CAPUT XVIII.

DE MOTU CORPORUM BASI SPHAERICA PRAEDITORUM SUPER PLANO HORIZONTALI.

PROBLEMA. 104.

884. Si corpus basi sphaerica praeditum plano horizontali quomodocunque incumbat, definire vires, quibus sollicitatur, earumque effectum in motu corporis progressivo turbando.

SOLUTIO.

Sit EH planum horizontale et T punctum, ubi corpus ei insistit, in corpore autem notetur primo centrum baseos sphaericae MTN, quod sit in G, deinde centrum inertiae corporis I, ac tabula repraesentet planum, in quo haec tria puncta sunt sita. Ducatur radius GT, qui cum sit ad horizontem EH normalis, situm habebit verticalem, ideoque ipsum planum TGI erit verticale. Iam quia pro motu progressivo totam corpus massam, quae sit $= M$, tanquam in centro inertiae I collectam concipere licet, ducta IP ipsi GT parallela corpus primo ob gravitatem urgetur in directione IP vi $= M$; deinde vero ubi planum horizontale in T tangit, ab eo certa quadam vi urgebitur sursum in directione TG, et pressioni aequali, quae vis sit $= \Pi$. Quare nisi hae duae vires se destruant, corpus in quiete persistere nequit; ex quo perspicuum est, statum quietis exigere, ut producta recta GI in F corpus puncto F plano horizontali insistat, sicque recta DIGF fiat verticalis. Figura ergo repraesentat statum corporis inclinatum, et inclinatio indicatur angulo FGT, qui sit $= \varphi$, quo evanescente corpus in statu aequilibrii versatur. Ponamus porro radium basis sphaericae GF $= GT = e$, et intervallum punctorum G et I nempe $GI = f$, quatenus centrum inertiae I longius distat a puncto F quam centrum figurae G: ita ut si propius caderet, quantitas f negative esset accipienda. Hinc ergo erit $IP = e + f \cos \varphi$, quae est altitudo centri inertiae I supra planum horizontale EH, et quae a viribus sollicitantibus sola afficitur. Translata autem vi $TG = \Pi$ in centrum inertiae I, punctum I deorsum sollicitatur vi $= M - \Pi$; et quia ejus celeritas deorsum directa est

Fig. 113.

$= \frac{f d\varrho \sin\varrho}{dt}$, posita ea $= u$ erit $du = \frac{2g(M-\Pi)dt}{M}$, denotante dt elementum temporis, ex quo habetur $f(dd\varrho \sin\varrho + d\varrho^2 \cos\varrho) = 2g\left(1 - \frac{\Pi}{M}\right) dt^2$ sumto dt constante: neque aliter motus progressivus afficietur.

COROLL. 1.

885. Vicissim ergo si ratio motus progressivi detur, vel saltem ut data consideretur; inde pressio Π definietur, cum sit $\frac{\Pi}{M} = 1 - \frac{f(dd\varrho \sin\varrho + d\varrho^2 \cos\varrho)}{2g dt^2}$ seu $\frac{\Pi}{M} = 1 + \frac{f dd.\cos\varrho}{2g dt^2}$.

COROLL. 2.

886. Si fuerit $f = 0$, seu centrum inertiae I in ipsum centrum sphaerae G incidat, prodit $\Pi = M$, et corpus in omni situ aequilibrii proprietate gaudet.

COROLL. 3.

887. Si fuerit $f > 0$ seu FI $>$ FG, statim ac corpus tantillum inclinatur, a vi sollicitante inclinatio augebitur, sin autem sit $f < 0$ seu FI $<$ FG, inclinatio minuetur, corpusque in situm aequilibrii, quo punctum F plano insistit, restituetur: dum priori casu procumbit, alium quaerens aequilibrii situm.

SCHOLION. 1.

888. Quamcunque autem corpus habuerit figuram, in eo semper ad minimum duo dantur aequilibrii situs, quorum alter ita est comparatus, ut si corpus ex eo parumper declinetur, sponte sua se restituat, alter vero, ut penitus prolabatur: quorum prior *status aequilibrii stabilis*, posterior vero *labilis* vocari solet. Quodcunque enim corpus plano horizontali incumbit, in aequilibrio versatur, si recta a centro inertiae ad punctum contactus ducta fuerit verticalis: id quod semper duplici saltem modo evenire potest. Namque si ex centro inertiae ad omnia superficiei puncta rectae concipiantur ductae, quoniam nulla earum vel evanescit, vel fit infinita, inter eas necesse est dari

et

et maximam et minimam : utraque autem ad planum tangens erit normalis : quare si corpus alterutro eorum punctorum , a quibus centrum inertiae vel maxime vel minime distat , plano horizontali incumbat, recta ex centro inertiae ad punctum contactus ducta erit verticalis, idcoque situm aequilibrii dabit, eumque stabilem , si recta ista fuerit minima , contra vero labilem , si maxima : unde intelligitur, centrum inertiae semper infimum locum quaerere , ubi acquiescat. Saepenumero autem plures dantur aequilibrii situs , alii stabiles alii labiles , qui se alternatim excipere debent , quoniam corpus ex situ labili digressum in stabilem perveniat necesse est.

SCHOLION. 2.

889. In praesente casu , quo corporis superficiem sphaericam statuimus, recta per centrum inertiae I et centrum figurae G ducta dabit duo illa puncta F et D , quibus si corpus plano horizontali incumbat, situm aequilibrii teneat : ac dum puncto F planum horizontale tangit, situs aequilibrii erit stabilis, si FI $<$ FG seu $f < 0$, labilis autem *si* FI $>$ FG seu $f > 0$: neque praeter hos duos situs aequilibrii alius hic dabitur , *nisi fuerit* $f = 0$, quo casu subito omnes plane situs aequilibrii indolem recipiunt. Etsi autem hic totam corporis superficiem ut sphaericam considero , tamen ad institutum nostrum sufficit , si ea saltem portio , qua durante motu planum horizontale contingit , fuerit sphaerica : atque hinc ista tractatio etiam ad eos turbines patet , quorum axes inferius non in cuspidem , ut ante assumsimus , sed in hemisphaerium vel etiam minus sphaerae segmentum efformantur , ita ut forma supra considerata hinc prodeat , si radius sphaerae GF $= \epsilon$ evanescat, sicque haec tractatio superiorem in se complectatur. Recta igitur DIGF per centrum inertiae I et centrum basis sphaericae G ducta proprium turbinis axem exhibet , quae quidem, uti turbines construi solent, simul unus est axium principalium corporis , bini vero reliqui momenta inertiae habent paria , qualem formam jam supra statuimus. Verum quo haec tractatio latius pateat, simulque ad titubationes corporum quorumcunque basi sphaerica praeditorum accommodari queat, axes corporis principales utcunque ab axe proprio DF diversos considerabo, eorumque respectu momenta virium explorabo.

PROBLEMA. 105.

890. Data pressione Π , quo corpus basi sphaerica praeditum plano horizontali incumbit , definire momenta inde orta respectu axium princi-

principalium corporis, quomodocunque hi ratione axis proprii corporis fuerint dispositi.

SOLUTIO

Fig. 114. Circa corporis centrum inertiae I descripta sphaera, sit Z ejus punctum verticale, axisque proprius teneat jam situm DIG, ut ejus declinatio a situ verticali sit $DZ = \varrho$. Cum ergo directio pressionis Π sit verticalis et per punctum G transeat existente $IG = f$, referat recta verticalis GΠ hanc pressionem $= \Pi$, ita ut ZDGΠ sit planum verticale, in quo resolvatur vis G$\Pi = \Pi$ secundum directiones GI et GV, quarum haec ad illam sit normalis, et ob angulum DG$\Pi = \varrho$ prodit vis secundum $GI = \Pi \cos \varrho$ et vis secundum $GV = \Pi \sin \varrho$: quarum illa per centrum inertiae I transiens nulla suggerit momenta. Sint nunc IA, IB, IC, corporis tres axes principales, datum situm ratione axis proprii ID tenentes, ac per puncta A, B, C ex D ducantur semicirculi DAG, DBG, DCG. Quodsi axis IA esset normalis ad planum IGV, ejus respectu foret momentum vis $GV = \Pi f \sin \varrho$, quod autem nunc in ratione sinus totius tam ad sinum arcus GA quam ad sinum anguli VGA minui debet, ita ut ex vi pressionis resultet

mom. respectu axis $IA = \Pi f \sin \varrho . \sin GA . \sin VGA$ in sensum CB
mom. respectu axis $IB = \Pi f \sin \varrho . \sin GB . \sin VGB$ in sensum AC
mom. respectu axis $IC = \Pi f \sin \varrho . \sin GC . \sin VGC$ in sensum BA.

Haec autem terna momenta supra litteris P, Q, R indicavimus, quatenus quidem in sensum contrarium agere statuuntur, quare omnibus ad punctum D translatis habebimus:

$$P = -\Pi f \sin DZ . \sin DA . \sin ZDA = -\Pi f \sin ZD . \sin ZA . \sin DZA$$
$$Q = -\Pi f \sin DZ . \sin DB . \sin ZDB = -\Pi f \sin ZD . \sin ZB . \sin DZB$$
$$R = -\Pi f \sin DZ . \sin DC . \sin ZDC = -\Pi f \sin ZD . \sin ZC . \sin DZC.$$

COROLL. 1.

891. Assumsimus hic, centrum basis G propius esse termino imo F quam centrum inertiae I: sin autem secus eveniat, ut intervallum FI minus sit intervallo $FG = e$, intervallum $GI = f$ negative capi debet. At si fuerit $GI = 0$, momenta inventa evanescunt, seu corpus in omni situ aequilibrium tenebit.

COROLL. 2.

892. Si pro situ axis proprii ID respectu axium corporis principalium ponatur arcus $AD = \zeta$, $BD = \eta$, $CD = \vartheta$, tum vero angulus ZDA

$ZDA = \varphi$, existente arcu $ZD = \varrho$, ob $cof\, ADB = -\frac{cof\,\zeta\, cof\,\eta}{fi\,\zeta\, fi\,\eta}$ et fi $ADB = \frac{-cof\,\theta}{fi\,\zeta\, fi\,\eta}$, quia fi ADB : fi DAB seu fi ADB : $-cof$ DAC $= 1 : fin$ BD $= 1 : fin\,\eta$, erit fi $ZDB = \frac{-cof\,\zeta\, cof\,\eta\, fi\,\varphi + cof\,\theta\, cof\,\varphi}{fi\,\zeta\, fi\,\eta}$, at cof DAC $= \frac{cof\,CD}{fi\,AD} = \frac{cof\,\theta}{fi\,\zeta}$, ideoque $P = -\Pi f\, fi\,\varrho\, fi\,\zeta\, fi\,\varphi$, atque $Q = \frac{\Pi f fi\varrho(cof\,\zeta\, cof\,\eta\, fi\,\varphi - cof\,\theta\, cof\,\varphi)}{fi\,\zeta}$ et $R = \frac{\Pi f fi\varrho(cof\,\zeta\, cof\,\theta\, fi\,\varphi - cof\,\eta\, cof\,\varphi)}{fi\,\zeta}$.

COROLL. 3.

893. Si axis proprius ID congrueret cum axe principali IA, foret $\zeta = 0$ atque $\eta = \theta = 90°$, ut esset $cof\,\eta = cof\,\theta = fi\,\zeta$ et angulus φ maneret indefinitus. At ex prioribus formulis fiunt momenta virium:

$P = 0$, $Q = -\Pi f\, fi\,\varrho\, fi\, ZAB$; et $R = -\Pi f\, fi\,\varrho\, fi\, ZAC$

seu $P = 0$, $Q = \Pi f\, cof\, ZC$ et $R = -\Pi f\, cof\, ZB$.

COROLL. 4.

894. Quodsi vero ut supra ponamus $ZA = l$, $ZB = m$ et $ZC = n$, reperiemus momenta virium in genere

$P = \Pi f(cof\,\theta\, cof\, m - cof\,\eta\, cof\, n)$; $Q = \Pi f\,(cof\,\zeta\, cof\, n - cof\,\theta\, cof\, l)$

atque $R = \Pi f\,(cof\,\eta\, cof\, l - cof\,\zeta\, cof\, m)$,

unde illa si $\zeta = 0$, et $\eta = \theta = 90°$ sponte sequuntur.

EXPLICATIO.

895. Ratio investigationis harum posteriorum formularum ita se habet: Primo cum sit fi DZ fi ZDA $= fi$ ZA fi ZAD erit $P = -\Pi f\, fi$ DA fi ZA fi ZAD; at est ZAD $=$ BAD $-$ BAZ, et

$$fin\, BAD = -cof\, CAD = \frac{-cof\,\theta}{fi\,DA}; \quad cof\, BAD = \frac{cof\,\eta}{fi\,DA}$$

$$fin\, BAZ = -cof\, CAZ = \frac{-cof\, n}{fi\, ZA}; \quad cof\, BAZ = \frac{cof\, m}{fi\, ZA}$$

unde fin $ZAD = \frac{-cof\, m\, cof\,\theta + cof\, n\, cof\,\eta}{fi\, ZA\, fi\, DA}$ et $P = +\Pi f(cof\,\theta\, cof\, m - cof\,\eta\, cof\, n)$.

Reliqua

Reliqua duo momenta Q et R praebet analogia ſine ulteriori calculo. Deinde vero eſt $cof\ DL = cof\ \varrho = cof\ \zeta\ cof\ l + cof\ \eta\ cof\ m + cof\ \theta\ cof\ n$, quae expreſſio uti unitatem nunquam ſuperare poteſt, ita unitati aequalis eſſe nequit, ſeu $DZ = 0$, niſi ſit $l = \zeta$, $m = \eta$ et $n = \theta$, ſcilicet has ternas determinationes ſimul ſuppeditat haec aequatio, $cof\ \zeta\ cof\ l + cof\ \eta\ cof\ m + cof\ \theta\ cof\ n = 1$. Cum enim praeterea ſit

$$cof\ \zeta^2 + cof\ \eta^2 + cof\ \theta^2 = 1 \text{ et } cof\ l^2 + cof\ m^2 + cof\ n^2 = 1,$$

ſi a ſumma harum illius duplum ſubtrahatur, prodit

$$(cof\ \zeta - cof\ l)^2 + (cof\ \eta - cof\ m)^2 + (cof\ \theta - cof\ n)^2 = 0,$$

trium autem quadratorum ſumma nihilo aequari nequit, niſi ſingula ſint nulla.

SCHOLION.

896. Cum neque in his expreſſionibus pro momentis virium, P, Q, R inventis, neque in preſſione $\Pi = M\left(1 + \frac{f\,dd.cof\,\varrho}{2g\,dt^2}\right)$ radius ſphaerae e baſin conſtituentis inſit, omnia quae ſupra de motu turbinis infra in cuſpidem deſinentis ſunt tradita, etiam valent de ejusmodi turbinibus, qui deſinunt in hemiſphaerium ſeu aliam ſphaerae partem, dummodo punctum F, quod ante cuſpidem denotabat, hic in centro figurae ſphaericae G conſtituatur. Perinde ergo eſt, ſive turben gyretur ſuper cuſpide, ſive ſuper hemiſphaerio, dummodo f ſit diſtantia centri inertiae I a centro baſis ſphaericae; quantuscunque enim fuerit radius hujus baſis e, is in computum non ingreditur, eo autem evaneſcente baſis turbinis abit in cuſpidem. Totum igitur caput praecedens hic inferi intelligatur, ita ut Theoria turbinum ſine ullo labore haud mediocriter amplificata ſit cenſenda. Baſin autem ſphaericam faciendo caſus occurrit ante excluſus, ſcilicet quo centrum inertiae I fundo propius eſt, quam centrum ſphaericitatis, hicque ſit quantitas f negativa. Sive jam tale corpus ſit globus completus, ſive baſin habeat

Fig. 115.

MEN portionem ſphaerae, centro G deſcriptae, qua plano horizontali incumbat, ejus motum, quatenus contactus in hanc baſin cadit, inveſtigemus. Hic autem cogimur corpori talem indolem tribuere, ut axis proprius AGIF, qui ſi fuerit verticalis, ſtatum quietis exhibet, ſimul ſit corporis axis principalis, reliqui vero bini axes principales habeant momenta inter ſe aequalia. Scilicet ſi reſpectu axis IA momentum inertiae ſit $= Maa$ reſpectu binorum reliquorum vero Mbb, et Mcc, ſtatuemus $bb = cc$. Hujusmodi ergo corpus quemcunque receperit motum impreſſum, quomodo motum ſit continuaturum, determinemus.

PRO-

PROBLEMA. 106.

897. Si corpus basi sphaerica MFN instructum, in quo axis aequilibrii AGIF sit axis principalis, ejusque respectu momentum inertiae = M*aa*, respectu binorum reliquorum autem aequalia = M*cc*, motum acceperit quemcunque, determinare motus continuationem. Fig. 115.

SOLUTIO.

Sit radius basis sphaericae FG = *e*, centrum autem inertiae I cadat infra centrum basis G ad intervallum GI = *f*. Pro motu progressivo, si centrum inertiae I habuerit motum secundum directionem horizontalem, eum constantem in directum conservabit, quatenus autem motu verticali cietur, is cognita pressione, quae sit = Π, inde definietur, quod si declinatio axis AF a situ verticali ponatur = ϱ, sit $\frac{\Pi}{M} = 1 - \frac{f\,dd.\,cos\varrho}{2g\,dt^2}$, existente M corporis massa seu pondere. Verum ipsa haec pressio Π, quam corpus in planum horizontale exerit, non nisi ex motu gyratorio cognosci potest. Teneat ergo corpus nostrum respectu sphaerae fixae, in qua Z est punctum verticale, nunc elapso tempore *t* ejusmodi situm, ut ejus axes principales in A, B, C pertingant, ponanturque arcus ZA = *l*, ZB = *m*, ZC = *n*, tum vero anguli XZA = λ, XZB = μ, XZC = ν, ita ut sit *l* = ϱ. Nunc autem gyretur circa polum O in sensum ABC celeritate angulari = ϑ, ac positis arcubus OA = α, OB = ϐ, OC = γ sit brevitatis gratia ϑ *cos* α = *x*, ϑ *cos* ϐ = *y*, ϑ *cos* γ = *z*. Momenta autem virium ex pressione Π orta sunt P = 0, Q = − Π *f cos n*; R = + Π *f cos m*, unde colligimus sequentes aequationes: Fig. 111.

$$dx = 0;\quad dy + \frac{aa-cc}{cc}\,xz\,dt = \frac{-2\Pi f g\,dt\,cos\,n}{Mcc};\quad dz + \frac{cc-aa}{cc}\,xy\,dt = \frac{2\Pi f g\,dt\,cos\,m}{Mcc}$$

dl sin l = *dt* (*y cos n* − *z cos m*); *dλ sin l*² = − *dt* (*y cos m* + *z cos n*)
dm sin m = *dt* (*z cos l* − *x cos n*); reliqui anguli μ et ν hinc sponte
dn sin n = *dt* (*x cos m* − *y cos l*); dantur.

Si porro ponamus *cos l* = *p*, *cos m* = *q*, *cos n* = *r*, quoniam hae aequationes congruunt cum iis, quas supra probl. 99. integravimus, nisi quod *f* capiatur negative, habebimus in finitis has aequationes: *x* = ∫Λ et

I. $qy + rz = \mathrm{B} - \frac{Aaap}{cc}$

II. $(qz - ry)^2 = \frac{(Ccc+4fgp)(1-pp)-cc(B-\frac{Aaap}{cc})^2}{cc+ff(1-pp)}$

III. $yy + zz = \frac{(Ccc+4fgp+ff(B-\frac{Aaap}{cc})^2}{cc+ff(1-pp)}$

IV. $\frac{\Pi}{M} = \frac{2gcc-Afaa(B-\frac{Aaap}{cc})}{2g(cc+ff(1-pp)} + \frac{fccp(Ccc+4fgp+ff(B-\frac{Aaap}{cc})^2)}{2g(cc+ff(1-pp))^2}$

V. $dt = \frac{dp\,r(cc+ff(1-pp))}{r\left((Ccc+4fgp)(1-pp)-cc(B-\frac{Aaap}{cc})^2\right)}$

VI. $d\lambda = \frac{-dt}{1-pp}\left(B - \frac{Aaap}{cc}\right)$

VII. $ss = \mathrm{AA} + \frac{Ccc+4fgp+ff(B-\frac{Aaap}{cc})^2}{cc+ff(1-pp)}$

VIII. $\frac{ydz-zdy}{yy+zz} = \frac{A(aa-cc)dt}{cc} + \frac{2\Pi fg\,dt(B-\frac{Aaap}{cc})(cc+ff-ffpp)}{Mcc\left(Ccc+4fgp+ff(B-\frac{Aaap}{cc})^2\right)}$

nbi conſtantes A, B, C et reliquae per integrationem ingreſſurae ex ſtatu corporis initiali debent definiri.

COROLL. 1.

898. Si corpus initio quieverit, axisque principalis A fuerit in *a* declinatione ejus exiſtente $Za = l$ et $cos\, l = p$, initio erat $x = 0$, $y = 0$ et $z = 0$, ob $s = 0$; atque $p = p$. Fit ergo $A = 0$; $B = 0$, et $Ccc = -4fgp$: Hinc elapſo tempore t erit $x = 0$; $qy + rz = 0$;

$$qz - ry = \frac{2rfg(p-p)(1-pp)}{r(cc+ff(1-pp))}; \quad yy + zz = \frac{4fg(p-p)}{cc+ff(1-pp)} = ss, \text{ et } \frac{\Pi}{M} = \frac{cc}{cc+ff(1-pp)} + \frac{2ffccp(p-p)}{(cc+ff(1-pp))^2}$$

CO-

COROLL. 2.

