TOPOLOGICAL REPRESENTATIONS OF MATROIDS AND THE CD-INDEX

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Zhexiu Tu

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Zhexiu Tu, Ph.D.

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A fundamental achievement in the theory of matroids is the Topological Representation Theorem which says that every oriented matroid arises from an arrangement of pseudospheres. In 2003 Swartz extended this result to arbitrary matroids by using homotopy spheres [14]. Later, Anderson [1] and Engstrom [7] also constructed topological representations of matroids by homotopy sphere arrangements. Inspired by Swartz's work, this thesis will show an explicit fully partitioned homotopy sphere d-arrangement $\mathcal{S}$ that is a CW-complex whose intersection lattice is the geometric lattice of the corresponding matroid for matroids of rank ≤ 4. Moreover $\mathcal{S}$ has a d-sphere in it that is a regular CW-complex. This will allows us to look at how the flag f-vector formula of Billera, Ehrenborg and Readdy (BER) for oriented matroids applies to arbitrary matroids.

BIOGRAPHICAL SKETCH

Zhexiu was born on June 5 in 1989, a day of Sorrow and Penance, in a small fishing village in the southeast coast of China. As his name signifies, he was raised up to be a philosopher. The philosophy of mathematics is the deepest philosophy, and that is what attracted mathematicians from generation to generation in human history.

Compared to his peers, Zhexiu was never a talented kid in math competitions of different levels, especially after he settled in Shanghai, a city that he had never imagined before whose population overwhelmed him tremendously. Because of a piece of news from one of his high school friends, he applied to U.S. colleges from Hong Kong, and was lucky enough to continue his study in the New World. At Bard College, Zhexiu learned that mathematics is not only a philosophy, but also an art, an art that people pursued with passion and principles.

Zhexiu's Cornell life is the seedling that the Bardian gardeners planted with care and share. From Sam Hsiao, the Belks, Ethan, Lauren, and Greg, Zhexiu found the trunk at Cornell, and studied combinatorics delightfully with Ed, Lou, and Bob, who, together with other faculty members and friends, developed the stem into a young tree. The tree is to move to the Centre of the beautiful country, with combinatorial fertilizer and the Blue Grass.

Dedicated to the Almighty

who has carried me, comforted me, guided me, and forgiven me,

countless times.

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kindheartedness from you two respectively. I felt deeply guilty when I heard my grandma passed away and yet wasn't able to leave Ithaca for complicated reasons. May peace be upon you forever. You accompanied me when I was in Shanghai and now you are with me again.

We love because God loves us; we trust because God trusts us; we remember because God never forgets us.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

A fundamental achievement in the theory of matroids is the Topological Representation Theorem [8], which says that every oriented matroid arises from an arrangement of pseudospheres. Until 2003, there were no similar results available for ordinary matroids. In 2003 Swartz extended this Topological Representation Theorem to general matroids by relaxing the requirement of homeomorphism to homotopy equivalence [14, Theorem 6.1]. However, Swartz didn't propose a specific algorithm to construct the homotopy sphere arrangements of matroids. Moreover, although the homotopy sphere arrangements of Swartz are CW-complexes, they may or may not be regular CW-complexes. Some enumerative properties of cell complexes do require a regular CW-complex. For example, Bayer and Sturmfels [3] showed that the geometric lattice of the underlying matroid of an oriented matroid completely determines the flag f-vectors of the associated pseudosphere arrangement. Billera, Ehrenborg, Readdy (BER) [4, Theorem 3.1] computed the flag f-vectors of the face poset of any oriented matroid in terms of the flag f-vectors of the geometric lattice. We will call this the BER formula. The topological representations of oriented matroids are always regular CW-complexes. Is there an analog of the BER formula for homotopy sphere arrangements of matroids?

After Swartz' work, Anderson and Engstrom each gave a different topological representation of matroids by homotopy sphere arrangements. Anderson's construction [1] is elegant in that it gives a simplicial complex, a stronger restriction than being a regular CW-complex. However her simplicial complex may have much greater dimension than expected. Engstrom's construction [7] via diagrams of spaces gives the correct dimension. However, it may provide extra cells. This thesis gives an explicit fully partitioned homotopy sphere d-arrangement $\mathcal{S}$ that is a CW-complex whose intersection lattice is the geometric lattice of the corresponding matroid for rank ≤ 4. Moreover $\mathcal{S}$ has a d-sphere in it that is a regular CW-complex. Our work has no problems in matching the dimensions of the arrangements, nor in the cell numbers that are completely determined by Zaslavsky's enumerative theory [15] or Las Vergnas's formula [11], either of which works for pseudosphere arrangements. In Chapter 3 we propose how to extend the algorithm to all matroids.

The next chapter deals with the enumerative problems. The **cd**-index is another way to convey the information that flag f-vectors carry. The **cd**-index is only defined for Eulerian posets [2]. A graded poset is called Eulerian if every nontrivial interval has the same number of elements of even rank as of odd rank. The **cd**-index gives a compact way of presenting the flag f-vector data since it removes the linear redundancies in the flag f-vectors inherent in Eulerian posets. For example, **cd**-index of a rank n Eulerian poset has F_n entries (where F_n is the nth Fibonacci number) while the flag f-vector has 2^n entries. Another interesting fact is the non-negativity of the **cd**-index of Gorenstein* complexes, which Stanley conjectured in 1994 [13] and was later proved by Karu [10].

The face poset of the pseudosphere arrangement associated to an oriented matroid is an Eulerian poset. So we can define the **cd**-index of an oriented matroid as the **cd**-index of this face poset. Bayer and Sturmfels [3] showed the **cd**-index only depends on the geometric lattice of the underlying matroid. Billera, Ehrenborg, and Readdy [4, Theorem 3.1] found an explicit formula of the **cd**-index of any oriented matroid in terms of flag f-vectors of the geometric lattice. However, when it comes to ordinary matroids, it is not clear how to interpret the formula in [4, Theorem 3.1]. It was asked by Swartz if it can still hold as a bound for the **cd**-index of some poset related to the ordinary matroid. In this thesis, we constructed a similar formula for matroids, and prove it for matroids of ranks less than 4. We also showed that if the equality of the BER formula holds, then the matroid must be orientable.

The following sections of this chapter contains background knowledge and necessary information on matroids, geometric lattices, and broken circuits. Chapter 2 talks about different types of arrangements.

1.2 Matroids

In this section we give the basic definitions from matroid theory that we need. It is devoted to providing the minimal background knowledge to understand this thesis. A matroid is a combinatorial structure that generalizes the notion of linear independence in vector spaces. Formally, a matroid M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of $\mathcal{I}$ having the following three properties:

1. $\emptyset \in \mathcal{I}$.
2. If $I' \subseteq I \subseteq E$, and $I \in \mathcal{I}$, then $I' \in \mathcal{I}$.
3. If $I_1, I_2 \in \mathcal{I}$, and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

In order, each of the properties says the following: the empty set is independent; every subset of an independent set is independent; we can augment the smaller of a pair of independent sets with different cardinalities. A finite set of vectors in a vector space with the usual independent sets is an example of a matroid.

A minimal dependent set in an arbitrary matroid M is called a *circuit* of M and we shall denote the set of circuits of M by C or $C(M)$. We call a maximal independent set in M a *basis* or a *base* of M. The set of all bases of M is $\mathcal{B}$ or $\mathcal{B}(M)$. As in any vector space, the members of $\mathcal{B}$ all have the same cardinality [12, Lemma 1.2.4]. Suppose $X \subseteq E$. Let $I|X$ be $\{I \subseteq X \mid I \in \mathcal{I}\}$. Then it is easy to see that $(X, I|X)$ is also a matroid. That matroid is called the *restriction* of M to X, denoted by $M|X$. Since all the bases of $M|X$ are equicardinal, we let the function $\rho : 2^E \to \mathbb{N}$ be the cardinality of a basis B of $M|X$, and call it the *rank* of X or $\rho(X)$. When $X = M$, $\rho(M)$ measures the rank of the matroid. The rank function has the following properties: [12]

1. $\rho(X) \geq 0$.
2. $\rho(X) \leq |X|$.
3. $\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$, i.e., the rank is a submodular function.

Let the closure function $Cl : 2^E \to 2^E$, for all $X \subseteq E$ be defined by

$$Cl(X) = \{e \in E \mid \rho(X \cup e) = \rho(X)\}.$$

For any subset $X \subseteq E$, if $Cl(X) = X$, it is called a *flat* of M. We say that X *spans* a subset Y of $E(M)$ if $Y \subseteq Cl(X)$. There are different cryptomorphic characterizations of matroids, for example, in terms of bases, circuits, flats, etc. Matroids are frequently associated to hyperplane arrangements, matrices, and graphs.

Many matroid ideas are derived from graph theory. Recall that a circuit of a graph is a connected subgraph all of whose vertices have degree two.

Proposition 1.1 *[12, Proposition 1.1.7] Let E be the set of edges of a graph G and C be the set of edge sets of circuits of G. Then C is the set of circuits of a matroid on E.*

We denote the matroid described in the above proposition by $M(G)$.

1.3 Geometric Lattices

As mentioned earlier, for any matroid M of rank r, a maximal subset of a given rank is called a *flat*. If a set $\{x_1, \ldots, x_l\}$ spans a flat X, we write $X = (x_1, \ldots, x_l)$ or $X = (x_1 \cdots x_l)$. The flats of rank 1 are called *atoms* while the flats of rank $r-1$ are called *coatoms*. The flats of M, together with the inherited rank function $\rho(M)$, form a graded partially ordered set (or poset), under inclusion, denoted by $L(M)$. The *meet* of two elements is their intersection and their *join* is the closure of their union. Thus we get a lattice from a given matroid; in fact, that lattice is a *geometric lattice* as the following proposition shows. A geometric lattice is a finite atomic semi-modular graded lattice [12]. A poset with least element $\hat{0}$ is called *atomic* if every element other than $\hat{0}$ is the least upper bound of a set of atoms. A semimodular lattice is a lattice that satisfies the following semimodular law:

$$a \wedge b <: a \Rightarrow b <: a \vee b$$

where $a <: b$ means that b *covers* a, i.e. $a < b$ and there is no element c such that $a < c < b$.

Proposition 1.2 *[12] If M is matroid, then $L(M)$ is a geometric lattice. Conversely, suppose L is a geometric lattice. Let E be the atoms of L. For any $A \subseteq E$ let $\bigvee A$ be the poset join of all the atoms in A. Then $\mathcal{I} = \{A \subseteq E\colon \rho(\bigvee A) = |A|\}$ are the independent subsets of a matroid M such that $L = L(M)$.*

1.4 Broken Circuits

Geometric lattices are frequently considered in the realm of matroid theory because they are cryptomorphic to finite simple matroids, which are matroids without *loops* (1-element circuits) or *parallel elements* (2-element circuits) [12]. Assume the ground set $E = [n] = \{1, 2, \ldots, n\}$ has the usual order. Depending on the context, we denote the atoms of M by $e_1, e_2, \ldots, e_n$ or $1, 2, \ldots, n$. Let C be a circuit of M. Deleting the minimal element of C gives a *broken circuit*. If a basis B of M does not have a broken circuit as a subset, we call it a *non-broken-circuit basis*, or **nbc**-basis. The *broken-circuit complex $BC(M)$* is the simplicial complex of all subsets of $[n]$ that do not contain a broken circuit. $BC(M)$ is a pure, $(r-1)$-dimensional simplicial complex, and the facets of $BC(M)$ are the **nbc**-bases of M [5, Proposition 7.4.2]. Every facet of $BC(M)$ contains 1, which makes the broken circuit complex a cone.

