4 KINEMATIC ANALYSIS OF PLANAR MOTION

4-1 INTRODUCTION

Kinematics, the study of geometry in motion, covers two broad and interrelated areas that are subject to separate study. An existing or specified mechanism or hypothetical situation may be investigated for whatever characteristics and properties it possesses; this is formally called analysis. The inverse of such a procedure is synthesis: in this, a mechanism is created meeting the specification of certain desirable characteristics and properties. Many of the operations forming a synthesis are directly related to or make use of procedures stemming from analysis.

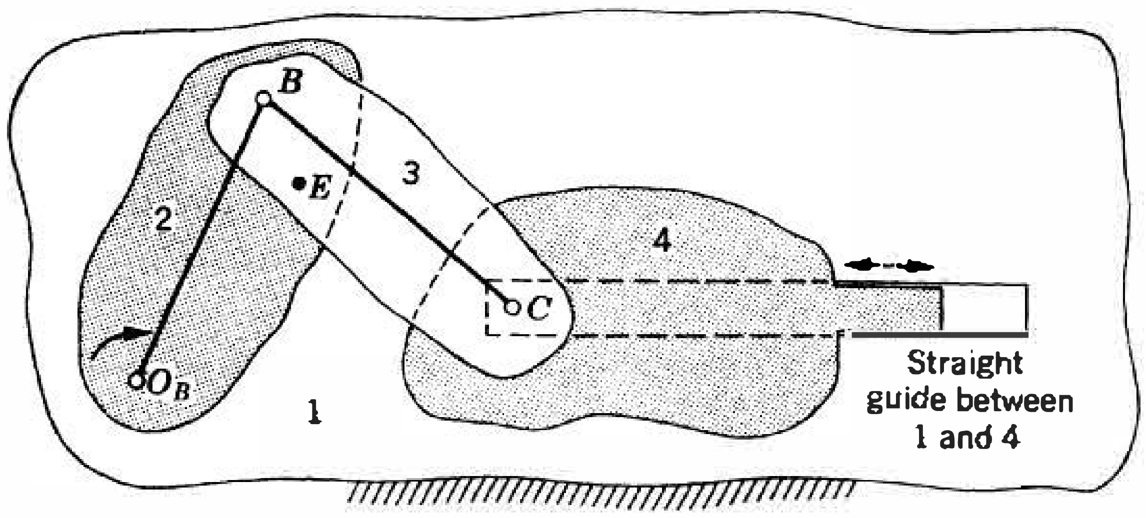
This chapter presents some generally useful aspects of velocity and acceleration analysis for planar mechanisms, based on geometrical (graphical) procedures. No previous acquaintance with kinematic analysis is assumed; this allows a treatment in closed form, but due attention is given to details and concepts deemed important even to the mature reader who may have fallen into careless habits. Sophisticated procedures and special methods for particular types of mechanisms are not a direct part of the material to be discussed.

4-2 COINCIDENT POINTS

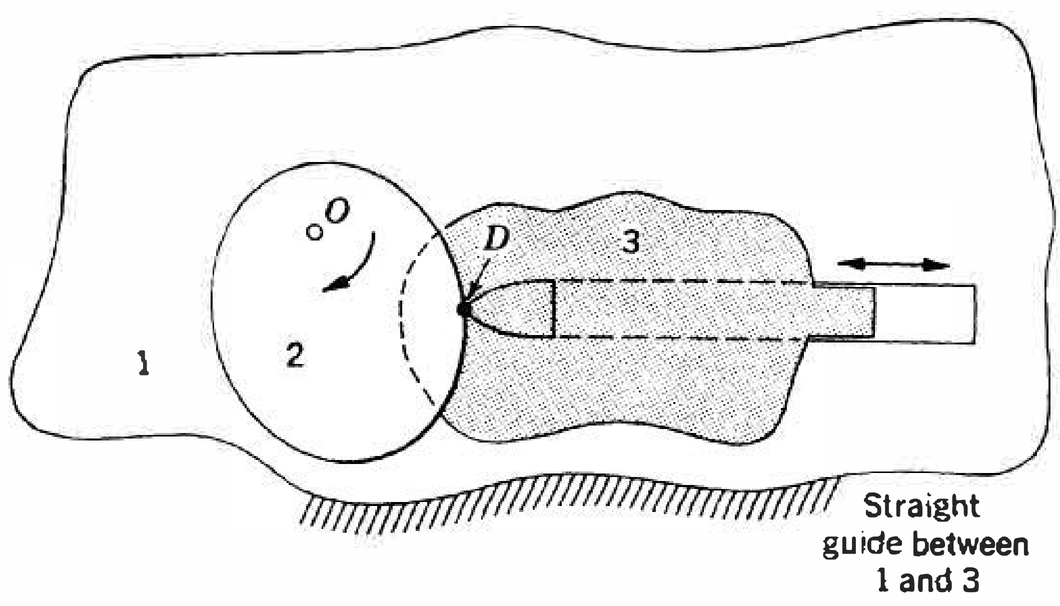
The properties of coincident points are best discussed by considering a mechanism to consist of as many superposed planes as there are links. Each plane is of infinite extent, and any physical link or machine member is then only a portion of a plane. The superposed planes of two mechanisms are visualized in Fig. 4-1.

The relative motion of one plane with respect to its neighbor is dictated by the nature of the connection or contact between planes. Thus, a revolute or pin connection will allow only rotation, as for example at O_B, B, and C in Fig. 4-1a. A prismatic pair or rectilinear slide, as between links 1 and 4 in Fig. 4-1a, limits the relative motion to rectilinear translation. Turning to the cam mechanism (Fig. 4-1b), we find a motion between link 2 (cam) and 3 (follower) that is a combination of rolling and sliding. The latter would also occur if two links were connected by a pin riding in a curved slot.

Consider the pin located at O_B (Fig. 4-1a). Whether it is large



(a) The four planes of a slider-crank mechanism



(b) The three planes of a cam and follower mechanism

FIGURE 4-1 Coincident points.

or small, roller bearing or not will have nothing to do with its kinematic function of limiting motion to only rotation: kinematically it is only the axis about which rotation between links 1 and 2 takes place that is important. This axis pierces both planes at O_B; and this point, common to both planes, is identified equally well as O_{B1} or O_{B2}, points on links 1 and 2 at O_B, but clearly coincident or superposed. Each of these coincident points is firmly attached to its own link at the axis of rotation, superposition being always maintained. We shall call these permanent coincident points permanent centers.

Consider next point E, shown on link 3 (Fig. 4-1a). We shall call it E_3. Directly under it at this moment—or this position of the mechanism—lies point E_2 of link 2, and under this E_1 of link 1. These three superposed points are fixed to their links at the locations at which we see them; with motion of the links, they retain their positions *on their respective links*, but not with respect to each other, each going with its link. Motion of link 2 will cause separation of the coincident points. Each no longer superposed point will trace a curve on each of the planes different from its own. Sometimes the shapes of the curves are obvious, but not always We see that E_1 will trace a circular arc (about O_B) on plane 2; E_2 will trace a circular arc (about O_B) on plane 1. These two curves are traced in opposite directions; i.e., if link 2 is rotating counterclockwise, E_1's trace on plane 2 will be developing in the opposite sense.

E_2 will also trace a circular arc on plane 3. E_3 will trace a circular arc on plane 2 and some sort of curve on plane 1. The shape of this curve—actually a coupler-point curve—is usually difficult to imagine. It is of the fourth order, since it is derived from a slider-crank mechanism (the corresponding curve of a four-bar linkage is of the sixth order). Point E_1 will trace a different fourth-order curve on plane 3.

It should be remarked that while our sketch does not attempt to show it, there is also a point E_4 busy tracing curves. Thus, E_4 makes a straight-line trace on plane 1, a circular arc on plane 3, and some kind of curve on plane 2. And, of course, points E_1, E_2, and E_3 leave their traces on plane 4.

As drawn, Fig. 4-1a evidences three coincident points at C. Points C_3 and C_4 of the pin axis are identical in their actions with respect to plane 1, tracing straight coincident lines on it. Meanwhile C_1 is tracing a straight line on plane 4 and a fourth-order curve on plane 3.

The motion between cam profile and follower point (D of Fig. 4-1b) is a combination of roll and slide, as we have noted earlier. An observer stationed at D_3 would notice the cam rolling about D_3 and simultaneously sliding by it; i.e., the point D_2 (fixed to the cam profile) would trace some curve on the observer's plane 3. D_3's trace on plane 2 is the cam profile itself. Here, too, the originally superposed points move away from each other.

Coincident points of moving systems are thus seen to be of two kinds: (1) those which remain permanently coincident, occurring only along a permanent axis of rotation; and (2) those which separate on motion, having been coincident only instantaneously, as at the moment of a particular configuration.

4-3 NOTATION

The discussion of physical events is dependent upon a notation of some sort, preferably associative to convey meaning. In the absence of a universally accepted notation, we shall frequently use the symbols noted below and shall also improvise as necessary. The context will serve to distinguish the meanings of symbols having several qualities.

$\mathbf{a}$	total acceleration (vector)
$\mathbf{a}^n$, $\mathbf{a}^t$	normal and tangential components (vectors)
$\mathbf{a}^r$, $\mathbf{a}^\theta$	radial and transverse components (vectors)
$\mathbf{a}^{\text{cor}}$	Coriolis component (vector)
$a = \lvert\mathbf{a}\rvert$	magnitude of acceleration (scalar)
$\mathbf{a}_{CB}$	acceleration difference of points C and B, defined by $\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{CB}$
$A, B, C, \ldots$	points of a linkage, usually joints
O	point of zero motion
$P, Q, R, \ldots$	points on links other than joints
$\mathbf{PQ}$	vector representing the directed distance PQ
$\mathbf{v}$	total velocity (vector)
$\mathbf{v}^r$, $\mathbf{v}^\theta$	radial and transverse components (vectors)
$v = \lvert\mathbf{v}\rvert$	magnitude of velocity (scalar)
$\mathbf{v}_{CB}$	velocity difference between points C and B, defined by $\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{CB}$
$1, 2, 3, \ldots$	links comprising the chain or mechanism
ϑ	("tetta"), angle defining the ϑ line of velocity distribution
$\boldsymbol{\rho}$	position vector, variable
$\rho = \lvert\boldsymbol{\rho}\rvert$	magnitude of the position vector
r	radius of a circle, fixed
$O_BB, BC, \ldots$	distances
θ	angle of position vector with respect to reference line; ρ and θ are polar coordinates defining the position of a point
x, y	rectangular position coordinates
ω	angular velocity
α	angular acceleration

Consistent units are implied, with radians for angular measure unless otherwise specified.

4-4 LINEAR AND ANGULAR VELOCITIES

Motion refers to an event requiring a certain amount of time for its completion. Motion includes at least three distinctly different yet related characteristics—displacement, velocity, and acceleration. To these may be added higher derivatives, such as jerk and those following. To describe the motion of anything, we need a reference system to which we refer the motion or from which we observe it. In kinematics, any arbitrary reference system will do, but, of the lot, one may be much more convenient than the others. The motions of the various links of a mechanism are generally referred to a link called the *frame*, and the motions with respect to it, i.e., relative to it, are called *absolute* motions. Motions referred to a link other than the frame are called *relative* motions. We see that absolute motion is merely a special case of relative motion.

When only planar motion is considered, as in this chapter, we may speak of a reference plane. Following the selection of a convenient origin O and rectangular axes Ox and Oy fixed in the reference plane, the position of a point may be defined by cartesian (x, y) or polar (ρ, θ) coordinates; the angular position of a line may be given by an angle such as θ between the axis Ox and the line.

Displacement is defined as the difference between the position coordinates of the final and initial locations. Dependence on only position coordinates makes displacement a vector quantity; it is the most direct route from here to there. Distance, on the other hand, depends upon the actual route or path; it is not a vector, but only a magnitude. For example, a man (point) driving from Chicago to New York suffers a displacement expressible by the differences in the latitudes and longitudes of the two cities. The distance traveled depends upon the highways chosen and how often the man got lost and had to retrace or correct his route. Had the man been able to go "as the crow flies," his displacement and distance would have been the same. On the man's completion of a round trip, his displacement was zero (same position coordinates after as before), but the distance was the odometer reading of his car.

The same ideas apply to the angular motion of a line. If a line rotates directly to 270°, then both angular displacement and distance will be 270°. If the line oscillates on the way to the 270° mark, the angular distance will be greater than the 270° displacement. When the line rotates continuously in the same direction, the angular displacement will be zero on passing its original position; the angular distance is of course 360ℓ. With continuous rotation the angular distance mounts by 360ℓ for each revolution, while the displacement can never exceed 360°.

Linear velocity is defined as the time rate of change of position of a point. Since this change of position or displacement is a vector quantity, velocity is also a vector quantity, having the sense of the displacement being taken on.

Angular velocity is the time rate of change of angular position of a line; it is a positive or negative number for planar mechanisms.

Speed is the time rate of covering distance; it is not a vector, but only a number, for it has all the varied directions of the path. Our Chicago–New York driver would have had an average velocity given by dividing the displacement by the trip time, whereas his average speed would have been the result of dividing the actual distance by the time. The speedometer showed the instantaneous speed, which was also the magnitude of the instantaneous velocity. Angular speed is usually given in rpm; when there is a designation of direction, as "1,800 rpm, clockwise when facing the output shaft," we speak of angular velocity.

We consider the details of linear velocity—or what happens when a point of any plane changes its position—with the aid of Fig. 4-2. The plane of the paper is the reference plane on which the point B of the moving link k traces the path s shown; the point changes its location from s_i to s_f in a time interval Δt.

We are at liberty to select an arbitrary origin and reference line for the polar coordinates ρ and θ. We shall consider the path s to be traced by the head of the bound vector $\boldsymbol{\rho}$ as the time t varies. The vectors $\boldsymbol{\rho}(t)$ and $\boldsymbol{\rho}(t + \Delta t)$ for positions s_i and s_f will also be called position vectors. For analytical reasons they must of course be functions of time t and must possess at least a second derivative. We also establish two sets of rectangular directions at s_i. They are the r (radial) and θ (transverse) directions, related to the *reference coordinate system;* the t (tangential) and n (normal) directions related to the *path.*

The finite vector change $\Delta\boldsymbol{\rho}(t) = \boldsymbol{\rho}(t + \Delta t) - \boldsymbol{\rho}(t)$ corresponding to a small but finite time interval Δt is directed along the chord s_is_f. If we form $\Delta\boldsymbol{\rho}(t)/\Delta t$, we recognize this as the average velocity vector $\mathbf{v}_{B,\text{av}}$, also lying along the chord. This says that we get from s_i to s_f by proceeding along the chord $\Delta\boldsymbol{\rho}(t)$ at a constant (i.e., average) rate (speed or magnitude of velocity) numerically equal to $|\Delta\boldsymbol{\rho}(t)|/\Delta t$.

As the time interval Δt approaches zero, we have

$$\lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{\rho}(t)}{\Delta t} = \mathbf{v}_B(t)$$

the velocity of the point B at the instant t defining the position s_i. The vector $\mathbf{v}_B(t)$ becomes tangent to the curve at s_i. To find the magnitude

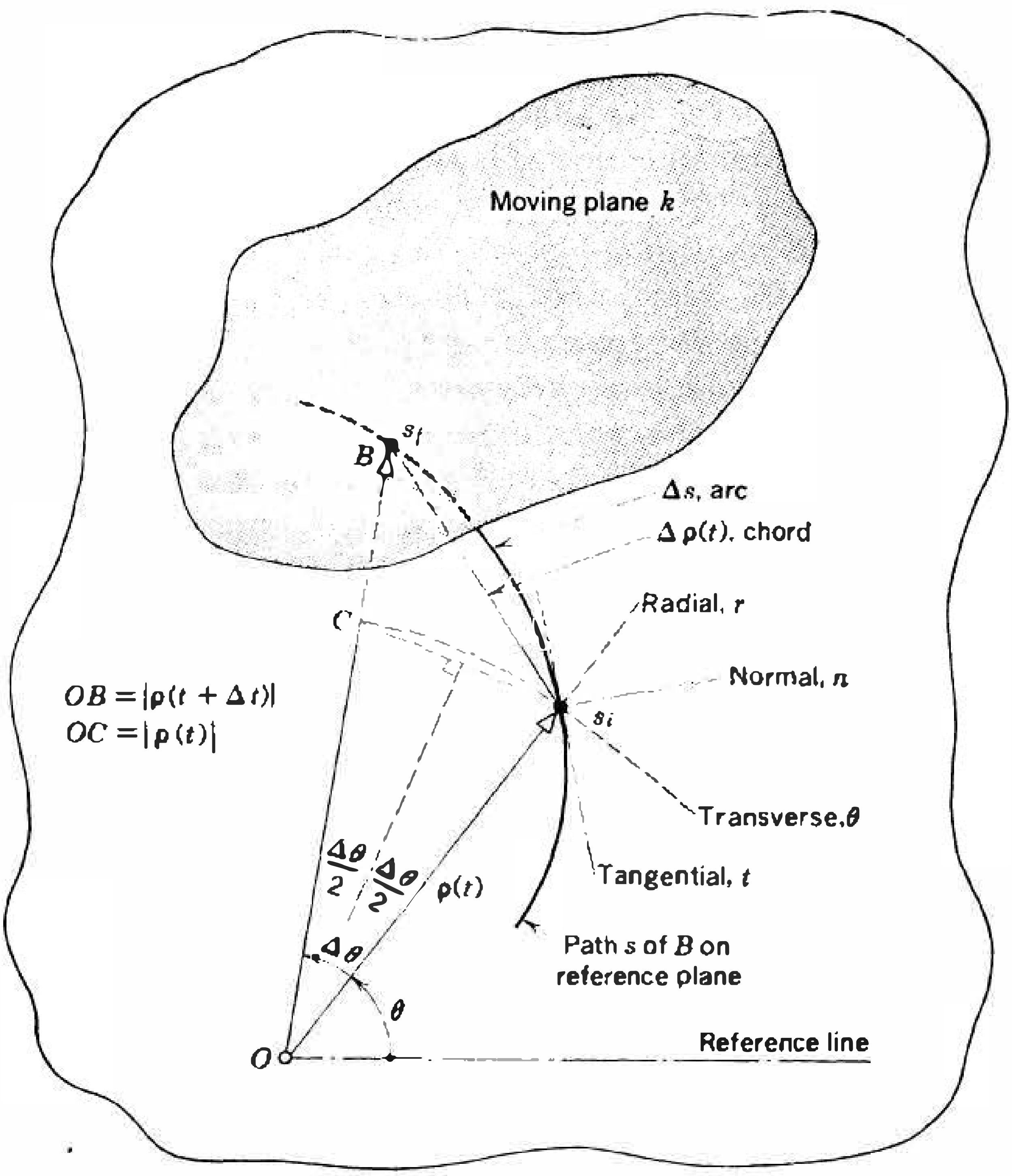


FIGURE 4-2 Velocity of a point of a plane moving with respect to a reference plane.

of $\mathbf{v}_B(t)$, we note that $\Delta\boldsymbol{\rho}(t) \neq 0$ for small $\Delta t > 0$. We may then write

$$\frac{\Delta s}{\Delta t} = \frac{\Delta s}{|\Delta\boldsymbol{\rho}(t)|}\frac{|\Delta\boldsymbol{\rho}(t)|}{\Delta t} = \frac{\Delta s}{|\Delta\boldsymbol{\rho}(t)|}\left|\frac{\Delta\boldsymbol{\rho}(t)}{\Delta t}\right|$$

In this, $|\Delta\boldsymbol{\rho}(t)|$ is the length of the chord, and the ratio $\Delta s/|\Delta\boldsymbol{\rho}(t)|$ = arc/chord tends to 1, whence in the limit

$$|\mathbf{v}_B(t)| = \frac{ds}{dt} = \left|\frac{d\boldsymbol{\rho}(t)}{dt}\right|$$

The velocity vector $\mathbf{v}_B$ thus has the magnitude $|\mathbf{v}_B| = ds/dt$, where s is the arc length along the path.

Inspection of the figure shows that $\mathbf{v}_B$ has no component in the direction n of the normal to the path, since it is directed along the tangent, lying wholly in it. However, components are seen to exist for the radial r and transverse θ directions. We shall investigate these.

