NONLINEAR WAVE EQUATIONS FOR RELATIVITY

Maurice H.P.M. van Putten
CRSR & Cornell Theory Center
Cornell University, Ithaca, NY 14853-6801, and
Douglas M. Eardley,
I.T.P., UCSB, Santa Barbara, CA 93106

Abstract

Gravitational radiation is described by canonical Yang-Mills wave equations on the curved space-time manifold, together with evolution equations for the metric in the tangent bundle. The initial data problem is described in Yang-Mills scalar and vector potentials, resulting in Lie-constraints in addition to the familiar Gauss-Codacci relations.

1 Introduction and outline of the approach

The asymptotic gravitational wave structure resulting from the coalescence of astrophysical black holes or neutron stars is receiving wide attention in connection with the gravitational wave detectors currently under construction. A compact binary system produces strongly nonlinear gravitational waves of which a small residue escapes to infinity as radiation. The asymptotic wave structure at a distant observer is in quantitative relation to the system parameters of compact binaries, which makes gravitational radiation a new spectrum for astronomical observations. The prediction of gravitational wave forms is presently pursued by numerical simulation. The current approaches are based the 3+1 Hamiltonian formulation by Arnowitt, Deser and Misner [2], with the notable exception of strategies employing null-coordinates (e.g. [7]) or systems of conservation laws [5].

The significance of gravity waves in compact binaries, both in the process of coalescence and in radiation, suggests to focus on a description of relativity by nonlinear wave equations. Wave equations provide a proper framework for the physics of nonlinear wave motion, establish connection with electromagnetism (abelian and linear) and Yang-Mills theory (non-abelian and nonlinear; see, e.g., [1]) and may complement present approaches in numerical relativity by way of understanding or circumventing some of the computational difficulties. This paper establishes a first step in this direction. Generally, wave equations provide a setting appropriate for numerical implementation via cylindrical or spherical coordinate systems, including outgoing boundary conditions at grid boundaries, and offer a possible setting for numerical treatment of ingoing horizon boundary conditions.

Pirani [4] presented compelling arguments to concentrate on the Riemann tensor in describing gravitational radiation. One can proceed by proposing to establish a numerical algorithm based on the divergence of the Riemann tensor [11], \( R_{abcd} \), in which the tensor

\[
\tau_{abcd} = 16\pi (\nabla_a T_{db} - \frac{1}{2} g_{da} \nabla_b T)
\]  (1)

acts as a source term. A tetrad \( \{ (e^a)_b \} \)-formalism takes the Riemannian formulation into equations of Yang-Mills type. With Lorentz gauge on the connection 1-forms, \( \omega_{\mu
u} \),

\[
e_{\mu
u} := \nabla_c \omega_{\mu\nu} = 0
\]  (2)

(\( \nabla_a \) denotes the covariant derivative associated with the metric, \( g_{ab} \)), fully nonlinear, canonical Yang-Mills wave equations are obtained

\[
\Box \omega_{\mu\nu} - R^c_a \omega_{c\mu\nu} - [\omega^c, \nabla_a \omega_c]_{\mu\nu} = \tau_{a\mu\nu}.
\]  (3)

Here, \( \Box \) is the non-abelian generalization of the Laplace-Beltrami wave operator. The wave equations are also derived for the scalar, Ricci rotation coefficients. Complemented by the equations of structure for the evolution of the tetrad legs, a complete system.
of evolution equations is obtained which is amenable to numerical implementation.

This interwoveness of wave motion and causal structure distinguishes gravity from the other field theories. In the present description, this two-fold nature of gravity has thus been made explicit. Essentially, gravitational waves are now propagated by wave equations on the (curved) manifold, while the metric is evolved in the (flat) tangent bundle by the equations of structure. Of course, such two-fold description is only meaningful for wave motion with wave lengths above the Planck scale, below which the causal structure is not well-defined and wave motion can not be distinguished from quantum fluctuations.

Because the wave equations are second order in time, conditions on the initial data involve both \( \omega_{a\mu\nu} \) and their normal Lie derivatives \( L_\alpha \omega_{a\mu\nu} \). In addition to the Gauss-Codacci relations, \( \omega_{a\mu\nu} \) and \( L_\alpha \omega_{a\mu\nu} \) are related to the initial energy-momentum and \( \tau_{\mu\nu} \). The latter follow from Gauss-Riemann relations, as non-abelian generalizations of Gauss’s law.

