Hedging Guarantees in Variable Annuities (Under Both Market and Interest Rate Risks)

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Abstract

In order to prevent possibly very large losses, insurance companies have to devise risk management strategies for the guarantees provided by variable annuities. When hedging the options embedded in these guarantees, due to their long maturities and sensitivity to the underlying accounts tail distributions, it is important to use an appropriate model for the stochastic evolution of the account values as well as the stochastic interest rates. This paper illustrates the discrete hedging of lookback options embedded in guarantees with ratchet features, under jump risk and interest rate risk. Since discrete hedging and the model considered for the underlying price dynamics lead to an incomplete financial market, we compute hedging strategies using local risk minimization.

Our numerical results show that computing the hedging strategies under a joint model for the real-world underlying price dynamics and the short interest rates, leads to effective risk reduction. We investigate the performance of hedging using underlying assets and hedging using liquid options. We illustrate that the additional effectiveness in risk reduction, achieved by hedging using options instead of the underlying, can be lost if the interest risk is not accurately modeled in the hedging computation. However, when both equity and interest rate risks are appropriately modeled, hedging with options is superior to hedging with the underlying assets. We also analyze the sensitivity of the hedging performance to the correlation between the underlying asset and the short interest rate.

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1. Introduction

Annuities are contracts designed to provide payments to the holder at specified intervals, usually after retirement. Traditionally, insurance companies offered fixed annuities which guarantee a stream of fixed payments over the life of the contract. In order to meet these liabilities, the insurance companies invested their premiums in bonds and mortgages. This type of annuities were attractive in the context of high interest rates and high cost of investment in the equity market. However, bullish markets and low interest rate environments motivate the investors to look for higher returns than those provided by the conventional annuities. Variable annuities, whose future benefits are based on the performance of a portfolio of securities including equities, have proved to be very attractive for investors, since they not only provide participation in the stock market, but they also have some protection against the downside movements in the market. The more aggressive equity-indexed annuities, first offered by Keyport Life Insurance Co. in 1995, have been the fastest growing annuity products in the US. The returns of these annuities depend on the appreciation of an equity market index, such as S&P 500, DJIA or NASDAQ, while still being protected from losses. Variable annuities in the U.S. are similar to the unit-linked annuities in the U.K. and the segregated funds in Canada.

Variable annuities are appealing to investors because they are tax-deferred and they offer guaranteed minimum death benefits (GMDB). Until the beginning of 1990's, the death benefits were just simple principal guarantees (the original investment was guaranteed in case of death) or rising floor guarantees (at least the original investment accrued at a minimally guaranteed interest rate, possibly capped at a predetermined level). In the circumstances of the bullish market of the 1990's, insurance companies have started to offer GMDB with more attractive features, such as the ratchet, which guarantees a death benefit based upon the highest anniversary account value. The anniversary step-up dates are typically annual and there may exist an initial period when no reset is allowed.

The simultaneous occurrence of death and market downturn seemed unlikely during the strong bullish market of 1990's, however, in the following market crash, insurance companies realized that they may face extremely large losses. Devising good risk management strategies has become of crucial importance. The traditional actuarial methods adopt a passive strategy of holding a sufficient reserve in risk-free instruments in order to meet the liabilities with high probability. Recent research applies methods from finance for computing the fair price of a guaranteed minimum death benefit in a variable annuity and meeting the contract liabilities. The risk management strategies in this case consist in holding positions in stocks and bonds and dynamically rebalance these positions in order to cover the guarantees. The financial engineering approach is based on the fact that the guaranteed minimum death benefit can be viewed as a put option with a stochastic maturity date. This put option has a strike equal to the initial investment for a GMDB with principal guarantee, or a strike increasing at the minimum guaranteed rate in the case of a rising floor feature. For a GMDB with ratchet features, the corresponding option is a lookback put for which the strike price is equal to a running maximum of the account value. Since the exercise of these options can only be triggered by the involuntary event of the holder's death, these options are in between European and American options and have been named "Titanic options" by Milevski and Possner ([22]).