We take the vector $\Delta\boldsymbol{\rho}(t)$ to be composed of two other vectors, namely, $\Delta\boldsymbol{\rho}(t) = \mathbf{s}_i\mathbf{C} + \mathbf{C}\mathbf{s}_f$. On dividing by Δt,

$$\frac{\Delta\boldsymbol{\rho}(t)}{\Delta t} = \frac{\mathbf{s}_i\mathbf{C}}{\Delta t} + \frac{\mathbf{C}\mathbf{s}_f}{\Delta t} = \mathbf{v}_{B,\mathrm{av}}$$

That is, $\mathbf{v}_{B,\mathrm{av}}$ is composed of two velocity vectors each having a different direction. This situation is maintained to the limit,

$$\lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{\rho}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{s}_i\mathbf{C}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\mathbf{C}\mathbf{s}_f}{\Delta t} = \mathbf{v}_B$$

The vector $\mathbf{s}_i\mathbf{C}/\Delta t$ has a direction perpendicular to the bisector of the angle $s_iOs_f = \Delta\theta$. and its magnitude is

$$\frac{s_iC}{\Delta t} = \left|\frac{2\rho(t)\sin(\Delta\theta/2)}{\Delta t}\right| = \left|\boldsymbol{\rho}(t)\frac{\Delta\theta}{\Delta t}\right|$$

the sine of a small angle being equal to itself (expressed in radians). With decreasing Δt, $\Delta\theta$ also decreases, and, in the limit $\Delta t \to 0$, s_f coincides with s_i, and the bisector of the angle coincides with Os_i. The vector $\mathbf{s}_i\mathbf{C}/\Delta t$ thus becomes perpendicular to the position vector $\boldsymbol{\rho}(t)$, and its magnitude is $|\boldsymbol{\rho}(t)\,\Delta\theta/\Delta t| = |\boldsymbol{\rho}\,d\theta/dt|$. This component of the velocity is denoted as $\mathbf{v}_B{}^\theta$.

The second component vector $\mathbf{C}\mathbf{s}_f/\Delta t$ has the direction of the line Os_f; its magnitude is the increment in magnitude (i.e., length) of the position vector during the time Δt, namely, $|\boldsymbol{\rho}(t + \Delta t)| - |\boldsymbol{\rho}(t)|$. As Δt is taken smaller and smaller, $\mathbf{O}\mathbf{s}_f$ approaches $\mathbf{O}\mathbf{s}_i$, whence the vector $\mathbf{C}\mathbf{s}_f/\Delta t$ lies in the direction of the position vector $\boldsymbol{\rho}(t)$ of location s_i; its magnitude is the rate of change of length of the position vector, or $d\rho/dt$. This radial component of velocity is written $\mathbf{v}_B{}^\rho$.

In summary, the velocity $\mathbf{v}_B$ of a point at a given instant is composed of two rectangular components,

$$\mathbf{v}_B = \mathbf{v}_B{}^\theta + \mathbf{v}_B{}^\rho$$

where the vectors $\mathbf{v}_B{}^\theta$ and $\mathbf{v}_B{}^\rho$ are as follows:

Transverse component $\mathbf{v}_B{}^\theta$

Magnitude: $\rho\,d\theta/dt$

Direction: perpendicular to position vector $\boldsymbol{\rho}$

Radial component $\mathbf{v}_B{}^\rho$

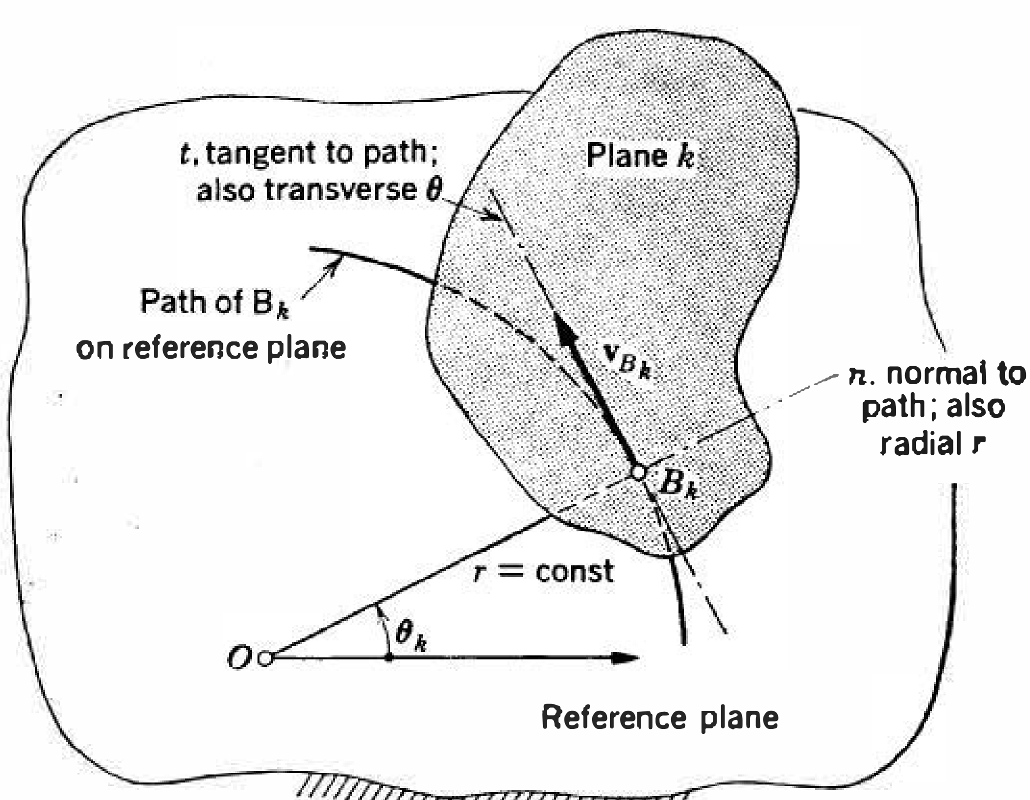
Magnitude: $d\rho/dt$

Direction: along position vector $\boldsymbol{\rho}$

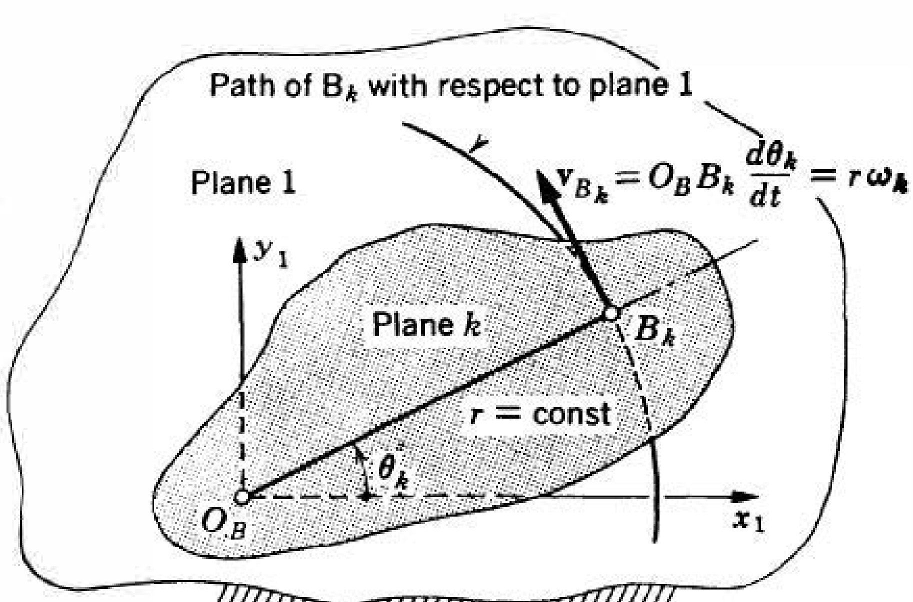
As Δt approaches zero, the direction of the vector $\Delta \boldsymbol{\rho}$ approaches the tangent to the path, which means that there is never a normal component, namely, $\mathbf{v}_B{}^t = \mathbf{v}_B$ and $\mathbf{v}_B{}^n = 0$.

It is important to note again that the velocity of a moving point is with respect to a *plane* (not to a point in the plane); the existence of a velocity requires a path. For example, an aircraft's velocity is reckoned with respect to the earth, not with respect to a point such as a city.

A particular case in which a moving point describes a circular arc about an already specified point of the reference plane occurs so often that special comment is necessary (Fig. 4-3a). Since the origin has been arbitrarily (but by specification) placed at the center of the circular path,



(a)



(b) Link k pinned to link 1 at O_B

FIGURE 4-3 Point of moving plane k describing circular-arc path on reference plane 1.

$\rho = r =$ const, whence the velocity reduces to

$$\mathbf{v}_{B_k} = \mathbf{v}_{B_k}{}^{\theta} = r\,\frac{d\theta_k}{dt}$$

As before, $\mathbf{v}_{B_k}$ lies on the tangent to the circle (path). The r and θ directions, although identifiable, are usually suppressed in favor of the n and t directions, with which they coincide.

Thus far we have discussed the motion of a point with respect to a reference plane in general terms. We had no mechanism for guiding the point on the path. We may now consider a specific and very common type of guidance, namely, that occasioned by a pin connection.

The physical situation is sketched in Fig. 4-3b, in which plane k (link k) is pin-connected to the reference plane 1 (link 1) at O_B. Link point B_k then describes a circular arc on plane 1 from whose reference system x_1y_1 the θ_k of the moving link may be measured.o Then

$$|\mathbf{v}_{B_k}| = r\,\frac{d\theta_k}{dt} = r\omega_k = (O_BB)\omega_k$$

We remark again that the above equation is a consequence of two circumstances, (1) the circular path, i.e., a link of constant length, and (2) the fortuitous specification of the origin O_B. But even with all this, the velocity of B_k is with respect to the reference *plane;* we observe that O_B was convenient for calculation, the velocity being defined by one vector component instead of the two that would have resulted from the specification of any other origin.

4-5 RELATIVE VELOCITY AND VELOCITY DIFFERENCE

The concept of relative velocity is best understood by considering a plane 2 moving with respect to a reference plane 1. A point[1] called B moving over plane 2 has an *absolute velocity* referred to plane 1, denoted by $\mathbf{v}_B$ (Fig. 4-4). Seen by an observer moving with plane 2, the point B has a *relative velocity* with respect to plane 2 denoted by $\mathbf{v}_{B/2}$. Finally, the point B_2 coincident with B at the instant considered, but moving with plane 2, has an absolute velocity $\mathbf{v}_{B_2}$.

The relation connecting the three velocities—the absolute velocity $\mathbf{v}_B$, the relative velocity $\mathbf{v}_{B/2}$, and the absolute velocity of the coincident point $\mathbf{v}_{B_2}$—is obtained by considering small displacements as shown in Fig. 4-5. Here, plane 2 moves from position 2 to position 2′ during the time interval Δt. At the beginning of the interval, B_2 and B are coinci-

[1] This point B may be either (1) a point of another plane or (2) a point defined by a certain geometry, e.g., the intersection of two moving lines.

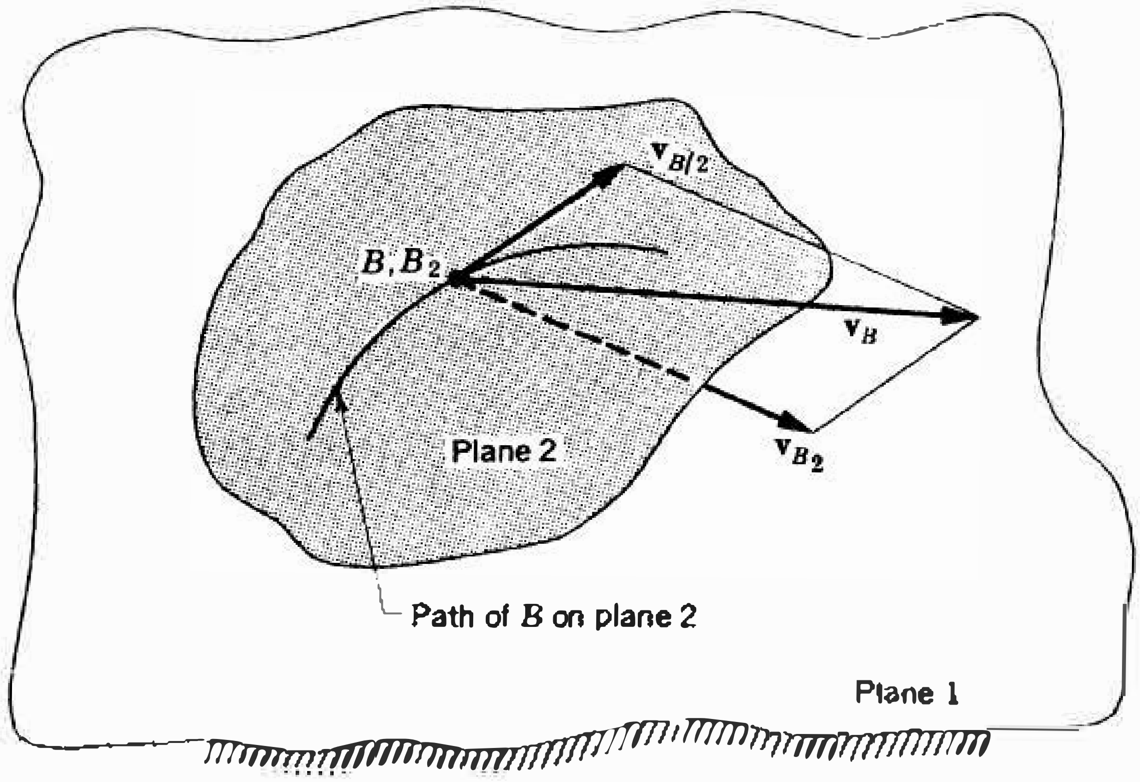


FIGURE 4-4 Relative velocity.

dent; during the interval Δt, B_2 moves with plane 2 to position B_2'. Point B, however, moves with respect to plane 2 and at the end of the interval will occupy position B' distinct from B_2'. Now, from the definition of velocity,

$$\mathbf{v}_B = \lim_{\Delta t \to 0} \frac{\mathbf{BB'}}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\mathbf{B_2B_2'}}{\Delta t} + \frac{\mathbf{B_2'B'}}{\Delta t} \right)$$
$$= \lim_{\Delta t \to 0} \frac{\mathbf{B_2B_2'}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\mathbf{B_2'B'}}{\Delta t}$$

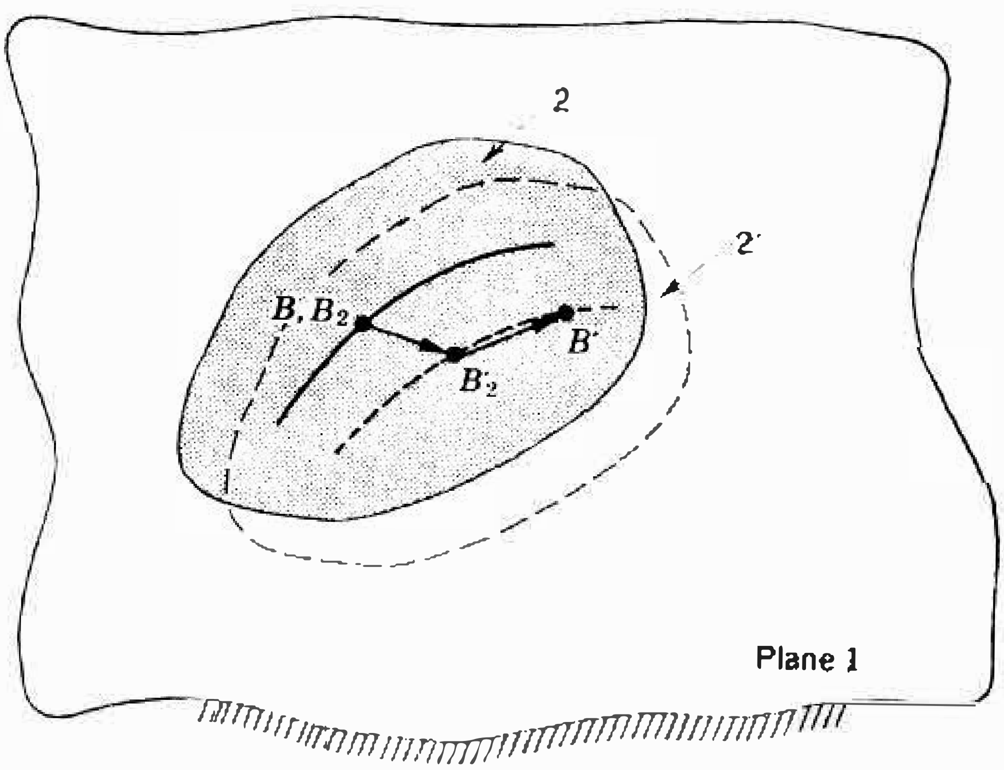


FIGURE 4-5 Proof of relative-velocity theorem.

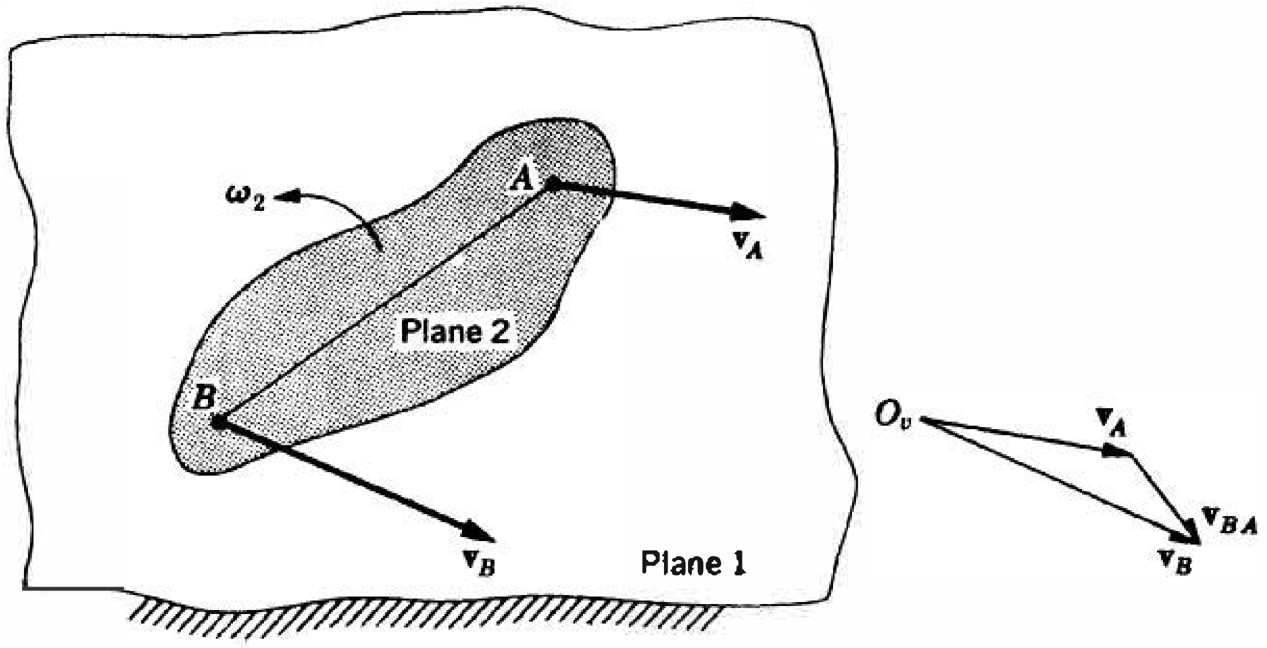


FIGURE 4-6 Velocity difference of two points on the same link.

But the first limit in the second equation above is simply the velocity of point B_2, and the second limit is the velocity of B as seen by an observer moving with plane 2, i.e., the relative velocity of B, whence

$$\mathbf{v}_B = \mathbf{v}_{B_2} + \mathbf{v}_{B/2} \tag{4-1}$$

The above equation thus relates the absolute velocities of two coincident points B_2 and B moving independently of one another. The relative velocity $\mathbf{v}_{B/2}$ is the velocity that point B would appear to have to any observer moving with plane 2.