# 2 Equations for \( R_{abcd} \)

We will work on a four-dimensional manifold, \( M \), with hyperbolic metric \( g_{ab} \). In a given coordinate system \( \{ x^a \} \) the line-element is given by

\[
ds^2 = g_{ab} dx^a dx^b. \tag{4}\]

The natural volume element on \( M \) is \( \epsilon_{abcd} = \sqrt{-g} \Delta_{abcd} \), where \( g \) denotes the determinant of the metric in the given coordinate system, and \( \Delta_{abcd} \) denotes the completely antisymmetric symbol. Covariant differentiation associated with \( g_{ab} \) will be referred to by \( \nabla_a \). The geometry of \( M \) is contained in the Riemann tensor, \( R_{abcd} \), which satisfies the Bianchi identity

\[
3 \nabla[e R_{abcd}] = \nabla[e R_{abcd}] + \nabla[a R_{bced}] + \nabla[b R_{eacd}] = 0. \tag{5}\]

Using the volume element \( \epsilon_{abcd} \), the dual \( \ast R \) is defined as \( \frac{1}{2} \epsilon_{abcd} R_{efcd} \). The Bianchi identity then takes the form

\[
\nabla^a \ast R_{abcd} = 0. \tag{6}\]

Einstein proposed that the Ricci tensor \( R_{ab} = R_{a[b} \) and scalar curvature \( R = R^c_c \) are connected to energy-momentum, \( T_{ab} \), through

\[
R_{ab} = \frac{1}{2} g_{ab} R = 8 \pi T_{ab}. \tag{7}\]

Geometry now becomes dynamical with (5) (see, e.g., [13])

\[
\nabla^d R_{abcd} = 2 \nabla [b R_c^c]. \tag{8}\]

In the presence of Einstein’s equations (7), therefore, (8) becomes

\[
E : \nabla^a R_{abcd} = 16 \pi (\nabla_e T_{d0} - \frac{1}{2} g_{[d} \nabla_e T). \tag{9}\]

The quantity on the right-hand side shall be referred to as \( \tau_{bcd} \). In vacuo, \( E \) has been discussed by Kleinerman [8], who refers to \( E \) (with \( \tau_{bcd} = 0 \)) together with (5) as the spin-2 equations. The tensor \( \tau_{bcd} \) is divergence free:

\[
\nabla^b \tau_{bcd} = 0, \tag{10}\]

in consequence of the conservation laws \( \nabla^a T_{ab} = 0 \), and in agreement with \( \nabla^b \nabla^a R_{abcd} = 0 \).

## 2.1 Yang-Mills equations

In the language of tetrads, additional invariance arises due to the liberty of choosing the tetrad position at each space-time point. This invariance is described by the Lorentz group, and introduces a vector gauge invariance in the form of the connection 1-forms, governed by Yang-Mills equations as outlined below. The formal arguments can be found in the theory of supersymmetry (see, e.g., [1, 6, 3]).

Let \( \{(e_\mu)^b \} \) denote a tetrad satisfying

\[
\begin{align*}
(e_\mu)^b (e_\nu)^c &= \eta_{\mu\nu}, \\
\eta^{\mu\nu} (e_\mu)^a (e_\nu)^b &= \delta^b_a, \\
\eta_{\mu
u} &= \delta_{\mu\nu} \text{sign}(\eta_{\mu\nu}).
\end{align*} \tag{11}\]

Neighboring tetrads introduce their connection 1-forms, \( \omega_{a\mu\nu} \),

\[
\omega_{a\mu\nu} := (e_\mu)^c \nabla_a (e_\nu)^c. \tag{12}\]

The connection 1-forms \( \omega_{a\mu\nu} \) serve as Yang-Mills connections in the gauge covariant derivative

\[
\nabla_a = \nabla_a + [\omega_a, \cdot], \tag{13}\]

satisfying \( \nabla_a (e_\mu)^b = 0 \). Here, the commutator is defined by its action on tensors \( \phi_{a_1...a_k a_1'...a_1} \) as

\[
[\omega_{a_1} \cdot \phi_{a_1...a_k a_1'...a_1}]_{a_2'...a_j'...a_1} = \sum_i \omega_{a_i a_1} \phi_{a_1...a_k a_1'...a_1}. \tag{14}\]
In particular, we have
\begin{equation}
[\omega_a, \omega_b]_{\mu\nu} = \omega_a^\nu \omega_{b\mu} - \omega_a^\mu \omega_{b\nu}.
\end{equation}

In what follows, Greek indices stand for contractions with tetrad elements: if \( v^b \) is a vector field, then \( v_\mu = v^b (e_\mu)_b \), and \( v^\mu = \eta^{\mu\nu} v_\nu \).

