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The Economic Threshold With a Stochastic Pest Population: An Application to the European Red Mite

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THE ECONOMIC THRESHOLD

WITH A STOCHASTIC PEST POPULATION

AN APPLICATION TO THE EUROPEAN RED MITE

by

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ABSTRACT

This paper applies option value theory to determine the timing and value of pesticides in controlling crop damage. Entomologists and economists have both sought to determine "action" or "economic" thresholds which would trigger the application of a pesticide, or an alternative pest control policy. These concepts are reviewed, and the differences arising in each discipline are noted. Two mathematical models are developed to more precisely define these concepts. The first model is deterministic, and builds on the work of Headley (1972) and Hall and Norgaard (1973). The second model is stochastic, and assumes that the pest population can be described by a "diffusion process." This formulation falls naturally into a class of models that have been used successfully in the field of finance to determine the optimal time to exercise a stock option. Within the pest control context, it is optimal to spray as soon as the stochastically evolving pest population hits a "stopping frontier." With a fixed harvest date this frontier is a positively-sloped, convex curve in time-pest space. This implies that the closer one gets to harvest, the larger the pest population must be to trigger a pesticide application.

Both models are applied to the European Red Mite, a foliar pest of apples. Geometric Brownian motion is used to model red mite density. In the stochastic model, the pesticide application is delayed until the pest population reaches a larger size. This stopping value increases with the volatility of the pest density process because of the increased likelihood that predators or other random factors might "naturally" reduce the pest population, thus avoiding the cost of pesticide application. To our knowledge, option value theory has not been applied to the problem of pest control. The theory, however, is well suited to the problem of optimal timing in a stochastic environment and is certainly consistent with the integrated pest management (IPM) philosophy. Optimal timing may reduce the overall amount of chemicals applied to crops and soils.

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I. Introduction and Overview

The concepts of economic thresholds (ET) and economic injury level (EIL) are still "the backbone of progressive concepts in insect control, namely, integrated control, insect pest management" (Poston, Pedigo, and Welch (1983)). These concepts were introduced by entomologists to provide practical decision rules for using pesticides in a context of complex interactions between plants and pests. The most widely accepted definition of economic threshold was first given by Stern et al. (1959) as the "density at which control measures should be initiated to prevent an increasing pest population from reaching the economic injury level." The ET is thus an operating rule intimately tied to the EIL, which is "the lowest population density that will cause economic damage", where the later was taken to be the amount of injury which justifies the cost of artificial control measures.

Although quite useful, these definitions needed some clarification regarding the amount of injury that should trigger some action to control a pest population.

Hillebrant (1960) was apparently the first economist to apply a marginal analysis to the pest control problem; she showed that, for a profit maximizing farmer, there is an optimal dose of pesticide which is such that the marginal benefit of pesticide treatment just equals its marginal cost. Headley (1972) formalized these results, but he redefined the economic threshold as the level to which a given pest should be reduced in order for the marginal revenue from the application of a pesticide to just equal its marginal cost. This new definition has unfortunately resulted in some misunderstandings between economics and entomologists. As explained by Mumford and Norton (1984), the root of this confusion lies in a difference of approach: while entomologists want to find the pest density level at which a given control action should be taken, economists are looking for the most profitable control level for a particular pest density. Headley's (1972) model was expanded by Hall and Norgaard (1973) who specifically examined optimal timing of a pesticide application.

These economic models, however, have remained mostly theoretical tools of limited practical value. On one hand, they have been criticized for being too stylized. Some attempts[1] to better describe the dynamics of pest and host interaction, while providing some insights in the factors affecting optimal control strategies, have shown that more realistic mathematical models can quickly become almost untractable. On the other hand, the economic threshold, as defined by economists, has not been used widely in practice (Moffit et al., 1984) because, it requires the simultaneous calculation

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[1] See for example Shoemaker (1973).

of both the optimal spraying time and the optimal pesticide dosage. Farmers prefer to use simpler rules for applying pesticides, such as the action threshold (Edwards and Heath, 1964) which is defined as the critical pest density prior to taking a specific action, such as applying a fixed dose of pesticide.

Another explanation for the resistance to the adoption of the economic threshold is that it relies on the assumption of perfect information. In practice, farmers face considerable uncertainty regarding potential pest damage, the efficacy of the pesticide, or the value of the crop. Feder (1979) investigated qualitatively the effect of uncertainty on the dosage of a pesticide by a risk averse farmer; he considered uncertainty in the rate of damage per pest, in the size of the pest population, and in the efficacy of the pesticide. Moffit et al. (1984) and Moffit (1986) considered uncertainty in the initial rate of pest infestation to derive more efficient action-thresholds-type-rules for risk averse farmers. Other researchers have suggested the use of Bayesian decision theory: while we do not know in advance which state of nature will occur, a probability weighted average can be calculated for each action if it is possible to evaluate the probability of each outcome from past experience. The action with the highest selected return is then selected. The earliest application of Bayesian decision theory to pest control decisions is due to Carlson (1969), who analyzed the peach brown rot, a fungal pathogen which attacks peaches. To date, the issue of uncertainty in pest population and pesticide efficacy does not appear to have been solved satisfactorily.

There are a number of other important issues in pest management, such as increasing pest resistance, issues of coordination between adjacent farms, inter-seasonal pest dynamics, or environmental side-effects of pesticides for both field workers or consumers of food containing pesticide residues. Entomologists have also been focusing on systems of pests complexes, where several types of pests compete for the same crop. To decrease the reliance on chemicals and to reduce crop damage, integrated pest management (IPM) has been promoted in an attempts to control pests that cause damage by a variety of chemical, biological and managerial strategies, thus diversifying the pest-control portfolio. For a survey of the economic and biological issues attending pest control see Carlson and Wetzstein (1993).

In this paper, we first reconsider the issue of optimal timing when a pest population evolves deterministically, thus revisiting some of the issues addressed in the earlier studies by Headley (1972) and Hall and Norgaard (1973). Our exposition is aided by the specification of some tractable functional forms which allow for closed-form results. The deterministic model sets the stage for a stochastic model where the right to spray is seen as a financial option. We apply stochastic dynamic programming to formulate the optimal application problem when the pest density can be modeled by a diffusion process. In the stochastic case, it is optimal to spray the pesticide as soon as the pest density hits a "stopping barrier" (or "stopping frontier"), while for the deterministic case the optimal spraying time is determined uniquely from the initial pest density and other parameters. We then apply both models to the European Red Mite, a

foliar pest of apples. We find that modeling explicitly the randomness of the mite density may lead to decreased pesticide use. To our knowledge this approach has not been applied to the problem of pest control, yet it is a compelling extension of the existing literature which is consistent with the IPM philosophy of better monitoring and coordination to reduce pest damage, while at the same time reducing the amount of chemicals applied to crops and soils.

II   The Case of a Deterministic Pest Population

II.1   Optimal Timing with a Fixed Pesticide Dose

Let $Q(t)$ and $X(t)$ denote, respectively, the plant biomass of a crop and the pest density at instant t. The initial plant biomass Q_0 and pest density X_0 are given. For simplicity, we assume that, while the pest may destroy a sizable percentage of the crop, it will not wipe it out. We also assume that a regulatory agency allows a single application of a specified dose of a pesticide which costs K per application per unit area and causes the instantaneous death of a fraction, M, of the pest population. This pesticide can be applied between t=0, which is "some time" after the start of the growing season, and t=T, one day before harvest. The crop is thus harvested on day T+1. In reality, the impact of a pesticide is not known with certainty; it may depend on weather conditions (such as wind or precipitation) and pest resistance. We ignore these issues to focus on the effect of stochastic pest growth on the economic threshold.

We define p as the per unit harvest price, net of harvest cost, and allow for discounting at the instantaneous rate δ. We postulate that the objective of the farmer is to maximize discounted net revenue, given that a specified amount of pesticide can be sprayed once at any time between 0 and T. The joint dynamics of the plant biomass and the pest population are:

$$
(1) \quad \begin{cases} \dfrac{dQ}{dt} = G(t, X) \\[2mm] \dfrac{dX}{dt} = F(X) \end{cases}
$$

In the above, G(t,X) and F(X) are, respectively, the crop growth and insect growth functions. They are assumed to be both continuously differentiable. Moreover, G(t,X) is assumed to be increasing in t and decreasing in X, while F is increasing in X. Note that the absence of Q as an argument of F(•) may overstate pest growth if the crop is the pest's only food source.