899. Praeterea vero in eodem caſu eſt $d\lambda = 0$, ideoque axis, qui initio in a erat, per ipſum arcum aZ movebitur, eritque $dt = \frac{dp\,r\,(cc+ff(1-pp))}{2\,r\,fg(p-\mathfrak{p})(1-pp)}$, unde quia $p > \mathfrak{p}$ ſeu $l < 1$ axis ab a recta ad Z progreditur. Denique ob $ydz - zdy = 0$, fit $z = \delta y$, et $y = \frac{2\,r\,fg(p-\mathfrak{p})}{r\,(1+\delta\delta)(cc+ff(1-pp))}$: atque $q\,(yy+zz) = \frac{2z\,r\,fg(p-\mathfrak{p})(1-pp)}{r\,(cc+ff(1-pp))}$ ſeu $q = \frac{\delta\,r\,(1-pp)}{r\,(1+\delta\delta)}$ et $r = \frac{-r\,(1-pp)}{r\,(1+\delta\delta)}$, unde fit $cof\,ZAB = \frac{q}{r\,(1-pp)} = \frac{\delta}{r\,(1+\delta\delta)}$, qui ergo angulus manet conſtans.

COROLL. 3.

900. Si ergo corpus initio quieſcat, ejusque axis principalis IA tenuerit ſitum inclinatum Ia, inde recta ſe eriget ex a ad Z aſcendens, gyrabitur autem circa punctum O, ut ob $x = \varkappa\,cof\,\alpha = 0$ arcus AO ſit quadrans, et quia $cof\,ZO = cof\,\alpha\,cof\,l + cof\,\mathfrak{G}\,cof\,m + cof\,\gamma\,cof\,n = \frac{qy+rz}{\varkappa} = 0$, erit etiam ZO quadrans, ſicque O erit polus circuli XZY.

Et cum axis in Z pervenerit, erit celeritas angularis $\varkappa = \frac{2\,r\,fg(a-p)}{c}$.

SCHOLION. 1.

901. Si corpus initio non quieverit, ſed motum quemcunque acceperit, continuatio motus ex iisdem formulis determinatur, dummodo conſtantes A, B, C ſtatui initiali convenienter definiantur; ubi autem ad ejusmodi formulas integrandas devenitur, quae nonniſi conceſſis quadraturis ſuperioris ordinis expediri poſſunt. Quin etiam caſus hic ſimpliciſſimus, quo corpus initio in ſitu inclinato quievit, ab integratione formulae hujus $dt = \frac{dp\,r\,(cc+ff(1-pp))}{2\,r\,fg(p-\mathfrak{p})(1-pp)}$ pendet, quae neque per logarithmos neque arcus circulares abſolvi poteſt. At ſi declinatio initialis Za fuerit quaſi infinite parva, negotium ad arcus circulares perducitur: ſit enim initio $Za = l$, et elapſo tempore t declinatio $ZA = l$, ob l et l arcus minimos, erit $p = 1 - \frac{1}{2}ll$, $dp = -ldl$, et $\mathfrak{p} = 1 - \frac{1}{2}$ ll

ſi unde $dt = \frac{-c\,dl}{\sqrt{2fg(\Pi - l)}}$, et $t = \frac{c}{\sqrt{2fg}} \mathrm{A}\cos\frac{l}{\mathfrak{l}}$ ſeu $l = \mathfrak{l}\cos\frac{t\sqrt{2fg}}{c}$. Quare axis IA fiet verticalis elapſo tempore $= \frac{\pi c}{2\sqrt{2fg}}$, et corpus titubationes iſochronas conficiet; uti pendulum ſimplex longitudinis $= \frac{cc}{f}$.

SCHOLION. 2.

902. Niſi corpori ejusmodi indolem tribuiſſemus, ut ejus axis naturalis FD, qui in ſtatu quietis ſit verticalis, ſimul eſſet ejus axis principalis, binique reliqui haberent momenta inertiae aequalia, formulas quidem differentiales motum ejus continentes aſſignare, nullo autem modo ob analyſeos defectum ipſum motum definire potuiſſemus. Interim tamen, quemadmodum in caſu tractato, ubi corpori infinite parvam declinationem tribuimus, uſu venit, ut motus fieret ſatis ſimplex motuique penduli conformis, id adeo in genere locum habet, quomodocunque axes principales reſpectu axis naturalis fuerint diſpoſiti. In ſitu ſcilicet aequilibrii, ubi axis naturalis DF ſitum tenet verticalem, aſſumo centrum inertiae I infra centrum baſis ſphaericae G ad intervallum $GI = f$ cadere: tum vero hoc corpus infinite parum de ſitu ſuo quietis declinari ponamus, ut arcus $ZD = \varrho$ ſit infinite parvus, atque evidens eſt, corpus ſe reſtituendo oſcillationes ſeu titubationes eſſe peracturum, donec tandem motu ob reſiſtentiam extincto in ſtatu aequilibrii aequieſcat. Quoniam declinatio corporis hic perpetuo eſt minima, non opus eſt, ut tota corporis figura ſit ſphaerica, ſed ſufficit, ſi infima ejus portio eaque minima, qua plano horizontali applicatur, ſit pars ſuperficiei ſphaericae, cujus centrum eſt in G. Hunc igitur motum titubatorium inveſtigaturi primo diſpiciamus, quomodo formulae ſupra in genere erutae pro hoc caſu, quo axis corporis naturalis DF quam minime a ſitu verticali declinat, contrahi, indeque momenta virium P, Q, R ita commode definiri queant, ut deinceps ex iis motum aſſignare valeamus.

Fig. 114.

PROBLEMA. 107.

Fig. 114. 902. Si corpus baſi ſphaerica inſtructum infinite parum a ſitu aequilibrii declinet, definire momenta virium reſpectu ternorum ejus axium principalium.

SO-

SOLUTIO.

Circa corporis centrum inertiae I descripta sphaera, in qua Z sit punctum verticale, teneat axis corporis naturalis ID situm a verticali minime declinans, ut sit arcus $ZD = \varrho$ minimus: axes autem corporis principales respondeant punctis A, B, C, quorum situs ratione puncti D ita se habeat, ut sint arcus $DA = \zeta$, $DB = \eta$, $DC = \theta$, qui sunt constantes. Nunc autem respectu puncti verticalis Z sint arcus $ZA = l$, $ZB = m$, et $ZC = n$, qui ob arcum $ZD = \varrho$ minimum vix discrepabunt ab illis ζ, η, θ: quare si ponamus:

$$\cos l = \cos \zeta + p \; ; \; \cos m = \cos \eta + q \; ; \; \cos n = \cos \theta + r$$

quantitates p, q, r erunt minimae. Quia vero est tam $\cos \zeta^2 + \cos \eta^2 + \cos \theta^2 = 1$; quam $\cos l^2 + \cos m^2 + \cos n^2 = 1$, fiet

$$2p \cos \zeta + 2q \cos \eta + 2r \cos \theta + pp + qq + rr = 0.$$

Deinde autem cum sit $\cos \varrho = \cos \zeta \cos l + \cos \eta \cos m + \cos \theta \cos n$, erit $\cos \varrho = 1 + p \cos \zeta + q \cos \eta + r \cos \theta$, ideoque

$$p \cos \zeta + q \cos \eta + r \cos \theta = -\tfrac{1}{2} \varrho\varrho \text{ et } pp + qq + rr = \varrho\varrho.$$

Nunc igitur, posita pressione corporis in planum horizontale $= \Pi$, ex §. 894. tribuendo ipsi f valorem negativum, obtinebimus momenta virium respectu axium principalium:

$$P = \Pi f (r \cos \eta - q \cos \theta) \; ; \; Q = \Pi f (p \cos \theta - r \cos \zeta)$$
$$\text{et } R = \Pi f (q \cos \zeta - p \cos \eta),$$

tum vero vidimus esse $\Pi = M \left(1 - \frac{f d d . \cos \varrho}{2 g d t^2}\right)$; quia autem $\cos \varrho$ proxime est $= 1$, et minimas variationes subit, erit satis exacte $\Pi = M$, ita ut corpus toto suo pondere planum horizontale premere sit censendum: sicque habebimus

$$P = M f (r \cos \eta - q \cos \theta)$$
$$Q = M f (p \cos \theta - r \cos \zeta)$$
$$R = M f (q \cos \zeta - p \cos \eta).$$

PROBLEMA. 108.

903. Si corpus basi sphaerica praeditum de situ quietis, in quo axis DI est verticalis, parumper declinetur, iterumque dimittatur, ut ex quiete versus statum aequilibrii revertatur, determinare ejus motum.

SOLUTIO.

Elapso tempore t teneat corpus situm in fig. 114, repraesentatum, Fig. 114. maneantque omnes denominationes in probl. praecedente stabilitae:

tum vero ſint corporis momenta inertiae Maa, Mbb, Mcc reſpectu axium principalium IA, IB, IC. Nunc autem corpus gyretur circa axem IO in ſenſum ABC celeritate angulari $= \varkappa$, ſintque arcus AO $= \alpha$, BO $= \mathfrak{G}$, CO $= \gamma$: ac ponatur $\varkappa \cos \alpha = x$, $\varkappa \cos \mathfrak{G} = y$, $\varkappa \cos \gamma = z$. Quoniam igitur initio ubi $t = 0$, corpus ex quiete motum incipere aſſumitur, erat tum $x = 0$, $y = 0$, et $z = 0$. Tum vero quia motus corporis perpetuo manet tardiſſimus, quantitates x, y, z ſemper manebunt minimae, ita ut binarum producta xy, xz, et yz prae ſingulis pro evaneſcentibus haberi queant. Cum ergo momenta virium ſollicitantium P, Q, R, modo ſint definita ex §. 810. ſequentes adipiſcimur aequationes;

$$dx = \frac{2fg\,dt}{aa}(r\cos\eta - q\cos\theta)$$

$$dy = \frac{2fg\,dt}{bb}(p\cos\theta - r\cos\zeta)$$

$$dz = \frac{2fg\,dt}{cc}(q\cos\zeta - p\cos\eta).$$

Deinde quia eſt $\cos l = \cos\zeta + p$, $\cos m = \cos\eta + q$, et $\cos n = \cos\theta + r$ ob ζ, η, θ conſtantes erit $dl \sin l = -dp$; $dm \sin m = -dq$ et $dn \sin n = -dr$, unde inſuper hae ternae aequationes accedunt

$$-dp = dt(y\cos\theta - z\cos\eta);$$
$$-dq = dt(z\cos\zeta - x\cos\theta);$$
$$-dr = dt(x\cos\eta - y\cos\zeta);$$

ubi producta yr, zq, zp, xr, xq, yp ut minima prae terminis hic exhibitis omittimus.

Denique ſi arcus ZA a circulo quodam verticali fixo nunc declinare ſtatuatur angulo λ, ob $\sin l^2 = \sin\zeta^2 - 2p\cos\zeta$ habebimus hanc aequationem: $d\lambda = \dfrac{-dt(y\cos\eta + z\cos\theta)}{\sin\zeta^2 - 2p\cos\zeta}$. Quia autem in ſuperioribus aequationibus quantitates x, y, z et p, q, r ubique unam dimenſionem occupant, atque x, y, z poſito $t = 0$ evaneſcere debent, manifeſtum eſt, tam huic conditioni, quam ſex illis aequationibus ſatisfieri poſſe ponendo:

$$x = A\sin\delta t,\ y = B\sin\delta t;\ z = C\sin\delta t;$$
$$p = D\cos\delta t;\ q = E\cos\delta t,\ r = F\cos\delta t,$$

tum enim ternae priores aequationes per $\cos\delta t$ diviſae, et ternae poſteriores per $\sin\delta t$ diviſae dabunt.

$$A\delta = \frac{2fg}{aa}(F\cos\eta - E\cos\theta);\ D\delta = B\cos\theta - C\cos\eta$$

$$B\delta = \frac{2fg}{bb}(D\cos\theta - F\cos\zeta);\ E\delta = C\cos\zeta - A\cos\theta$$

Cδ

$$C\delta = \frac{2fg}{cc}(E\ cof\ \zeta - D\ cof\ \eta);\quad F\delta = A\ cof\ \eta - B\ cof\ \zeta.$$

Ex poſterioribus ſubſtituantur valores coefficientium D, E, F in prioribus et obtinebimus:

$$\frac{A\delta\delta aa}{2fg} = A\ cof\ \eta^2 - B\ cof\ \zeta\ cof\ \eta - C\ cof\ \zeta\ cof\ \theta + A\ cof\ \theta^2$$

$$\frac{B\delta\delta bb}{2fg} = B\ cof\ \theta^2 - C\ cof\ \eta\ cof\ \theta - A\ cof\ \zeta\ cof\ \eta + B\ cof\ \zeta^2$$

$$\frac{C\delta\delta cc}{2fg} = C\ cof\ \zeta^2 - A\ cof\ \zeta\ cof\ \theta - B\ cof\ \eta\ cof\ \theta + C\ cof\ \eta^2.$$

Quodſi jam brevitatis gratia ponamus A $cof\ \zeta$ + B $cof\ \eta$ + C $cof\ \theta$ = G, ob $cof\ \zeta^2 + cof\ \eta^2 + cof\ \theta^2 = 1$ erit

$$A\left(1 - \frac{\delta\delta aa}{2fg}\right) = G\ cof\ \zeta,\quad B\left(1 - \frac{\delta\delta bb}{2fg}\right) = G\ cof\ \eta$$

$$\text{et}\quad C\left(1 - \frac{\delta\delta cc}{2fg}\right) = G\ cof\ \theta.$$

Ponamus brevitatis cauſa $\frac{\delta\delta}{2fg} = u$; ut fiat

$$A = \frac{G\ cof\ \zeta}{1 - aau};\quad B = \frac{G\ cof\ \eta}{1 - bbu};\quad C = \frac{G\ cof\ \theta}{1 - ccu}.$$

Cum autem ſit A $cof\ \zeta$ + B $cof\ \eta$ + C $cof\ \theta$ = G, erit

$$\frac{cof\ \zeta^2}{1 - aau} + \frac{cof\ \eta^2}{1 - bbu} + \frac{cof\ \theta^2}{1 - ccu} = 1,$$

qua aequatione evoluta conſequimur per u dividendo,

$$\left.\begin{aligned} aa\,bb\,cc\,uu - bb\,cc\,u\,ſi\,\zeta^2 + aa\ cof\ \zeta^2 \\ - aa\,cc\,u\,ſi\,\eta^2 + bb\ cof\ \eta^2 \\ - aa\,bb\,u\,ſi\,\theta^2 + cc\ cof\ \theta^2 \end{aligned}\right\} = 0.$$

Statuantur quantitates cognitae

$$bb\,cc\,ſi\,\zeta^2 + aa\,cc\,ſi\,\eta^2 + aa\,bb\,ſi\,\theta^2 = K\,aa\,bb\,cc$$

$$aa\ cof\ \zeta^2 + bb\ cof\ \eta^2 + cc\ cof\ \theta^2 = L\,aa\,bb\,cc$$

ut ſit $uu - Ku + L = 0$, hincque

$$u = \frac{\delta\delta}{2fg} = \tfrac{1}{2}K + \sqrt{(\tfrac{1}{4}KK - L)}$$

et quantitas G manet indefinita ex ſtatu initiali definienda dum contra

tra quantitates K et L ſunt ex natura corporis datae. Cum igitur hinc inventus ſit valor ipſius u, inde habemus $\delta = r^{-2fgu}$, et

$$A = \frac{G\,cof\zeta}{1-aau}; \quad B = \frac{G\,cof\eta}{1-bbu}; \quad C = \frac{G\,cof\theta}{1-ccu}$$

$$D = \frac{Gu(bb-cc)\,cof\eta\,cof\theta}{\delta(1-bbu)(1-ccu)}; \quad E = \frac{Gu(cc-aa)\,cof\zeta\,cof\theta}{\delta(1-ccu)(1-aau)}$$

$$\text{et } F = \frac{Gu(aa-bb)\,cof\zeta\,cof\eta}{\delta(1-aau)(1-bbu)}.$$

Si jam initio fuerit arcus ZD = r, qui nunc eſt ϱ, cum ſit initio $p = D$: $q = E$; $r = F$, habebimus

$$DD + EE + FF = rr,$$

unde per r invenitur conſtans G. Denique pro angulo λ inveniendo prodit $d\lambda = \frac{-dt\,(B\,cof\eta + C\,cof\theta)\,ſin\,\delta t}{ſi\zeta^2}$, ideoque $d\lambda = \frac{(B\,cof\eta + C\,cof\theta)(cof\delta t - 1)}{\delta ſi\zeta^2}$ ſi quidem arcus ZA initio fuerit in verticali ſixo, indeque in ſenſum XOY moveri ſumatur, quatenus ergo haec expreſſio pro λ eſt negativa, in ſenſum contrarium axis IA circa Z gyrari eſt cenſendus.

Denique cum ſit $pp + qq + rr = \varrho\varrho$, erit $\varrho = r\,cof\,\delta t$, ob $r = r^{-(DD + EE + FF)}$, unde patet axem ID in ſitum verticalem erigi elapſo tempore $= \frac{\pi}{2\delta}$ et titubationes iſochronas fore oſcillationibus penduli, cujus longitudo eſt $= \frac{2g}{\delta\delta} = \frac{1}{fu} = \frac{K - r^{-(KK-4L)}}{2Lf}$.

COROLL. 1.

904. Cum ſit $DD + EE + FF = rr$, erit

$$\delta\delta\,rr = AA\,(cof\eta^2 + cof\theta^2) + BB\,(cof\zeta^2 + cof\theta^2) + CC\,(cof\zeta^2 + cof\eta^2) - 2BC\,cof\eta\,cof\theta - 2AC\,cof\zeta\,cof\theta - 2AB\,cof\zeta\,cof\eta$$

et quia $G = A\,cof\zeta + B\,cof\eta + C\,cof\theta$, hujus quadratum eo additum dabit

$$\delta\delta\,rr + GG = AA + BB + CC,$$

ubi

ubi ſi brevitatis gratia ponatur $\frac{1}{1-aau} = \mathfrak{P}$; $\frac{1}{1-bbu} = \mathfrak{Q}$; $\frac{1}{1-ccu} = \mathfrak{R}$, ob $\mathfrak{P} \cos\zeta^2 + \mathfrak{Q} \cos\eta^2 + \mathfrak{R} \cos\theta^2 = 1$ et $A = G\mathfrak{P} \cos\zeta$; $B = G\mathfrak{Q} \cos\eta$ et $C = G\mathfrak{R} \cos\theta$, fiet

$$\delta\delta\, rr = GG\,(\mathfrak{P}\mathfrak{P} \cos\zeta^2 + \mathfrak{Q}\mathfrak{Q} \cos\eta^2 + \mathfrak{R}\mathfrak{R} \cos\theta^2 - 1)$$

ideoque ob $\mathfrak{P}\mathfrak{P} - \mathfrak{P} = \frac{aau}{(1-aau)^2}$ habebitur

$$\int\!\!\int rr = GG\,u\left(\frac{aa\cos\zeta^2}{(1-aau)^2} + \frac{bb\cos\eta^2}{(1-bbu)^2} + \frac{cc\cos\theta^2}{(1-ccu)^2}\right).$$

COROLL. 2.

905. Quia porro eſt $\delta\delta = 2fgu$, ſi in ſubſidium vocetur aequatio $uu - Ku + L = 0$, reperietur

$$GG = \frac{-2fgrr(1-aau)(1-bbu)(1-ccu)}{aa\cos\zeta^2 + bb\cos\eta^2 + cc\cos\theta^2 - aabbccuu}.$$

EXPLICATIO.

906. Haec expreſſio pro GG ſatis concinna ſequenti modo eruitur: Poſito brevitatis gratia $\frac{1}{aa} = a$, $\frac{1}{bb} = b$, $\frac{1}{cc} = c$, habemus:

I. $K = a + b + c - a\cos\zeta^2 - b\cos\eta^2 - c\cos\theta^2$

II. $L = bb\cos\zeta^2 + aa\cos\eta^2 + ab\cos\theta^2$

III. $1 = \cos\zeta^2 + \cos\eta^2 + \cos\theta^2$.

Hinc deducitur ob $uu - Ku + L = 0$

$$\cos\zeta^2 = \frac{aK - L - aa}{(a-b)(c-a)}, \text{ et } u\cos\zeta^2 = \frac{(a-u)(L-au)}{(a-b)(c-a)}$$

ideoque

$$\frac{\delta\delta rr}{GG} = \frac{a(L-au)}{(a-b)(c-a)(a-u)} + \frac{b(L-cu)}{(b-c)(a-b)(b-u)} + \frac{c(L-cu)}{(c-a)(b-c)(c-u)}$$

ex qua aequatione reducta illa expreſſio obtinetur.

SCHOLION.

