# Continuity and Monotonicity Properties of an Optimally Controlled Double Integrator with State and Input Constraints

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**Abstract**

This paper presents minimum-time solutions for driving a double integrator to a desired state with zero final velocity. The presence of constraints on the magnitude of the control input and the velocity state is assumed. The derived solutions are piecewise constant control efforts, which have to be executed in sequence. Finally, it is shown that the required times are strictly monotone and continuous functions of the constraints.

## 1 Introduction

Double integrators are very common dynamical systems in science and engineering. In some applications it is desirable to change the state of a system in an optimal way, i.e., to drive the system to a particular state in minimum time. For most physical systems there are restrictions on the amount of available control input and the speed at which the system state can change. This paper presents minimum-time solutions for a double integrator with rate constraints. It is shown that the required final time is a continuous and strictly monotone function of the input and state constraints. These important properties can be exploited by algorithms to find motion primitives, as presented in [1].

Section 2 presents the formal problem and shows an approach to solve it by breaking it up into small control intervals. Section 3 enumerates all possible sequences of control cases and shows that the final time of the solution is a strictly monotone and continuous function of the input and state constraints. Section 4 shows that the properties of monotonicity and continuity even hold if the structure of the solution changes significantly due to modified constraints, i.e., if the solution consists of a different sequence of control cases.

## 2 The Optimal Control Problem

The objective is to find a minimum-time $t_{\mathrm{f},w}$ solution for the double integrator

$$\ddot{w}(t) = q_w(t) \tag{1}$$

subject to

$$w(0) = 0, \quad w(t_{\mathrm{f},w}) = w_\mathrm{f}, \quad \dot{w}(0) = \dot{w}_0, \quad \dot{w}(t_{\mathrm{f},w}) = 0 \tag{2}$$

$$|\dot{w}(t)| \leq v_{w,\max}, \quad |q_w(t)| \leq a_{w,\max} \tag{3}$$

where $w$ is the degree of freedom (DOF) of the system, $q_w$ is the system input or control effort, the subscript $\bullet_0$ denotes the initial state, and the subscript $\bullet_\mathrm{f}$ denotes the final state. The constants $v_{w,\max}$ and $a_{w,\max}$ are the rate and input constraints. Note that $v_{w,\max}$ is not necessarily a hard physical constraint. The system is physically capable of exceeding $v_{w,\max}$, but for system specific reasons it is desired to keep the velocity state below this constant if possible.

