CHAPTER 2

Graphical Mechanics and Intermittent Motion

Several concepts in basic mechanics were reviewed in Chapter 1, as well as methods of handling force and velocity vectors when analyzing mechanism designs. But all this involved only instantaneous velocities or instantaneous forces. Most design situations are concerned with a continuous spectrum of forces and velocities. If we had to compute these things separately for each instant of a long and complex machine cycle using the techniques of Chapter 1, alone, it would soon prove to be too difficult. Fortunately, in most design situations, approximate "motion curves" of force and velocity can be constructed, showing the complete and continuous variation of these parameters, and the need to find exact instantaneous values of force and velocity will exist only for a few special points on the curves (maximums and minimums, for example). Frequently, these approximate curves will be constructed using design information; e.g., when a spring is to be supplied capable of producing a certain force to drive given mechanisms. Some curves will result from experimental information, as described in Chapter 6, where a transducer, fastened to the machine, produces electrical signals that are "plotted" by a strip chart recorder or oscilloscope. And finally, there are curves derived from others, in some situations. In this chapter we will examine these motion curves, and see how they relate to each other.

This will also lead into a discussion of acceleration and displacement. These parameters will be handled as scalar quantities rather than vector quantities (which is why they were not discussed in Chapter 1), since only their magnitudes are of interest here. In conclusion, the annoying, but essential, subject of "units" will be discussed.

Forces on Mechanisms

Every element in a machine or instrument is acted upon by forces produced either by springs; electrical, pneumatic, or hydraulic motors; gravitational or magnetic fields; or by interactions with other machine elements and structures. In general, such forces can be considered to be either driving forces or load forces, and can be expressed as functions of time or functions of displacement. The discussion will be restricted almost exclusively to forces which are described as functions of time, but will also briefly cover the case of force versus displacement.

Forces, as have been shown, are vector quantities and are, therefore, directional in nature. It is useful to adopt "plus- or minus-sign rules" to describe force vectors so that our theoretical models can account for the fact that some forces act together while others act to oppose each other. For the purpose of this text, forces will be considered to be *positive* if they are directed toward the right, or vertically upwards. They will be considered to be *negative* if they are directed towards the left, or vertically downwards. These and other sign and symbol conventions used in this text are illustrated in Appendix 1.

Force and Intermittent Motion

Let us start our study of force with a very simple and tangible case of intermittent motion. In Fig. 2–1, a finger applies a brief push to a small block resting on a table. This produces a drive force F, on the stone and inevitably, a friction reaction force f_r. After the finger has stopped pushing (comes to rest), the block coasts for a while until the frictional force brings it to a rest. This action, plus the resulting graph of net horizontal force (drive force minus friction force) on the block, is shown in Fig. 2–2, A and B.

It would be quite easy to measure the drive force involved here (with a push scale for example). Measuring the friction force would be a slightly more difficult problem, but it can be done by tipping the table to determine the sliding angle, for example, and a graph can be produced of the force as a function of time, such as is shown in Fig. 2–2B.

This would immediately disclose one important fact: the positive and negative areas enclosed by the force curve (and time axis) must be equal to each other if the velocity of the block is to be zero at the beginning and end of the period being studied. In other words, if the block is to be a true "intermittent motion device" it must start with zero velocity and end there. This equality of the positive and negative areas enclosed by a force (or torque) curve is an important and basic concept in the study of intermittent motion.

Notice that in the graph, Fig. 2–2B, force is expressed in pounds, and time in seconds. These are two quantities of the so-called English system of units which will be used throughout this text and which is tabulated in the appendix. Also given in the appendix, however, is a table of alternate sets of units involving ounces, grams, inches, kilograms, etc. Be consistent and stay within one set of units (one column in the table) for a given problem; it is

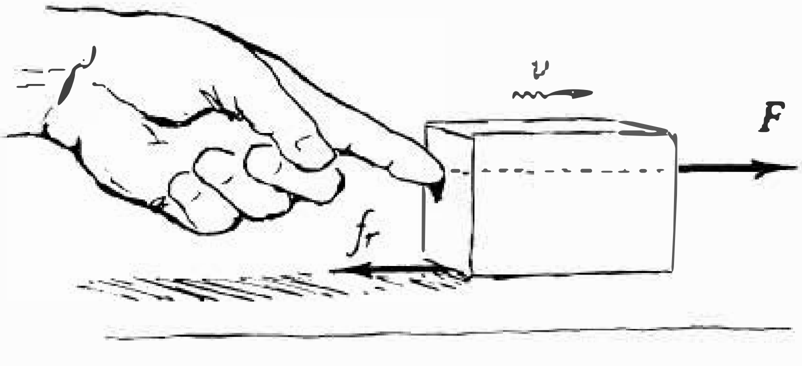


Fig. 2–1. Forces acting on a block being pushed along a flat surface.

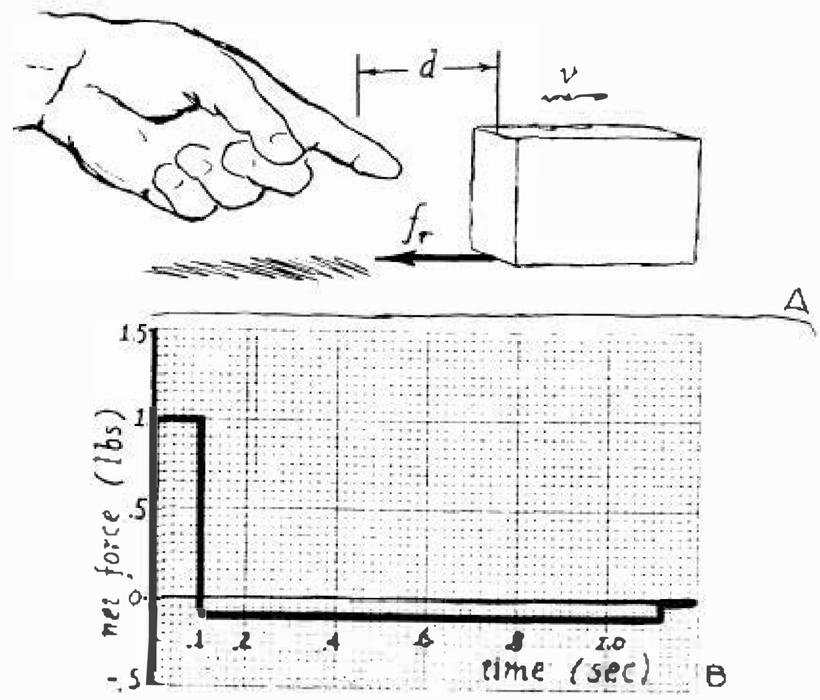


Fig. 2–2. (A) Forces acting on a sliding block; and (B) Graph showing the net or resultant horizontal force on the block as a function of time.

only when the sets of units are intermixed that you will get into difficulty.

Consider now, another topic, the acceleration of the body and its relationship to the forces on that body.

Acceleration

Whenever an object changes its rate of motion; whenever it speeds up or slows down, it experiences positive acceleration (an *increase* in speed), or negative acceleration (a *decrease* in speed). The latter, of course, is sometimes called deceleration but "negative acceleration" is the more accurate term for theoretical model builders. Acceleration is defined as the rate-of-change of velocity as a function of time. It is a very important parameter in machine design and is, therefore, worthy of our attention. Fortunately, once the forces on a body are known, its acceleration can readily be determined. The acceleration (as a function of time) is found by using Newton's famous Second Law:

$$F = ma$$

force = mass $\times$ acceleration

(units: lbs = slugs $\times$ ft/sec/sec = slugs $\times$ ft/sec^2)

This law tells us that the acceleration of a body at any instant is equal to the force on the body at that instant divided by the inertial mass of the body. A force curve, then, can be converted to an acceleration curve merely by dividing each point on the force

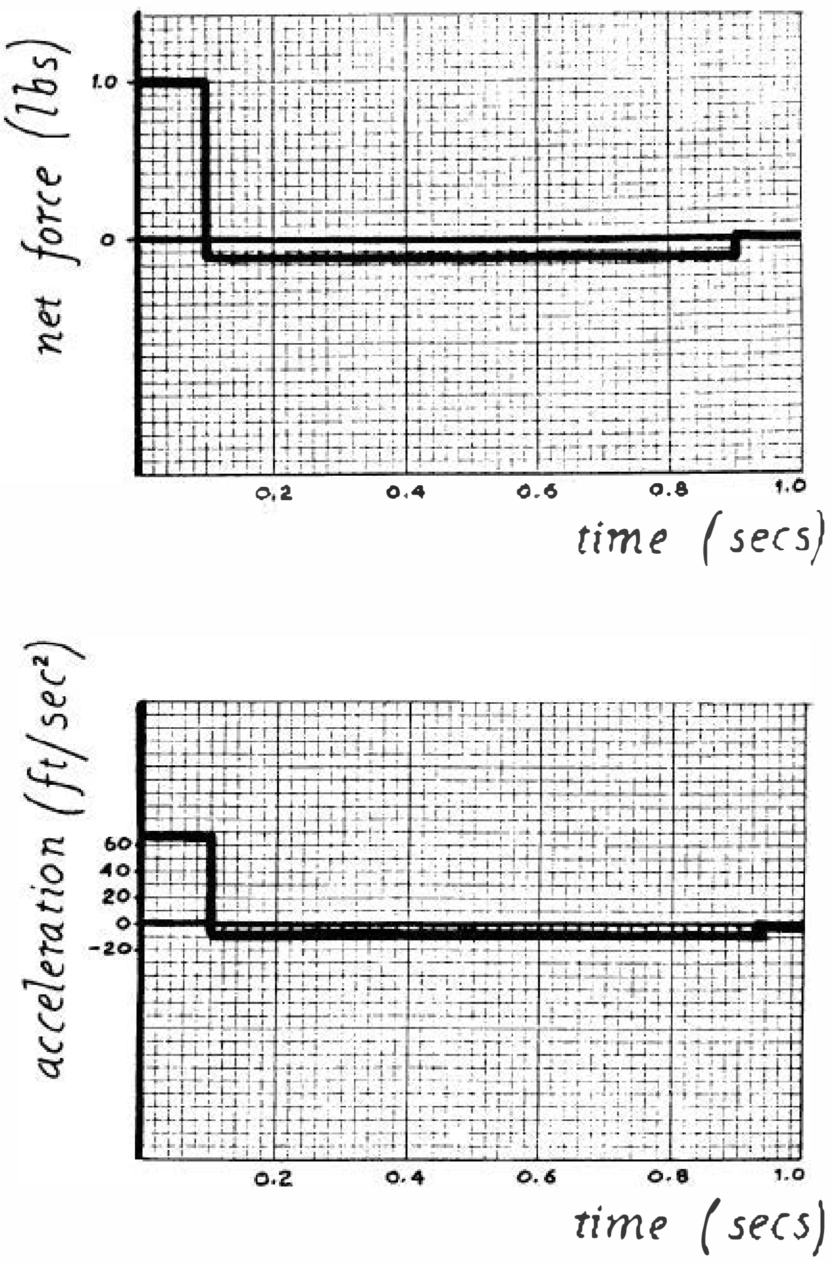


Fig. 2-3. Relationship between acceleration and force curves for the block-on-the-table problem.

curve by the mass of the body being studied. This can be done by drawing a new acceleration curve using information obtained from the force curve; or by merely changing the vertical scale on the original force curve. Figure 2-3 shows the relationship between the forces and accelerations on the block. Note that the positive and negative areas enclosed by the acceleration curve are equal for intermittent motion.

The units for acceleration are feet per second per second, which equals feet per second, squared; again, in the English system of units. Time is still given in seconds. Mass has been expressed in slugs. I do not know the origin of the name "slugs" nor why the word was adopted. By definition (and using Newton's equation of $F = ma$), a unit force of 1 pound will produce an acceleration of 1 foot per second, squared, in a 1-slug mass: thus a slug is related to more familiar units by the following equation:

$$M = f/a$$

$$\text{One slug} = \text{one}\,\frac{\text{lb-sec}^2}{\text{ft}}$$

—if that is any help! We can find the mass of a body by dividing its sea-level weight, in pounds, by 32. (32 ft/sec² being the acceleration of gravity in English units. 32.16 or 32.2 are sometimes used but 32 is close enough for our purposes.)

Once an acceleration curve is achieved, a velocity curve can be constructed. Velocity is also of interest in design work, and is the next topic to be discussed.

Velocity

Velocity is defined as the rate-of-change of position of a body with time. It is a measure of the speed at which a body is moving; and the direction in which it is moving. Because velocity, like force, is a vector, positive and negative sign conventions must be adopted. In this text we will assume that velocity to the right, or upward, is positive: and velocity to the left, or downward, is negative.

If the acceleration of a body as a function of time is known, a velocity curve can be constructed by measuring the area enclosed by the acceleration curve and the horizontal (time) axis on the acceleration diagram. This area could be found by a mathematical process called *integration* if the acceleration curve could be described mathematically. There are many special cases in which this is possible, and some students struggle through a course in integral calculus to learn them. With a drawing of an acceleration curve, however, the integration process can be performed graphically, merely by counting the squares enclosed by the curve and the time axis, by using an instrument called the planimeter, or by a series of algebraic computations in which the area under a curve is approximated by a series of rectangles, triangles, trapezoids, and other simple forms whose areas are easy to compute. The same techniques can be used for finding the areas under complex curves as for finding those areas under simple curves. It is never necessary to learn special procedures for special cases; an advantage of the graphical over the mathematical approach.