Brennan and Schwartz ([9]), Boyle and Schwartz ([8]), Aase and Persson ([1]), Persson ([28]), Bacinello and Ortu ([2]), use option theory to price and hedge the embedded options in variable annuities. With the main assumption that the market is complete under both financial and mortality risk, the option price is equal to the expected value of the payoff with respect to a risk-neutral probability measure. Moreover, the option can be exactly replicated using delta hedging. The number of shares of the underlying held in a delta hedging strategy is given by the sensitivity (delta) of the option value to the underlying. We remark that this strategy is computed under the risk-neutral measure and the hedging portfolio has to be adjusted continuously in time.

Typically, if the number of policyholders is large enough, it can be assumed that the market is complete under mortality risk. By the Law of Large Numbers, for a large enough pool of insured lives, the total liability will be close to its expected value. An insurance company can, therefore, diversify away its mortality risk by selling enough policies. In this context, the embedded put options can be assumed to have a deterministic maturity. However, Moller ([24], [25], [26]) investigates the pricing and hedging of insurance contracts under mortality risk.

Assuming market completeness under financial risk is, however, more problematic, especially for life insurance contracts which generally have long maturities. One issue is that the benefits are sensitive to the tail distributions of the underlying account values. While empirical market data shows that the distributions of equity returns exhibit fat tails, this behavior cannot be explained by the simple Black-Scholes model for equity prices. Unfortunately, as soon as one allows for stochastic volatility, or if one adds a jump component to the model in order to capture the fat tails, the market becomes incomplete. Moreover, liquidity constraints and also the impossibility of hedging continuously in time, coupled with the need to rebalance as little as possible due to the impact of transaction costs, lead to an incomplete market. Another problem with modeling the life insurance contracts is that, because of the long maturities of these contracts, stochastic interest rates may be more appropriate than a constant rate.

The main emphasis of the literature is on pricing the options embedded in the life insurance contracts, however, hedging is also very important for risk management purposes. In this paper we investigate the computation of hedging strategies under interest rate risk and with the assumption that the market is complete under mortality risk, but the financial market is incomplete, due to a suitable equity model for fat tails or to discrete hedging. We have analyzed the modeling of volatility risk in [12].

We remark that Bacinello and Ortu ([3], [4]), Nielsen and Sandmann ([27]), Miltersen and Persson ([23]), Bacinello and Persson ([5]) also investigate stochastic interest rates, however these authors focus on pricing and they assume a complete financial market which leads to the existence of a unique equivalent martingale measure for the equity price. In an incomplete market, however, if we assume the existence of an equivalent martingale measure in order to exclude arbitrage opportunities, this equivalent martingale measure is not unique. The expected value of the discounted payoff under any such equivalent martingale measure provides a price for the option which excludes arbitrage opportunities. However, there exists no best choice when trying to consider one of these prices as the fair price of the option. Therefore, options cannot be priced by arbitrage considerations alone. Moreover, since there exists no self-financing strategy that replicates the option payoff, the intrinsic risk of an option cannot be eliminated and there is much uncertainty regarding the choice of an optimal hedging strategy.

Delta hedging is often used by practitioners for hedging the options embedded in a GMDB ([7], [17]). In theory, delta hedging assumes continuous rebalancing of the hedging portfolio. In practice, a natural problem that occurs is the impossibility of hedging continuously and the necessity to rebalance as little as possible due to transaction costs. If only discrete hedging times are allowed, achieving a risk-free position at each time is no longer possible since this instantaneous hedging will not last till the next rebalancing time. Moreover, for evaluating the hedging performance and the riskiness of a hedging strategy, one has to work with the real world model of the underlying asset. Since the delta hedging strategy is computed in a risk neutral framework, it is typically not optimal under the real world dynamics when the market is incomplete.

