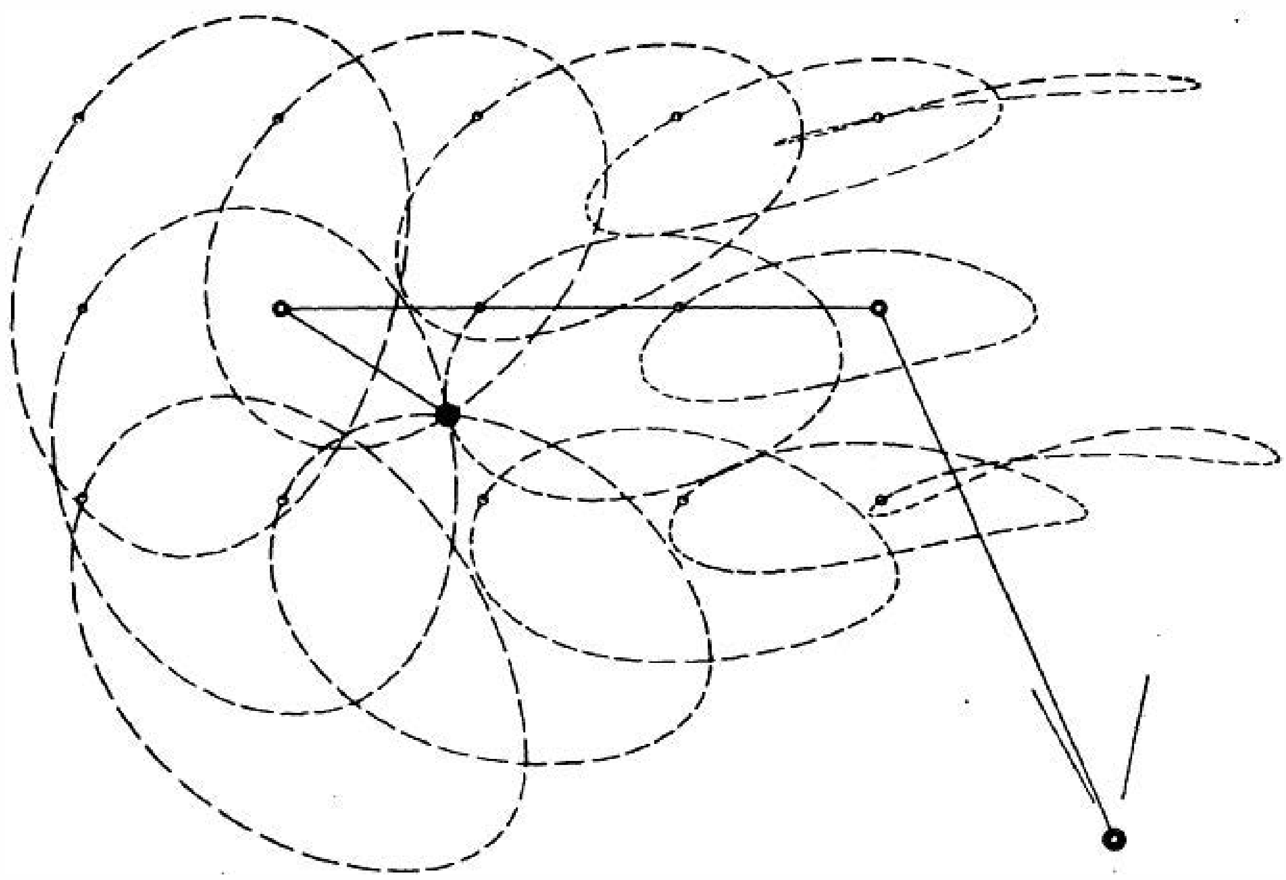


FIGURE 6-36 $A = 3, B = 3, C = 4.$

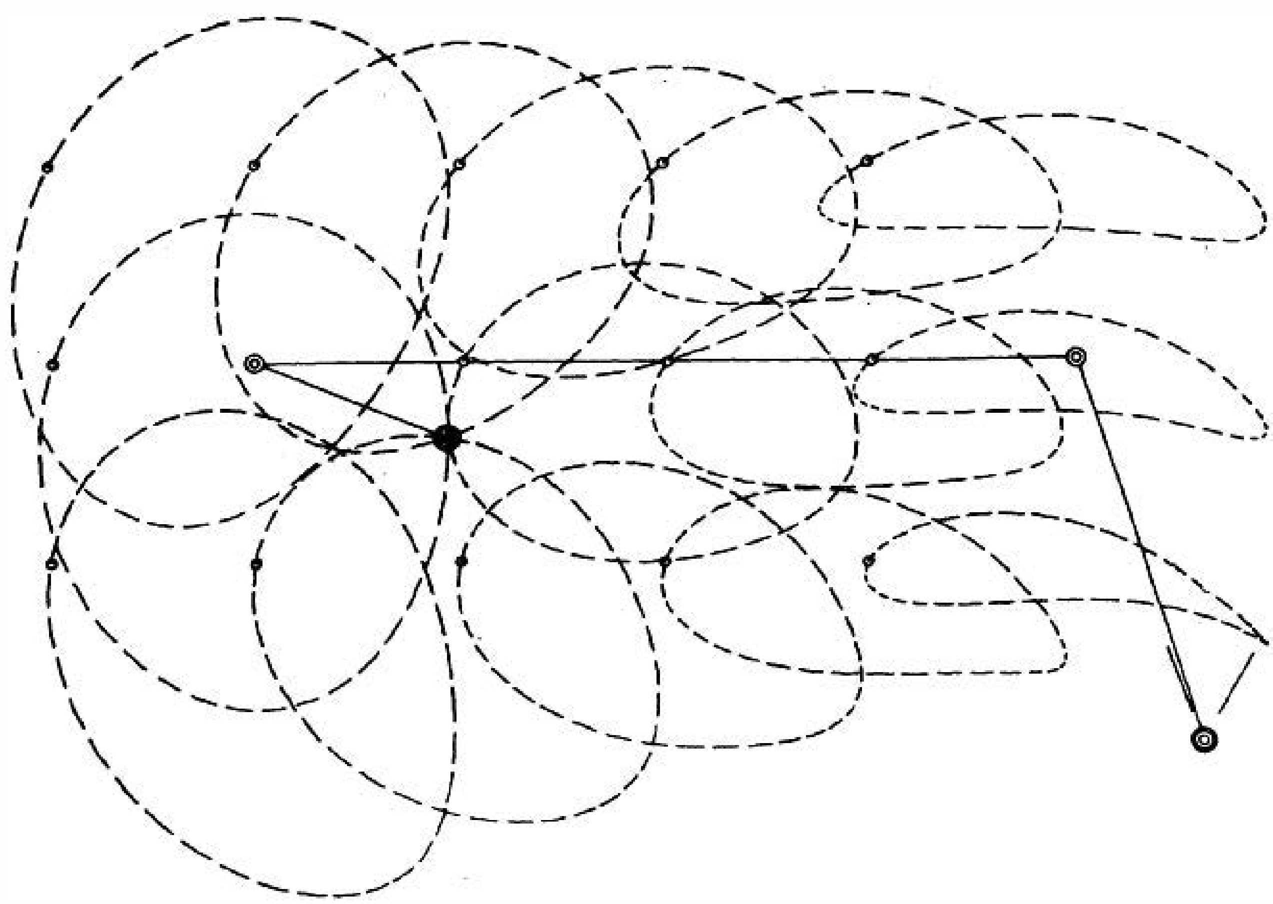


FIGURE 6-37 $A = 4, B = 2, C = 4.$

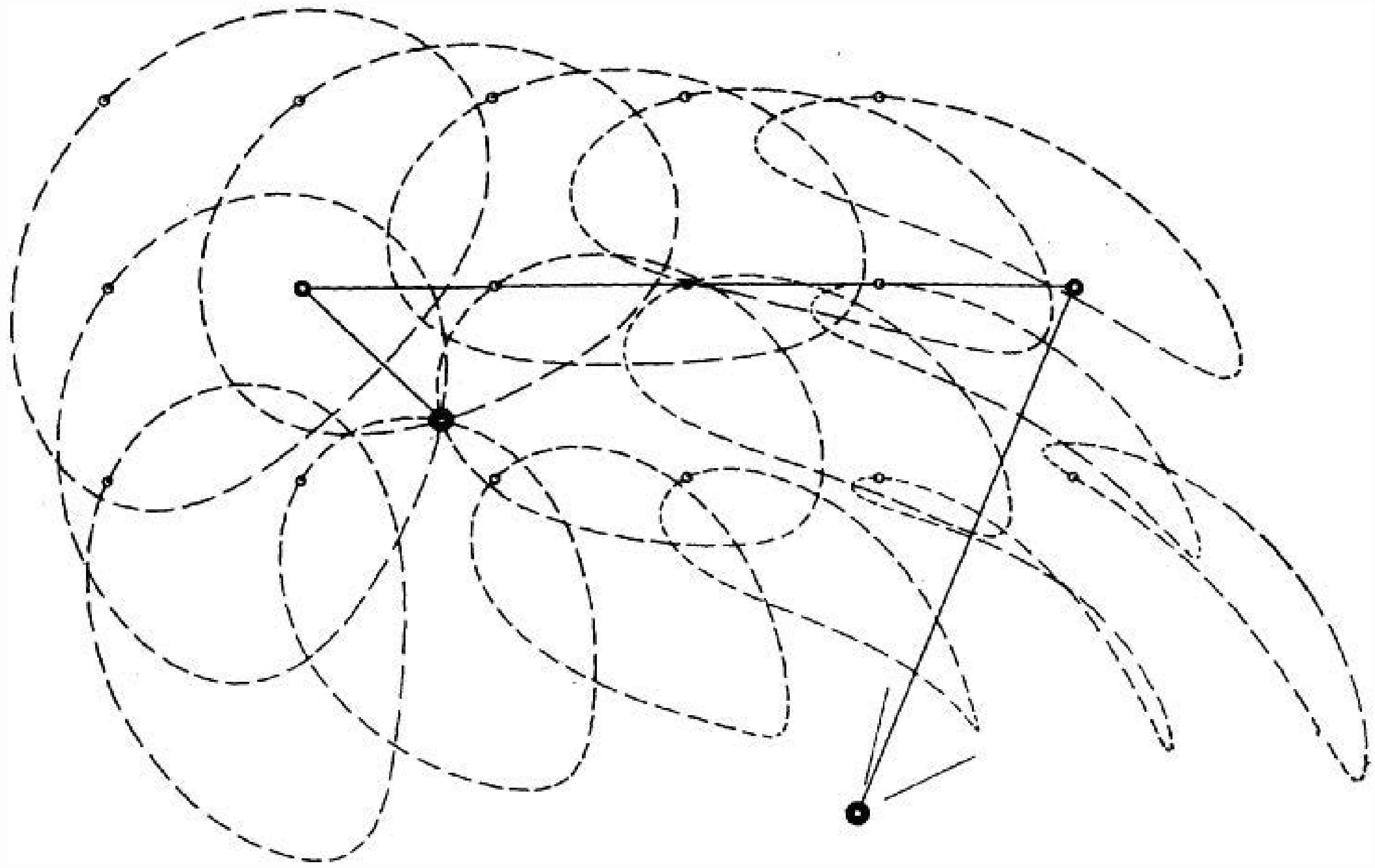


FIGURE 6-38 $A = 4, B = 3, C = 3.$

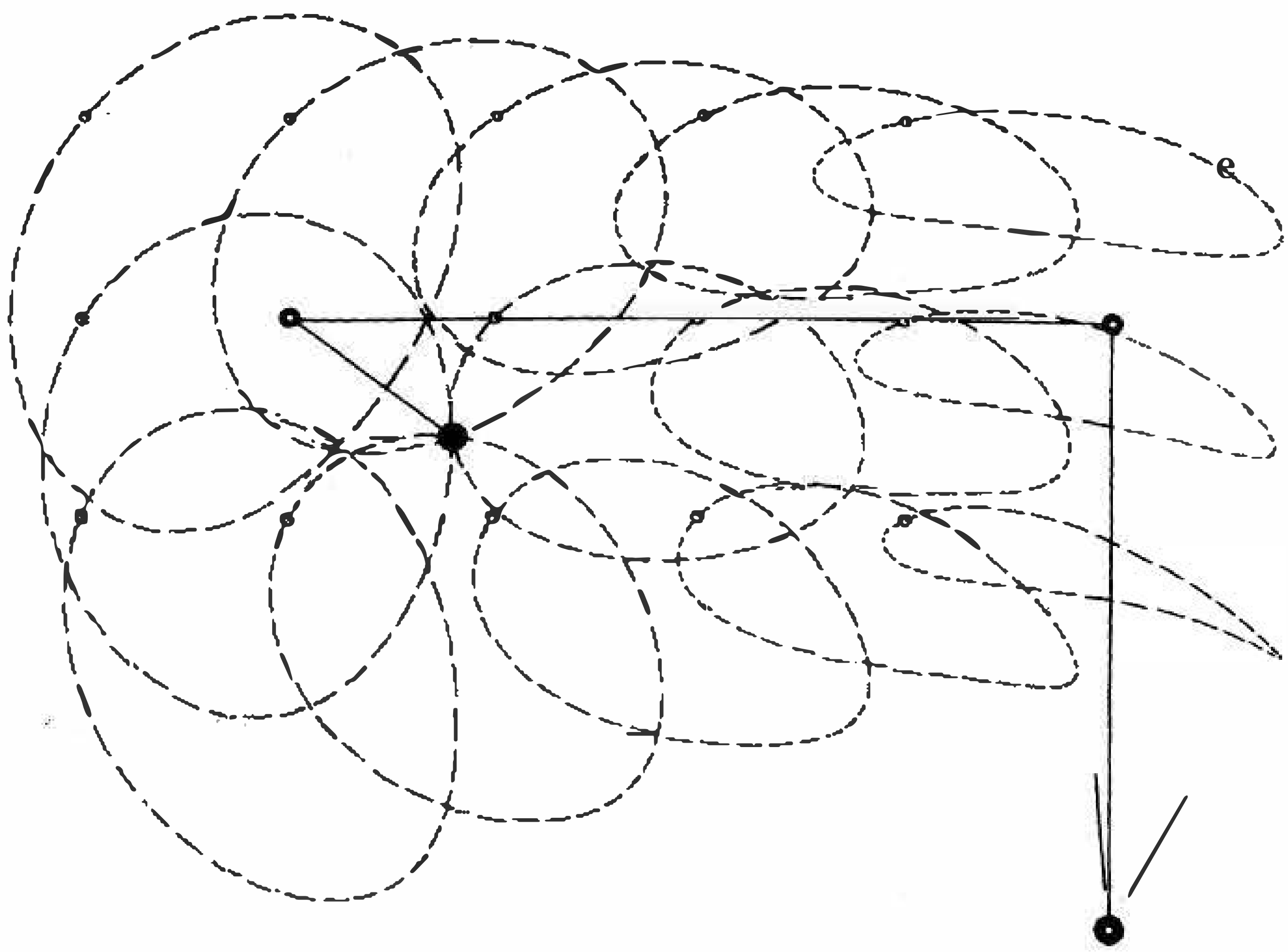


FIGURE 6-39 $A = 4, B = 3, C = 4.$

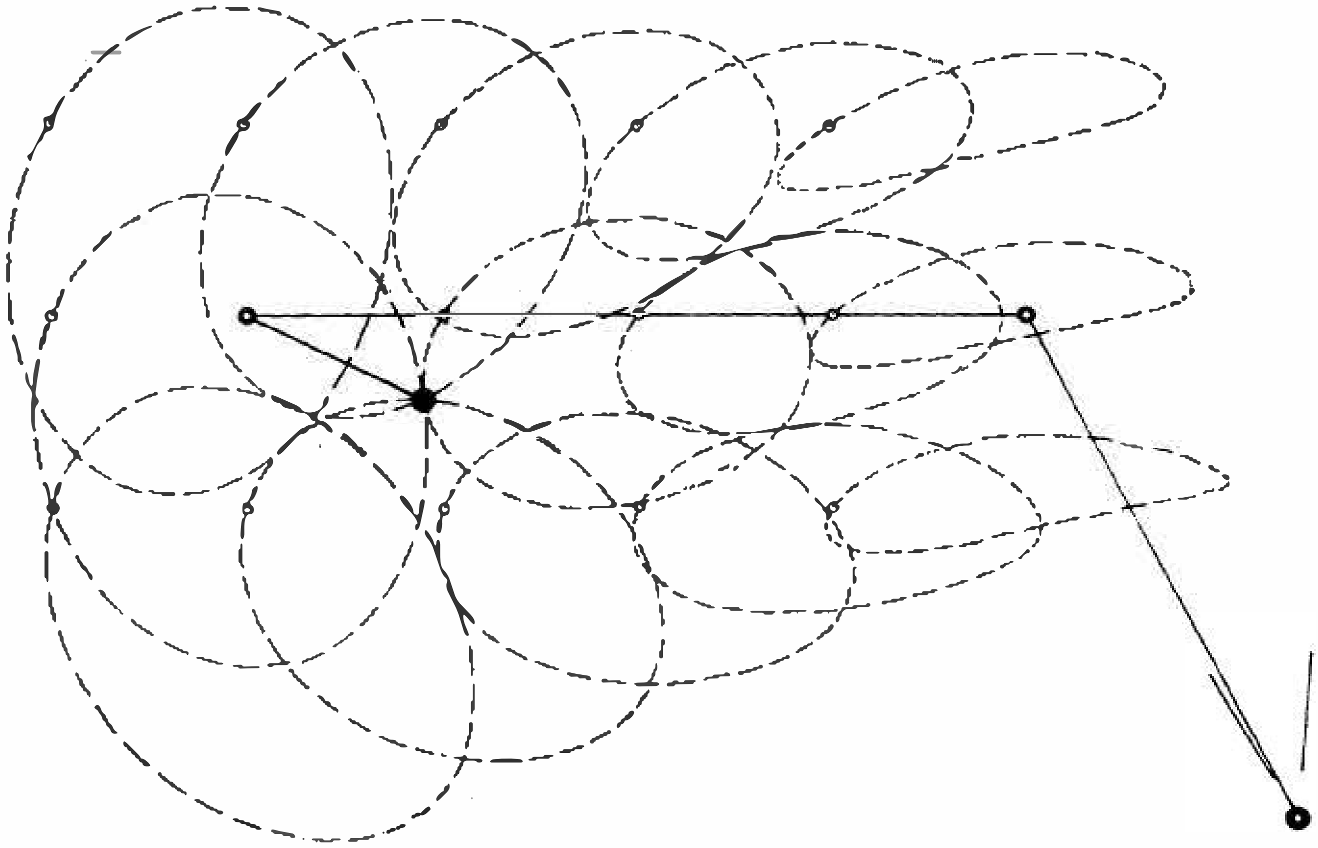


FIGURE 6-40 $A = 4$, $B = 3$, $C = 5$.

where B and C are plotted on the horizontal and vertical axes and A is at a 45° angle. The three diagrams correspond to $A = 2$, 3, and 4; larger