Suppose that the differential equation for the pest density can be solved, so that X=ϕ(t;X₀), where X_0 is the initial pest density. Then, the discounted net revenue from the crop with <u>no</u> pesticide application, π_N, is given by the expression:

$$
(2) \quad \pi_N = pe^{-\delta(T+1)}Q(T+1)
$$

where Q(T+1) is the biomass at the time of harvest. For any t between 0 and T+1, we have:

(3) $Q(t) = Q_0 + \int\limits_0^t G(u, \phi(u; X_0)) du$

If the pesticide is applied at time τ, the discounted net harvest value, π_P, is

(4) $\pi_P(\tau) = pe^{-\delta(T+1)} \left[Q(\tau) + \int\limits_\tau^{T+1} G(u, \phi(u - \tau, X_\tau^+)) du \right] - Ke^{-\delta\tau}$

where X_τ^+, the pest density immediately after τ, is related to X_τ^-, the pest density just before τ, by $X_\tau^+ = (1-M) X_\tau^-$. Let τ^* be the value of $\tau \in [0, T]$ which maximizes $\pi_P(\tau)$. Then, the farmer's maximum discounted net revenue is the largest of π_N and $\pi_P(\tau^*)$.

As an example, consider the functional forms:

(5a) $\dfrac{dQ}{dt} = a - bX$

(5b) $\dfrac{dX}{dt} = rX$

where a is the (linear) growth rate of the crop biomass, b is a damage coefficient relating the size of the pest population to crop loss, and r is the (exponential) growth rate of the pest population. If no pesticide is applied, Equation (5b) implies:

(6a) $X = \phi(t; X_0) = X_0 e^{rt}$

Substituting this expression into Equation (5a) and integrating yields:

(6b) $Q(t) = Q_0 + at + \dfrac{b}{r} X_0 (1 - e^{rt})$

We can now write the expressions for π_N and π_P as

$$(7a) \quad \pi_N = pe^{-\delta(T+1)}\left[Q_0 + a(T+1) - \frac{b}{r}X_0\left(e^{r(T+1)} - 1\right)\right]$$

$$(7b) \quad \pi_P(\tau) = \pi_N + pe^{-\delta(T+1)}\frac{bMX_0}{r}\left(e^{r(T+1)} - e^{r\tau}\right) - Ke^{-\delta\tau}$$

In the above, $Q_0 + a(T+1)$ is the final crop biomass at harvest time; $-\frac{b}{r}X_0\left(e^{r(T+1)} - 1\right)$ is the biomass consumed by the pest if left unchecked; and $\frac{bMX_0}{r}\left(e^{r(T+1)} - e^{r\tau}\right)$ is the biomass saved by spraying the pesticide at time τ.

Maximizing π_P with respect to τ yields an explicit solution for τ^*:

$$(8) \quad \tau^* = \begin{cases} T, & \text{if } X_0 < \dfrac{\delta Ke^{\delta}e^{-rT}}{Mbp} \\[3ex] 0, & \text{if } X_0 > \dfrac{\delta Ke^{\delta(T+1)}}{Mbp} \\[3ex] \dfrac{1}{\delta+r}Ln\left(\dfrac{\delta Ke^{\delta(T+1)}}{bMpX_0}\right) & \text{otherwise} \end{cases}$$

However, it is not worth applying the pesticide if the value of the crop saved is less than the discounted cost of the pesticide. Solving for the values of τ^* for which $\pi_P(\tau^*)<\pi_N$, we find that the pesticide should not be applied when

$$(9) \quad \tau^* > T+1-\frac{1}{r}Ln\left(1+\frac{r}{\delta}\right), \quad \text{for } \tau^* \in (0,T)$$

From Equation (8), it is clear, that the optimal spraying time τ^* decreases as b, p, M, r, or X_0 increase, everything else being the same: the farmer sprays earlier if pest damage is more severe (higher b), or if the crop is more valuable (higher p); a more effective pesticide (higher M), a higher rate of pest growth (higher r), or a higher initial pest density (higher X_0) also lead to an earlier application because this reduces the cost of pest damage. On the opposite, τ^* increases when K, T, or δ increase: if the cost of an application increases, the pesticide will be applied only for larger pest populations (thus less frequently); finally, an increase in T (corresponding to a longer growing season) or δ reduces the present value of the crop. Equation (8) shows that X_0, and model parameters δ, r, b, M, p, and K uniquely determine τ^*, if spraying takes place. We will see that this is not the case when the pest density varies stochastically. A numerical illustration is provided in Section IV.

II.2   Optimal Timing with a Variable Pesticide Dose

So far, we have assumed that the farmer was restricted to applying a single fixed dose of pesticide. We now relax this constraint and allow the amount of pesticide per unit area, noted Y, to be chosen by the farmer. Both the mortality rate and the cost of application now become increasing functions of Y; they are noted M(Y) and K(Y) respectively. Other assumptions are unchanged.

We break down the cost function, K(Y) in three components: the fixed cost to the farmer of applying the pesticide, denoted K_F; the corresponding variable cost,

denoted K_V; and external costs due to the side-effects of the pesticide, which may include water pollution and adverse health-effects. We note these costs $f(Y)$, where f is an increasing function of Y. Thus, we write:

(10a) $K(Y) = K_F + YK_V + f(Y)$

A possible form for f is:

(10b) $f(Y) = \alpha Y^2$

The pesticide efficacy function, M(Y), should be zero when Y is zero, and it should increases asymptotically towards 1 as $Y \to +\infty$. A convenient functional form for M is:

(10b) $M(Y) = 1 - e^{-\beta Y}$

To find the deterministic economic threshold, we repeat the above analysis. If we adopt the functional forms given by Equations (5a-b), the optimal spraying time τ^*, which now has to be determined jointly with Y^*, verifies Equation (8). In addition, Y^* must verify the first order condition with respect to Y, if $Y^*>0$:

(11) $\quad pe^{-\delta(T+1)} \dfrac{bM'(Y^*)X_0}{r}(e^{r(T+1)} - e^{r\tau^*}) = (K_V + f'(Y^*))e^{-\delta\tau^*}$

τ^* and Y^* are thus obtained by solving a pair of non-linear equations which do not, in general, yield explicit expressions. Note that Equation (11) is only a necessary condition for $Y^*>0$ as there may be several local maxima for π_P depending on the functional forms of M(Y) and K(Y). More specific results have to be obtained numerically. In practice, however, pesticide dosage cannot be selected by the farmer

since, by law, many pesticides can only be applied in a fixed dosage specified by the manufacturer. Pesticide efficacy also depends on a number of factors, such as the built-up resistance of targeted pest, or weather conditions: for example, rain may wash away the pesticide and thus reduce pest mortality. Usually, data on pesticide efficacy in operating conditions is not available, and the quantification of external damage is open to debate. This lack of information considerably reduces the usefulness of the concept of economic threshold. In section IV, we illustrate how it could affect pesticide usage based on hypothetical data.

III   The Case of a Stochastic Pest Population

The stochastic model assumes that the pest density, X, follows a diffusion process with drift rate r(t,X) and infinitesimal variance $\sigma^2(t,X)$, so that an increment in X can be written:

(12) $dX = r(t,X)dt + \sigma(t,X)dz$

where dz is the increment of a standard Wiener process. Because the pest density varies randomly, future crop biomass and thus net revenue are uncertain until harvest time.

To find the optimal pest density that will trigger spraying at any given time between 0 and T, we use stochastic dynamic programming and concepts from the theory of investment under uncertainty. We view the crop as an investment and the

right to spray as an option which can be exercised to improve the expected value of this investment. To emphasize this point, we write the value in time zero dollars of the crop "asset" before pesticide application, noted J^B, as the sum of the expected value of the crop at time t plus the value of the option to spray, noted $\varphi(t,X)$:

$$(13a) \quad J^B(t,X) = \varphi(t,X) + pe^{-\delta(T+1-t)}\left(Q(t) + E_t\left[\int_t^{T+1} G(u,X)du \mid X(t)\right]\right)$$

In the above, E_t stands for the expectation at t, and $\dfrac{dQ}{dt} = G(u,X)$ is the rate of change of plant biomass at time u given a pest density of X. After spraying, the value of the crop (also referred to as the "value function" in the finance literature) is just the expected revenue from harvesting the crop because we assume that the pesticide can only be sprayed once. Since spraying instantly reduces the pest density by M%, the value function immediately after the optimal spraying time , at τ, is:

$$(13b) \quad J^A(\tau,X) = pe^{-\delta(T+1-\tau)}\left(Q(\tau) + E_\tau\left[\int_\tau^{T+1} G(u,X)du \mid X_\tau^+ = X_\tau^-(1-M)\right]\right)$$

where X_τ^- is the size of the pest density just before spraying.