The integration process is started at the "velocity equals 0" point. This is not an arbitrary choice, it is necessary to start at the zero velocity point to avoid what mathematicians call the "constant of

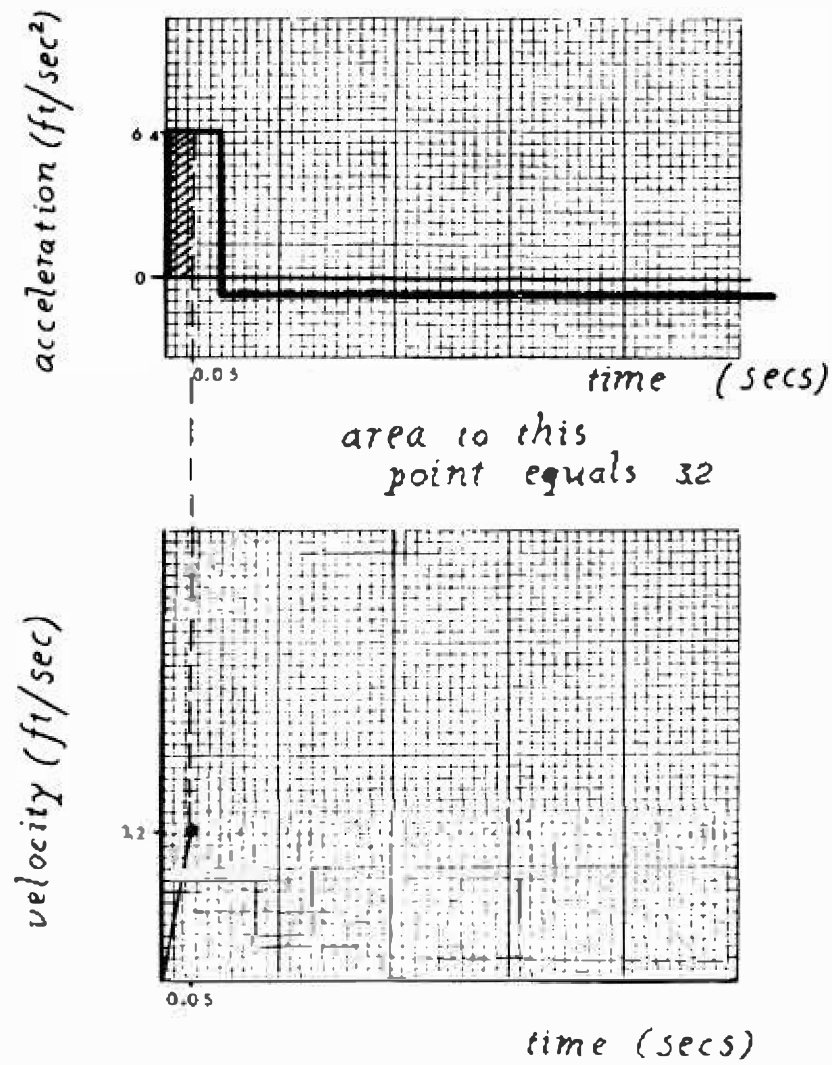


Fig. 2-4. Deriving a velocity curve from an acceleration curve—first step.

integration." The velocity of the body after 0.05 second of acceleration is equal to the total area enclosed under the acceleration curve from the time (or velocity) equals zero point to the time equals 0.05 second point. The velocity of the body after 0.1 second of acceleration is equal to the total area under the acceleration curve from time = 0, to time = 0.1 second, etc., as shown in Figs. 2-4 and 2-5.

When the acceleration curve passes below the time axis it produces the "negative area" which must be subtracted from the total area accumulated to that point, as shown in Fig. 2-6. In an intermittent motion system, the velocity curve starts and stops at zero; thus, as shown in Fig. 2-7, as much area is eventually subtracted as was added. This process of graphical integration is very important as the methods of this text cannot be used unless the process is thoroughly understood. Re-read the above and study the illustrations until the process of converting the "force-on-the-block" curve into a velocity curve is clearly understood.

The integration process can also be used to shed light on the "units" that should be used for the function being derived. Acceleration, as was seen, has the units of feet per second, squared. Time is measured in seconds. The area enclosed by an acceleration curve and the time axis, therefore, would have the dimensions of:

$$\text{Area} = \text{height} \times \text{base}$$

$$\text{Units: area} = \frac{\text{ft}}{\text{sec}^2} \times \text{sec}$$

It is a very useful, but little-known fact that "units" can be treated as algebraic quantities in trying to simplify an expression or to see where an equation is heading. For example, just as:

$$\frac{A}{B^2} \times B = \frac{A}{B}$$

so does the original "units" expression given above,

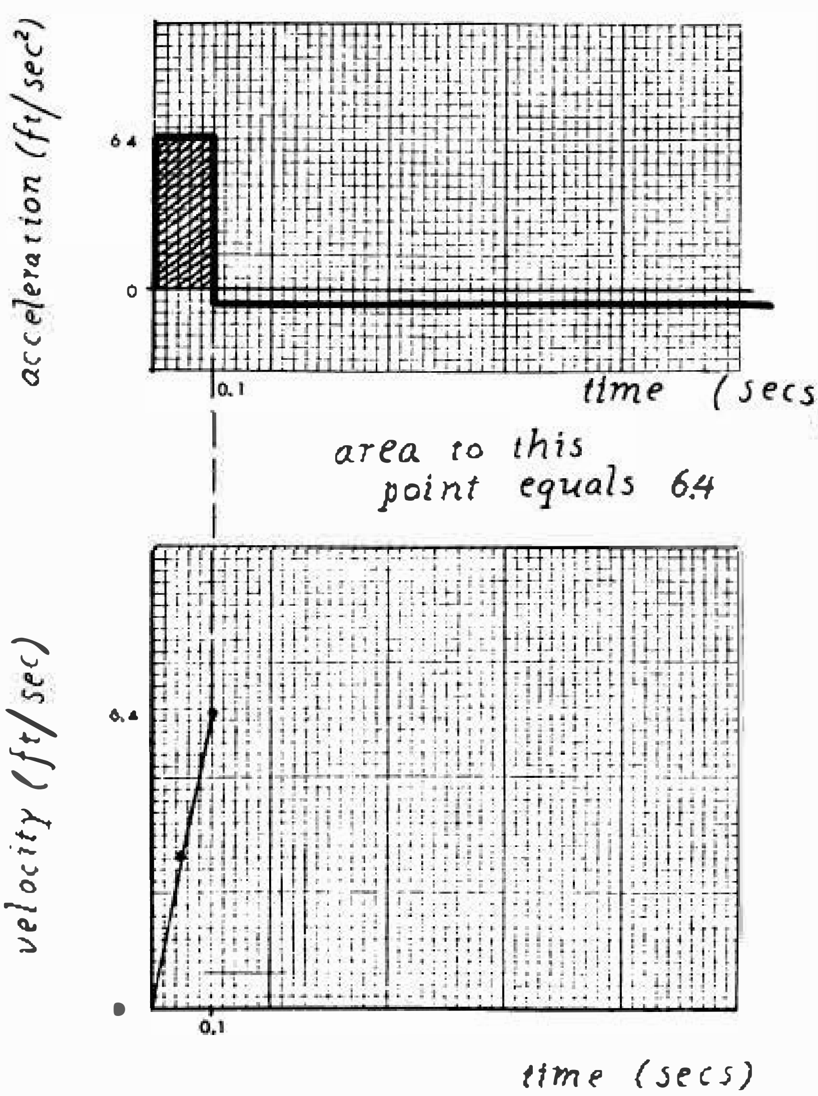


Fig. 2-5. Deriving a velocity curve from an acceleration curve—second step.

reduce to feet per second, as shown below:

$$\frac{\text{ft}}{\text{sec}^2} \times \text{secr} = \frac{\text{ft}}{\text{sec}}$$

And feet per second is indeed the correct "unit" for linear velocity. Therefore, although it may seem strange, the "area under the acceleration curve" really *is* "velocity."

Units as Algebraic Quantities

Again, this treatment of units should be reviewed before leaving this section. Nothing causes the beginner more grief than units, and this method of handling them will be helpful. I believe it comes from that branch of mathematics called dimensional analysis.

This method works with any group of units. For example, suppose there is a truck that holds 500 pies; each pie takes up 200 cubic inches; has a density of 100 ounces per cubic foot; and it is necessary to know the total weight of the 500 pies in

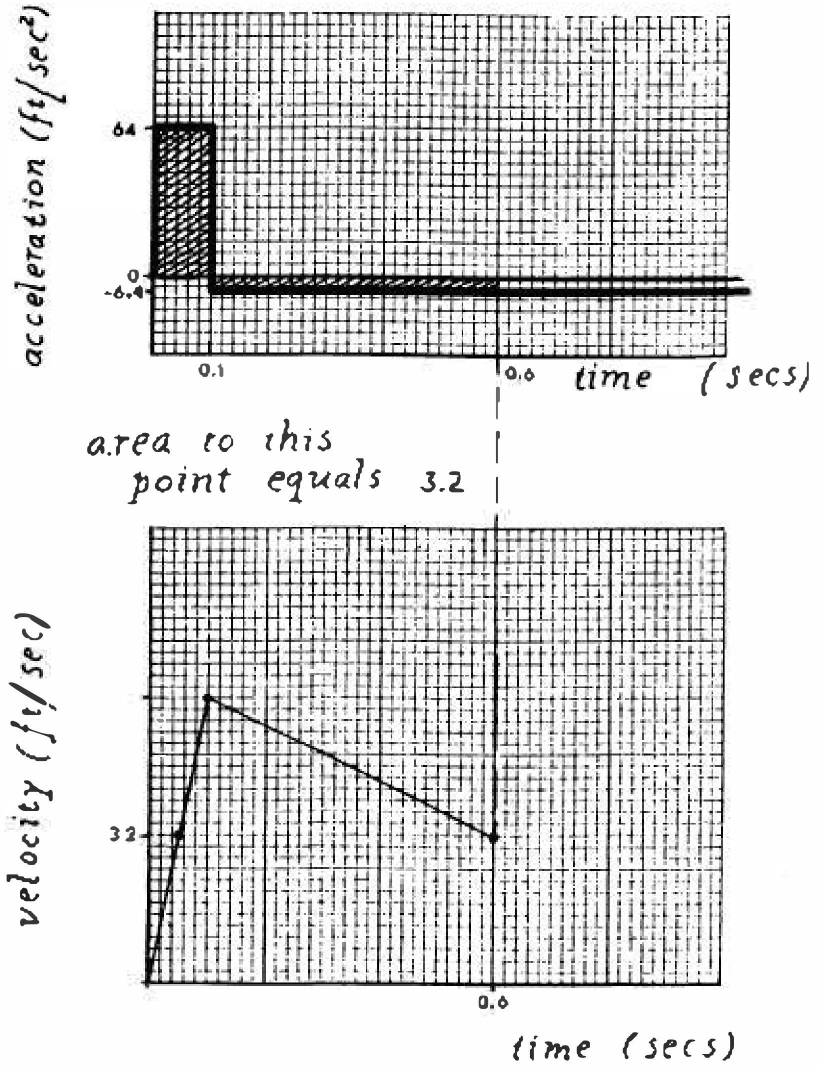


Fig. 2-6. Deriving a velocity curve from an acceleration curve—third step.

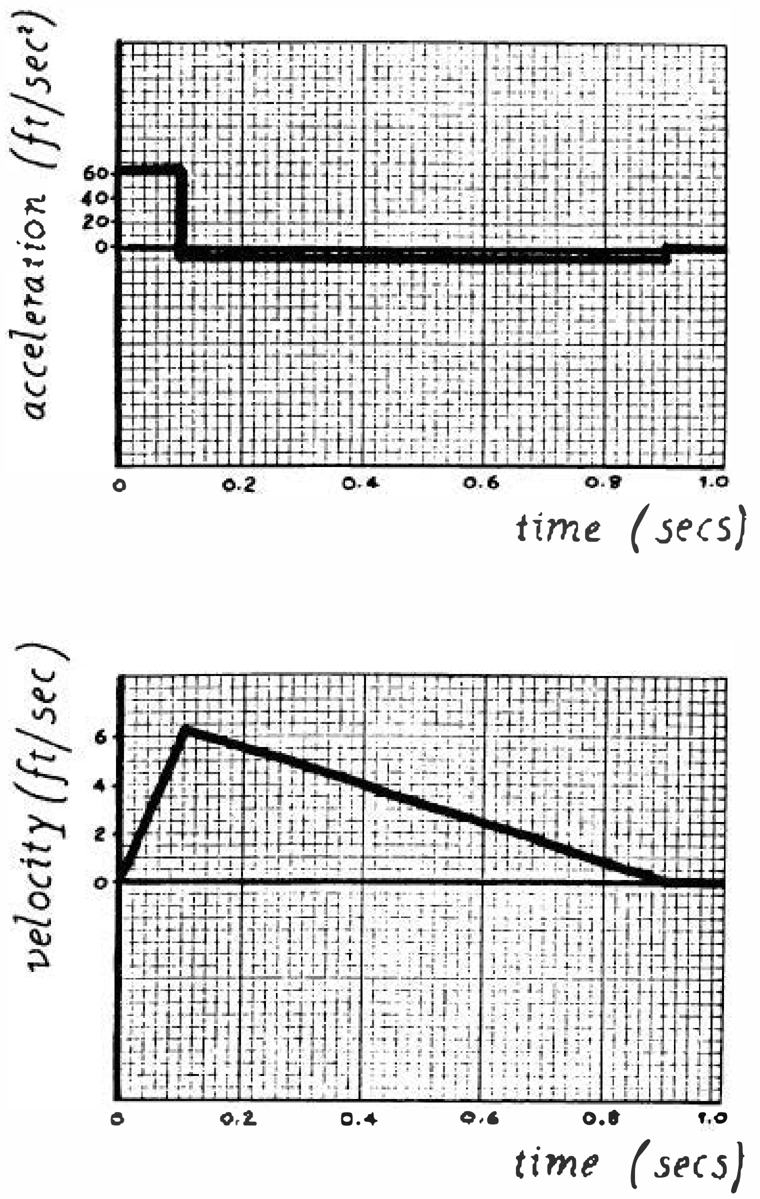


Fig. 2-7. Completed relationship between acceleration and velocity curves for the block-on-the-table.

pounds! If the three known "things" are multiplied together and algebra is used to "cancel" the "pies" unit the result is:

$$500 \text{ pies} \times 200 \frac{\text{in}^3}{\text{pies}} \times 100 \frac{\text{ounces}}{\text{ft}^3}$$

$$= 10{,}000{,}000 \frac{\text{in}^3 - \text{ounces}}{\text{ft}^3}$$

—which sounds meaningless. Somehow we must get rid of the in^3 and ft^3 terms and convert OUNCES to POUNDS. To do this multiply the original expression by conversion factors:

We know that $1 \text{ ft}^3 = 1728 \text{ in}^3$, or that

$$\frac{1 \text{ ft}^3}{1728 \text{ in}^3} = 1$$

We also know that 16 oz = 1 lb, or $\dfrac{1\text{ lb}}{16\text{ oz}} = 1$

The "pie" equation can now be multiplied by these two expressions for "1" to change its units without changing its true value:

$$500\text{ pies} \times 200\,\frac{\text{in}^3}{\text{pie}} \times 100\,\frac{\text{oz}}{\text{ft}^3}$$

$$\times\,\frac{1\text{ ft}^3}{1728\text{ in}^3} \times \frac{1\text{ lb}}{16\text{ oz}} = 362\text{ lbs}$$

or 0.72 pound per pie.

All units have cancelled algebraically except the one wanted, pounds, so there is the answer. Practice this method of checking units. It is extremely worthwhile.

Displacement

Getting back to graphical mechanics: once there is a velocity-versus-time curve we can graphically integrate again—finding the area under the velocity curve this time—to get a curve of the displacement of the block on the table as a function of time. Use exactly the same graphical procedure as in going from the acceleration curve to the velocity curve, first finding the area between "time = zero," and "time = 0.1 sec," and setting this equal to the displacement at 0.1 sec, etc.