Given that, generally, in an incomplete market, the intrinsic risk of an option cannot be fully hedged, one idea for computing an optimal hedging strategy is to minimize a particular measure of this intrinsic risk. Föllmer and Sondermann ([16]), Föllmer and Schweizer ([15]), Schäll ([29]), Schweizer ([30], [31]), Mercurio and Vorst ([21]), Heath et al. ([18], [19]), Bertsimas et al. ([6]) study quadratic criteria for risk minimization. Alternatively, Coleman, Li and Patron ([13], [14]) investigate piecewise linear measures. There are two main criteria: local risk minimization and total risk minimization. Local risk minimization consists in choosing an optimal hedging strategy that exactly matches the option by its final value and, since such a strategy cannot be self-financing, it minimizes the intermediate cashflows for rebalancing the hedging portfolio. Alternatively, total risk minimization computes an optimal self-financing strategy that best matches the option payoff by its final value. Unfortunately, total risk minimization is a dynamic stochastic programming problem which generally leads to very expensive computations. However, local risk minimizing hedging strategies can be easily computed. Coleman, Li, Kim and Patron ([12]) illustrate numerically that, for hedging variable annuities with ratchet features, risk minimization hedging is superior to delta hedging in an incomplete market framework.

Moller ([24], [25], [26]), Lin and Tan ([20]) use risk minimization to compute the hedging strategies under mortality risk. Lin and Tan ([20]) consider, in addition, a model with stochastic interest rates. However, in the above papers the market is assumed complete from a financial point of view.

The main contribution of this paper is to illustrate the importance of modeling of stochastic interest rates in the computation of hedging strategies for the options embedded in a GMDB. Specifically, we evaluate hedging effectiveness under both equity risk and interest risk with the more realistic assumption of an incomplete financial market due to discrete hedging or to an appropriate equity model for fat tails, using quadratic local risk minimization, an optimal hedging criterion in this framework. The paper focuses on the hedging of GMDB with a ratchet feature, since this is the most difficult benefit to hedge. We investigate hedging with the underlying asset, as well as hedging with standard options. As illustrated in [12], using standard options as hedging instruments can be significantly more effective than using the underlying, especially under jump risk. We demonstrate that the superiority of hedging using standard options, compared with the hedging using the underlying, is greatly compromised if the interest risk is not appropriately modeled. Moreover, by modeling the interest rate risk explicitly, local risk minimization is able to achieve hedging effectiveness comparable to the performance obtained under no interest risk; this is true even when correlation risk is ignored in the risk minimization hedging computation.

The paper is structured as follows: in Section 2, we introduce the mathematical framework and we review some background notions related to quadratic risk minimization in the context of discretely hedging a lookback option embedded in a GMDB with ratchet feature. In Section 3 we describe the computation of hedging strategies under a Vasicek stochastic interest rate model. We illustrate numerically the hedging effectiveness using the underlying and the sensitivity of the hedging performance to the correlation between the underlying asset and the stochastic interest rates. The hedging performance of strategies using options as hedging instruments is analyzed in Section 4. Section 5 presents the conclusions of the paper.

2. Quadratic risk minimization

The traditional criteria for hedging in an incomplete market are quadratic risk minimizing criteria. This section reviews the main definitions in the context of discretely hedging the option embedded in a GMDB with ratchet feature. Since in this paper we assume that mortality risk can be diversified away, we analyze the hedging of an embedded option with fixed maturity. Moreover, we do not address the problem of basis risk in this paper and consequently, we assume, that the underlying account determining the benefit is linked to a market index, such as S&P500.

Let $T > 0$ be the maturity of the embedded option and denote by

$$0 = \tilde{t}_0 < \tilde{t}_1 < ... < \tilde{t}_{\tilde{M}-1} < \tilde{t}_{\tilde{M}} = T \tag{1}$$

the anniversary account step-up dates. A GMDB with ratchet feature guarantees a payoff of $\max(H, S_T)$, where H is the maximum anniversary account value, $H = \max(S_{\tilde{t}_0}, ..., S_{\tilde{t}_{\tilde{M}-1}})$. Since

$$\max(H, S_T) = S_T + \max(H - S_T, 0)$$

the GMDB payoff corresponds to the underlying account plus an embedded lookback option with payoff given by