values of A are not considered. Each point in the plane of the diagrams corresponds to a four-bar linkage, and values satisfying the above conditions are located in the "rectangles." Limiting B to values no larger than 3 (a follower no longer than three times the crank length) and taking unit increments for B and C gives a total of 10 linkages, shown as dots, for which coupler curves are drawn in Figs. 6-31 to 6-40. The curves of a number of coupler points are shown for each linkage, and each dash corresponds to 10° of crank rotation, thus giving a representation of the coupler-point velocity. Much more detailed information is given in the original atlas of Hrones and Nelson, but only at the expense of a rather overwhelming effect. The curves presented here will be sufficient for preliminary designs, which may later be refined by the geometric or analytical methods of synthesis presented in later chapters.

BIBLIOGRAPHY

Beyer, Rudolf: "Technische Kinematik," Johann Ambrosius Barth, Munich, 1931. Also, J. W. Edwards Publisher, Incorporated, Ann Arbor, Mich., 1948.

——: "Kinematische Getriebesynthese," Springer-Verlag OHG, Berlin, 1953. English translation by H. Kuenzel, "The Kinematic Synthesis of Mechanisms," McGraw-Hill Book Company, New York, 1963.

Bloch, S. S.: "Angenaeherte Synthese von Mechanismen," translation from Russian, VEB Verlag Technik, Berlin, 1951.

Bricard, R.: "Leçons de cinématique," vol. II. Gauthier-Villars, Paris, 1927.

Cayley, A.: On Three-bar Motion, *Proc. London Math. Soc.*, vol. 7, 1876.

Chebyshev, P. L.: Les plus simples systèmes de tiges articulées (1878), from "Oeuvres de P. L. Tchehychef," vol. 2, Markoff et Sonin, St. Petersburg, 1907. Reprint, Chelsea Publishing Company, New York, 1962.

Hartenberg, R. S., and J. Denavit: The Fecund Four-bar, *Trans. Fifth Conf. on Mechanisms*, Penton Publishing Co., Cleveland, 1958. Also, Cognate Linkages, *Machine Design*, vol. 31, 1959.

Primrose, E. J. F.: "Plane Algebraic Curves," The Macmillan Company, New York, 1955.

Prony, G. F. de: "Nouvelle architecture hydraulique," vol. II, Paris, 1796.

Roberts, S.: On Three-bar Motion in Plane Space, *Proc. London Math. Soc.*, vol. 7, 1876.