Again, the units of the area of the original curve (in this case, velocity) become the units for the vertical axis of the derived curve (in this case, the displacement curve), as can be seen by the following equation:

Area under the velocity curve = height × base

$$\left(\text{area} = \frac{\text{ft}}{\text{sec}} \times \text{sec} = \text{ft}\right)$$

and feet, of course, is the correct English unit for displacement.

Figure 2-8 shows the velocity curve of Fig. 2-7, repeated, and the displacement curve that is derived from it. Again, if graphical integration is still not understood, please re-read the discussion. It is essential that you understand it if you are to use the methods of this text.

Note that the displacement curve does not return to zero in this case. Although the block has started from rest, moved, and stopped again, it retains its new displacement, or position. This is typical of many intermittent motion devices.

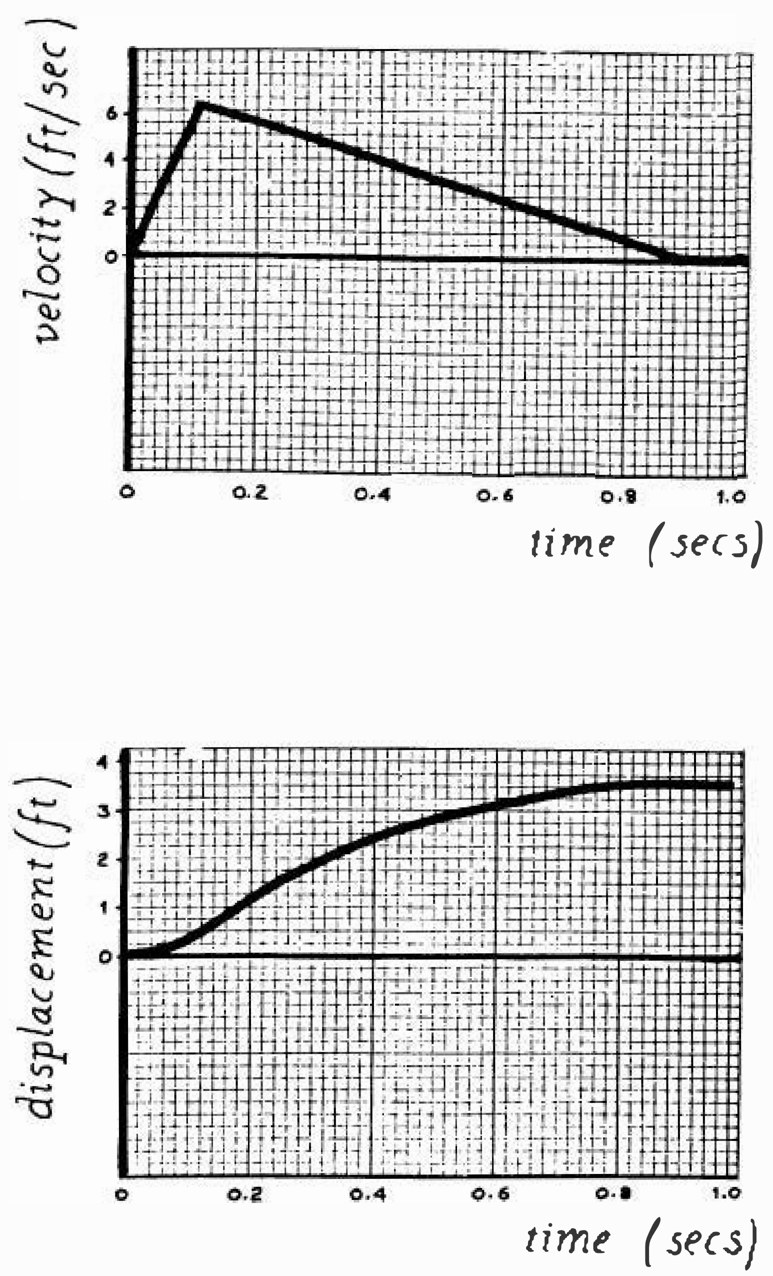


Fig. 2-8. Relationship between velocity and displacement curves for the block-on-the-table.

Exact Numerical Values by Graphical Integration

The process of graphical integration has now led from a force curve (acceleration curve after the vertical scale was changed), to a velocity, and then a displacement curve. This provides all of the necessary information about a mechanism, to analyze its performance. However, the techniques of Chapter 1 must be used to find critical values on the curves of force and velocity if exact numerical values for the various parameters are wanted, because the graphical integration process is a method for obtaining approximate results only. It is used for insight, and to reduce the number of points for which exact numbers are calculated, rather than for definite numerical data.

To get exact accelerations the methods of Chapter

1 must be applied to find instantaneous forces, then Newton's Second Law is used to convert these to accelerations. Or the methods of Chapter 1 can be used to find exact values for velocity just before and just after the point at which it is desired to determine the acceleration; and then use the equation:

$$\text{average acceleration} = \frac{v_2 - v_1}{t_2 - t_1}$$

to determine the average acceleration between times t_1 and t_2. If t_1 and t_2 are nearly equal, the average acceleration will equal the instantaneous acceleration at that point.

There is no really good way to find exact numerical values for displacement by graphical integration. Here it is better to study the geometry of the parts involved and to attempt to relate displacement to the motion of the input link or shaft of the machine. If this input is running at a constant velocity (and only then!), it can be used as a "time base" and output displacement plotted versus input displacement, as shown in Fig. 2–9. Input displacement can be used as a base for all curves; displacement, velocity, and acceleration; and graphical integration can still be used to get the *shape* of lower curves. Cam designers frequently use displacement of driver for the horizontal axis. But be sure input displacement is converted to time before equating area under a curve to height of the next curve, in numerical terms. Ft/sec times secs = ft; but ft/sec times radians (input displacement) has no real meaning.

Deriving a Velocity Curve from a Displacement Curve

So far, we have seen how to derive acceleration, velocity, and displacement curves when starting with an experimental or predicted curve of force on a body as a function of time. It is often difficult or impossible, however, to determine what the force "picture" should be, so a procedure must be developed for going in the other direction; for starting, let us say, with a displacement-versus-time curve and from it deriving velocity, acceleration, and force curves. In a calculus course this process is called "differentiation." It can be done mathematically only if we can describe the original curve mathematically. For our purposes, however, it will be done graphically and without the mathematical description of the curve. As a result, extremely complex curves can be handled nearly as readily as simple curves. The procedure will always be the same, at any rate.

A curve, is differentiated by "measuring its slope" at a number of points. The slope at each point is found by drawing a line tangent to the curve at each point, and applying the equation: slope = H/B, as shown in Fig. 2–10. Note that the "slope" is equal to the tangent of the angle θ. Slopes, like areas, can be considered to be positive or negative in sign. They are *positive* if the tangent line is directed *upward* from left to right, as in Fig. 2–10; and are *negative* if the tangent line is directed *downward*, from left to right.

Notice that the process of calculating the slope can be used to determine the units for the quantity being derived; just as it was possible to use the process of taking areas under a curve to determine the units for the quantity being derived. In taking the derivative of a displacement-versusetime curve, for example, the vertical axis has the units of feet, and the horizontal axis has the units of seconds; treating these quantities algebraically, we determine

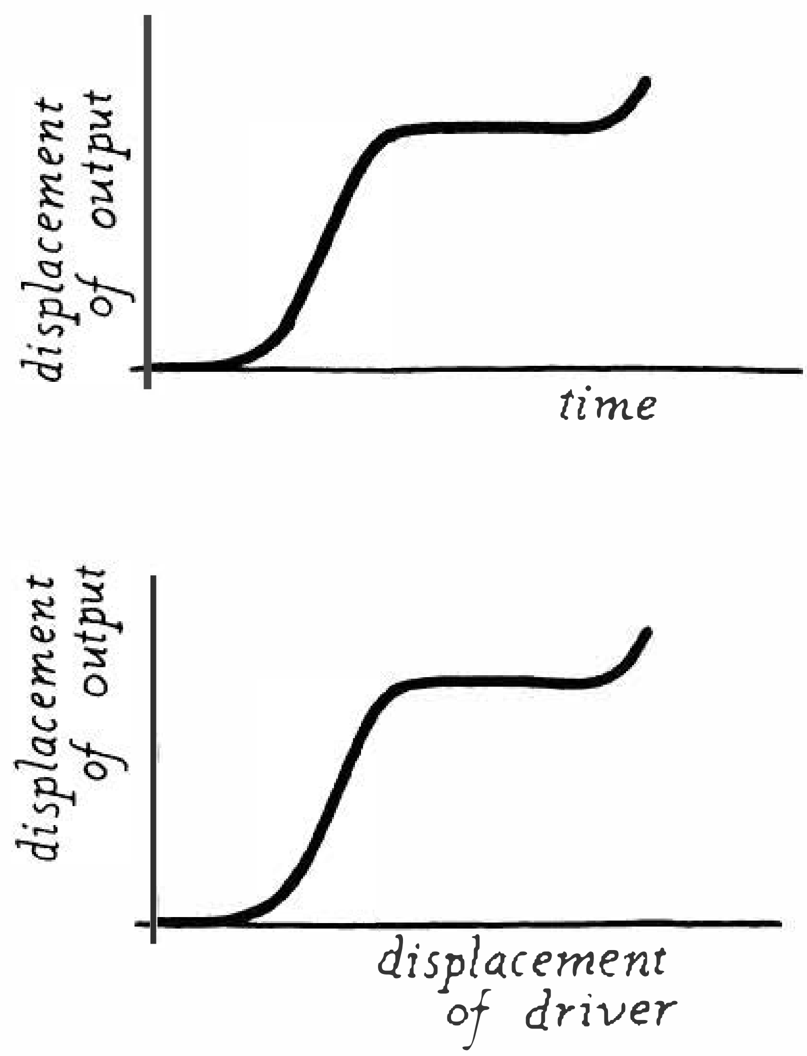


Fig. 2–9. Displacement of the "driver" can be used for a horizontal axis instead of time; but only if the driver is moving with a constant velocity.

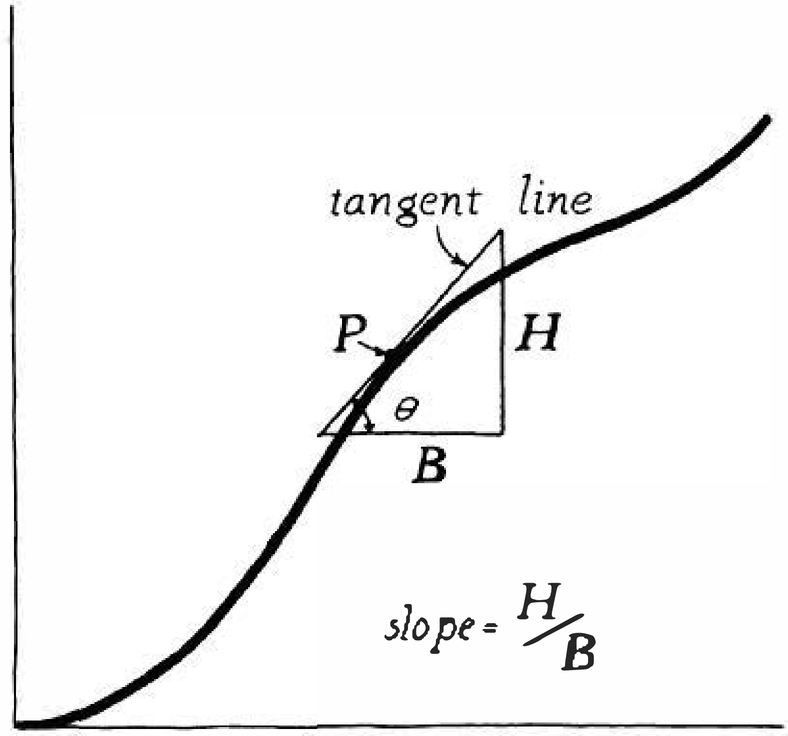


Fig. 2-10. Graphical differentiation—Determining the slope at point P.

that the units for velocity are feet per second, as shown below:

$$\text{slope} = H/B \text{ (from Fig. 2-10)}$$

$$\left(\text{units: slope} = \frac{\text{ft}}{\text{sec}}\right)$$

In Fig. 2-11 is seen the derivation of a complete velocity curve from a displacement curve. These graphs should be studied until their meaning is really apparent, since this process of taking derivatives is just as basic and important as is the process of taking areas. Notice that whenever the displacement curve is horizontal (parallel to the time axis), the derived or velocity curve is 0, for example. Notice that the maximum peaks on the velocity curve do not occur at times of maximum displacement, but at times of maximum rate-of-change of displacement. After some experience in graphical differentiation is gained, a derivative curve can often be roughed-in just by an inspection of the first curve, using a few special points such as those described above. This is discussed further, in a later section of this chapter.

In practice, slopes of curves can be taken by drawing tangent lines (by eye) on displacement curves and computing the slopes algebraically. There are instruments that can be used to draw accurate tangent lines in some conditions, and also other instruments for rapid calculation of the slopes of these tangent lines. Both are described in the appendix.

If one of these instruments is not available and, therefore, arithmetic or a slide rule must be used to compute numerical values for the slopes drawn, it will be a convenience to use the same base distance (B), for each little slope "triangle" drawn. This will simplify the many divisions needed; or the division steps can be skipped and the various triangles' altitudes (H values) can be plotted directly, to sketch the derived curve. Its shape will be correct as long as all B's are equal. Then the correct vertical scale can be computed and added, to complete the drawing.