$$\Pi_T = \max(H - S_T, 0) \tag{2}$$

We suppose that there are only a finite number of hedging dates

$$0 = t_0 < t_1 < ... < t_{M-1} < t_M = T \tag{3}$$

For simplicity, the anniversary step-up dates in (1), form a subset of the above trading dates. The financial market is described by a filtered probability space $(\Omega, \mathcal{F}, P)$, with filtration $(\mathcal{F}_k)_{k=0,1,...,M}$, where $\mathcal{F}_k$ corresponds to the hedging time t_k and w.l.o.g. $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is trivial. We suppose that the stock price follows a stochastic process $S = (S_k)_{k=0,1,...,M}$, with S_k being $\mathcal{F}_k$-measurable for all $0 \leq k \leq M$. We consider the bond price $B \equiv 1$ by assuming that the stock price is normalized by the bond price.

At each hedging time, the instruments available for trading are n risky assets with values $U_k \in \Re^n$ and a riskless asset, e.g., a bond. The risky assets can include the underlying, liquid options, and bonds, if more than one bond is used to hedge the interest risk. The values of the risky assets at time t_{k+1}, are given by $U_k(t_{k+1})$. In the literature on hedging in incomplete markets, the hedging instrument is usually the underlying asset. In this case, $n = 1$ and $U_n = S_n$. With the expansion of the option trading, it has become attractive to use options for hedging. We also analyze the case when the n risky instruments available for hedging are standard options. Because of liquidity considerations, we only use options with short maturity (e.g., 1 year), thus the hedging instruments may exist for only a sub-period of the hedging horizon. In these conditions, the hedging portfolio constructed at time t_k is liquidated at time t_{k+1}, when a new portfolio is formed.

A hedging strategy is a sequence of trading positions $\{(\xi_k, \eta_k)|k = 0, 1, ..., M\}$, where ξ_k is a vector of positions in the risky assets at time t_k and η_k is the amount invested in the riskless asset at t_k. We assume $\xi_M \equiv 0$. This condition corresponds to the fact that at time T we liquidate the hedging portfolio in order to cover for the lookback option payoff.

The value of the hedging portfolio at any time t_k, $0 \leq k \leq M$, is given by:

$$P_k = U_k \cdot \xi_k + \eta_k. \tag{4}$$

The change in value of the hedging portfolio due to changes in the risky asset values at time t_{j+1}, before any rebalancing, is given by $(U_j(t_{j+1}) - U_j) \cdot \xi_j$. Therefore, the *accumulated gain* at time t_k is given by:

$$G_k = \sum_{j=0}^{k-1} (U_j(t_{j+1}) - U_j) \cdot \xi_j \tag{5}$$

and $G_0 = 0$.

The *cumulative cost* at time t_k, C_k, is defined as:

$$C_k = P_k - G_k \tag{6}$$

A strategy is called *self-financing* if the cumulative cost $(C_k)_{k=0,1,...,M}$ is constant over time, i.e. $C_0 = C_1 = ... = C_M$. This is equivalent to $U_{k+1} \cdot \xi_{k+1} + \eta_{k+1} - (U_k \cdot \xi_k + \eta_k) = 0$, for all $0 \leq k \leq M - 1$. In other words, the hedging portfolio can be rebalanced with no inflow or outflow of capital. The value of the portfolio for a self-financing strategy is then given by $P_k = P_0 + G_k$ at any time $0 \leq k \leq M$.

In an incomplete market there does not exist, in general, a self-financing strategy that matches exactly the option payoff. Under these conditions, a hedging strategy has to be chosen based on some optimality criterion.

One approach to hedging in an incomplete market is to first impose $P_M = \Pi_T$. Since there exists no self-financing strategy satisfying this condition, a reasonable idea is to choose the optimal trading strategy to minimize the incremental cost incurred from adjusting the portfolio at each hedging time. This is the *local risk minimization*. The traditional criterion for local risk minimization is the quadratic criterion, given by minimizing:

$$\begin{aligned} E((C_{k+1} - C_k)^2 | \mathcal{F}_k) &= E((U_{k+1} \cdot \xi_{k+1} + \eta_{k+1} - (U_k \cdot \xi_k + \eta_k))^2 | \mathcal{F}_k) \\ &= E((P_{k+1} - U_k(t_{k+1}) \cdot \xi_k - \eta_k)^2 | \mathcal{F}_k), \end{aligned} \tag{7}$$

for all $0 \leq k \leq M - 1$, starting from the final condition $P_M = \Pi_T$. The quadratic local risk minimizing strategy is not self-financing, but it is mean self-financing, that is

$$E(C_{k+1} | \mathcal{F}_k) = C_k \tag{8}$$

(i.e., the cost process is a martingale).