907. Quoniam haec ad titubationes omnium corporum, quorum baſis eſt portio ſphaerica, patent, quomodocunque ejus axes principales ratione axis naturalis DGIF fuerint diſpoſiti, eorumque reſpectu momenta inertiae inaequalia, ne in tanta amplitudine confundamur,

conveniet primo formulas noſtras ad ſpecies corporum ſimpliciores accommodari, quo inde facilius ad ſpecies magis complicatas progredi liceat. Ac primo quidem caſus, quo omnia momenta inertiae ſunt inter ſe aequalia, ſeu $aa = bb = cc$, omnium eſt ſimpliciſſimus, quia tum etiam axis DF pro principali haberi poteſt, et titubationes eaedem prodire debent, quas jam ante definivimus. Tum vero duo ſaltem momenta inertiae aequalia ſtatuamus, ſcilicet $bb = cc$.

CASUS I.

quo $aa = bb = cc$.

908. Hoc ergo caſu habemus:

$$A = \frac{G\,cof\zeta}{1-aau}; \quad B = \frac{G\,cof\eta}{1-aau}; \quad C = \frac{G\,cof\theta}{1-aau},$$

hincque $G = \frac{G\,cof\zeta^2 + G\,cof\eta^2 + G\,cof\theta^2}{1-aau} = \frac{G}{1-aau}$, ita ut ſit $u = 0$.

Verum iisdem quoque formulis ſatisfit ponendo $u = \frac{1}{aa}$ et $G = 0$, ut ſit $A\,cof\xi + B\,cof\eta + C\,cof\theta = 0$, neque quicquam praeterea determinetur, ſicque habebimus $\delta = \frac{\Gamma 2fg}{a}$: tum vero

$$D = \frac{B\,cof\theta - C\,cof\eta}{\delta}; \quad E = \frac{C\,cof\zeta - A\,cof\theta}{\delta}; \quad F = \frac{A\,cof\eta - B\,cof\zeta}{\delta}$$

atque $\int\!\!\int rr = AA + BB + CC$: ut ſit $\varrho = r\,cof\,\delta t$. Videamus jam circa quemnam polum O corpus ſit gyraturum, ac primo habemus $cof\,OD = cof\,\alpha\,cof\,\zeta + cof\,\beta\,cof\,\eta + cof\,\gamma\,cof\,\theta$, ſeu

$\nu\,cof\,OD = x\,cof\,\zeta + y\,cof\,\eta + z\,cof\,\theta = 0$, ſicque arcus OD quadrans. Deinde eſt $cof\,OZ = cof\,\alpha\,cof\,l + cof\,\beta\,cof\,m + cof\,\gamma\,cof\,n$ ſeu

$$\nu\,cof\,OZ = x\,cof\,l + y\,cof\,m + z\,cof\,n = 0 + px + qy + rz = 0$$

ob $AD + BE + CF = 0$, eritque ergo etiam OZ quadrans. Ex quo perſpicitur corpus circa punctum O, quod eſt polus circuli verticalis ZDX gyrari, ſicque axem ex D recta in ſitum verticalem Z erigi, ita ut elapſo tempore t ſit $\varrho = r\,cof\,\frac{t\Gamma 2fg}{a}$. Quare hae titubationes iſochronae erunt oſcillationibus penduli, cujus longitudo eſt $= \frac{aa}{f}$.

CA-

CASUS. II.

quo duo tantum momenta principalia sunt aequalia seu $bb = cc$.

909. Hoc ergo casu est $K = \frac{cc\,si\,\zeta^2 + aa\,si\,\eta^2 + aa\,si\,\theta^2}{aacc} = \frac{cc\,si\,\zeta + aa + aa\,cos\,\zeta^2}{aacc}$, et $L = \frac{aa\,cos\,\zeta^2 + cc\,si\,\zeta^2}{aac^4}$: sive cum aequatio, unde u definiri debet, sit $\frac{cos\,\zeta^2}{1-aau} + \frac{si\,\zeta^2}{1-ccu} = 1$, erit

$$aa\,cos\,\zeta^2 + cc\,si\,\zeta^2 = aaccu, \text{ ideoque } u = \frac{aa\,cos\,\zeta^2 + cc\,si\,\zeta^2}{aacc}$$

qui valor etiam ex generali forma elicitur, nisi quod hoc modo radix inutilis $u = \frac{1}{cc}$ excluditur. Quamobrem habebimus

$$\delta = \frac{\sqrt{2fg}(aa\,cos\,\zeta^2 + cc\,si\,\zeta^2)}{ac}, \text{ tum vero}$$

$$A = \frac{Gcc}{(aa+cc)\,cos\,\zeta};\ B = \frac{Gaa\,cos\,\eta}{(aa-cc)\,si\,\zeta^2};\ C = \frac{Gaa\,cos\,\theta}{(aa-cc)\,si\,\zeta^2}.$$

Deinde pro G ex $\mathfrak{r}$ inveniendo fit

$$\delta\delta\,\mathfrak{r}\mathfrak{r} + GG = \frac{GG(a^4\,cos\,\zeta^2 + c^4\,si\,\zeta^2)}{(aa-cc)^2\,si\,\zeta^2\,cos\,\zeta^2} \text{ sive}$$

$$\delta\delta\,\mathfrak{r}\mathfrak{r} = \frac{GG(aa\,cos\,\zeta^2 + cc\,si\,\zeta^2)^2}{(aa-cc)^2\,si\,\zeta^2\,cos\,\zeta^2}, \text{ et } G = \frac{(aa-cc)\,\delta\mathfrak{r}\,\zeta\,si\,cos\,\zeta}{aa\,cos\,\zeta^2 + cc\,si\,\zeta^2},$$

vel $G = \frac{(aa-cc)\,\mathfrak{r}\,si\,\zeta\,cos\,\zeta\sqrt{2fg}}{ac\sqrt{(aa\,cos\,\zeta^2 + cc\,si\,\zeta^2)}}$. Deinde vero obtinemus

$$D = 0,\ E = \frac{\mathfrak{r}\,cos\,\theta}{si\,\zeta};\ F = \frac{-\mathfrak{r}\,cos\,\eta}{si\,\zeta}; \text{ atque}$$

$$A = \frac{-c\mathfrak{r}\,si\,\zeta\sqrt{2fg}}{a\sqrt{(aa\,cos\,\zeta^2 + cc\,si\,\zeta^2)}};\ B = \frac{a\mathfrak{r}\,cos\,\zeta\,cos\,\eta\sqrt{2fg}}{c\,si\,\zeta\sqrt{(aa\,cos\,\zeta^2 + cc\,si\,\zeta^2)}};$$

$$C = \frac{a\mathfrak{r}\,cos\,\zeta\,cos\,\theta\sqrt{2fg}}{c\,si\,\zeta\sqrt{(aa\,cos\,\zeta^2 + cc\,si\,\zeta^2)}}$$

ex quibus consequimur

$x = \varkappa \cos\alpha = A\sin\delta t$; $y = \varkappa \cos\mathfrak{C} = B\sin\delta t$; $z = \varkappa \cos\gamma = C\sin\delta t$
$p = \cos\lambda - \cos\zeta = D\cos\delta t$; $q = \cos\mu - \cos\eta = E\cos\delta t$; $r = \cos\nu - \cos\theta = F\cos\delta t$,

atque $\lambda = \dfrac{-\alpha\gamma\sin\zeta\cos\zeta(1-\cos\delta t)\gamma 2fg}{\delta c\gamma(aa\cos\zeta^2+cc\sin\zeta^2)} = \dfrac{-aa\gamma\sin\zeta\cos\zeta(1-\cos\delta t)}{aa\cos\zeta^2+cc\sin\zeta^2}$

eſque λ angulus VZA, exiſtente ZV circulo verticali fixo, a quo declinationem poli A computamus. Deinde vero eſt $\varrho = r\cos\delta t$, et ut obtineamus angulum DZV, quaeramus angulum DZA, ex formula $\cos DZA = \dfrac{\cos\zeta - \cos\lambda\cos\varrho}{\varrho\sin\lambda} = \dfrac{\cos\zeta - \cos\zeta\cos\varrho - p\cos\varrho}{\varrho\sin\zeta} = \frac{1}{2}\varrho\dfrac{\cos\zeta}{\sin\zeta}$,

ob $D = 0$ ideoque $p = 0$, ergo $\cos DZA = \dfrac{r\cos\zeta\cos\delta t}{2\sin\zeta}$, qui cum ſit infinite parvus, patet angulum DZA eſſe rectum proxime et angulo ZDA aequalem. Quare cum initio angulus ZDA fuerit non rectus, haec ſolutio, quippe quae manifeſto tantum eſt particularis, eo non extenditur. Ceterum vero et hae titubationes erunt iſochronae oſcillationibus penduli, cujus longitudo eſt $= \dfrac{aacc}{f(aa\cos\zeta^2+cc\sin\zeta^2)}$.

Denique cum ſit $\varkappa = \sin\delta t.\gamma(AA + BB + CC)$, prodibit

$$\varkappa = \frac{r\gamma 2fg(a^4\cos\zeta^2+c^4\sin\zeta^2)}{ac\gamma(aa^3\cos\zeta^2+cc\sin\zeta^2)}.\ \sin\delta t.$$

Pro polo autem gyrationis O invenimus:

$\varkappa\cos OD = (A\cos\zeta + B\cos\eta + C\cos\theta)\sin\delta t = G\sin\delta t$ et
$\varkappa\cos OZ = (A\cos\lambda + B\cos\mu + C\cos\nu)\sin\delta t = G\sin\delta t$.

ita ut ſit $OD = OZ$, ob $Ap + Bq + Cr = 0$.

SCHOLION.

910. Mirum non eſt, hanc ſolutionem non eſſe generalem, cum enim ex data indole corporis, quantitatibus aa, bb, cc et angulis ζ, η, θ comprehenſa, et ex ſitu axis DF initiali ejusve declinatione r a ſitu verticali omnes coefficientes A, B, C, D, E, F cum numero δ determinentur, ex iis angulus ADZ, quo arcus DA initio ab arcu DZ deviabat, ſponte determinatur, neque amplius arbitrio noſtro, uti natura rei poſtulat, relinquitur. Verum cum in genere pro quantitate u geminum valorem elicuerimus, quorum neutrum prae altero rejicere fas eſt, ſi utrumque ſimul adhibeamus, ſolutionem ampliorem obtinebimus,

bimus, unde ſimul effici poteſt, ut angulus ADZ initio fuerit dato angulo aequalis. Cum enim in aequationibus differentialibus quantitates x, y, z, et p, q, r ubique unicam habeant dimenſionem, ſi iis duplici modo ſatisfieri queat, pro qualibet quantitate ſumma binorum ejus valorum ſtatui poterit, hincque ſolutionem generalem impetrabimus, quam hic exponamus.

PROBLEMA. 109.

911. Si corpus baſi ſphaerica praeditum de ſitu aequilibrii quomodocunque infinite parum declinetur, ſubitoque dimittatur, definire motum titubatorium, quo agitabitur.

SOLUTIO.

Retentis denominationibus ſuperioris problematis, quoniam poſito

$$\frac{\sin\zeta^2}{aa} + \frac{\sin\eta^2}{bb} + \frac{\sin\theta^2}{cc} = \mathrm{K} \text{ et } \frac{\cos\zeta^2}{bbcc} + \frac{\cos\eta^2}{aacc} + \frac{\cos\theta^2}{aabb} = \mathrm{L}$$

pro u geminum invenimus valorem; ſint ii

$$u = \tfrac{1}{2}\mathrm{K} + r\sqrt{(\tfrac{1}{4}\mathrm{KK} - \mathrm{L})} \text{ et } u' = \tfrac{1}{2}\mathrm{K} - r\sqrt{(\tfrac{1}{4}\mathrm{KK} - \mathrm{L})}$$

unde pro δ etiam binos adipiſcimur valores, qui ſint $\delta = r\sqrt{2fgu}$ et $\delta' = r\sqrt{2fgu'}$

atque hinc pro ſenis quantitatibus x, y, z et p, q, r ſequentes impetrabimus valores

$$x = \varkappa\cos\alpha = \frac{G\cos\zeta\sin\delta t}{1-aau} + \frac{H\cos\zeta\sin\delta' t}{1-aau'}$$

$$y = \varkappa\cos\mathfrak{b} = \frac{G\cos\eta\sin\delta t}{1-bbu} + \frac{H\cos\eta\sin\delta' t}{1-bbu'}$$

$$z = \varkappa\cos\gamma = \frac{G\cos\theta\sin\delta t}{1-ccu} + \frac{H\cos\theta\sin\delta' t}{1-ccu'}$$

tum vero porro

$$p = \cos\lambda - \cos\zeta = \frac{Gu(bb-cc)\cos\eta\cos\theta\cos\delta t}{\delta(1-bb)(1-ccu)} + \frac{Hu'(bb-cc)\cos\eta\cos\theta\cos\delta' t}{\delta'(1-bbu')(1-ccu')}$$

$$q = \cos m - \cos \eta = \frac{G u (cc - aa) \cos\zeta \cos\theta \cos\delta t}{\delta (1 - cc u)(1 - aa u)} +$$
$$\frac{H u' (cc - aa) \cos\zeta \cos\theta \cos\delta' t}{\delta' (1 - cc u')(1 - aa u')}$$
$$r = \cos n - \cos\theta = \frac{G u (aa - bb) \cos\zeta \cos\eta \cos\delta t}{\delta (1 - aa u)(1 - bb u)} +$$
$$\frac{H u' (aa - bb) \cos\zeta \cos\eta \cos\delta' t}{\delta' (1 - aa u')(1 - bb u')}.$$

Hic jam habemus binas quantitates conſtantes arbitrarias G et H, atque hi valores ita ſatisfaciunt, ut facta ſubſtitutione in aequationibus differentialibus termini tam per G quam per H affecti ſeorſim ſe deſtruant. Verum ſi initio arcus ZD fuerit $= r$, poſito $t = 0$, fieri debet $pp + qq + rr = rr$. Deinde vero ſi initio fuerit angulus ZDA $= f$, ob

$$\cos f = \frac{\cos l - \cos\zeta \cos r}{\sin\zeta \sin r} = \frac{\cos\zeta (1 - \cos r) + p}{\sin\zeta \sin r} = \frac{r \cos\zeta}{\sin\zeta} + \frac{p}{r \sin\zeta}$$ et ob

r infinite parvum, erit $p = r \sin\zeta \cos f$. Si hic ergo pro p ejus valor ſuperior poſito $t = 0$ ſubſtituatur, habebitur alia aequatio ex qua cum illa conjuncta binae conſtantes G et H determinabuntur. At poſito angulo VZA $= \lambda$ erit $d\lambda = \frac{-dt(y \cos\eta + z \cos\theta)}{\sin\zeta^2}$, cujus integrale facile exhibetur. Simili autem modo poſitis angulis VZB $= \mu$ et VZC $= \nu$, erit

$d\mu = \frac{-dt(z \cos\theta + x \cos\zeta)}{\sin\eta^2}$; et $d\nu = \frac{-dt(x \cos\zeta + y \cos\eta)}{\sin\theta^2}$.

Hic autem notari convenit, ſi ſit $bb = cc$, fore binos valores $u = \frac{1}{cc}$ et $u' = \frac{\sin\zeta^2}{aa} + \frac{\cos\zeta^2}{cc}$, ideoque pro priore fractionum ſuperiorum quasdam numeratores ac denominatores ſimul evaneſcere. Ad earum ergo valores inveſtigandos ponatur $\frac{1}{bb} = \frac{1}{cc} + \omega$, exiſtente ω quantitate evaneſcente, reperieturque

$$u = \frac{1}{cc} + \frac{\omega \cos\theta^2}{\sin\zeta^2} \text{ et } u' = \frac{\sin\zeta^2}{aa} + \frac{\cos\zeta^2}{cc}; \text{ hincque ſi}$$

$\frac{G}{\omega}$

$\frac{G}{\omega}$ ponatur $= I$, ut fit $G = I\omega = o$, fiet

$$x = \varkappa \cos\alpha = \frac{H \cos\zeta \sin\delta^N t}{1 - aau'} = \frac{-Hcc\sin\delta^N t}{(aa-cc)\cos\zeta}$$

$$y = \varkappa \cos\mathfrak{C} = \frac{I\sin\zeta^2 \sin\delta t}{cc\cos\eta} + \frac{H\cos\eta\sin\delta^N t}{1-cc u'} = \frac{I\sin\zeta^2\sin\delta t}{cc\cos\eta} + \frac{Haa\cos\eta\sin\delta^N t}{(aa-cc)\sin\zeta^2}$$

$$z = \varkappa \cos\gamma = \frac{-I\sin\zeta^2\sin\delta t}{cc\cos\theta} + \frac{H\cos\theta\sin\delta^N t}{1-cc u'} = \frac{-I\sin\zeta^2\sin\delta t}{cc\cos\theta} + \frac{Haa\cos\theta\sin\delta^N t}{(aa-cc)\sin\zeta^2}$$

deinde vero

$$p = \cos l - \cos\zeta = \frac{I\sin\zeta^4\cos\delta t}{\delta cc\cos\eta\cos\theta}$$

$$q = \cos m - \cos\eta = \frac{-I\sin\zeta^2\cos\zeta\cos\delta t}{\delta cc\cos\theta} - \frac{Hu'(aa-cc)\cos\zeta\cos\theta\cos\delta^N t}{\delta(1-aau')(1-cc u')}$$

$$r = \cos n - \cos\theta = \frac{-I\sin\zeta^2\cos\zeta\cos\delta t}{\delta cc\cos\eta} + \frac{Hu'(aa-cc)\cos\zeta\cos\eta\cos\delta^N t}{\delta'(1-aau')(1-cc u')}$$

ubi est $\frac{aa-cc}{(1-aau')(1-cc u')} = \frac{-aacc}{(aa-cc)\sin\zeta^2\cos\zeta^2}$.

Vel si ponamus $I = \frac{\mathfrak{G}\delta cc\cos\eta\cos\theta}{\sin\zeta^2}$ et $H = \frac{\mathfrak{H}\delta^N(aa-cc)\sin\zeta^2\cos\zeta}{aacc}$

ob $u' = \frac{1}{cc}$, et $u' = \frac{aa\cos\zeta^2 + cc\sin\zeta^2}{aacc}$, ideoque $\delta = \frac{r^2 fg}{c}$ et $\delta' = \frac{r^2 fg(aa\cos\zeta^2+cc\sin\zeta^2)}{ac}$

erit $x = \varkappa\cos\alpha = \frac{-\mathfrak{H}\delta^N\sin\zeta^2\sin\delta t}{aa}$

$$y = \varkappa \cos 6 = \frac{\mathfrak{H}\,\delta' \cos\zeta \cos\eta \sin \delta' t}{cc} + \mathfrak{G}\,\delta' \cos\theta \sin \delta t$$

$$z = \varkappa \cos \gamma = \frac{\mathfrak{H}\,\delta' \cos\zeta \cos\theta \sin \delta' t}{cc} - \mathfrak{G}\,\delta \cos\eta \sin \delta t$$

$$p = \cos l - \cos\zeta = \mathfrak{G}\,\mathfrak{f}\,\zeta^2 \cos \delta t$$
$$q = \cos m - \cos\eta = -\mathfrak{G} \cos\zeta \cos\eta \cos \delta t + \mathfrak{H}\,u' \cos\theta \cos \delta' t$$
$$r = \cos n - \cos\eta = -\mathfrak{G} \cos\zeta \cos\theta \cos \delta t - \mathfrak{H}\,u' \cos\eta \cos \delta' t$$

quae formulae jam ſine ulla difficultate ad omnes caſus accommodari poſſunt.

COROLL. 1.

912. Haec integralia adhuc latius extendi poſſunt, cum x, y, z et p, q, r partes conſtantes recipiant; ac forma litterarum G et H mutata, habebimus:

$$x = \cos\zeta\,(\mathfrak{E} + \mathfrak{G}\,(1 - bbu)(1 - ccu) \sin \delta t + \mathfrak{H}\,(1 - bbu')(1 - ccu') \sin \delta' t)$$

$$y = \cos\eta\,(\mathfrak{E} + \mathfrak{G}\,(1 - aau)(1 - ccu) \sin \delta t + \mathfrak{H}\,(1 - aau')(1 - ccu') \sin \delta' t)$$

$$z = \cos\theta\,(\mathfrak{E} + \mathfrak{G}\,(1 - aau)(1 - bbu) \sin \delta t + \mathfrak{H}\,(1 - aau')(1 - bbu') \sin \delta' t)$$

atque

$$p = \mathfrak{F} \cos\zeta + (bb - cc) \cos\eta \cos\theta \left(\frac{\mathfrak{G}\,u(1 - aau) \cos \delta t}{\delta} + \frac{\mathfrak{H}\,u'(1 - aau') \cos \delta' t}{\delta'} \right)$$

$$q = \mathfrak{F} \cos\eta + (cc - aa) \cos\zeta \cos\theta \left(\frac{\mathfrak{G}\,u(1 - bbu) \cos \delta t}{\delta} + \frac{\mathfrak{H}\,u'(1 - bbu') \cos \delta' t}{\delta'} \right)$$

$$r = \mathfrak{F} \cos\theta + (aa - bb) \cos\zeta \cos\eta \left(\frac{\mathfrak{G}\,u(1 - ccu) \cos \delta t}{\delta} + \frac{\mathfrak{H}\,u'(1 - ccu') \cos \delta' t}{\delta'} \right).$$

CO-

COROLL. 2.