Graphical differentiation, just as graphical integration, is not a precise process, and will not lend itself to exact answers except in special cases, but again, this need not be considered a serious problem. Exact answers for intermittent motion situations cannot be obtained anyway—usually, for example,

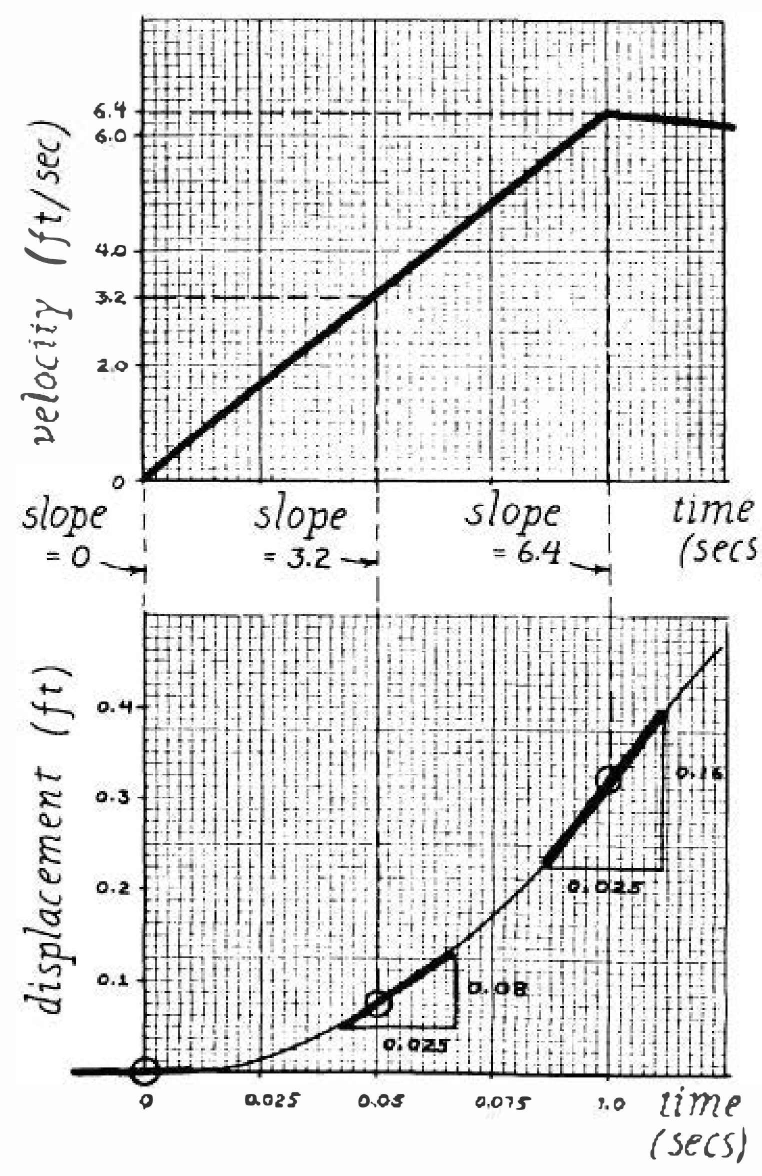
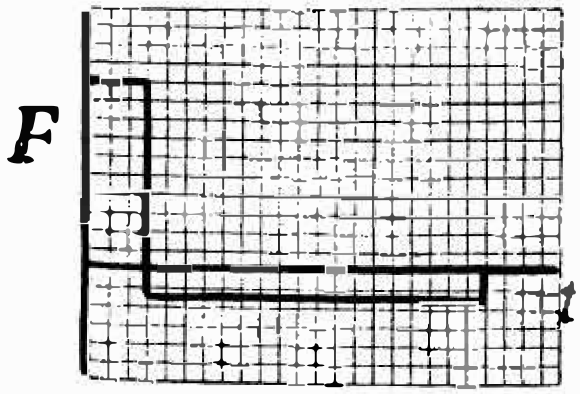
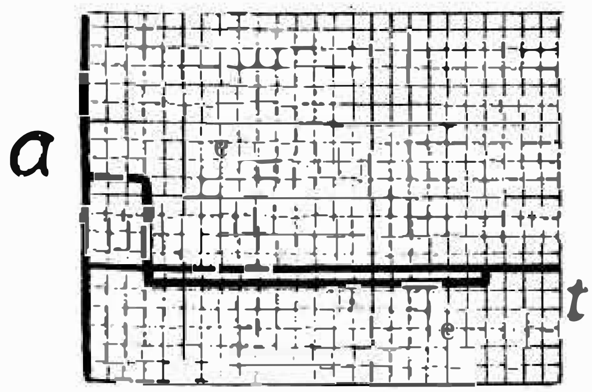


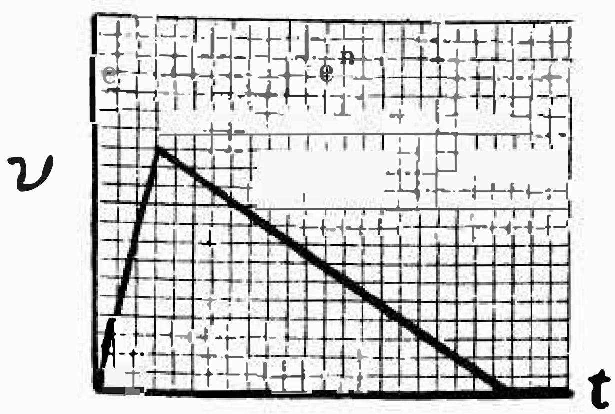
Fig. 2-11. Deriving a velocity curve from a displacement curve by graphical differentiation.



Divide F by "mass" m, to get a curve, below, or multiply a by m to get F curve, above.



Take areas under a to get v, below, or take slopes of v to get a, above.



Find areas under v curve to get d curve, below, or find slopes of d curve to get v curve, above.

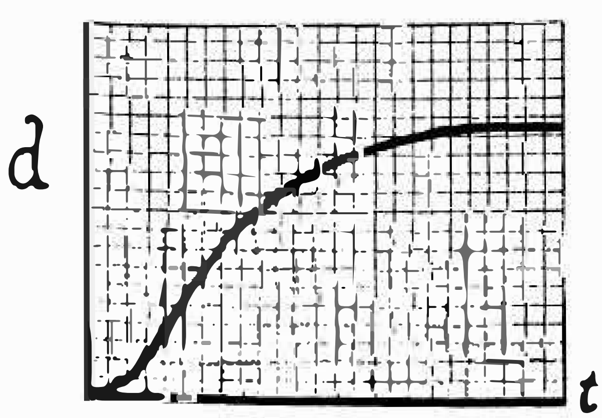


Fig. 2-12. The relationship between the force (F), linear acceleration (a), linear velocity (v), and linear displacement (d) curves.

it is impossible to write the equations of displacement as a function of the time which would allow exact mathematical solutions. The best that can be hoped for is to get an approximate feel for various situations, and this the graphical process does. Sometimes one can get "fairly exact" solutions for a few points on the curves, using the techniques of Chapter 1, for example, but even this is virtually impossible with many types of intermittent devices, as will be seen in later chapters. But the curves do permit a valuable insight to be obtained and they should be drawn.

From Velocity, to Acceleration, to Force

Using the process of graphical differentiation, an acceleration curve can also be plotted from the previously derived velocity curve. Then, using the familiar $F = ma$, each point of the acceleration curve can be multiplied by a constant factor (the mass of the body being studied) and a force-versus-time curve derived; reversing the process shown in Fig. 2–3.

Summary of Motion Curves—FAVD!

The text has now shown how to proceed from a force-versus-time curve to acceleration, to velocity, and then to displacement curves; and also how to proceed from a displacement-versus-time curve to velocity, acceleration, and force curves. It is now possible to enter the force-acceleration-velocity-displacement (FAVD) sequence at any point and to derive the rest. For example, if an experiment or design prediction gave a velocity curve as a function of time, graphical integration could be used to derive the displacement curve, and graphical differentiation to derive the acceleration curve. The acceleration curve would then be multiplied by mass to get a force curve. All that is needed in any problem is that first curve. Graphics will give approximate solutions for the rest. The vector methods of Chapter 1, or design data, will produce exact numerical values for critical points on these curves.

Figure 2–12 summarizes the relationships between the force, acceleration, velocity, and displacement curves for the block sliding on the table. Again, make certain that these relationships are understood before going further. This is not a difficult procedure once the sequence is apparent; but it is an essential development for the graphical understanding of intermittent motion devices. Perhaps the following will prove helpful in remembering the procedures:

> Remember the relationships between the F-a-v-d curves and how to start with any one and derive the others.
> "F A V D"—*F*ind *A*reas *V*en going *D*own.
> "D V A F"—*D*ifferentiate *V*ery *A*ctively *F*riend, on going up!

But Be Careful!

Graphs, followed blindly, can cause as much trouble as a problem in mathematics followed blindly. Even though only one variable may be measured (force, or acceleration, or velocity, or displacement), there is probably some knowledge of the other factors in a given design situation. Make sure that your graphs appear to represent what you think the situation might indeed be.

Infinite Accelerations

Figure 2–13 shows a typical trap. It begins with a curve of displacement versus time and measurements

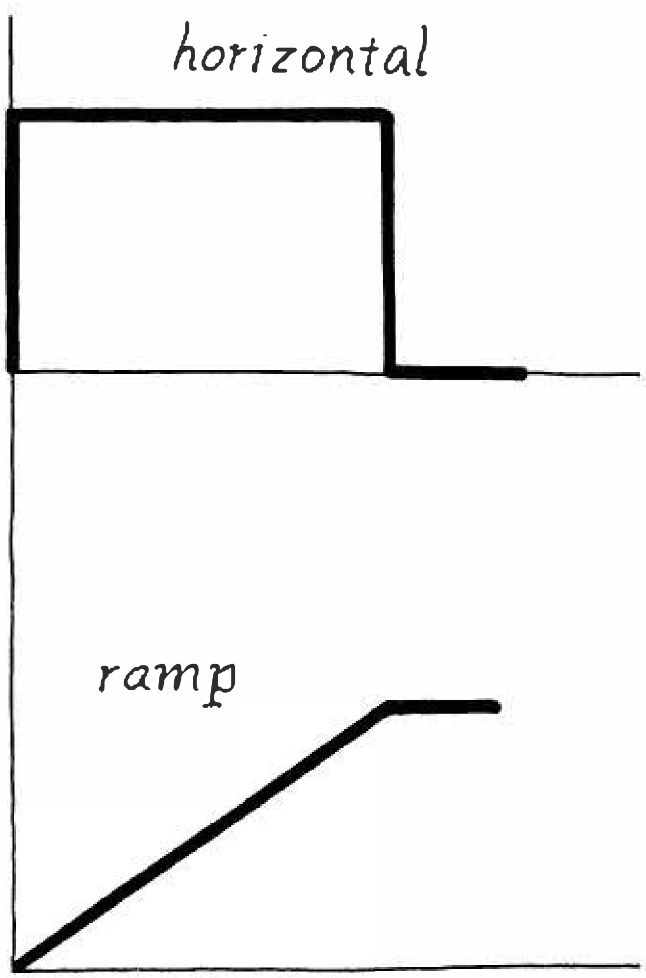


Fig. 2–13. Relationship between a ramp displacement curve and a step velocity curve.

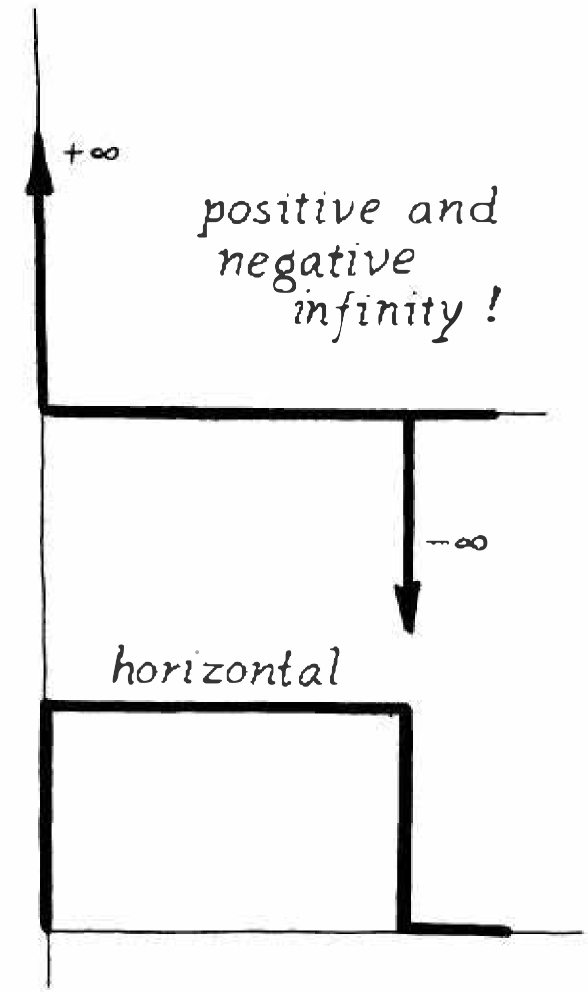


Fig. 2–14. A step velocity curve produces plus and minus infinite accelerations.

or design information indicating that this curve would be shaped like a ramp, as shown in the figure; with an upward slope from time equals zero to the inflection point of the curve. If this were the case, the velocity curve would consist of a straight line parallel to the time axis shown in the same figure. Notice that the velocity curve is vertical at its beginning and end: there was a finite velocity at the beginning and end of motion (just after, and just before, the body was at rest) says the displacement curve.

If we now go on to plot the acceleration curve (Fig. 2–14), remembering that acceleration is defined as the rate-of-change of velocity with time, we find that when the body starts into motion it must have an *infinite* positive acceleration. The velocity change was not very great (from zero to some finite level), but since it occurred in zero time (instantaneously) the *rate*-of-change, the acceleration, must be infinite.

$$\frac{\text{Change in velocity}}{0} = \infty$$

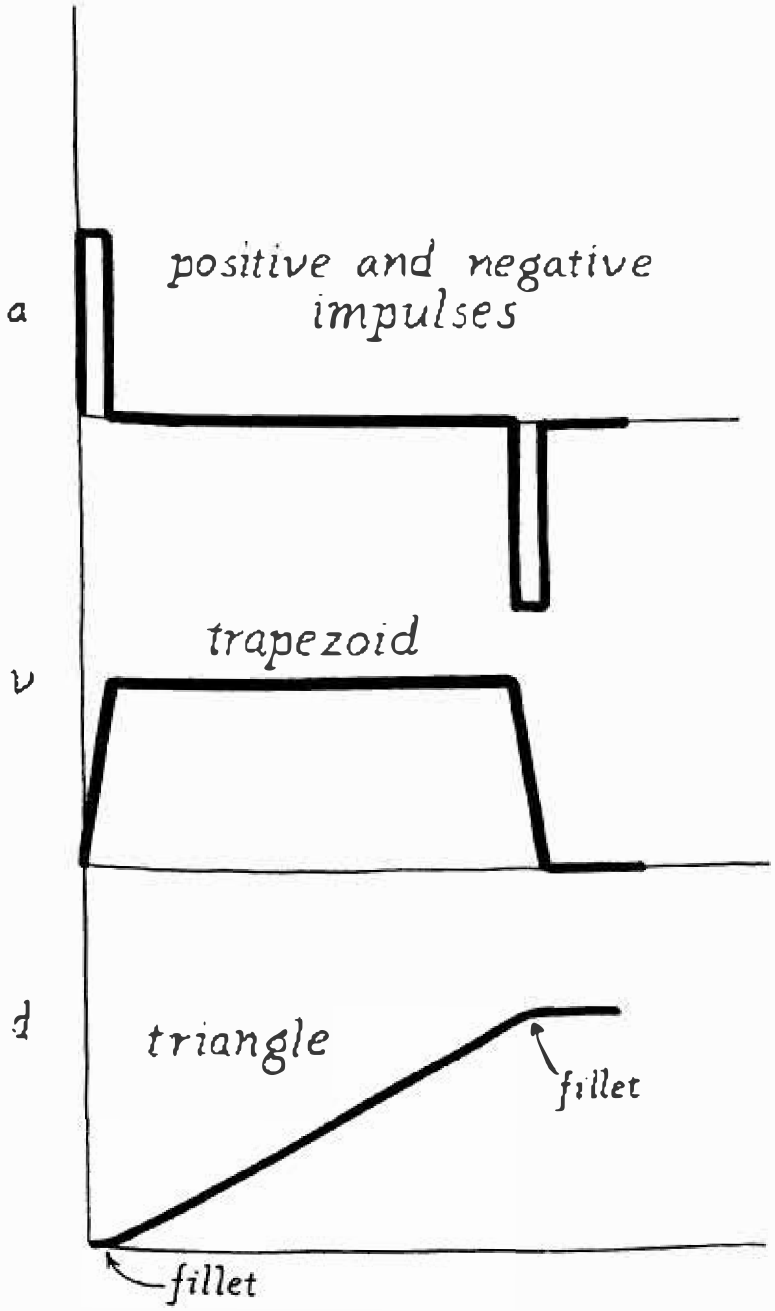


Fig. 2-15. Small fillets at the beginning and end of a ramp displacement curve produce a velocity trapezoid and finite positive and negative accelerations.