Denote the set of **nbc**-bases as follows.

$$\mathbf{nbc}(M) = \{B \text{ is an } \mathbf{nbc}\text{-basis of } M\}.$$

We call a set B_0 an **nbc**-set if $1 \in B_0$ and $B_0 \subsetneq B$ for some $B \in \mathbf{nbc}(M)$. The smallest **nbc**-set for any geometric lattice is $\{1\}$. Notice any **nbc**-set is always an **nbc**-basis of the sub-lattice $[\hat{0}, z]$ for $z = Cl(B_0)$ and $1 \le z < \hat{1}$.

The *dual of a matroid* M is another matroid M^* that has the same elements as M and whose bases are the complements of the bases of M. We can also approach the notion of **nbc**-bases in the following manner. Suppose B is a basis of M, $p \in B$ and $q \notin B$. Let $c(B, q)$ denote the basic circuit, which is the unique circuit in the set $B \cup q$. We have $q \in c(B, q)$. Let M^* be the dual matroid of M. We call an element $p \in B$ *internally active* if $p = \min c(E - B, p)$ in M^*. The set of all internally active elements with respect to B is denoted by $\mathrm{IA}(B)$. Similarly, we call an element $q \notin B$ *externally active* if $q = \min c(B, q)$ in M, and call the set of all externally active elements with respect to B, $\mathrm{EA}(B)$. We call the minimal element in EA the *minimal external activity* of B, denoted by $\min EA$ or $\min EA(B)$. Thus comes an equivalent definition of **nbc**-bases, those with no external activity.

$$\mathbf{nbc}(M) = \{B \mid \mathrm{EA}(B) = \emptyset\}.$$

Notice that 1 is always either internally active or externally active. A *pointed geometric lattice* is a pair (L, e) where e is a specific atom of L. We let L/e denote the sub-lattice that represents the upper interval $[e, \hat{1}]$.

Here is an example.

Example 1 *Let $M(G)$ be the matroid given by the graph G as shown in Figure 1.1. We list the 8 bases of M in lexicographic order, and indicate for each basis which elements are internally or externally active in it.*

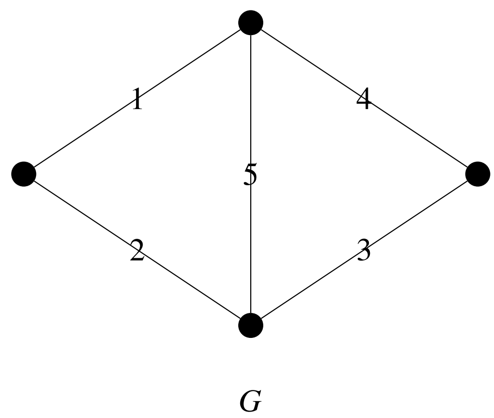


Figure 1.1:

Basis	*Internally Active*	*Externally Active*
$\{1,2,3\}$	1, 2, 3	-
$\{1,2,4\}$	1, 2	-
$\{1,3,4\}$	1	-
$\{1,3,5\}$	1, 3	-
$\{1,4,5\}$	1	3
$\{2,3,4\}$	-	1
$\{2,3,5\}$	3	1
$\{2,4,5\}$	-	1, 3

$\{1,3,4\}$ *is an* ***nbc****-basis.* $\{1,4\}$ *is an* ***nbc****-set because* $\{1,4\} \subseteq \{1,3,4\}$. $\{3,4\}$ *is not an* ***nbc****-set, but is an independent set.*

1.5 Möbius Function

Let Int(L) denote the set of all closed intervals of a lattice L. A *locally finite poset* is a poset whose intervals $[x,y]$ are all finite. We now define the *Möbius function* as follows.

Definition 1.3 *Let L be a locally finite poset. The Möbius function of L, $\mu : Int(L) \to \mathbb{Z}$ is defined recursively by the conditions:*

$$\mu(X, X) = 1, \text{ for all } x \in L,$$
$$\mu(X, Y) = -\sum_{X \leq Z < Y} \mu(X, Z), \text{ for all } X < Y \text{ in } L.$$

The *characteristic polynomial* of a geometric lattice L is defined by

$$p(L, t) = \sum_{X \in L} \mu(\hat{0}, X) t^{\rho(\hat{1}) - \rho(X)}.$$

Proposition 1.4 *Let L be a geometric lattice. The number of* ***nbc****-sets and* ***nbc****-bases of L is $\frac{1}{2}|p(L, -1))|$.*

Proof. See [6]. ■

Example 2 *Let M be the same matroid as in Example 1. The* ***nbc****-bases of M are* $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$. *The* ***nbc****-sets of M are* $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{1\}$. *On the other hand, we have*

$$\frac{1}{2}|p(L, -1)| = \frac{1}{2}(1 + 5 + 8 + 4) = 9.$$

The number of ***nbc****-sets and* ***nbc****-bases of L matches $\frac{1}{2}|p(L, -1))|$.*

CHAPTER 2

ARRANGEMENTS

2.1 Arrangements of Subspaces

We follow [16] closely in this section. An *arrangement* is a finite collection $\mathcal{A} = \{A_1, \ldots, A_m\}$ of closed subspaces of a topological space U such that $\mathcal{A}$ is closed under intersection and for $A_i, A_j \in \mathcal{A}$ and $A_i \subseteq A_j$ the inclusion map $A_i \hookrightarrow A_j$ is a cofibration. Let P be the poset $(\mathcal{A}, \leq)$ where $A_i \leq A_j$ if and only if $A_j \subseteq A_i$. A functor $\mathcal{D} : S \to A$ from a small category S to an arbitrary category A is called an *S-diagram of objects in A*. Any partially ordered set $(P, \leq)$ can be considered a small category with arrows pointing downwards, i.e., $p \to q$ if and only if $p \geq q$. Note that the arrangement $\mathcal{A}$ gives rise to an intersection poset P of $\mathcal{A}$. Hence there is an associated P-diagram $\mathcal{D} = \mathcal{D}(\mathcal{A}) : (P, \leq) \to$ CW-Top. If $\mathcal{D}$ is a P-diagram of spaces, then the space related to $p \in P$ is denoted by D_p, and the morphism corresponding to $p \geq q$ for $p, q \in P$ is denoted by d_{pq}. The union of the arrangement $\mathcal{A}$ is the direct limit of its diagram of spaces. Provided that the inclusion maps are all inclusion maps of cells in a finite CW complex and hence are cofibrations, this limit is homotopy equivalent to the homotopy direct limit $||\mathcal{D}||$, defined by [16, Definition 1.3], i.e., $D(\mathcal{A}) \simeq ||\mathcal{D}||$. For any $p \in P$, let $\Delta(P_{<p})$ be the order complex of the poset $P_{<p} = \{q \in P\colon q < p\}$. We present the following theorem from [16, Lemma 1.8].

Theorem 2.1 *[16, Lemma 1.8] Let P be a poset with a unique maximal element $\hat{1}$, and let $\mathcal{D}$ be a P-diagram so that there exist points $c_p \in D_p$ for all $p < \hat{1}$ such that d_{pq} is* *homotopic to the constant map* $d_{pq} : x \mapsto c_q \in D_q$ *for all* $p > q$*. Then*

$$\|\mathcal{D}\| \simeq \bigvee_{p \in P} (\Delta(P_{<p}) * D_p), \tag{2.2}$$

where the wedge is formed by identifying $c_p \in \Delta(P_{<p}) * D_p$ *with the point* $p \in \Delta(P_{<\hat{1}}) * D_p$*, for every* $p < \hat{1}$*.*

2.2 Arrangements of Homotopy Spheres

A *homotopy d-sphere* is a d-dimensional CW-complex that is homotopy equivalent to S^d. Our homotopy spheres are not required to be manifolds. The empty set is considered a (-1)-sphere. We define d-arrangements of homotopy spheres as follows.

Definition 2.3 *A d-arrangement of homotopy spheres consists of a finite d-dimensional CW-complex* $\mathcal{S}$ *homotopy equivalent to the d-sphere and a finite set of* $(d-1)$*-dimensional subcomplexes* $\mathcal{A} = \{S_1, \ldots, S_n\}$ *of* $\mathcal{S}$*, each of which is a homotopy* $(d-1)$*-sphere, satisfying*

- *every intersection of elements in* $\mathcal{A}$ *is a homotopy sphere,*
- *if* $X \simeq S^{d'}$ *is an intersection in* $\mathcal{A}$*, and* $X \nsubseteq S_i$*, then* $X \cap S_i \simeq S^{d'-1}$*, where* $\simeq$ *denotes homotopy equivalence.*
- *there exists a fixed-point free involution of the arrangement; specifically there exists a free involution* ι *of* $\mathcal{S}$ *such that if* X *is an intersection in* $\mathcal{A}$*,* $\iota : X \to X$*.*

Definition 2.4 *Let* $\mathcal{A}$ *be an arrangement of homotopy spheres. The intersection lattice of A is the intersection poset of the elements of* $\mathcal{A}$ *ordered by reverse inclusion, denoted by* $L(\mathcal{A})$*.*

Proposition 2.5 *Suppose $\mathcal{A}$ is a d-arrangement of homotopy spheres. Then $L(\mathcal{A})$ is a geometric lattice with rank function $\rho(X) = d - \dim X$.*

Proof. See [14, Proposition 5.3].

Suppose X is a non-empty intersection in $\mathcal{A}$, then the contraction $\mathcal{A}/X$ is the arrangement with ambient space X and elements the intersections of the old elements with X. $L(\mathcal{A}/X)$ is isomorphic to that of $[X, \hat{1}]$. The deletion $\mathcal{A} - S_j$ is the arrangement $\{S_1, \ldots, \hat{S}_j, \ldots, S_n\}$ with S_j removed. $\mathcal{A}$ is called *essential* if $\bigcap_{j=1}^{n} S_j = \emptyset$. The *link* of $\mathcal{A}$ is $V := \cup_{j=1}^{n} S_j$. The homotopy type of the link of $\mathcal{A}$ only depends on $L(\mathcal{A})$.

Proposition 2.6 *If $\mathcal{A}$ is a d-arrangement of homotopy spheres, then*

$$V \simeq \bigvee_{i=1}^{|p(L(\mathcal{A},-1))|-1} S^{d-1}.$$

Proof. The proof is an application of Theorem 2.1. See [14, Proposition 5.4] for details. ■

Before we cite some propositions as applications of Zaslavsky's enumerative theory [15], we introduce the idea of partitionedness.

Definition 2.7 *An arrangement of homotopy spheres is partitioned if the $(d-1)$-skeleton of $\mathcal{S}$ is contained in V. If every contraction of $\mathcal{A}$. is partitioned, then $\mathcal{A}$ is fully partitioned.*

Suppose $\mathcal{S}$ is partitioned. Since V is homotopy equivalent to $|p(L(\mathcal{A}, -1))| - 1$ $(d-1)$-spheres, we can expect to have $|p(L(\mathcal{A}, -1))|$ d-cells in $\mathcal{S}$. Suppose V is a union of two symmetric parts attached along S_1, called respectively the upper and lower halves of V. A straightforward Mayer-Vietoris argument shows that the upper half is homotopy equivalent to $\frac{1}{2}|p(L(\mathcal{A}, -1))|$ $(d-1)$-spheres. We fill in the $(d-1)$-spheres by d-cells, and call it $\mathcal{S}_{\text{up}}$. By Proposition 1.4, $\frac{1}{2}|p(L(\mathcal{A}, -1))|$ is the number of **nbc**-sets and **nbc**-bases of L. So there exists a bijection between the d-cells of $\mathcal{S}_{\text{up}}$ and the **nbc**-sets and **nbc**-bases.

If the homotopy sphere arrangement $\mathcal{S}$ is fully partitioned, then an i-cell in $\mathcal{S}_{\text{up}}$ lives in a unique $d-i-1$-flat y of $L(\mathcal{A})$, i.e., $\rho(y)+i = d-1$ where d is the rank of the geometric lattice. Hence to specify an i-cell in $\mathcal{S}_{\text{up}}$, we can expect two pieces of data, the $d-i-1$-flat y and an **nbc**-set or **nbc**-basis B which lives in the contraction L/y. Zaslavsky's formulas (and their equivalent formulations by Las Vergnas [11]) for the number of cells still hold for arrangements of homotopy spheres as they do for pseudosphere arrangements.