Alternative to local risk minimization, which computes an optimal hedging strategy that replicates exactly the option payoff, but is no longer self-financing, one could instead choose to work only with self-financing hedging strategies. Since it is not possible to match exactly the option payoff in this case, an optimal self-financing strategy minimizes the L^2-norm:

$$E((\Pi_T - P_M)^2) = E((\Pi_T - P_0 - \sum_{j=0}^{M-1} (U_j(t_{j+1}) - U_j) \cdot \xi_j)^2). \tag{9}$$

The existence and uniqueness of the optimal strategies for the above criteria have been extensively studied by [29], [30], [21] and [6], for the case when the hedging instruments are the underlying assets. A local risk minimizing strategy exists under the assumption that the discounted underlying asset price has a bounded mean-variance tradeoff. The same assumption being sufficient for the existence of a total risk minimizing strategy. When the underlying asset price has a deterministic mean-variance tradeoff, the initial values of the local and total risk minimizing hedging portfolios are equal. In the spirit of option pricing in complete markets, Schäl ([29]) suggest to interpret this value as a "fair hedging price" or a "fair value" for the option. However, Mercurio and Vorst ([21]) show that this interpretation may not always make sense from an economic point of view. Moreover, the initial value of the hedging portfolio depends, in general, on the subjective criterion for measuring the risk. As Bertsimas et. al ([6]) remark, in a dynamically incomplete market an option cannot be priced by arbitrage considerations alone and has to be the result of a market equilibrium based on supply and demand. The focus our paper is on computing effective hedging strategy; this computation also provides cost information, e.g., initial hedging cost as well as average hedging cost.

As mentioned in the introduction, total risk minimization is a dynamic stochastic programming problem which is, in general, computationally challenging to solve. In this paper we use local risk minimization to compute the hedging strategies. Schweizer ([31]) remarks that, since local risk minimization tries to control the riskiness of a hedging strategy as measured by its incremental risk, it is, by its very nature, a hedging approach.

A local risk minimizing hedging strategy is not self-financing, however, we can define a self-financing portfolio related to this strategy. If $\{(\xi_k, \eta_k) | k = 0, 1, ..., M\}$ are the optimal holdings for the local risk minimizing strategy, then the time T value of the associated self-financing portfolio is given by:

$$P_M^{sf} = P_0 + G_M = U_0 \cdot \xi_0 + \eta_0 + \sum_{j=0}^{M-1} (U_j(t_{j+1}) - U_j) \cdot \xi_j \tag{10}$$

We investigate the hedging efectiveness of the local risk minimizing strategy in terms of the *total risk* and *total cost* computed at the maturity T of the option:

- Total risk: $\Pi_T - P_M^{sf}$, is the amount of money the hedging portfolio is short of meeting the liability.
- Total cost: $\Pi_T - G_M$, is the total amount of money needed for setting up the hedging portfolio and covering the option payoff.