Since the crop asset pays no dividend, J^A and both components of J^B must satisfy the H-J-B equation:

$$(14) \quad \delta J = J_t + r(t,X)J_X + \frac{\sigma^2(t,X)}{2}J_{XX}$$

From standard results in optimal stopping theory (Malliaris and Brock, 1982), the points in (t,X) space where spraying is optimal form a curve, often called the stopping frontier, which separates the "continuation region", where the pest is uncontrolled, from the "stopping region," where spraying should take place. The stopping frontier is characterized by the "value-matching" and "smooth-pasting" conditions, which require continuity of the value function and its first derivative on the stopping frontier. These conditions are:

(15a) $J^B(\tau, X(\tau)) = J^A(\tau, X(\tau)) - K$

(15b) $J^B_X(\tau, X(\tau)) = J^A_X(\tau, X(\tau))$

To solve this boundary and initial value problem, we need, in addition to Equations (14), (15a) and (15b), two more conditions on the boundaries of the domain of variation of t and X. First, we assume that the option to spray expires at time T, irrespective of the size of the pest population at that time. At that time, the farmer can spray if the value of the crop that would thus be saved is less than the cost of spraying. This gives us a terminal condition. Moreover, we assume for simplicity that, if anytime between 0 and T+1 the pest population becomes extinct, it remains extinct. This implies that the value of the option to spray vanishes when the pest population falls to zero, which leads to the boundary condition:

(16) $\forall t \in [0, T], \; \varphi(t, 0) = 0$

To prepare for the numerical application to the European Red Mite, consider again the case where $\frac{dQ}{dt} = a - bX$. We now assume that X follows a geometric Brownian motion with infinitesimal drift r and variance σ, so that:

(17) $\quad dX = rXdt + \sigma Xdz$

This process satisfies our assumption if the pest population vanishes between 0 and T, it remains extinct for the season. For this choice of diffusion process for X, Equation (14) becomes:

(18) $\quad \delta J = J_t + rX J_X + \frac{\sigma^2}{2} X^2 J_{XX}$

The value functions in the continuation and in the stopping regions are respectively:

(19a) $\quad J^B(t, X) = \varphi(t, X) + pe^{-\delta(T+1-t)} \left(Q(t) + a(T+1-t) - \frac{bX}{r}(e^{r(T+1-t)} - 1) \right)$

(19b) $\quad J^A(t, X) = pe^{-\delta(T+1-t)} \left(Q(t) + a(T+1-t) - \frac{b(1-M)X}{r}(e^{r(T+1-t)} - 1) \right)$

so that the continuity and smooth-pasting conditions become

(20a) $\quad \varphi(\tau, X(\tau)) = pe^{-\delta(T+1-\tau)} \frac{bMX(\tau)}{r}(e^{r(T+1-\tau)} - 1) - K$

(20b) $\quad \frac{\partial \varphi(\tau, X(\tau))}{\partial X} = pe^{-\delta(T+1-\tau)} \frac{bM}{r}(e^{r(T+1-\tau)} - 1)$

The terminal condition says that, at T, the farmer will spray if:

(21) $\quad pe^{-\delta} \frac{bMX}{r}(e^r - 1) \geq K$

so that the terminal condition is:

$$(22a) \quad \forall X \in [0,+\infty), \quad \varphi(T,X) = pe^{-\delta} \frac{bM}{r}(e^r - 1)Max(X - X_T^*, 0)$$

where

$$(22b) \quad X_T^* = \frac{Kr}{pbM} \frac{e^{\delta}}{e^r - 1}$$

Equation (18) together with Equations (16), (20a-b) and (22a-b) uniquely define the optimal stopping frontier (t,X*(t)). Intuitively, we expect X*(t) to increase with t, since as t increases, given a pest density value, we have to pay the same amount, K, for a shorter time interval during which the pest might cause crop damage. Before applying this model to the European Red Mite, we outline the procedure to follow if the farmer is free to select the dosage of pesticide.

III.2 Optimal Timing with a Variable Pesticide Dose

If the pesticide dosage can be chosen by the farmer, the formulation described by Equations (15-17c) still holds but the optimal spraying time is now a function of Y. For any given Y, we can go through the steps outlined above and calculate $\tau^*(Y)$. The optimal Y can then be calculated by maximizing $\pi_P(\tau^*(Y),Y)$ with respect to Y. The stopping frontiers calculated this way are the stochastic counterpart of the "economic threshold" identified by Headley (1972) and Hall and Norgaard (1973).

IV. Application to the European Red Mite

IV.1 Assumptions and Model Data

The functional forms selected, especially the assumption that the pest density follows a geometric Brownian motion, are appropriate to model pests which have multiple generations per year and thus pose a risk of outbreak, such as mites, aphids, or trips. We chose to apply our model to the European red mite (ERM) (Panonychus ulme (Koch)), which has become a problematic foliar pest of apples. Indeed, the ERM has developed a resistance to many pesticides (Croft et al. 1987) and pesticides applied to control other pests destroy natural enemies of the ERM, often leading to outbreaks which can damage apple crops.

Typically, in New York State, ERM eggs hatch during April and May. Preventive treatment, in the form of oil, can be applied during that period. The first adults appear in early June; they grow and reproduce until harvest time, in early September. In the absence of predators, ERM grows exponentially; Nyrop et al. (1994) report that the ERM growth rate per day varies between 0.025 and 0.065 for slowly growing populations, and between 0.065 and 0.125 for fast growing populations. From June to August, pesticides can be applied as a rescue treatment to limit mite damage. An often used pesticide is Kethane, at a total cost (material and labor) of approximately $30 per acre per application; we assume a mortality rate of 90% for the recommended dosage, although the efficacy of a pesticide can be affected by atmospheric conditions. We thus pick T=90 days, K=$30/acre, and M=0.9.

The primary effect of mite infestation is to reduce yield through fruit size but it can also affect fruit appearance thus reducing the value of fresh fruits. Some studies (see Francesconi et al, 1996) have shown that mites damage apples by reducing leaf net CO_2 exchange rate. In general, quantifying the damage foliar pests inflict on perennial crops is difficult because of a high variation among plants. Also, fruit growth reduction due to mites is more severe and noticeable earlier on normally cropped trees than on lightly cropped trees (Francesconi et al. 1996). To simplify our analysis, we ignore appearance damage and we focus only on crop weight loss. Although it explains less than half of the variation in fresh fruit weight, we adopt the relationship between fresh fruit weight and cumulative mite days (CMD) developed in Francesconi et al. (1996):

(23a) $W(g) = 190 - 0.016\, CMD$

Since treatment is commonly expressed in $ per acre, we normalize this damage relationship to a standard crop per acre. Averaging good and bad years leads to an average crop of 18,000 lb./acre, which leads to:

(23b) $Q(lb) = 18,000 - 1.5\, CMD$

From this relationship, we deduce that a reasonable value for the damage coefficient b, if our assumptions hold, is 1.5 lb. per (acre cumulative mite day).

Another important model parameter is the price of a pound of apple. We look only at fresh fruit which is much more valuable to the farmer than fruit destined to be processed. The price structure of fresh fruit price is actually quite complex, because it

depends on fruit size and appearance, not to mention apple variety. Half the farms in New York State, however, sell their crop to wholesalers who deal with packaging and distribution. In recent years, a bushel of apples has brought approximately $6 to the farmer, net of harvesting. If we further assume that a bushel weights approximately 42 lb., we end up with a net farm price of $0.15 per lb. of apples.

Two more parameters are needed: the discount rate δ, and the infinitesimal variance of the process of X, σ^2, when the pest density varies stochastically. For δ, we consider values between 0.0001 per day (3.72% per year) and 0.0002 per day (7.57% per year). Since adequate time series data on mite density are not available, we use arbitrary but plausible values for σ^2, ranging between 0.0025, and 0.04 per day. Table 4.1 summarizes the parameter values considered.