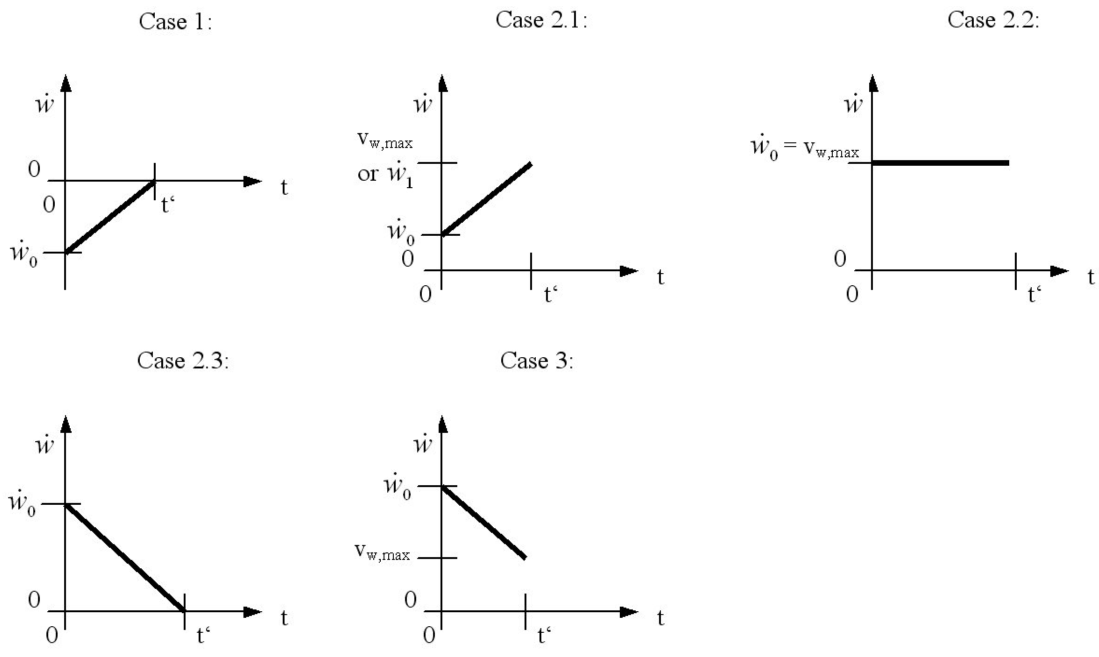


Figure 1: Possible optimal control cases

Table 1: Control efforts for every control case

| Case | 1 | 2.1 | 2.2 | 2.3 | 3 |
|---|---|---|---|---|---|
| $q_w(t)$ | $a_{w,\max}$ | $a_{w,\max}$ | 0 | $-a_{w,\max}$ | $-a_{w,\max}$ |

The minimum time solution to this problem occurs on the boundary of the velocity/acceleration constraints (see [2]). At any time the system is either accelerating with $q_w(t) = a_{w,\max}$, decelerating with $q_w(t) = -a_{w,\max}$, or has reached the velocity constraint and is coasting with $|\dot{w}(t)| = v_{w,\max}, \quad q_w(t) = 0$.

Without loss of generality, assume that $w_{\mathrm{f}} \geq 0$ (the problem can easily be normalized that way). It follows that at any given time, the system can be in one of five distinct states or cases, depending on $\dot{w}_0$, $w_{\mathrm{f}}$, $v_{w,\max}$, and $a_{w,\max}$, see Figure 1. Every case has a control effort associated with it, see Table 1. The system has to execute its current case until it either switches into a different case or reaches the final destination with zero final velocity. The complete solution is therefore a sequence of cases, which in the following will be written using arrows "$\rightarrow$". Example:

> The system is far away from the destination and moving slowly towards it. It will accelerate until it reaches $v_{w,\max}$ (case 2.1), cruise at $v_{w,\max}$ (case 2.2), and finally decelerate such that it reaches the destination with exactly zero final velocity (case 2.3). Shorthand notation: $2.1 \rightarrow 2.2 \rightarrow 2.3$

A particular set of conditions $\dot{w}_0$, $w_{\mathrm{f}}$, $v_{w,\max}$, and $a_{w,\max}$ will call for one particular sequence of cases. It should be noted that only certain cases can follow each other. For example, it is impossible for case 2.1 to follow after case 3, since after case 3 the system moves with $v_{\max}$ and it cannot accelerate beyond this velocity. Note that all sequences must end with case 2.3, since the vehicle must come to a stop at the end of a trajectory. Sequences where case 2.3 is followed by other cases means that the system overshot its destination and has to reverse direction in order to get to the desired position. This can happen when the desired final position is set too close to a system that was already in motion.

# 3 Sequences of Cases

This section contains all possible sequences with the required execution time $t_{\mathrm{f},w,\mathrm{seq}}$ and the initial conditions that have to be met for that particular sequence. Further, it will be shown that for a given set of normalized initial conditions, $w_\mathrm{f}$ and $\dot{w}_0$, the minimum required time $t_{\mathrm{f},w}$ for the optimal control problem is a continuous and strictly monotone function of $v_{w,\max}$ and $a_{w,\max}$. It is assumed that the constraints on the state and control effort have the form:

$$v_{w,\max} = kv_{\max}, \quad v_{\max} > 0, \quad 0 < k < 1 \tag{4}$$
$$a_{w,\max} = ka_{\max}, \quad a_{\max} > 0, \quad 0 < k < 1 \tag{5}$$

The proof of continuity and monotonicity begins with computing the execution times $t_{\mathrm{f},w,\mathrm{seq}}$ of all possible sequences in closed form. Equations (4) and (5) are substituted into the closed form solutions of the sequences in order to see the effect of the factor $k$ on $t_{\mathrm{f},w,\mathrm{seq}}$. It will be shown that all $t_{\mathrm{f},w,\mathrm{seq}}$ are continuous and strictly increasing with decreasing $k$. The proof concludes by showing that when transitioning from one sequence to another due to a change in $v_{w,\max}$ or $a_{w,\max}$, the solution will still be strictly monotone and continuous.