Quite a different situation will now be considered (Fig. 4-6)t again plane 2 moves with respect to the reference plane 1, but now plane 2 contains two distinct points A and B with absolute velocities $\mathbf{v}_A$ and $\mathbf{v}_B$. The *velocity difference* between these two points is defined as $\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$, from which

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA} \tag{4-2}$$

The velocity difference $\mathbf{v}_{BA}$ will now be related to the distance AB and the angular velocity ω_2 of plane 2. Again, velocities will be considered as limits of vanishingly small displacements, as shown in Fig. 4-7. The displacement of plane 2 during the time interval Δt is considered to consist of two steps: a translation from AB to $A'B''$, followed by a rotation about A' from $A'B''$ to $A'B'$. Then,

$$\begin{aligned}\mathbf{v}_B &= \lim_{\Delta t\to 0} \frac{\mathsf{BB'}}{\Delta t} = \lim_{\Delta t \to 0}\left(\frac{\mathsf{BB''}}{\Delta t} + \frac{\mathsf{B''B'}}{\Delta t}\right)\\ &= \lim_{\Delta t\to 0} \frac{\mathsf{AA'}}{\Delta t} + \lim_{\Delta t\to 0}\frac{\mathsf{B''B'}}{\Delta t}\end{aligned}$$

since $\mathsf{BB''} = \mathsf{AA'}$. The first limit in the last equation above is $\mathbf{v}_A$, the

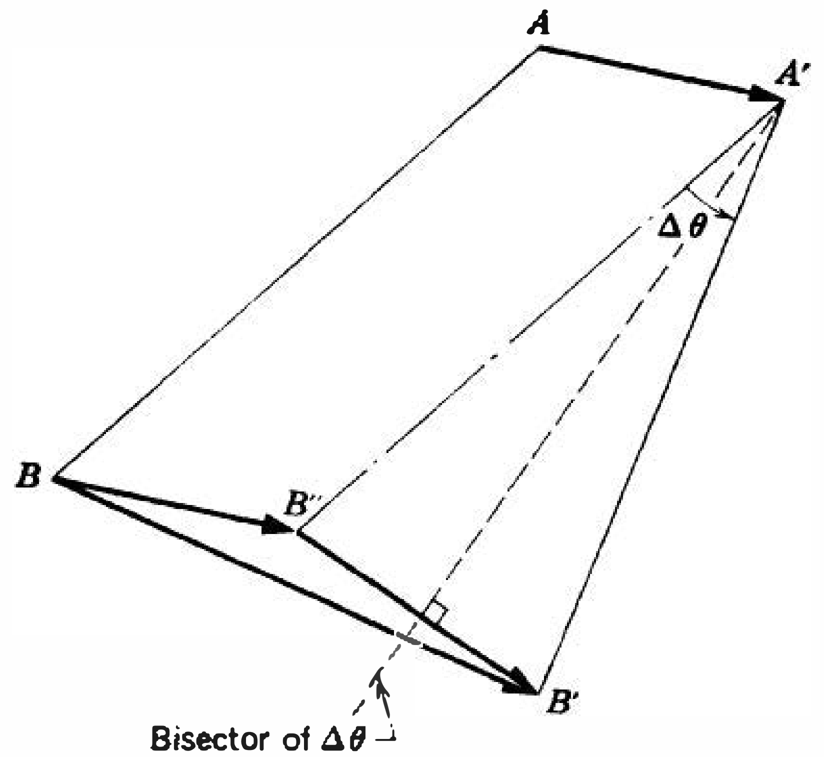


FIGURE 4-7 Determination of the velocity difference of A and B.

velocity of point A; the second limit may therefore be identified with the velocity difference of the two points,

$$\mathbf{v}_{BA} = \lim_{\Delta t \to 0} \frac{\mathbf{B''B'}}{\Delta t}$$

The vector $\mathbf{B''B'}$ has a direction perpendicular to the bisector of the angle $B''A'B' = \Delta\theta$ and a magnitude $B''B' = (AB)\,\Delta\theta$, $\Delta\theta$ being a small angle. As Δt approaches zero, the bisector approaches the direction AB, and $B''B/\Delta t = AB/(\Delta\theta/\Delta t)$ approaches $AB/(d\theta/dt)$. Noting that $\Delta\theta$ is the angle of rotation of plane 2 during the time interval Δt, we conclude that the velocity difference $\mathbf{v}_{BA}$ is perpendicular to AB and has a magnitude equal to $(AB)\omega_2$.

Because of the similarity between Eqs. (4-1) and (4-2), the velocity difference is often called the "relative velocity" of B with respect to A. It should be noted, however, that the two situations described by these equations are different, and it is misleading to identify relative velocity ($\mathbf{v}_{B/2}$) and velocity difference ($\mathbf{v}_{BA}$) only on the basis of a similarity in equations. We have already noticed that both velocity and displacement are vectors and can be referred only to a plane (for planar motion) in which a reference direction such as Ox has been defined. A velocity cannot be referred to a point, since its direction cannot be specified. The difference between relative velocity and velocity difference may be further explained in terms of an example. A person (point B) pacing the interior of a railroad car (plane 2) riding through the night would be able, by simple observation, to determine his velocity with respect to the car ($\mathbf{v}_{B/2}$), for this is independent of the motion of the car. It would be impossible for him, however, to establish the velocity difference between the two ends of the car, for this velocity difference is the difference of two (absolute) velocities referred to the earth, which the person cannot see; the velocity difference depends on the motion of the car with respect to the earth.

Velocity analysis—the determination of the linear velocities of points and the angular velocities of links of mechanisms—is conveniently handled by the method of vector polygons. To establish the fundamentals of velocity analysis based on vector polygons, we shall examine the velocity condition of the four-bar linkage of Fig. 4-8*a*, the angular velocity ω_2 of link 2 being known. This will involve finding the linear velocities of the several points, the angular velocities of links 3 and 4, and the use of the velocity image.

Suppose that in our awkwardness we attempt first the velocity of D. We would write the equation $\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{DB}$ and proceed to evaluate the three terms as best we can. Each of the vector terms has two endowments, magnitude and direction, each of which may be treated as an "unknown" in the same sense as the unknowns of algebraic equations. Algebraic equations are scalars, and one such equation cannot be solved if the number of unknowns exceeds one. A vector equation (such as we have) may be viewed as consisting of two scalar equations, whence the limit of unknowns is two per vector equation. In algebra,

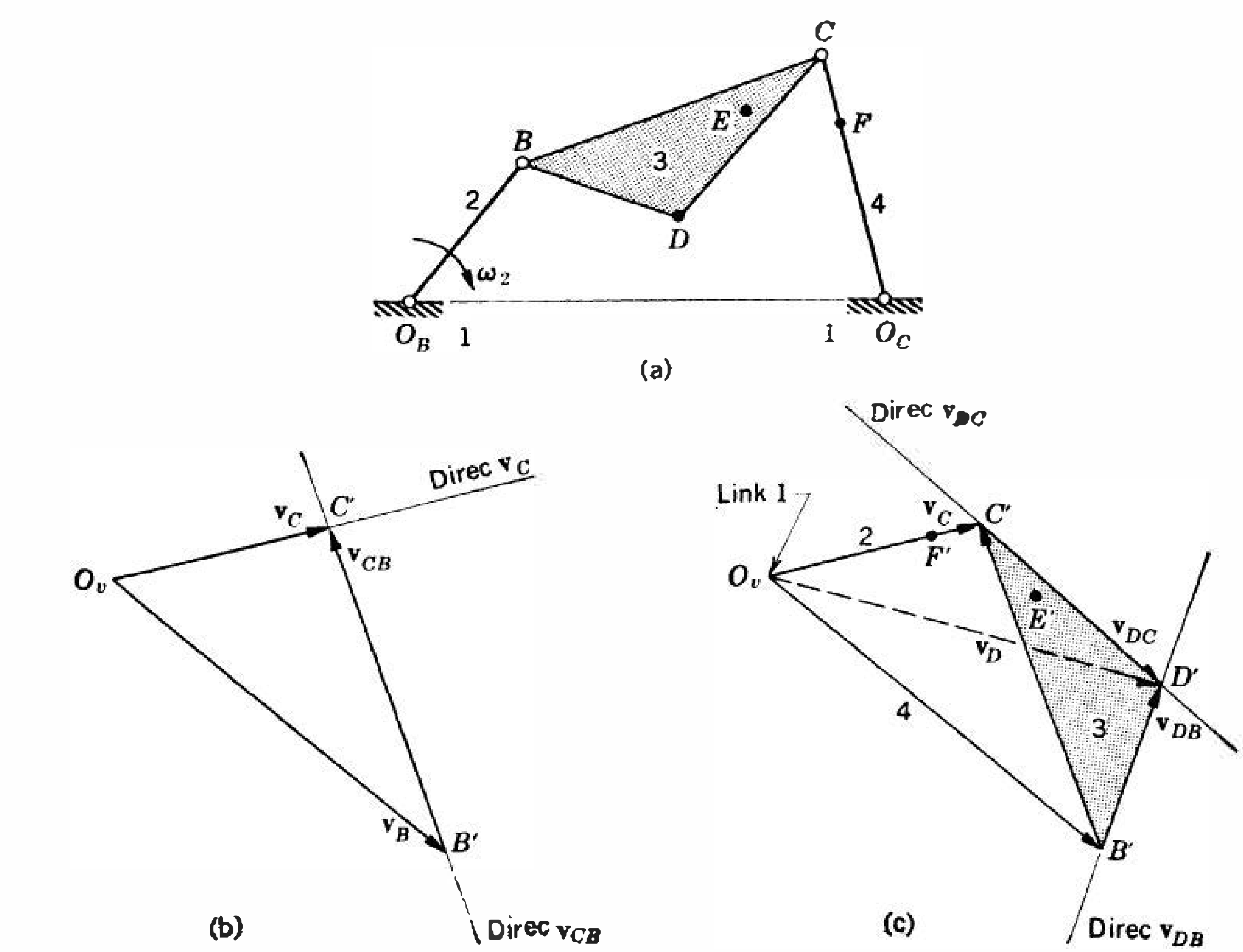


FIGURE 4-8 Velocity analysis of the four-bar linkage.

as many simultaneous equations as there are unknowns are needed for solution; with vector equations, only half as many equations as unknowns are needed.

To return to the evaluation of the vectors, we shall examine each in turn for (1) direction and (2) magnitude and shall hope that not more than two unknowns exist for the entire equation:

Term $\mathbf{v}_D$ both magnitude and direction are unknown
Term $\mathbf{v}_B$ completely known in magnitude $(O_BB)\omega_2$ and direction
Term $\mathbf{v}_{DB}$ only direction is known

This information is conveniently summarized on the equation itself by a system of checks (knowns) and x's (unknowns) as follows:

$$\overset{x_1 x_2}{\mathbf{v}_D} = \overset{\checkmark\checkmark}{\mathbf{v}_B} + \overset{\checkmark x_3}{\mathbf{v}_{DB}} \tag{4-3}$$

With three unknowns the situation is hopeless, whence another tactic—another point—must be looked at, with hope for better luck.

Turning to the alternative point C, we write $\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{CB}$. Termwise examination allows the equation to be marked as

$$\overset{x_1 \checkmark}{\mathbf{v}_C} = \overset{\checkmark\checkmark}{\mathbf{v}_B} + \overset{\checkmark x_2}{\mathbf{v}_{CB}} \tag{4-4}$$

Here x_1 and x_2 indicate unknown velocity magnitudes. But since the unknowns total only two, the vector equation can be solved. We choose an origin O_v and from it lay off the completely known $\mathbf{v}_B$ to some convenient scale (Fig. 4-8b). The equation instructs us to add $\mathbf{v}_{CB}$ to $\mathbf{v}_B$; all that can be done now is to lay off the direction of $\mathbf{v}_{CB}$. Only $\mathbf{v}_C$ is left; it is an absolute velocity and hence originates from O_v, going out to meet the "other side" of the equation. On laying off the direction of $\mathbf{v}_C$ the line intersects the $\mathbf{v}_{CB}$ direction line. Since the tail of the vector $\mathbf{v}_{CB}$ must start from the head of $\mathbf{v}_B$—the cue comes from the equation—the sense (as different from the direction) and magnitude of each of the partially known vectors are established; i.e., vectors $\mathbf{v}_{CB}$ and $\mathbf{v}_C$ are completely defined. The angular velocity of link 3 is given by $\omega_3 = v_{CB}/BC$. Similarly, $\omega_4 = v_C/O_CC$. The senses of the angular velocities are deduced from those of the linear velocities.

With $\mathbf{v}_C$ at hand, the velocity $\mathbf{v}_D$ becomes approachable, for we may involve $\mathbf{v}_D$ in two equations, which, after being checked off, show an acceptable four unknowns:

$$\overset{x_1 x_2}{\mathbf{v}_D} = \overset{\checkmark\checkmark}{\mathbf{v}_B} + \overset{\checkmark x_3}{\mathbf{v}_{DB}}$$

$$\overset{x_1 x_2}{\mathbf{v}_D} = \overset{\checkmark\checkmark}{\mathbf{v}_C} + \overset{\checkmark x_4}{\mathbf{v}_{DC}}$$

For clarity, the first vector polygon has been redrawn (Fig. 4-8c) and the solution of the last two equations added. Common to both equations, $\mathbf{v}_D$ is shown by the broken-line vector O_vD'. The shaded figure $B'C'D'$ is known as the velocity image of link 3; it is geometrically similar to the figure BCD that is link 3. The line $B'C'$ stands at right angles to the line BC, the image having been rotated 90° in the sense of ω_3. Note also that D' is on the side of $B'C'$ corresponding to the location of D with respect to BC.

The velocity image provides a great convenience, for the velocity of any other point of link 3, such as E, may be found without recourse to further equations. It is necessary only to locate the corresponding E' on the image, and the vector O_vE' (not shown) will be $\mathbf{v}_E$.

It follows now, of course, that the velocity image of a link may be drawn directly after establishing the velocity of two points of the link, e.g., the points B and C. Had we done this, there would have been no need for the simultaneous solution of the two vector equations relating point D to points B and C. However, there are situations in which the procedure of simultaneous solution is necessary.

4-6 INSTANTANEOUS CENTERS OF VELOCITY

By instantaneous center is understood a pair of coincident points having either zero relative velocity or acceleration at the moment of observation. The instantaneous centers of velocity and acceleration do not occur at the same point except in trivial situations. That two points of different planes have momentarily equal velocities or accelerations provides a means for taking the known motion of one plane into another.

While the position of the velocity center is in many cases intuitive or is found without too much difficulty, the same cannot be said for the acceleration center. As an analytical device, the instantaneous center of velocity has wide application, and the following remarks will be directed to the velocity center, sometimes abbreviated simply to IC.

Consider a plane 2 moving with respect to a reference plane 1, and assume that the velocities of two points A_2 and B_2 are known, as in Fig. 4-9a. Note that, while the velocity of one point, say A_2, could be chosen arbitrarily, the velocity of the other point B_2 has to be such that the velocity difference $\mathbf{v}_{B_2A_2} = \mathbf{v}_{B_2} - \mathbf{v}_{A_2}$ is perpendicular to the line A_2B_2. Having chosen an origin O_v, we can draw a velocity diagram for the motion of plane 2 with respect to plane 1, as shown in Fig. 4-9b. Now the velocity of any point C_2 of plane 2 may be found by locating its image C_2' on the velocity diagram, similar triangles being used as shown in the preceding section.

Since there is a one-to-one correspondence between the points of plane 2 and their images on the velocity diagram, we may also solve the reverse problem by the same method and find which point C_2 of plane 2 has a specified velocity $\mathbf{v}_{C_2}$. The specified velocity locates the image C_2' such that $\mathbf{O_vC_2'} = \mathbf{v}_{C_2}$, and a triangle $A_2B_2C_2$ similar to $A_2'B_2'C_2'$ is constructed over the line segment A_2B_2 to locate C_2. In particular, we may find by this method the point O_2 of plane 2 having zero velocity at the instant considered. This point is the instantaneous center of velocities of plane 2 with respect to plane 1; its image must be the origin O_v of the velocity diagram. The instantaneous center O_2 is therefore located as shown in Fig. 4-9, such that triangles $A_2B_2O_2$ and $A_2'B_2'O_v$ are similar. Note that this means that O_2 is the intersection of the lines A_2u and B_2v, respectively perpendicular to the velocities at points A_2 and B_2. Note further, by comparing plane 2 with the velocity polygon, that the velocity of any point such as C_2 is perpendicular to the radius vector O_2C_2 from the IC to the point, and the magnitude of the velocity is $v_{C_2} = \omega_2(O_2C_2)$.

In a mechanism consisting of a number of links, an IC relates to two links; with a mechanism of n links the total number N of possible ICs is the number of combinations of n things taken two at a time, or $N = n(n-1)/2$. Thus a four-link mechanism will have an N of 6; for a six-link mechanism, $N = 15$; an eight-link mechanism will have $N = 28$; and so on.

Clearly some sort of bookkeeping system is needed when the

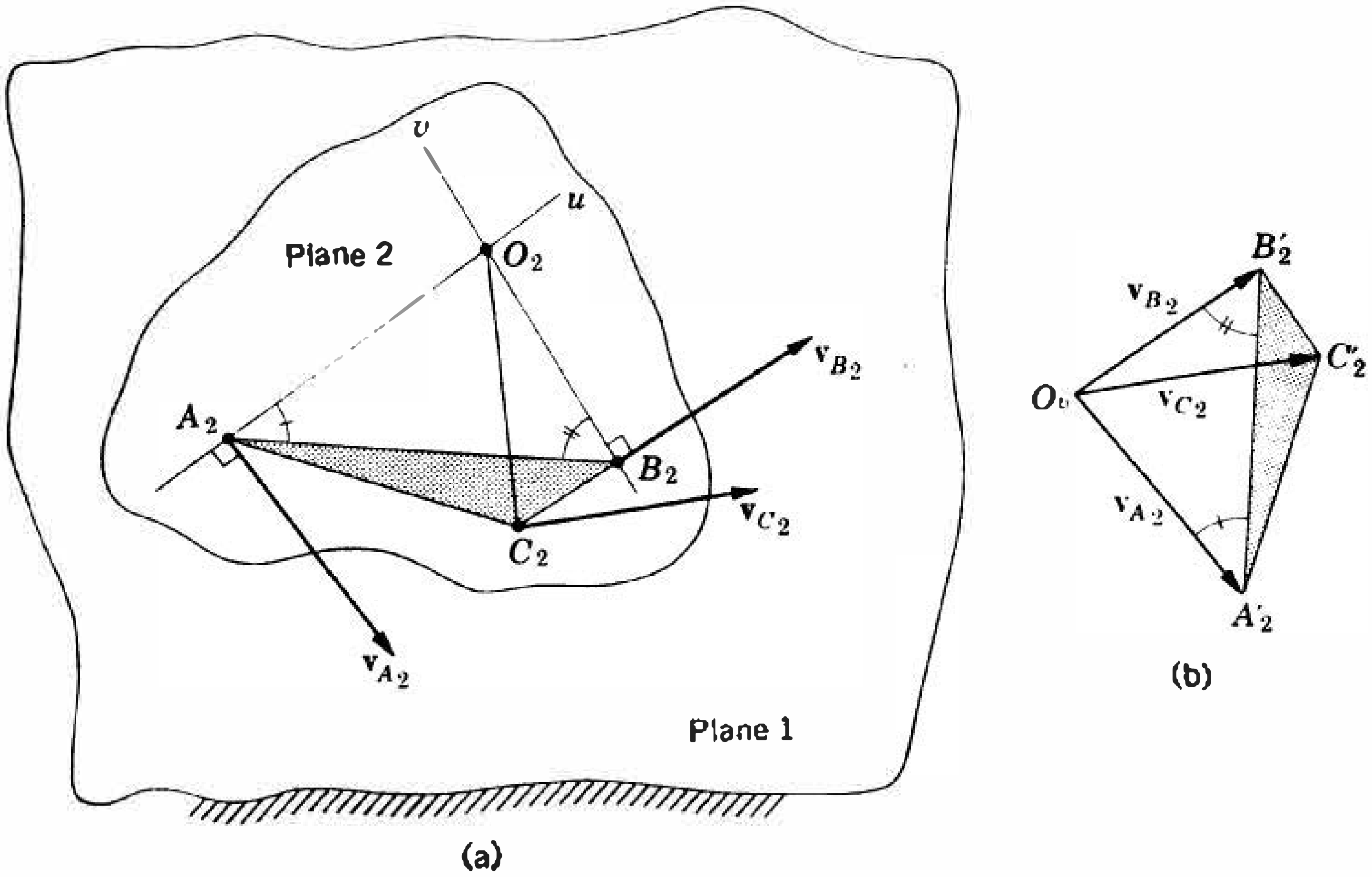


FIGURE 4-9 Existence and properties of the instantaneous center of two planes in relative motion.