Table 4.1: Summary of Model Parameters

Parameter	Unit	Values Considered	Parameter Meaning
p	$/lb.	0.10, 0.15, 0.20	Price/lb. of apples
b	lb./(acre*mite*day)	1.0, 1.5, 2.0	Mite damage coefficient
M	--	0.8, 0.9, 0.99	Pesticide efficacy
K	$/acre	30	Cost of pesticide appl.
δ	/day	10^{-4}, $1.5*10^{-4}$, $2*10^{-4}$	Discount rate
r	/day	0.04, 0.065, 0.09	Mite growth rate
T	days	90	Period of mite damage
σ	$1/\sqrt{\text{day}}$	0.05, 0.1, 0.2	Mite density std. dev.

IV.2 Results

IV.2.1 Deterministic Case

Figure 4.1 shows the optimal "stopping point," i.e. the point in (t,X) space where it is optimal to spray for an initial pest density of 5.83E-03 and a density growth rate of r=0.09. Other parameter values are at their medium value. For these parameters, the spraying time is τ^*=15 days and the critical pest density is X^*=2.25E-2 mite per leaf, or a mite for every 40 leaves, which is very low.

Figure 4.1: Deterministic Action Threshold
r=0.09, δ=0.00015, Xo=5.83E-03
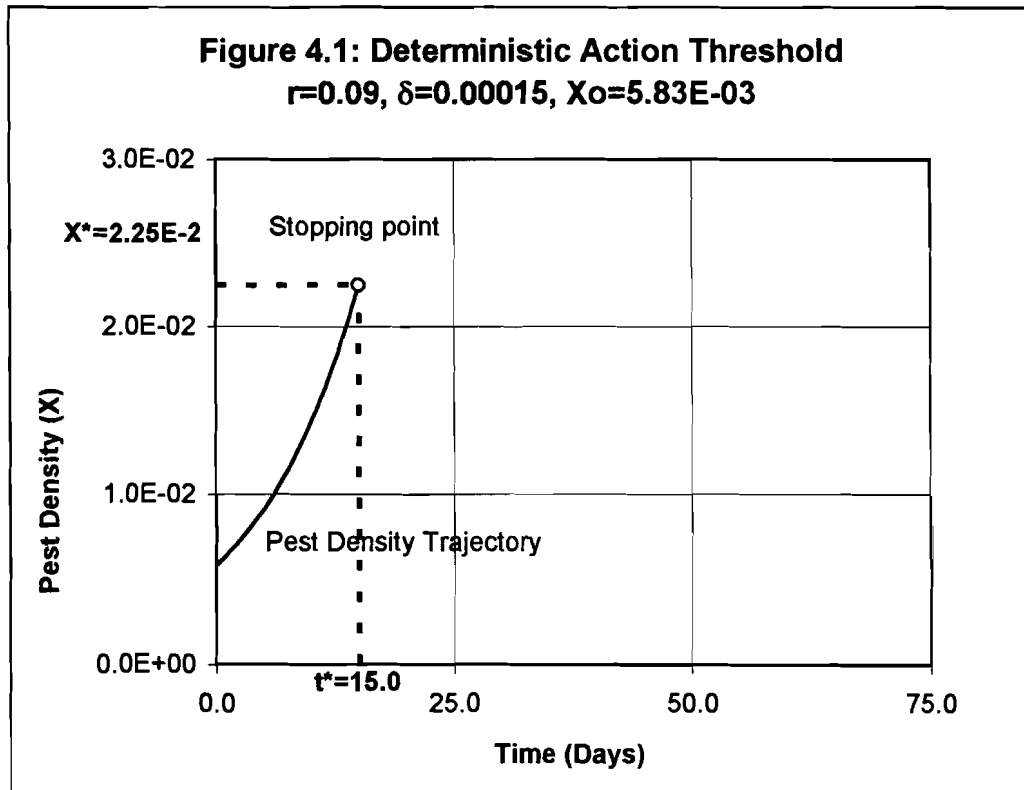
To explore the sensitivity of the model to the value of the parameters, Table 4.2 shows the optimal spraying time τ^* and the critical mite density X^* at which spraying

Table 4.2: Optimal Spraying Time and Mite Density

δ /day	r /day	p $/lb.	b lb./(acre*mite*day)	t* days	X* (mite*day/leaf)
1.5E-04	0.090	0.15	1.5	12.1	2.23E-02
1.5E-04	0.040	0.15	1.5	N.S. [2]	N.S.
1.5E-04	0.065	0.15	1.5	N.S.	N.S.
1.0E-04	0.090	0.15	1.5	7.6	1.49E-02
1.5E-04	0.090	0.15	1.5	12.1	2.23E-02
1.5E-04	0.090	0.10	1.5	16.6	3.34E-02
1.5E-04	0.090	0.20	1.5	8.9	1.67E-02
1.5E-04	0.090	0.15	1.0	16.6	3.34E-02
1.5E-04	0.090	0.15	2.0	8.9	1.67E-02

Table 4.3: Determination of the Economic Threshold

Y/Yo[3]	M %	K $/acre	τ* day	X* mite day/leaf	Net Savings $/acre
0.50	0.6838	$17.50	9.3	1.73E-02	$28.09
0.60	0.7488	$20.00	9.7	1.80E-02	$29.93
0.70	0.8005	$22.50	10.3	1.90E-02	$30.87
0.80	**0.8415**	**$25.00**	**10.9**	**2.00E-02**	**$31.11**
0.90	0.8741	$27.50	11.6	2.12E-02	$30.79
1.00	0.9000	$30.00	12.2	2.25E-02	$30.02
1.10	0.9206	$32.50	12.8	2.38E-02	$28.89

---

[2] N.S. stands for "no spraying".
[3] Y_0 is the dose of pesticide corresponding to M=0.9 and K=$30/acre.

should take place for combinations of the model parameters, for an initial mite density of 7.5E-3, M=0.9, and K=$30/acre. At low or medium values of r, spraying never takes place (indicated by "N.S." in Table 4.2). For the cases presented, when there is spraying, it takes place in the first twenty days after the appearance of the mite, at very low mite densities.

Table 4.3 presents results used to determine the economic threshold for the following model parameters: r=0.09 per day, T=90 days, δ=1.5E-4 per day, b=1.5 lb./(acre*mite*day), p=$0.15 per lb., K_f=$5, K_v=$25/pesticide dosage unit, β=2.30, and X_0=7.5E-3 mite per leaf. Environmental damages were ignored in the calculations shown in Table 4.3 because of the difficulty to quantify them; results are thus biased toward higher pesticide usage. As expected, in addition to leading to a decrease pesticide use (80% of the original dose), consideration of the economic threshold results in an increased benefit for the farmer. However, the extra benefit per acre is quite small in comparison to the net value of the crop. Using the economic threshold over the action threshold has thus the potential of decreasing pesticide usage, but it requires information on pesticide efficacy and external costs which are seldom, if ever, available.

IV.2.2 Stochastic Case

We now assume that the pest density can be modeled by a geometric Brownian motion. The problem facing the farmer is fully described by Equation (18) together

with Equations (16), (20a-b) and (22a-b). We solve these equations numerically using finite difference techniques[4]. Figure 4.2 shows optimal stopping frontiers for different values of the infinitesimal standard deviation σ and the following model parameters: b=1.5 lb./(acre*mite*day), M=0.9, K=$30 /acre, δ=1.5E-4 per day, r=0.065 per day, and T=90 days. In contrast to the deterministic case, there is no unique optimal spraying time when the pest density varies stochastically. The initial pest density X_0, now determines the probability of hitting various points of the stopping frontier, noted $(\tau,X^*(\tau))$. These stopping frontiers are the stochastic counterpart of the "action threshold" (Edwards and Heath,1964) often used in practice.