All rational functions $p(x)/q(x)$, where $p$ and $q$ are polynomials, are continuous on the domain $\{x : q(x) \neq 0\}$, see [3]. Further, continuity is preserved under addition and subtraction. All times $t_{\mathrm{f},w,\mathrm{seq}}$ meet these requirements, therefore they are continuous functions of $k$.

For some sequences, monotonicity follows straight from the fact that $k$ enters linearly or quadratically with the coefficient being constant and larger than zero. Whenever it is not obvious, additional comments will be provided.

**Sequence 1c:** $1 \to 2.1 \to 2.2 \to 2.3$

$$t_{\mathrm{f},w,1\mathrm{c}} = -\frac{\dot{w}_0}{a_{w,\max}} + \frac{2v_{w,\max}^2 + 2\dot{w}_0^2 + 2w_\mathrm{f}a_{w,\max}}{2a_{w,\max}v_{w,\max}} \tag{6}$$
$$= -\frac{\dot{w}_0}{a_{\max}k} + \frac{2v_{\max}^2 + 2\dot{w}_0^2/k^2 + 2w_\mathrm{f}a_{\max}/k}{2a_{\max}v_{\max}} \tag{7}$$
$$\text{Conditions:} \quad \dot{w}_0 < 0, \quad v_{w,\max} < \sqrt{\dot{w}_0^2 + w_\mathrm{f}a_{w,\max}} \tag{8}$$

Monotonicity: With $\dot{w}_0 < 0$, all terms of (7) containing $k$ are constant and positive and therefore $t_{\mathrm{f},w,1\mathrm{c}}$ is strictly increasing with decreasing $k$.

**Sequence 1d:** $1 \to 2.1 \to 2.3$

$$t_{\mathrm{f},w,1\mathrm{d}} = -\frac{\dot{w}_0}{a_{w,\max}} + \frac{\sqrt{4a_{w,\max}w_\mathrm{f} + 4\dot{w}_0^2}}{a_{w,\max}} \tag{9}$$
$$= -\frac{\dot{w}_0}{a_{\max}k} + \frac{\sqrt{4a_{\max}w_\mathrm{f}/k + 4\dot{w}_0^2/k^2}}{a_{\max}} \tag{10}$$
$$\text{Conditions:} \quad \dot{w}_0 < 0, \quad v_{w,\max} \geq \sqrt{\dot{w}_0^2 + w_\mathrm{f}a_{w,\max}} \tag{11}$$

Monotonicity: With $\dot{w}_0 < 0$, all terms of (10) containing $k$ are constant and positive and therefore $t_{\mathrm{f},w,1\mathrm{d}}$ is strictly increasing with decreasing $k$.

**Sequence 2a:** 2.3

$$t_{\mathrm{f},w,2\mathrm{a}} = \frac{\dot{w}_0}{a_{w,\max}} = \frac{\dot{w}_0}{ka_{\max}} \tag{12}$$
$$\text{Conditions:} \quad 0 < \dot{w}_0 \leq v_{w,\max}, \quad w_\mathrm{f} = \frac{\dot{w}_0^2}{2a_{w,\max}} \tag{13}$$

**Sequence 2b:** $2.2 \to 2.3$

$$t_{\mathrm{f},w,2\mathrm{b}} = \frac{2w_\mathrm{f}a_{w,\max} + v_{w,\max}^2}{2a_{w,\max}v_{w,\max}} = \frac{2w_\mathrm{f}a_{\max}/k + v_{\max}^2}{2a_{\max}v_{\max}} \tag{14}$$

$$\text{Conditions:}\quad \dot{w}_0 = v_{w,\max},\quad w_\mathrm{f} > \frac{v_{w,\max}^2}{2a_{w,\max}} \tag{15}$$