913. Angulorum etiam δt et $\delta' t$ uterque quantitate conſtante augeri poteſt, ac ſi eorum loco ſcribamus $\delta t + \mathfrak{g}$ et $\delta t + \mathfrak{h}$, integralia continebunt ſex conſtantes arbitrarias $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, ideoque erunt integralia completa harum ſex aequationum differentialium:

$$aadx = 2fgdt\,(r\,cof\,\eta - q\,cof\,\theta);\quad dp = dt\,(z\,cof\,\eta - y\,cof\,\theta)$$
$$bbdy = 2fgdt\,(p\,cof\,\theta - r\,cof\,\zeta);\quad dq = dt\,(x\,cof\,\theta - z\,cof\,\zeta)$$
$$ccdz = 2fgdt\,(q\,cof\,\zeta - p\,cof\,\eta);\quad dr = dt\,(y\,cof\,\zeta - x\,cof\,\eta).$$

COROLL. 3.

914. Si corpus initio quieverit, ut in problemate aſſumſimus, ita ut tum fuerit $x = 0$, $y = 0$ et $z = 0$; poni debet $\mathfrak{E} = 0$, $\mathfrak{g} = 0$, et $\mathfrak{h} = 0$, reliquas autem conſtantes ex ſitu corporis initiali definiri oportet

COROLL. 4.

915. Nempe ſi pro initio, quo $t = 0$, ponantur anguli $ZDA = \lambda$, $ZDB = \mu$, et $ZDC = \nu$; ut ſit

$$fi\,(\lambda - \mu) = \frac{-cof\,\theta}{fi\,\zeta\,fi\,\eta};\quad fi\,(\mu - \nu) = \frac{-cof\,\zeta}{fi\,\eta\,fi\,\theta};\quad fi\,(\nu - \lambda) = \frac{-cof\,\eta}{fi\,\zeta\,fi\,\theta}$$

$$cof\,(\lambda - \mu) = \frac{-cof\,\zeta\,cof\,\eta}{fi\,\zeta\,fi\,\eta};\quad cof\,(\mu - \nu) = \frac{-cof\,\eta\,cof\,\theta}{fi\,\eta\,fi\,\theta};$$

$$cof\,(\nu - \lambda) = \frac{-cof\,\zeta\,cof\,\theta}{fi\,\zeta\,fi\,\theta}$$

pro initio $t = 0$, conſtantes ita definiri oportet, ut ſi tum fuerit $ZD = r$, fiat

$$p = r\,fi\,\zeta\,cof\,\lambda;\quad q = r\,fi\,\eta\,cof\,\mu;\quad r = r\,fi\,\theta\,cof\,\nu.$$

EXPLICATIO.

916. Ad conſtantes $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ in genere ex ſtatu initiali modo deſcripto definiendas, ponamus brevitatis gratia

$$aa\,cof\,\zeta^2 + bb\,cof\,\eta^2 + cc\,cof\,\theta^2 = \mathfrak{A}$$
$$bb\,cc\,cof\,\zeta^2 + aa\,cc\,cof\,\eta^2 + aa\,bb\,cof\,\theta^2 = \mathfrak{B}$$

ſitque

fitque $\frac{\mathfrak{G} u \cos \mathfrak{g}}{\mathfrak{d}} = X$, et $\frac{\mathfrak{H} u' \cos \mathfrak{h}}{\mathfrak{d}} = Y$, quo calculus facilius expediatur.

Eo autem absoluto reperietur

$$\mathfrak{F} = r \sin \zeta \cos \zeta \cos \mathfrak{l} + r \sin \eta \cos \eta \cos \mathfrak{m} + r \sin \theta \cos \theta \cos \mathfrak{n}$$

$$X + Y = \frac{+r\sin\zeta\cos\mathfrak{l}}{\cos\eta\cos\theta}(\mathfrak{B} - bbcc) + \frac{r\sin\eta\cos\mathfrak{m}}{\cos\zeta\cos\theta}(\mathfrak{B} - aacc) + \frac{r\sin\theta\cos\mathfrak{n}}{\cos\zeta\cos\eta}(\mathfrak{B} - aabb)$$
$$\overline{(bb-cc)(cc-aa)(aa-bb)}$$

$$uX + u'Y = \frac{-r\sin\zeta\cos\mathfrak{l}}{\cos\eta\cos\theta}(\mathfrak{A} - aa) - \frac{r\sin\eta\cos\mathfrak{m}}{\cos\zeta\cos\theta}(\mathfrak{A} - bb) - \frac{r\sin\theta\cos\mathfrak{n}}{\cos\zeta\cos\theta}(\mathfrak{A} - cc)$$
$$\overline{(bb-cc)(cc-aa)(aa-bb)}.$$

Ex his autem valoribus $\mathfrak{A}$ et $\mathfrak{B}$ est pro superioribus

$$L = \frac{\mathfrak{A}}{aabbcc} \text{ et } K = \frac{aabb + aacc + bbcc - \mathfrak{B}}{aabbcc}$$

ex quibus fit

$$u \tfrac{1}{2} K - r(\tfrac{1}{4} KK - L) \text{ et } u' = \tfrac{1}{2} K + r(\tfrac{1}{4} KK - L)$$

ita ut sit $u + u' = K$ et $u' - u = r(KK - 4L)$.

Haec analysis in genere valet, etiamsi corpori initio motus fuerit impressus, quoniam loco angulorum δt et $\delta' t$ hic adhibuimus $\delta t + \mathfrak{g}$ et $\delta' t + \mathfrak{h}$. Simili modo, quo hic ex situ initiali constantes $\mathfrak{F}$, $\mathfrak{G}$ et $\mathfrak{H}$ definivimus, ex motu initio impresso, quantitates x, y, z datos obtinebunt valores, quibus si formulae coroll. 1. traditae et pro δt et $\delta' t$ scribendo $\delta t + \mathfrak{g}$ et $\delta' t + \mathfrak{h}$ extensae, posito $t = 0$ aequentur, determinabuntur reliquae constantes $\mathfrak{E}$, $\mathfrak{g}$ et $\mathfrak{h}$: quae quidem, uti jam ante notavimus evanescunt, si motus a quiete incipiat.

SCHOLION.

917. Pro casu ergo ejusmodi corporum pro quibus est $bb = cc$, erit $u = \frac{1}{cc}$, $u' = \frac{aa\cos\zeta^2 + cc\sin\zeta^2}{aacc}$, atque $\delta = r\sqrt{2fgu}$, $\delta' = r\sqrt{2fgu'}$, integralia in genere ita se habebunt:

$$x = \mathfrak{E}\cos\zeta - \frac{\mathfrak{H}\delta'\sin\zeta^2\sin(\delta' t + \mathfrak{h})}{aa}$$

$$y = \mathfrak{E}\cos\eta + \mathfrak{G}\delta\cos\theta\sin(\delta t + \mathfrak{g}) + \frac{\mathfrak{H}\delta'\cos\zeta\cos\eta\sin(\delta' t + \mathfrak{h})}{cc}$$

$$z =$$

$$z = \mathfrak{E}\cos\theta - \mathfrak{G}\delta\cos\eta\sin(\delta t + g) + \frac{\mathfrak{H}\delta'\cos\zeta\cos\theta\sin(\delta' t + h)}{cc}$$

atque

$$p = \mathfrak{F}\cos\zeta + \mathfrak{G}\sin\zeta^2\cos(\delta t + g)$$
$$q = \mathfrak{F}\cos\eta - \mathfrak{G}\cos\zeta\cos\eta\cos(\delta t + g) + \mathfrak{H}u'\cos\theta\cos(\delta' t + h)$$
$$r = \mathfrak{F}\cos\theta - \mathfrak{G}\cos\zeta\cos\theta\cos(\delta t + g) - \mathfrak{H}u'\cos\eta\cos(\delta' t + h).$$

Quare si initio $t = 0$ fuerit

$$p = r\sin\zeta\cos l\,;\; q = r\sin\eta\cos m\,,\; r = r\sin\theta\cos n$$

reperitur

$$\mathfrak{H} = \frac{r\sin\eta\cos\theta\cos m - r\cos\eta\sin\theta\cos n}{u'\sin\zeta^2\cos h}$$

$$\mathfrak{G} = \frac{r\sin\zeta^3\cos l - r\sin\eta\cos\zeta\cos\eta\cos m - r\sin\theta\cos\zeta\cos\theta\cos n}{\sin\zeta^2\cos g}$$

$$\mathfrak{F} = r\sin\zeta\cos\zeta\cos l + r\sin\eta\cos\eta\cos m + r\sin\theta\cos\theta\cos n.$$

At datis angulis l, m, n, simul dantur ζ, η, θ

$$\cos\zeta^2 = \frac{\cos(l-m)\cos(n-l)}{\sin(l-m)\sin(n-l)};\quad \cos\eta^2 = \frac{\cos(m-n)\cos(l-m)}{\sin(m-n)\sin(l-m)};$$
$$\cos\theta^2 = \frac{\cos(n-l)\cos(m-n)}{\sin(n-l)\sin(m-n)}$$

$$\sin\zeta^2 = \frac{-\cos(m-n)}{\sin(l-m)\sin(n-l)};\quad \sin\eta^2 = \frac{-\cos(n-l)}{\sin(m-n)\sin(l-m)};$$
$$\sin\theta^2 = \frac{-\cos(l-m)}{\sin(n-l)\sin(m-n)}.$$

Ex his autem formulis colligitur, esse

$$\sin\zeta\cos\zeta\cos l + \sin\eta\cos\eta\cos m + \sin\theta\cos\theta\cos n = 0,$$

ita ut constans supra definita $\mathfrak{F}$ semper sit $= 0$. Simili vero modo est

$$\frac{\sin\zeta\cos l}{\cos\eta\cos\theta} - \frac{\sin\eta\cos m}{\cos\zeta\cos\theta} + \frac{\sin\theta\cos n}{\cos\zeta\cos\eta} = 0$$

unde valores coefficientium supra definiti multo simplicius determinantur, ita ut litterae $\mathfrak{A}$ et $\mathfrak{B}$ ex illis prorsus elidantur. Valent haec in genere etiamsi non sit $bb = cc$.

PROBLEMA. 110.

918. Si corpus basi sphaerica praeditum habeat duos axes principales pares, eique cum de situ quietis infinite parum fuerit declinatum

tum, motus minimus quicunque fuerit impreſſus, definire motus continuationem.

SOLUTIO.

Fig. 116. Sit ID axis corporis aequilibrii per centrum inertiae I et centrum baſis G tranſiens, ſitque hoc illo altius ſitum exiſtente intervallo GI $= f$. Sit porro IA axis corporis ſingularis principalis ejusque reſpectu momentum inertiae $= Maa$, reſpectu axium omnium autem ad hunc normalium $= Mcc$, quos omnes cum aeque pro principalibus habere liceat, ſumatur alter IB in arcu DA producto, eritque alter IC, ut quadrans AC ſit ad AD normalis, idcoque DC etiam quadrans ad AD normalis. Poſito ergo $DA = \zeta$, erit $DB = \eta = \zeta + 90°$ et $DC = \theta = 90°$. Initio autem quo $t = 0$, fuerit arcus $DZ = r$, et angulus $ZDA = l$, erit $ZDB = m = l$ et $ZDC = n = l + 90°$. Ex formulis ergo praecedentibus habebimus

$$u = \frac{1}{cc}; u' = \frac{aa\,cof\zeta^2 + cc\,ſi\zeta^2}{aacc}; \vartheta = \frac{r\,2fg}{c}; \vartheta' = \frac{r\,2fg\,(aa\,cof\zeta^2 + cc\,ſi\zeta^2)}{ac}$$

tum vero ex hoc ſitu initiali fiet primo $\mathfrak{F} = 0$, tum vero

$$r\,ſi\zeta\,cof\,l = \mathfrak{G}\,ſi\zeta^2\,cof\,g; \quad \text{ergo}$$

$$r\,cof\zeta\,cof\,l = \mathfrak{G}\,ſi\zeta\,cof\zeta\,cof\,g; \quad \mathfrak{G} = \frac{r\,cof\,l}{ſi\zeta\,cof\,g}$$

$$-r\,ſi\,l = \mathfrak{H}\,u'\,ſi\,\zeta\,cof\,h; \quad \mathfrak{H} = \frac{-r\,ſi\,l}{u'\,ſi\zeta\,cof\,h}.$$

Deinde initio corpori motus ſit impreſſus circa axem IO celeritate angulari $= \varepsilon$ in ſenſum ABC, exiſtente $OA = a$, $OB = b$, $OC = c$, fierique debet

$$\varepsilon\,cof\,a = \mathfrak{E}\,cof\zeta - \frac{\mathfrak{H}\vartheta'\,ſi\zeta^2\,ſi\,h}{aa}$$

$$\varepsilon\,cof\,b = -\mathfrak{E}\,ſi\,\zeta - \frac{\mathfrak{H}\vartheta'\,ſi\zeta\,cof\zeta\,ſi\,h}{cc}$$

$$\varepsilon\,cof\,c = \mathfrak{G}\vartheta\,ſi\zeta\,ſi\,g$$

unde concludimus

$$\varepsilon\,(aa\,cof\,a\,cof\zeta - cc\,cof\,b\,ſi\zeta) = \mathfrak{E}\,(aa\,cof\zeta^2 + cc\,ſi\zeta^2)$$

$$\text{et } \varepsilon\,(cof\,a\,ſi\zeta + cof\,b\,cof\zeta) = -\mathfrak{H}\vartheta'\,ſi\zeta\,ſi\,h\left(\frac{ſi\zeta^2}{aa} + \frac{cof\zeta^2}{cc}\right)$$

Ergo

Ergo $\mathfrak{E} = \dfrac{\varepsilon(aa\cos\alpha\cos\zeta - cc\cos\mathfrak{b}\sin\zeta)}{aa\cos\zeta^2 + cc\sin\zeta^2}$

$$\mathfrak{G} = \frac{\varepsilon\cos\mathfrak{c}}{\delta\sin\zeta\sin\mathfrak{g}}$$

$$\mathfrak{H} = \frac{-\varepsilon(\cos\alpha\sin\zeta + \cos\mathfrak{b}\cos\zeta)}{\delta' u'\sin\zeta\sin\mathfrak{h}}$$

hinc erit $tang\ \mathfrak{g} = \dfrac{\varepsilon\cos\mathfrak{c}}{\delta r\cos l}$, et $tang\ \mathfrak{h} = \dfrac{\varepsilon(\cos\alpha\sin\zeta + \cos\mathfrak{b}\cos\zeta)}{\delta' r\sin l}$ unde anguli $\mathfrak{g}$, et $\mathfrak{h}$ hincque numeri $\mathfrak{G}$ et $\mathfrak{H}$ innotescunt.

His definitis teneat corpus elapso tempore t situm in figura repraesentatum, sitque $ZD = \varrho$, $ZA = l$, $ZB = m$, $ZC = n$: ac ponatur $\cos l = \cos\zeta + p$, $\cos m = \cos\eta + q$; $\cos n = \cos\theta + r$ seu $\cos m = -\sin\zeta + q$ et $\cos n = r$. Deinde gyretur nunc circa axem IO celeritate angulari $= \varkappa$ in sensum ABC existentibus arcubus $OA = \alpha$, $OB = \mathfrak{C}$, $OC = \gamma$, ac ponendo $\varkappa\cos\alpha = x$, $\varkappa\cos\mathfrak{C} = y$ et $\varkappa\cos\gamma = z$, habebimus ex §. 917.

$$\varkappa\cos\alpha = \frac{\varepsilon\cos\zeta(aa\cos\alpha\cos\zeta - cc\cos\mathfrak{C}\sin\zeta)}{aa\cos\zeta^2 + cc\sin\zeta^2} + \frac{\delta' r\sin\zeta\sin l\sin(\delta' t + \mathfrak{h})}{aau'\cos\mathfrak{h}}$$

$$\varkappa\cos\mathfrak{C} = \frac{-\varepsilon\sin\zeta(aa\cos\alpha\cos\zeta - cc\cos\mathfrak{C}\sin\zeta)}{aa\cos\zeta^2 + cc\sin\zeta^2} + \frac{\delta' r\cos\zeta\sin l\sin(\delta' t + \mathfrak{h})}{aau'\cos\mathfrak{h}}$$

$$\varkappa\cos\gamma = \frac{\delta r\cos l\sin(\delta t + \mathfrak{g})}{\cos\mathfrak{g}}$$

tum vero praeterea:

$$p = \frac{r\sin\zeta\cos l\cos(\delta t + \mathfrak{g})}{\cos\mathfrak{g}}$$

$$q = \frac{r\cos\zeta\cos l\cos(\delta t + \mathfrak{g})}{\cos\mathfrak{g}}$$

$$r = \frac{-r\sin l\cos(\delta t + \mathfrak{h})}{\cos\mathfrak{h}}.$$

Ex his si ponatur arcus $ZD = \varrho$, erit

$$\varrho = r\,r\left(\frac{\cos l^2\cos(\delta t + \mathfrak{g})^2}{\cos\mathfrak{g}^2} + \frac{\sin l^2\cos\delta' t + \mathfrak{h})^2}{\cos\mathfrak{h}^2}\right).$$

Porro ex triangulo AZD eſt $cof\,ADZ = \frac{cos\,l - cos\,\zeta\,cos\,\varrho}{ſi\,\zeta\,ſi\,\varrho} = \frac{p + \frac{1}{2}\varrho\varrho\,cos\,\zeta}{\varrho ſi\,\zeta}$

$= \frac{p}{\varrho ſi\,\zeta}$ evaneſcente termino $\frac{\varrho\,cos\,\zeta}{2 ſi\,\zeta}$, hinc ergo erit

$$cof\ ADZ = \frac{cos\,l\,cos\,(\delta t + \mathfrak{g})}{cos\,\mathfrak{g}} : r\left(\frac{cos\,l^2\,cos\,(\delta t + \mathfrak{g})^2}{cos\,\mathfrak{g}^2} + \frac{ſi\,l^2\,cos\,(\delta' t + \mathfrak{h})^2}{cos\,\mathfrak{h}^2}\right)$$

ideoque $tang\ ADZ = \frac{cos\,\mathfrak{g}\,tang\,l\,cos\,(\delta' t + \mathfrak{h})}{cos\,\mathfrak{h}\,cos\,(\delta t + \mathfrak{g})}$.

Praeter arcum autem $DZ = \varrho$ et angulum ADZ noſſe oportet angulum XZD a circulo verticali fixo ZX computatum ; eſt vero $DZA = 180° - ADZ$, ſeu $tang\ DZA = \frac{-cof\,\mathfrak{g}\,cof\,(\delta t + \mathfrak{h})}{cof\,\mathfrak{h}\,cof\,(\delta t + \mathfrak{g})}\ tang\ l$, cum initio eſſet $DZA = 180° - l$ et $tang\ DZA = -\,tang\ l$. Deinde vero poſito angulo $XZA = \lambda$, eſt $d\lambda = \frac{-dt\,(y\,cof\,\eta + z\,cof\,\theta}{ſi\,\zeta^2} = \frac{y\,dt}{ſi\,\zeta}$, hincque $\lambda =$

Conſt. $- \frac{\varepsilon t\,(a\,a\,cof\,a\,cof\,\zeta - c\,c\,cof\,b\,ſi\,\zeta)}{a\,a\,cof\,\zeta^2 + c\,c\,ſi\,\zeta^2} - \frac{r\,cof\,\zeta\,ſi\,l\,cof\,(\delta' t + \mathfrak{h})}{c\,c\,u'\,ſi\,cof\,\mathfrak{h}}$. Quodſi ponamus initio angulum XZD evaniuiſſe, initio fuerat $\lambda = 180° - l$, ſicque conſtans hic ingreſſa eſt:

$$\text{Conſt.} = 180° - l + \frac{r\,cof\,\zeta\,ſi\,l}{c\,c\,u'\,ſi\,\zeta}$$

unde habebimus:

$$\lambda = 180° - l - \frac{\varepsilon t\,(a\,a\,cof\,a\,cof\,\zeta - c\,c\,cof\,b\,ſi\,\zeta)}{a\,a\,cof\,\zeta^2 + c\,c\,ſi\,\zeta^2} + \frac{r\,cof\,\zeta\,ſi\,l}{c\,c\,u'\,ſi\,\zeta}\left(1 - \frac{cof\,(\delta' t + \mathfrak{h})}{cof\,\mathfrak{h}}\right)$$

hincque $XZD = \lambda - DZA$, ex quibus ad tempus t ſitus corporis perfecte cognoſcitur, in hacque determinatione ſimul motus continetur.

COROLL. 1.

919. Si motus corpori initio impreſſus evaneſcat, eſt $\mathfrak{g} = 0$ et $\mathfrak{h} = 0$; hincque

$x =$

$$x = \gamma \cos \alpha = \frac{\delta' r \sin \zeta \sin l \sin \delta' t}{a a u'}$$

$$y = \gamma \cos \beta = \frac{\delta' r \cos \zeta \sin l \sin \delta' t}{c c u'}$$

$$z = \gamma \cos \gamma = \delta r \cos l \sin \delta t$$

$$p = r \sin \zeta \cos l \cos \delta t; \quad q = r \cos \zeta \cos l \cos \delta t; \quad r = -r \sin l \cos \delta' t$$

$tang\ \mathrm{ADZ} = tang\ (180° - \mathrm{DZA}) = \frac{\cos \delta' t}{\cos \delta t} tang\ l$, et $\lambda = 180° - l + \frac{r \cos \zeta \sin l}{c c u' \sin \zeta} (1 - \cos \delta' t)$.

COROLL. 2.