Proposition 2.8 *[14] If $\mathcal{A}$ is a partitioned d-arrangement of homotopy spheres, then S has $|p(L(\mathcal{A})), -1|$ d-dimensional cells.*

Corollary 2.9 *[14] If every contraction $\mathcal{A}/X$, $\rho(X) \leq i$, is partitioned, then the number of $(d-i)$-dimensional cells is*

$$\sum_{\rho(X)=i,\, X\leq Y} |\mu(X, Y)|.$$

CHAPTER 3

TOPOLOGICAL REPRESENTATIONS

3.1 Topological Representations of lower ranks

There exist different topological representations of geometric lattices, among which Swartz contributed one using arrangements of homotopy spheres whose class of intersection lattices matches the corresponding geometric lattices [14, Theorem 6.1]. We restate it as follows.

Theorem 3.1 *[14] If L is the lattice of flats of a rank $d+1$ matroid, then there exists a d-arrangement of homotopy spheres, $\mathcal{A}$, such that $L = L(\mathcal{A})$. Furthermore, $\mathcal{A}$ can be constructed so that there is a fixed-point free involution of S which preserves $\mathcal{A}$.*

Swartz's proof was not algorithmic nor was there a guarantee that $\mathcal{S}$ would be a regular CW-complex. Let L be a rank r geometric lattice of rank ≤ 4. The algorithm we have gives an explicit fully partitioned homotopy sphere r-arrangement $\mathcal{S}$ that is a CW-complex whose intersection lattice is L. Moreover $\mathcal{S}$ has a $(r-1)$-sphere in it that is a regular CW-complex. Our work has no problems in matching the dimensions of the arrangements, nor in the cell numbers that are completely determined by Zaslavsky's enumerative theory [15] or Las Vergnas's formula [11], either of which works for pseudosphere arrangements. The construction is demonstrated below in detail.

Theorem 3.2 *If L is the lattice of flats of a rank $d+1$ matroid, $d \leq 3$, then there exists a d-arrangement of homotopy spheres, $\mathcal{A}$, which is a CW-complex and contains a sphere* *that is a regular CW-complex, such that $L = L(\mathcal{A})$. Furthermore, $\mathcal{A}$ can be constructed so that there is a fixed-point free involution of S which preserves $\mathcal{A}$.*

Since the construction process is inductive by rank, we begin with rank 1 matroids.

3.1.1 Rank 1

This case is trivial because we only have one matroid with one element $\hat{1} = 1$. Two discrete vertices $\mathcal{S} = \tau_1, \tau_1'$ will be its topological representation $\mathcal{S}$. Here τ_1 is an abbreviation of $\tau_{\{1\}}$. From now on, we write either τ_1 or $\tau_{\{1\}}$, which means the same cell. The upper half is $\{\tau_1\}$ whereas the lower half is $\{\tau_1'\}$. Let $\phi_{\{1\}} : \text{pt} \to \mathcal{S}$ be defined such that $\phi_{\{1\}}(\text{pt}) = \tau_1$ and symmetrically we define $\phi'_{\{1\}} : \text{pt} \to \mathcal{S}$ by $\phi'_{\{1\}}(\text{pt}) = \tau_1'$.

3.1.2 Rank 2

Let M be a rank 2 simple matroid whose geometric lattice is L. Let $a_1, \ldots, a_m$ be the coatoms of L. Here coatoms and atoms coincide, i.e., $a_i = e_i$ for $i = 1, \ldots, m$. In $M/1$ the **nbc**-basis is $\{2\}$ whereas in M/i where $i \geq 2$ the **nbc**-basis is $\{1\}$. Hence in $M/1$, We have two vertices $\{\tau_2^1, \tau_2^{1'}\}$ as the CW-complex by the rank 1 process where 1 in τ_2^1 and $\tau_2^{1'}$ signifies $M/1$ and $\{2\}$ is the **nbc**-basis. In all we have $\{\tau_2^1, \tau_2^{1'}\}, \{\tau_1^2, \tau_1^{2'}\}, \ldots, \{\tau_1^m, \tau_1^{m'}\}$ as the $2m$ 0-spheres. The link $V := \cup_{i=1}^m \mathcal{S}_{e_i}$ is the union of the representations $\mathcal{S}_{e_i}$ of contraction $[e, \hat{1}]$ where e runs over the atoms. So V is a union of $2m$ discrete vertices. Let the upper link be all of $\mathcal{S}_{e_1}$ with the upper halves of $\mathcal{S}_{e_i}$ where $i = 2, 3, \ldots, m$. Thus $V_{\text{up}} := \{\tau_2^1, \tau_1^2, \ldots, \tau_1^m, \tau_2^{1'}\}$.

Symmetrically, let $V_{\text{low}} := \{\tau_2^{1'}, \tau_1^{2'}, \ldots, \tau_1^{m'}, \tau_2^1\}$. We denote the n-ball by D^n. For D^1 we use the specific 1-ball $[0,1] \subseteq \mathbb{R}$.

For each **nbc**-basis $B = \{1, i\}$, we let $\kappa_B : \partial D^1 \to V$ be defined by

$$\kappa_B(0) = \tau_2^1, \quad \kappa_B(1) = \tau_1^i.$$

For the **nbc**-set $B = \{1\}$, we let

$$\kappa_B(0) = \tau_2^1, \quad \kappa_B(1) = \tau_2^{1'}.$$

Symmetrically, for each **nbc**-basis $B = \{1, i\}$, we let $\kappa'_B : \partial D^1 \to V$ be defined by

$$\kappa'_B(0) = \tau_2^{1'}, \quad \kappa'_B(1) = \tau_1^{i'}.$$

For the **nbc**-set $B = \{1\}$, we let

$$\kappa'_B(0) = \tau_2^{1'}, \quad \kappa'_B(1) = \tau_2^1.$$

Then we construct $\mathcal{S}$ as

$$\mathcal{S} = V \cup_{\kappa_B} D^1 \cup_{\kappa'_B} D^1$$

over all **nbc**-sets and **nbc**-bases B where D^1's are all attached via the κ_B and κ'_B maps. Let the upper and lower $\mathcal{S}$ be respectively as follows,

$$\mathcal{S}_{\text{up}} = V \cup_{\kappa_B} D^1, \quad \mathcal{S}_{\text{low}} = V \cup_{\kappa'_B} D^1$$

when B runs over all **nbc**-sets and **nbc**-bases. See Figure 3.1 for an example where $m = 4$. Observe that $\mathcal{S} = \mathcal{S}_{\text{up}} \cup \mathcal{S}_{\text{low}}$. The intersection $\mathcal{S}_{\text{up}} \cap \mathcal{S}_{\text{low}}$ is the homotopy sphere constructed for $M/1$. Also note that $\mathcal{S}_{\text{up}}, \mathcal{S}_{\text{low}}$ are both contractible. Thus $\mathcal{S}$ is a homotopy sphere 1-arrangement and contains the sphere $\tau_2^1 \cup \tau_2^{1'}$ that is a regular CW-complex.

For any **nbc**-set or **nbc**-basis B, let τ_B and τ'_B be the cell whose associated attaching map is κ_B and κ'_B. As discussed earlier, a 1-cell of $\mathcal{S}_{\text{up}}$ corresponds to an **nbc**-set or **nbc**-basis B of $L = L/\emptyset$. In general, we denote any i-cell by τ_B^y where y is the corresponding flat and B the corresponding basis. For convenience, we write τ_B for $\tau_B^{\emptyset}$ where B is an **nbc**-set or basis on $[\hat{0}, \hat{1}]$. This notation applies to all future higher ranks.

For higher rank matroids, we may need attaching maps ϕ_B in rank 2 contractions in which any independent set can be considered. We now describe how to do this. For a map $\phi : D^n \to X$, we denote $\phi|_{\partial D^n}$ by $\partial\phi$. When B is not an **nbc**-set or **nbc**-basis ϕ_B is defined in terms of ϕ_C's for lexicographically earlier C's.

For each **nbc**-set or **nbc**-basis B, let $\phi_B : D^1 \to \mathcal{S}$ be defined such that $\partial\phi_B = \kappa_B$ and ϕ_B is a homeomorphism to the open cell τ_B on $(0,1)$. Symmetrically we define $\phi'_B : D^1 \to \mathcal{S}$. For any non-**nbc** independent set $B = \{i, j\}$, we define $\phi_B : D^1 \to \mathcal{S}$ as follows. If i is parallel to 1, then $\phi_{\{i,j\}} = \phi_{\{1,j\}}$. Similarly If j is parallel to 1, then $\phi_{\{i,j\}} = \phi_{\{1,i\}}$. Otherwise $\{1, i, j\}$ is a circuit and $\phi_{\{i,j\}}$ is defined by

$$\phi_{\{i,j\}} = \begin{cases} \phi_{\{1,i\}}(1-2x) & \text{if } 0 \le x \le \frac{1}{2}, \\ \phi_{\{1,j\}}(2x-1) & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

For singleton non-**nbc** sets $B = \{i\}$, we define $\phi_B : D^1 \to \mathcal{S}$ as follows. If i is parallel to 1, then $\phi_{\{i\}} = \phi_{\{1\}}$. Otherwise $\phi_B : D^1 \to \mathcal{S}$ is defined by

$$\phi_{\{i\}} = \begin{cases} \phi_{\{1,i\}}(1-3x) & \text{if } 0 \le x \le \frac{1}{3}, \\ \phi_{\{1\}}(3x-1) & \text{if } \frac{1}{3} \le x \le \frac{2}{3}, \\ \phi'_{\{1,i\}}(3x-2) & \text{if } \frac{2}{3} \le x \le 1. \end{cases}$$

See Figure 3.2 for an illustration where the green path represents $\phi_{\{2,4\}}$ and $\phi_{\{3\}}$, respectively. For any non-**nbc** independent set B, $\phi'_B : D^1 \to \mathcal{S}$ is defined symmetrically. Observe that $\partial\phi_{\{i,j\}} = \phi^i_{\{j\}} \cup \phi^j_{\{i\}}$, $\partial\phi_{\{i\}} = \phi^i_{\{1\}} \cup \phi^{i'}_{\{1\}}$ where $i \neq 1$, and

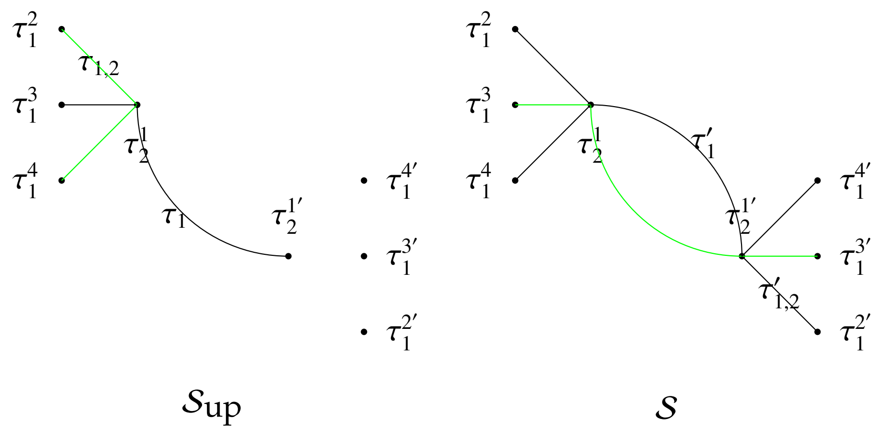


Figure 3.1:

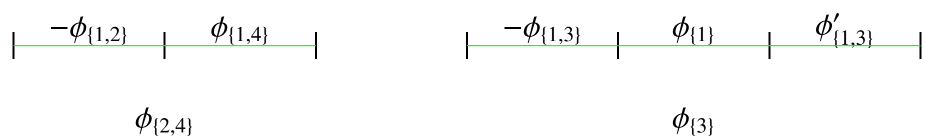


Figure 3.2:

$\partial\phi_{\{1\}} = \phi^1_{\{2\}} \cup \phi^{1'}_{\{2\}}$.