The Yang-Mills construction thus obtains for the Bianchi identity
\begin{equation}
\hat{\nabla}^a \ast R_{ab\mu\nu} = 0,
\end{equation}
and for the equivalent of \( E \)
\begin{equation}
E' : \hat{\nabla}^a R_{ab\mu\nu} = \tau_{\mu\nu}.
\end{equation}

The Bianchi identity (16) introduces the representation (c.f. [13])
\begin{equation}
R_{ab\mu\nu} = \nabla_a \omega_{b\mu\nu} - \nabla_b \omega_{a\mu\nu} + [\omega_a, \omega_b]_{\mu\nu},
\end{equation}
which will be central in our discussion.

The antisymmetries in the Riemann tensor introduce conditions on initial data on an initial hypersurface, \( \Sigma \). If \( \nu^b \) denotes the normal to \( \Sigma \), and
\begin{equation}
\nabla_a = - \nu_a (\nu^c \nabla_c) + D_a \text{ on } \Sigma,
\end{equation}
we obtain the \textit{Gauss-Riemann} relations
\begin{equation}
\begin{cases}
\nu^b D^a R_{abcd} = - \rho_{cd}, \\
\nu^b D^a \ast R_{abcd} = 0,
\end{cases}
\end{equation}
where \( \rho_{cd} = \nu^b \tau_{b\mu\nu} \). These may be considered generalizations of Gauss’s law in electromagnetism. Conditions (20) find their equivalents in the tetrad formulation. To this end, we write, analogous to (19), on \( \Sigma \) the derivative as
\begin{equation}
\nabla_a = - \nu_a (\nu^c \nabla_c) + D_a.
\end{equation}
The Gauss-Riemann relations (20) thus become
\begin{equation}
\begin{cases}
\nu^b D^a R_{ab\mu\nu} = \rho_{\mu\nu}, \\
\nu^b D^a \ast R_{ab\mu\nu} = 0.
\end{cases}
\end{equation}

### 2.2 Evolution equations for the tetrads

The definition of the connection 1-forms gives the equations of structure [13]
\begin{equation}
\partial_{[a} (e_{\mu])_b = (e^c)_b \omega_{a[c} e_{\mu]}.
\end{equation}

In (23), \( \partial_{[a} (e_{\mu})_b \) is left undefined. Defining \( \xi^a = (\partial_i)^a \), the four time-components
\begin{equation}
N_\mu := (e_\mu)_a \xi^a
\end{equation}
become freely specifiable functions. The evolution equations for the tetrad legs thus become
\begin{equation}
\partial_t (e_\mu)_b + \omega_{i\mu}^\nu (e_\nu)_b = \partial_b N_\mu + \omega_{by}^\nu N_\nu.
\end{equation}
The tetrad lapse functions \( N_\mu \) are related to the familiar lapse and shift functions in the Hamiltonian formalism through
\begin{equation}
g_{at} = N_a (e^a)_t = (N_t N^a - N^2, N_t).
\end{equation}

We will now turn to evolution equations for the connection 1-forms.

### 3 Equations for \( \omega_{a\mu\nu} \)

We define a \textit{Lorentzian} cross-section of the tangent bundle of the space-time manifold by [11]
\begin{equation}
c_{\mu\nu} := \nabla^d \omega_{d\mu\nu} = 0.
\end{equation}
The \textit{Lorentz gauge} (27) provides\(^1\) a complete, six-fold connection between neighboring tetrads. The six \textit{constraints} \( c_{\mu\nu} = 0 \) are incorporated in \( E' \) by application of the divergence technique [10, 12]:
\begin{equation}
E'' : \hat{\nabla}^a \{ R_{ab\mu\nu} + g_{ab} e_{\nu\mu} \} = \tau_{\mu\nu}.
\end{equation}

Recall the transformation rule for the connection (see, e.g. [3])
\begin{equation}
\omega_{a\rho\sigma} = \Lambda_\rho^\alpha \Lambda_\sigma^{b\beta} \omega_{a\alpha\beta} + \Lambda_\rho^\alpha \nabla_a \Lambda_{\beta a}.
\end{equation}

\(^1\)Conceptually, \( c_{\mu\nu} = f(\omega_{a\mu\nu}, g_{ab}) \) will also serve its purpose, where \( f(\cdot, \cdot) \) depends analytically on its arguments.
3.1 Existence of Lorentzian cross-section