920. Sin autem corpori initio motus fuerit impressus celeritate angulari ε, haec non multo major esse debet quam r. Si enim $\frac{\varepsilon}{r}$ esset numerus praemagnus, anguli g et h prodirent fere recti, eorumque cosinus fere evanescentes, ficque numeri p, q, r nimis fierent magni, quam ut tanquam valde parvae, uti natura solutionis exigit, considerari possent. Namque arcus $ZD = \varrho$ semper minimus esse debet.

COROLL. 3.

921. Cum sit $\gamma = \int (xx + yy + zz)$, nisi ternae quantitates x, y, z seorsim evanescant, fieri nequit, ut corpus unquam ad quietem redigatur. Atque etiamsi corpus a quiete moveri inceperit, tamen fieri potest, ut corpus deinceps nunquam ad quietem revertatur, hocque adeo semper eveniet, nisi fuerit vel $\sin l = 0$ vel $\cos l = 0$; quin etiam tum axis corporis naturalis DF nunquam in situm verticalem perveniet.

COROLL. 4.

922. Cum sit r quantitas valde exigua, si corpus initio nullum motum acceperit, ut sit $\varepsilon = 0$, erit satis exacte $\lambda = 180 - l$; scilicet angulus XZA manebit constans, motusque axis IA ita comparatus, ut in arcu ZA modo ad punctum verticale Z propius accedat modo ab eo longius recedat, erit autem $AZ = \zeta - r \cos l \cos \delta' t$, et ang. $ZAD = \frac{r \sin l \cos \delta' t}{\sin \zeta}$.

CO-

COROLL. 5.

923. In genere autem quicunque motus corpori initio fuerit impressus, erit

$$XZA = \lambda = 180^\circ - \mathfrak{l} - \frac{\varepsilon t(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\,\mathrm{si}\,\zeta)}{aa\cos\zeta^2 + cc\,\mathrm{si}\,\zeta^2}$$

sicque arcus ZA uniformiter circa punctum verticale Z circumferetur: deinde vero cum sit si AD : si DZA = ZD (ϱ) : ZAD, erit ZAD = $\frac{\mathfrak{r}\,\mathrm{si}\,\mathfrak{l}\cos(\delta' t + \mathfrak{h})}{\mathrm{si}\,\zeta\cos\mathfrak{h}}$ atque arcus ZA $= \zeta - \frac{\mathfrak{r}\cos\mathfrak{l}\cos(\delta t + \mathfrak{g})}{\cos\mathfrak{g}}$. Seu pro $\mathfrak{g}$ et $\mathfrak{h}$ substituendis valoribus:

$$\text{angulus ZAD} = \frac{\mathfrak{r}\,\mathrm{si}\,\mathfrak{l}\cos\delta' t}{\mathrm{si}\,\zeta} - \frac{\varepsilon(\cos\mathfrak{a}\,\mathrm{si}\,\zeta + \cos\mathfrak{b}\cos\zeta)\,\mathrm{si}\,\delta' t}{\delta\,\mathrm{si}\,\zeta}$$

et arcus ZA $= \zeta - \mathfrak{r}\cos\mathfrak{l}\cos\delta t + \frac{\varepsilon\cos\mathfrak{c}\,\mathrm{si}\,\delta t}{\delta}$.

SCHOLION. 1.

924. Hae tres postremae formulae, angulos XZA, ZAD cum arcu ZA exhibentes, totam problematis solutionem complectuntur. Quodsi enim has res ad quodvis tempus assignare possimus, situm corporis perfecte cognoscimus. Quare si pro δ et δ' valores supra inventos substituamus, universa problematis solutio his formulis continebitur:

$$\text{ang. XZA} = 180^\circ - \mathfrak{l} - \frac{\varepsilon t(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\,\mathrm{si}\,\zeta)}{aa\cos\zeta^2 + cc\,\mathrm{si}\,\zeta^2}$$

$$\text{arc. ZA} = \zeta - \mathfrak{r}\cos\mathfrak{l}\cos\frac{t\sqrt{2fg}}{c} + \frac{\varepsilon c\cos\mathfrak{c}}{\sqrt{2fg}}\,\mathrm{si}\,\frac{t\sqrt{2fg}}{c}$$

$$\text{ang. ZAD} = \frac{\mathfrak{r}\,\mathrm{si}\,\mathfrak{l}}{\mathrm{si}\,\zeta}\cos\frac{t\sqrt{2fg}(aa\cos\zeta^2 + cc\,\mathrm{si}\,\zeta^2)}{ac} - \frac{\varepsilon ac(\cos\mathfrak{a}\,\mathrm{si}\,\zeta + \cos\mathfrak{b}\cos\zeta)}{\mathrm{si}\,\zeta\sqrt{2fg}(aa\cos\zeta^2 + cc\,\mathrm{si}\,\zeta^2)}$$
$$\mathrm{si}\,\frac{t\sqrt{2fg}(aa\cos\zeta^2 + cc\,\mathrm{si}\,\zeta^2)}{ac}.$$

Quodsi ergo omnia momenta inertiae fuerint aequalia, scilicet $aa = cc$, erit

$$\text{ang. XZA} = 180^\circ - \mathfrak{l} - \varepsilon t(\cos\mathfrak{a}\cos\zeta - \cos\mathfrak{b}\,\mathrm{si}\,\zeta)$$

$$\text{arc. ZA} = \zeta - \mathfrak{r}\cos\mathfrak{l}\cos\frac{t\sqrt{2fg}}{c} + \frac{\varepsilon c\cos\mathfrak{c}}{\sqrt{2fg}}\,\mathrm{si}\,\frac{t\sqrt{2fg}}{c}$$

ang.

$$\text{ang. ZAD} = \frac{r \sin l}{\sin \zeta} \cos \frac{tr\sqrt{2fg}}{c} - \frac{ec(\cos a \sin \zeta + \cos b \cos \zeta)}{\sin \zeta r \sqrt{2fg}} \sin \frac{tr\sqrt{2fg}}{c}$$

quo casu positio puncti A est plane arbitraria.

SCHOLION. 2.

925. Argumentum, quod in hoc capite potissimum evolvendum suscepimus, motum scilicet titubatorium corporum basi sphaerica praeditorum, perfecte pertractavimus, dummodo titubationes fuerint quam minimae, quae hypothesis etiam in doctrina oscillationum statui solet: formulae enim §. 912. et seqq. exhibitae perfectam continent hujus quaestionis solutionem, si quidem ibi anguli δt et $\delta' t$ constantibus g et h augeantur. Constantes autem ex statu initiali definire docuimus in §. 916. quae operatio vehementer sublevatur annotatione sub finem §. 917. adjuncta; quare ad motum corporum cylindricorum explicandum progrediamur.

CAPUT XIX.

DE MOTU CORPORUM CYLINDRICORUM SUPER PLANO HORIZONTALI.

THEOREMA. 11.

926. Dum corpus cylindricum plano horizontali movetur, pressio, qua plano innititur, est verticalis, et per centrum cujusdam sectionis cylindri ad longitudinem normaliter factae transit.

DEMONSTRATIO.

Corpus cylindricum plano horizontali incumbit secundum lineam rectam axi cylindri parallelam, in qua vires existunt cylindrum sustentantes, fierique potest ut hae vires per totam illam rectam sint dispersae. Cum autem istae vires omnes sint ad planum horizontale normales, ideoque verticales, ac parallelae inter se, una dabitur vis iis omnibus aequivalens: cujus ergo directio pariter erit verticalis, certoque rectae contactus puncto insistet. Quodsi igitur in hoc puncto cylindrus ad longitudinem normaliter secetur, sectio erit circulus, et vis omnibus pressionibus aequivalens, quia est in hac sectione ad punctum con-

tactus

tactus verticalis per ejus centrum transibit. Nisi ergo haec sectio transeat per centrum inertiae corporis, directio media pressionis non in sectione per centrum inertiae ad longitudinem normaliter facta versabitur.

EXPLICATIO.

927. Corpora igitur hic consideramus cylindrica, in quibus primo notetur eorum axis quasi geometricus, ad quem omnes sectiones normaliter factae, sint circuli aequales, ita ut corpus sit cylindrus rectus, cujus motus, dum plano horizontali perpetuo incumbit, sumus exploraturi. Fig. 117. Si centrum inertiae in ipso axe geometrico esset, in omni situ cylindrus aequilibrium teneret: sin autem secus, consideretur sectio cylindri ad axem normalis per centrum inertiae I facta, cujus centrum sit in G, atque ad statum aequilibrii requiritur, ut recta GI sit verticalis, ex quo duplex datur aequilibrii situs, alter quo centrum inertiae I infra G alter quo supra G versatur, quorum ille *stabilis*, hic *labilis* vocetur. In omni autem cylindri situ axis geometricus est horizontalis, rectae contactus verticaliter imminens. Deinde ternos axes mechanicos cylindri nosse oportet, qui si qualemcunque materiae distributionem admittimus, utcunque ab axe geometrico seu secundum longitudinem ducto differre possunt, a quorum positione motus determinatio potissimum pendet. Per centrum inertiae I etiam ducta concipiatur recta axi geometrico parallela, quae plerumque axis principalis esse solet, et semper manet horizontalis. His igitur notatis, quomodocunque cylindrus plano horizontali incumbat, in sectione LMFN per centrum inertiae I facta notetur punctum T, ubi hic circulus planum horizontale tangit, deinde etiam illa sectio huic parallela probe notetur, in qua media directio pressionum versatur, quae rectae TG erit parallela; ac tam quam quantitas pressionis, quam distantia sectionis, in qua versatur, a sectione per centrum inertiae facta, erit incognita, demum ex motu deinceps determinanda: siquidem haec vi effici debet, ut axis cylindri longitudinalis perpetuo maneat horizontalis, et cylindrus plano horizontali incumbat.

SCHOLION.

928. In his motibus investigandis non opus est, ut totum corpus in figuram cylindri sit efformatum, sed sufficit, si in locis, quibus plano horizontali incumbit, talem habuerit formam. Huc ergo pertinent motus omnium eorum corporum, quae in terminos cylindricos

desinunt,

deſinunt, quibus tantum planis horizontalibus aeque elevatis utrinque incumbant, dum intra eos moles corporis utcunque fuerit extenſa, quemadmodum evenit in cunis, quae motu vacillatorio ſuper terminis, quos tanquam circulos ſpectare licet, agitantur: deinde etiam huc referendus eſt motus pendulorum, quae non circa axem linearem, uti ſupra aſſumſimus, ſed materialem utrinque in cylindrum abeuntem oſcillantur, dum his binis cylindris planis horizontalibus incumbunt, intra quae maſſa penduli dependet. Etſi igitur his caſibus ſectiones mediae non ſunt circuli, tamen binas illas ſectiones, in quarum altera centrum inertiae verſatur, in altera vero vis preſſionis, tanquam circulos ſpectare licet, quoniam figura ~~eatenus tantum~~ in cenſum venit, quatenus corpus plano horizontali incumbit. Tum vero etiam niſi hujusmodi corpora integras circumvolutiones peragant, ne opus quidem eſt, ut toti termini ſint cylindrici, ſed ſufficit, ſi eorum portio, qua fit contactus, talem habeat figuram, cujus axem longitudinalem per totum corpus extenſum probe notaſſe convenit. Quocirca haec tractatio ad plurimos caſus extenditur, de quo motu primum tenendum, centrum inertiae a viribus ſollicitantibus alium motum niſi in eadem recta verticali recipere non poſſe; ita ut nullum conſequatur motum horizontalem, niſi extrinſecus talem acceperit, quem autem deinceps uniformiter eſſet proſecuturum, in quo cum nulla inſit difficultas, ad eum hic non attendimus.

PROBLEMA. III.

929. Si corpus cylindricum ſuper plano horizontali moveatur, deturque preſſio, qua plano innititur, definire motum progreſſivum, quo centrum inertiae corporis incedet.

SOLUTIO.

Ad axem cylindri fiat ſectio normalis per centrum inertiae I, quae ſive corpus ſit cylindrus continuus, ſive tantum terminis cylindricis ſit praeditum, ſpectetur tanquam circulus LMFN baſibus aequalis, cujus centrum ſit in G, centrum autem inertiae corporis in I, exiſtente intervallo $GI = f$, ita ut in ſtatu quietis recta LIGF teneat ſitum verticalem, in quo centrum inertiae I ſupra centrum circuli G in figura repraeſentatur, quod ſi fuerit profundius, intervallum $GI = f$ negative eſt capiendum. Nunc autem contactus reſpondeat puncto T, ita ut recta ad planum circuli in T normalis ſit linea contactus in planum horizontale Hh cadens. Ducta igitur per centrum circuli recta ΠGT ad contactum Fig. 118.

T, parallela erit directioni pressionis, qua corpus a plano horizontali repellitur, quae vis, in quacunque alia sectione versetur, ponatur $= \Pi$, quam tanquam cognitam spectamus. Sit porro pondus cylindri $= M$, radius circuli $GF = GT = e$, et angulus declinationis $\Pi GL = \varphi$; erit elevatio centri inertiae I supra horizontem $IP = v + f \cos \varphi$, quae ponatur $= v$. Quoniam igitur in motum progressivum inquirimus, et gravitas deorsum urget secundum IP vi $= M$, pressionis vis Π ipsi centro inertiae I sursum secundum IV applicata concipiatur, ita ut jam tota vis deorsum sollicitans sit $= M - \Pi$, et massa movenda $= M$: unde ex principiis motus habetur $ddv = \frac{-2g(M-\Pi)}{M} dt^2$ hincque $\frac{\Pi}{M} = 1 + \frac{ddv}{2g\,dt^2}$ seu $\Pi = M\left(1 + \frac{f\,dd.\cos\varphi}{2g\,dt^2}\right)$, qua aequatione angulus $\Pi GL = \varphi$, ex eoque elevatio centri inertiae $IP = e + f \cos \varphi$ innotescit: ac nisi corpori motus horizontalis fuerit impressus, punctum I in recta PIV agitaretur, in ea vel ascendens vel descendens, ita ut punctum P maneret fixum, ex quo punctum contactus T definitur, quia est $PT = f \sin \varphi$. Sin autem corpus acceperit motum horizontalem, eum constanter servaret uniformem in directum, motusque puncti I ex hoc aequabili rectilineo horizontali et illo verticali foret compositus.

SCHOLION.

930. Praeterea autem in hoc corpore motus gyratorius generari potest, ita tamen ut tam punctum G quam recta ad circulum LMFN in G normalis, quae est axis proprius cylindri, perpetuo maneat in eodem plano horizontali. Ad hunc motum gyratorium investigandum, sepositio motu progressivo centrum inertiae I tanquam in quiete spectabimus, circa quod sphaera descripta, in ea circulus LMFN perpetuo erit verticalis, ad quem si recta normalis per I ducatur, erit ea axis cylindri longitudinalis. Qua conditione observata, omnes motus gyratorii, quorum cylindrus est capax, facile in figura repraesentari possunt. Hic autem ante omnia ad situm axium principalium probe attendi oportet, quorum respectu momenta ex vi pressionis nata sunt definienda.

PROBLEMA. 112.

931. Si corpus cylindricum plano horizontali incumbens habeat situm quemcunque, deturque tam pressio Π quam sectio cylindri transversa-

verſa, in qua verſatur, invenire ejus momenta reſpectu axium principalium.

SOLUTIO.

Sectio cylindri per centrum inertiae I ad longitudinem normaliter facta cadat in planum tabulae, in qua recta IZ ſit normalis, et recta LIG per centrum hujus ſectionis G tranſeat, ita ut ſit intervallum IG $= f$, et angulus ZIL $= \varphi$. Ex G erigatur ad planum tabulae normalis GH usque ad ſectionem, in qua vis preſſionis Π verſatur, ſitque intervallum GH $= b$, ac ſupra vidimus eſſe $\Pi = M\left(\alpha + \frac{fdd.\cos\varphi}{2gdt^2}\right)$ cujus vis directio erit HΠ verticalis ideoque parallela ipſi IZ. Iam radio arbitrario $= 1$, circa centrum inertiae I ſphaera concipiatur deſcripta, ad cujus ſuperficiei puncta A, B, C axes corporis principales dirigantur, vocenturque arcus pro horum punctorum determinatione LA $= \zeta$, LB $= \eta$, LC $= \theta$; item ZA $= l$, ZB $= m$, ZC $= n$ exiſtente ZL $= \varphi$; tum vero anguli ZLA $= \mathfrak{f}$, ZLB $= \mathfrak{g}$, ZLC $= \mathfrak{h}$, ut ſit

$$\cos l = \cos\zeta\cos\varphi + \cos\mathfrak{f}\sin\zeta\sin\varphi;\quad \cos m = \cos\eta\cos\varphi + \cos\mathfrak{g}\sin\eta\sin\varphi;$$
$$\cos n = \cos\theta\cos\varphi + \cos\mathfrak{h}\sin\theta\sin\varphi.$$

Ac primo quidem vis HΠ $= \Pi$ reſolvatur ſecundum directiones axibus principalibus parallelas, quae reſolutio perinde inſtituitur, ac ſi vis haec in centro F ſecundum directionem IZ eſſet applicata: inde autem naſcitur

vis ſec. IA $= \Pi\cos l$; vis ſec. IB $= \Pi\cos m$; vis ſec. IC $= \Pi\cos n$,

quae autem vires jam in puncto H applicatae ſunt intelligendae. Ducatur recta IH quae erit $= r(ff + hh)$, ſecans ſphaeram in F, erit

$\tang$ GIH $= \frac{h}{f}$, et arcus LF cum arcu ZL faciet angulum ZLF rectum. Ponatur arcus LF $= e$, erit $h = -f\tang e$ et IH $= \frac{-f}{\cos e}$, ita ut loco intervalli GH $= h$ commode arcum LF $= e$ in calculo retineamus. Nunc autem inveſtigari oportet, quomodo recta IF ad axes principales inclinetur, quae inclinatio per arcus FA, FB et FC definitur. Reperitur autem

$$\cos AF = \cos\zeta\cos e + \sin\mathfrak{f}\sin\zeta\sin e$$
$$\cos BF = \cos\eta\cos e + \sin\mathfrak{g}\sin\eta\sin e$$
$$\cos CF = \cos\theta\cos e + \sin\mathfrak{h}\sin\theta\sin e.$$

Repraeſentent jam rectae inter ſe normales IA, IB, IC axes principa- Fig. 120.
les

los corporis, inter quos existat recta $IH = \frac{-f}{\cos e}$ eruntque coordinatae pro puncto H axibus parallelae

$$IN = IH \cos AF \; ; \; NM = IH \cos BF \; ; \; MH = IH \cos CF$$

et vires in H applicatae axibusque parallelae erunt

$$\text{vis } Ha = \Pi \cos l \; ; \; \text{vis } Hb = \Pi \cos m \; ; \; \text{vis } Hc = \Pi \cos n,$$

unde respectu axium principalium nascuntur momenta:

Mom. respectu axis IA in sensum $BC = \Pi . IH (\cos n \cos BF - \cos m \cos CF)$

Mom. respectu axis IB in sensum $CA = \Pi . IH (\cos l \cos CF - \cos n \cos AE)$

Mom. respectu axis IC in sensum $AB = \Pi . IH (\cos m \cos AF - \cos l \cos BF)$.

Quae momenta cum supra litteris P, Q, R indicaverimus, si valores supra exhibitos substituamus, obtinebimus:

$$P = \frac{-\Pi f}{\cos e}\Big(\sin e \cos\varrho(\sin g \sin\eta \cos\theta - \sin\mathfrak{h}\cos\eta\sin\theta) + \cos e \sin\varrho(\cos\mathfrak{h}\cos\eta\sin\theta - \cos g \sin\cos\theta) + \sin\eta\sin\theta\sin e\sin\varrho(\sin g\cos\mathfrak{h} - \cos g\sin\mathfrak{h})\Big)$$

At est $\sin g \cos\mathfrak{h} - \cos g \sin\mathfrak{h} = \sin(g-\mathfrak{h}) = \frac{-\cos\zeta}{\sin\eta\sin\theta}$, tum vero

$$\sin g \sin\eta\cos\theta - \sin\mathfrak{h}\cos\eta\sin\theta = \cos f \sin\zeta$$
$$\cos\mathfrak{h}\cos\eta\sin\theta - \cos g\sin\eta\cos\theta = \sin f\sin\zeta$$

ita ut tam pro P quam pro Q et R ex analogia habeamus

$$P = \frac{-\Pi f}{\cos e}(\cos f \sin\zeta \sin e \cos\varrho + \sin f \sin\zeta \cos e \sin\varrho - \cos\zeta \sin e \sin\varrho)$$

$$Q = \frac{-\Pi f}{\cos e}(\cos g \sin\eta \sin e \cos\varrho + \sin g \sin\eta \cos e \sin\varrho - \cos\eta \sin e \sin\varrho)$$

$$R = \frac{-\Pi f}{\cos e}(\cos\mathfrak{h} \sin\theta \sin e \cos\varrho + \sin\mathfrak{h}\sin\theta\cos e\sin\varrho - \cos\theta\sin e\sin\varrho).$$

COROLL. 1.

932. Cum sit $-f \operatorname{tang} e = h$, et h denotet intervallum GH, quo sectio, in quam cadit pressio, antrorsum distat a sectione, in qua est centrum inertiae, erit

$$P = -\Pi f \sin f \sin\zeta \sin\varrho + \Pi h(\cos f \sin\zeta\cos\varrho - \cos\zeta\sin\varrho)$$
$$Q = -\Pi f \sin g \sin\eta\sin\varrho + \Pi h(\cos g \sin\eta\cos\varrho - \cos\eta\sin\varrho)$$
$$R = -\Pi f \sin\mathfrak{h}\sin\theta\sin\varrho + \Pi h(\cos\mathfrak{h}\sin\theta\cos\varrho - \cos\theta\sin\varrho).$$

CO-

COROLL. 2.