3. Hedging under interest rate risk, using the underlying asset

Hedging the embedded options in a GMDB is difficult due to the long maturity of these options and to their sensitivity to the tail distributions of the underlying assets. Bacinello and Ortu ([3], [4]), Nielsen and Sandmann ([27]), Miltersen and Persson ([23]), Baccinelo and Persson ([5]), Lin and Tan ([20]) address the issue of the long maturity of the options by jointly modeling the underlying asset (which follows an extended Black-Scholes SDE) and the stochastic short interest rate (given by an extended Vasicek or CIR model) or the term structure of interest rate (HJM). However, these models do not explain the fat tails exhibited by equity returns. Moreover, the authors focus on pricing the embedded options in a continuous trading framework using risk-neutral probability measure arguments. Although Lin and Tan ([20]) discuss delta hedging in a financially complete market and risk minimization hedging under mortality risk, the paper mainly focuses on pricing instead of hedging; it does not provide any hedging results. The present paper focuses on hedging the embedded options in GMDB, in an incomplete financial market due to discrete hedging or to the underlying asset following an MJD model. We also illustrate numerically hedging with standard options. The semi-static hedging proposed in [11, 10] uses options as hedging instruments, it assumes, however, the existence of a continuum of options. In practice, the availability of only a finite number of options leads to incompleteness. Moreover, semi-static hedging, like delta hedging, requires the underlying price dynamics in a risk-neutral framework.

This section investigates the hedging under stochastic interest rates of a lookback option embedded in a GMDB with ratchet feature, using the underlying asset as hedging instrument.

We first analyze the performance of a risk minimizing hedging strategy when there is no interest rate risk, that is, the strategy is computed under the assumption that the real-world has a deterministic term structure of interest rates. We assume that the real-world price dynamics of the underlying asset follows the Black-Scholes stochastic differential equation:

$$\frac{dS_t}{S_t} = (\mu - q)dt + \sigma dW_t \tag{11}$$

where μ is the instantaneous asset return, σ is the volatility and q is the continuous dividend yield.

Even if the above Black-Scholes model is not suitable for calibrating the fat tails of equity returns, this framework is still interesting, since market incompleteness comes only from the assumption of discrete hedging. We will later analyze a MJD model for the underlying.

We emphasize that equation (11) describes the underlying price dynamics in a real-world framework, which is required for the computation and hedging performance analysis of a risk minimizing strategy in an incomplete market. As mentioned before, a delta hedging strategy, or a semi-static hedging strategy are computed using risk-neutral price dynamics.

We remark that, when computing the risk minimizing hedging strategy, we only need to use the transitional density function of the underlying asset.

The results illustrating the hedging performance of a risk minimizing strategy in the above framework, when there is no interest rate risk, are presented in Table 1, in the columns labeled “ann”, corresponding to annual rebalancing, and “month”, corresponding to monthly rebalancing.

As discussed before, a deterministic term structure of interest rates is not suitable for a long maturity option and the stochastic interest rate evolution needs to be appropriate modeled. We further assume that the real-world is governed by a joint model for the underlying asset and the short rate:

$$\begin{cases} \frac{dS_t}{S_t} & = & (\mu - q)dt + \sigma_1 \; dW_{1,t} \\ dr_t & = & a(\bar{r}_t - r_t)dt + \sigma_2 \; dW_{2,t} \end{cases} \tag{12}$$

We remark that in (12), the short rate is given by an extended Vasicek model, with $\bar{r}$ the mean reversion, a the rate of reversion and σ_2 the volatility of the interest rate.

Even if the real-world is modeled by (12), we first ignore the stochastic interest rates in the computation of the risk minimizing hedging strategy assuming that the term structure of interest rates is deterministic and the underlying asset dynamics are given by the first equation in (11). Exploring the hedging effectiveness of this risk minimizing strategy in the real-world (12) illustrates the effect of the interest rate risk on the hedging performance. The numerical results for annual and monthly hedging are presented in Table 1, in the columns labeled "ann-r" and "month-r", respectively.

The hedging strategy described in the above paragraph does not take into account the interest rate risk. In fact, it is possible to model this risk in the computation of the hedging strategy by considering the short rate r as a state variable. The hedging results for this strategy are illustrated in the columns "ann-$\check{r}$" - annual hedging, and "month-$\check{r}$" - monthly hedging, in Table 1.

We notice that when computing the risk minimizing hedging strategy under the joint model (12), we need the joint transitional density function of the underlying and the short rate. Since modeling the correlation between the change of the interest rate and the change of the account value introduce significantly more complex and costly hedging strategy computation, we assume that the Brownian motions W_1 and W_2 are independent when computing the hedging strategy "month-$\check{r}$". We will later illustrate that the hedging strategy computed by appropriately modeling the interest rate changes, but assuming no correlation, leads to effective hedging under typically observed correlations between the changes in interest rates and equity values.