From Figure 4.2, we first notice that the stochastic stopping frontiers are almost linear in (t, log(X)) space, except in the neighborhood of T where they curve up: thus, near harvest time, spraying takes place only if there is a mite outbreak. Figure 4.2 also shows that an increasing volatility in X, everything else being the same, leads to spraying at higher pest densities: although pest density is expected to rise exponentially, the more it fluctuates randomly, the larger the probability that it may go down to zero. This effect is not linked to the risk preference of the farmer, who is assumed to be risk
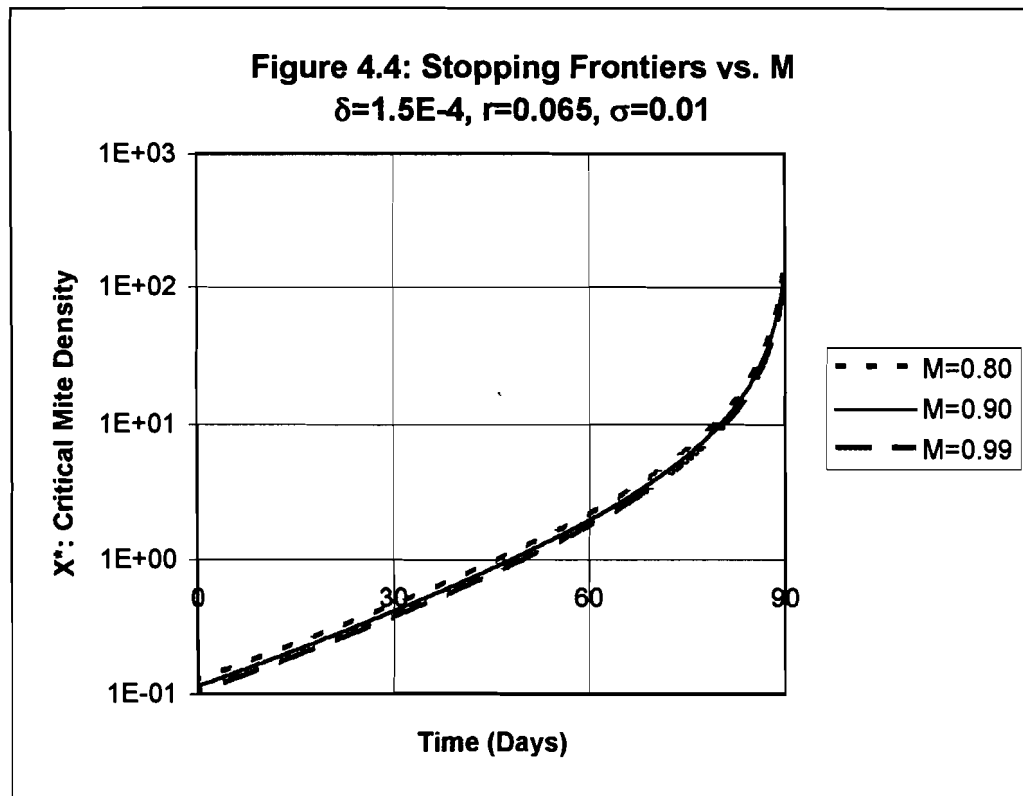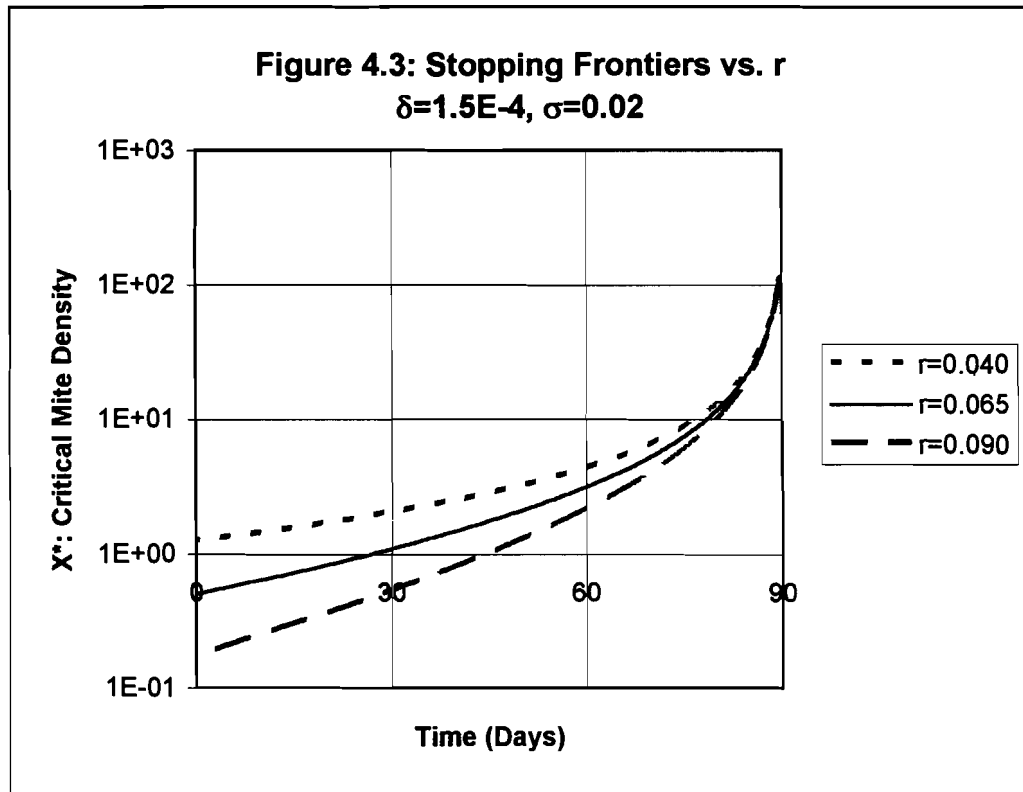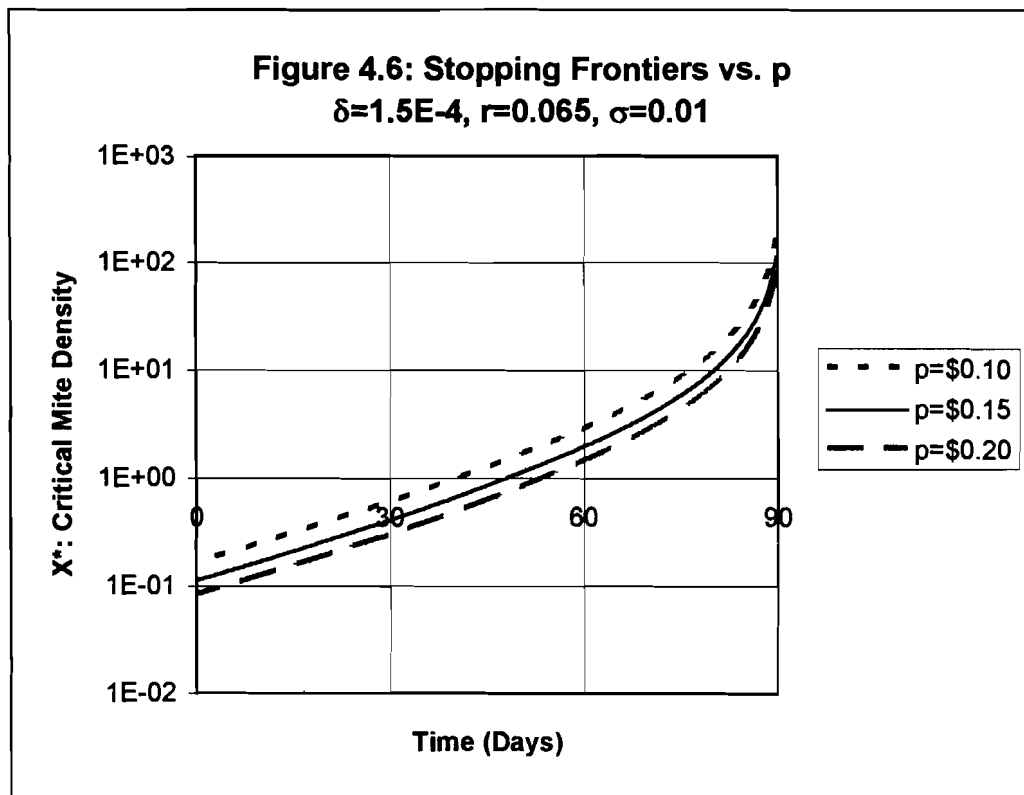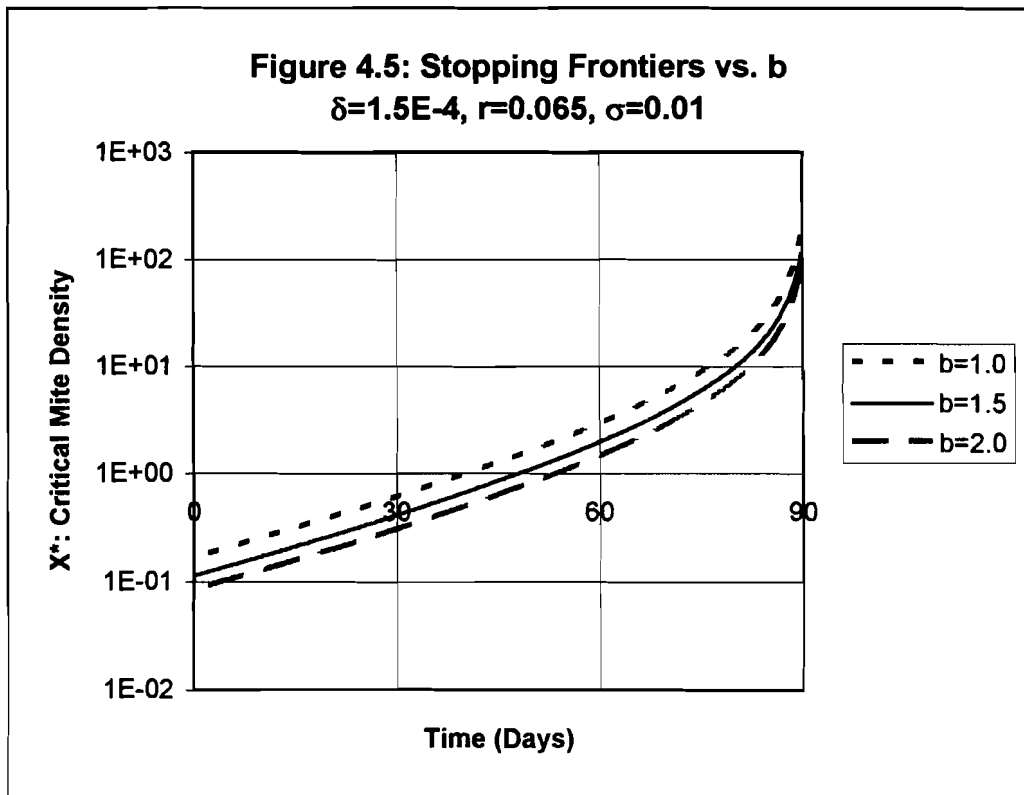
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[4] An explicit finite difference scheme, tailored to this problem, was developed to compute the stopping frontiers for the various parameter combinations. Results were checked by a Crank-Nicholson scheme coupled with an SOR solver. Highlights of the numerical procedures are presented in the appendix.