933. Dum ergo motus corporis determinatur, non ſolum quantitatem preſſionis Π ſed etiam intervallum $GH = b$ definiri oportet, ut habeatur locus, ubi media directio preſſionum eſt applicata.

EXPLICATIO.

934. Relatio inter arcus ζ, η, θ et angulos $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ inſignes ſuppeditat proprietates, inter quas ſubſtitutiones in ſolutione adhibitae continentur. Primo enim pro illorum angulorum differentiis invenimus

$$\cos(\mathfrak{f}-\mathfrak{g}) = \frac{-\cos\zeta\cos\eta}{\sin\zeta\sin\eta}; \quad \cos(\mathfrak{g}-\mathfrak{h}) = \frac{-\cos\eta\cos\theta}{\sin\eta\sin\theta}; \quad \cos(\mathfrak{h}-\mathfrak{f}) = \frac{-\cos\zeta\cos\theta}{\sin\zeta\sin\theta}$$

$$\sin(\mathfrak{f}-\mathfrak{g}) = \frac{-\cos\theta}{\sin\zeta\sin\eta}; \quad \sin(\mathfrak{g}-\mathfrak{h}) = \frac{-\cos\zeta}{\sin\eta\sin\theta}; \quad \sin(\mathfrak{h}-\mathfrak{f}) = \frac{-\cos\eta}{\sin\zeta\sin\theta}$$

Hinc jam anguli $\mathfrak{g}$ et $\mathfrak{h}$ ad angulum $\mathfrak{f}$ reduci poſſunt, ob $\mathfrak{g} = \mathfrak{f} - (\mathfrak{f}-\mathfrak{g})$ et $\mathfrak{h} = \mathfrak{f} + (\mathfrak{h}-\mathfrak{f})$, unde colligitur

$$\sin\mathfrak{g} = \frac{-\sin\mathfrak{f}\cos\zeta\cos\eta + \cos\mathfrak{f}\cos\theta}{\sin\zeta\sin\eta}; \quad \sin\mathfrak{h} = \frac{-\sin\mathfrak{f}\cos\zeta\cos\theta - \cos\mathfrak{f}\cos\eta}{\sin\zeta\sin\theta}$$

$$\cos\mathfrak{g} = \frac{-\cos\mathfrak{f}\cos\zeta\cos\eta - \sin\mathfrak{f}\cos\theta}{\sin\zeta\sin\eta}; \quad \cos\mathfrak{h} = \frac{-\cos\mathfrak{f}\cos\zeta\cos\theta + \sin\mathfrak{f}\cos\eta}{\sin\zeta\sin\theta}.$$

Quodſi binis conjungendis vel $\cos\mathfrak{f}$ vel $\sin\mathfrak{f}$ elidatur, obtinentur ſequentes formulae:

I. $\sin\mathfrak{f}\sin\zeta\cos\zeta + \sin\mathfrak{g}\sin\eta\cos\eta + \sin\mathfrak{h}\sin\theta\cos\theta = 0$
II. $\cos\mathfrak{f}\sin\zeta\cos\zeta + \cos\mathfrak{g}\sin\eta\cos\eta + \cos\mathfrak{h}\sin\theta\cos\theta = 0$
III. $\sin\mathfrak{f}\sin\zeta = -\cos\mathfrak{g}\sin\eta\cos\theta + \cos\mathfrak{h}\cos\eta\sin\theta$
IV. $\sin\mathfrak{g}\sin\eta = -\cos\mathfrak{h}\sin\theta\cos\zeta + \cos\mathfrak{f}\cos\mathfrak{h}\sin\zeta$
V. $\sin\mathfrak{h}\sin\theta = -\cos\mathfrak{f}\sin\zeta\cos\eta + \cos\mathfrak{g}\cos\zeta\sin\eta$
VI. $\cos\mathfrak{f}\sin\zeta = \sin\mathfrak{g}\sin\eta\cos\theta - \sin\mathfrak{h}\cos\eta\sin\theta$
VII. $\cos\mathfrak{g}\sin\eta = \sin\mathfrak{h}\sin\theta\cos\zeta - \sin\mathfrak{f}\cos\theta\sin\zeta$
VIII. $\cos\mathfrak{h}\sin\theta = \sin\mathfrak{f}\sin\zeta\cos\eta - \sin\mathfrak{g}\cos\zeta\sin\eta$
IX. $\sin\mathfrak{f}\cos\mathfrak{f}\sin\zeta^2 + \sin\mathfrak{g}\cos\mathfrak{g}\sin\eta^2 + \sin\mathfrak{h}\cos\mathfrak{h}\sin\theta^2 = 0$
X. $\sin\mathfrak{f}^2\sin\zeta^2 + \sin\mathfrak{g}^2\sin\eta^2 + \sin\mathfrak{h}^2\sin\theta^2 = 1$; XI. $\cos\mathfrak{f}^2\sin\zeta^2 + \cos\mathfrak{g}^2\sin\eta^2 + \cos\mathfrak{h}^2\sin\theta^2 = 1$

quarum

quarum ope aequationes, ad quas motus determinatio perducitur, simpliciores reddi possunt.

PROBLEMA. 113.

Fig. 121. 935. Si corpus cylindricum quodcunque super plano horizontali moveatur utcunque, aequationes exhibere, quibus ad quodvis tempus ejus situs et motus gyratorius determinetur.

SOLUTIO.

Manentibus denominationibus in praecedente problemate factis, consideretur centrum inertiae I ut quiescens, circa quod descripta sit sphaera, cujus punctum verticale Z, et circulus verticalis fixus ZDX, in quo recta centralis IL initio situm ID tenuerit. Elapso autem tempore t ea pervenerit in L, ac ponatur arcus $ZL = \varrho$ et angulus $XZL = \varphi$, atque hinc situs axium principalium, quorum poli sunt A, B, C ita definitur, ut sint arcus $LA = \zeta$, $LB = \eta$, $LC = \theta$, et anguli $ZLA = \mathfrak{f}$, $ZLB = \mathfrak{g}$, $ZLC = \mathfrak{h}$, qui sunt quantitates constantes, ex quibus cum arcu variabili $ZL = \varrho$, ita definiuntur arcus $ZA = l$, $ZB = m$, $ZC = n$ ut sit:

$$\cos l = \cos\zeta\cos\varrho + \cos\mathfrak{f}\sin\zeta\sin\varrho;\ \cos m = \cos\eta\cos\varrho + \cos\mathfrak{g}\sin\eta\sin\varrho;\ \cos n = \cos\theta\cos\varrho + \cos\mathfrak{h}\sin\theta\sin\varrho.$$

Quodsi jam momenta inertiae corporis respectu axium principalium IA, IB, IC sint Maa, Mbb, Mcc existente M massa corporis, Π autem sit pressio, et sectio in qua ea versatur ab I antrorsum distet intervallo $= s$, quod quia est variabile, in superioribus formulis loco b scribi debet s. Gyretur nunc corpus circa polum O celeritate angulari $= \gamma$ in sensum ABC, positisque arcubus $OA = \alpha$, $OB = \mathfrak{G}$, $OC = \gamma$ sit $\gamma\cos\alpha = x$, $\gamma\cos\mathfrak{G} = y$, $\gamma\cos\gamma = z$, ac primo habemus $\Pi = M\left(1 + \frac{fdd.\cos\varrho}{2gdt^2}\right)$, tum vero has tres aequationes

$$aadx + (cc - bb)\,yzdt = \frac{-2\Pi fg}{M}\,dt\sin\mathfrak{f}\sin\zeta\sin\varrho + \frac{2\Pi gs}{M}\,dt\,(\cos\mathfrak{f}\sin\zeta\cos\varrho - \cos\zeta\sin\varrho)$$

$$bbdy + (aa - cc)\,xzdt = \frac{-2\Pi fg}{M}\,dt\sin\mathfrak{g}\sin\eta\sin\varrho + \frac{2\Pi gs}{M}\,dt\,(\cos\mathfrak{g}\sin\eta\cos\varrho - \cos\eta\sin\varrho)$$

$$ccdz + (bb - aa)\,xydt = \frac{-2\Pi fg}{M}\,dt\,\mathit{sin}\,\mathfrak{h}\,\mathit{sin}\,\theta\,\mathit{sin}\,\varrho + \frac{2\Pi gs}{M}\,dt\,(\mathit{cos}\,\mathfrak{h}\,\mathit{sin}\,\theta\,\mathit{cos}\,\varrho - \mathit{cos}\,\theta\,\mathit{sin}\,\varrho).$$

Praeterea habemus has tres aequationes

$$dl\,\mathit{sin}\,l = dt\,(y\,\mathit{cos}\,n - z\,\mathit{cos}\,m) = d\varrho\,(\mathit{cos}\,\zeta\,\mathit{sin}\,\varrho - \mathit{cos}\,\mathfrak{f}\,\mathit{sin}\,\zeta\,\mathit{cos}\,\varrho)$$
$$dm\,\mathit{sin}\,m = dt\,(z\,\mathit{cos}\,l - x\,\mathit{cos}\,n) = d\varrho\,(\mathit{cos}\,\eta\,\mathit{sin}\,\varrho - \mathit{cos}\,\mathfrak{g}\,\mathit{sin}\,\eta\,\mathit{cos}\,\varrho)$$
$$dn\,\mathit{sin}\,n = dt\,(x\,\mathit{cos}\,m - y\,\mathit{cos}\,l) = d\varrho\,(\mathit{cos}\,\theta\,\mathit{sin}\,\varrho - \mathit{cos}\,\mathfrak{h}\,\mathit{sin}\,\theta\,\mathit{cos}\,\varrho)$$

quarum autem binas tantum sumsisse sufficit, ita ut superfint sex aequationes, ex quibus variabiles totidem x, y, z, Π, s et ϱ ad datum tempus t determinari oporteat. Denique vero positis angulis $XZA = \lambda$, $XZB = \mu$, $XZC = \nu$, fit

$$d\lambda\,\mathit{sin}\,l^2 = -dt\,(y\,\mathit{cos}\,m + z\,\mathit{cos}\,n)$$

quam unam resolvisse sufficit. At cum sit $LZA = \lambda - \Phi$, erit $\mathit{cos}\,(\lambda - \Phi) = \frac{\mathit{cos}\,\zeta - \mathit{cos}\,l\,\mathit{cos}\,\varrho}{\mathit{sin}\,l\,\mathit{sin}\,\varrho}$ et $\mathit{sin}\,(\lambda - \Phi) = \frac{\mathit{sin}\,\mathfrak{f}\,\mathit{sin}\,\zeta}{\mathit{sin}\,l}$, unde $(d\lambda - d\Phi)$

$$\mathit{cos}\,(\lambda - \Phi) = \frac{-dl\,\mathit{sin}\,\mathfrak{f}\,\mathit{sin}\,\zeta\,\mathit{cos}\,l}{\mathit{sin}\,l^2} = \frac{(d\lambda - d\Phi)(\mathit{cos}\,\zeta - \mathit{cos}\,l\,\mathit{cos}\,\varrho)}{\mathit{sin}\,l\,\mathit{sin}\,\varrho},$$ ideoque

$$d\Phi = \frac{-dt\,(y\,\mathit{cos}\,m + z\,\mathit{cos}\,n)}{\mathit{sin}\,l^2} + \frac{dl\,\mathit{sin}\,\mathfrak{f}\,\mathit{sin}\,\zeta\,\mathit{cos}\,l\,\mathit{sin}\,\varrho}{\mathit{sin}\,l\,(\mathit{cos}\,\zeta - \mathit{cos}\,l\,\mathit{cos}\,\varrho)},$$

hincque etiam ad datum tempus angulus Φ definitur: ex quibus rebus motus corporis perfecte cognoscitur.

COROLL. 1.

936. Cum sit $\mathit{cos}\,\zeta - \mathit{cos}\,l\,\mathit{cos}\,\varrho = \mathit{sin}\,\varrho\,(\mathit{cos}\,\zeta\,\mathit{sin}\,\varrho - \mathit{cos}\,\mathfrak{f}\,\mathit{sin}\,\zeta\,\mathit{cos}\,\varrho) = \frac{dl\,\mathit{sin}\,l\,\mathit{sin}\,\varrho}{d\varrho}$ erit $d\Phi = \frac{-dt\,(y\,\mathit{cos}\,m + z\,\mathit{cos}\,n)}{\mathit{sin}\,l^2} + \frac{d\varrho\,\mathit{sin}\,\mathfrak{f}\,\mathit{sin}\,\zeta\,\mathit{cos}\,l}{\mathit{sin}\,l^2}$, hincque

$$d\Phi\,\mathit{sin}\,l^2 = -dt\,(y\,\mathit{cos}\,m + z\,\mathit{cos}\,n) + d\varrho\,\mathit{cos}\,\varrho\,\mathit{sin}\,\mathfrak{f}\,\mathit{sin}\,\zeta\,\mathit{cos}\,\zeta + a\varrho\,\mathit{sin}\,\varrho\,\mathit{sin}\,\mathfrak{f}\,\mathit{cos}\,\mathfrak{f}\,\mathit{sin}\,\zeta^2.$$

Similes autem expressiones pro $d\Phi\,\mathit{sin}\,m^2$ et $d\Phi\,\mathit{sin}\,n^2$ reperiuntur, quae in unam summam collectae, ob $\mathit{sin}\,l^2 + \mathit{sin}\,m^2 + \mathit{sin}\,n^2 = 2$, dabunt

$d\Phi = -dt\,(x\,\mathit{cos}\,l + y\,\mathit{cos}\,m + z\,\mathit{cos}\,n)$ per n° 1 et IX. §. 934. ubi $x\,\mathit{cos}\,l + y\,\mathit{cos}\,m + z\,\mathit{cos}\,n$ denotat cosinum arcus ZO per a multiplicatum.

COROLL. 2.

937. Ex aequationibus pro $dl\,\mathit{sin}\,l$, $dm\,\mathit{sin}\,m$, $dn\,\mathit{sin}\,n$ inventis colligimus,

$$dl\,\mathfrak{si}\,l\,\mathfrak{cos}\,\zeta + dm\,\mathfrak{si}\,m\,\mathfrak{cos}\,\eta + dn\,\mathfrak{si}\,n\,\mathfrak{cos}\,\theta = d\varrho\,\mathfrak{si}\,\varrho$$

ac valoribus per dt ſubſtitutis, impetrabimus,

$$d\varrho = -dt\,(x\,\mathfrak{si}\,\mathfrak{f}\,\mathfrak{si}\,\zeta + y\,\mathfrak{si}\,\mathfrak{g}\,\mathfrak{si}\,\eta + z\,\mathfrak{si}\,\mathfrak{h}\,\mathfrak{si}\,\theta)$$

ope reductionum ſupra traditarum.

COROLL. 3.

938. Ex tribus autem prioribus aequationibus deducimus, ob $xdl\,\mathfrak{si}\,l + ydm\,\mathfrak{si}\,m + zdn\,\mathfrak{si}\,n = 0$, hanc aequationem

$$aaxdx + bbydy + cczdz = \frac{-2\Pi fg}{M}\,dt\,\mathfrak{si}\,\varrho\,(x\,\mathfrak{si}\,\mathfrak{f}\,\mathfrak{si}\,\zeta + y\,\mathfrak{si}\,\mathfrak{g}\,\mathfrak{si}\,\eta + z\,\mathfrak{si}\,\mathfrak{h}\,\mathfrak{si}\,\theta)$$

$$= \frac{2\Pi fg}{M}\,d\varrho\,\mathfrak{si}\,\varrho = -2fgd.\,\mathfrak{cos}\,\varrho\,\left(1 + \frac{ffdd.\mathfrak{cos}\,\varrho}{2gdt2}\right),$$

cujus ergo integrale eſt

$$aaxx + bbyy + cczz = C - 4fg\,\mathfrak{cos}\,\varrho - \frac{ff\,d\varrho^2\,\mathfrak{si}\,\varrho^2}{dt^2}.$$

SCHOLION.

939. Si in ſphaera noſtra dicatur circulus maximus horizontalis YMX, in eo perpetuo axis cylindri longitudinalis reperiatur neceſſe eſt. Pertingat ejus terminus anterior in M, et quia tam ML quam MZ ſunt quadrantes, erunt anguli MZL et MLZ recti, ideoque angulus ZML $= \varrho$ et arcus XM = angulo XZM $= 90° + \Phi$. Tum vero quia punctum M aliter niſi in circulo XY moveri nequit, polus gyrationis O neceſſario in quadrante ZM ſitus ſit neceſſe eſt. Hinc ſi arcus OM ponatur $= \omega$, ob celeritatem angularem $= \varkappa$ in ſenſum ABC tendentem, punctum M tempuſculo dt regreditur verſus X per arculum $= \varkappa dt\,\mathfrak{si}\,\omega$: eſt vero $\mathfrak{si}\,\omega = \mathfrak{cos}\,OZ = \mathfrak{cos}\,\alpha\,\mathfrak{cos}\,l + \mathfrak{cos}\,\mathfrak{b}\,\mathfrak{cos}\,m + \mathfrak{cos}\,\gamma\,\mathfrak{cos}\,n$, ideoque $\varkappa\,\mathfrak{si}\,\omega = x\,\mathfrak{cos}\,l + y\,\mathfrak{cos}\,m + z\,\mathfrak{cos}\,n$, ita ut ſit $-d\Phi = dt\,(x\,\mathfrak{cos}\,l + y\,\mathfrak{cos}\,m + z\,\mathfrak{cos}\,n)$ uti in coroll. 1. invenimus. Deinde cum in triangulo OZL ſit ZO $= 90° - \omega$, ZL $= \varrho$ et OZL $= 90°$, erit arcus $\mathfrak{cos}$ OL $= \mathfrak{si}\,\omega\,\mathfrak{cos}\,\varrho$, $\mathfrak{si}$ OLZ $= \frac{\mathfrak{cos}\,\omega}{\mathfrak{si}\,OL}$ et $\mathfrak{cos}$ OLZ $= \frac{\mathfrak{si}\,\omega\,\mathfrak{si}\,\varrho}{\mathfrak{si}\,OL}$ ob $\mathfrak{cot}$ OLZ $= \frac{\mathfrak{si}\,\varrho\,\mathfrak{si}\,\omega}{\mathfrak{cos}\,\omega}$. Quare ſi tempuſculo dt punctum L circa O gyretur in l, erit $Ll = \varkappa dt\,\mathfrak{si}$ OL, et angulus OLl rectus; hinc ducto circulo $l\lambda$ ad Zl perpendiculari fiet $L\lambda = Ll\,\mathfrak{cos}\,ZLl = Ll\,\mathfrak{si}\,OLZ = \varkappa dt\,\mathfrak{cos}\,\omega$, at eſt $L\lambda = -d\varrho$ ideoque $d\varrho = -\varkappa dt\,\mathfrak{cos}\,\omega$. Quae formula comparata cum ea, quam §. 937. invenimus, dat

Fig. 122.

$\varkappa\,\mathfrak{cos}$

$$\varkappa \cos\omega = x \sin\mathfrak{f} \sin\zeta + y \sin\mathfrak{g} \sin\eta + z \sin\mathfrak{h} \sin\theta$$

at est $xx + yy + zz = \varkappa\varkappa = (x \cos l + y \cos m + z \cos n)^2 + (x \sin\mathfrak{f} \sin\zeta + y \sin\mathfrak{g} \sin\eta + z \sin\mathfrak{h} \sin\theta)^2$ quae aequalitas per aequationem $xdl \sin l + ydm \sin m + zdn \sin n = 0$ confirmatur. Verum ne multitudine litterarum obruamur, evolvamus casum, quo axis cylindri longitudinalis simul est axis principalis.

PROBLEMA. 114.

940. Si corporis cylindrici axis longitudinalis per ejus centrum inertiae ductus simul fuerit axis principalis, idque super plano horizontali utcunque moveatur, **definire ejus motum.** Fig. 123.

SOLUTIO.

Cum puncta A et M in unum incidant, bini reliqui poli principales B et C in circulo verticali ZL existent, eritque propterea: $LA = \zeta = 90°$; $LB = \eta$; $LC = \theta = 90° - \eta$: $ZLA = \mathfrak{f} = 90°$, $ZLB = \mathfrak{g} = 180°$; $ZLC = \mathfrak{h} = 0$; hincque $ZA = l = 90$; $ZB = m = \eta + \varrho$ et $ZC = n = \varrho - \theta = \eta + \varrho - 90°$. Quibus valoribus substitutis, habebimus istas aequationes:

$$\frac{\Pi}{M} = 1 + \frac{fdd.\cos\varrho}{2gdt^2}$$

$$aadx + (cc - bb)\,yzdt = \frac{-2\Pi fg}{M}\,dt \sin\varrho$$

$$bbdy + (aa - cc)\,xzdt = \frac{-2\Pi gs}{M}\,dt \sin(\eta + \varrho)$$

$$ccdz + (bb - aa)\,xydt = \frac{2\Pi gs}{M}\,dt \cos(\eta + \varrho)$$

$$y \sin(\eta + \varrho) - z \cos(\eta + \varrho) = 0$$

$$-xdt \sin(\eta + \varrho) = d\varrho \sin(\eta + \varrho) \text{ seu } d\varrho = -xdt$$

$$\text{et } d\Phi = -dt\,(y \cos(\eta + \varrho) + z \sin(\eta + \varrho)).$$

Ponatur $y = u \cos(\eta + \varrho)$ et $z = u \sin(\eta + \varrho)$, ac pro dt scribatur $-\frac{d\varrho}{x}$, seu $x = \frac{-d\varrho}{dt}$, quo facto nostrae aequationes erunt

I. $\frac{\Pi}{M} = 1 + \frac{fdd.\cos\varrho}{2gdt^2}$

II. $-aaddq + \frac{1}{2}(cc - bb)\,uudt^2 \sin 2(\eta + \varrho) + \frac{2\Pi}{M} fgdt^2 \sin\varrho = 0$

III. $bbdu\, cos(\eta+\varrho) - (aa+bb-cc)\, ud\varrho\, si(\eta+\varrho) = \frac{-2\Pi}{M}$
$gsdt\, si(\eta+\varrho)$

IV. $ccdu\, si(\eta+\varrho) + (aa-bb+cc)\, ud\varrho\, cos(\eta+\varrho) = \frac{2\Pi}{M}$
$gsdt\, cos(\eta+\varrho)$

et V. $d\varphi = -udt$.