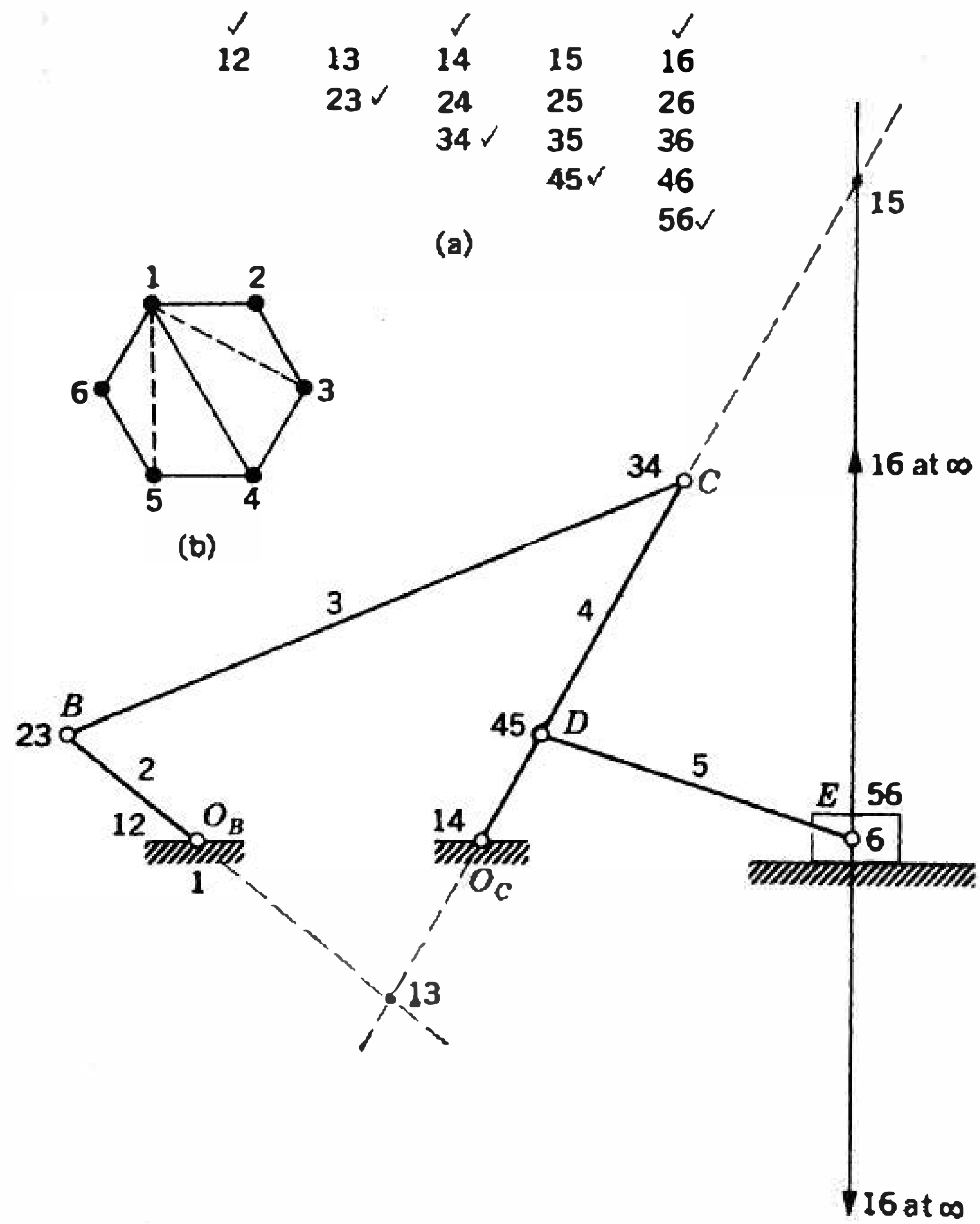


FIGURE 4-10 Determination of ICs of a six-link mechanism.

number of links exceeds four. A six-link mechanism (Fig. 4-10) will serve as example. The ICs may be tabulated as in Fig. 4-10*a*, each link with every other, and checked off as they are found. However, the circle diagram of Fig. 4-10*b* has greater utility. In this, each dot represents a link, and a line between any two dots represents the IC of the two links (dots) so connected. Starting with the dots ordered around a circle (the latter for the sake of neatness), the obvious ICs are entered in the diagram by drawing full lines between appropriate dots.

The obvious ICs are those associated with permanent connections between the links. The term permanent means pin connections and also includes the infinitely distant IC of a slider, located on a normal to the direction of rectilinear translation. This idea is based on the philosophy that such translation is regarded as the limiting case of rotation about an infinitely distant permanent center of rotation, a center common to the two links bearing the kinematic elements of the sliding pair.

In our example (Fig. 4-10) the obvious or directly available ICs are 12,* 23, 34, 45, 14, 56, and 16, the last being involved with translation. The normal on which IC 16 lies is drawn through IC 56 because of the implication of the theorem of three centers, to be stated below. Actually, any line parallel to it will point to IC 16.

The remaining centers are found from deductions based on the theorem of three centers:[1] *The ICs of any three links (connected or not) having planar motion lie on the same straight line.* (We leave the proof as an exercise.) It is in the application of this theorem that the circle diagram is more aggressive than the tabular array in suggesting what to do next. Since a point such as an IC is defined by the intersection of at least two lines, an unknown IC is specified if it is a line in the circle diagram that is the common side of two triangles. Thus, IC 13 (shown as a broken line in Fig. 4-10*b*) is rendered by the intersection of the line 12-23 with the line 14-34. Similarly, IC 15 is at the intersection of 14-45 and 56-16. The remaining ICs are found in like fashion.

We may now distinguish between *absolute* and *relative* ICs. Absolute ICs are those carrying the frame numeral or digit; they are the top row of the tabular array, Fig. 4-10*a*, characterized by the zero velocity of their coincident points. All other ICs are called relative; their coincident points have velocities different from zero, being in motion with respect to the frame. The significance of these ICs is that the coincident points on the two links involved in a relative IC have equal velocities, for example, $\mathbf{v}_{(23)_2} = \mathbf{v}_{(23)_3}$, $\mathbf{v}_{(34)_3} = \mathbf{v}_{(34)_4}$, etc.

It will be noticed that permanent ICs are distributed among both the absolute and the relative ICs. The use of the IC as a medium for determining both velocity distribution in a given link and motion transmission between links will be discussed in terms of a four-bar linkage (Fig. 4-11*a*). A line drawn from 13 to the tip of $\mathbf{v}_{23}$ (which is also $\mathbf{v}_{B_2}$) defines an angle ϑ_3. (The symbol ϑ is the cursive θ and may be pronounced "tetta" to distinguish it from theta.) This ϑ_3 line gives the velocity distribution along any radial line of link 3 emanating from 13, whence the velocity of any other point, such as $\mathbf{v}_{C_3} = \mathbf{v}_{34}$, may be found by duplicating ϑ_3 on a radial line from 13 passing through the point C_3. Also, $\omega_3 = v_{23}/(13\text{-}23) = v_{34}/(13\text{-}34)$, and $\omega_4 = v_{34}/(14\text{-}34)$.

Action at a relative center such as IC 24 may be visualized as in Fig. 4-11*b*. IC 24, as remarked, lies at the intersection of lines 23-34 and 12-14. The sketch shows planes 2 and 4 extended to cover the

* Read "IC one-two." If the smaller digit is always placed first, possible dualities will be avoided.

[1] This is known as the Aronhold theorem in Germany, and sometimes as the Kennedy theorem in the United States, after the independent discoverers, Aronhold, 1872, and Kennedy, 1886.

center 24; each plane, of course, retains its absolute rotation about its own center, as 2 about IC 12, etc. From $\mathbf{v}_{B_3} = \mathbf{v}_{23}$ the ϑ_2 line for link 2 is found and then set off on the radial line 14-24. The perpendicular to it at IC 24 defines $\mathbf{v}_{(24)_2}$; but since this is at the center 24, the coincident point of plane 4 has a velocity precisely equal to that of its mate on plane 2, namely, $\mathbf{v}_{(24)_2} = \mathbf{v}_{(24)_4} = \mathbf{v}_{24}$. In this manner a known velocity of plane 2 is transferred to plane 4.

The ϑ_4 line may now be established and then reconstructed with

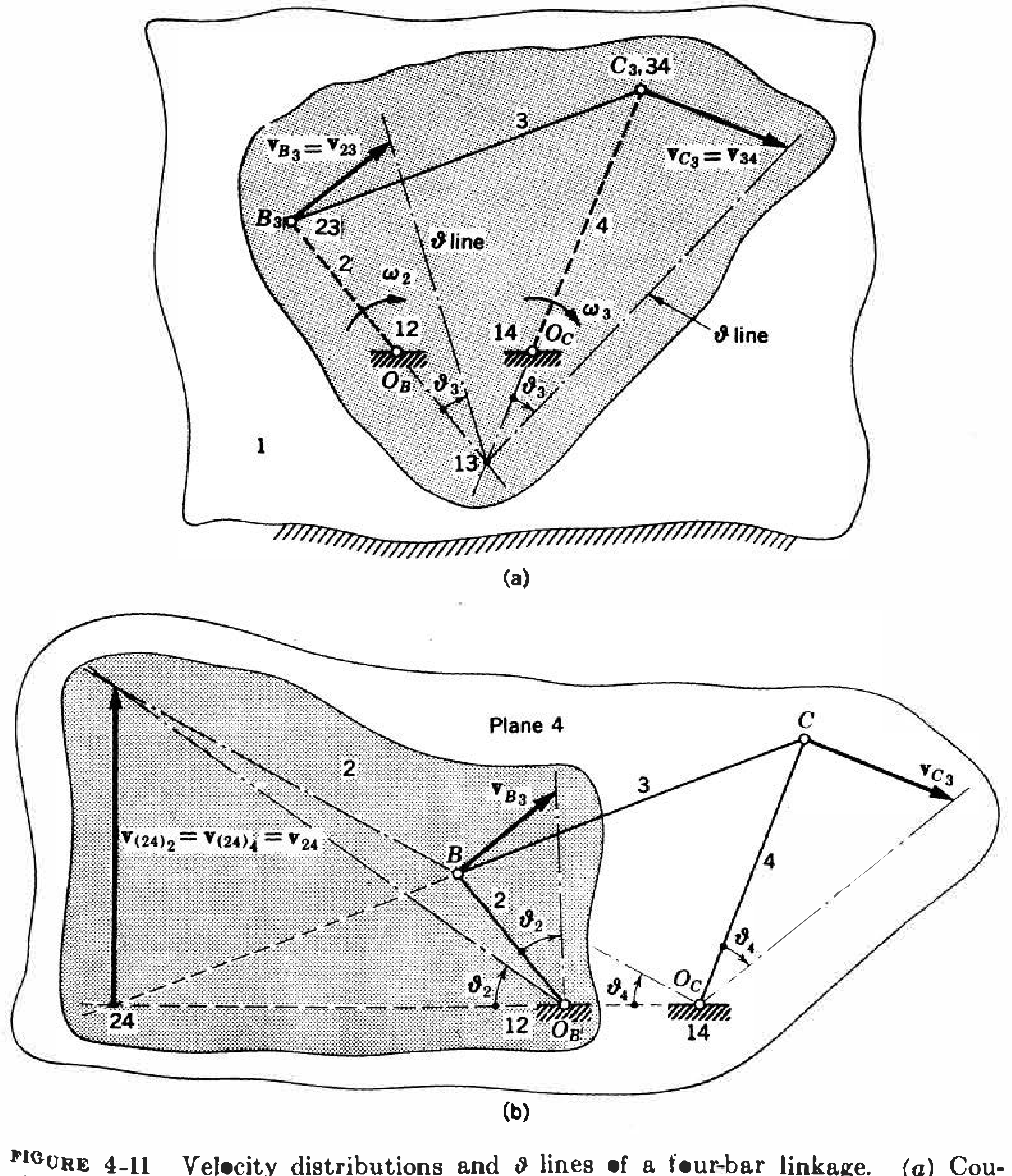


FIGURE 4-11 Velocity distributions and ϑ lines of a four-bar linkage. (*a*) Coupler and frame, absolute IC 13; (*b*) crank and follower, relative IC 24.

respect to line 14-34 to define $\mathbf{v}_{C_4} = \mathbf{v}_{34}$. From this $\omega_4 = v_{34}/(14\text{-}34)$. We should note that we were able to get from link 2 to link 4 without involving the intermediate link 3.

The case of IC 24 is typical of all relative IC situations. The action at such an IC can always be visualized (if the need arises) by considering the pertinent planes to be extended far enough to cover the center, after which the ϑ lines are brought into play.

Relative centers allow one to go directly from one link to any other, provided that the IC of the two links is at hand. For example, if we had an eight-link mechanism we could make the direct transition from link 2 to link 8 via IC 28. However, IC 28 might be troublesome to find: it could be the last to disclose its location $[N = (8 \times 7)/2 = 28]$.

Four-bar and slider-crank linkages were at one time called indirect contact mechanisms, since the coupler link 3 is the intermediary of links 2 and 4, one of which is the input link, with the other the output. Mechanisms in which the input and output links are in immediate connection on curved or camlike surfaces, as in Fig. 4-12*a*, are now often called "direct-contact" mechanisms. Other examples include meshed gear teeth and many conventional cam and follower setups.

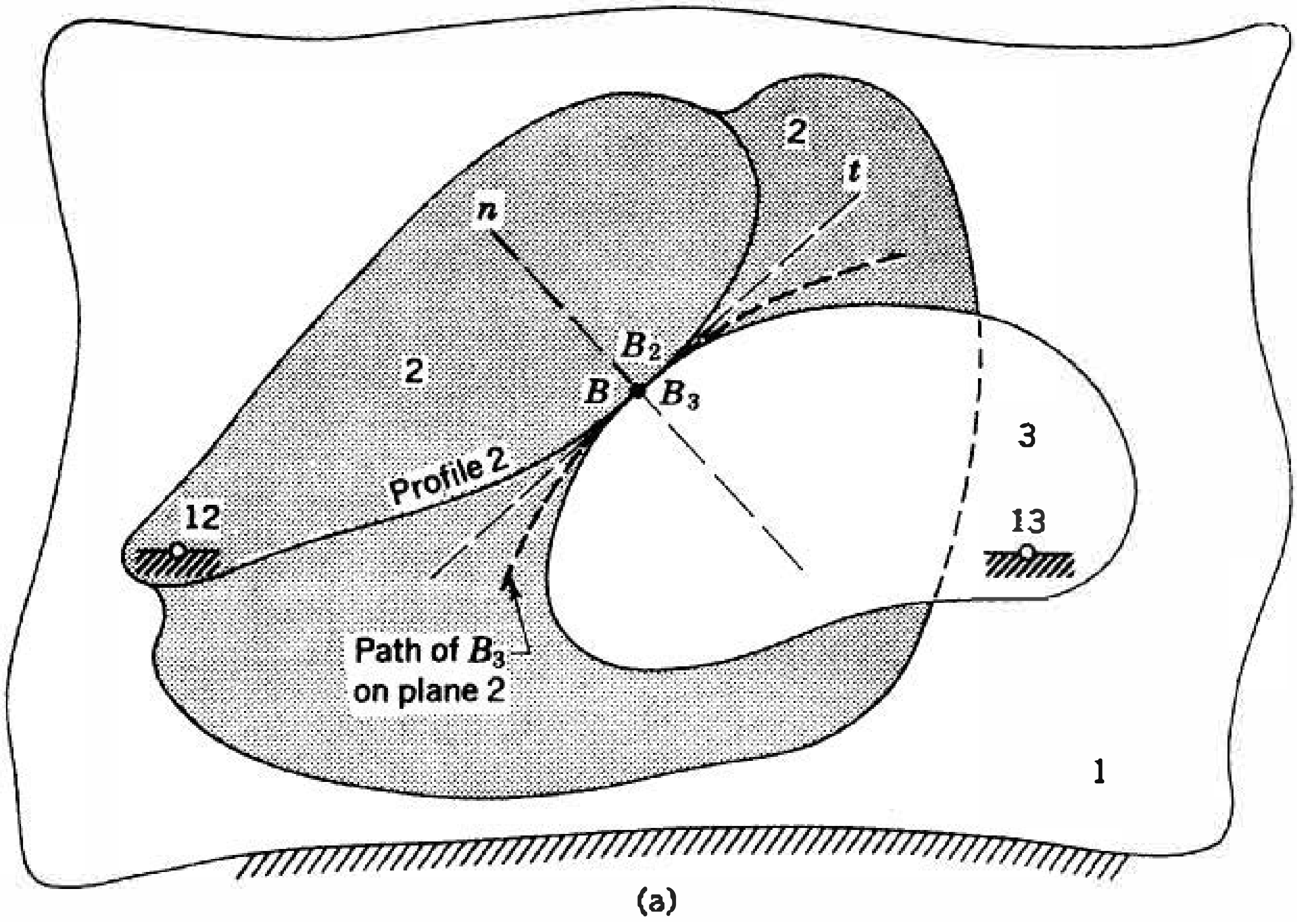
For the purposes of this discussion, the plane of link 2 has been extended beyond the limits of the profile it carries, and link 3 lies over it. At the point of contact, B, the two curves have a common tangent t and normal n. The point B is comprised of two coincident points B_2 and B_3. The path that B_3 would describe on plane 2 is not a simple curve; as the figure shows, it does not coincide with the profile of link 2, and its shape and curvature cannot be found without some trouble. Similarly, B_2's path on plane 3 is not an obvious curve.

As links 2 and 3 move while remaining in contact along their profiles, the instantaneous contact point B runs along the profiles and thus moves with respect to both links 2 and 3. By application of the theorem of relative velocities [Eq. (4-1)], the velocity of this point may be written in two different ways,

$$\mathbf{v}_B = \mathbf{v}_{B/2} + \mathbf{v}_{B_2} \qquad \text{and} \qquad \mathbf{v}_B = \mathbf{v}_{B/3} + \mathbf{v}_{B_3}$$

from which
$$\mathbf{v}_{B/2} - \mathbf{v}_{B/3} = \mathbf{v}_{B_3} - \mathbf{v}_{B_2} = \mathbf{v}_{B_3/2} = -\mathbf{v}_{B_2/3}$$

Since B moves along the profiles of links 2 and 3, both $\mathbf{v}_{B/2}$ and $\mathbf{v}_{B/3}$ are along the common tangent t and from the last equalities above we conclude that the relative velocities $\mathbf{v}_{B_3/2}$ and $\mathbf{v}_{B_2/3}$ are also along this tangent. With $\mathbf{v}_{B_3/2}$ along the common tangent t, the IC 23 must lie on the common normal n, and with the aid of the theorem of three centers we can now locate 23 as the intersection of 12-13 with the normal n as shown in Fig. 4-12*b*. The figure also shows the ϑ lines of the velocity distribution of links 2 and 3 with $\mathbf{v}_{(23)_2} = \mathbf{v}_{(23)_3}$.



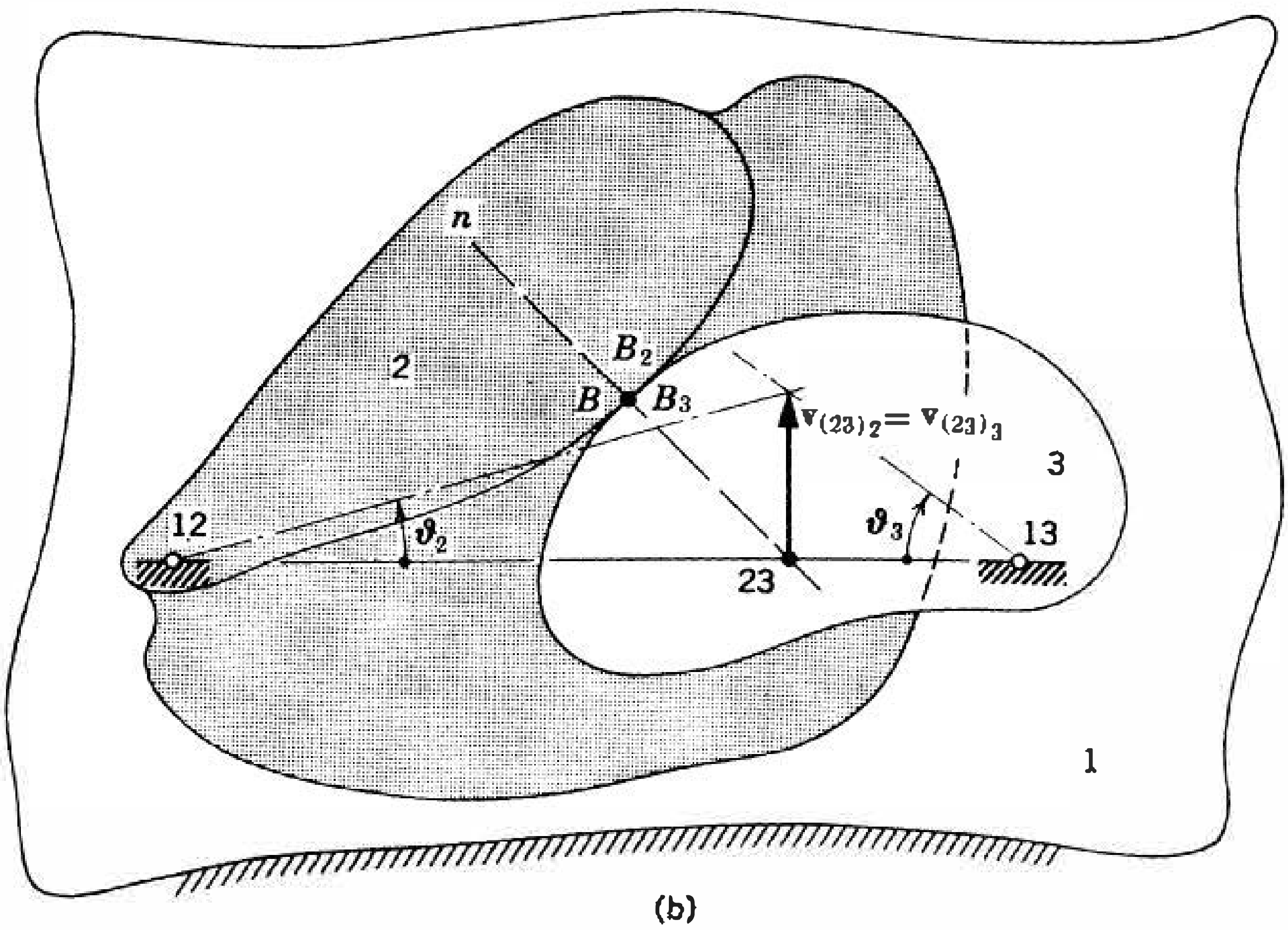


FIGURE 4-12 ICs of direct-contact mechanisms.