Figure 4.2: Stopping Frontiers vs. σ
δ=1.5E-4, r=0.065
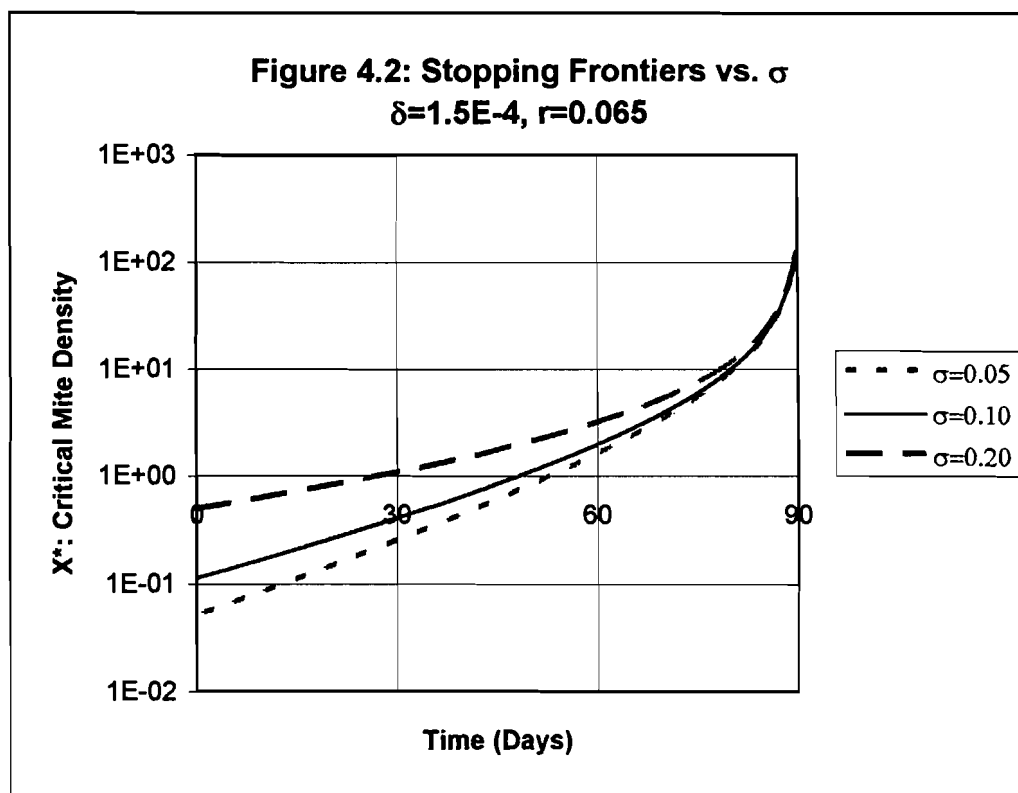
neutral in our formulation. Thus, taking into account large random fluctuations in pest density should lead to a reduction in pesticide application.

Figures 4.3 to 4.6 explore the impact of changes in parameter values on the stopping frontiers. Figure 4.3 shows the stopping frontier for three values of the pest density growth rate. As expected, spraying takes place at higher density values for low growth rates (r=0.04) than at high growth rates (r=0.09): the farmer waits for higher pest density at low growth rates because the expected value of pest damage is lower. Figure 4.4 shows the impact of M on the stopping frontier. For the range of values selected, the stopping frontier does not change drastically when M goes from 0.8 to 0.99, and for practical purposes it is almost unchanged.

Figure 4.3: Stopping Frontiers vs. r
δ=1.5E-4, σ=0.02

X*: Critical Mite Density

- - - - r=0.040
——— r=0.065
— — r=0.090

Time (Days)

Figure 4.4: Stopping Frontiers vs. M
δ=1.5E-4, r=0.065, σ=0.01

X*: Critical Mite Density

- - - - M=0.80
——— M=0.90
— — M=0.99

Time (Days)

Figure 4.5: Stopping Frontiers vs. b
δ=1.5E-4, r=0.065, σ=0.01

X*: Critical Mite Density — Time (Days)

Legend:
- b=1.0
- b=1.5
- b=2.0

Figure 4.6: Stopping Frontiers vs. p
δ=1.5E-4, r=0.065, σ=0.01

X*: Critical Mite Density — Time (Days)

Legend:
- p=$0.10
- p=$0.15
- p=$0.20

We also need to investigate the sensitivity of the stopping frontier to changes in the damage coefficient, b, and the farm price of a pound of apples. Figures 4.5 and 4.6 show that they have the same effect on the stopping frontier: if p or b are higher, the value of pest damage increases which leads the farmer to spray at lower pest densities. Finally, we could wonder about the impact of the discount rate, δ, on the location of the stopping frontier. Solving the problem for δ ranging between 10^{-4} and $2*10^{-4}$, we find that the discount rate has a very small effect on the stopping boundary, which can be explained by the relative shortness of the growing season (90 days) and the small magnitude of δ compared to the pest density growth rate. This brings out another difference with the deterministic case where δ is a key parameter for the timing of the pesticide application, if it takes place.

Finally, we compute the stopping frontier corresponding to the economic threshold in the simplified case where the farmer has the choice only between three doses of pesticide. Model parameters for this case are: b=1.5 lb./(acre*mite*day), p=$0.15 /lb., δ=1.5E-4 per day, r=0.065 per day, σ^2=0.01 /day, and T=90 days. If we normalize pesticide dosage and say that a unit dose of pesticide has an application cost of K=$30/acre and results in the elimination of M=90% of the mites, then $\beta\approx2.30$. If, in addition, K_f=$5 and K_v=25, the application cost of doses of pesticide of 0.8 and 0.9 are then $25.00 and $27.50 per acre respectively, and the corresponding pesticide efficacy is M$\approx$0.84 and M$\approx$0.87. Combining the results from each run, the stopping

frontier corresponding to the economic threshold is a combination of the three stopping

frontiers, with small jumps[5] corresponding to the changes in pesticide dosage. Results

are shown on Figure 4.7. We find that, until day 16, the farmer should apply a unit

dose of pesticide if mite density reaches the stopping frontier for this case; from day 17

to day 24, a dose of 0.9 should be used; and afterwards, only a dose of 0.8 should be

used, if the mite density reaches the stopping frontier for this parameter. This exercise

illustrates again that a better modeling of uncertainty could lead to a more efficient use

of pesticide. However, as mentioned before, practical applications of the economic

threshold are limited because the efficacy of the pesticide is not well known.