To proceed, we consider in an open neighborhood \( \mathcal{N}(\Sigma) \) of \( \Sigma \) with Gaussian normal coordinates \( \{ \tau, x^a \} \) (\( \nu^c \nabla_c \tau = 1 \)) the infinitesimal Lorentz transformation

\[
\Lambda_{\mu}^\nu = \delta_{\mu}^\nu + \frac{1}{2} \tau^2 \sigma_{\mu}^\nu \quad \text{in} \quad \mathcal{N}(\Sigma). \tag{30}
\]

It is convenient to employ language of scalar and vector potentials, \( \Phi_{\mu\nu} \) and \( \Lambda_{a\mu\nu} \), respectively, defined in

\[
\omega_{a\mu\nu} = \nu_a \Phi_{\mu\nu} + \Lambda_{a\mu\nu}. \tag{31}
\]

Denoting the effect of (30) via (29) by a superscript \((r)\), we have

\[
\omega_{a\mu\nu}^{(r)} = \omega_{a\mu\nu} + \tau \delta_a^c \sigma_{\mu\nu} \quad \text{in} \quad \mathcal{N}(\Sigma), \tag{32}
\]

so that

\[
\Phi_{\mu\nu}^{(r)} = \Phi_{\mu\nu} - \tau \sigma_{\mu\nu} \quad \text{in} \quad \mathcal{N}(\Sigma). \tag{33}
\]

Using a geodesic extension of the normal \( \nu^b \) off \( \Sigma \), constraints (27) become

\[
c_{\mu\nu} = \nabla_a \omega^a_{\mu\nu} = \hat{\Phi}_{\mu\nu} + D_c \omega^c_{\mu\nu} \quad \text{on} \quad \Sigma. \tag{34}
\]

In \( \Lambda_{a\mu\nu} \), this obtains

\[
c_{\mu\nu} = \Phi_{\mu\nu} + D_c (\nu^a \Phi_{\mu\nu} + \Lambda^c_{\mu\nu}) = \Phi_{\mu\nu} + K \Phi_{\mu\nu} + D_c \Lambda^c_{\mu\nu}. \tag{35}
\]

on \( \Sigma \). Consequently, we have the transformations

\[
\begin{align*}
\Phi_{\mu\nu}^{(r)} &= \Phi_{\mu\nu} \\
\Phi_{a\mu\nu}^{(r)} &= \Phi_{a\mu\nu} - \sigma_{a\mu\nu} \\
\Lambda_{a\mu\nu}^{(r)} &= \Lambda_{a\mu\nu}
\end{align*}
\quad \text{on} \quad \Sigma. \tag{36}
\]

By choice of \( \sigma_{a\mu\nu} = c_{a\mu\nu} \), the constraints (27) transform into

\[
c_{a\mu\nu}^{(r)} = c_{a\mu\nu} - \sigma_{a\mu\nu} = 0. \tag{37}
\]

Notice that bringing the tetrad in Lorentz gauge (37) by (30) is achieved by proper second time-derivative \( \partial^2_{\tau} (E_\mu)_{\tau=0} \) of its legs, leaving the tetrad position and its first time-derivative as invariants at \( \tau = 0 \).

We shall now establish that \( E''_\alpha \) maintains the Lorentz gauge (27) in the future domain of dependence of \( \Sigma \). The inhomogeneous Gauss-Riemann relation (20) is implied by antisymmetry of the Riemann tensor in its coordinate indices, and gives

\[
0 = \nu^a (\nabla^a (R_{\alpha b\mu\nu} + g_{ab} c_{\mu\nu}) - \tau_{b\mu\nu}) = \nu^a (D^a R_{\alpha b\mu\nu} - \tau_{b\mu\nu}) + (\nu^a \nabla_b) c_{\mu\nu} = (\nu^a \nabla_b) c_{\mu\nu}. \tag{38}
\]

The inhomogeneous Gauss-Riemann relations are gauge covariant, so that we are at liberty to consider initial data satisfying both (38) and the gauge choice (37), whence

\[
c_{\mu\nu} = (\nu^c \nabla_c) c_{\mu\nu} = 0 \quad \text{on} \quad \Sigma. \tag{39}
\]

In (28) \( c_{\mu\nu} \) satisfies a homogeneous Yang-Mills wave equation (cf. [10, 12]),

\[
\Box c_{\mu\nu} := \nabla^a \nabla_a c_{\mu\nu} = 0. \tag{40}
\]

It follows that the scalar fields \( c_{\mu\nu} \) are solutions to an initial value problem for homogeneous wave equations (40) with trivial Cauchy data (39). Consequently,

\[
c_{\mu\nu} = 0 \quad \text{in} \quad D^+ (\Sigma), \tag{41}
\]

which establishes that solutions to \( E''_\alpha \) are solutions to \( E'_\alpha \).