**Sequence 2c:** 2.1 → 2.2 → 2.3

$$t_{\mathrm{f},w,2\mathrm{c}} = \frac{2v_{w,\max}^2 - 2v_{w,\max}\dot{w}_0 + 2w_\mathrm{f}a_{w,\max} + \dot{w}_0^2}{2a_{w,\max}v_{w,\max}} \tag{16}$$

$$= \frac{2v_{\max}^2 + (2w_\mathrm{f}a_{\max} - 2v_{\max}\dot{w}_0)/k + \dot{w}_0^2/k^2}{2a_{\max}v_{\max}} \tag{17}$$

$$\text{Conditions:}\quad 0 \le \dot{w}_0 < v_{w,\max},\quad w_\mathrm{f} > \frac{\dot{w}_0^2}{2a_{w,\max}}, \tag{18}$$

$$v_{w,\max} < \sqrt{\dot{w}_0^2/2 + w_\mathrm{f}a_{w,\max}} \tag{19}$$

Monotonicity: The derivative of (17) w.r.t. $k$ has to be strictly smaller than zero:

$$\frac{\partial t_{\mathrm{f},w,2\mathrm{c}}}{\partial k} = \frac{1}{2a_{\max}v_{\max}}\left(\frac{2v_{\max}\dot{w}_0 - 2w_\mathrm{f}a_{\max}}{k^2} - \frac{2\dot{w}_0^2}{k^3}\right) < 0 \tag{20}$$

With $\dot{w}_0 < kv_{\max}$ and (19) it follows that

$$\frac{2v_{\max}\dot{w}_0 - 2w_\mathrm{f}a_{w,\max}}{k^2} - \frac{2\dot{w}_0^2}{k^3} < \frac{2v_{\max}^2}{k} - \frac{2(kv_{\max}^2 - \dot{w}_0^2/(2k))}{k^2} - \frac{2\dot{w}_0^2}{k^3} < 0 \tag{21}$$

This makes $\partial t_{\mathrm{f},w,2\mathrm{c}}/\partial k < 0$ and completes the proof.

**Sequence 2d:** 2.1 → 2.3

$$t_{\mathrm{f},w,2\mathrm{d}} = \frac{\sqrt{4w_\mathrm{f}a_{w,\max} + 2\dot{w}_0^2} - \dot{w}_0}{a_{w,\max}} \tag{22}$$

$$= \frac{\sqrt{4w_\mathrm{f}a_{\max}k + 2\dot{w}_0^2} - \dot{w}_0}{ka_{\max}} \tag{23}$$

$$\text{Conditions:}\quad 0 \le \dot{w}_0 < v_{w,\max},\quad w_\mathrm{f} > \frac{\dot{w}_0^2}{2a_{w,\max}}, \tag{24}$$

$$v_{w,\max} \ge \sqrt{\dot{w}_0^2/2 + w_\mathrm{f}a_{w,\max}} \tag{25}$$

Monotonicity: The derivative of (23) w.r.t. $k$ has to be strictly smaller than zero:

$$\frac{\partial t_{\mathrm{f},w,2\mathrm{d}}}{\partial k} = \frac{1}{a_{\max}}\left[\frac{1}{2\sqrt{4w_\mathrm{f}a_{\max}/k + 2\dot{w}_0^2/k^2}}\left(-4w_\mathrm{f}a_{\max}/k^2 - 4\dot{w}_0^2/k^3\right) + \frac{\dot{w}_0}{k^2}\right] < 0 \tag{26}$$

Since all variables are larger than 0, this is equivalent to

$$-\frac{1}{2k}\left(\sqrt{4w_\mathrm{f}a_{\max}/k + 2\dot{w}_0^2/k^2} + H\right) + \frac{\dot{w}_0}{k^2} < 0,\quad H = \frac{2\dot{w}_0^2}{k^2\sqrt{4w_\mathrm{f}a_{\max}/k + 2\dot{w}_0^2/k^2}} \tag{27}$$

Using (24) the following inequality holds

$$-\frac{1}{2k}\sqrt{4w_\mathrm{f}a_{\max}/k + 2\dot{w}_0^2/k^2} - \frac{H}{2k} + \frac{\dot{w}_0}{k^2} < -\frac{1}{2k}\sqrt{4\dot{w}_0^2/k^2} - \frac{H}{2k} + \frac{\dot{w}_0}{k^2} = -H < 0 \tag{28}$$

With $H > 0$ it follows that $\partial t_{\mathrm{f},w,2\mathrm{d}}/\partial k < 0$ and this completes the proof.