3.1.3 Rank 3

Now let M be a rank 3 simple matroid whose geometric lattice is L with coatoms $a_1, a_2, \ldots, a_m$ and atoms $e_1, \ldots, e_n$ in lexicographic order. Let the coatoms containing e_1 be respectively $a_1, \ldots, a_l$ where $l < m$. We induce from rank 2 cases. Each M/e_i for $i = 1, 2, \ldots, n$ is a rank 2 contraction, and we call its homotopy sphere arrangement $\mathcal{S}_{e_i}$. Denote $\mathcal{S}_{\text{up}}$ in M/e_i as $\mathcal{S}^{e_i}_{\text{up}}$ and denote $\mathcal{S}_{\text{low}}$ in M/e_i as $\mathcal{S}^{e_i}_{\text{low}}$. Then the link is $V := \cup^n_i \mathcal{S}_{e_i}$. Let the upper link be $V_{\text{up}} := \mathcal{S}_{e_1} \cup \bigcup^n_{i=2} \mathcal{S}^{e_i}_{\text{up}}$. Symmetrically we define the lower link V_{low}. We consider V_{up} and how to add

2-cells to V_{up} to construct $\mathcal{S}_{\text{up}}$ then construct S_{low} symmetrically.

For B an independent set in M/e_i, we denote the ϕ_B map in $\mathcal{S}_{e_i}$ as $\phi_B^{e_i}$.

For each **nbc**-basis $B = \{1, a, b\}$, we let $\kappa_B : \partial D^2 \to V$ where $\partial D^2 = [0, 1]$ with 0 identified with 1 be defined by

$$\kappa_B = \begin{cases} \phi^1_{\{a,b\}}(3x) & \text{if } 0 \le x \le \frac{1}{3}, \\ \phi^a_{\{1,b\}}(3x-1) & \text{if } \frac{1}{3} \le x \le \frac{2}{3}, \\ \phi^b_{\{1,a\}}(2-3x) & \text{if } \frac{2}{3} \le x \le 1. \end{cases}$$

For **nbc**-set $B = \{1, a\}$, we let

$$\kappa_B = \begin{cases} \phi^1_{\{a\}}(2x) & \text{if } 0 \le x \le \frac{1}{2}, \\ \phi^a_{\{1\}}(1-2x) & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

For the **nbc**-set $B = \{1\}$, we let

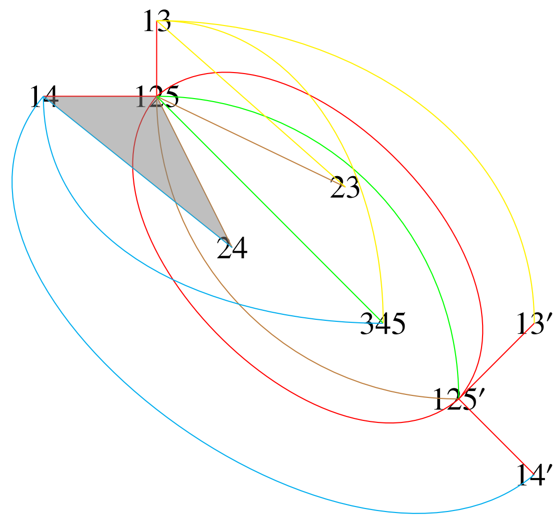
$$\kappa_B = \begin{cases} \phi^1_{\{2\}}(2x) & \text{if } 0 \le x \le \frac{1}{2}, \\ \phi^{1'}_{\{2\}}(1-2x) & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

We define $\kappa'_B : \partial D^2 \to V$ symmetrically.

Example 3 *Let M be the same matroid as in Example 1. Let the topological representations of contractions $[e_i, \hat{1}]$ be in five different colors, respectively. $\mathcal{S}_1$ in red, $\mathcal{S}^2_{up}$ in brown, $\mathcal{S}^3_{up}$ in yellow, $\mathcal{S}^4_{up}$ in cyan, and $\mathcal{S}^5_{up}$ in green. They make up V_{up}. In Figure 3.3, the boundary of the shaded area is $\kappa_{\{1,2,4\}}$. In Figure 3.4, the boundary of the shaded area is $\kappa_{\{1,3,4\}}$. In Figure 3.5, the boundary of the shaded area is $\kappa_{\{1,2\}}$. In Figure 3.6, the boundary of the shaded area is $\kappa_{\{1,4\}}$. In Figure 3.7, the boundary of the shaded area is $\kappa_{\{1\}}$.*

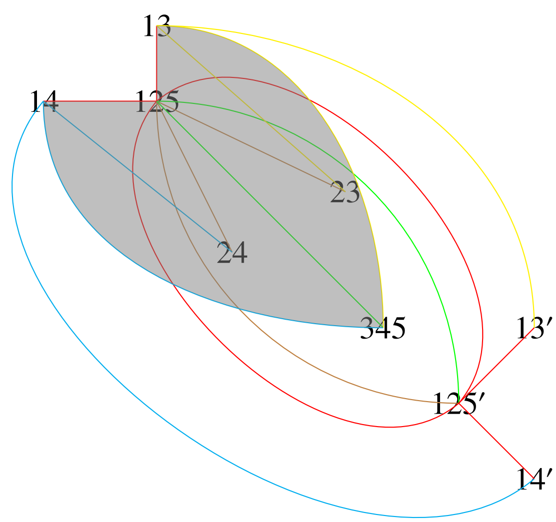
Then we construct $\mathcal{S}$ as

$$\mathcal{S} = V \cup_{\kappa_B} D^2 \cup_{\kappa'_B} D^2$$



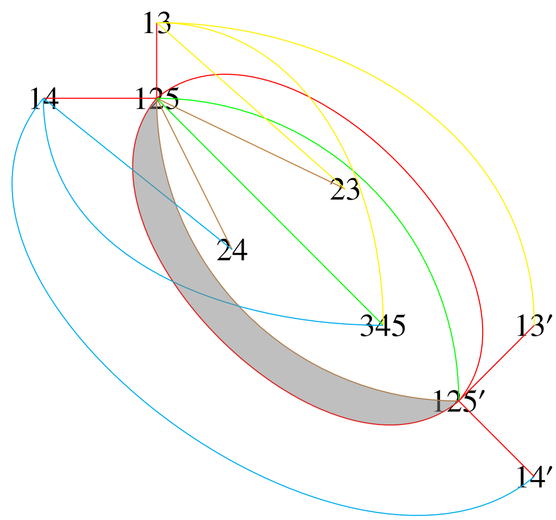
$\kappa_{\{1,2,4\}}$ in Vup

Figure 3.3:



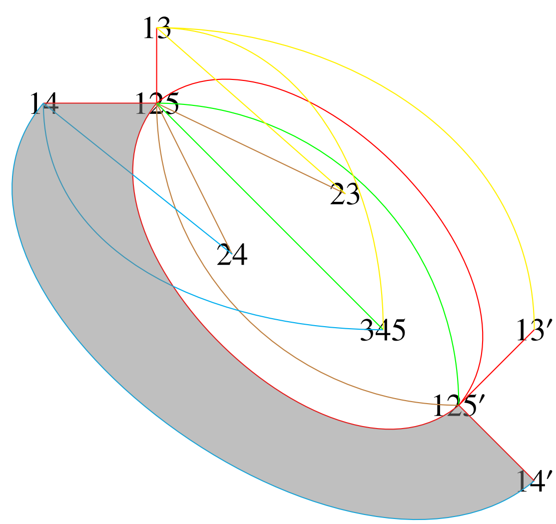
$\kappa_{\{1,3,4\}}$ in Vup

Figure 3.4:



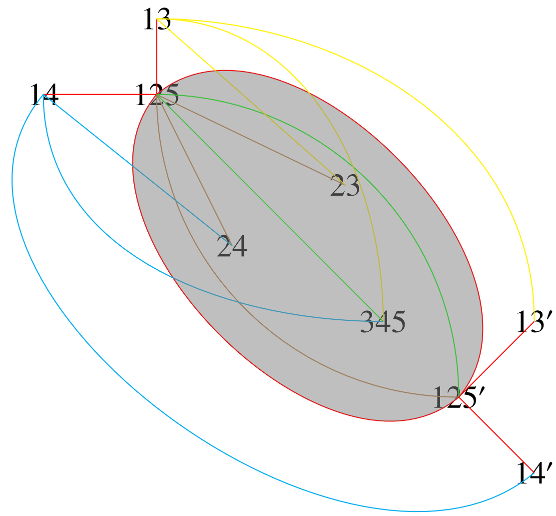
$\kappa_{\{1,2\}}$ in V_{up}

Figure 3.5:



$\kappa_{\{1,4\}}$ in V_{up}

Figure 3.6:



$\kappa_{\{1\}}$ in V_{up}

Figure 3.7:

over all **nbc**-sets and **nbc**-bases B where D^2's are all attached via the κ_B and κ'_B maps. Let the upper and lower $\mathcal{S}$ be respectively as follows,

$$\mathcal{S}_{\text{up}} = V \cup_{\kappa_B} D^2, \quad \mathcal{S}_{\text{low}} = V \cup_{\kappa'_B} D^2$$

when B runs over all **nbc**-sets and **nbc**-bases. Observe that $\mathcal{S} = \mathcal{S}_{\text{up}} \cup \mathcal{S}_{\text{low}}$. The intersection $\mathcal{S}_{\text{up}} \cap \mathcal{S}_{\text{low}}$ is the homotopy sphere constructed for $M/1$.

For any **nbc**-set or **nbc**-basis B, let τ_B and τ'_B be the cell whose associated attaching map is κ_B and κ'_B.

For higher rank matroids, we may need attaching maps ϕ_B in rank 3 contractions in which any independent set can be considered. We now describe how to do this. When B is not an **nbc**-set nor **nbc**-basis ϕ_B is defined in terms of ϕ_C's for lexicographically earlier C's.

Now for each **nbc**-set or **nbc**-basis B, let $\phi_B : D^2 \to \mathcal{S}$ be defined such that

$\partial\phi_B = \kappa_B$ and ϕ_B is a homeomorphism to the open cell τ_B on the interior of D^2. Symmetrically we define $\phi'_B : D^2 \to \mathcal{S}$. For any non-**nbc** independent set $B = \{i, j, k\}$ of types as follows where $i < j < k$, let $\phi_B : D^2 \to \mathcal{S}$ be defined as illustrated in Figure 3.8 and Figure 3.9. For any non-**nbc** independent set B, $\phi'_B : D^2 \to \mathcal{S}$ is defined symmetrically.

It is possible that in an independent set B there can be an elements parallel to and smaller than an element in B. Then that element of B can be replaced by the smaller parallel one. Also, there can be external activities in a subset of B. In these cases the ϕ maps in the diagram involve lexicographically earlier independent sets and hence are well-defined. In general if $EA(B) \neq 0$ and $\min EA = c$, then $\phi_{\{i,j,k\}}$ is defined to be the union of $\phi_{\{c,j,k\}}$, $\phi_{\{c,i,k\}}$, and $\phi_{\{c,i,j\}}$, each of which a union of $\phi_{B'}$ maps until $EA(B') = 0$.