We now elaborate further on (28).

3.2 Canonical wave equations

In Lorentz gauge (27), the divergence equation \( E''_\alpha \) obtains wave equations for the connection 1-forms through the representation of the Riemann tensor (18). Indeed, by explicit calculation, we have

\[
\Box \omega_{a\mu\nu} - R_a^{\gamma\beta} \omega_{\gamma\beta\mu\nu} - \omega_{a\mu\nu} = \tau_{a\mu\nu}. \tag{42}
\]

Here, we have used \( c_{a\mu\nu} = 0 \), so that \( \nabla_a c_{\mu\nu} = \nabla_a c_{\mu\nu} = 0 \) is used to denote the Yang-Mills wave operator \( \nabla^a \nabla_a \). The Ricci tensor \( R_{ab} \) in (42) is understood in terms of \( T_{\alpha\beta} \) using Einstein’s equations.

Similarly, we can obtain a system of scalar equations for the Ricci rotation coefficients, \( \omega_{a\mu\nu} = (\omega_{a\mu\nu})^2 \omega_{a\mu\nu} \). This involves a number of manipulations, the result of which is

\[
\Box \omega_{a\mu\nu} - R_a^{\gamma\beta} \omega_{\gamma\beta\mu\nu} - \omega_{a\mu\nu} = \tau_{a\mu\nu}. \tag{43}
\]

The two-fold nature of gravity can now be expressed as

**Separation Theorem.** 1 Gravitational wave motion is governed by canonical wave equations on \( M \). In response to the wave motion, the metric on \( M \) evolves in the tangent bundle of \( M \) by the equations of structure.
The coupling of matter to the connections is accounted for by \( \tau_{abcd} \). It is of interest to note that \( \tau_{abcd} \) contains vorticity; a detailed enumeration of the nature of \( \tau_{abcd} \) falls outside the scope of this discussion.

The Theorem suggests some computational approximations for weakly nonlinear gravity waves. Firstly, consider equations with uncoupled (prescribed or fixed) metric, with given Laplace-Beltrami wave operator, \( \Box_g \). Thus, we have (in vacuo)

\[
\Box_g \omega_{\mu\nu} - [\omega^c, \nabla_c \omega_{\mu\nu}] = 0,
\]

with the metric \( g_{ab} \) as a background field providing the underlying causal structure. The approximate radiation then follows from the fluctuations in the metric as would follow from the equations of structure. Secondly, small amplitude waves allow us to linearize, thereby obtaining the purely abelian wave equations

\[
\Box_\eta \omega_{\mu\nu} = 0,
\]

where \( \eta \) refers to the Lorentz metric. Numerical experiments must show the validity of such approximations.

4 Initial value problem

Initial data for the wave equations are \( \omega_{\mu\nu} \) and its Lie derivative \( \mathcal{L}_n \omega_{\mu\nu} \) on \( \Sigma \). These data must satisfy certain constraints on \( \Sigma \), commensurate with the initial distribution of energy-momentum and the Gauss-Riemann equations (22). We shall express these equations in terms of scalar and vector potentials (31).

The projection tensor, \( h_{ab} \), onto \( \Sigma \) is

\[
h_{ab} = g_{ab} + \nu_a \nu_b.
\]

The covariant derivative induced by \( h_{ab} \) shall be denoted by \( \tilde{\nabla}_a \), i.e., \( \tilde{\nabla}_a h_{cd} = 0 \). In what follows, \((\nu^c \nabla_c) f \) is also denoted by \( f \). The unit normal is extended geodesically off \( \Sigma \), so that \( \nu^c \nabla_c \nu^b = 0 \) and \( (\nu^c \nabla_c) h_{ab} = h_{ab}(\nu^c \nabla_c) \). Thus, \( \Phi = (\nu^c \nabla_c) \Phi \) and \( A_{a\mu} = (\nu^c \nabla_c) A_{a\mu} \). We further define

\[
\left\{ \begin{array}{l}
\Phi' := \Phi_{\nu} + K \Phi_{\nu}, \\
A_{a\mu}' := \mathcal{L}_n A_{a\mu} = A_{a\mu} + K \cdot A_{a\mu}.
\end{array} \right.
\]

4.1 Expressions for \( A_{a\mu} \)

Consider a normal tetrad, \( \{(E_\mu)^b\} \), in which one of the legs is (initially) everywhere normal to the initial hypersurface:

\[
(E_{\mu=0})^b = \nu^b \text{ on } \Sigma.
\]