**Sequence 2e:** 2.3 → 2.1 → 2.2 → 2.3

$$t_{\mathrm{f},w,2\mathrm{e}} = \frac{2v_{w,\max}\dot{w}_0 + 2v_{w,\max}^2 - 2w_\mathrm{f}a_{w,\max} + \dot{w}_0^2}{2a_{w,\max}v_{w,\max}} \tag{29}$$

$$= \frac{2v_{\max}^2 + (2v_{\max}\dot{w}_0 - 2w_\mathrm{f}a_{\max})/k + \dot{w}_0^2/k^2}{2a_{\max}v_{\max}} \tag{30}$$

$$\text{Conditions:} \quad 0 < \dot{w}_0 \le v_{w,\max}, \quad w_\mathrm{f} < \frac{\dot{w}_0^2}{2a_{w,\max}}, \tag{31}$$

$$v_{w,\max} < \sqrt{\dot{w}_0^2/2 - w_\mathrm{f}a_{w,\max}} \tag{32}$$

Monotonicity: The derivative of (30) w.r.t. $k$ has to be strictly smaller than zero:

$$\frac{\partial t_{\mathrm{f},w,\mathrm{2e}}}{\partial k} = \frac{1}{2a_{\max}v_{\max}}\left[\frac{-2v_{\max}\dot{w}_0 + 2w_\mathrm{f}a_{\max}}{k^2} - \frac{\dot{w}_0^2}{k^3}\right] < 0 \tag{33}$$

With (31) it follows that

$$\frac{2w_\mathrm{f}a_{\max}}{k^2} - \frac{\dot{w}_0^2}{k^3} < \frac{2}{k^2}\frac{\dot{w}_0^2}{2k} - \frac{\dot{w}_0^2}{k^3} = 0 \tag{34}$$

This makes $\partial t_{\mathrm{f},w,\mathrm{2e}}/\partial k < 0$ and completes the proof.

**Sequence 2f:** 2.3 → 2.1 → 2.3

$$t_{\mathrm{f},w,\mathrm{2f}} = \frac{\sqrt{-4w_\mathrm{f}a_{w,\max} + 2\dot{w}_0^2} + \dot{w}_0}{a_{w,\max}} \tag{35}$$

$$= \frac{\sqrt{-4w_\mathrm{f}a_{\max}k + 2\dot{w}_0^2} + \dot{w}_0}{a_{\max}k} \tag{36}$$

$$\text{Conditions:} \quad 0 < \dot{w}_0 \le v_{w,\max}, \quad w_\mathrm{f} < \frac{\dot{w}_0^2}{2a_{w,\max}}, \tag{37}$$

$$v_{w,\max} \ge \sqrt{\dot{w}_0^2/2 - w_\mathrm{f}a_{w,\max}} \tag{38}$$

Monotonicity: Similar to the proof of sequence 2d, the derivative of (36) w.r.t. $k$ has to be strictly smaller than zero:

$$\frac{\partial t_{\mathrm{f},w,\mathrm{2f}}}{\partial k} = \frac{1}{a_{\max}}\left[\frac{1}{2\sqrt{-4w_\mathrm{f}a_{\max}/k + 2\dot{w}_0^2/k^2}}\left(4w_\mathrm{f}a_{\max}/k^2 - 4\dot{w}_0^2/k^3\right) - \frac{\dot{w}_0}{k^2}\right] < 0 \tag{39}$$

Since all variables are larger than 0, this is equivalent to

$$-\frac{1}{2k}\left(\sqrt{-4w_\mathrm{f}a_{\max}/k + 2\dot{w}_0^2/k^2} + \frac{2\dot{w}_0^2}{k^2\sqrt{-4w_\mathrm{f}a_{\max}/k + 2\dot{w}_0^2/k^2}}\right) - \frac{\dot{w}_0}{k^2} < 0 \tag{40}$$

Using (37) it is possible to show that the radicands are larger than zero:

$$\sqrt{-4w_\mathrm{f}a_{\max}/k + 2\dot{w}_0^2/k^2} > \sqrt{-4\dot{w}_0^2/(2k^2) + 2\dot{w}_0^2/k^2} = 0 \tag{41}$$

which makes $\partial t_{\mathrm{f},w,\mathrm{2f}}/\partial k < 0$ and completes the proof.