4-7 POLES AND CENTRODES

The instantaneous center of velocity, IC, of a plane 2 moving with respect to a reference plane 1 was defined as the point of plane 2 having zero velocity at the instant considered. The location and properties of this point were deduced by velocity images. The purpose of the present section is to reconsider instantaneous centers of velocity from a different point of view, which will be useful for further developments in Chap. 8. This new approach is based on the awareness that every displacement of a moving plane over a reference plane is equivalent to either a unique translation or a unique rotation. For a rotation in which a finite angle $\Delta\theta$ is involved, the center of rotation is called a *pole*. A succession of positions of a body may then be likened to a succession of finite rotations about properly selected poles. The use of the pole furnishes a powerful tool for synthesis as developed in Chaps. 8 and 9.

Let AB of Fig. 4-13 represent a line in a moving link. Successive positions are shown as A_1B_1 and A_2B_2. In Fig. 4-13a the link is moved from position 1 to position 2 by turning about the pole P_{12}. This pole

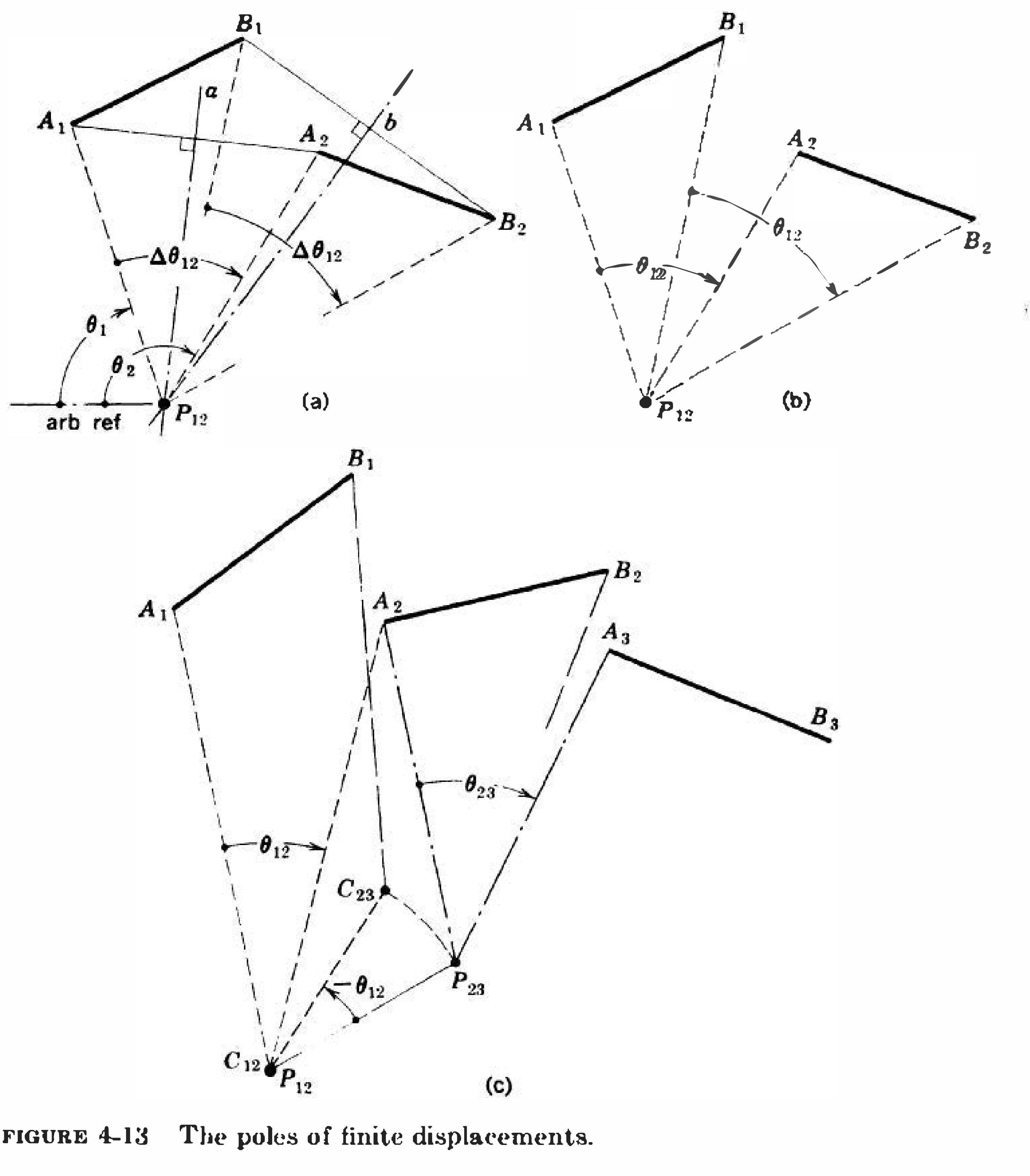


FIGURE 4-13 The poles of finite displacements.

was defined by the intersection of the midnormals a and b, that is, the perpendicular bisectors of the distances A_1A_2 and B_1B_2. The A and B rays have been rotated through the finite angle $\Delta\theta_{12}$, the increment in angle between θ_1 and θ_2. The same situation, but without construction lines and by now superfluous identification, is shown in Fig. 4-13b. Note that $\Delta\theta_{12}$, the finite angle of rotation between positions 1 and 2, is written simply as θ_{12}. It is understood that the pole was found from the midnormals' intersection.

In Fig. 4-13c a third position A_3B_3 is shown, the pole P_{23} having been determined from the midnormals to A_2A_3 and B_2B_3. For convenience, the pole P_{12} is also labeled C_{12}. Suppose the line $P_{12}P_{23}$ to be rotated by an amount $-\theta_{12}$, thus defining point C_{23}, and that $C_{23}B_1$ is drawn. The line $C_{12}C_{23}$ may now be considered to be part of the moving plane containing AB. When A_1B_1 is swung into A_2B_2 about P_{12}, $C_{12}C_{23}$ swings down to become coincident with $P_{12}P_{23}$; figures $C_{12}A_1B_1C_{23}$ and $P_{12}A_2B_2P_{23}$ are identical, and so are, of course, the triangles $P_{12}A_1B_1$ and $P_{12}A_2B_2$. With the move to position 3, the triangles $P_{23}A_2B_2$ and $P_{23}A_3B_3$ are identical.

The foregoing allows us to take a different view of things. Although we are really concerned with moving AB from position 1 (A_1B_1) to position 2 (A_2B_2), and although we have no idea of the mechanism that is to do this, we begin to see how it might be accomplished: the clue lies with the lines $C_{12}C_{23}$ and $P_{12}P_{23}$. $P_{12}P_{23}$ was established from the poles; these poles are fixed in the reference plane, and so is the line $P_{12}P_{23}$. $C_{12}C_{23}$ was found by a rotation of $-\theta_{12}$; it is a line now attached to the moving plane carrying AB. We are thus able to imagine going from position 1 to position 2 by rolling the line $C_{12}C_{23}$ of the moving plane into congruence with the fixed line $P_{12}P_{23}$. We still have the moving plane rotating, but we think now of two lines—one moving, one fixed—providing the action by coming into congruence and effecting the position change of AB by a rolling action.

Figure 4-14 shows the moving plane in six successive positions; the poles were determined from the intersections of the appropriate midnormals. By joining the poles we obtain the beginnings of a polygon; it is incomplete, since AB (or the moving plane) does not return to its original position. This polygon is called the fixed polygon, for it is in the reference plane.

Another polygon, but one that moves because it is attached to the moving plane AB, must next be determined. The point C_{23} is located as before; and, as we have seen, when $C_{12}C_{23}$ swings down to coincide with $P_{12}P_{23}$, the link moves into position A_2B_2. The line $C_{23}C_{34}$ has the length of $P_{23}P_{34}$ and makes an angle $(\widehat{23} + \theta_{23})$ with $C_{12}C_{23}$. Note that the θ_{23} added to $\widehat{23}$ is of opposite sense to the finite

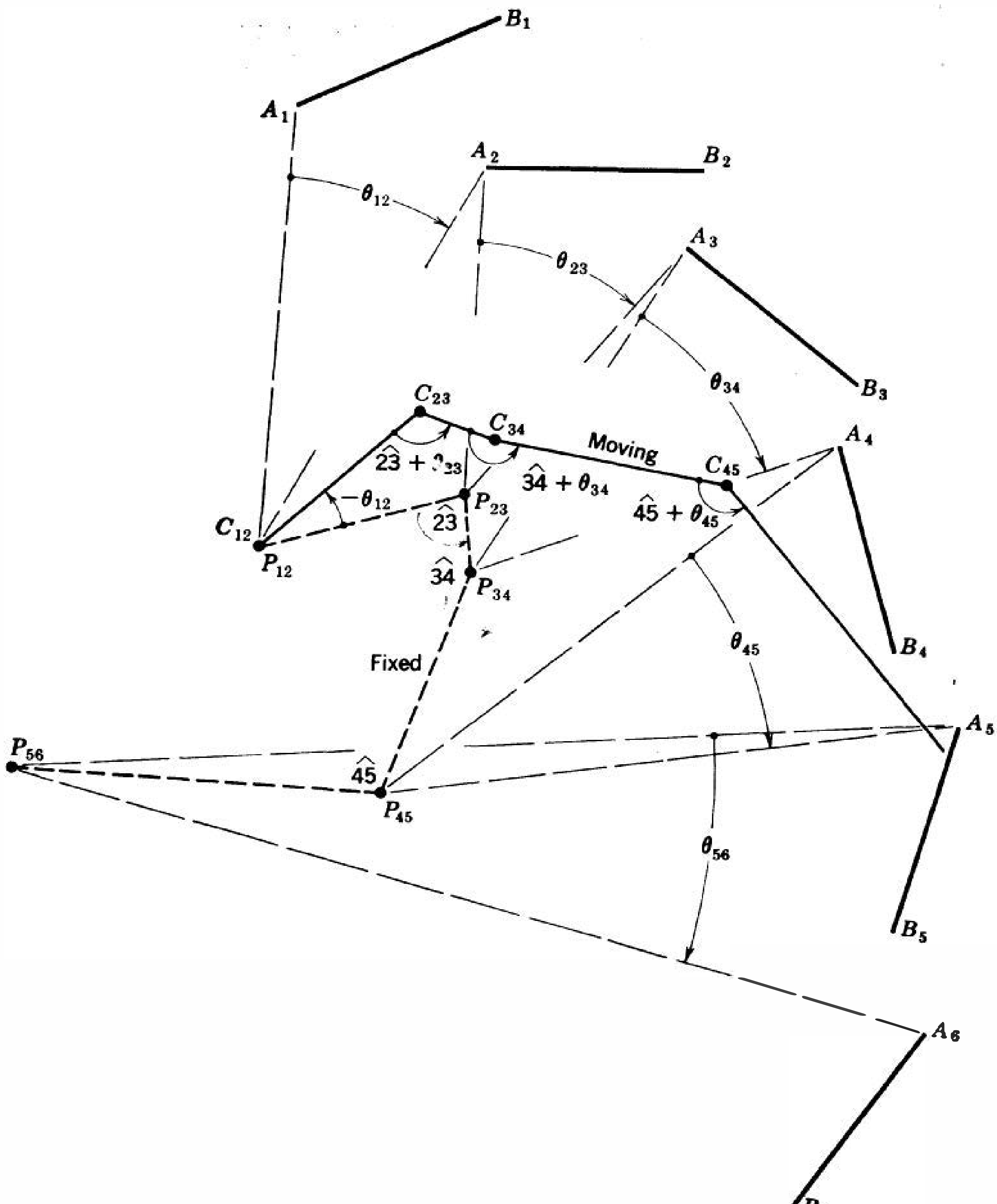


FIGURE 4-14 Poles, fixed and moving polygons for finite rotations.

rotation θ_{23} of the link between positions 2 and 3. This process is continued; thus $C_{34}C_{45} = P_{34}P_{45}$, and the angle with its predecessor line is $(\widehat{34} + \theta_{34})$, and so on. The moving polygon is created in this manner; when it rolls around the corners of the fixed polygon, the link AB will assume the successive positions. This is easily demonstrated by drawing A_1B_1 and the moving polygon on a transparent overlay and rolling the latter's polygon on the fixed polygon. We remark that these polygons are independent of the actual method of moving the link AB; all that they manage to do is to position the link for a succession of finite rotations.

Ordinarily the motion of a body is continuous. The six positions of AB may be likened to six flashlight photographs of a continuously moving link, the finite angles $\Delta\theta = \theta$ corresponding to finite times Δt between successive photographs. With our poles we have been able to account for the positions at the beginning and end of any interval. Were we to think of photographing the procedure at shorter and shorter intervals, we should have to provide more poles, until at the end, when Δt between pictures approached zero, the infinity of individual poles would have to be called instantaneous centers. The polygons would no longer have corners because of the infinitesimal spacing of the defining points; instead there would be smooth curves with the special name of *centrode.* The centrodes are actually the loci of the instantaneous centers and will be identified by the symbol π (lowercase p of the Greek alphabet; think of *path*). These curves, or centrodes, retain the roll property of the polygons from which they sprang. By rolling the centrode attached to the body about the centrode fixed to the reference plane, the motion of the link is faithfully represented. Any point on either is the instantaneous center for any virtual motion. We shall distinguish between the curves by calling them the moving and fixed centrodes; they are also known as the body (moving) and space (fixed) centrodes.

Figure 4-15 shows a line AB of a moving plane 2 somehow guided continuously along the two paths shown. The velocity directions (but

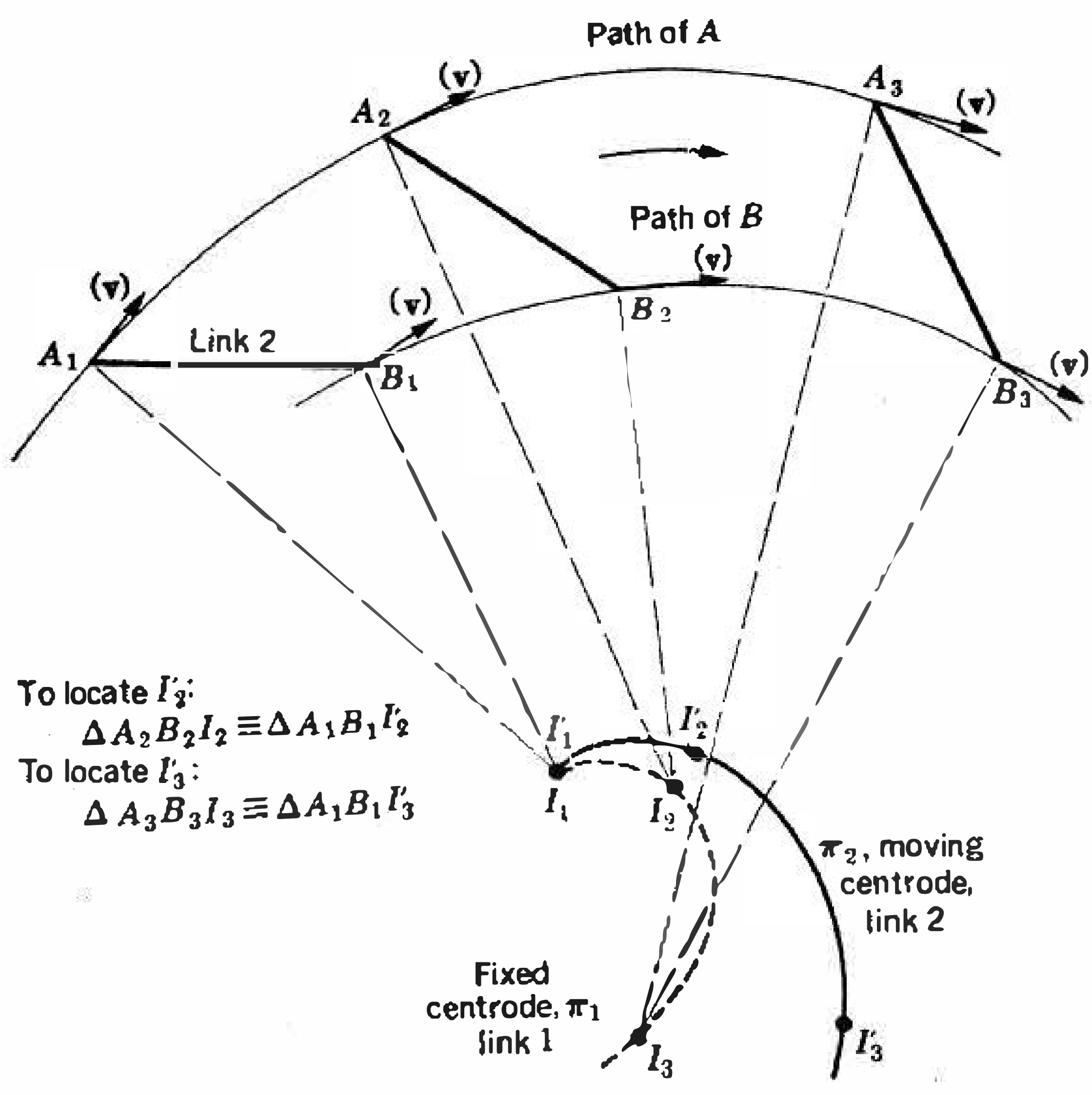


FIGURE 4-15 Determination of moving centrode.

not their magnitudes) are indicated by (v) at A and B for successive positions; these directions are both tangents. The intersections of the normals from the pair of velocities at each of the three positions define the instantaneous centers for each, namely, I_1, I_2, and I_3. A curve drawn through these points is then the locus of the instantaneous centers fixed to the frame link 1: this is the fixed centrode, labeled π_1 for the link (plane) to which it is attached.

It is considerably more trouble to establish the moving centrode π_2. The use of a transparent overlay on which the line AB has been drawn will prove convenient. Upon laying this over A_2B_2 (position 2), I_2 is marked on the overlay. The overlay's line AB is then laid over A_1B_1, and I_2' is located by pricking through the overlay's point I_2. By repetition of this operation at A_3B_3, at which I_3 is picked up, and from which the overlay is moved back to A_1B_1, I_3' is found. We may say that the triangle $A_nB_nI_n$ must always be returned to A_1B_1 to locate I_n', the point on the moving centrode. The arc lengths, such as $I_1'I_2'$ of the moving centrode π_2, are of course equal to the corresponding arc lengths I_1I_2, just as the mating side of the former polygons were of equal length. Without this equality of the vis-à-vis arcs no rolling would be possible. It will be instructive to make an overlay having on it A_1B_1 and the moving centrode π_2; when π_2 is rolled over π_1, the various positions of AB will be indicated.

A four-bar linkage (double-crank) is shown in Fig. 4-16, with portions of the fixed π_1 and moving π_3 centrodes. Successive positions of the coupler are indicated by the number of primes on each I; thus, I''' means position 3. We may now imagine removing the actual physical guides (links 2 and 4) between the coupler (link 3) and the frame (link 1) and achieving the proper displacement of the coupler by rolling π_3 on π_1. The proper velocity is dependent on supplying the rolling motion with the proper angular velocity. The two centrodes amount to connecting the planes 3 and 1 by a kinematic pair having but one degree of freedom, viz., roll. The elements of this pair are the centrodes π_3 and π_1.

In like manner, but by the use of IC 24, two centrodes π_2 and π_4 could be developed. Since both centrodes are attached to the moving links 2 and 4, they are denoted as relative centrodes.

Centrodes simulating any planar motion other than translation may be found. With translation, the centrodes move to infinity. In the double-crank linkage of Fig. 4-16, the cranks never become parallel, whence the centrodes π_3 and π_1 will be closed curves. This is not true for crank-rocker mechanisms, for the crank and rocker become parallel at two phases, IC 13 going to infinity for each.