Extrinsic curvature, \( K_{ab} \), of \( \Sigma \) can be defined ‘static’ by projection of the unit normal following parallel transport over \( \Sigma \),

\[
K_{ab} = \tilde{\nabla}_a \nu_b,
\]

and ‘dynamic’ by the Lie derivative, \( \frac{1}{2} \mathcal{L}_n h_{ab} \), of the projection operator \( h_{ab} \) with respect to a Gaussian normal coordinate, \( n \). With a normal tetrad, its (extrinsic) helicity, i.e., the twist in a strip swept out by the integral curves of a leg \((E_\mu)^b \) passing through an integral curve of leg \((E_\mu)^b \) is

\[
H_{\mu\nu} := \nu_b (E_\mu)^c (\nu^c \nabla_c (E_\nu)^b) \text{ on } \Sigma.
\]

For \( \mu, \nu \neq n \), \( H_{\mu\nu} \) describes twist in \( \Sigma \) when \( \mu \neq \nu \), and bending when \( \mu = \nu \). It follows that

\[
H_{\mu\nu} = \nu_b (E_\mu)^c [\omega_{\nu} (E_\nu)^b]
\]

\[
= \begin{cases} 
-\Phi_{\nu} & \text{if } \mu = n, \\
A_{a\mu} & \text{if } \mu \neq n.
\end{cases}
\]

By (51), the symmetry of \( K_{ab} \) and \( A_{a\mu} = -A_{\mu a} \), we have for \( \mu, \nu \neq n \) the symmetry

\[
A_{a\mu} = 0.
\]

If \( \mu \neq n \), \( (E_\mu)_a = h_{ab} (E_\nu)^b \), and so

\[
A_{a\mu} = h_{ab} (E_\mu)_c \nabla_b (E_\nu)^c
\]

\[
= (E_\mu)_c [\nabla_a h_{cb} (E_\nu)^b] (E_\nu)^d
\]

\[
= (E_\mu)_c \tilde{D}_a (E_\nu)^c.
\]

If also \( \nu \neq n \), \( \tilde{\nabla}_a \) acts in (54) on tangent legs \((E_\mu)^b \), the result of which is determined by \( h_{ab} \). We shall write \( A_{a\mu} = A_{a\mu} h_{ab} h_{bc} \) for this intrinsic part of \( A_{a\mu} \).

In the normal tetrad, therefore, \( A_{a\mu} \) falls into two groups: (i) the extrinsic part \( A_{\mu=0} = K_{\mu\nu} \), and the (ii) intrinsic part \( A_{a\mu} \), with which we have

\[
A_{a\mu} = A_{a\mu} + 2K_{a\mu} \delta_{\nu}^n \text{ on } \Sigma.
\]

4.2 Expressions for \( R_{ab\mu\nu} \)

On \( \Sigma \), \( R_{ab\mu\nu} \) contains intrinsic, three-dimensional, and extrinsic parts by substitution of (31):

\[
R_{ab\mu\nu} = \nabla_a \omega_{b\mu} - \nabla_b \omega_{a\mu} + [\omega_a, \omega_b]_{\mu\nu}
\]

\[
= 2\eta_{b} [\mathcal{L}_n \Phi_{\mu} + 2\eta_{a} A_{a\mu}]
\]

\[
+ 2D_a A_{b\mu} + [A_a, A_b]_{\mu\nu}.
\]
Using (55), the latter may be reduced to
\[
R_{ab\mu
u} = 2\eta_0 \{ D_\mu \Phi_{b\nu} + \dot{A}_b \mu \} \\
+ 2D_\alpha A_{b\mu} + [\dot{A}_a, \dot{A}_b]_{\mu \nu} \\
+ 4D_\alpha [K_{b\mu}\delta^n_\nu] + 2K_{a\nu}[K_{b\mu}]
\]
(57)
where \( \dot{D}_a K_{\mu \nu} = \dot{D}_a K_{\mu \nu} + A_{a\mu} K_{\nu b} \). Notice that the intrinsic, three-curvature \( (3) R_{ab\mu \nu} \) of \( \Sigma \) associated with \( h_{ab} \) satisfies
\[
(3) R_{ab\mu \nu} = \dd\alpha \dot{A}_{b \mu \nu} - \dot{D}_b \dot{A}_{a \mu \nu} + [\dot{A}_a, \dot{A}_b]_{\mu \nu}.
\]
(58)
It follows that
\[
h_a^e h_b^f R_{\mu \nu} = 2\dd\alpha \dot{A}_{b \mu \nu} + [\dot{A}_a, \dot{A}_b]_{\mu \nu} \\
+ 2K_{a \mu}[K_{b \nu}] + 4\dd\alpha [K_{b \mu}\delta^n_\nu] \\
+ 2K_{a \nu}[K_{b \mu}]
\]
(59)
This last equation generalizes a similar expression obtained in [9], where \( \mu, \nu \neq n \) is considered.