By the same token, an infinite negative acceleration must later be applied, to bring the body to rest. Since we obtain force curves by multiplying the acceleration curve by the mass of the body, it is apparent that an infinite force would have to be applied to the body in order to accomplish this apparent displacement as a function of time.

Of course, infinite forces and accelerations are not possible. What actually happens in a machine design situation is that the basically triangular displacement-versus-time curve will have small "fillets" at the beginning and end, as shown in Fig. 2-15. There would be, in other words, a finite period of time during which the velocity changed from zero to the essentially constant positive value, and a second finite period of time during which the body was brought to rest. Very frequently, these tiny fillets will be virtually impossible to measure, especially with intermittent motion mechanisms where motion reversals, etc., occur under impact conditions, in small fractions of a second. As will be seen later, this results in a great deal of difficulty in estimating maximum stress levels, etc., in such devices. In many cases, however, it is sufficient for the designer to realize that such fillets must exist. He must estimate their possible duration and put them into the measured or derived curves, accordingly.

Another aspect of this, however, would be to remember that in an intermittent motion device everything starts at zero. The output link of the mechanism is at rest, which means, for graphical evaluation purposes, that the initial velocity and acceleration are zero. The initial net force on the body is also at zero to begin with. The previous graph of Fig. 2-14 indicated that the velocity had reached a finite value at time zero and this resulted in infinite accelerations to get started. If a time at which the whole mechanism is at rest is picked as the zero time point, and the fact that it takes a finite period of time for it to start moving is recognized, these difficulties will be avoided.

In graphical mechanics, all calculations begin with everything at rest in order to avoid what the mathe-

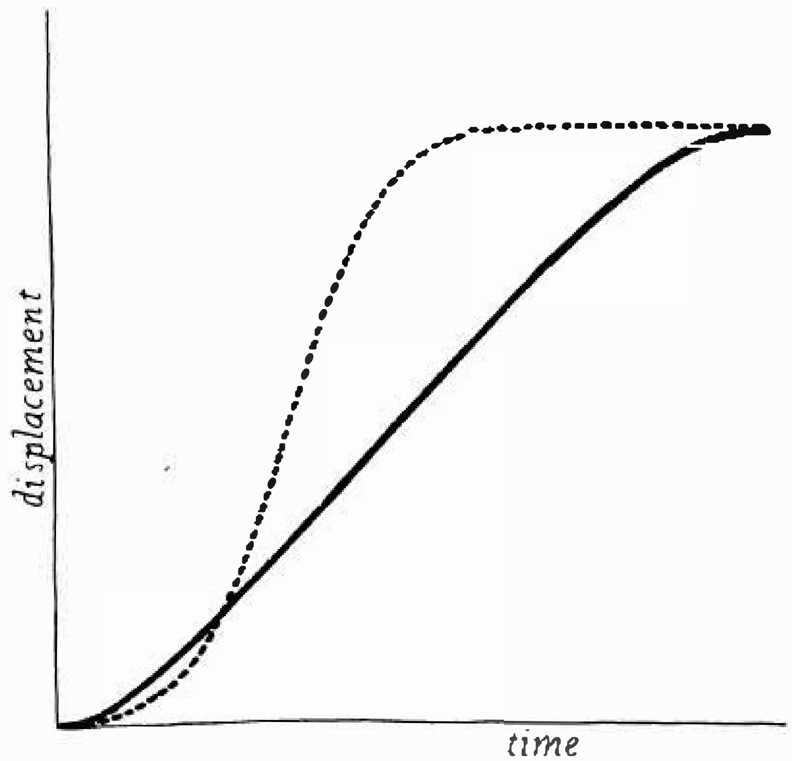


Fig 2-16. Similar displacement curves for an internal Geneva (solid line), and external Geneva (dotted line).

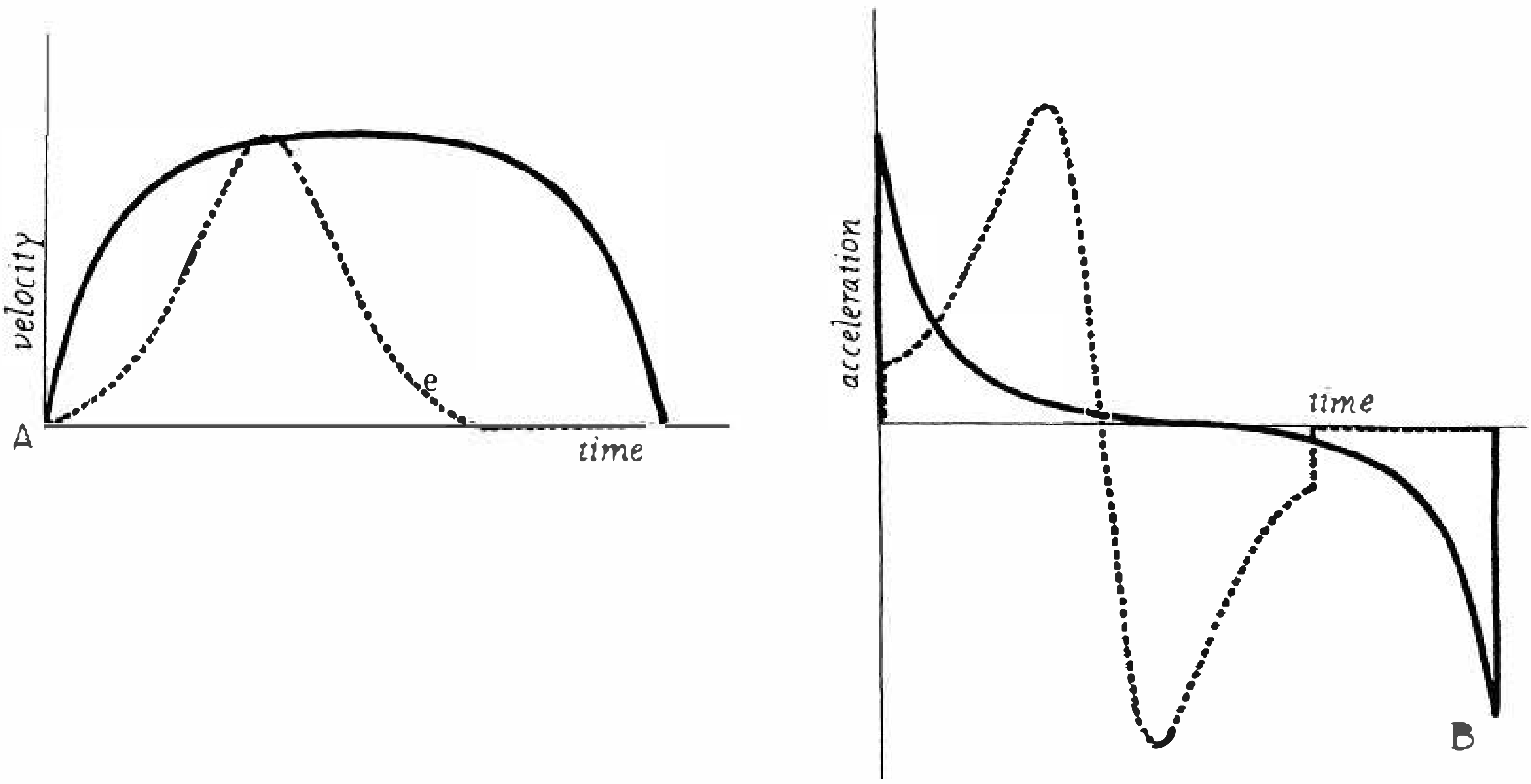


Fig. 2–17. (A) Velocity and (B) acceleration curves for the internal and external Geneva. The curves for each mechanism are drawn to different vertical scales to emphasize their shape differences. (See Fig. 9–6 which shows the same curves drawn to a common scale.)

maticians call "constants of integration," as mentioned earlier. The above results show why this is true.

Another Caution—Watch Out for Small Differences!

In the previous section it was noted that a small difference in the displacement curve (the addition of nearly invisible fillets at the beginning and end of a displacement curve) turned two infinite accelerations into two finite accelerations. Other small changes in curve shape can make a surprising difference in the shapes of related curves. The beginner is advised to be very careful, and to plot more points on his derived curves that he would be inclined to do, at first, until he has had a good deal of experience with graphical methods. Figure 2–16 shows superimposed displacement curves for internal and external Geneva mechanisms (described in detail in Chapter 10). The curves are drawn to different vertical scales (the internal Geneva has lower acceleration than the external Geneva for a given input speed) in order to emphasize their different shapes. Notice that both displacement curves seem to be "ramps" with fillets at the beginning and end of the ramps. The dotted curve shows larger fillets and a steeper slope than the solid curve, but otherwise they seem very similar in basic shape.

Figures 2–17a and 2–17b show the velocity and acceleration curves of internal (dotted curves) and external Genevas. The velocity curves may have a distant relationship to each other, perhaps; although some obvious differences have begun to appear. The acceleration curves though are very different. Small differences between the basic shapes of the original curves failed to suggest the drastic differences in the acceleration patterns of these two similar mechanisms.

Jerk

One more topic must be considered before this discussion of linear motion is completed; that is, the subject of *jerk*. The subject of jerk does not appear in physics books, but machine designers insist that it exists and that it must be considered in design situations involving heavy loads and/or high speeds. Jerk is defined as the rate-of-change of acceleration of a body. As such, it is computed by differentiating (graphically or mathematically) the acceleration-versus-time curve, as shown in Fig. 2–18. Although infinite acceleration would require an infinite force and is, therefore, impossible; infinite jerk simply means "an infinite desire to start in motion once the force is suddenly applied." Although from a purely theoretical point of view, any vertical step in the acceleration curve produces infinite jerk, designers assure us that a smaller step change in acceleration produces less jerk than a larger step. Machine tool designers worry about this parameter, and we will consider it from time to time as we compare the relative merits of various intermittent motion devices.

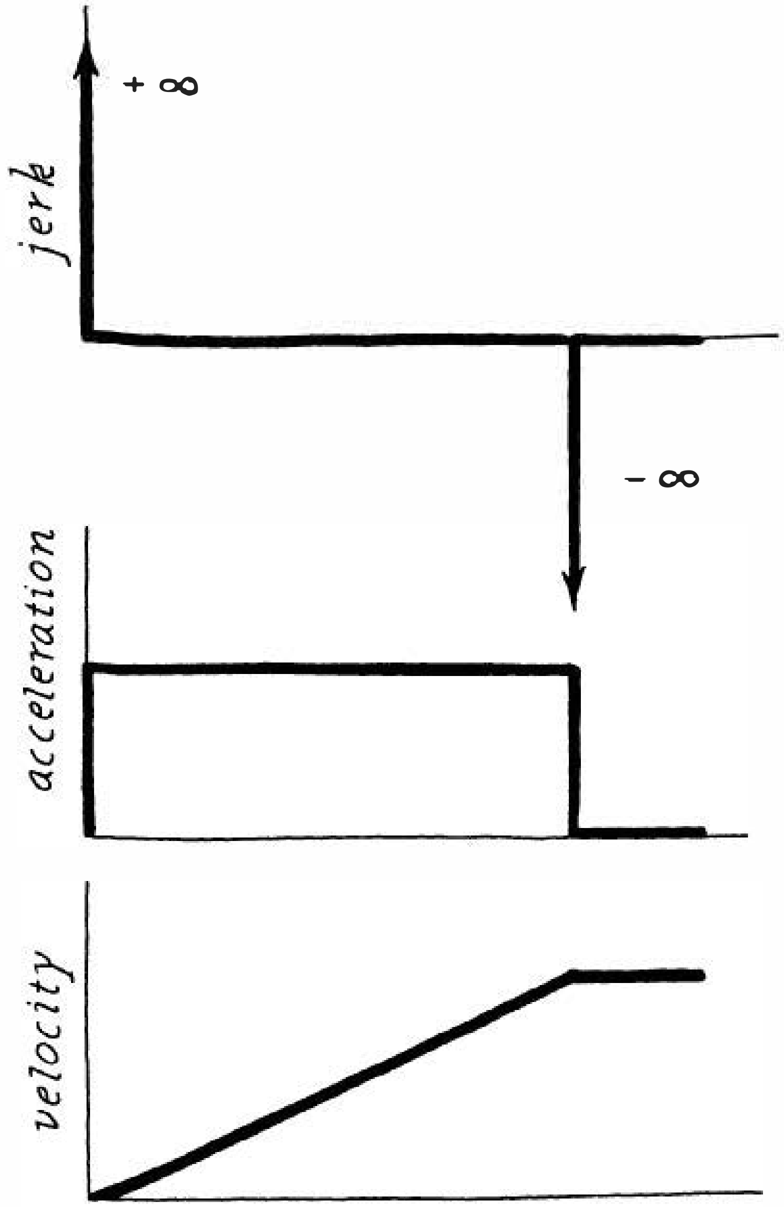


Fig. 2-18. Relationship between velocity, acceleration, and jerk curves.

ROTATIONAL MOTION

Thus far, only the case of linear motion has been considered. In most machine design situations rotational motion will be of far more interest. Fortunately, now that the linear ease is understood, only a new set of terms and units is needed to handle the rotational case.