Table 1 illustrates the hedging performance over 20000 simulated scenarios for the risk minimizing strategies described above. The numerical results show the initial cost of the hedging portfolio, the average hedging cost, the first two moments of the total risk, the Value-at-Risk (VaR) and the Conditional Value-at-Risk (CVaR) for a 95% confidence interval. VaR(95%) corresponds to the maximum amount of money the self-financing hedging portfolio, P_M^{sf}, is short of meeting the liability with 95% probability. CVaR(95%) gives information about the right tail of the total risk, it is the expected amount of money the self-financing portfolio is short of meeting the liability, conditional on the total risk being larger than VaR(95%). Also presented in Table 1, under the label "Unhedged", is information about the liability. This corresponds to no hedging.

We remark from Table 1 that hedging using the underlying asset can significantly reduce risk. Moreover, the effectiveness of the hedging strategies improves as we rebalance more frequently.

Comparing the hedging performance of the risk minimizing strategies under the assumption of no interest rate risk ("ann", "month") with the performance of the same strategies in the presence of interest rate risk ("ann-r" and "month-r", respectively), we notice that annual hedging with the underlying is relatively insensitive to interest rate risk, while monthly hedging is more sensitive to this risk. Due to the interest risk, the standard deviation of the total risk, VaR and CVaR are larger for the strategy "month-r" than for the strategy "month". For example, the standard deviation of monthly hedging is increased from 7.95,

Table 1: Hedging using the underlying asset in a Black-Scholes framework

	No hedge	Annually			Monthly		
	Π_T	ann	ann-r	ann-$\check{r}$	month	month-r	month-$\check{r}$
C_0	0	13.85	13.85	13.99	14.59	14.59	15.07
$\mathrm{E}(\Pi_T - G_M)$	24.41	26.62	25.59	26.91	27.98	26.86	28.94
$\mathrm{E}(\Pi_T - P_T^{\mathrm{sf}})$	24.41	0.04	-0.98	0.06	-0.007	-1.12	0.02
$\mathrm{std}(\Pi_T - P_T^{\mathrm{sf}})$	36.83	23.44	24.17	23.48	7.95	11.65	7.97
VaR(95%)	96.92	39.65	39.82	39.75	12.54	17.04	12.59
CVaR(95%)	136.60	62.76	63.15	62.82	19.54	25.35	19.66

BS: #scenarios = 20000, $\sigma_1 = 0.2$, $\mu = 0.1$, $S_0 = 100$
Vasicek: $r_0 = 0.05$, $\bar{r} = 0.08$, $a = 0.2$, $\sigma_2 = 0.02$

in the absence of interest rate, to 11.65 under the prescribed interest risk. This suggests that interest risk, if not modeled in the hedging computation, can significantly deteriorate hedging effectiveness. However, when the risk minimizing hedging strategy is computed under the joint model (12), monthly rebalancing ("month-$\check{r}$") effectively hedges the risk introduced by the stochastic interest rates. The standard deviation of the total risk, the VaR(95%) and CVaR(95%) in the column "month-$\check{r}$" are very close to the corresponding values under no interest rate risk ("month"). We remark that strategy "month-$\check{r}$" achieves this reduction in risk, for a slightly larger average hedging cost than the average hedging cost for strategy "month".

It is important to note that, in order to obtain an effective risk reduction when taking into account the stochastic interest rates in the computation of the hedging strategies, the hedging portfolio has to contain not one, but two bonds, besides the underlying asset. In our computational experience, using a single bond cannot eliminate the additional interest rate risk in the dynamic discrete hedging setting. For subsequent results of the hedging strategies computed by modeling interest risk, we have used a bond with maturity 1 year and a bond with maturity T. Intuitively, the amount invested in one bond at each hedging time is controlled by equation (8) which ensures the strategy is mean self-financing, while the second bond can be used to hedge against the fluctuations of the interest rate.