**Sequence 3a:** 3 → 2.3

$$t_{\mathrm{f},w,\mathrm{3a}} = \frac{\dot{w}_0}{a_{w,\max}} = \frac{\dot{w}_0}{a_{\max}k} \tag{42}$$

$$\text{Conditions:} \quad \dot{w}_0 > v_{w,\max}, \quad w_\mathrm{f} = \frac{\dot{w}_0^2}{2a_{w,\max}} \tag{43}$$

**Sequence 3b:** 3 → 2.2 → 2.3

$$t_{\mathrm{f},w,\mathrm{3b}} = \frac{2v_{w,\max}\dot{w}_0 + 2w_\mathrm{f}a_{w,\max} - \dot{w}_0^2}{2a_{w,\max}v_{w,\max}} \tag{44}$$

$$= \frac{(2v_{\max}\dot{w}_0 + 2w_\mathrm{f}a_{\max})/k - \dot{w}_0^2/k^2}{2a_{\max}v_{\max}} \tag{45}$$

$$\text{Conditions:} \quad \dot{w}_0 > v_{w,\max}, \quad w_\mathrm{f} > \frac{\dot{w}_0^2}{2a_{w,\max}} \tag{46}$$

Monotonicity: The derivative of (45) w.r.t. $k$ has to be strictly smaller than zero:

$$\frac{\partial t_{\mathrm{f},w,3\mathrm{b}}}{\partial k} = \frac{1}{2a_{\max}v_{\max}} \left[ \frac{-2v_{\max}\dot{w}_0 - 2w_\mathrm{f}a_{\max}}{k^2} + \frac{\dot{w}_0^2}{k^3} \right] < 0 \tag{47}$$

With (46) it follows that

$$-\frac{2w_\mathrm{f}a_{\max}}{k^2} + \frac{\dot{w}_0^2}{k^3} < -\frac{2}{k^2}\frac{\dot{w}_0^2}{2k} + \frac{\dot{w}_0^2}{k^3} = 0 \tag{48}$$

This makes $\partial t_{\mathrm{f},w,3\mathrm{b}}/\partial k < 0$ and completes the proof.

**Sequence 3e:** $3 \to 2.3 \to 2.1 \to 2.2 \to 2.3$

$$t_{\mathrm{f},w,3\mathrm{e}} = \frac{2v_{w,\max}\dot{w}_0 + 2v_{w,\max}^2 - 2w_\mathrm{f}a_{w,\max} + \dot{w}_0^2}{2a_{w,\max}v_{w,\max}} \tag{49}$$

$$= \frac{2v_{\max}^2 + (2v_{\max}\dot{w}_0 - 2w_\mathrm{f}a_{\max})/k + \dot{w}_0^2/k^2}{2a_{\max}v_{\max}} \tag{50}$$

$$\text{Conditions:} \quad \dot{w}_0 > v_{w,\max}, \quad w_\mathrm{f} < \frac{\dot{w}_0^2}{2a_{w,\max}}, \tag{51}$$

$$v_{w,\max} < \sqrt{\dot{w}_0^2/2 - w_\mathrm{f}a_{w,\max}} \tag{52}$$

Monotonicity: The proof is analogous to sequence 2e.

**Sequence 3f:** $3 \to 2.3 \to 2.1 \to 2.3$

$$t_{\mathrm{f},w,3\mathrm{f}} = \frac{\sqrt{-4w_\mathrm{f}a_{w,\max} + 2\dot{w}_0^2} + \dot{w}_0}{a_{w,\max}} \tag{53}$$

$$= \frac{\sqrt{-4w_\mathrm{f}a_{\max}k + 2\dot{w}_0^2} + \dot{w}_0}{a_{\max}k} \tag{54}$$

$$\text{Conditions:} \quad \dot{w}_0 > v_{w,\max}, \quad w_\mathrm{f} < \frac{\dot{w}_0^2}{2a_{w,\max}}, \tag{55}$$

$$v_{w,\max} \geq \sqrt{\dot{w}_0^2/2 - w_\mathrm{f}a_{w,\max}} \tag{56}$$

Monotonicity: The proof is analogous to sequence 2f.