For some situations the centrodes are simple curves. When a wheel rolls along a rail, the IC is at the point of contact: the moving centrode is the rim of the wheel, and the rail is the fixed centrode (Fig.

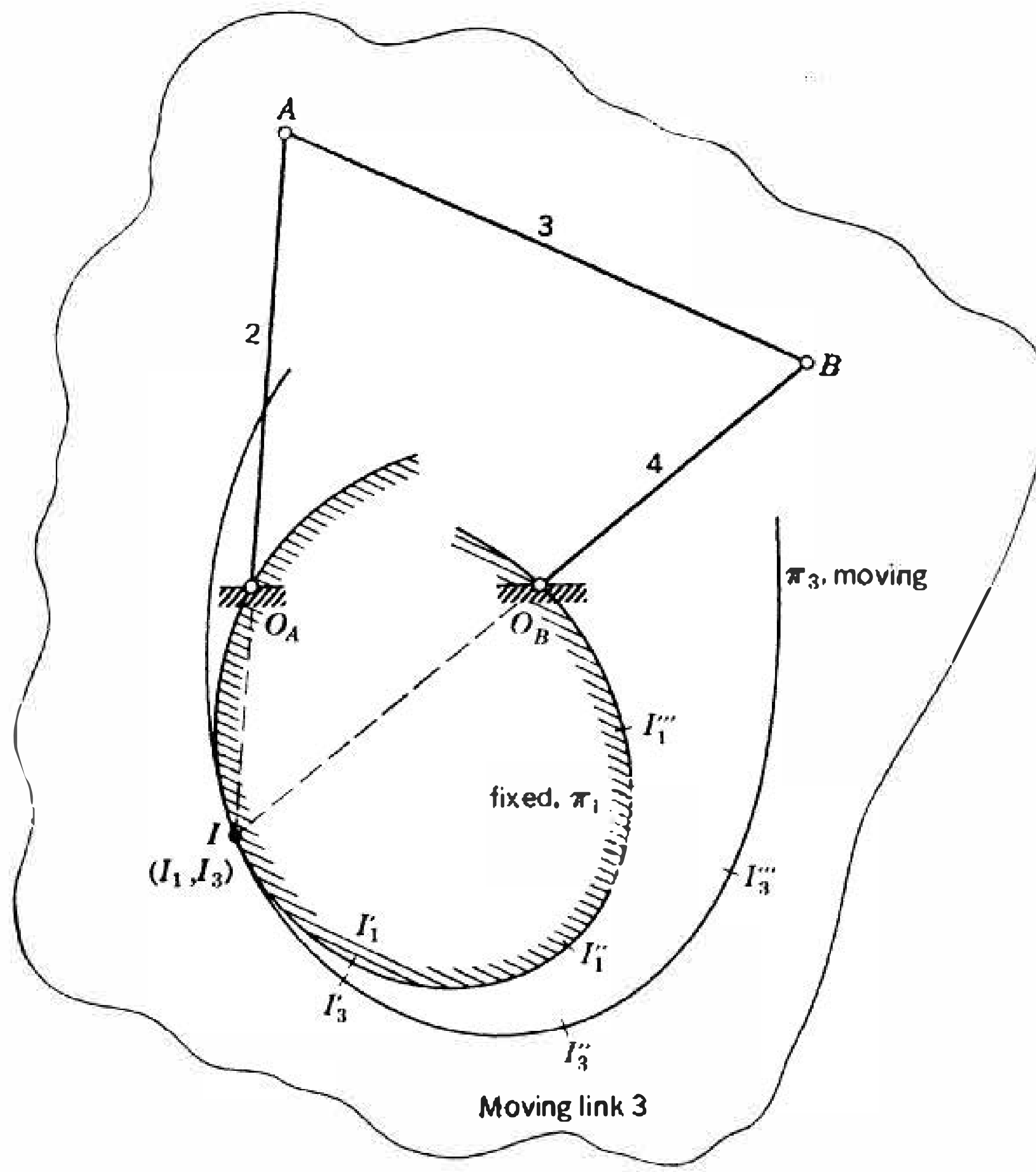


FIGURE 4-16 Centrodes of the coupler (3) with respect to the frame (1) of a four-bar linkage.

4-17a). With two circular gears in contact (Fig. 4-17b), the relative IC is always at the point of contact (pitch point) of the pitch circles; the two relative centrodes are the pitch circles. Also, as shown in Fig. 4-17c, displaying a crossed parallelogram linkage, both the relative centrodes are ellipses.

The direction of the common tangent (or the common normal) to a pair of centrodes at a given instant will have a particular significance in the studies of path curvature to be undertaken in Chap. 7. One method to determine these directions, without actually drawing the centrodes themselves, will now be presented as an application of the velocity analysis. The direction of the common tangent is along the IC velocity,[1] $\mathbf{v}_I$, whence our objective becomes the determination of that velocity.

Again, for the sake of simplicity, the example chosen is that of the four-bar linkage O_AABO_B, shown in Fig. 4-18a. In order to apply

[1] See page 196.

the method of velocity diagrams, we "materialize" the point I, IC of link 3 with respect to 1, as the center of a pin riding between two forks as shown, so that for all positions of the links this point is at the IC. Assume for convenience an angular velocity $\omega_3 = 1$ rad/sec counterclockwise. With the assumed dimensions we have

$$v_A = IA\omega_3 = 2.33 \text{ fps} \qquad v_B = IB\omega_3 = 1.74 \text{ fps}$$

Then, using the theorem of relative velocities [Eq. (4-1)],

$$\mathbf{v}_I = \mathbf{v}_{I/2} + \mathbf{v}_{I_2} \qquad \text{and} \qquad \mathbf{v}_I = \mathbf{v}_{I/4} + \mathbf{v}_{I_4}$$

or

$$\overset{\checkmark x_1}{\mathbf{v}_{I/2}} + \overset{\checkmark\checkmark}{\mathbf{v}_{I_2}} = \overset{\checkmark x_2}{\mathbf{v}_{I/4}} + \overset{\checkmark\checkmark}{\mathbf{v}_{I_4}}$$

The velocities $\mathbf{v}_{I_2}$ and $\mathbf{v}_{I_4}$ may be found by using ϑ lines and the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ as shown, so that both their direction and magnitude may be checked as known. The directions of $\mathbf{v}_{I/2}$ and $\mathbf{v}_{I/4}$ are along the forks of links 2 and 4, so that their directions may be checked. The remaining two unknowns, the magnitudes of $\mathbf{v}_{I/2}$ and $\mathbf{v}_{I/4}$, are found by drawing

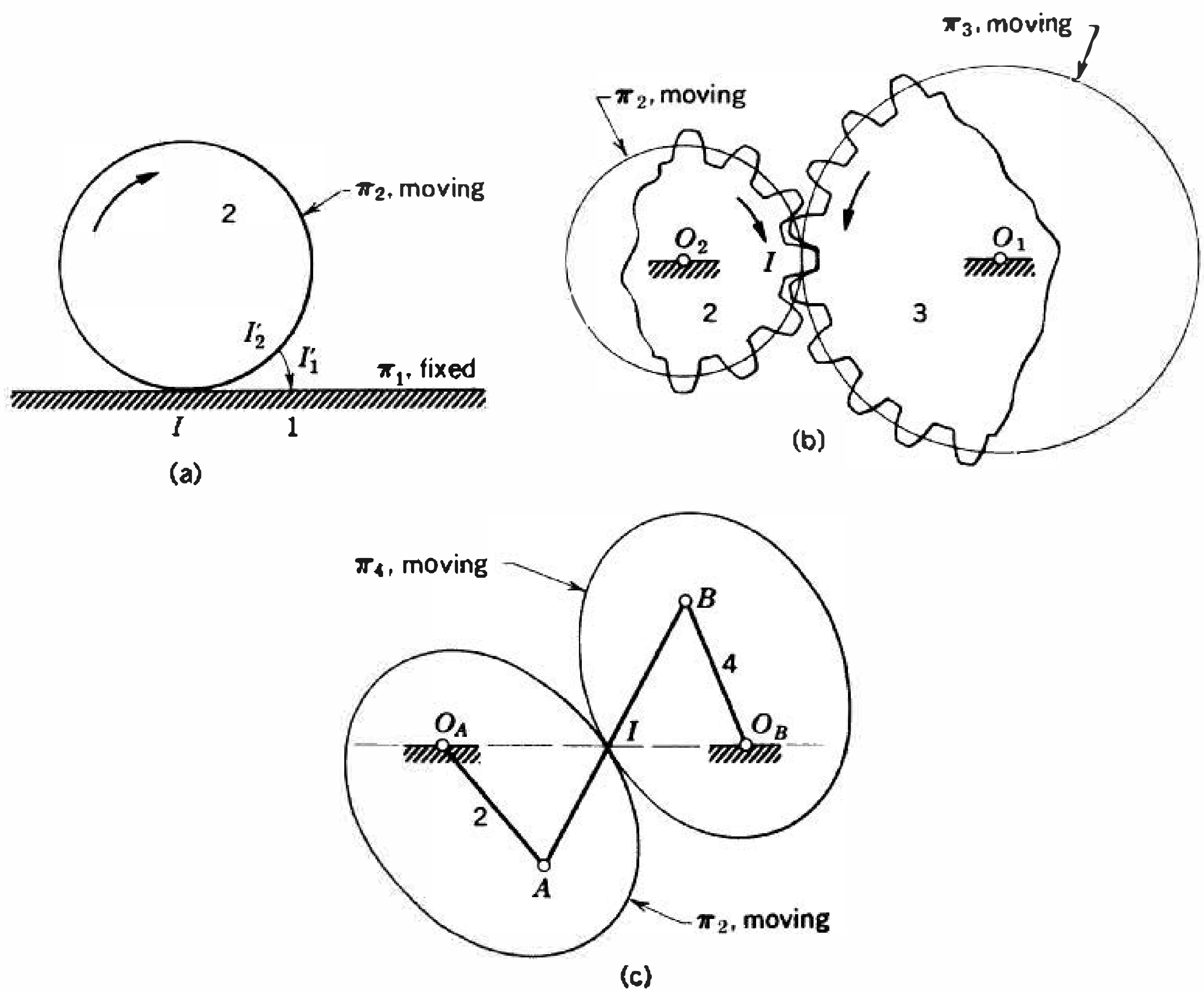


FIGURE 4-17 Situations for which centrodes are simple curves.

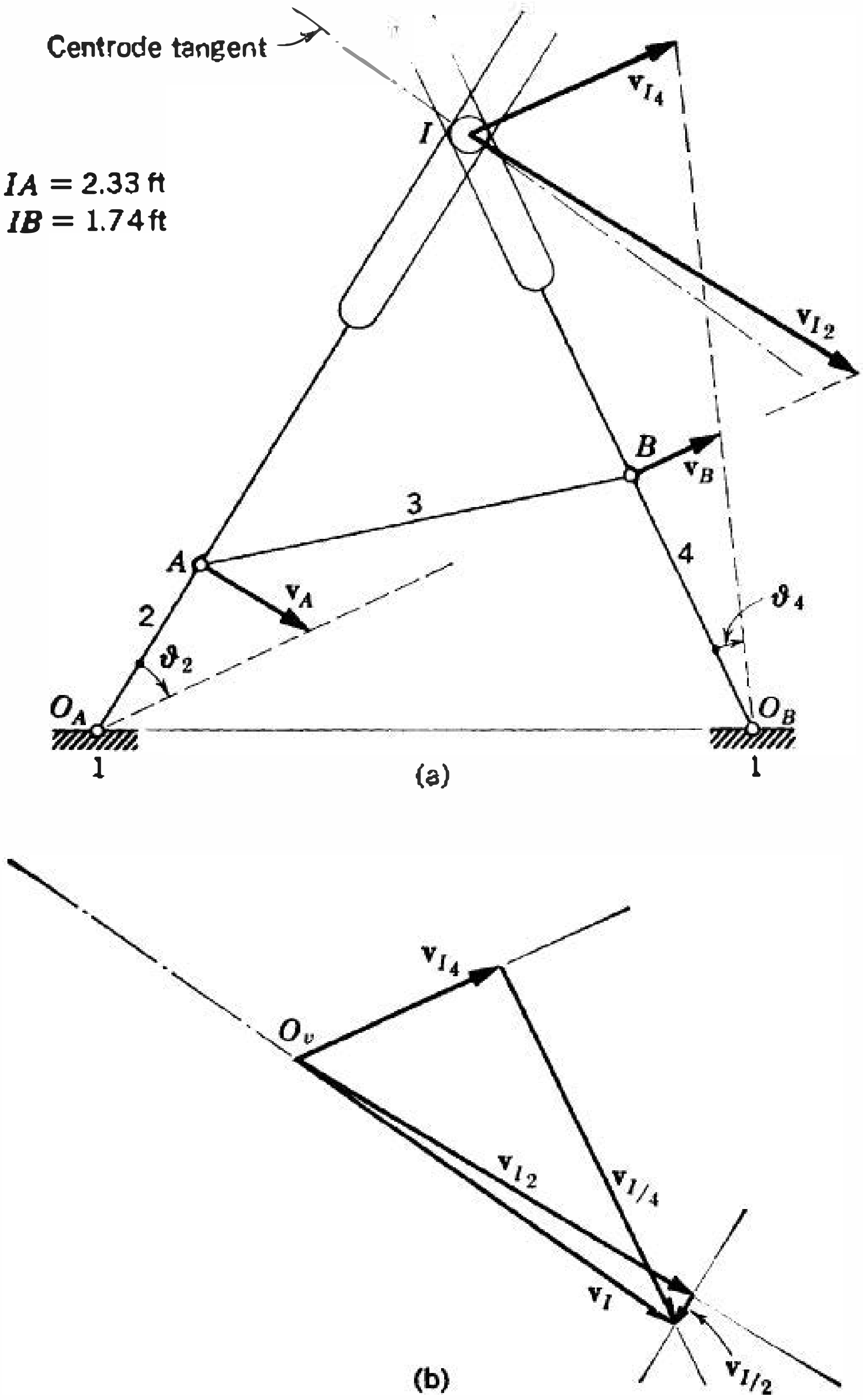


FIGURE 4-18 Determination of IC velocity. $O_AO_B = 3$ ft, $O_AA = 0.9$ ft, $AB = 2$ ft, $O_BB = 1.3$ ft.

the velocity diagram shown in Fig. 4-18*b*. An IC velocity of magnitude

$$v_I = 8.5 \text{ fps}$$

is found. Its direction is that of the common tangent to the centrodes at the instant considered.

4-8 ACCELERATION

The concept of acceleration is more difficult than the motion aspects discussed so far. Position, displacement, path, velocity, and speed are all qualities we can observe directly. We can judge the

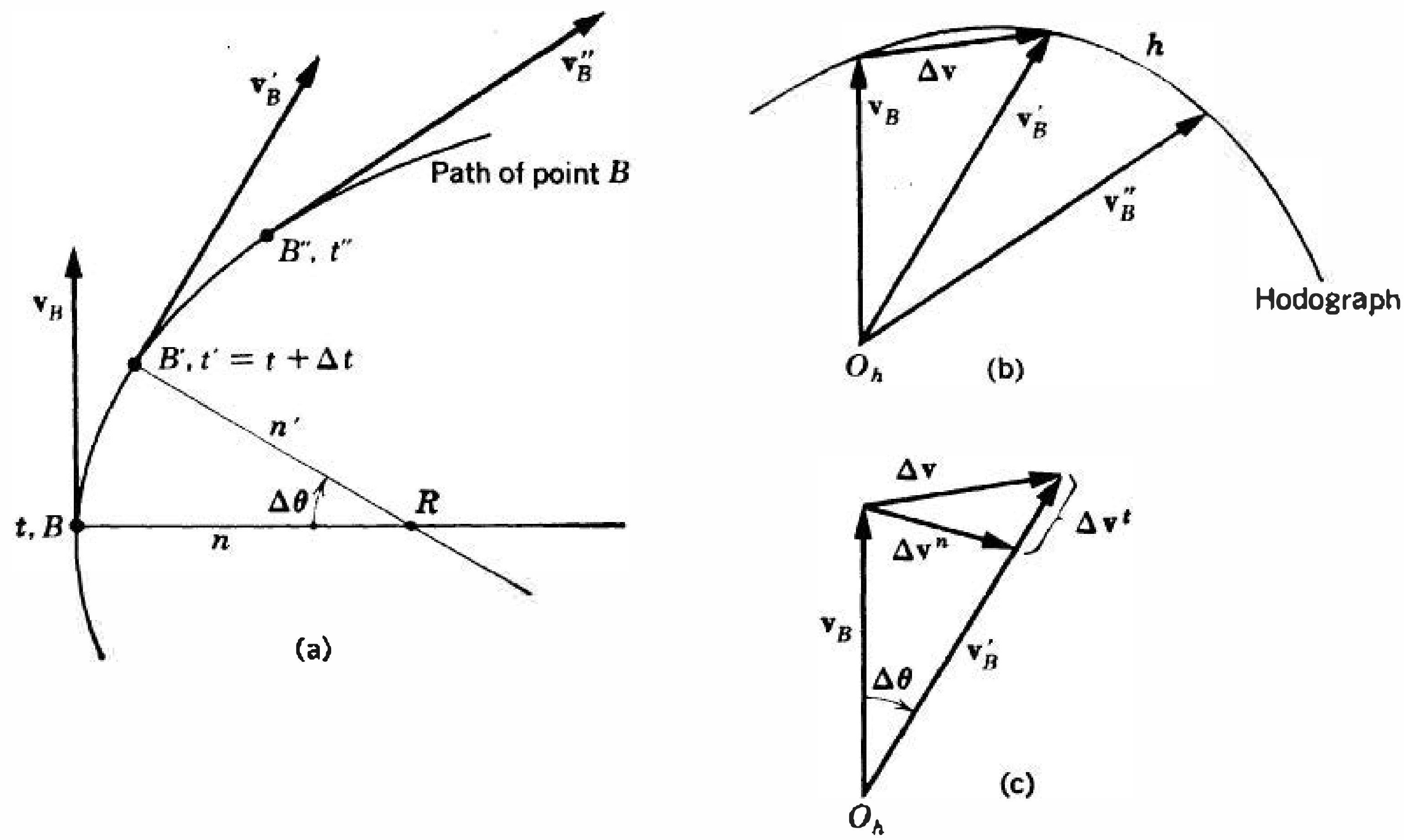


FIGURE 4-19 Definition of acceleration and hodograph.

implications of these qualities because of our considerable experience with them. For example, we are experts at integrating velocity vectors and are alive only because of our successes in doing so. The avoidance of collision between two moving bodies depends on an integration which it is hoped will produce different position coordinates of the moving bodies at all times, as in driving a car or crossing a street.

Acceleration is much more subtle and complex in its nature than the velocity change from which it comes. It may be said that our personal experience has been gained indirectly, for we observe the effect of acceleration, force, rather than the acceleration itself. For the layman there are no instruments providing numerical values of acceleration, in the sense that there are devices, as speedometers, that display information about distance and speed.

Consider (Fig. 4-19a) the path traced by a moving point occupying positions B and B' at instants t and $t' = t + \Delta t$. The velocities of the point at the instants considered are $\mathbf{v}_B$ and $\mathbf{v}'_B = \mathbf{v}_B + \Delta\mathbf{v}$, where $\Delta\mathbf{v}$ is the velocity increment during the time interval Δt (Fig. 4-19b). The average acceleration during this time interval Δt is

$$\mathbf{a}_{B,\mathrm{av}} = \frac{\Delta\mathbf{v}}{\Delta t}$$

and the instantaneous acceleration, or simply acceleration at time t, is

$$\mathbf{a}_B = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t}$$

The acceleration is therefore a vector representing the instantaneous rate of change of velocity; its sense is that of the velocity being acquired.

A useful aid to the understanding of acceleration is the *hodograph*. The path traced by a moving point is shown in Fig. 4-19*a*, with several positions corresponding to times t, $t' = t + \Delta t$, etc., and their corresponding velocities. If these and all intermediate velocities are laid off from an origin O_h as shown in Fig. 4-19*b*, the curve h joining the tips of the velocity vectors is known as the hodograph. We may think of the hodograph as being generated by the tip of a rotating and stretching velocity vector. The chord between the tips of any two velocity vectors of the hodograph is the velocity increment $\Delta\mathbf{v}$ for the associated time interval Δt, and the "velocity" with which the hodograph is being traversed is the acceleration of point B at the instant considered. The acceleration of B is therefore at all times tangent to the hodograph at the corresponding point.