1. $B = \{1, a, b\}$ where $\{c, a, b\}$ is a circuit and $\min EA = c \in (12)$.
2. $B = \{1, a, b\}$ where $\{c, a, b\}$ is a circuit and $\min EA = c \notin (12)$.
3. $B = \{a, b, c\}$ where $\{1, a, b, c\}$ is a circuit and $a \notin (12)$.
4. $B = \{a, b, c\}$ where $\{1, a, b, c\}$ is a circuit and $a \in (12)$ (or $b \in (12)$ or $c \in (12)$).
5. $B = \{a, b, c\}$ where $\{1, a, b\}$ (or $\{1, a, c\}$ or $\{1, b, c\}$) is a circuit.
6. $B = \{a, b\}$ where $a \in (12)$ and $b \notin (12)$ and $EA = \emptyset$.
7. $B = \{a, b\}$ where $a \notin (12)$ and $b \notin (12)$ and $EA = \emptyset$.
8. $B = \{a, b\}$ where $\{1, a, b\}$ is a circuit.
9. $B = \{a, b\}$ where $\{c, a, b\}$ is a circuit and $\min EA = c \notin (12)$.
10. $B = \{a, b\}$ where $\{c, a, b\}$ is a circuit and $\min EA = c \in (12)$.
11. $B = \{a\}$ where $\{1, a\}$ is a circuit and $a \in (12)$.

12. $B = \{a\}$ where $a \notin (12)$.

Observe that $\partial\phi_{\{i,j,k\}} = \phi^i_{\{j,k\}} \cup \phi^j_{\{i,k\}} \cup \phi^k_{\{i,j\}}$, $\partial\phi_{\{i,j\}} = \phi^i_{\{j\}} \cup \phi^j_{\{i\}}$, $\partial\phi_{\{i\}} = \phi^i_{\{1\}} \cup \phi^{i'}_{\{1\}}$ where $i \neq 1$, and $\partial\phi_{\{1\}} = \phi^1_{\{2\}} \cup \phi^{1'}_{\{2\}}$. In $\partial\phi_{\{i,j,k\}}$, $\phi^i_{\{j,k\}} \cap \phi^j_{\{i,k\}} = \phi^{ij}_{\{k\}}$; in $\partial\phi_{\{i,j\}}$, $\phi^i_{\{j\}} \cap \phi^j_{\{i\}} = \phi^{ij}_{\min(E-(i,j))} \cup \phi^{ij'}_{\min(E-(i,j))}$; in $\partial\phi_{\{i\}}$, $\phi^i_{\{1\}} \cap \phi^{i'}_{\{1\}} = \phi^{1i}_{\min(E-(1,i))} \cup \phi^{1i'}_{\min(E-(1,i))}$; and in $\partial\phi_{\{1\}}$, $\phi^1_{\{2\}} \cap \phi^{1'}_{\{2\}} = \phi^{12}_{\min(E-(1,2))} \cup \phi^{12'}_{\min(E-(1,2))}$.

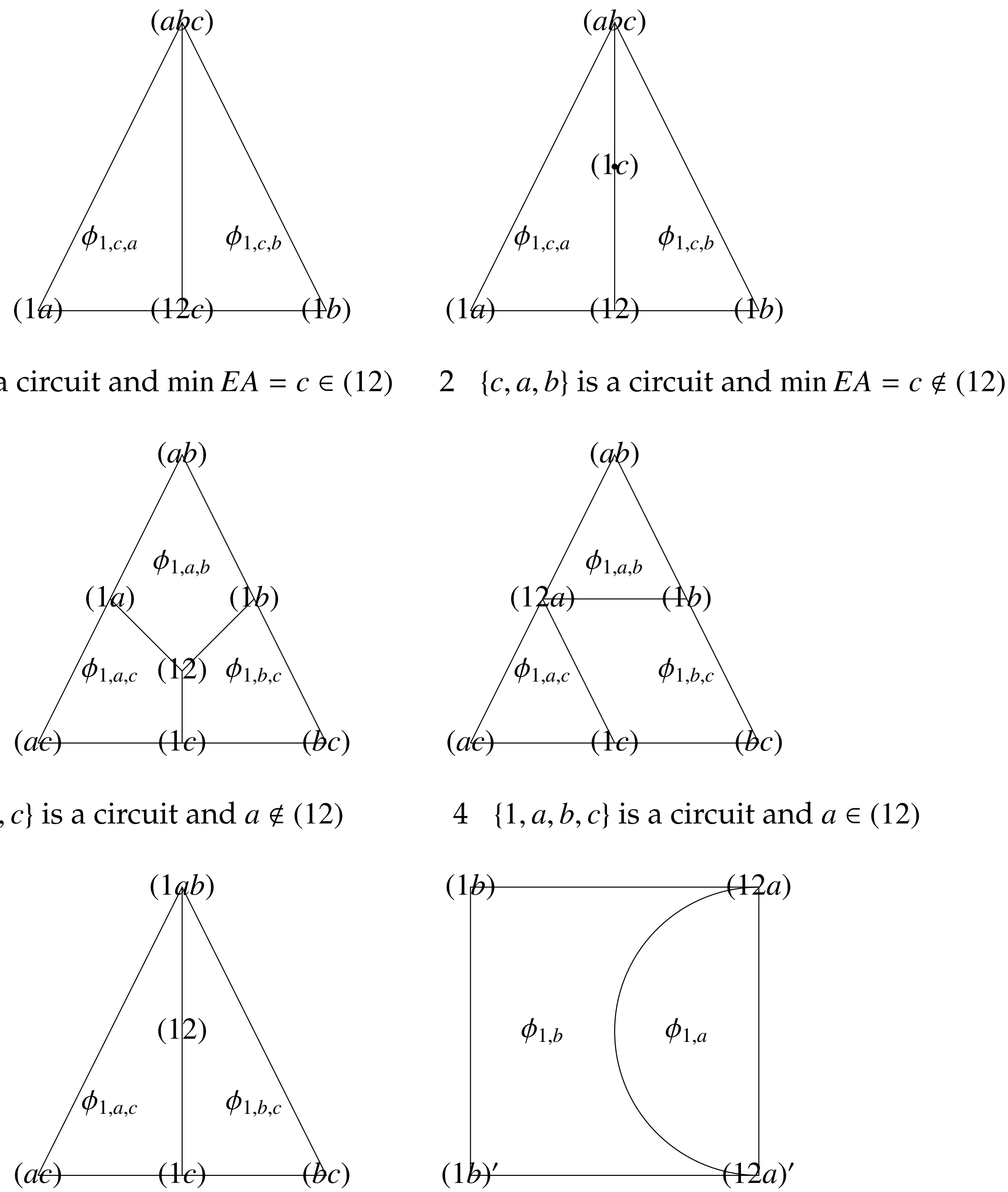
Recall $\mathcal{S} := \mathcal{S}_{\text{up}} \cup \mathcal{S}_{\text{low}}$ being glued along $\mathcal{S}_{e_1}$. The following lemma shows $\mathcal{S}_{\text{up}}$ is contractible.

Lemma 3.3 *Let M be a matroid on $[n]$ of rank 3. Then $\mathcal{S}_{up}$ is contractible.*

Proof. A unique facet of a cell τ^z_B is a facet which is on the boundary of τ^z_B and is in no other $\tau^{z_0}_{B_0}$. The unique facet for $\tau_{\{1,a,b\}}$ is $\phi^b_{\{1,a\}}$; for $\tau_{\{1,a\}}$, $\phi^a_{\{1\}}$; and for $\tau_{\{1\}}$, $\phi^{1'}_{\{2\}}$. To show $\mathcal{S}_{\text{up}}$ is contractible, it is sufficient to show when we collapse the cells by their unique facets, we are left with a tree. It is equivalent to show $\bar{V}$ is a tree, where $\bar{V}$ is obtained by removing all the unique facets in V_{up}. First $\bar{V}$ is connected because every vertex is connected to (12). Second we show the number of edges is 1 less than the number of vertices. The number of vertices here is $m + l$ where m is the number of coatoms and l is the number of coatoms above e_1. The number of edges is

$$\begin{aligned} & \sum_{Y \in L(M),\, r(X)=1} |\mu(X,Y)| + \sum_{Y \in L(M),\, X=e_1} |\mu(X,Y)| - \sum_{Y \in L(M),\, r(X)=0} |\mu(X,Y)| \\ = \quad & f_{\{1,2\}}(M) + l - \frac{2 - 2f_{\{2\}}(M) + f_{\{1,2\}}(M)}{2} \\ = \quad & f_{\{2\}}(M) + l - 1 \\ = \quad & m + l - 1. \end{aligned}$$

Hence $\bar{V}$ is a tree. ■



1 $\{c, a, b\}$ is a circuit and $\min EA = c \in (12)$

2 $\{c, a, b\}$ is a circuit and $\min EA = c \notin (12)$

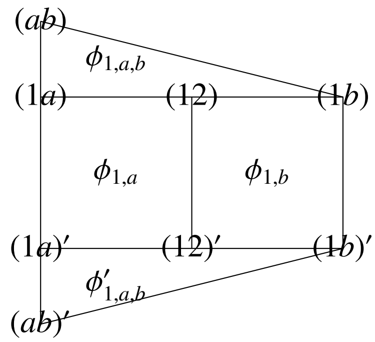
3 $\{1, a, b, c\}$ is a circuit and $a \notin (12)$

4 $\{1, a, b, c\}$ is a circuit and $a \in (12)$

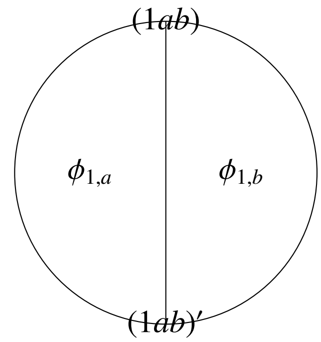
5 $\{1, a, b\}$ or $\{1, a, c\}$ or $\{1, b, c\}$ is a circuit

6 $a \in (12)$ and $b \notin (12)$ and $EA = \emptyset$

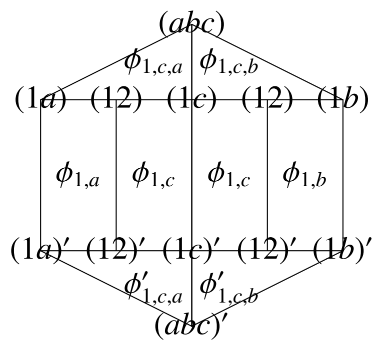
Figure 3.8:



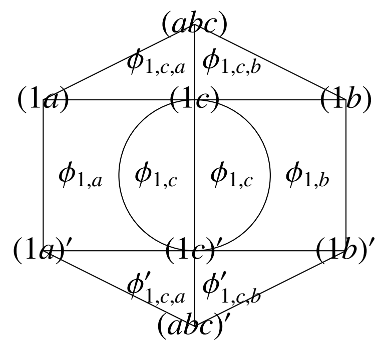
7 $a \notin (12)$ and $b \notin (12)$ and $EA = \emptyset$



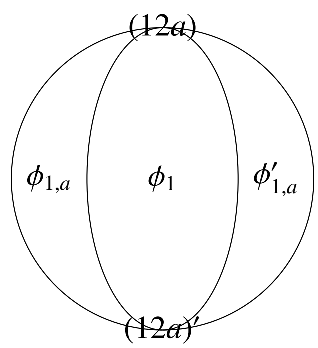
8 $\{1, a, b\}$ is a circuit



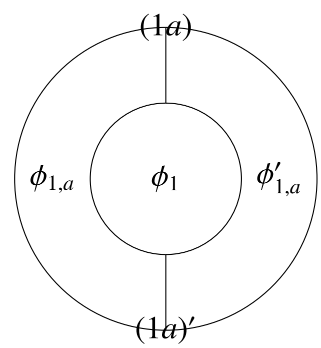
9 $\{c, a, b\}$ is a circuit and $\min EA = c \notin (12)$



10 $\{c, a, b\}$ is a circuit and $\min EA = c \in (12)$



11 $\{1, a\}$ is a circuit and $a \in (12)$



(12) $a \notin (12)$

Figure 3.9:

Thus $\mathcal{S}_{\text{up}}, \mathcal{S}_{\text{low}}$ are both contractible. Since $\mathcal{S}_{\text{up}} \cap \mathcal{S}_{\text{low}}$ is a homotopy 1-sphere, $\mathcal{S}$ is a homotopy sphere 2-arrangement that is a CW-complex and contains the 2-sphere $\tau^1_{\{2\}} \cup \tau^{1'}_{\{2\}}$ that is a regular CW-complex.