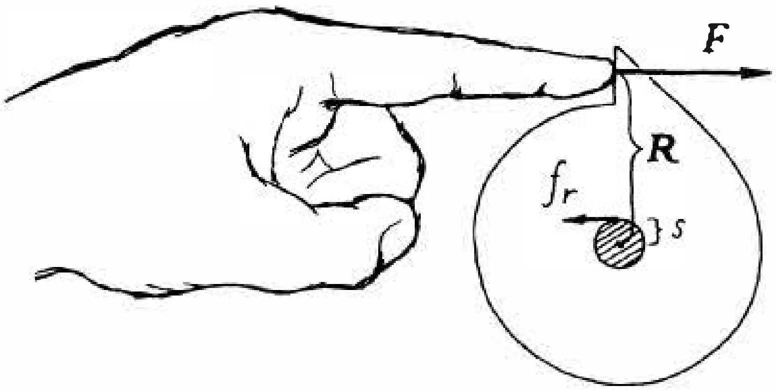


Fig. 2-19. Forces acting on a disc.

In Fig. 2-19, a finger pushes a ratchet wheel. This produces a drive torque on the wheel $F \times R$ and the inevitable friction reaction torque $f_r \times s$. If the finger pushes the wheel briefly and then stops, the wheel will eventually coast to a rest under the action of the frictional torque, as in Fig. 2-20. This is an exact analogue of the block-on-the-table example studied earlier, and the curves which we would draw of torque (τ), angular acceleration (α), angular velocity (ω), and angular displacement (θ), are identical to those drawn for the earlier example. We use different symbols and units for the various factors involved, as shown in Fig. 2-21. Compare this illustration to Fig. 2-12 to see that they are, indeed, exact analogues of each other. We move down in the curve set by taking areas; we move up by measuring slopes; just as before.

The second curve in Fig. 2-21 describes the angular acceleration of the wheel as a function of time and as a result of the applied torque. The angular acceleration curve is found by dividing the torque on the body at any instant, by the moment of inertia of the body; just as, previously, linear acceleration

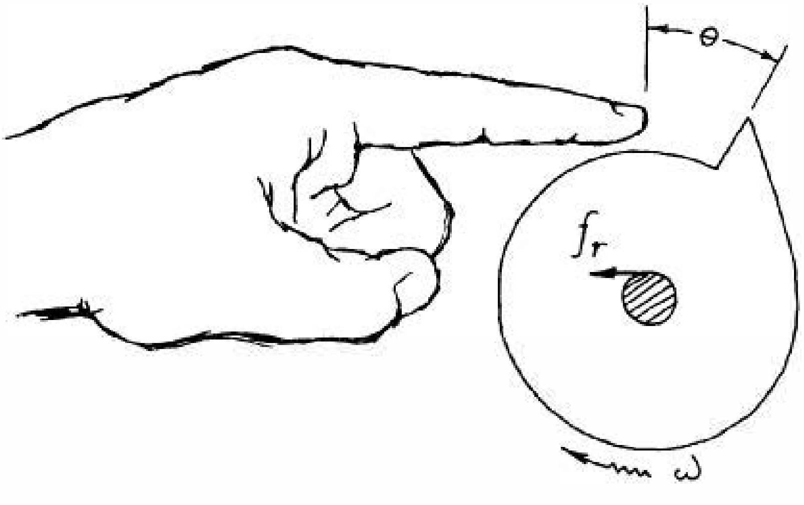
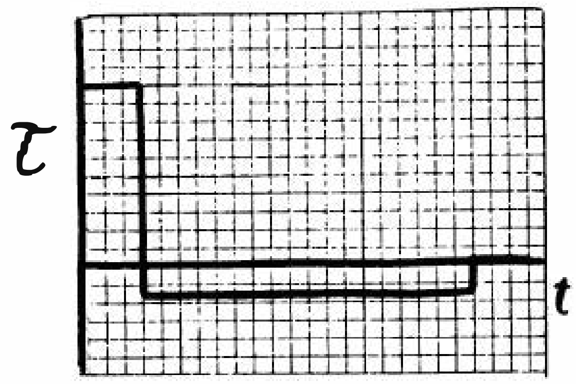
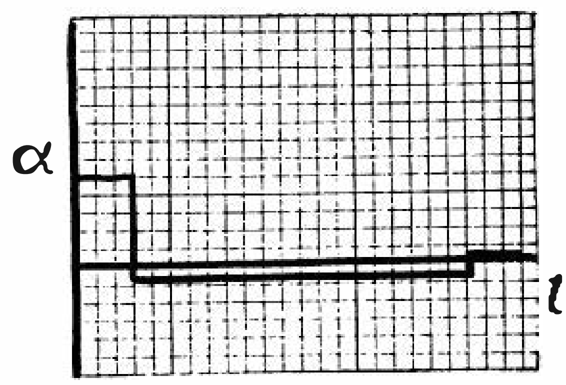


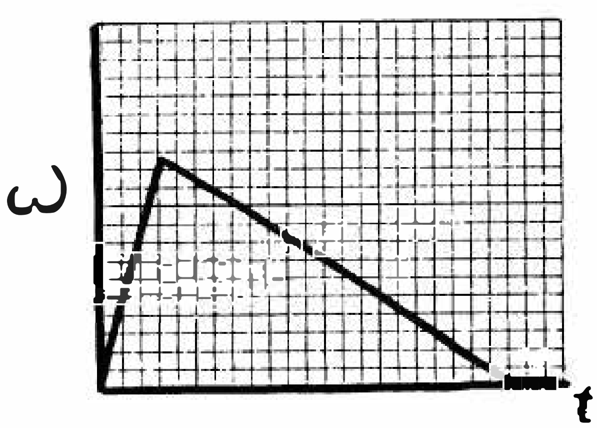
Fig. 2-20. Frictional force acting on a coasting disc.



Divide τ by "moment of inertia," I to get α curve, below, or multiply α by I to get τ curve, above.



Take areas under α to get ω, below, or take slopes of ω to get α, above.



Find areas under ω curve to get θ curve, below, or find slopes of θ curve to get ω curve, above.

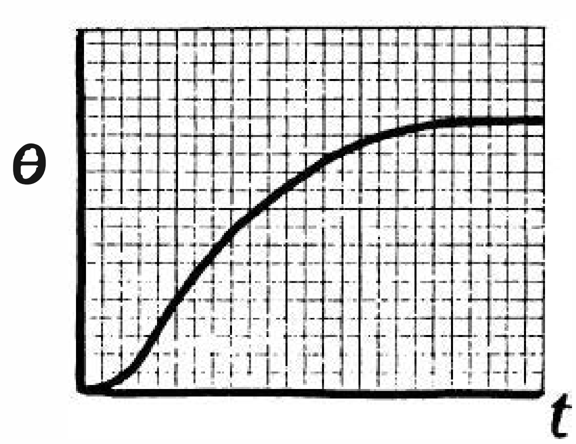


Fig. 2-21. Relationship between torque, angular acceleration, angular velocity, and angular displacement curves for the finger-on-a-disc case.

was found by dividing force by mass. The equation:

$$T = I\alpha$$

is the rotational analogue of $F = ma$.

It should be noted for accuracy's sake that $T = I\alpha$ is only an analogue (a mathematical equivalent, not a physical equivalent) of $F = ma$ and is not, in itself, a basic law. $T = I\alpha$ only applies when a body is constrained by bearings to rotate about a principal axis through its center of gravity. It does not, therefore, describe the complete motion of unbalanced bodies or of free bodies rotating in space. However, to gain an insight into the design problems of intermittent motion mechanisms without attempting to gain exact solutions, this analogue will serve the purpose very well (and most machine bodies obey the above restrictions).

Rotational Units

The unit of angular displacement in the English system is the radian. There are 2π radians in a complete circle, thus one radian is equal to 57.3 degrees. The unit of angular velocity is, therefore, radians per second; since velocity is rate-of-change of displacement. The unit of angular acceleration, similarly, is radians per second per second, or radians per second, squared. And torque comes in lb-ft. The unit of moment of inertia is slug-ft^2 or ft-lb-sec^2. Moment of inertia is a measure of the resistance of the rotating body to angular acceleration. It is discussed at some length in the appendix.

Angular Acceleration

The angular acceleration (α) of a rotating body is its rate-of-change of angular velocity (ω). Exact numerical values for certain points on an angular acceleration curve are often wanted when plotting sets of motion curves. There are two ways to obtain these. If the instantaneous applied torque (drive torque, less friction and load torques, in many cases) on a given body is known, its instantaneous angular acceleration can be found by applying the equation: $\alpha = \tau/I$. Sometimes, friction torque can be ignored, which simplifies things.

Another method is to use the techniques of Chapter 1 to calculate two values of ω at two different times, t_2 and t_1. Since angular acceleration is the rate-of-change of angular velocity it is then possible to use the equation:

$$\text{Average } \alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1}$$

to compute the average value of α in the time interval, t_1 to t_2. If t_1 and t_2 are close to each other, there will be little difference between average acceleration and instantaneous acceleration for that interval of time.

To be complete, it should be mentioned that there are other accelerations associated with rotational motion; such things as Coriolis acceleration and radial acceleration. These account for the changes in tangential (linear) velocities that occur as an object rotates, or moves to a different radius. However, at this point, our concern will be only with angular acceleration: the increase or decrease in the number of radians per second of angular velocity of the body.

Angular Displacement

Frequently, a set of motion curves is started by plotting angular displacement versus time; differentiating this graphically, to derive velocity and then again to obtain acceleration curves. The displacement curve is produced with the aid of mylar or paper templates and the original design drawings; determining output displacement as a function of input displacement, perhaps, and then converting this to a time-based curve by using other design information such as the speed of the drive motor.

OTHER ASPECTS OF GRAPHICAL MECHANICS

How to Handle More than One Force

The force plotted in studying linear motion was the net horizontal force on the body. In most machine design problems interest will be in the motion of a body under the action of a single resultant force or of some component of the resultant force, as discussed in Chapter 1. Fortunately, the methods already derived can be used to determine the response to either a resultant or to a component. How can the complete motion of a body be analyzed by evaluating its motion for several components separately?

The diagram at the top of Fig. 2-22 shows a hand that is pushing a block along a book on a table. The hand exerts a positive driving force of F_d on the block. The book exerts a negative friction force of f_f, as shown in the illustration. At time, $t = 0.1$ sec, the block runs off the edge of the book, but the hand keeps on pushing it in the same direction until it lands on the table where it eventually slides to a halt. This might be a little difficult to do, but it definitely would be possible.

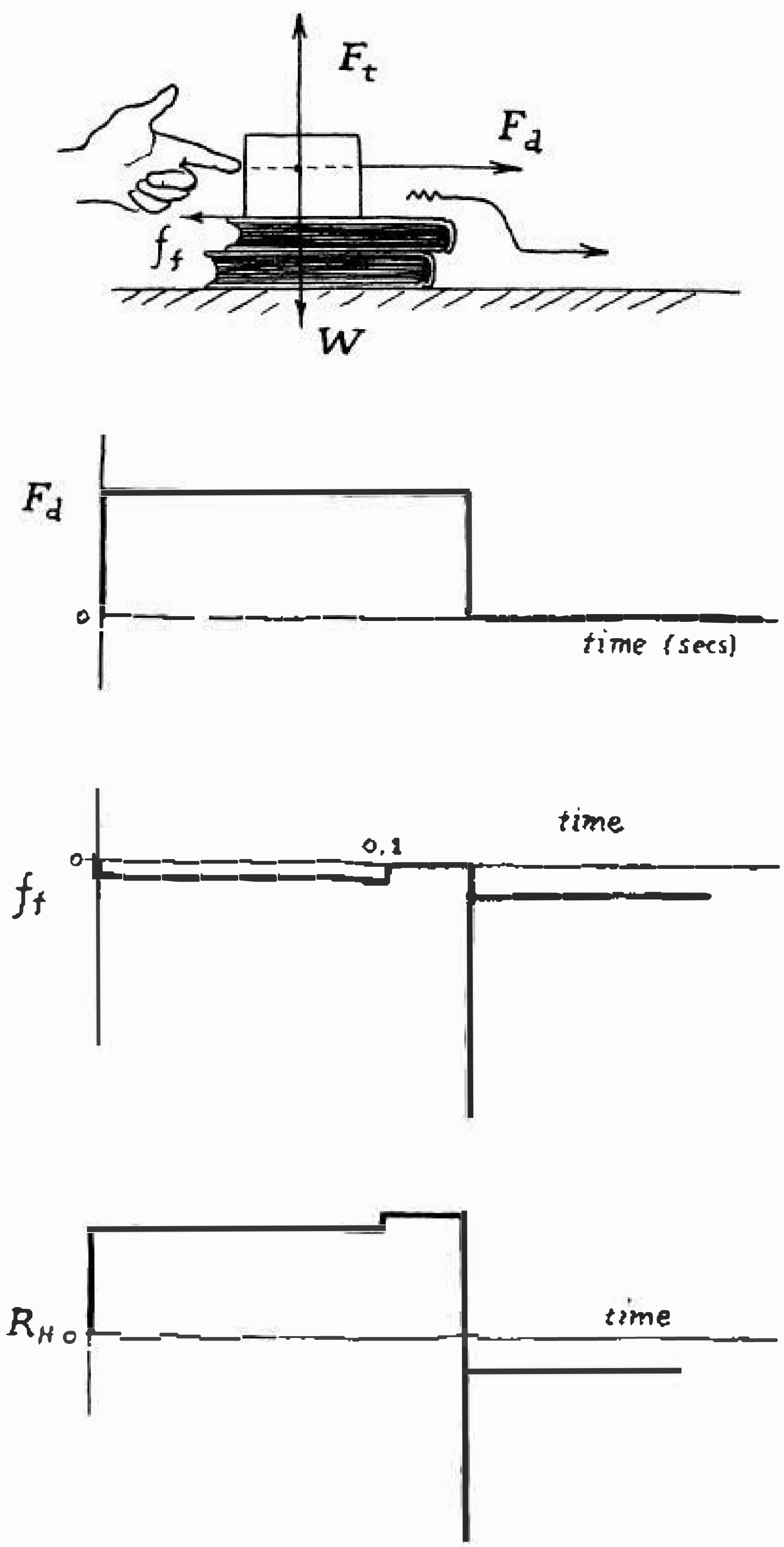


Fig. 2-22. Forces acting on a block which is being pushed off a pile of books; and graphs of the component and net horizontal forces on the block.

In the graphs below the diagram, in Fig. 2-22, two different horizontal component forces are plotted; the force of friction and the force exerted by the finger on the block, also, the net horizontal resultant (which was used in the original block-on-a-table analysis to determine the horizontal acceleration, velocity, and displacement of the block as a function of time). The negative spike in friction force occurs when the block first strikes the table and impact increases the normal force on the table.

Plotting finger and friction forces separately in this manner, can be useful. This allows for estimating the amount of work done against friction; for example, the impulse imparted to the block by the finger, etc., as will be seen later on. Finger and friction forces, furthermore, would be easier to measure on an experimental basis than would be the resultant. They could be measured and plotted separately, and then added graphically, to derive the resultant, if this were the case.