The velocity increment $\Delta\mathbf{v}$ may be resolved into the sum of two components, $\Delta\mathbf{v}^n$ and $\Delta\mathbf{v}^t$, as shown in Fig. 4-19*c*. On considering these two components separately, the acceleration is given by

$$\mathbf{a}_B = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}^t}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}^n}{\Delta t}$$

The direction of $\Delta\mathbf{v}^n$ is perpendicular to the bisector of the angle $\Delta\theta$ between $\mathbf{v}$ and $\mathbf{v}'$, and its magnitude is $\Delta v^n = v\,\Delta\theta$, $\Delta\theta$ being assumed to be a small angle. The angle $\Delta\theta$ also appears between the normals n and n' to the path at positions B and B' (see Fig. 4-19*a*). For a small interval Δt the distance BB' will be small, and the normals n and n' will intersect at R, close to the center of curvature of the path, so that $RB \simeq RB' = \rho$. Then considering triangle RBB', $\Delta\theta = BB'/\rho = v\,\Delta t/\rho$ and $\Delta v^n = v^2\,\Delta t/\rho$. As Δt approaches zero, $\Delta\mathbf{v}^n/\Delta t$ becomes perpendicular to $\mathbf{v}$, that is, normal to the path, and its magnitude becomes v^2/ρ, where ρ is the radius of curvature of the path at B. This vector, called the *normal acceleration* and denoted as $\mathbf{a}_B{}^n$, always points toward the center of curvature of the path and is zero when the center of curvature is at infinity (rectilinear path, or inflection point).

The direction of $\Delta\mathbf{v}^t$ is along the velocity vector $\mathbf{v}'$, and its magnitude is the change in magnitude Δv of the velocity during the time interval Δt. As Δt approaches zero, the direction of $\Delta\mathbf{v}^t$ approaches that of $\mathbf{v}$; that is, it becomes tangent to the path, and its magnitude is dv/dt, the rate of change of the speed of the point along the path. This last vector gives the *tangential acceleration*, denoted as $\mathbf{a}_B{}^t$.

In summary, we have shown that the acceleration of a moving point is the sum of two rectangular components,

$$\mathbf{a}_B = \mathbf{a}_B{}^n + \mathbf{a}_B{}^t$$

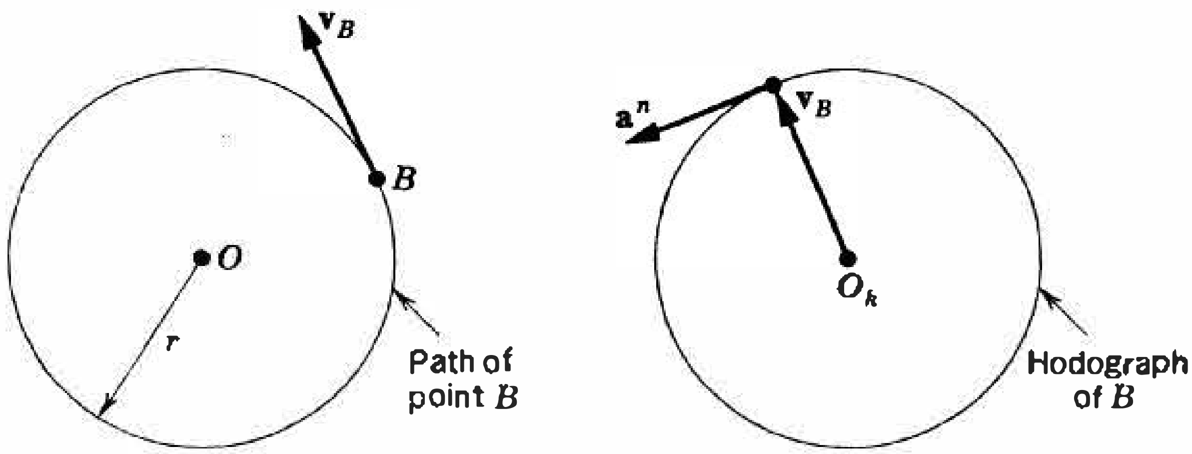


FIGURE 4-20 Hodograph of a point moving at constant speed around a circular path, $|\mathbf{v}_B| = v$ (const), $a^n = v^2/r$.

where the vectors $\mathbf{a}_B{}^n$ and $\mathbf{a}_B{}^t$ are as follows:

Normal acceleration $\mathbf{a}_B{}^n$

Magnitude: v^2/ρ

Direction: normal to path, pointing toward center of curvature

Tangential acceleration $\mathbf{a}_B{}^t$

Magnitude: dv/dt

Direction: tangential to path, pointing in direction of increasing speed

The hodographs of points having particular motions are interesting to note. If a point is moving along a straight line with constant speed, its hodograph is just a point, for its acceleration is zero. If a point moves around a circle of radius r with constant speed v, its hodograph is also a circle, but of radius v (Fig. 4-20). The acceleration of the point is then only the normal with a magnitude $a^n = v^2/r$.

4-9 RELATIVE ACCELERATION AND CORIOLIS ACCELERATION

In dealing with accelerations we shall follow the same pattern as with velocities and shall establish the distinction between relative acceleration and acceleration difference. Relative acceleration will be considered in this section, together with its necessary complement, the Coriolis acceleration component. Acceleration difference will be considered in the next section.

As in the study of relative velocity, consider a plane 2 moving with respect to a reference plane 1 (Fig. 4-21). A point[1] called B moves with respect to plane 2 with a relative velocity $\mathbf{v}_{B/2}$ and a *relative acceleration* $\mathbf{a}_{B/2}$. This relative acceleration is that which would be measured

[1] As in the case of relative velocity, this point B may be either (1) a point of another plane or (2) a point defined by a certain geometry, e.g., the intersection of two moving lines.

by an observer stationed on plane 2. It is related to the velocity of B with respect to plane 2 and the curvature of the path traced by B on plane 2. Point B, however, also has an absolute velocity $\mathbf{v}_B$ and an *absolute acceleration* $\mathbf{a}_B$ referred to plane 1. Finally, the point B_2 fixed in plane 2 and coincident with B at the instant considered has velocity $\mathbf{v}_{B_2}$ and acceleration $\mathbf{a}_{B_2}$ with respect to plane 1. From the theorem on relative velocity we have that

$$\mathbf{v}_B = \mathbf{v}_{B/2} + \mathbf{v}_{B_2}$$

and one might expect something analogous to hold for acceleration. This, however, is not the case, for whenever plane 2 has an angular velocity ω_2 different from zero,

$$\mathbf{a}_B = \mathbf{a}_{B/2} + \mathbf{a}_{B_2} + \mathbf{a}^{cor} \tag{4-5}$$

where $\mathbf{a}^{cor}$ is the *Coriolis acceleration* component. There is nothing directly obvious about this component, also known as the compound supplementary acceleration. The term supplementary is descriptive, for the Coriolis component results from changes of two linear velocity vectors occasioned by the rotation ω_2 of the path that B traces upon plane 2. The Coriolis acceleration has a magnitude $2v_{B/2}\omega_2$; its *direction* is always normal to the path of the relative motion, and its *sense* is that of $\mathbf{v}_{B/2}$ after that vector has been turned 90° in the sense of the angular velocity. Different situations are shown in Fig. 4-22.

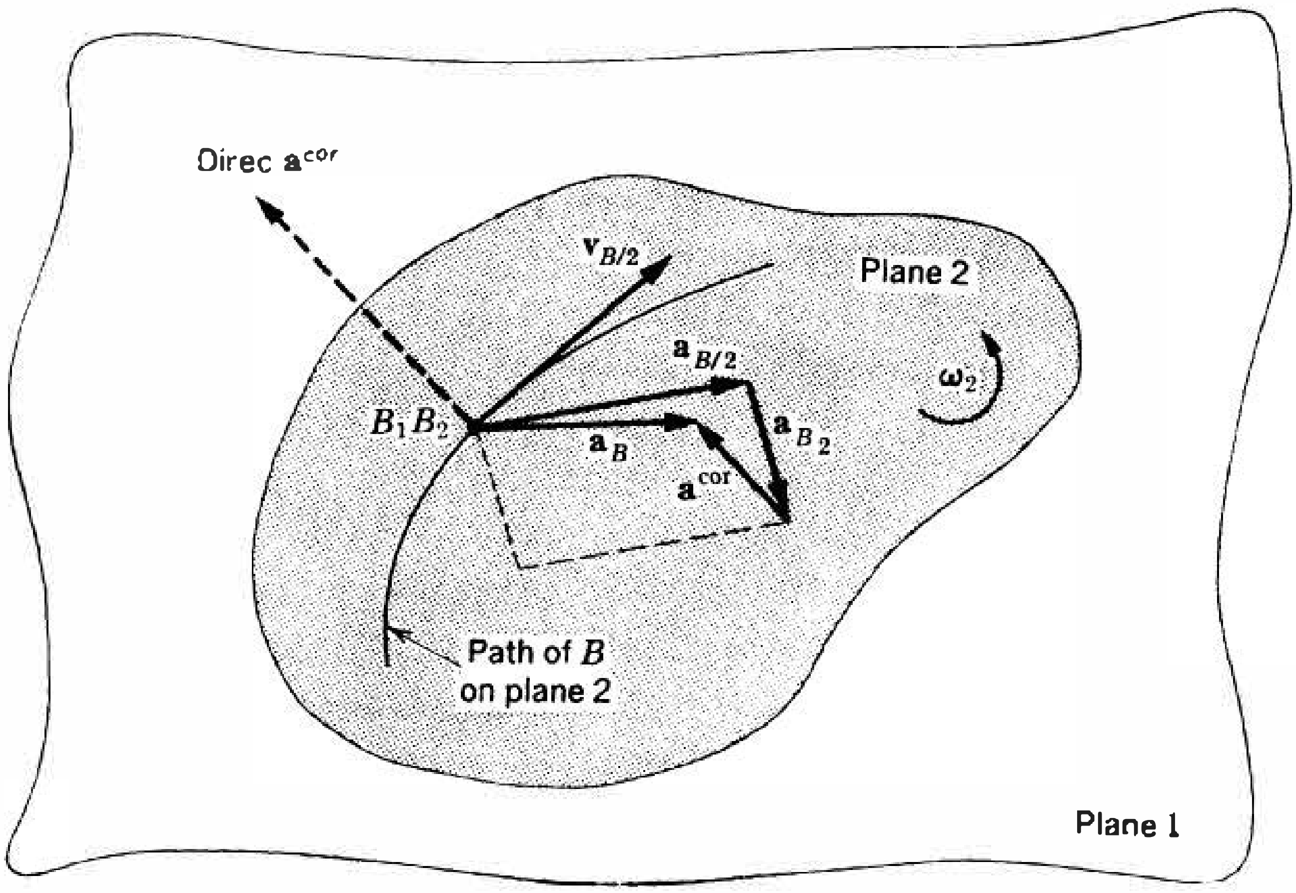


FIGURE 4-21 Relative acceleration and Coriolis acceleration.

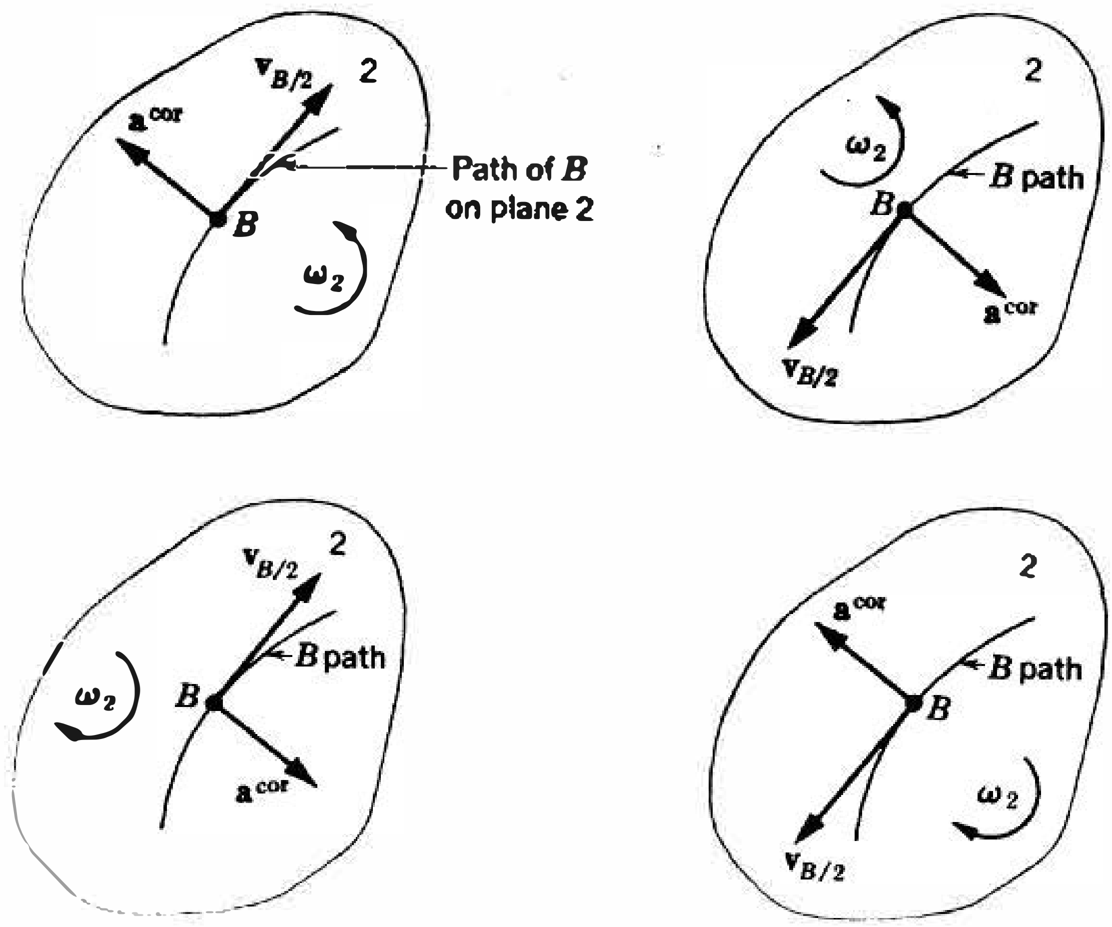


FIGURE 4-22 Direction of Coriolis acceleration for different cases.

To prove the Coriolis theorem [Eq. (4-5)] in its complete generality (even within the realm of planar motion) by the direct methods of the present chapter would be tedious and serve no useful purpose with its complexity.[1] The proof that follows is based upon a special situation in which the center of rotation is fixed. As implied above, the general case, in which the center of rotation itself is moving, will answer to the Coriolis theorem as developed here.

The situation shown in Fig. 4-23*a* represents a plane 2 rotating about O_B with respect to the fixed plane 1 with an angular velocity ω_2. The point[2] B of block 3 is guided by the slot in plane 2 and moves with respect to this plane with a velocity $\mathbf{v}_{B/2}$ and an acceleration $\mathbf{a}_{B/2}$. Position vectors and velocity components are shown in Fig. 4-23*b* for two positions of the slider corresponding to times t and $t + \Delta t$. It is clear that both the velocity components (radial and transverse) have undergone changes of magnitude and direction. These changes are best seen by considering separately what has happened to each velocity component.

The radial component is treated in Fig. 4-23*c*, the increment $\Delta \mathbf{v}_B{}^r = \mathbf{DF}$ being resolved into two components, a radial $\mathbf{DE}$ and a transverse $\mathbf{EF}$. The diagram in Fig. 4-23*d* shows the resolution of the increment $\Delta \mathbf{v}_B{}^\theta = \mathbf{GJ}$ into its components, a radial $\mathbf{HJ}$ and a transverse $\mathbf{GH}$.

[1] See Prob. 4-9 for a complete (planar) proof using complex number notation.

[2] This point B is associated with plane 3 and should therefore be identified as B_3. However, the subscript 3 has been suppressed to reduce unwieldiness.

The total change of velocity in the radial direction is given by DE − JH. The radial component of acceleration is then

$$a_B{}^r = \lim_{\Delta t \to 0} \frac{\mathsf{DE}}{\Delta t} - \lim_{\Delta t \to 0} \frac{\mathsf{JH}}{\Delta t}$$

By analogy with similar vector situations we may write

$$a_B{}^r = \frac{d}{dt} v_B{}^r - v_B{}^\theta \frac{d\theta_2}{dt}$$

With $v_B{}^r = d\rho/dt$, $v_B{}^\theta = \rho(d\theta_2/dt)$, and $d\theta_2/dt = \omega_2$, this magnitude of the radial component of the acceleration is

$$a_B{}^r = \frac{d^2\rho}{dt^2} - \rho\omega_2{}^2 = a_{B/2} - \rho\omega_2{}^2 \tag{4-6}$$

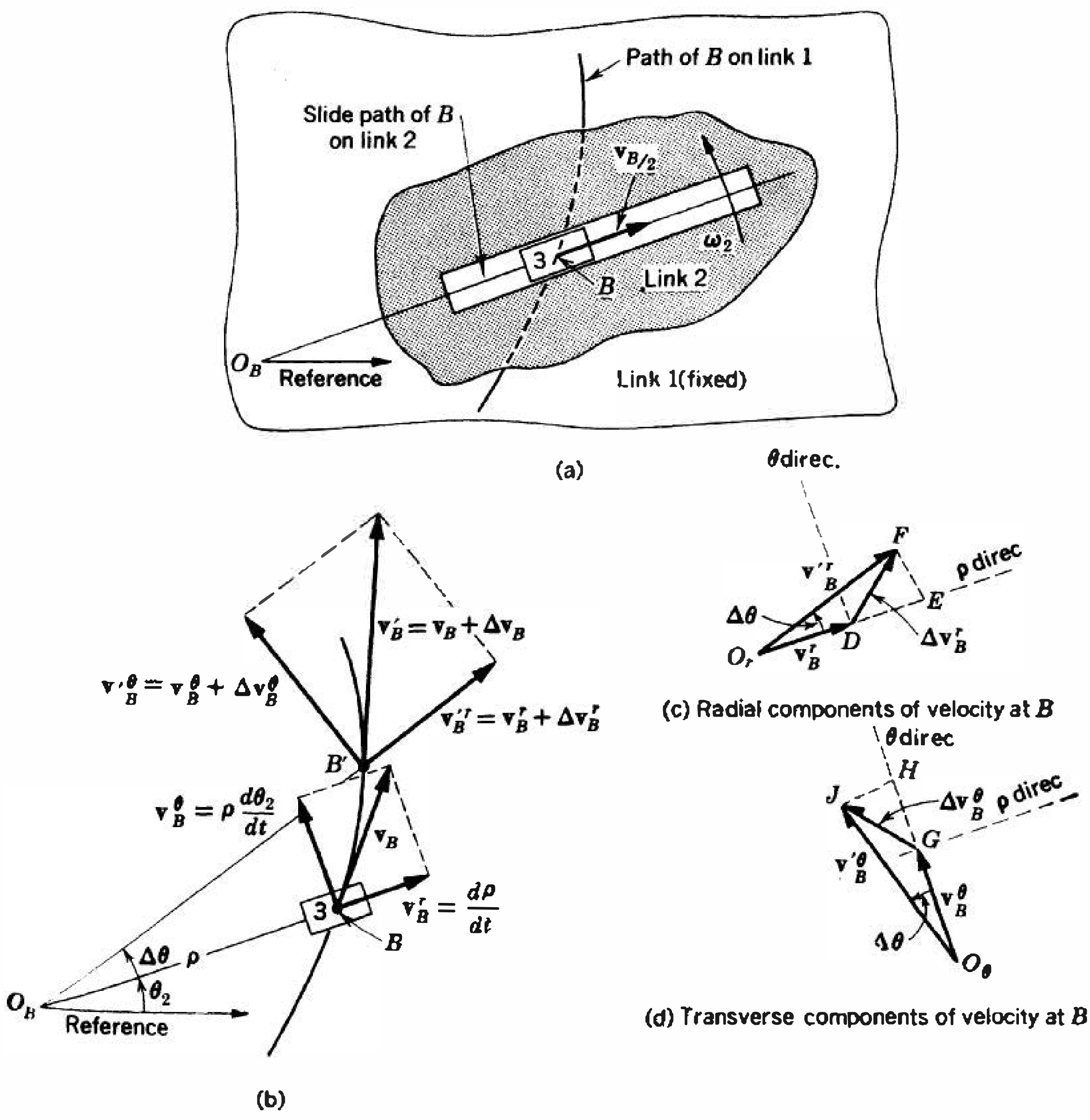


FIGURE 4-23 Proof of Coriolis theorem in a special case.

For this radial, or slide-path, direction we recognize $a_{B/2}$ as the magnitude of the relative linear acceleration of B with respect to link 2; it is a measure of the speed change of the sliding velocity that would be present even if link 2 were not rotating. The $\rho\omega_2^2$ term is the normal acceleration of the coincident point B_2, obviously a consequence of rotation of link 2.