3.1.4 Rank 4

Now let M be a rank 4 simple matroid whose geometric lattice is L with coatoms $a_1, a_2, \ldots, a_m$ and atoms $e_1, \ldots, e_n$ in lexicographic order. Let the coatoms containing e_1 be respectively $a_1, \ldots, a_l$ where $l < m$. We induce from rank 3 cases. Each M/e_i for $i = 1, 2, \ldots, n$ is a rank 3 contraction, and we call its homotopy sphere arrangement $\mathcal{S}_{e_i}$. Denote $\mathcal{S}_{\text{up}}$ in M/e_i as $\mathcal{S}^{e_i}_{\text{up}}$ and denote $\mathcal{S}_{\text{low}}$ in M/e_i as $\mathcal{S}^{e_i}_{\text{low}}$. Then the link is $V := \cup^n_i \mathcal{S}_{e_i}$. Let the upper link be $V_{\text{up}} := \mathcal{S}_{e_1} \cup \bigcup^n_{i=2} \mathcal{S}^{e_i}_{\text{up}}$. Symmetrically we define the lower link V_{low}. We consider V_{up} and how to add 3-cells to V_{up} to construct $\mathcal{S}_{\text{up}}$ then construct S_{low} symmetrically.

Recall ∂D^3 is homeomorphic to a tetrahedron that is a union of four triangles. Denote them by Δ_i, $i = 1, 2, 3, 4$. ∂D^3 is also homeomorphic to a union of three bigons, or a union of two bigons, all of which share the same south and north pole. Denote them as γ_i as shown in Figure 3.10, seen downwards above the north pole where $i = 1, 2, 3, 4, 5$. For each **nbc**-basis $B = \{1, a, b, c\}$, we let $\kappa_B : \partial D^3 \to V$ be defined such that

$$\kappa_B|_{\Delta_1} = \phi^1_{\{a,b,c\}},\ \kappa_B|_{\Delta_2} = \phi^a_{\{1,b,c\}},\ \kappa_B|_{\Delta_3} = \phi^b_{\{1,a,c\}},\ \kappa_B|_{\Delta_4} = \phi^c_{\{1,a,b\}}.$$

For **nbc**-set $B = \{1, a, b\}$, we let $\kappa_B : \partial D^3 \to V$ be defined such that

$$\kappa_B|_{\gamma_1} = \phi^1_{\{a,b\}},\ \kappa_B|_{\gamma_2} = \phi^a_{\{1,b\}},\ \kappa_B|_{\gamma_3} = \phi^b_{\{1,a\}}.$$

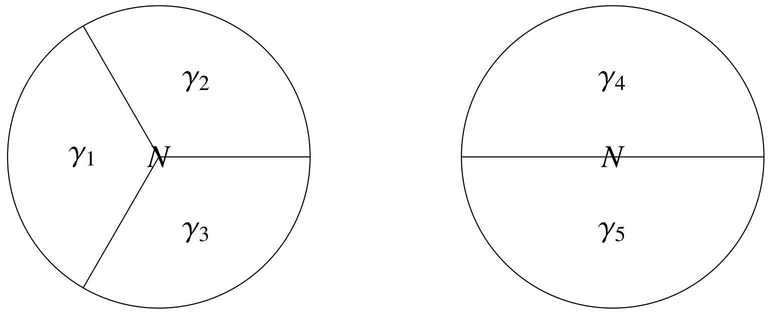


Figure 3.10:

For **nbc**-set $B = \{1, a\}$, we let $\kappa_B : \partial D^3 \to V$ be defined such that

$$\kappa_B|_{\gamma_4} = \phi^1_{\{a\}},\ \kappa_B|_{\gamma_5} = \phi^a_{\{1\}}.$$

For the **nbc**-set $B = \{1\}$, we let $\kappa_B : \partial D^3 \to V$ be defined such that

$$\kappa_B|_{\gamma_4} = \phi^1_{\{2\}},\ \kappa_B|_{\gamma_5} = \phi^{1'}_{\{2\}}.$$

The previously observed values of $\partial\phi$ insure that the various piecewise parts of κ_B are compatible with each other.

We define $\kappa'_B : \partial D^3 \to V$ symmetrically.

Then we construct $\mathcal{S}$ as

$$\mathcal{S} = V \cup_{\kappa_B} D^3 \cup_{\kappa'_B} D^3$$

over all **nbc**-sets and **nbc**-bases B where D^3's are all attached via the κ_B and κ'_B maps. Let the upper and lower $\mathcal{S}$ be respectively as follows,

$$\mathcal{S}_{\text{up}} = V \cup_{\kappa_B} D^3, \quad \mathcal{S}_{\text{low}} = V \cup_{\kappa'_B} D^3$$

when B runs over all **nbc**-sets and **nbc**-bases. Observe that $\mathcal{S} = \mathcal{S}_{\text{up}} \cup \mathcal{S}_{\text{low}}$. The intersection $\mathcal{S}_{\text{up}} \cap \mathcal{S}_{\text{low}}$ is the homotopy sphere constructed for $M/1$.

For any **nbc**-set or **nbc**-basis B, let τ_B and τ'_B be the cell whose associated attaching map is κ_B and κ'_B.

The following lemma shows $\mathcal{S}_{\text{up}}$ is contractible.

Lemma 3.4 *Let M be a matroid on $[n]$ of rank 4. Then $H_3(\mathcal{S}_{up}) = H_2(\mathcal{S}_{up}) = H_1(\mathcal{S}_{up}) = \pi_1(\mathcal{S}_{up}) = 0$, and $\mathcal{S}_{up}$ is contractible.*

Proof. A unique facet of a cell τ^z_B is a facet which is on the boundary of τ^z_B and is in no other $\tau^{z_0}_{B_0}$. The unique facet for $\tau_{\{1,a,b,c\}}$ is $\phi^c_{\{1,a,b\}}$; for $\tau_{\{1,a,b\}}$, $\phi^b_{\{1,a\}}$; for $\tau_{\{1,a\}}$, $\phi^a_{\{1\}}$; and for $\tau_{\{1\}}$, $\phi^{1'}_{\{2\}}$. Hence those τ_B's for B being **nbc**-sets and **nbc**-bases are all independent to each other and they all have a vertex (12). There are $|\frac{1}{2}p(L(\mathcal{A},-1))|$ of them. Similar to Proposition 2.6, we can show

$$V_{\text{up}} \simeq \bigvee_{i=1}^{\frac{1}{2}|p(L(\mathcal{A},-1))|} S^2_i.$$

Since $\mathcal{S}_{\text{up}}$ is V_{up} with several 3-cells attached, we have $\pi_1(\mathcal{S}_{\text{up}}) = H_1(\mathcal{S}_{\text{up}}) = 0$. To show $H_i(\mathcal{S}_{\text{up}}) = 0$ for $i = 2, 3$, it is equivalent to show that $\{\partial\tau_B\}$ for B being **nbc**-sets and **nbc**-bases is a basis for $H_i(V_{\text{up}})$, which can be seen via their unique facets. Therefore

$$H_3(\mathcal{S}_{\text{up}}) = H_2(\mathcal{S}_{\text{up}}) = H_1(\mathcal{S}_{\text{up}}) = \pi_1(\mathcal{S}_{\text{up}}) = 0.$$

By Whitehead's Theorem [9, Theorem 4.5], $\mathcal{S}_{\text{up}}, \mathcal{S}_{\text{low}}$ are both contractible. ■

Since $\mathcal{S}_{\text{up}} \cap \mathcal{S}_{\text{low}}$ is a homotopy 2-sphere, $\mathcal{S}$ is a homotopy sphere 3-arrangement that is a CW-complex and contains the 3-sphere $\tau^1_{\{2\}} \cup \tau^{1'}_{\{2\}}$ that is a regular CW-complex.

3.2 Rank r

Conjecture 3.5 *If L is the lattice of flats of a rank $d+1$ matroid, $d > 4$, then the inductive algorithm produces a d-arrangement of homotopy spheres, $\mathcal{A}$, which is a CW-complex and contains a sphere that is a regular CW-complex, such that $L = L(\mathcal{A})$. Furthermore, there is a fixed-point free involution of S which preserves $\mathcal{A}$.*

As we see in the cases of ranks ≤ 4 our construction is inductive on rank. Let M be a rank r simple matroid whose geometric lattice is L with coatoms $a_1, a_2, \ldots, a_m$ and atoms $e_1, \ldots, e_n$ in lexicographic order. Let the coatoms containing e_1 be respectively $a_1, \ldots, a_l$ where $l < m$. We induce from rank $r-1$ cases. Each M/e_i for $i = 1, 2, \ldots, n$ is a rank $r-1$ contraction, and we call its homotopy sphere arrangement $\mathcal{S}_{e_i}$. Denote $\mathcal{S}_{\text{up}}$ in M/e_i as $\mathcal{S}^{e_i}_{\text{up}}$ and denote $\mathcal{S}_{\text{low}}$ in M/e_i as $\mathcal{S}^{e_i}_{\text{low}}$. Then the link is $V := \cup^n_i \mathcal{S}_{e_i}$. Let the upper link be $V_{\text{up}} := \mathcal{S}_{e_1} \cup \bigcup^n_{i=2} \mathcal{S}^{e_i}_{\text{up}}$. Symmetrically we define the lower link V_{low}. We consider V_{up} and how to add $(r-1)$-cells to V_{up} to construct $\mathcal{S}_{\text{up}}$ then construct S_{low} symmetrically.

For each **nbc**-basis or set $B = \{1, u_1, \ldots, u_q\}$, $q \leq r-1$, we let $\kappa_B : \partial D^{r-1} \to V$ be defined such that

$$\kappa_B = \phi^1_{\{u_1, u_2, \ldots, u_q\}} \cup \phi^{u_1}_{\{1, u_2, \ldots, u_q\}} \cup \cdots \cup \phi^{u_q}_{\{1, u_1, \ldots, u_{q-1}\}}. \tag{3.6}$$

For the **nbc**-set $B = \{1\}$, we let $\kappa_B : \partial D^{r-1} \to V$ be defined such that

$$\kappa_B = \phi^1_{\{2\}} \cup \phi^{1'}_{\{2\}}.$$

We define $\kappa'_B : \partial D^{r-1} \to V$ symmetrically.

Then we construct $\mathcal{S}$ as

$$\mathcal{S} = V \cup_{\kappa_B} D^{r-1} \cup_{\kappa'_B} D^{r-1}$$

over all **nbc**-sets and **nbc**-bases B where D^{r-1}'s are all attached via the κ_B and κ'_B maps. Let the upper and lower $\mathcal{S}$ be respectively as follows,

$$\mathcal{S}_{\text{up}} = V \cup_{\kappa_B} D^{r-1}, \quad \mathcal{S}_{\text{low}} = V \cup_{\kappa'_B} D^{r-1}$$

when B runs over all **nbc**-sets and **nbc**-bases. Observe that $\mathcal{S} = \mathcal{S}_{\text{up}} \cup \mathcal{S}_{\text{low}}$. The intersection $\mathcal{S}_{\text{up}} \cap \mathcal{S}_{\text{low}}$ is the homotopy sphere constructed for $M/1$.

For any **nbc**-set or **nbc**-basis B, let τ_B and τ'_B be the cell whose associated attaching map is κ_B and κ'_B.