Ex tertia et quarta eliminando s nanciscimur,

$$bbdu\, cos(\eta+\varrho)^2 + ccdu\, si(\eta+\varrho)^2 - 2(bb-cc)\, ud\varrho\, si(\eta+\varrho)\, cos(\eta+\varrho) = 0$$

cujus integrale est

$$u = \frac{C}{bb+cc+(bb-cc)\, cos(2\eta+2\varrho)}$$

qui valor in II substitutus praebet

$$-2aadd\varrho + \frac{CC(cc-bb)\, dt^2\, si\, 2(\eta+\varrho)}{(bb+cc+(bb-cc)\, cos\, 2(\eta+\varrho))^2} + dt^2\, si\, \varrho\, \left(4fg + \frac{2ffdd.\, cos\, \varrho}{dt^2}\right) = 0$$

quae aequatio per $d\varrho$ multiplicata et integrata dat,

$$-aad\varrho^2 - \frac{\frac{1}{2}CCdt^2}{bb+cc+(bb-cc)\, cos\, 2(\eta+\varrho)} - 4fgdt^2\, cos\, \varrho - ff\, d\varrho^2\, si\, \varrho^2 + Ddt^2 = 0$$

seu $d\varrho^2(aa+ff\, si\, \varrho^2) = dt^2\left(D - 4fg\, cos\, \varrho - \frac{\frac{1}{2}CC}{bb+cc+(bb-cc)\, cos\, 2(\eta+\varrho)}\right)$

unde fit

$$dt = \frac{d\varrho\, \sqrt{(aa+ff\, si\, \varrho^2)(bb+cc+(bb-cc)\, cos\, 2(\eta+\varrho))}}{\sqrt{((D-4fg\, cos\, \varrho)(bb+cc+(bb-cc)\, cos\, 2(\eta+\varrho)) - \frac{1}{2}CC)}}.$$

Nunc dato tempore t per ϱ, pariter ac u, inde colligimus pressionem π porroque intervallum s ex hac aequatione

$$\frac{2\Pi}{M} gsdt = (cc-bb)\, du\, si(\eta+\varrho)\, cos(\eta+\varrho) + aaud\varrho + (cc-bb)\, ud\varrho\, cos\, 2(\eta+\varrho).$$

Tum vero obtinemus $x = \frac{-d\varrho}{dt}$; $\eta = u\, cos(\eta+\varrho)$ et $z = u\, si(\eta+\varrho)$ ac denique $\varphi = -\int udt$.

CO-

COROLL. 1.

941. Si initio punctum L fuerit in D ut sit ZD $= r$, ibique quieverit, posito $t = 0$ erat $u = 0$, $\frac{d\varrho}{dt} = 0$, ob $\gamma = 0$; ideoque constantes ita definiri oportet, ut sit $C = 0$; et $D = 4fg \cos r$: unde fit $dt = d\varrho \sqrt{\frac{aa + ff \sin \varrho^2}{4fg(\cos r - \cos \varrho)}}$, sicque $\varrho > r$. Porro est $u = 0$, hinc $\Phi = 0$, et $s = 0$; pressio autem Π hinc facile innotescit, et cum ϱ ad 90° augeri possit, corpus quasi procumbet. Hic ergo motus neque a positione axium principalium IB et IC, neque a radio basium cylindri e pendet.

COROLL. 2.

942. Si initio recta IL fuerit verticalis seu $\varrho = 0$, et corpus circa eam gyrari coeperit celeritate angulari ε in sensum AB, ut fuerit O in L, ideoque $\alpha = 90°$, $\mathfrak{G} = \eta$ et $\gamma = 90° - \eta$: initio erat $x = \frac{-d\varrho}{dt} = 0$, $y = \varepsilon \cos \eta$ et $z = \varepsilon \sin \eta$. Hinc fiunt constantes $C = \varepsilon(bb + cc + (bb - cc)\cos 2\eta)$ et $D = 4fg + \frac{1}{2}\varepsilon\varepsilon(bb + cc + (bb - cc)\cos 2\eta)$: unde colligitur $u = \frac{\varepsilon(bb + cc + (bb - cc)\cos 2\eta)}{bb + cc + (bb - cc)\cos 2\eta(\eta + \varrho)}$ atque $\frac{d\varrho^2(aa + ff \sin \varrho^2)}{dt^2} = 4fg(1 - \cos \varrho) + \frac{\frac{1}{2}\varepsilon\varepsilon(bb - cc)(bb + cc + (bb - cc)\cos 2\eta)(\cos 2(\eta + \varrho) - \cos 2\eta)}{bb + cc + (bb - cc)\cos 2(\eta + \varrho)}$.

COROLL. 3.

943. Si esset $bb = cc$, fieret $dt = d\varrho \sqrt{\frac{aa + ff \sin \varrho^2}{4fg(1 - \cos \varrho)}}$, et recta IL perpetuo maneret verticalis: corpusque circa eam uniformiter gyrari pergeret: cum enim denominator contineat $r(1 - \cos \varrho) = \sin \frac{1}{2}\varrho\, r2$ nonnisi tempore elapso infinito arcus ϱ finitus evaderet: quod idem evenit, si fuerit vel $\eta = 0$, vel $\theta = 0$, hoc est si recta IL fuerit axis principalis.

SCHOLION.

944. Nisi axis longitudinalis simul sit axis principalis corporis, ob multitudinem literarum vix patet, quomodo formulae supra erutae

rutae generaliter evolvi queant, quod tamen inferius ſuſcipiemus. Verum ſi hujusmodi corporum cylindricorum tantum motus quaſi infinite parvos conſideremus, ad quod neceſſe eſt, ut in recta centrali LF (ſig. 118.) centrum inertiae I infra centrum circuli G cadat, corpusque infinite parum de ſtatu quietis deturbetur, oſcillationes vel vacillationes minimae orientur, quarum indolem ex formulis noſtris generalibus determinare licebit. Fig. 124. Hic non opus eſt, ut totum corpus ſit cylindricum, ſed ſufficit, ſi ejus termini circa M et N ſint cylindrici, quibus ſuper planis horizontalibus firmis P et Q ſuſtentetur, quin etiam ſufficit, ſi tantum circa contactum utriusque termini figura fuerit cylindrica, ſiquidem motus tantum admittimus infinite parvos. Deinde inter ſuſtentacula P et Q annexum eſſe poteſt corpus pendulum quodcunque FMH*n*, ut oriatur pendulum non circa axem fixum linearem, ſed circa terminos cylindricos planis horizontalibus incumbentes mobile, cujus motum oſcillatorium definiri oporteat. In tali ergo pendulo primo notetur ejus centrum inertiae I, per quod ducatur recta *mn* axi geometrico cylindri MN parallela, quae eſt axis longitudinalis jugiter manens horizontalis. Ducatur porro ex I ad MN recta perpendicularis IGL, quae ſi fuerit verticalis, corpus in quiete verſabitur: ac ſi intervallum GI ponamus $= f$, in ſuperioribus formulis litteram f negative ſumere debemus. Tum pro figura cylindrica terminorum ſit radius baſis $= e$, qui autem, ut vidimus, prorſus non in computum ingreditur, ita ut perinde ſit ſive termini ſint craſſiores ſive graciliores. Quodſi recta IG $= f$ minor fuerit, quam GF $= e$, totumque corpus ſupra ſuſtentacula P et Q verſetur, motus prodit ſimilis ei, quo cunae agitari ſolent. Quicquid autem ſit centrum inertiae I, perpetuo in eadem recta verticali manebit, unde tota inveſtigatio ad motum gyratorium definiendum perducitur, in quo centrum inertiae I ut quieſcens conſideramus.

PROBLEMA. 115.

945. Si corpus, quod baſibus cylindricis ſuper planis horizontalibus incumbit, infinite parum de ſitu quietis deturbetur, eique forte ſimul motus infinite parvus imprimatur, determinare motum vacillatorium, quo agitabitur.

SOLUTIO.

Fig. 121. In formulis noſtris generalibus primo intervallum GI $= f$ negativum ſtatuatur: deinde arcus ZL $= \varrho$, quo recta centralis LGI a ſitu verticali declinat, ut infinite parvus ſpectari debet, perinde atque celeritas

ritas angularis $\varkappa$: unde quantitates $x = \varkappa \cos \alpha$, $y = \varkappa \cos \mathfrak{G}$; $z = \varkappa \cos \gamma$, ut evanescentes tractari debent. Quomodocunque ergo axes principales IA, IB, IC respectu rectae centralis GI et axis longitudinalis mn fuerint dispositi, quorum situs cum arcubus $LA = \zeta$, $LB = \eta$ $LC = \theta$, tum angulis $ZLA = \mathfrak{f}$, $ZLB = \mathfrak{g}$, $ZLC = \mathfrak{h}$ definitur, primo habebimus $\sin \varrho = \varrho$, $\cos \varrho = 1$, deinde producta xy, xz et yz omitti poterunt; unde fiet $\cos l = \cos \zeta$, $\cos m = \cos \eta$ et $\cos n = \cos \theta$; et aequationes solutionem continentes ex probl. 113. ob $\frac{\Pi}{M} = 1 - \frac{fdd.\cos \varrho}{2gdt^2}$ $= 1$ erunt:

I. $aadx = 2fg\varrho dt \sin \mathfrak{f} \sin \zeta + 2gs\, dt \cos \mathfrak{f} \sin \zeta$

II. $bbdy = 2fg\varrho dt \sin \mathfrak{g} \sin \eta + 2gs\, dt \cos \mathfrak{g} \sin \eta$

III. $ccdz = 2fg\varrho dt \sin \mathfrak{h} \sin \theta + 2gs\, dt \cos \mathfrak{h} \sin \theta$

unde ex §. 938. haec integralis est derivata

$$aaxx + bbyy + cczz = C - 2fg\varrho\varrho - \frac{ff\varrho\varrho d\varrho^2}{dt^2}$$

ob $\cos \varrho = 1 - \frac{1}{2}\varrho\varrho$, quia hic infinite parvum $\varrho\varrho$ negligere non licet: Deinde habemus:

IV. $y \cos \theta - z \cos \eta = -\frac{d\varrho}{dt} \cos \mathfrak{f} \sin \zeta$

V. $z \cos \zeta - x \cos \theta = -\frac{d\varrho}{dt} \cos \mathfrak{g} \sin \eta$

VI. $x \cos \eta - y \cos \zeta = -\frac{d\varrho}{dt} \cos \mathfrak{h} \sin \theta$

atque ex §. §. 936. et 937.

$$d\varrho = -dt\,(x \sin \mathfrak{f} \sin \zeta + y \sin \mathfrak{g} \sin \eta + z \sin \mathfrak{h} \sin \theta)$$
$$d\Phi = -dt\,(x \cos \zeta + y \cos \eta + z \cos \theta).$$

Cum nunc sit IV. x + V. y + VI. $z = 0$, erit

$$x \cos \mathfrak{f} \sin \zeta + y \cos \mathfrak{g} \sin \eta + z \cos \mathfrak{h} \sin \theta = 0.$$

Deinde ex n°. I. II. III. in subsidium vocando formulas n°. I. et II. ex §. 934. colligimus

$$aax \cos \zeta + bb\, y \cos \eta + cc\, z \cos \theta = A.$$

et pro intervallo s determinando

$$aadx \cos \mathfrak{f} \sin \zeta + bbdy \cos \mathfrak{g} \sin \eta + ccdz \cos \mathfrak{h} \sin \theta = 2gs\, dt.$$

Statuamus $d\varrho = -u dt$ et $d\Phi = -v dt$, atque ob $x \cos \mathfrak{f} \sin \zeta + y \cos \mathfrak{g} \sin \eta$ $+ z \cos \mathfrak{h} \sin \theta = 0$ consequimur

$\varkappa = u$

$$x = u \sin f \sin \zeta + v \cos \zeta ; \; y = u \sin g \sin \eta + v \cos \eta ; \; z = u \sin h \sin \theta + v \cos \theta$$ hincque

$$A = u (aa \sin f \sin \zeta \cos \zeta + bb \sin g \sin \eta \cos \eta + cc \sin h \sin \theta \cos \theta) + v (aa \cos \zeta^2 + bb \cos \eta^2 + cc \cos \theta^2).$$

Ponamus ad abbreviandum

$$\left.\begin{array}{l} bb \cos h \cos \eta \sin \theta - cc \cos g \sin \eta \cos \theta = \mathfrak{A} \\ cc \cos f \cos \theta \sin \zeta - aa \cos h \sin \theta \cos \zeta = \mathfrak{B} \\ aa \cos g \cos \zeta \sin \eta - bb \cos f \sin \zeta \cos \eta = \mathfrak{C} \end{array}\right\}$$ eritque $\mathfrak{A} \cos f \sin \zeta + \mathfrak{B} \cos g \sin \eta + \mathfrak{C} \cos h \sin \theta = 0$

$$aa \cos \zeta^2 + bb \cos \eta^2 + cc \cos \theta^2 = \mathfrak{D} : aa \sin f \sin \zeta \cos \zeta + bb \sin g \sin \eta \cos \eta + cc \sin h \sin \theta \cos \theta = \mathfrak{F}$$

et habebimus

$$v = \frac{A - \mathfrak{F} u}{\mathfrak{D}}$$

$$x = \frac{A \cos \zeta + \mathfrak{A} u}{\mathfrak{D}} ; \; y = \frac{A \cos \eta + \mathfrak{B} u}{\mathfrak{D}} : \; z = \frac{A \cos \theta + \mathfrak{C} u}{\mathfrak{D}}$$

qui valores, in aequatione integrali vim vivam complectente substituti, ob $\frac{d\varrho}{dt} = -u$ dabunt:

$$\frac{AA\mathfrak{D} + 2Au(\mathfrak{A} a^2 \cos \zeta + \mathfrak{B} b^2 \cos \eta + \mathfrak{C} c^2 \cos \theta) + uu(\mathfrak{A}^2 a^2 + \mathfrak{B}^2 b^2 + \mathfrak{C}^2 c^2)}{\mathfrak{D}\mathfrak{D}} = C - 2fg\varrho\varrho - ff\varrho\varrho uu$$

quae aequatio ob $\mathfrak{A} aa \cos \zeta + \mathfrak{B} bb \cos \eta + \mathfrak{C} cc \cos \theta = 0$ abit in hanc

$$AA\mathfrak{D} + (\mathfrak{A}^2 a^2 + \mathfrak{B}^2 b^2 + \mathfrak{C}^2 c^2) uu = C\mathfrak{D}\mathfrak{D} - 2\mathfrak{D}\mathfrak{D} fg\varrho\varrho - \mathfrak{D}\mathfrak{D} ff \varrho\varrho uu$$

ubi si loco $C\mathfrak{D}\mathfrak{D} - AA\mathfrak{D}$ ponatur $B\mathfrak{D}\mathfrak{D}$, fiet

$$u = \frac{\mathfrak{D} r (B - 2fg\varrho\varrho)}{r (\mathfrak{A}^2 a^2 + \mathfrak{B}^2 b^2 + \mathfrak{C}^2 c^2 + \mathfrak{D}\mathfrak{D} ff \varrho\varrho)}.$$

Statuamus porro $\mathfrak{A}^2 a^2 + \mathfrak{B}^2 b^2 + \mathfrak{C}^2 c^2 = \mathfrak{D}\mathfrak{D}\mathfrak{H}\mathfrak{H}$, et rejecto termino infinite parvo $\mathfrak{D}\mathfrak{D} ff \varrho\varrho$ habebimus

$$u = \frac{r (B - 2fg\varrho\varrho)}{\mathfrak{H}} \text{ et } dt = \frac{-\mathfrak{H} d\varrho}{r (B - 2fg\varrho\varrho)},$$

unde colligimus $t = \text{Const.} + \frac{\mathfrak{H}}{r 2fg} \text{Arc.} \cos \frac{\varrho r 2fg}{rB}$, seu

$$\varrho = \frac{rB}{r 2fg} \cos \frac{(t + \delta) r 2fg}{\mathfrak{H}}, \text{ et } u = \frac{rB}{\mathfrak{H}} \sin \frac{(t + \delta) r 2fg}{\mathfrak{H}}$$

tum

tum vero $v = \frac{A}{\mathfrak{D}} - \frac{\mathfrak{F} r B}{\mathfrak{D}\mathfrak{H}} \int \frac{(t+\delta) r^{2fg}}{\mathfrak{H}}$; hincque

$$\varphi = \mathrm{D} - \frac{At}{\mathfrak{D}} - \frac{\mathfrak{F} r B}{\mathfrak{D} r^{2fg}} \cos \frac{(t+\delta) r^{2fg}}{\mathfrak{H}} = \mathrm{D} - \frac{At}{\mathfrak{D}} - \frac{\mathfrak{F}\varrho}{\mathfrak{D}}.$$

Deinde reperiemus:

$$s = \frac{-r B f}{\mathfrak{D}\mathfrak{H}\mathfrak{H} r^{2g}} (aa\,bb \sin \mathfrak{h} \cos \mathfrak{h} \sin \theta^2 + aa\,cc \sin \mathfrak{g} \cos \mathfrak{g} \sin \eta^2 + bb\,cc \sin \mathfrak{f} \cos \mathfrak{f} \sin \zeta^2) \cos \frac{(t+\delta) r^{2fg}}{\mathfrak{H}}$$

Denique vero erit

$$\mathfrak{s}\mathfrak{s} = xx + yy + zz = \frac{AA - 2A\mathfrak{F}u + (\mathfrak{A}\mathfrak{A} + \mathfrak{B}\mathfrak{B} + \mathfrak{C}\mathfrak{C})uu}{\mathfrak{D}\mathfrak{D}}$$

ficque omnia ad datum tempus sunt definita. Ceterum hic notasse juvat, esse $\mathfrak{A}\mathfrak{A} + \mathfrak{B}\mathfrak{B} + \mathfrak{C}\mathfrak{C} = \mathfrak{D}\mathfrak{D} + \mathfrak{F}\mathfrak{F}$, ita ut sit $\mathfrak{s}\mathfrak{s} = vv + uu$.

COROLL. 1.

946. Cum sit $\varrho = \frac{r B}{r^{2fg}} \cos \frac{(t+\delta) r^{2fg}}{\mathfrak{H}}$; patet arcum $ZL = \varrho$ seu declinationem rectae LI a situ verticali ad similitudinem penduli variari, hujusque lineae LI vacillationes isochronas fore oscillationibus penduli, cujus longitudo est $= \frac{\mathfrak{H}\mathfrak{H}}{f}$, quae longitudo est $= \frac{\mathfrak{A}^2 a^2 + \mathfrak{B}^2 b^2 + \mathfrak{C}^2 c^2}{\mathfrak{D}\mathfrak{D} f}$.

COROLL. 2.

947. Deinde cum sit $\varphi = \mathrm{D} - \frac{At}{\mathfrak{D}} - \frac{\mathfrak{F}\varrho}{\mathfrak{D}}$, punctum L motu medio revolvitur circa verticem Z celeritate angulari $= \frac{A}{\mathfrak{D}}$; verum locus medius corrigi debet particula $\frac{\mathfrak{F}\varrho}{\mathfrak{D}}$. Sin autem sit constans $A = 0$, angulus DZL parumper mutatur, nisi sit $\mathfrak{F} = 0$.

COROLL. 3.

948. Si ergo revolutiones corporis circa axem verticalem IZ excludantur, ut sit $A = 0$, atque initio fuerit $\varphi = 0$; $\varrho = t$ et

et celeritas angularis $\mathfrak{x} = \varepsilon$, constantes ita definientur, ut sit

$$D = \frac{\mathfrak{F}\mathfrak{r}}{\mathfrak{D}}; \quad \mathfrak{r} = \frac{\mathfrak{r}B}{\mathfrak{r}^{2fg}} \cos\frac{\delta\mathfrak{r}^{2fg}}{\mathfrak{H}} \text{ et } \varepsilon\varepsilon = \frac{(\mathfrak{D}\mathfrak{D}+\mathfrak{F}\mathfrak{F})B}{\mathfrak{D}\mathfrak{D}\mathfrak{H}\mathfrak{H}}\left(\sin\frac{\delta\mathfrak{r}^{2fg}}{\mathfrak{H}}\right)^2$$

$$\text{Ergo } \mathfrak{r}B = \frac{\mathfrak{r}\mathfrak{r}^{2fg}}{\cos\frac{\delta\mathfrak{r}^{2fg}}{\mathfrak{H}}} = \frac{\varepsilon\mathfrak{D}\mathfrak{H}}{\sin\frac{\delta\mathfrak{r}^{2fg}}{\mathfrak{H}}\,\mathfrak{r}(\mathfrak{D}\mathfrak{D}+\mathfrak{F}\mathfrak{F})}, \text{ ideoque}$$

$$\tan\frac{\delta\mathfrak{r}^{2fg}}{\mathfrak{H}} = \frac{\varepsilon\mathfrak{D}\mathfrak{H}}{\mathfrak{r}\mathfrak{r}^{2fg}(\mathfrak{D}\mathfrak{D}+\mathfrak{F}\mathfrak{F})}, \text{ unde et constans } \mathfrak{B} \text{ innotescit.}$$

Sin autem fuerit $\varepsilon = 0$; prodit $\mathfrak{r}B = \mathfrak{r}\mathfrak{r}^{2fg}$, et $\delta = 0$.