As for the transverse direction, the total velocity change is given by $\mathbf{EF} + \mathbf{GH}$, whence

$$\mathbf{a}_B^\theta = \lim_{\Delta t \to 0} \frac{\mathbf{EF}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\mathbf{GH}}{\Delta t}$$

or

$$a_B^\theta = v_B^r \frac{d\theta_2}{dt} + \frac{dv_B^\theta}{dt} = \frac{d\rho}{dt}\frac{d\theta_2}{dt} + \frac{d}{dt}\left(\rho \frac{d\theta_2}{dt}\right)$$

$$= \frac{d\rho}{dt}\omega_2 + \rho \frac{d^2\theta_2}{dt^2} + \frac{d\rho}{dt}\omega_2$$

$$= v_{B/2}\omega_2 + \rho\alpha_2 + v_{B/2}\omega_2$$

In the θ direction, which is perpendicular to the slide path, the component $\rho\alpha_2$ is the angular acceleration of point B_2. The first of the $v_{B/2}\omega_2$ terms is a result of the direction change of the radial velocity component; the second $v_{B/2}\omega_2$ has to do with the changing magnitude of the transverse velocity component. Gathering the two like terms into one, we realize that this supplementary term is compounded from a sliding velocity along a rotating path. We write

$$a_B^\theta = \rho\alpha_2 + 2v_{B/2}\omega_2 \tag{4-7}$$

Assembling the acceleration components to form $\mathbf{a}_B$, we have

$$\mathbf{a}_B = \mathbf{a}_B^r + \mathbf{a}_B^\theta$$
$$= \mathbf{a}_{B_2} + \mathbf{a}_{B/2} + \mathbf{a}^{cor}$$

where we have

Coriolis acceleration $\mathbf{a}^{cor}$

Magnitude: $2v_{B/2}\omega_2$

Direction: rotated 90° from $\mathbf{v}_{B/2}$ in same sense as ω_2

It will be observed that the Coriolis acceleration was derived from the $\mathbf{EF}$ and $\mathbf{GH}$ components (Fig. 4-23c and d). As remarked, these components are in the θ direction, whence they are normal to the path (and $\mathbf{v}_{B/2}$).

4-10 ACCELERATION DIFFERENCE

Two points A and B of a moving plane 2 are shown in Fig. 4-24. The reference plane is 1, and the angular velocity and acceleration of plane 2 with respect to plane 1 at the instant considered are ω_2 and α_2.

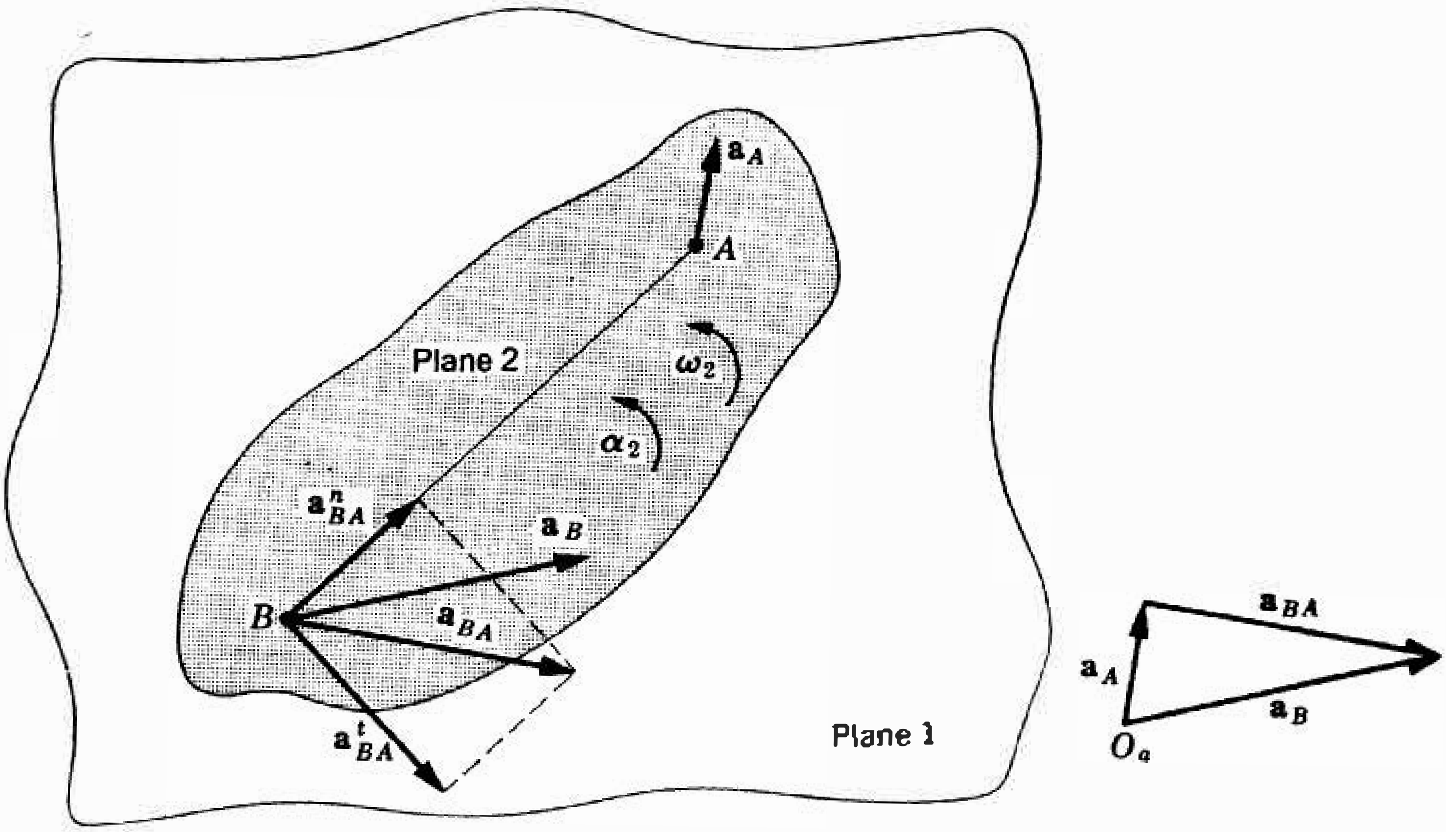


FIGURE 4-24 Acceleration difference of two points on the same moving plane.

With $\mathbf{a}_A$ and $\mathbf{a}_B$ representing the accelerations of points A and B, the acceleration difference of these two points is defined as

$$\mathbf{a}_{BA} = \mathbf{a}_B - \mathbf{a}_A$$

from which

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA} \tag{4-8}$$

The remarks made concerning the concepts of relative velocity and velocity difference apply equally well to the relative acceleration and acceleration difference. It is inconsistent to call $\mathbf{a}_{BA}$ a relative acceleration, as is often done, because an acceleration cannot be referred to a point but must be referred to a plane (in the case of planar motion). Furthermore, since Eq. (4-8) deals with the difference of two absolute accelerations, no component such as the Coriolis can be distinguished. We emphasize that A and B are points of the *same* moving plane 2; they are not points of two different planes, as in the preceding section.

The acceleration difference will now be related to the distance AB and the angular motion of plane 2 as described by ω_2 and α_2. Again, accelerations are considered in terms of vanishingly small velocity increments. During a small time interval Δt, points A and B are assumed to move to positions A' and B' as shown in Fig. 4-25a. The diagrams in Fig. 4-25b and c show the situations so far as velocities are concerned at the beginning and end of the time interval; or in equation form we have

$$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A \qquad \text{and} \qquad \mathbf{v}'_{BA} = \mathbf{v}'_B - \mathbf{v}'_A$$

But $\quad \mathbf{v}'_A = \mathbf{v}_A + \Delta\mathbf{v}_A \qquad \mathbf{v}'_B = \mathbf{v}_B + \Delta\mathbf{v}_B \qquad \mathbf{v}'_{BA} = \mathbf{v}_{BA} + \Delta\mathbf{v}_{BA}$

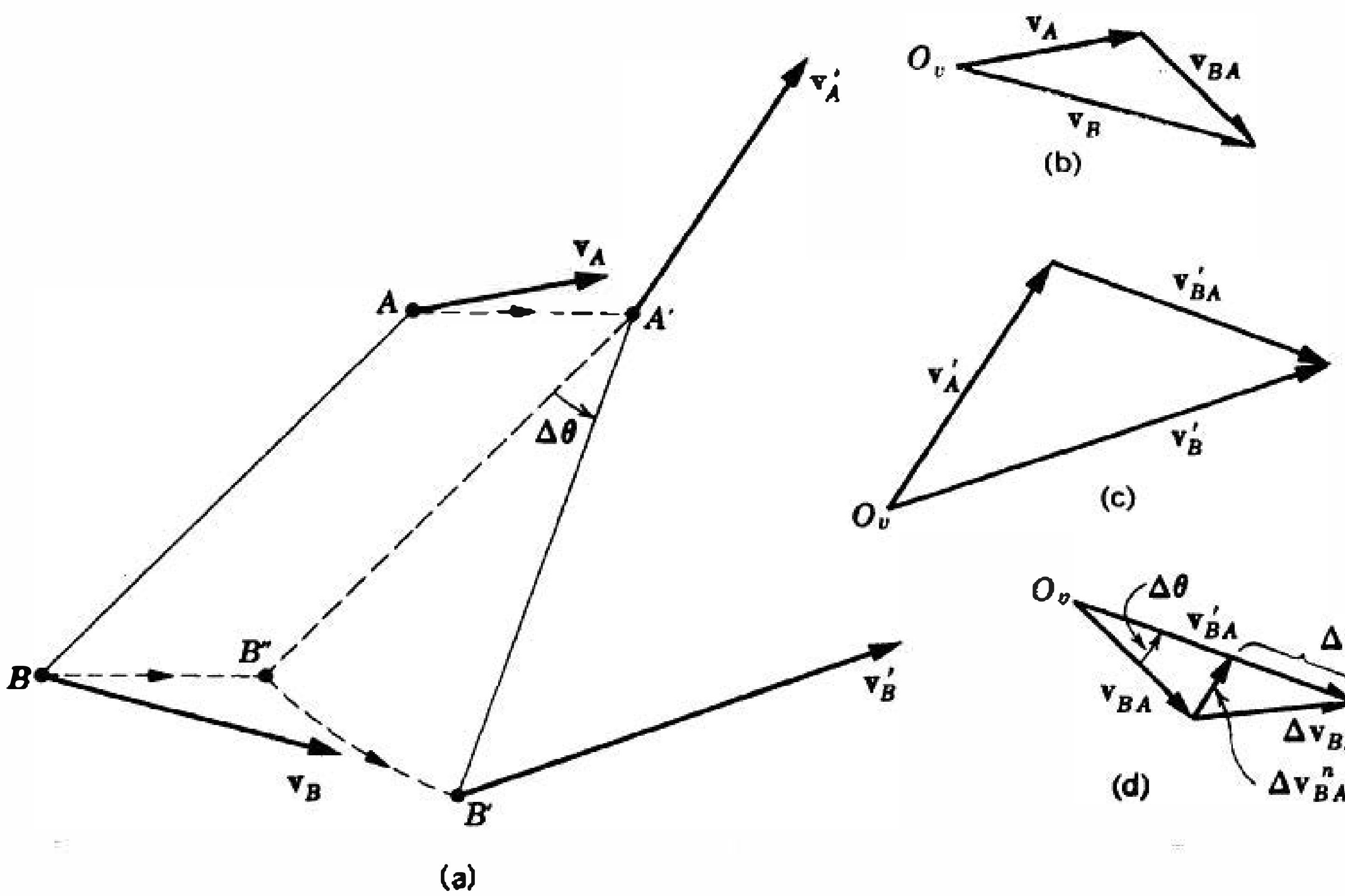


FIGURE 4-25 Determination of the acceleration difference of A and B.

where $\Delta\mathbf{v}_A$, $\Delta\mathbf{v}_B$, and $\Delta\mathbf{v}_{BA}$ are the increments in the velocity of A, the velocity of B, and the velocity difference between A and B during the time interval Δt. Then

$$\Delta\mathbf{v}_{BA} = \Delta\mathbf{v}_B - \Delta\mathbf{v}_A = (\mathbf{v}'_B - \mathbf{v}'_A) - (\mathbf{v}_B - \mathbf{v}_A)$$

The acceleration difference, which may be written as

$$\mathbf{a}_{BA} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}_B}{\Delta t} - \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}_A}{\Delta t}$$

is then

$$\mathbf{a}_{BA} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}_{BA}}{\Delta t}$$

The velocity differences $\mathbf{v}_{BA}$ and $\mathbf{v}'_{BA}$ are shown in Fig. 4-25d, which also shows the increment $\Delta\mathbf{v}_{BA}$ as the sum of two components $(\Delta\mathbf{v}_{BA})^n$ and $(\Delta\mathbf{v}_{BA})^t$. The angle $\Delta\theta$ shown in that diagram is also the angle of rotation of plane 2 during Δt. Bypassing intermediate steps which are now familiar, we state the final result: the acceleration difference is the sum of two rectangular components,

$$\mathbf{a}_{BA} = \mathbf{a}_{BA}{}^n + \mathbf{a}_{BA}{}^t$$

where the vectors $\mathbf{a}_{BA}{}^n$ and $\mathbf{a}_{BA}{}^t$ are as follows (Fig. 4-24):

Normal acceleration difference $\mathbf{a}_{BA}{}^n$

Magnitude: $v_{BA}{}^2/AB$

Direction: along AB, pointing from B toward A

Tangential acceleration difference $\mathbf{a}_{BA}{}^t$

Magnitude: $(AB)\alpha_2$

Direction: rotated 90° from AB; in the same direction as α_2

As an example, we shall investigate the acceleration condition of the slider-crank mechanism of Fig. 4-26, assuming the angular velocity ω_2 and angular acceleration α_2 of link 2 to be known.

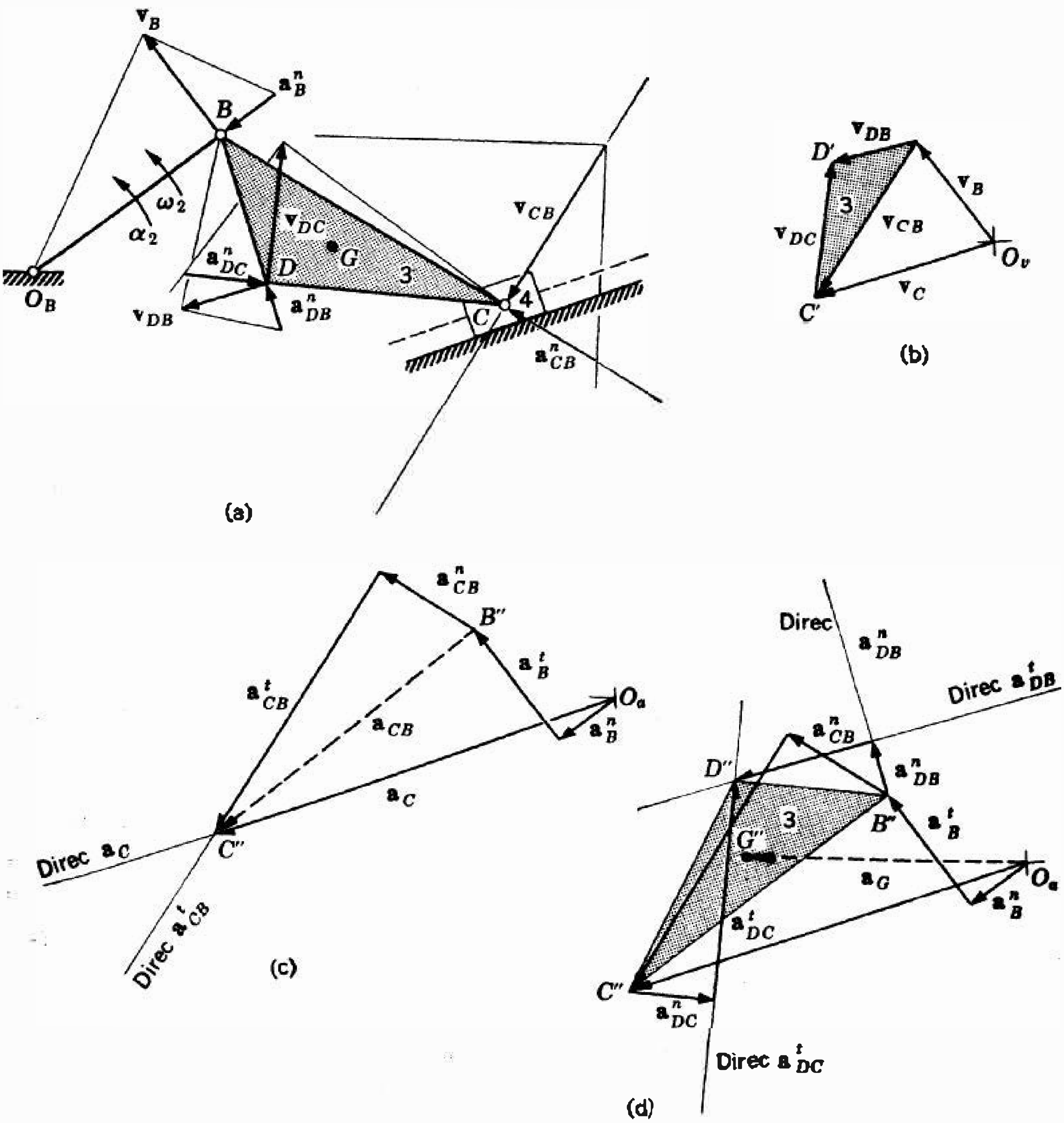


FIGURE 4-26 Acceleration analysis of the slider-crank mechanism.

It will be convenient to employ scales conforming to the relation $k_a = k_v^2/k_s$.* The velocity polygon (Fig. 4-26*b*) is then constructed from the single equation $\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{CB}$ and the velocity image. With point D' at hand, the velocity differences $\mathbf{v}_{DB}$ and $\mathbf{v}_{DC}$ are directly available.

Full possession of the velocity vectors allows the graphical determination of all a^n components at points B, C, and D, as shown in Fig. 4-26*a*. Only $a_B{}^t$ needs external calculation, as by slide rule.

The diagram in Fig. 4-26*c* shows the vector polygon of the equation

$$\overset{\surd x_1}{\mathbf{a}_C} = \overset{\surd\surd}{\mathbf{a}_B{}^n} + \overset{\surd\surd}{\mathbf{a}_B{}^t} + \overset{\surd\surd}{\mathbf{a}_{CB}{}^n} + \overset{\surd x_2}{\mathbf{a}_{CB}{}^t}$$

in which the magnitude of $\mathbf{a}_C$ and the direction of $\mathbf{a}_{CB}{}^t$ are the unknowns. After completion of the polygon, the absolute angular acceleration of link 3 is calculated from $\alpha_3 = a_{CB}{}^t/BC$ (clockwise).

The acceleration of D can be established from the simultaneous solution of two equations in four unknowns,

$$\overset{x_1 x_2}{\mathbf{a}_D} = \overset{\surd\surd}{\mathbf{a}_B} + \overset{\surd\surd}{\mathbf{a}_{DB}{}^n} + \overset{\surd x_3}{\mathbf{a}_{DB}{}^t}$$

$$\overset{x_1 x_2}{\mathbf{a}_D} = \overset{\surd\surd}{\mathbf{a}_C} + \overset{\surd\surd}{\mathbf{a}_{DC}{}^n} + \overset{\surd x_4}{\mathbf{a}_{DC}{}^t}$$

For clarity, the first vector polygon has been redrawn in Fig. 4-26*d* and the solutions of the last two equations added, defining the homologous point D''. The vector $\mathbf{O_aD''}$, common to both equations, is the acceleration of D; it is not shown because of the already many lines. The shaded figure $B''C''D''$ is known as the acceleration image of link 3; it is geometrically similar to the figure BCD that is link 3. Note that D'' is on the same side of $B''C''$ as D is with respect to BC. The acceleration image has been turned through an angle $\pi - \beta$ with respect to its link in the sense of α_3; $\beta = \arctan\,(\alpha_3/\omega_3{}^2)$.

In use, the performance of the acceleration image is similar to that of the velocity image: the simultaneous solution of the two vector equations for $\mathbf{a}_D$ could have been avoided. Thus, the acceleration of point G is given by the vector $\mathbf{O_aG''}$ of the acceleration diagram (Fig. 4-26*d*), since G'' is the homologous point of G.

* The scale factors are defined in consistent units, e.g.:
1 in. on drawing = a distance of k_s ft on real mechanism.
1 in. of velocity vector = k_v fps.
1 in. of acceleration vector = k_a fps^2.

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