Now for each **nbc**-set or **nbc**-basis B, let $\phi_B : D^{r-1} \to \mathcal{S}$ be defined such that $\partial\phi_B = \kappa_B$ and ϕ_B is a homeomorphism to the open cell τ_B on the interior of D^{r-1}. Symmetrically we define $\phi'_B : D^{r-1} \to \mathcal{S}$. For any non-**nbc** independent set $B = \{u_1, \ldots, u_q\}$, $q \leq r-1$, we let $\phi_B : D^{r-1} \to \mathcal{S}$ be defined such that

$$\partial\phi_B = \phi^1_{\{u_1,u_2,\ldots,u_q\}} \cup \phi^{u_1}_{\{1,u_2,\ldots,u_q\}} \cup \cdots \cup \phi^{u_q}_{\{1,u_1,\ldots,u_{q-1}\}}. \tag{3.7}$$

As before, when B is not an **nbc**-set or **nbc**-basis ϕ_B is defined in terms of ϕ_C's for lexicographically earlier C's. If we can show $H_i(\mathcal{S}_{\text{up}}) = \pi_1(\mathcal{S}_{\text{up}}) = 0$ for $i = 1, 2, \ldots, r-1$, then $\mathcal{S}_{\text{up}}$ is contractible. That might be done by showing each cell τ^z_B has a unique facet. The main obstacle is to prove that Equation 3.6 and Equation 3.7 can be met on a sphere.

CHAPTER 4

ORIENTED MATROIDS AND ITS CD-INDEX

4.1 Oriented Matroids

We have a set $\{+, -, 0\}$ with the order relations

$$+ > 0, \ - > 0.$$

Then the set $\{+, -, 0\}^E$ is a poset with the usual product ordering of posets. Within this poset, there is no maximal element. We call an element $X \in \{+, -, 0\}^E$ a *sign vector*.

For any $X \in \{+, -, 0\}^E$, we let its negative $-X \in \{+, -, 0\}^E$ be the sign vector such that $-X(e) = -(X(e))$ for all $e \in E$. For a collection $\mathcal{F}$ of sign vectors, $-\mathcal{F}$ denotes $\{-X \mid X \in \mathcal{F}\}$.

For any $X, Y \in \{+, -, 0\}^n$, define their *composition* $X \circ Y \in \{+, -, 0\}^n$ by

$$X \circ Y(e) = \begin{cases} X(e) & \text{if } X(e) \neq 0, \\ Y(e) & \text{otherwise.} \end{cases}$$

The *support* of $X \in \{+, -, 0\}^n$ is $\{x \in [n] \mid X(x) \neq 0\}$.

Definition 4.1 *An oriented matroid M on a finite set E is a collection of sign vectors $\mathcal{V}$ from $\{+, -, 0\}^E$, called covectors, which satisfies the covector axioms as follows,*

1. $0 \in \mathcal{V}$
2. *(Symmetry)* $-\mathcal{V} = \mathcal{V}$

3. *(Composition) for every* $X, Y \in \mathcal{V}$ *we have* $X \circ Y \in \mathcal{V}$
4. *(Vector Elimination) for every* $X, Y \in \mathcal{V}$ *and* $e \in X^+ \cap Y^-$ *there exists* $Z \in \mathcal{V}$ *such that*

$$Z^+ \subseteq (X^+ \cup Y^+) - \{e\}, \; Z^- \subseteq (X^- \cup Y^-) - \{e\},$$

and every $f \in E$ *such that* $\{X(f), Y(f)\} \neq \{0\}$ *and* $\{X(f), Y(f)\} \neq \{+, -\}$ *is in the support of* Z.

Like ordinary matroids, oriented matroids can be defined by several cryptomorphic systems of axioms [8].

4.2 Pseudosphere Arrangements

Recall that $S \subset S^d$ is a *pseudosphere* if there is a self-homeomorphism h sending S to the equator of S^d. A *signed pseudosphere* in S^d is a triple $S = (S^0, S^+, S^-)$ where S^0 is the pseudosphere in S^d and S^+, S^- are the two hemispheres bounded by S. S^+ and S^- are called positive and negative sides of S, respectively.

Definition 4.2 *An arrangement of signed pseudospheres in* S^d *is a collection of signed pseudospheres* $\{S_e\}_{e \in E}$ *in* S^d *where* $S_e = (S_e^0, S_e^+, S_e^-)$ *such that*

1. *For every* $A \subseteq E$*,* $S_A := \cap_{e \in A} S_e^0$ *is a topological sphere.*
2. *For every* $e \in E$ *and* $A \subseteq E - \{e\}$*, either* $S_A \subseteq S_e^0$ *or* $S_A \cap S_e^0$ *is a pseudosphere in* S_A *with sides* $S_A \cap S_e^+$ *and* $S_A \cap S_e^-$*.*

Every pseudosphere arrangement has a corresponding regular CW-complex. The *face poset* of $\mathcal{A}$ is the set of all faces ordered by inclusion. Given an arrangement $\mathcal{A} = \{S_e\}_{e \in E}$ of signed pseudospheres in S^d, to each $x \in S^d$, we have an associated sign vector $X : E \to \{0, +, -\}$, where $X(e)$ determines on which sides of S_e the point x lies. Let $\mathcal{V}^*(\mathcal{A})$ be the set of all of these sign vectors, together with 0. A loop in an oriented matroid is an element $e \in E$ such that $X(e) = 0$ for all $x \in \mathcal{V}^*(\mathcal{A})$. We have the following Topological Representation Theorem [8].

Theorem 4.3 *Let E be a finite set. For $\mathcal{L} \in \{0, +, -\}^E$, the following are equivalent.*

1. *$\mathcal{L}$ is the set of signed covectors of a loop-free oriented matroid of rank r with elements E.*
2. *$\mathcal{L} = \mathcal{V}^*(\mathcal{A})$ for some arrangement of signed pseudospheres $\mathcal{A} = \{S_e\}_{e \in E}$ in S^{r-1}.*
3. *As a poset, $\mathcal{L}$ is the face poset of an arrangement of signed pseudospheres.*

This Topological Representation Theorem tells us that the set of covectors of an oriented matroid and the face poset of the corresponding arrangement of signed pseudospheres are equivalent. From now on, when we say the face poset of an oriented matroid, we actually mean the face poset of the corresponding arrangement of signed pseudospheres.

4.3 ab-indices and flag vectors

Let P be a graded poset of rank $n + 1$. For $S \subseteq [n]$, let P_S be the sub-poset of P which consists of all the elements from P whose rank is in S. Let $f_S(P)$ be the number of maximal chains in P_S. The collection $\{f_S\}_{S \subseteq [n]}$ is called the *flag f-vector*. The flag h-vector $h_S(P)$ is the collection $\{h_S\}_{S \subseteq [n]}$ where $h_S(P)$ is defined by

$$h_S(P) = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_S(P).$$

Any flag h-vector of a poset P can be displayed via a polynomial in non-commutating variables $\mathbf{a}$ and $\mathbf{b}$. For a subset $S \subseteq [n]$, let the $\mathbf{ab}$-monomial u_S be defined by $u_S = u_1 \cdots u_n$ where $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. Then the $\mathbf{ab}$-index $\Psi(P)$ of a poset P is defined by

$$\Psi(P) = \sum_{S \subseteq [n]} h_S(P) \cdot u_S.$$

Observe that if we assign $\mathbf{a}$ and $\mathbf{b}$ to have degree 1, then $\Psi(P)$ is a homogeneous polynomial of degree n.

Bayer and Billera found the most general linear relations that hold between the components of the flag h-vector of an *Eulerian poset* P, a poset where $\mu(x, y) = (-1)^{\rho(x)-\rho(y)}$ holds for each interval $[x, y]$. When P is Eulerian, Fine gave an elegant way to transform the $\mathbf{ab}$-index from non-commuting variables $\mathbf{a}, \mathbf{b}$ to non-commutative variables $\mathbf{c}, \mathbf{d}$ where $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. When $\Psi(P)$ is written in terms of $\mathbf{c}$ and $\mathbf{d}$, we call it the $\mathbf{cd}$-index of P.

Following [4], let $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ be the ring of polynomials with integer coefficients in the variables $\mathbf{a}$ and $\mathbf{b}$, and let the degree of $\mathbf{a}$ and $\mathbf{b}$ be 1. Let $\mathbb{Z}\langle \mathbf{c}, 2\mathbf{d} \rangle$ denote the subring of $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ spanned by the elements $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $2\mathbf{d} = 2\mathbf{ab} + 2\mathbf{ba}$ and let $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ denote $\mathbf{c} - 2\mathbf{d}$-polynomials.

For an $\mathbf{ab}$-monomial $v = v_1 v_2 \cdots v_n$ let $*$ be the dual function such that $v* = v_n \cdots v_2 v_1$. By linearity we extend this operation to be an involution on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. Since $\mathbf{c}^* = \mathbf{c}$ and $2\mathbf{d}^* = 2\mathbf{d}$, the involution restricts to $\mathbb{Z}\langle \mathbf{c}, 2\mathbf{d} \rangle$ by reading the $\mathbf{c} - 2\mathbf{d}$-monomial backwards. We define a linear function $\omega : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \to \mathbb{Z}\langle \mathbf{c}, 2\mathbf{d} \rangle$ as follows: replace each occurrence of $\mathbf{ab}$ in the given monomial with $2\mathbf{d}$, replace the rest with $\mathbf{c}$'s, and extend this definition by linearity to $\mathbf{ab}$-polynomials. Thus ω always maps an $\mathbf{ab}$-polynomial of degree n to a $\mathbf{c} - 2\mathbf{d}$-polynomial of degree

n. For example,

$$\omega(\mathbf{aaa} + 3\mathbf{aab} + 2\mathbf{abb}) = \mathbf{c}^3 + 3 \cdot \mathbf{c} \cdot 2\mathbf{d} + 2 \cdot 2\mathbf{d} \cdot \mathbf{c}.$$

4.4 Oriented Matroids and cd-indices

Introduced by Bayer and Klapper [2], the **cd**-index is only defined for Eulerian posets. Because the face poset of an oriented matroid is an Eulerian poset, the **cd**-index of an oriented matroid is well-defined. Every oriented matroid has an underlying matroid, which is a geometric lattice and can be described by its lattice of flats. The flats of the underlying matroid are the 0-sets of covectors. We denote the lattice of flats of an oriented matroid M by L.

Billera, Ehrenborg, and Readdy's explicit formula for the **cd**-index of any oriented matroid is restated as follows.

Theorem 4.4 *[4, Theorem 3.1] Let M be an oriented matroid, T the face poset of M, and L the lattice of flats of M. Then the **c-2d**-index of T is given by*

$$\Psi(T) = \omega(\boldsymbol{a} \cdot \Psi(L))^*.$$

Here $\Psi(L)$ is the **ab**-index of the lattice of flats of M, and $\Psi(T)$ is the **cd**-index of T. Briefly speaking, they computed the **cd**-index of the facet poset by replacing every **ab** in $\mathbf{a} \cdot \Psi(L)$ by $2\mathbf{d}$ and replacing everything else by **c**. It is also called the $\mathbf{c} - 2\mathbf{d}$-index of oriented matroids since every **d** has a factor of 2 attached to it.

4.5 Main Theorem for Ordinary Matroids

When it comes to ordinary matroids, the equation in Theorem 4.4 no longer makes sense. In fact, we have no clue what that equality means. First, the face poset of a matroid is something we need to re-define. Second, paralleling the equality in oriented matroids, under what conditions can equality still hold after we re-define the face poset? It was conjectured by Swartz that the right side of the equation in Theorem 4.4 can be an upper bound for the **cd**-index of some poset related to the ordinary matroid. The problem is how to define and give a geometric meaning of that poset.