# 4 Continuity of Transitions

A transition is the change of the applicable sequence due to a change of the constraints. Each sequence can be characterized by two or three conditions, which are based on $\dot{w}_0$, $w_\mathrm{f}$, $v_{w,\max}$, and $a_{w,\max}$. Figure 2 gives an overview of the presented sequences with their respective conditions. The nodes in the Figure represent the sequences, the connections represent all possible transitions (numbered from 1 to 16).

Changing the factor $k$ can change the choice of the sequence if the state of the system moves across the boundary between two conditions (marked by dotted lines in Figure 2). Next, we will show that the execution time $t_{\mathrm{f},w}$ is continuous when doing a transition from one sequence to another.

Note that a direct switch from sequence 2d) to sequence 2b) is not possible, since taking the limit $\dot{w}_0 \to v_{w,\max}$ violates (24) together with (25).

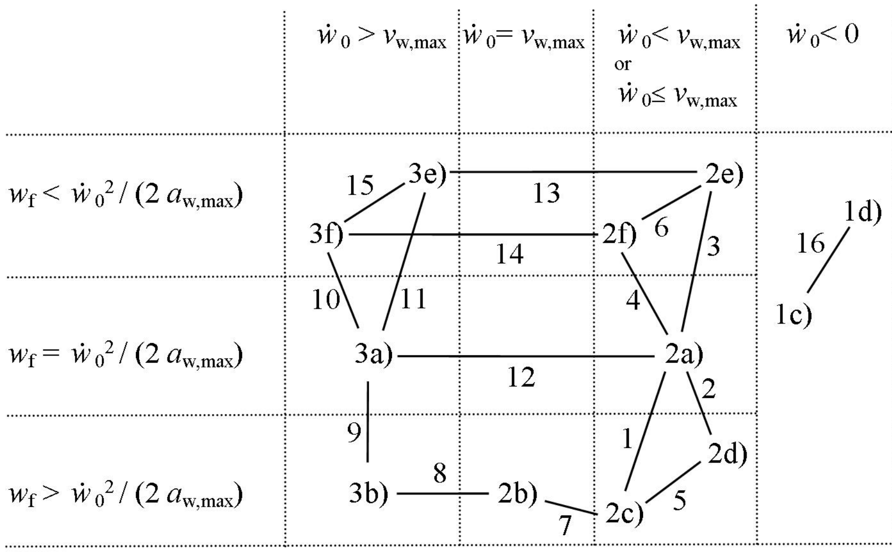


Figure 2: Sequences and Transitions

Conveniently defining the abbreviations

$$L_w = \frac{\dot{w}_0^2}{2a_{w,\max}} \tag{57}$$

$$L_{\mathrm{v},1} = \sqrt{\dot{w}_0^2/2 + w_\mathrm{f} a_{w,\max}} \tag{58}$$

$$L_{\mathrm{v},2} = \sqrt{\dot{w}_0^2/2 - w_\mathrm{f} a_{w,\max}} \tag{59}$$

$$L_{\mathrm{v},3} = \sqrt{\dot{w}_0^2 + w_\mathrm{f} a_{w,\max}} \tag{60}$$

the limits can be taken for all possible transitions:

**Transitions 1, 2, 3, 4**

$$\begin{aligned} \lim_{w_\mathrm{f} \to L_w^+} t_{\mathrm{f},w,\mathrm{2c}} = \lim_{w_\mathrm{f} \to L_w^+} t_{\mathrm{f},w,\mathrm{2d}} = \lim_{w_\mathrm{f} \to L_w^-} t_{\mathrm{f},w,\mathrm{2e}} \\ = \lim_{w_\mathrm{f} \to L_w^-} t_{\mathrm{f},w,\mathrm{2f}} = t_{\mathrm{f},w,\mathrm{2a}} = \frac{\dot{w}_0}{a_{w,\max}} \\ \text{Note for } t_{\mathrm{f},w,\mathrm{2c}}\text{:} \quad v_{w,\max} \to \dot{w}_0 \quad \text{due to (19)} \end{aligned} \tag{61}$$

**Transition 5**

$$\lim_{v_{w,\max} \to L_{v,1}^-} t_{\mathrm{f},w,\mathrm{2c}} = \lim_{v_{w,\max} \to L_{v,1}^+} t_{\mathrm{f},w,\mathrm{2d}} = \frac{2v_{w,\max} - \dot{w}_0}{a_{w,\max}} \tag{62}$$