Figure 4.7: Economic Threshold
Case of 3 Pesticide Doses

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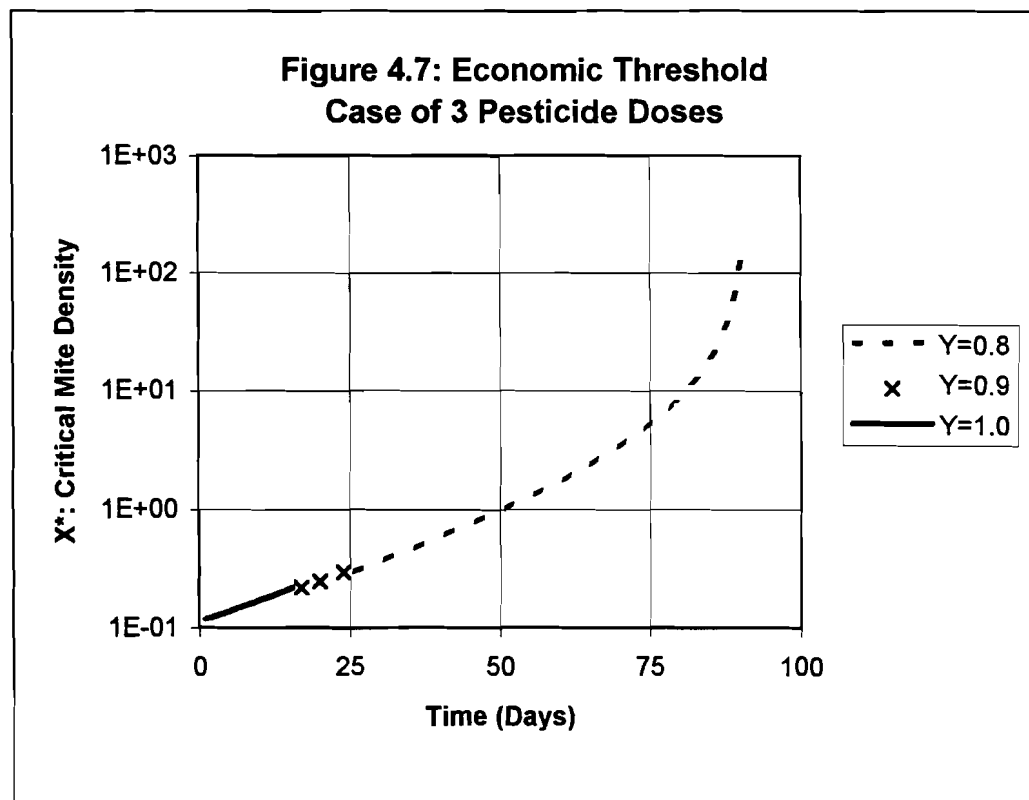
[5] These jumps are barely visible here because the stopping frontiers are very close to
one another.

V. Conclusions

Chemical, biological or other policies to control pest populations may be viewed as options. In this case study, we revisit the concepts of action threshold and economic threshold. After examining the deterministic case and using simple functional forms, we model the density of a pest population by a diffusion process and we present a formulation for finding the optimal spraying time of a pesticide based on stochastic dynamic programming. This leads to a boundary value problem which is solved numerically, using well-known finite difference techniques. We apply this approach to the European red mite, a foliar pest of apples which has shown increasing resistance to pesticides.

This approach offers a clean way of handling stochasticity in pest density growth. It has been known for some time (see Henry (1974), or Arrow and Fisher (1974), that ignoring stochasticity by replacing stochastic parameters with their mean value can lead to decision rules which differ greatly from the optimum. Application of our model to the European red mite illustrates this point. Whereas knowledge of the starting pest density and other model parameters leads to a prescribed spraying date in the deterministic model, spraying takes place only if the pest density reaches a critical level, specified by a stopping frontier, in the stochastic model. In addition, results show that an increase in the volatility of the pest density causes one to spray at higher pest

densities, another important finding in the context of increasing concerns about the environmental side-effects of pesticides.

A sensitivity analysis shows that the stopping frontier is not very sensitive to the values of the discount rate or the efficacy of the pesticide. More effort should be put in obtaining good estimates of the damage coefficient, b, or the expected price of apples, p. If these parameters exhibit a substantial volatility, the model should be modified to model their fluctuations using diffusion and jump processes. This would, however, increase the complexity of the model significantly.

The economic threshold was estimated for both the deterministic and the stochastic models. Results show that it could lead to significant decreases in pesticide use if parameters such as the damage coefficient or the efficacy of the pesticide were known. This is, however, very seldom the case in practice. In addition, the concept of economic threshold may be difficult to apply in more realistic biological situations which include several competing pests. Determining the economic threshold in the stochastic case can also be computer intensive.

Further research could investigate the impact of the risk preferences of the farmer. To include more realistic features, more than one pesticide application could also be considered. An alternative to the stochastic model examined above is to model

r, the pest rate of growth, using a diffusion model. Finally, it would also be of interest to set up a field experiment using the proposed stopping rule to see how it performs.

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Appendix

In this appendix, we briefly describe the two numerical schemes used to solve the optimal timing problem when the pest population follows a geometric Brownian motion. The first scheme is based on an explicit finite difference scheme; its main feature is that it keeps track of the boundary. Results obtained with this scheme, developed specifically for this problem, were checked using a more conventional and proven method: a semi-implicit scheme based on the Crank-Nicholson method coupled with an SOR solver. Derivations for implementing the implicit scheme clearly show the link between the stochastic pest problem and the propagation of heat in a long, insulated bar.

Explicit Algorithm

In the text, we have shown that, in order to find the stopping frontier, we have to solve the following equations for φ, the value of the option to spray:

(1) $\quad \delta\,\varphi = \dfrac{\partial \varphi}{\partial t} + rX\dfrac{\partial \varphi}{\partial X} + \dfrac{\sigma^2}{2}X^2\dfrac{\partial^2 \varphi}{\partial X^2}$: Arbitrage equation

(2) $\quad \forall t \in [0,T],\ \varphi\,(t,0) = 0$: Lower bound

(3a) $\quad \forall X \in [0,+\infty),\ \varphi\,(T,X) = \dfrac{pbM}{r}\dfrac{e^r - 1}{e^{\delta}}\,\mathrm{Max}(X - X_T^{*},0)$: Terminal condition

X_T^{*} is the pest density above which, at time T, spraying should take place. It is:

(3b) $\quad X_T^{*} = \dfrac{Kr}{pbM}\dfrac{e^{\delta}}{e^r - 1}$

(4) $\quad \varphi(\tau, X) = pe^{-\delta(T+1-\tau)} \dfrac{bMX}{r}(e^{r(T+1-\tau)} - 1) - K$: Continuity

(5) $\quad \dfrac{\partial \varphi(\tau, X)}{\partial X} = pe^{-\delta(T+1-\tau)} \dfrac{bM}{r}(e^{r(T+1-\tau)} - 1)$: Smooth-pasting

The value of the option to spray is given by Equation (1) in the continuation region and by Equation (4) in the stopping region. The continuity and smooth-pasting conditions require φ and its first derivative in X have to be continuous across the stopping frontier.