EXEMPLUM.

Fig. 123. 949. Ponamus rectam IM, quae per centrum inertiae I axi geometrico cylindri (MN fig. 124.) parallela ducitur, simul esse corporis axem principalem, et habebimus uti §. 940. $f = 90°$, $g = 180°$, $h = 0$ et $\zeta = 90°$, atque $\theta = 90° - \eta$. Hinc autem colligimus:

$\mathfrak{A} = bb\cos\eta^2 + cc\sin\eta^2$; $\mathfrak{B} = 0$; $\mathfrak{C} = 0$; $\mathfrak{D} = \mathfrak{A}$ et $\mathfrak{F} = 0$ ergo $aa = \mathfrak{H}\mathfrak{H}$: unde longitudo penduli simplicis isochroni fit $= \frac{aa}{f}$. Tum vero axis IA horizontalis manebit immotus. Ac si initio, ubi $\varrho = \mathfrak{r}$, corpus motum a quiete inceperit, erit $\delta = 0$, et $\mathfrak{r}B = \mathfrak{r}\mathfrak{r}^{2fg}$: ex quibus reliquae quantitates variabiles colliguntur,

$$\varrho = \mathfrak{r}\cos\frac{t\mathfrak{r}^{2fg}}{a}; \quad u = \frac{\mathfrak{r}\mathfrak{r}^{2fg}}{a}\sin\frac{t\mathfrak{r}^{2fg}}{a}; \quad v = 0; \text{ ob } A = 0$$

$$\text{et } x = u = \frac{\mathfrak{r}\mathfrak{r}^{2fg}}{a}\sin\frac{t\mathfrak{r}^{2fg}}{a}; \quad y = 0 \text{ et } z = 0, \text{ atque } \mathfrak{x} = x.$$

Fig. 124. Revera autem adjuncto motu progressivo centrum inertiae I in recta verticali alternatim ascendet ac descendet, cylindro superiore MN hunc motum sequente, dum super planis P et Q liberrime incedere potest, neque a frictione impediri assumitur.

SCHOLION.

950. Quia magnitudo cylindri MN in computum non ingreditur, eadem solutio valebit, si ejus crassities evanescat, corpusque annexum ab axe lineari esset suspensum. Ex quo hic motus convenire debere videtur cum motu oscillatorio supra definito, quod tamen longe aliter usu venit; quoniam pro motu oscillatorio vero longitudo penduli simplicis isochroni prodiit $= f + \frac{aa}{f} = \frac{aa+ff}{f}$, cum hic tantum sit $= \frac{aa}{f}$.

Causa

Cauſa hujus diſcriminis in eo eſt ſita, quod ſupra in doctrina oſcillationum axem MN fixum aſſumſimus, dum hic liberrime mobilis ſtatuitur. Hinc patet, ob libertatem axis, etſi plano horizontali incumbat, oſcillationes multo promtiores fieri, quam ſi axis in eodem loco firmiter detineretur. Atque hoc etiam Theoriae omnino eſt conforme, ſi enim (fig. 118.) circulus ΠMTN planum ſemper in eodem puncto T tangere debeat, praeter preſſionem Π vis quaedam horizontalis in calculum introduci debet, quae ſi ponatur $= \Theta$ ſecundum TH urgens, ut punctum T maneat conſtans, ob $TP = f\!\!\!/\varrho$ eſſe oportet $\frac{f\!\!\!/ d d f\!\!\!/ \varrho^2}{dt^2} = \frac{-2\Theta g}{M}$. Ex hac autem vi quoque naſcitur momentum reſpectu axium principalium, qua propterea motus gyratorius afficitur, ut talis prodeat, qualem ſupra in motus oſcillatorii inveſtigatione determinavimus. Ceterum hic probe notaſſe juvabit, ſi axiculi penduli planis horizontalibus politiſſimis incumbant, motum oſcillatorium plurimum diſcrepare poſſe ab eo, qui oriretur, ſi firmiter detinerentur, et multo quidem promtiorem eſſe futurum. Minima autem frictio hoc diſcrimen tollere, motumque ad oſcillationum legem reducere valebit. Hujus autem problematis ſolutio nos ad ſolutionem problematis generalis n°. 113. manuducet.

PROBLEMA. 116.

951. Si corpus cylindricum quodcunque ſuper plano horizontali moveatur utcunque, aequationes ſupra inventas, quibus ejus motus definitur, reſolvere atque ad integrationem perducere.

SOLUTIO.

Maneant hic omnia, uti ſupra in problemate 113. ſunt conſtituta, atque in recta centrali LIGF ſumamus ut ibi centrum inertiae I a puncto F magis remotum, quam centrum ſectionis cylindri G, ponendo intervallum $GI = f$. Ex aequationibus igitur differentialibus ibi exhibitis jam unam aequationem integralem eruimus, quae eſt: Fig. 121.

$$aaxx + bbyy + cczz = C - 4fg \cos\varrho - \frac{ff d\varrho \cdot f\!\!\!/\varrho^2}{dt^2}.$$

Praeterea vero ternae priores aequationes ope ternarum poſteriorum in poſtremis terminis applicatarum abeunt in has formas

I. $aadx + (cc-bb)\,yzdt = \frac{-2\Pi fg}{M}.\,dt\,\mathfrak{si}\,\mathfrak{f}\,\mathfrak{si}\,\zeta\,\mathfrak{si}\,\varrho - \frac{2\Pi gs}{M}.\frac{dtdl\mathfrak{si}l}{d\varrho}$

II. $bbdy + (aa-cc)\,xzdt = \frac{-2\Pi fg}{M}.\,dt\,\mathfrak{si}\,\mathfrak{g}\,\mathfrak{si}\,\eta\,\mathfrak{si}\,\varrho - \frac{2\Pi gs}{M}.\frac{dtdm\mathfrak{si}m}{d\varrho}$

III. $ccdz + (bb-aa)\,xydt = \frac{-2\Pi fg}{M}.\,dt\,\mathfrak{si}\,\mathfrak{h}\,\mathfrak{si}\,\theta\,\mathfrak{si}\,\varrho - \frac{2\Pi gs}{M}.\frac{dtdn\mathfrak{si}n}{d\varrho}.$

Hinc jam colligatur forma I. $\mathfrak{cos}\,l$ + II. $\mathfrak{cos}\,m$ + III. $\mathfrak{cos}\,n$, et quia $dl\,\mathfrak{si}\,l\,\mathfrak{cos}\,l + dm\,\mathfrak{si}\,m\,\mathfrak{cos}\,m + dn\,\mathfrak{si}\,n\,\mathfrak{cos}\,n = 0$, termini ultimi intervallum s involventes se destruent: tum vero etiam per relationes §. 934. traditas reperitur

$$\mathfrak{si}\,\mathfrak{f}\,\mathfrak{si}\,\zeta\,\mathfrak{cos}\,l + \mathfrak{si}\,\mathfrak{g}\,\mathfrak{si}\,\eta\,\mathfrak{cos}\,m + \mathfrak{si}\,\mathfrak{h}\,\mathfrak{si}\,\theta\,\mathfrak{cos}\,n = 0$$

ita ut quoque penultimi tollantur. Quocirca perveniemus ad hanc aequationem.

$$aadx\,\mathfrak{cos}\,l + bbdy\,\mathfrak{cos}\,m + ccdz\,\mathfrak{cos}\,n + aaxzdt\,\mathfrak{cos}\,m + bbxy\,dt\,\mathfrak{cos}\,n + ccyzdt\,\mathfrak{cos}\,l = 0$$
$$- aaxydt\,\mathfrak{cos}\,n - bbyzdt\,\mathfrak{cos}\,l - ccxzdt\,\mathfrak{cos}\,m$$

at ex ternis posterioribus est

$$z\,\mathfrak{cos}\,m - y\,\mathfrak{cos}\,n = \frac{-dl\mathfrak{si}\,l}{dt};\quad x\,\mathfrak{cos}\,n - z\,\mathfrak{cos}\,l = \frac{-dm\mathfrak{si}\,m}{dt};\quad y\,\mathfrak{cos}\,l - x\,\mathfrak{cos}\,m = \frac{-dn\mathfrak{si}\,n}{dt}$$

quibus valoribus substitutis obtinemus

$$aadx\,\mathfrak{cos}\,l + bbdy\,\mathfrak{cos}\,m + ccdz\,\mathfrak{cos}\,n - aax\,dl\,\mathfrak{si}\,l - bby\,dm\,\mathfrak{si}\,m - ccz\,dn\,\mathfrak{si}\,n = 0$$

cujus integralis est

$$aax\,\mathfrak{cos}\,l + bby\,\mathfrak{cos}\,m + ccz\,\mathfrak{cos}\,n = \mathrm{D}.$$

Deinde loco x, y et z introducamus novas variabiles hinc definiendas

$$x\,\mathfrak{cos}\,\zeta + y\,\mathfrak{cos}\,\eta + z\,\mathfrak{cos}\,\theta = p$$
$$x\,\mathfrak{cos}\,\mathfrak{f}\,\mathfrak{si}\,\zeta + y\,\mathfrak{cos}\,\mathfrak{g}\,\mathfrak{si}\,\eta + z\,\mathfrak{cos}\,\mathfrak{h}\,\mathfrak{si}\,\theta = q$$
$$x\,\mathfrak{si}\,\mathfrak{f}\,\mathfrak{si}\,\zeta + y\,\mathfrak{si}\,\mathfrak{g}\,\mathfrak{si}\,\eta + z\,\mathfrak{si}\,\mathfrak{h}\,\mathfrak{si}\,\theta = r$$

eritque primo $d\varrho = -rdt$; porro ob $x\,\mathfrak{cos}\,l + y\,\mathfrak{cos}\,m + z\,\mathfrak{cos}\,n = p\,\mathfrak{cos}\,\varrho + q\,\mathfrak{si}\,\varrho$; erit $d\Phi = -dt\,(p\,\mathfrak{cos}\,\varrho + q\,\mathfrak{si}\,\varrho)$. Praeterea ob $x\,dl\,\mathfrak{si}\,l + ydm\,\mathfrak{si}\,m + zdn\,\mathfrak{si}\,n = 0$, fit, $p\,\mathfrak{si}\,\varrho - q\,\mathfrak{cos}\,\varrho = 0$. Quam ob rem ponamus

$p = u\,\mathfrak{cos}\,\varrho$ et $q = u\,\mathfrak{sin}\,\varrho$ eritque $d\Phi = -udt$ et $d\varrho = -rdt$,

at ex illis aequationibus assumtis elicimus

$$x = r\,\mathfrak{si}\,\mathfrak{f}\,\mathfrak{si}\,\zeta + u\,\mathfrak{cos}\,l;\quad y = r\,\mathfrak{si}\,\mathfrak{g}\,\mathfrak{si}\,\eta + u\,\mathfrak{cos}\,m;\quad z = r\,\mathfrak{si}\,\mathfrak{h}\,\mathfrak{si}\,\theta + u\,\mathfrak{cos}\,n$$

hincque $xx + yy + zz = rr + uu = \mathfrak{ss}$.

Nunc aequatio integralis modo ante inventa praebet

$$\mathrm{D} = r$$

$$D = r\,(aa\,\mathfrak{si}\,\mathfrak{f}\,\mathfrak{s}\,\zeta\,\mathfrak{cos}\,l + bb\,\mathfrak{s}\,\mathfrak{g}\,\mathfrak{si}\,\eta\,\mathfrak{cos}\,m + cc\,\mathfrak{si}\,\mathfrak{h}\,\mathfrak{si}\,\theta\,\mathfrak{cos}\,n) + u\,(aa\,\mathfrak{cos}\,l^2 + bb\,\mathfrak{cos}\,m^2 + cc\,\mathfrak{cos}\,n^2)$$

qua u determinatur per r et ϱ; ideoque et x, y, z. Denique aequatio integralis primum inventa

$$aaxx + bbyy + cczz = \mathfrak{C} - 4fg\,\mathfrak{cos}\,\varrho - ff\,rr\,\mathfrak{si}\,\varrho^2$$

quia tantum r et ϱ continet, determinabit ϱ per ϱ, indeque aequatio $dt = \frac{-r\,d\varrho}{r}$ pro dato tempore t omnes quantitates motum continentes manifestabit.

Quodsi ad abbreviandum ponantur constantes:

$$aa\,\mathfrak{cos}\,\zeta^2 + bb\,\mathfrak{cos}\,\eta^2 + cc\,\mathfrak{cos}\,\theta^2 = \mathfrak{A}$$
$$aa\,\mathfrak{cos}\,\mathfrak{f}\,\mathfrak{si}\,\zeta\,\mathfrak{cos}\,\zeta + bb\,\mathfrak{cos}\,\mathfrak{g}\,\mathfrak{si}\,\eta\,\mathfrak{cos}\,\eta + cc\,\mathfrak{cos}\,\mathfrak{h}\,\mathfrak{si}\,\theta\,\mathfrak{cos}\,\theta = \mathfrak{B}$$
$$aa\,\mathfrak{cos}\,\mathfrak{f}^2\,\mathfrak{si}\,\zeta^2 + bb\,\mathfrak{cos}\,\mathfrak{g}^2\,\mathfrak{si}\,\eta^2 + cc\,\mathfrak{cos}\,\mathfrak{h}^2\,\mathfrak{si}\,\theta^2 = \mathfrak{C}$$
$$aa\,\mathfrak{s}\,\mathfrak{f}\,\mathfrak{si}\,\zeta\,\mathfrak{cos}\,\zeta + bb\,\mathfrak{si}\,\mathfrak{g}\,\mathfrak{si}\,\eta\,\mathfrak{cos}\,\eta + cc\,\mathfrak{si}\,\mathfrak{h}\,\mathfrak{si}\,\theta\,\mathfrak{cos}\,\theta = \mathfrak{D}$$
$$aa\,\mathfrak{si}\,\mathfrak{f}\,\mathfrak{cos}\,\mathfrak{f}\,\mathfrak{si}\,\zeta^2 + bb\,\mathfrak{si}\,\mathfrak{g}\,\mathfrak{cos}\,\mathfrak{g}\,\mathfrak{si}\,\eta^2 + cc\,\mathfrak{si}\,\mathfrak{h}\,\mathfrak{cos}\,\mathfrak{h}\,\mathfrak{si}\,\theta^2 = \mathfrak{E}$$
$$aa\,\mathfrak{si}\,\mathfrak{f}^2\,\mathfrak{s}\,\zeta^2 + bb\,\mathfrak{si}\,\mathfrak{g}^2\,\mathfrak{si}\,\eta^2 + cc\,\mathfrak{si}\,\mathfrak{h}^2\,\mathfrak{s}\,\theta^2 = \mathfrak{F}$$

nostrae aequationes integrales erunt

$$D = r\,(\mathfrak{D}\,\mathfrak{cos}\,\varrho + \mathfrak{E}\,\mathfrak{si}\,\varrho) + u\,(\mathfrak{A}\,\mathfrak{cos}\,\varrho^2 + 2\mathfrak{B}\,\mathfrak{si}\,\varrho\,\mathfrak{cos}\,\varrho + \mathfrak{C}\,\mathfrak{si}\,\varrho^2)$$
$$\mathfrak{C} - 4fg\,\mathfrak{cos}\,\varrho - ff\,rr\,\mathfrak{si}\,\varrho^2 = \mathfrak{F}\,rr + 2ru\,(\mathfrak{D}\,\mathfrak{cos}\,\varrho + \mathfrak{E}\,\mathfrak{si}\,\varrho) + uu\,(\mathfrak{A}\,\mathfrak{cos}\,\varrho^2 + 2\mathfrak{B}\,\mathfrak{si}\,\varrho\,\mathfrak{cos}\,\varrho + \mathfrak{C}\,\mathfrak{si}\,\varrho^2)$$

ex quibus concluditur

$$rr = \frac{DD - (\mathfrak{A}\,\mathfrak{cos}\,\varrho^2 + 2\mathfrak{B}\,\mathfrak{si}\,\varrho\,\mathfrak{cos}\,\varrho + \mathfrak{C}\,\mathfrak{si}\,\varrho^2)(\mathfrak{C} - 4fg\,\mathfrak{cos}\,\varrho)}{(\mathfrak{D}\,\mathfrak{cos}\,\varrho + \mathfrak{E}\,\mathfrak{si}\,\varrho)^2 - (\mathfrak{A}\,\mathfrak{cos}\,\varrho^2 + 2\mathfrak{B}\,\mathfrak{si}\,\varrho\,\mathfrak{cos}\,\varrho + \mathfrak{C}\,\mathfrak{si}\,\varrho^2)(\mathfrak{F} + ff\,\mathfrak{si}\,\varrho^2)}$$

Hinc pro tempore adipiscimur $t = \int \frac{-d\varrho}{r}$, et cum sit $u = \frac{D - r\,(\mathfrak{D}\,\mathfrak{cos}\,\varrho + \mathfrak{E}\,\mathfrak{si}\,\varrho)}{\mathfrak{A}\,\mathfrak{cos}\,\varrho^2 + 2\mathfrak{B}\,\mathfrak{si}\,\varrho\,\mathfrak{cos}\,\varrho + \mathfrak{C}\,\mathfrak{si}\,\varrho^2}$, erit angulus $\varphi = -\int u\,dt = l\,\frac{u\,d\varrho}{r}$.

Cum autem ad quodvis tempus t tam arcum ϱ quam angulum φ determinaverimus, totus motus erit perfecte cognitus.

COROLL. 1.

952. Quantitates ergo $\mathfrak{A}$, $\mathfrak{C}$, et $\mathfrak{F}$ necessario sunt positivae, et $\mathfrak{B}$ ad $\mathfrak{A}$ et $\mathfrak{B}$ ad $\mathfrak{A}$ et $\mathfrak{C}$ ita refertur, ut sit

$$\mathfrak{A}\mathfrak{C} - \mathfrak{B}\mathfrak{B} = aabb\,\mathfrak{si}\,\mathfrak{h}^2\,\mathfrak{si}\,\theta^2 + aacc\,\mathfrak{si}\,\mathfrak{g}^2\,\mathfrak{si}\,\eta^2 + bbcc\,\mathfrak{si}\,\mathfrak{f}^2\,\mathfrak{si}\,\zeta^2$$

unde patet, formam $\mathfrak{A}\,\mathfrak{cos}\,\varrho^2 + 2\mathfrak{B}\,\mathfrak{si}\,\varrho\,\mathfrak{cos}\,\varrho + \mathfrak{C}\,\mathfrak{si}\,\varrho^2$ in duos factores simplices resolvi non posse.

CO-

COROLL. 2.

953. Ex hac ſolutione generali caſus in praecedente problemate evolutus facile deducitur, ſumendo f negative, et arcum φ infinite parvum, unde fit

$$rr = \frac{DD - \mathfrak{A}C - 4\mathfrak{A}fg\cos\varphi}{\mathfrak{D}\mathfrak{D} - \mathfrak{A}\mathfrak{F}} = \frac{Conſt. + 4\mathfrak{A}fg\cos\varphi}{\mathfrak{A}f - \mathfrak{D}\mathfrak{D}}.$$

Reperitur autem valoribus evolutis

$$\mathfrak{A}\mathfrak{F} - \mathfrak{D}\mathfrak{D} = aabb\cos\mathfrak{h}^2 \sin\theta^2 + aacc\cos\mathfrak{g}^2 \sin\eta^2 + bbcc\cos\mathfrak{f}^2 \sin\zeta^2$$

unde longitudo penduli ſimplicis iſochroni ſimplicius quam ſupra ita exhibetur, ut ſit $= \dfrac{aabb\cos\mathfrak{h}^2 \sin\theta^2 + aacc\cos\mathfrak{g}^2 \sin\eta^2 + bbcc\cos\mathfrak{f}^2 \sin\zeta^2}{f(aa\cos\zeta^2 + bb\cos\eta^2 + cc\cos\theta^2)}$.

SCHOLION.

954. His de motu corporum cylindricorum ſuper plano horizontali expeditis, inſtitueram pauca de motu ſuper plano inclinato adjungere: verum ſi motus fuerit ſimplex, res nullam habet difficultatem, ſin autem ſit complicatus, in calculos incommodos incideremus. Quare cum in praxi frictionem ab his motibus ſeparare haud liceat, motus ſaltem ſimpliciores ſuper plano inclinato ita pertractabimus, ut ſimul frictionis rationem habeamus, ex quo peculiarem tractatum de motu corporum rigidorum a frictione perturbato adjungi conveniet.

SUP-