**Transition 6**

$$\lim_{v_{w,\max} \to L_{v,2}^-} t_{\mathrm{f},w,\mathrm{2e}} = \lim_{v_{w,\max} \to L_{v,2}^+} t_{\mathrm{f},w,\mathrm{2f}} = \frac{2v_{w,\max} + \dot{w}_0}{a_{w,\max}} \tag{63}$$

**Transitions 7, 8**

$$\lim_{\dot{w}_0 \to v_{w,\max}^-} t_{\mathrm{f},w,\mathrm{2c}} = \lim_{\dot{w}_0 \to v_{w,\max}^+} t_{\mathrm{f},w,\mathrm{3b}} = t_{\mathrm{f},w,\mathrm{2b}} = \frac{2w_\mathrm{f} a_{w,\max} + v_{w,\max}^2}{2a_{w,\max} v_{w,\max}} \tag{64}$$

**Transitions 9, 10, 11**

$$\lim_{w_\mathrm{f}\to L_w^+} t_{\mathrm{f},w,3\mathrm{b}} = \lim_{w_\mathrm{f}\to L_w^-} t_{\mathrm{f},w,3\mathrm{e}} = \lim_{w_\mathrm{f}\to L_w^-} t_{\mathrm{f},w,3\mathrm{f}} = t_{\mathrm{f},w,3\mathrm{a}} = \frac{\dot{w}_0}{a_{w,\max}} \tag{65}$$

$$\text{with (52) for } t_{\mathrm{f},w,3\mathrm{e}} \quad v_{w,\max} \to 0$$

**Transition 12**

$$\lim_{\dot{w}_0\to v_{w,\max}^-} t_{\mathrm{f},w,2\mathrm{a}} = \lim_{\dot{w}_0\to v_{w,\max}^+} t_{\mathrm{f},w,3\mathrm{a}} = \frac{v_{w,\max}}{a_{w,\max}} \tag{66}$$

**Transition 13**

$$\lim_{\dot{w}_0\to v_{w,\max}^-} t_{\mathrm{f},w,2\mathrm{e}} = \lim_{\dot{w}_0\to v_{w,\max}^+} t_{\mathrm{f},w,3\mathrm{e}} = \frac{5v_{w,\max} - 2w_\mathrm{f} a_{w,\max}}{2v_{w,\max} a_{w,\max}} \tag{67}$$

**Transition 14**

$$\lim_{\dot{w}_0\to v_{w,\max}^-} t_{\mathrm{f},w,2\mathrm{f}} = \lim_{\dot{w}_0\to v_{w,\max}^+} t_{\mathrm{f},w,3\mathrm{f}} = \frac{\sqrt{-4w_\mathrm{f} a_{w,\max} + 2v_{w,\max}^2} + v_{w,\max}}{a_{w,\max}} \tag{68}$$

**Transition 15**

$$\lim_{v_{w,\max}\to L_{v,2}^-} t_{\mathrm{f},w,3\mathrm{e}} = \lim_{v_{w,\max}\to L_{v,2}^+} t_{\mathrm{f},w,3\mathrm{f}} = \frac{2v_{w,\max} + \dot{w}_0}{a_{w,\max}} \tag{69}$$

**Transition 16**

$$\lim_{v_{w,\max}\to L_{v,3}^-} t_{\mathrm{f},w,1\mathrm{c}} = \lim_{v_{w,\max}\to L_{v,3}^+} t_{\mathrm{f},w,1\mathrm{d}} = \frac{2v_{w,\max}}{a_{w,\max}} \tag{70}$$

This shows that the transitions from one sequence to another are “smooth”, i.e., they do not cause jump discontinuities. This means that the execution time $t_{\mathrm{f},w}$ of the entire optimal control problem is a continuous function of $k$ and strictly increasing with decreasing $k$.

# 5 Conclusion

The problem of finding minimum-time solutions for a double integrator with rate constraints has been presented. The optimal solutions in form of control sequences have been derived for all possible initial conditions. Finally, it has also been shown that the solutions are continuous and strictly monotone functions of the state and input constraints.

# References


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