There are also other forces on the block which are not considered at all by the analysis just completed. While the block is sitting or sliding on the book, the book is exerting a vertical supporting force against it, labeled F_t, in Fig. 2-22. At the same time, the weight of the block is pressing downward against the book with an equal and opposite force (W). Once the block has been pushed over the edge of the book, the supporting force F_t, vanishes, and the only remaining vertical force is W, the weight of the block (ignoring any vertical frictional component between the finger and the block).

The graph in Fig. 2-23 shows F_t, F_b, W, and the net resultant vertical force R_v as a function of time. This resultant vertical force is 0 while the block is on the table, but becomes finite (and negative) once the block has gone over the edge of the book. The block at this point, then, will start to move (accelerate) in a downward direction under the influence of this unbalanced vertical force. Finally, it strikes the table which stops it with a vertical blow and then supports it with a force F_b, again equal and opposite to W. Figure 2-23 shows the entire process.

By using the previous procedures, both horizontal and vertical accelerations, velocities, and displacements of this block, could now be determined, considering the horizontal and vertical motions separately.

Once horizontal and vertical values for acceleration, velocity, and displacement are determined, nothing further is needed to predict the total position, motion, etc., of the body in the XY plane as a function of time.

In fact, total motion can be found in this case

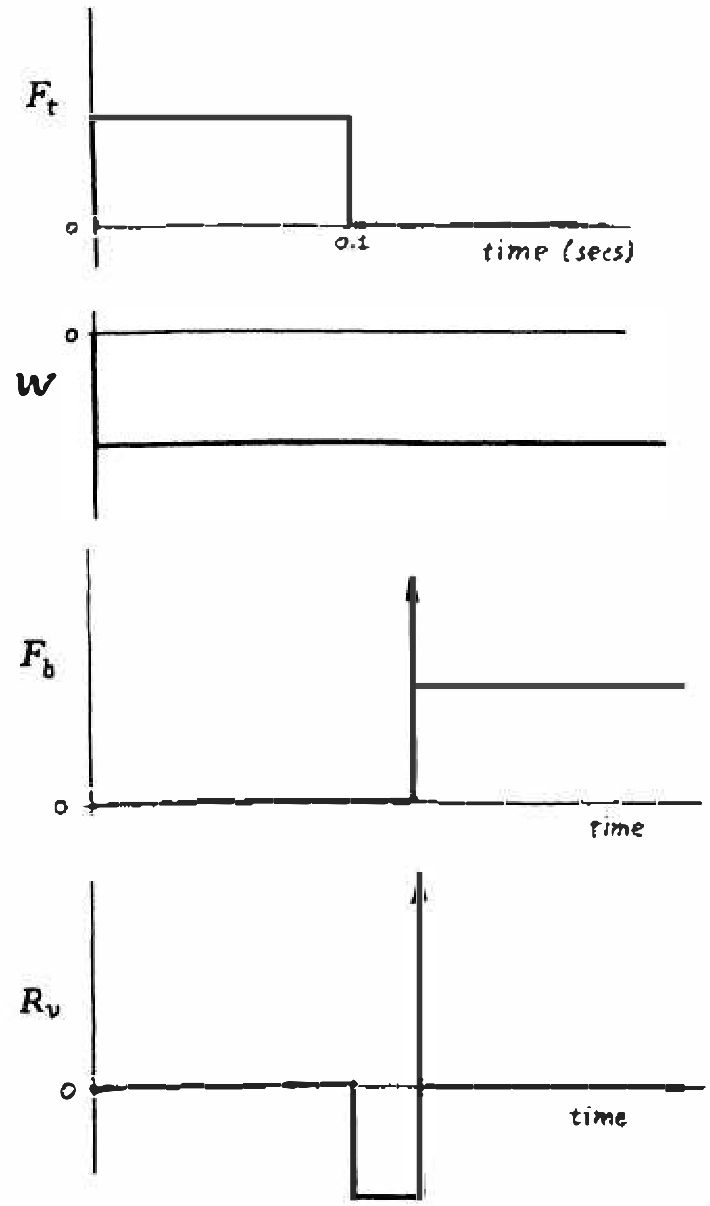


Fig. 2-23. Graphs of component and net vertical forces on the block being pushed off a pile of books.

only by calculating horizontal and vertical motions separately, then using these results as X and Y components of acceleration, velocity, and displacement to plot the total motion of the body. It is possible, with vector mathematics, to handle single body resultants that change magnitude and direction, but this cannot be done with the graphical techniques described here. The graphs *only* show variations in *magnitude* of vector quantities whose line-of-action directions do not change. (The vectors can reverse direction by 180 degrees, go from positive to negative, but the line of action must not change.) The resultant force on the block on a book, the single force that sums up all the X and Y forces on the block, starts as a pure X force, when the weight of the block is supported by the books; turns into an XY force as the block leaves the books; and ends as a pure X force as the block slides along the table. Each of these resultants could be plotted separately to find the total motion of the block, but "initial velocities," etc., would have to be accounted for at each switch from one resultant to another. This could cause serious trouble, as explained earlier. Thus it is necessary, when resultant forces change directions, to stick to components when making motion curves. In the first two examples, i.e., the original block on the table and the rotating disc, the motion can be analyzed with either components (separating friction and finger forces, for example) or resultants (adding finger and friction forces and figuring the response to this resultant), because the line-of-action direction of the resultant force in one case, or torque in the other, does not change.

Work, Energy, Momentum, Etc.

The science of mechanics deals with other things besides force, acceleration, velocity, and displacement, of course. Work, energy, impulse, and momentum, for example, are commonly used concepts and are all useful for mathematical solutions of many problems. It will not be necessary to examine all of these concepts in depth, in this book, since the goal of achieving a rough understanding of "what goes on" in intermittent motion mechanisms can be attained even though practice is restricted, primarily, to the FAVD approach.

Furthermore, concepts such as energy and momentum are more difficult to handle, graphically, than FAVD because they involve simultaneous treatment of several bodies in a machine. In FAVD a free body is isolated and studied alone, although, of course, it is necessary to know what forces other elements of the machine exert on the body being studied. In dealing with energy, on the other hand, laws such as that for the conservation of energy, which says that the total energy of a multibody system remains constant unless acted upon by external forces, must be respected. This means that during collisions or interactions between bodies, energy within the system is redistributed or altered in form; with potential energy in one body, for example, being converted to kinetic energy in another; with kinetic or potential energy being converted to useful work; or with mechanical energy being converted to heat energy and being lost as far as useful output is concerned. These are powerful concepts, but again, are really beyond the scope of this work and are unnecessary for our purposes. Included in Chapter 3, however, is a brief discussion of impulse, momentum, energy, etc., as related to impacts between machine members.

Table of Curves—and Some Basic Rules for Constructing Curves

Students of calculus make frequent reference to tables of differentials and tables of integrals when trying to do mathematically, what has been done here graphically. It would seem logical, therefore, to include in this text a pictorial table showing the relationships between different types of curves that will be encountered in many intermittent motion devices. Once the student learns to recognize these different shapes he can often integrate and differentiate curves almost by inspection.

First, however, some basic relationships between curves should be considered. Figure 2-24 shows the relationship between a zero slope on one curve, the axis crossover point on the related "derivative curve," and the change in slope on the related "integral curve." This relationship helps us find maximum

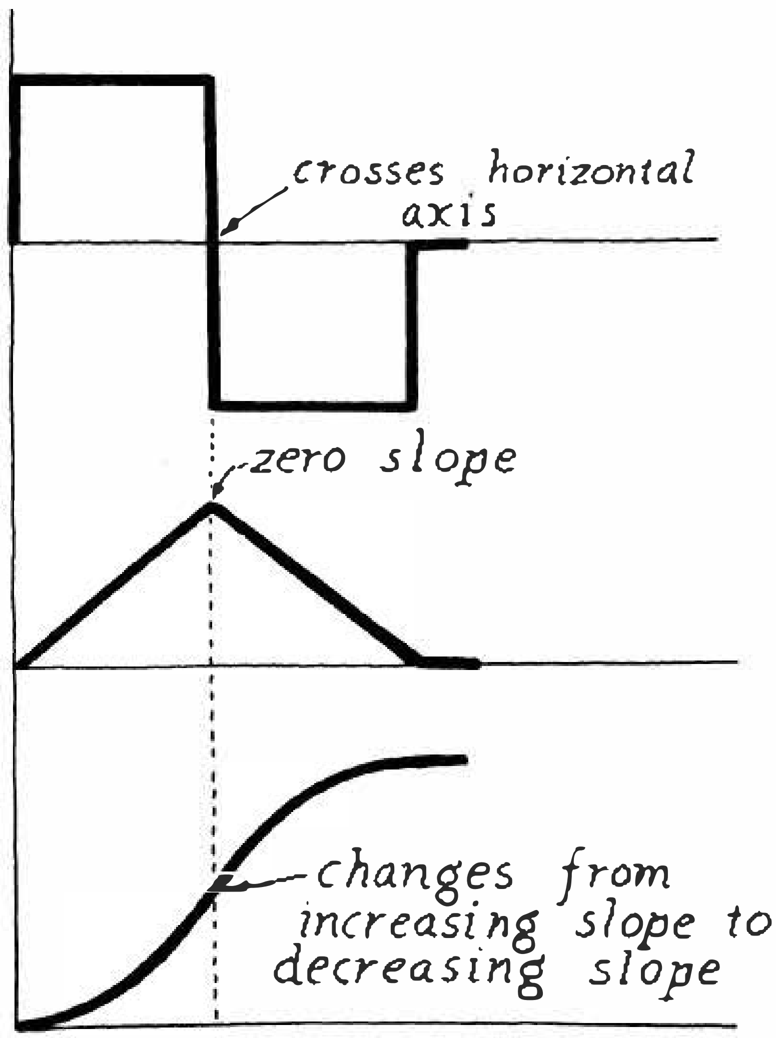


Fig. 2-24. The relationship between the axis crossing, and the zero slope in a series of related curves.

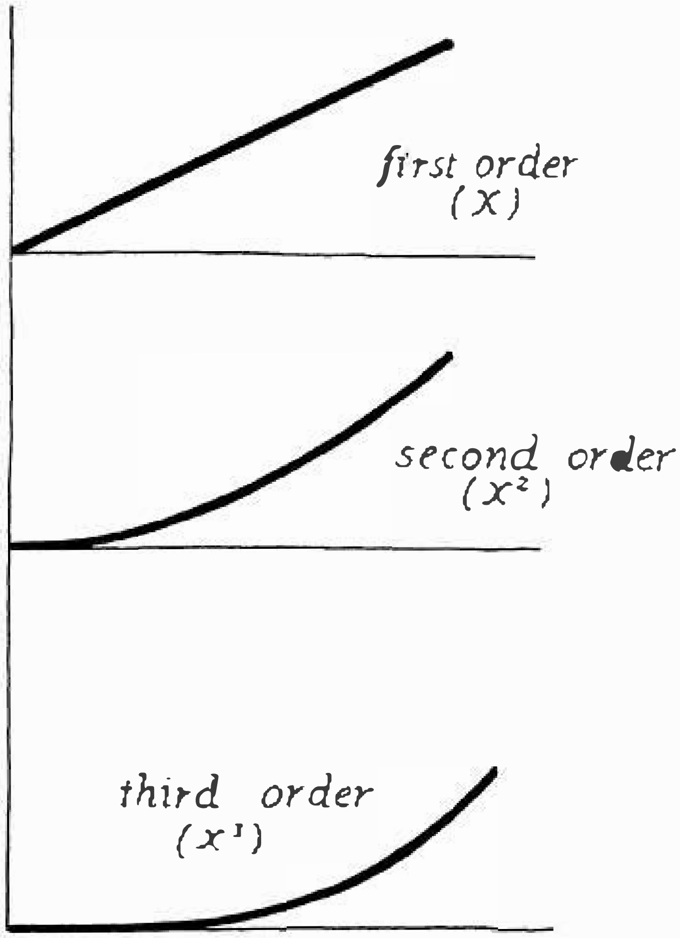


Fig. 2-25. Curves become more concave with successive integrations.

and minimum points on "integral curves." Figure 2–25 shows how the "order" of the curve (its degree of concavity) increases with multiple integrations (moving down): or decreases with multiple differentiations (moving up). Figures 2–26 and 2–27 show

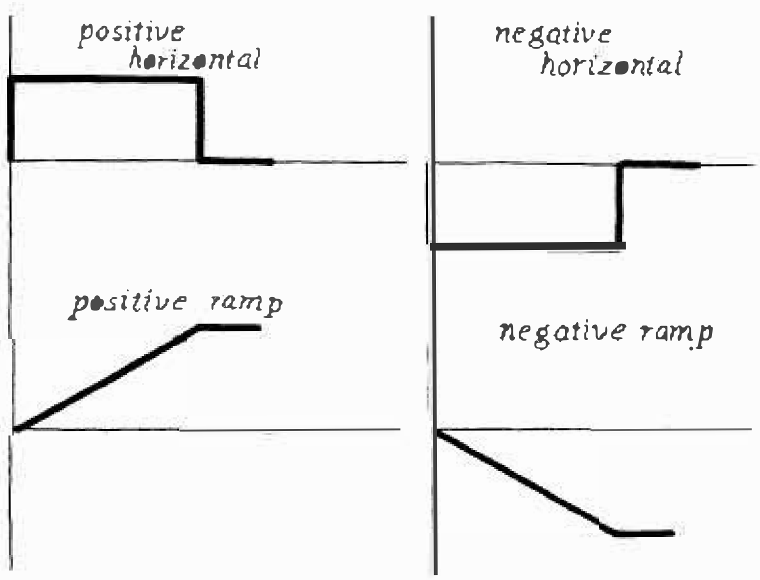


Fig. 2-26. Positive and negative curves of the same type.

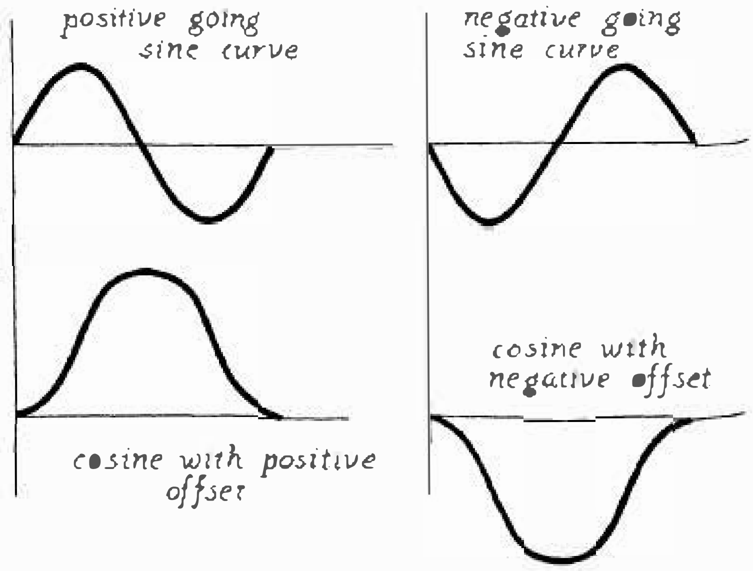


Fig. 2-27. More positive and negative curves.

how mirror-image curves produce mirror-image integral, or differential, curves.