To mimic the Folkman-Lawrence topological representation of oriented matroids, Swartz proved the Topological Representation Theorem [14, Theorem 6.1] for matroids, in which the homeomorphism condition was lessened to homotopy equivalences. The theorem shows that for any matroid M of rank $r+1$, there is always a r-arrangement of homotopy spheres $\mathcal{S}$ such that $L = L(\mathcal{S})$. In Chapter 3, we provided a specific algorithm to construct the homotopy sphere arrangement as a CW-complex in which there is an actual sphere $\mathcal{T}$ of $\mathcal{S}$ that is a regular CW-complex. We thus obtain a face poset of that $\mathcal{T}$, called T.

For an arbitrary homotopy sphere arrangement, there is no a priori reason such a $\mathcal{T}$ exists. Since T is the face poset of a regular CW-complex homeomorphic to a sphere, T must be an Eulerian poset. Hence, T has a **cd**-index. The following theorem shows an optimal upper bound for the **cd**-index of T when matroid rank $r = 3$ since rank 1 and 2 cases are trivial.

Theorem 4.5 *Let M be a rank 3 matroid. Let L be the lattice of flats of M, and suppose T is the face poset of a sub-complex of the fully partitioned homotopy sphere arrangement* *of M that is homeomorphic to S^2. Then the **c**-**2d**-index of T satisfies*

$$\Psi(T) \leq \omega(\boldsymbol{a} \cdot \Psi(L))^*.$$

For example, the Fano matroid is a minimal non-orientable matroid of rank 3. Let T be the face poset of a sub-complex of the homotopy sphere arrangement of M we constructed. Then

$$\Psi(T) = \mathbf{aaa} + \mathbf{aab} + \mathbf{aba} + \mathbf{baa} + \mathbf{abb} + \mathbf{bab} + \mathbf{bba} + \mathbf{bbb} = (\mathbf{a} + \mathbf{b})^3 = \mathbf{c}^3,$$

whereas

$$\begin{aligned} \omega(\mathbf{a} \cdot \Psi(L))^* &= \omega(\mathbf{a} \cdot (\mathbf{aa} + 6\mathbf{ba} + 6\mathbf{ab} + 8\mathbf{bb}))^* \\ &= \omega(\mathbf{aaa} + 6\mathbf{aba} + 6\mathbf{aab} + 8\mathbf{abb})^* \\ &= \mathbf{c}^3 + 28\mathbf{cd} + 12\mathbf{dc} \\ &\geq \Psi(T). \end{aligned}$$

Proof. The flag h-vectors of L are as follows,

$$h_{\emptyset}(L) = 1,\ h_{\{1\}} = f_{\{1\}}(L) - 1,\ h_{\{2\}}(L) = f_{\{2\}}(L) - 1$$
$$h_{\{1,2\}}(L) = f_{\{1,2\}}(L) - f_{\{2\}}(L) - f_{\{1\}}(L) + 1$$

Hence the **c**-**2d**-index of L is computed as follows,

$$\Psi(L) = \mathbf{aa} + (f_{\{1\}}(L) - 1)\ \mathbf{ba} + (f_{\{2\}}(L) - 1)\mathbf{ab} + (f_{\{1,2\}}(L) - f_{\{2\}}(L) - f_{\{1\}}(L) + 1)\mathbf{bb},$$

and thus

$$\begin{aligned} \omega(\mathbf{a} \cdot \Psi(L))^* &= \omega(\mathbf{aaa} + (f_{\{1\}}(L) - 1)\ \mathbf{aba} + (f_{\{2\}}(L) - 1)\mathbf{aab} \\ &+ (f_{\{1,2\}}(L) - f_{\{2\}}(L) - f_{\{1\}}(L) + 1)\mathbf{abb})^* \\ &= (\mathbf{c}^3 + (f_{\{1\}}(L) - 1) \cdot 2\mathbf{d} \cdot \mathbf{c} + (f_{\{2\}}(L) - 1) \cdot \mathbf{c} \cdot 2\mathbf{d} \\ &+ (f_{\{1,2\}}(L) - f_{\{2\}}(L) - f_{\{1\}}(L) + 1) \cdot 2\mathbf{d} \cdot \mathbf{c})^* \\ &= \mathbf{c}^3 + (f_{\{1,2\}}(L) - f_{\{2\}}(L))\mathbf{c} \cdot 2\mathbf{d} + (f_{\{2\}}(L) - 1)2\mathbf{d} \cdot \mathbf{c} \end{aligned}$$

According to Corollary 2.9, the numbers of 0,1-dimensional cells of the homotopy sphere arrangement of M are as follows,

$$\begin{aligned}
f_{\{1\}}(S) &= \sum_{r(X)=2,\ X\le Y} |\mu(X,Y)| = 2f_{\{2\}}(L),\\
f_{\{2\}}(S) &= \sum_{r(X)=1,\ X\le Y} |\mu(X,Y)| = 2f_{\{1,2\}}(L),\\
f_{\{3\}}(S) &= 2 - 2f_{\{2\}}(L) + 2f_{\{1,2\}}(L) \text{ (The Euler characteristic of } S \text{ is 2).}
\end{aligned}$$

Hence we have,

$$f_{\{1\}}(T) \le f_{\{1\}}(S) = 2f_{\{2\}}(L),$$

$$f_{\{2\}}(T) \le f_{\{2\}}(S) = 2f_{\{1,2\}}(L),$$

$$f_{\{3\}}(T) \le f_{\{3\}}(S) = 2 - 2f_{\{2\}}(L) + 2f_{\{1,2\}}(L).$$

Thus the flag h-vectors of T are as follows,

$$h_{\emptyset}(T) = 1,\ h_{\{1\}}(T) \le 2f_{\{2\}}(L) - 1,\ h_{\{2\}}(T) \le 2f_{\{1,2\}}(L) - 1,$$

$$h_{\{3\}}(T) \le 1 - 2f_{\{2\}}(L) + 2f_{\{1,2\}}(L),\ h_{\{1,2\}}(T) = h_{\{3\}}(T) \le 1 - 2f_{\{2\}}(L) + 2f_{\{1,2\}}(L),$$

$$h_{\{1,3\}}(T) = h_{\{2\}}(T) \le 2f_{\{1,2\}}(L) - 1,\ h_{\{2,3\}}(T) = h_{\{1\}}(T) \le 2f_{\{2\}}(L) - 1,\ h_{\{1,2,3\}}(T) = 1,$$

where $h_{\{1,2\}}(T) = h_{\{3\}}(T)$, $h_{\{2,3\}}(T) = h_{\{1\}}(T)$ and $h_{\{1,2,3\}}(T) = 1$ are obtained by the Dehn-Sommerville equations. Hence the **c**-**2d**-index of T is

$$\begin{aligned}
\Psi(T) &\le \mathbf{aaa} + (2f_{\{2\}}(L) - 1)\mathbf{baa} + (2f_{\{1,2\}}(L) - 1)\mathbf{aba}\\
&+ (1 - 2f_{\{2\}}(L) + 2f_{\{1,2\}}(L))\mathbf{aab} + (1 - 2f_{\{2\}}(L) + 2f_{\{1,2\}}(L))\mathbf{bba}\\
&+ (2f_{\{1,2\}}(L) - 1)\mathbf{bab} + (2f_{\{2\}}(L) - 1)\mathbf{abb} + \mathbf{bbb}\\
&= \mathbf{c}^3 + (2f_{\{1,2\}}(L) - 2f_{\{2\}}(L))\mathbf{cd} + (2f_{\{2\}}(L) - 2)\mathbf{dc}\\
&= \omega(\mathbf{a}\cdot\Psi(L))^*.
\end{aligned}$$

Therefore we have $\Psi(T) \le \omega(\mathbf{a}\cdot\Psi(L))^*$. ■

The following proposition and corollary show that what the matroid M will be once the equality in Theorem 4.5 holds.

Proposition 4.6 *If $(\mathcal{A}, \mathcal{S})$ is a homotopy sphere arrangement of a rank 3 matroid and $\mathcal{S}$ is homeomorphic to S^2, then $(\mathcal{A}, \mathcal{S})$ is a pseudosphere arrangement and M is orientable.*

Proof. Let X be a homotopy 1-sphere in S representing an atom of $L(\mathcal{A})$. Then X is a circle to which at least two disjoint trees attached. See Figure 4.1. The trees come in pairs in order to satisfy the involution condition. Suppose the extra edges t_0t_1 and $t_0't_1'$ are on the same side of X and t_0, t_0' are on X. Then the hemisphere of S^2 within S_1 is mapped to itself by involution. The Brouwer fixed point theorem gives a fixed point. Hence t_0t_1 and $t_0't_1'$ should be on different sides of X, as shown in Figure 4.1. Now any homotopy 1-sphere that intersects with Y at t_1, t_1' connects t_1 and t_1' by a path. This path must intersects Y again at some point other than t_1, t_1', making the entire intersection at least three points, which leads to a contradiction. Therefore the homotopy 1-spheres on $\mathcal{S}$ can only be pseudospheres.

Suppose there is a 1-sphere X_2 with two parts on the same side of X. Then the shaded area is mapped to itself by involution, which again gives a fixed-point by the Brouwer fixed point theorem. So no sphere can reside in a single side of any other sphere. Therefore $\mathcal{A}$ is a pseudosphere arrangement. ■

Corollary 4.7 *Suppose the equality in Theorem 4.5 holds for an arrangement of homotopy spheres of a rank 3 matroid M. Then M must be orientable.*

Proof. Since the flag f-vectors are equal, the f-vectors are equal. Hence $\mathcal{S}$ is a sphere and Proposition 4.6 applies. ■

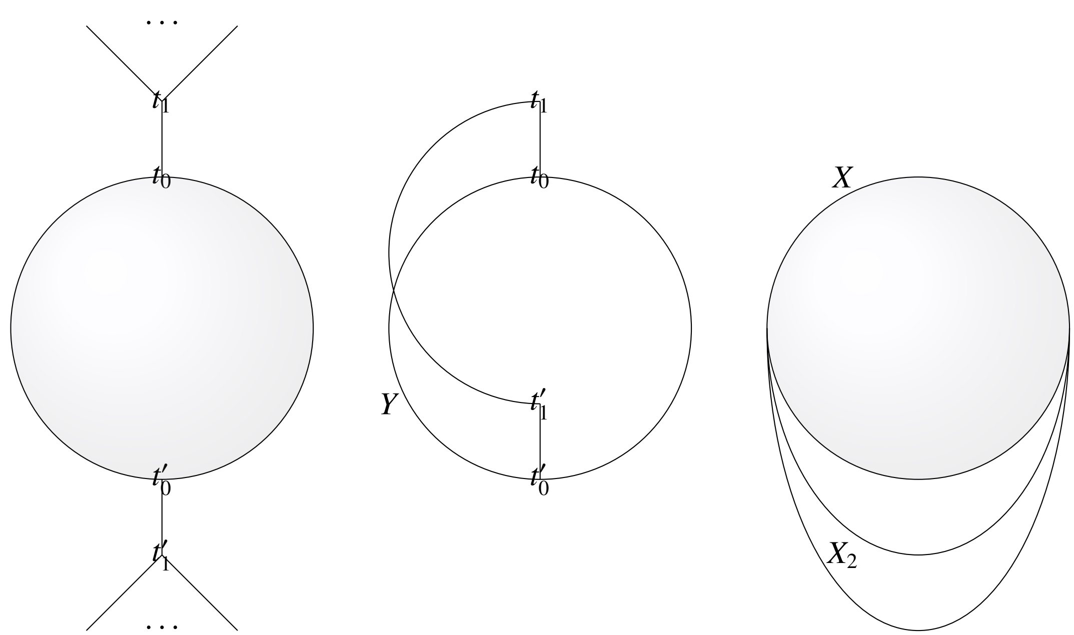


Figure 4.1:

Problem 4.8 *Does Corollary 4.7 hold for matroids of higher ranks?*

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