### 4.3 Energy-momentum relations

Constraints related to the initial distribution of energy-momentum are obtained by consideration of the Ricci tensor. To this end, consider the two expressions (56) and (59) for \( R_{ab\mu \nu} \) of the previous section in
\[
h_a^e \nu^b R_{e \mu \nu} = -\dd\alpha \Phi_{b \mu \nu} - A_{a \mu \nu},
\]
\[
h_a^e h_b^f R_{e \mu \nu} = 2\dd\alpha K_{b \mu \nu}.
\]
(60)
Consequently, the elements of the Ricci tensor become
\[
\nu^b R_{e \nu} = \nu^b h_a^e R_{e \mu \nu} \\
= -\dd\alpha \Phi_{b \mu \nu} - A_{a \mu \nu},
\]
\[
h_b^e R_{e \nu} = 2\dd\alpha [K_{b \mu \nu} K_{a \mu \nu}] \\
= \dd\alpha K_{a \nu} - \dd\alpha K_{b \nu}.
\]
(61)
The second expression (59) for the Riemann tensor also gives
\[
h_a^e h_b^d h_a^e h_b^d R_{e \alpha \beta} = (3) R_{ab \mu \nu} + K_{a \mu} K_{b \nu} \\
- K_{a \nu} K_{b \mu},
\]
resulting in the familiar identity
\[
16\pi T_{mn} = 2(R_{ab} - \frac{g_{ab}}{2}R)\nu^a \nu^b \\
= h_a^e h_b^d R_{e \mu \nu} \\
= h_a^e h_b^d R_{e \alpha \beta} \\
= (3) R + K^2 - K_{a \mu} K^{a \mu} \\
= (3) R + K^2 - K_{a \mu} K_{a \mu},
\]
(63)
By Einstein's equations, we have
\[
R_{ab} = 8\pi(T_{ab} - \frac{1}{2}g_{ab}T) = 8\pi \dd T_{ab}.
\]
(64)
Combining the results above, we have the constraints on the Lie derivative of \( A_{a \mu} \) in
\[
\dd D^a \Phi_{b \mu} + A^a_{b \mu} = -8\pi \dd T_{\nu a} \nu^b,
\]
(65)
together with the familiar Gauss-Codacci relations
\[
(3) R + K^2 - K_{a \mu} K^{a \mu} = 16\pi T_{mn},
\]
\[
\dd D^a K_{a \mu} - \dd D_b K_{b \mu} = 8\pi h_a^b T_{mn}.
\]
(66)
We shall refer to (65)-(66) as the energy-momentum constraints.

In going from a first order Hamiltonian description to a second order description, it is not surprising to encounter also constraints on the Lie derivative, reflecting the conditions that the Gauss-Codacci relations also must be satisfied on a future hypersurface in \( N(\Sigma) \).

### 4.4 Gauss-Riemann relations

The Gauss-Riemann relations (22) can similarly be obtained in terms of \( \Phi_{b \mu} \) and \( A_{a \mu} \). The antisymmetry of the Riemann tensor in its coordinate indices implies
\[
\nu^b D^a R_{ab \mu \nu} = D^a(\nu^b R_{ab \mu \nu}) - R_{ab \mu \nu} K_{a \mu} \\
= D^a(\nu^b R_{ab \mu \nu}).
\]
(67)
By (56), it follows that
\[
\nu^b R_{ab \mu \nu} = -D_a \Phi_{b \mu \nu} - A_{a \mu} K_{b \nu} \\
- [\omega_a, \Phi]_{b \mu \nu} \\
= -D_a \Phi_{b \mu \nu} - [A_a, \Phi]_{b \mu \nu} \\
- A_{a \mu} K_{b \nu} \\
- A_{a \mu} K_{b \nu}.
\]
(68)
Therefore, the first, inhomogeneous Gauss-Riemann constraint in (22) reads
\[
\nu^b \dd D^a R_{ab \mu \nu} = \nu^b D^a R_{ab \mu \nu} + [A_a, \nu^b R_{ab \mu \nu}] \\
= D^a(\nu^b R_{ab \mu \nu}) + [A_a, \nu^b R_{ab \mu \nu}] \\
= D^a(\nu^b R_{ab \mu \nu}) \\
= -\dd D_a \Phi_{b \mu \nu} - \dd D_b K_{a \mu} \\
- \rho_{a \mu}.
\]
(69)
where the symmetry of the extrinsic curvature tensor was used in going from the third to the fourth equation.
These six constraints on $\mathcal{L}_n A_{a\mu\nu}$ arise in view of the functional relationship
\[ \tilde{A}_{a\mu\nu} = A_{a\mu\nu}(h_{pq}, \partial_{r} h_{pq}). \] (70)
Because $\frac{1}{2} \mathcal{L}_n h_{ab} = K_{ab}$ and $K_{a\mu} = A_{a\mu\nu}$, six degrees of freedom in
\[ \mathcal{L}_n A_{a\mu\nu} = \mathcal{L}_n \tilde{A}_{a\mu\nu} + 2 A_{an}[\delta^n_{\mu}] \] (71)
are constraint by (70).
To summarize, the constraints on the initial data in terms of potentials are