When one basic shape of curve follows another, sequentially, in a single curve, the integrals and differentials also follow, with each portion of the derived curve being that which would be obtained for the related portion of the original curve if taken alone. For example, in Fig. 2-28a, the integral (going

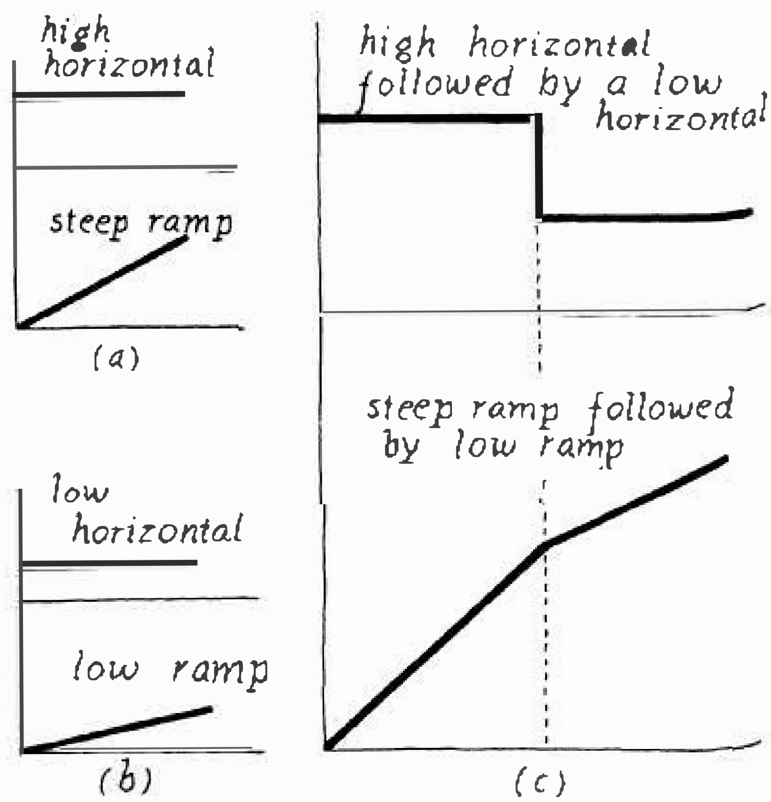


Fig. 2-28. When one type of curve follows another, the integral or differential curves also follow in the same order.

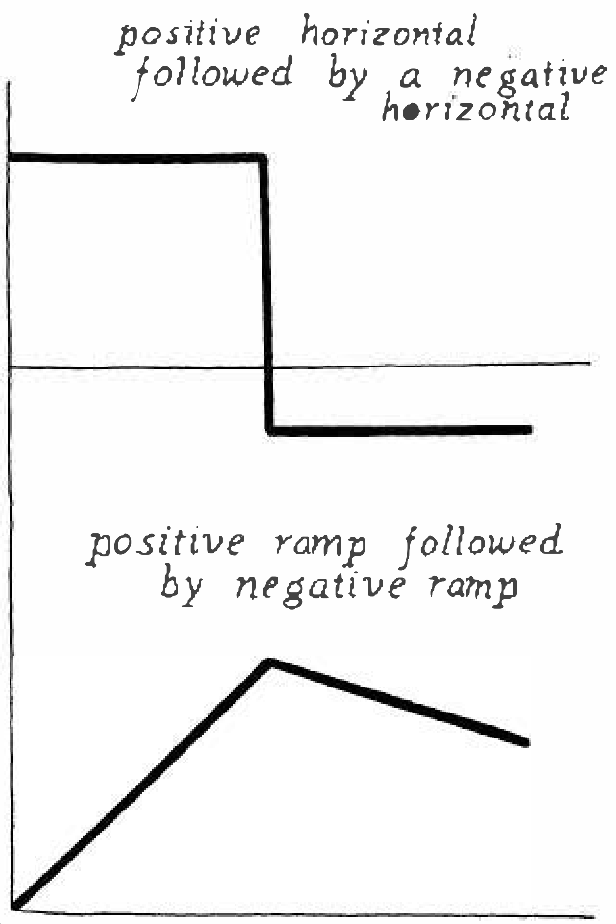


Fig. 2-29. Another illustration of "following" curves.

down) of a high horizontal curve would be a steep ramp. The integral of a low horizontal, Fig. 2–28b, would be a low ramp. If there is a single curve, as in Fig. 2–28c, in which a high horizontal is followed by a low horizontal, its differential curve will be a steep ramp followed by a low ramp. Notice, however, that the low ramp starts at the end of the high ramp rather than on the horizontal axis.

Similarly, in Fig. 2–29, a positive horizontal, followed by a negative horizontal, produces (going down) a positive ramp followed by a negative ramp.

Sometimes, curves are superimposed or added together, rather than following each other sequentially. In Fig. 2–30, for example, are a cosine curve and its integral, Fig. 2–30a, a horizontal and its integral, Fig. 2–30b; and an offset cosine curve (a cosine curve superimposed on a horizontal, as shown by the dotted line) and its integral (a sine curve superimposed on a ramp).

These are some of the "rules." There are undoubtedly others. Nothing beats experience, however. By sitting down and counting squares or measuring slopes to derive one curve from another, a feel for graphical mechanics is soon achieved. Nothing else really needs to be understood except:

(a) How to get that first curve of F, or A, or V, or D versus time
(b) How to measure areas
(c) How to measure slopes.

The rest follows automatically.

A series of ten drawings makes up Fig. 2–31, to show the relationships between common curve sets which are often found in machine design.

Figure 2-32 shows Da Vinci's famous perpetual motion machine. With a knowledge of how to "calculate" intermittent motion mechanisms, it should be possible to discover why it did not work.

In this device, a large ratchet wheel has a hub on which are mounted twelve arms. Each arm is free to rotate through an angle of approximately 60 degrees about a pivot in the surface of the drum. There is a weight at the outer end of each arm, and a series of stops on the outer ratchet wheel which catches the weights on the rim in the positions shown.

Presumably, it is necessary to give the wheel a shove to start it in motion. Once in motion (in a counterclockwise direction) it is supposed to keep

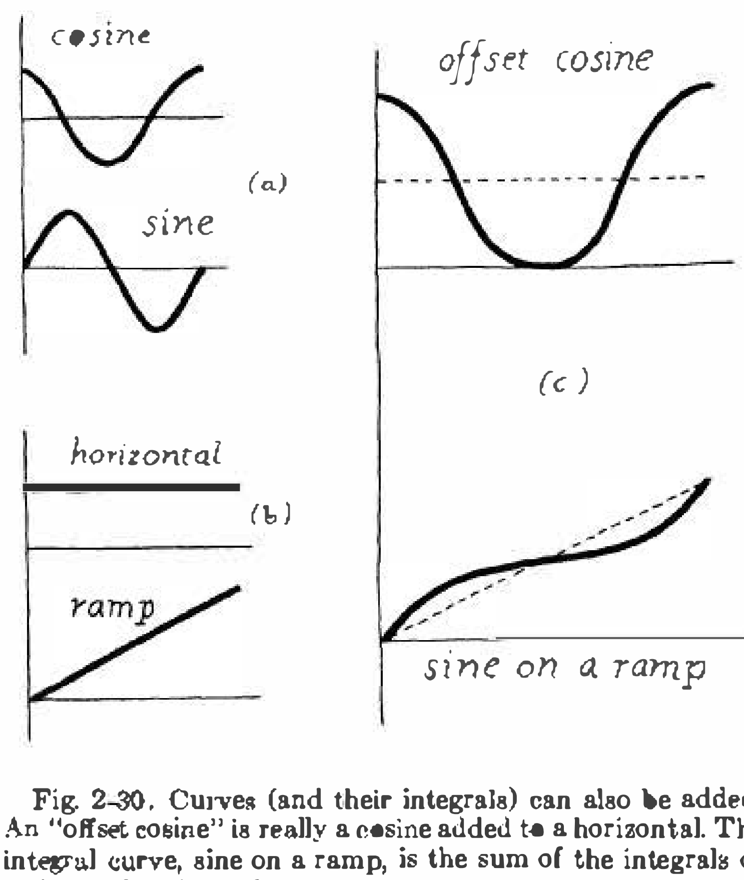


Fig. 2-30. Curves (and their integrals) can also be added. An "offset cosine" is really a cosine added to a horizontal. The integral curve, sine on a ramp, is the sum of the integrals of cosine and horizontal.

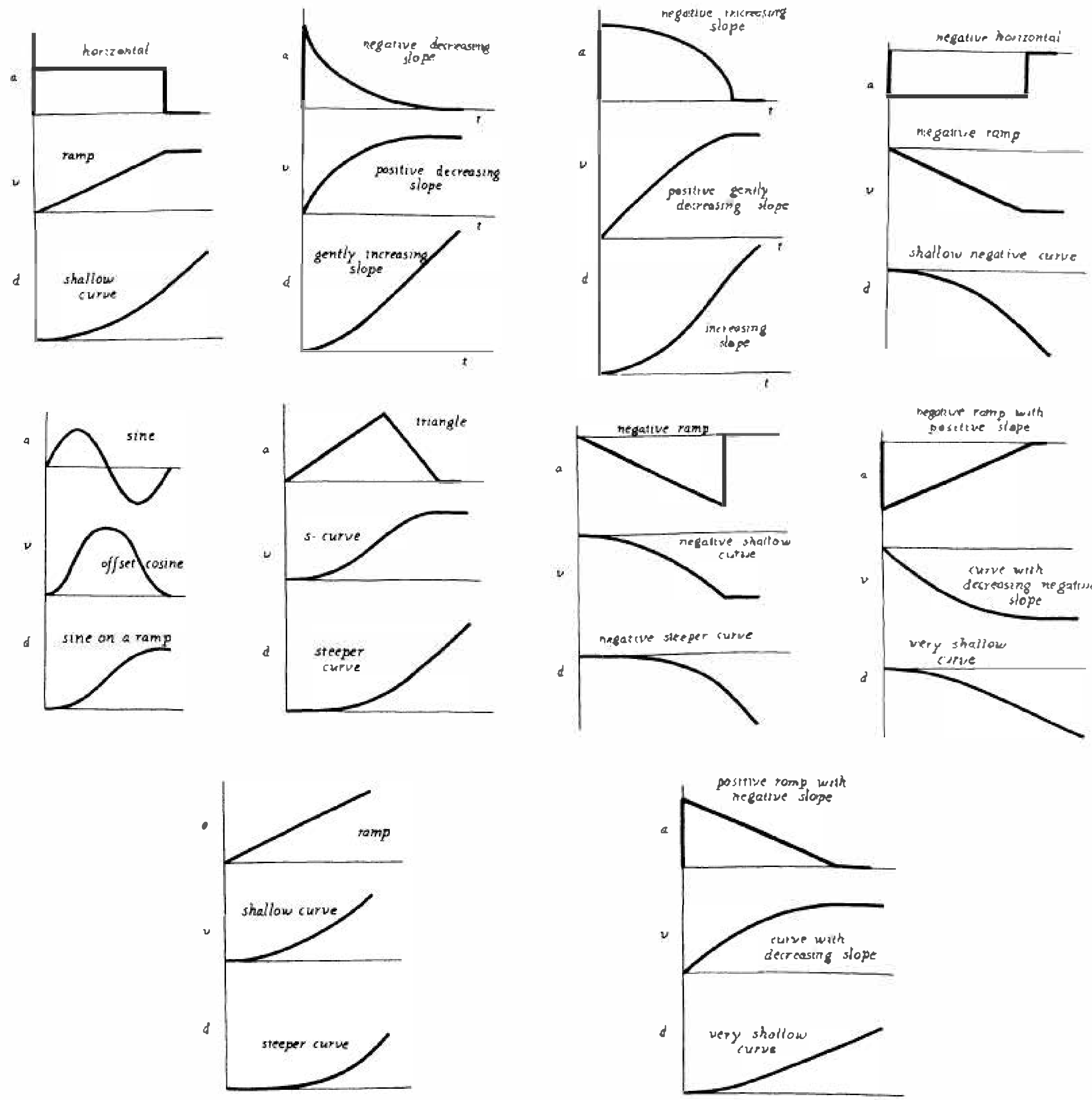


Fig. 2-31. Graphical Table of Integrals and Differentials. Shows relationships between common curve sets frequently encountered in machine design.

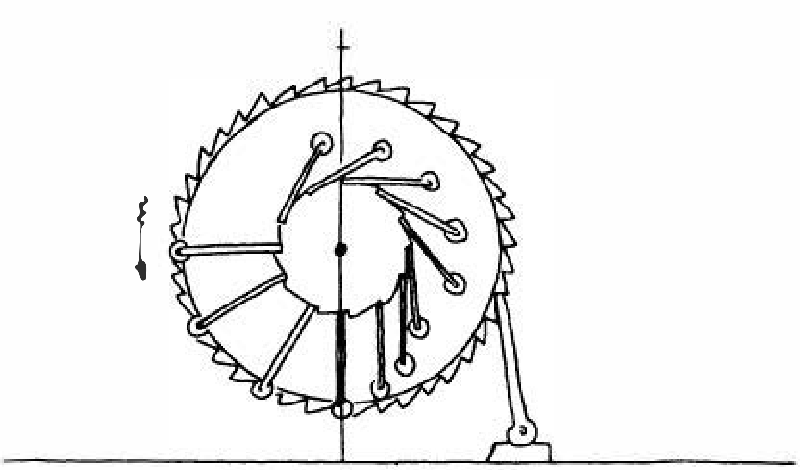


Fig. 2-32. Leonardo Da Vinci's perpetual motion machine, one of the earliest intermittent motion type designs.

itself going. As the wheel turns, the arms are lifted, as shown, over the top of the drum until they are out of balance and fall (rotate) until the weight on the end of the arm strikes the rim of the ratchet wheel. Leonardo presumably hoped that the fact that the weights are falling through a greater radius than they are being lifted would give his wheel a net torque and produce perpetual motion. A "no-back" pawl has also been provided, however, which indicates he knew better, or at least, he wasn't sure! It is not really an intermittent motion device, but is certainly an early and ingenious use of a ratchet wheel, which we will be studying at length, later on.