Since it is easier to deal with a partial differential equation with constant coefficients, we start by changing variables. Let:

(6a) $\quad Y = \ln(X)$

(6b) $\quad \varphi(t, X) = \psi(t, Y)$

Introducing (6a-b) into Equations (1)-(5) leads to the equivalent system of equations:

(7) $\quad \delta\,\psi = \dfrac{\partial \psi}{\partial t} + \left(r - \dfrac{\sigma^2}{2}\right)\dfrac{\partial \psi}{\partial Y} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 \psi}{\partial Y^2}$: Arbitrage equation

(8) $\quad \forall t \in [0, T],\ \psi(t, -\infty) = 0$: Lower bound

(9) $\quad \forall Y \in [-\infty, +\infty),\ \psi(T, Y) = \dfrac{pbM}{r}\dfrac{e^r - 1}{e^\delta}\mathrm{Max}(e^Y - X_T^*, 0)$: Terminal condition

(10) $\quad \psi(\tau, Y) = pe^{-\delta(T+1-\tau)}\dfrac{bMe^Y}{r}(e^{r(T+1-\tau)} - 1) - K$: Continuity condition

(11) $\quad \dfrac{\partial \psi(\tau, Y)}{\partial Y} = pe^{-\delta(T+1-\tau)}\dfrac{bMe^Y}{r}(e^{r(T+1-\tau)} - 1)$: Smooth-pasting condition

Implementing the explicit finite differences scheme is straightforward: we discretize t and Y over their domain and replace derivatives of the value function with truncated Taylor series expressions. To obtain an explicit scheme, we use a forward difference for the time derivative. Following conventions from the numerical literature, we note

(12a) $k = \Delta Y$

(12b) $h = \Delta t$.

(12c) $\varepsilon = \dfrac{h}{k^2}$

The parameter ε reflects how fine the grid is.

To solve numerically, we need to restrict the domain of variation of Y. We thus select upper and lower bounds for Y, noted $\underline{Y}$ and $\overline{Y}$ respectively. We split [0,T] and $[\underline{Y}, \overline{Y}]$ in n_t and n_y increments respectively, so that $k = \Delta Y = (\overline{Y} - \underline{Y})/n_y$ and $h = \Delta t = T/n_t$. We then define:

(13) $\psi_{i,j} = \psi(ih, \underline{Y} + jk) = \psi(t, Y)$, for $i \in \{0, n_t)$ and $j \in \{0, n_Y)$.

and replace partial derivatives of ψ in Equations (7)-(11) by the finite differences:

(14a) $\dfrac{\partial^2 \psi}{\partial Y^2} \approx \dfrac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{k^2}$

(14b) $\dfrac{\partial \psi}{\partial Y} \approx \dfrac{\psi_{i,j+1} - \psi_{i,j-1}}{2k}$

(14c) $\dfrac{\partial \psi}{\partial t} \approx \dfrac{\psi_{i,j} - \psi_{i-1,j}}{h}$

After simplification, we obtain a relationship that links ψ at time increment i-1 (we are solving the problem backward) to three values of ψ at time increment i:

(15) $\quad \psi_{i-1,j} = p_+ \, \psi_{i,j+1} + p_0 \, \psi_{i,j} + p_- \, \psi_{i,j-1}$

where:

(16a) $\quad p_0 = 1 - h\delta - \varepsilon\sigma^2$

(16b) $\quad p_+ = \dfrac{\varepsilon}{2}(\sigma^2 + (r - \dfrac{\sigma^2}{2})k)$

(16c) $\quad p_- = \dfrac{\varepsilon}{2}(\sigma^2 - (r - \dfrac{\sigma^2}{2})k)$

Following Smith (Chapter 2, 1985), we derive sufficient conditions for the discretization scheme to converge to the solution of the corresponding continuous problem. We obtain:

(17a) $\quad k \le \dfrac{\sigma^2}{r - \dfrac{\sigma^2}{2}}$ for $r - \dfrac{\sigma^2}{2} > 0$

(17b) $\quad h \le \dfrac{k^2}{\delta k^2 + \sigma^2}$

Discretizing Equations (8) to (11) leads to:

(18) $\forall i \in \{0, n_t\},\ \psi_{i,0} = 0$: Lower boundary

(19) $\forall j \in \{0, n_Y\},\ \psi_{n_t,j} = \dfrac{pbM}{r}\dfrac{e^r - 1}{e^\delta} \text{Max}(e^{Y+jk} - X_T^*, 0)$: Terminal boundary

(20) $\psi_{i,j} = pe^{-\delta(T+1-jh)} \dfrac{bMe^{Y+jk}}{r}(e^{r(T+1-jh)} - 1) - K$: Continuity

(21) $\psi_{i,j+1} = \psi_{i,j-1} + 2kpe^{-\delta(T+1-jh)} \dfrac{bMe^{Y+jk}}{r}(e^{r(T+1-jh)} - 1)$: Smooth-pasting

The idea behind the explicit scheme is simple. As we move back in time, we expect the critical value of the pest density at which spraying should take place to diminish. We know the location of the stopping frontier at time T. If the discretization of the (t,Y) space is fine enough, from one time step to the next, the stopping boundary will either stay at the same Y level, or go down by ΔY. We just need to test which of two points best meets a criterion defining the stopping boundary.

Thus, the algorithm goes as follows. First, we calculate the value function at time T based on Equation (19). We then iterate backward in time until t=0. At a given time t, the lower bound condition gives us a starting value for φ. Using Equation (15), we calculate φ between the lower boundary and the estimated location of the stopping frontier, based on its location at t+Δt. We then consider the point on the estimated stopping frontier and the one immediately below it; we check which of these two points is closest to the corresponding value of the option in the stopping region, based on Equation (20). If it is the point below the estimated stopping frontier, we upgrade the

location of the stopping frontier and go on to the next time step. Otherwise, we calculate the value of the option at the first grid point above the stopping frontier, in the stopping region, using the smooth pasting condition (Equation 21). We then move on to the next time step.

This algorithm is simple to implement. It is very stable, provided conditions (17a) and (17b) are respected, and it locates the stopping frontier precisely. Also, as we move back in time, the computational effort diminishes since we calculate the option value only in the continuation region and at one grid point in the stopping region. However, the explicit method is time consuming because it requires very small time steps. The results shown in the text were obtained with k=0.01 and h=0.0005. Comparisons which smaller grid sizes revealed no significant difference.

Implicit Algorithm

In an implicit method, the value of the unknown function is computed simultaneously at several grid points by solving simultaneously a system of equations. It allows, however, to select a larger time step, and thus to diminish the overall computational effort. Whereas the explicit method is specifically tailored to this problem, the implicit method used is standard. In the following, we show how to transform the stochastic pest problem so that an implicit method can be readily implemented.

We start by making the change of variables

(22) $\quad \tau = (T-t)\dfrac{\sigma^2}{2}$

(23) $\quad z = Ln(\dfrac{X}{X_T^*})$

(24) $\quad KU(z,\tau)e^{\alpha z + \beta \tau} = \varphi(X,t)$

where α and β are chosen so that the first-order partial derivatives in Equation (1) cancel out. Introducing the new variables in Equations (1)-(5), we obtain:

(25) $\quad \dfrac{\partial^2 U}{\partial z^2} = \dfrac{\partial U}{\partial \tau}$

(26) $\quad U(z,0) = Max(e^{0.5(\kappa+1)z} - e^{0.5(\kappa-1)z}, \; 0)$

(27) $\quad U(-\infty,0) = 0$

(28) $\quad U(z,\tau) = [e^{0.5(\kappa+1)z}\dfrac{e^{r+\kappa\tau}-1}{e^r-1} - e^{0.5(\kappa-1)z+\lambda\tau}]e^{0.25(1-\kappa)^2\tau}$

(29) $\quad \dfrac{\partial U(z,\tau)}{\partial z} = \dfrac{1}{2}[(\kappa+1)e^{0.5(\kappa+1)z}\dfrac{e^{r+\kappa\tau}-1}{e^r-1} - (\kappa-1)e^{0.5(\kappa-1)z+\lambda\tau}]e^{0.25(1-\kappa)^2\tau}$

where κ and λ are the two dimensionless parameters:

(30a) $\quad \kappa = \dfrac{2r}{\sigma^2}$

(30b) $\quad \lambda = \dfrac{2\delta}{\sigma^2}$

Moreover:

(31a) $\quad \alpha = 0.5(1-\kappa)$

(31b) $\beta = -0.25(1-\kappa)^2 - \lambda$

In equation (25), we recognize the standardized heat equation, which we solve using a Crank-Nicholson scheme combined with an SOR solver, as described in Chapter 9 of Wilmott, Howison, and Dewynne (1995).

This implicit method is faster than the explicit method described above. However, "far" from the initial conditions, for "low" values of the stopping density Y*, the quality of the solution decreases and the program finds slight oscillations in the stopping boundary. This problem can be remedied by refining the mesh, but so doing greatly increases the computational cost. Thus, the implicit method selected is good for finding a quick approximation to the stopping boundary. It serves to confirm the results obtained with the explicit method.