**Gauss-Codacci:**
\[ \begin{aligned}
(3) & R + K^2 - K_{ab} K^{ab} = 16\pi T_{nn} \\
\bar{D}^a K_{ab} - \bar{D}_b K &= 8\pi h_a^b T_{an} 
\end{aligned} \] (72)

**Lie-constraints:**
\[ \begin{aligned}
D^a \Phi_{\mu\nu} + A^a_{\mu\nu} &= -8\pi T_{\mu\nu} \\
\bar{D}_a \Phi_{\mu\nu} + \bar{D}^a A_{\mu\nu} &= -\rho_{\mu\nu}
\end{aligned} \] (73)

**Lorentz gauge:**
\[ \Phi_{\mu\nu} + \bar{D}^a A_{a\mu\nu} = 0. \] (74)

### 4.5 Restricted Lorentz gauge

Restricted gauge transformations for the connection in analogy to the same problem in electromagnetics, can be considered through the Lorentz transformation $\Lambda^\nu_{\mu}$
\[ \Lambda^\nu_{\mu} = \delta^\nu_{\mu} + \tau \lambda^\nu_{\mu} + \frac{1}{2} \tau^2 \sigma^\nu_{\mu} \text{ in } \mathcal{N}(\Sigma), \] (75)
where $\tau$ is a Gaussian coordinate of $\Sigma$ such that $\nu^\alpha \nabla_\alpha \tau = 1$, and $\partial_\tau \lambda^\nu_{\mu} = 0$. By (29), we have
\[ \begin{aligned}
\Phi_{\mu\nu}^{(r)} &= \Phi_{\mu\nu} - \lambda_{\mu\nu} \\
\dot{\Phi}_{\mu\nu}^{(r)} &= \dot{\Phi}_{\mu\nu} - \sigma_{\mu\nu} \\
A_{a\mu\nu}^{(r)} &= A_{a\mu\nu}
\end{aligned} \] on $\Sigma$. (76)
The first (energy-momentum) Lie-constraint is gauge invariant (invariant under transformations (29)), while the second is not. Therefore, we can choose $\lambda^\nu_{\mu}$ and $\sigma^\nu_{\mu}$ so that
\[ \begin{aligned}
\Phi_{\mu\nu}^{(r)} &= \Phi_{\mu\nu} - \lambda_{\mu\nu} = 0 \\
(\Phi_{\mu\nu}^{(r)})'_{\mu\nu} &= (\Phi_{\mu\nu}^{(r)})'_{\mu\nu} + K \Phi_{\mu\nu}^{(r)} \\
&= \Phi_{\mu\nu} - \sigma_{\mu\nu} = \Phi_{\mu\nu}'
\end{aligned} \] on $\Sigma$. (77)
The condition on $\sigma^\nu_{\mu}$ ensures that the second Lie-constraint remains satisfied. This shows that by choosing
\[ \begin{aligned}
\lambda_{\mu\nu} &= \Phi_{\mu\nu} \\
\sigma_{\mu\nu} &= -K \Phi_{\mu\nu}
\end{aligned} \] on $\Sigma$ (78)
we obtain
\[ \Phi_{\mu\nu}^{(r)} = 0 \text{ on } \Sigma. \] (79)

With this restricted Lorentz gauge, the Lie-constraints reduce to
\[ \begin{aligned}
\dot{\Phi}_{\mu\nu}^{(r)} &= -\rho_{\mu\nu} \\
\Phi_{\mu\nu} + \bar{D}^a A_{a\mu\nu} &= 0.
\end{aligned} \] on $\Sigma$. (80)

We remark it does not appear to be feasible to ensure $\Phi = 0$ throughout $D^a(\Sigma)$ by suitable choice of restricted Lorentz gauge on the initial data.

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