As mentioned above, in the computation of the risk minimizing strategies "ann-$\check{r}$" and "month-$\check{r}$" we supposed, for simplicity, that the underlying asset and the short rate are independent. Let us now investigate the sensitivity of the hedging performance to the correlation between the Brownian motions in (12). We compute the hedging strategies "ann-$\check{r}$" and "month-$\check{r}$" as before, assuming the independence of the underlying asset and the interest rates, however, we evaluate the hedging performance on simulated paths for different correlations. Annual hedging with the underlying in the framework given by (12) is relatively insensitive to the interest rate risk and the hedging performance is approximately unchanged for the different values of the correlation. Table 2 presents the numerical results for monthly hedging, corresponding to the strategies "month-r" and "month-$\check{r}$". We have omitted the results for strategy "month", since its hedging performance is evaluated under a deterministic term structure of interest rates and, therefore, it does not depend on the correlation value. The hedging results for strategy "month", on the simulated paths for different correlations, are very similar to the results illustrated in the column "month" of Table 1.

Table 2: Monthly hedging using the underlying for different correlations

		Correlation				
		-0.4	-0.2	0	0.2	0.4
Initial	month-r	14.59	14.59	14.59	14.59	14.59
	month-$\check{r}$	15.07	15.07	15.03	15.08	15.05
$\mathrm{E}(\Pi_T - G_M)$	month-r	25.36	26.16	27.01	27.63	28.32
	month-$\check{r}$	27.29	28.11	29.01	29.81	30.06
$\mathrm{E}(\Pi_T - P_T^{\mathrm{sf}})$	month-r	-2.62	-1.82	-0.97	-0.35	0.33
	month-$\check{r}$	-1.61	-0.79	0.17	0.88	1.72
$\mathrm{std}(\Pi_T - P_T^{\mathrm{sf}})$	month-r	11.30	11.44	11.44	11.58	12.11
	month-$\check{r}$	7.76	7.95	7.79	7.95	8.23
VaR(95%)	month-r	16.03	16.31	16.53	16.43	17.22
	month-$\check{r}$	10.39	11.41	12.60	13.10	14.41
CVaR(95%)	month-r	26.57	26.06	25.62	24.31	23.94
	month-$\check{r}$	17.42	18.58	19.41	20.32	21.62

BS: #scenarios = 20000, $\sigma_1 = 0.2$, $\mu = 0.1$, $S_0 = 100$
Vasicek: $r_0 = 0.05$, $\bar{r} = 0.08$, $a = 0.2$, $\sigma_2 = 0.02$

We remark that strategy "month-$\check{r}$" leads to an effective reduction in the interest rate risk for all the correlations in Table 2. The values of the standard deviation of the total risk, VaR(95%), CVaR(95%) for strategy "month-$\check{r}$" are close to the corresponding values under no interest rate risk, for strategy "month" (Table 1). However, the average total risk, $\mathrm{E}(\Pi_T - P_T^{\mathrm{sf}})$, which measures the amount of money the hedging portfolio is short of meeting the liability, suggests that strategy "month-$\check{r}$" tends to overhedge the option as the correlation becomes negative and to underhedge the options for positive correlations.

The results presented above have been obtained in a framework where the underlying asset follows a Black-Scholes model. We will further investigate risk minimization hedging with the underlying under a Merton's Jump Diffusion (MJD) model. This model is a better choice when trying to calibrate the fat tails exhibited by equity returns. In this framework, market incompleteness is not only due to discrete hedging, but also to the jump component of the underlying asset.

In a MJD model, the real-world underlying asset dynamics are given by:

$$\frac{dS_t}{S_t} = (\mu - q - k\lambda)dt + \sigma dW_t + (J - 1)d\pi_t \tag{13}$$

where μ is the instantaneous asset return, q is the continuous dividend yield, and σ is the volatility of the asset; λ is the jump intensity, while J models the jump amplitude, and $k = \mathrm{E}(J - 1)$. For simplicity, $\log J$ is a normal random variable with mean μ_J and volatility σ_J. We assume π_t is a Poisson process. When the underlying asset follows a MJD model, there exists an analytic formula for the transitional density function needed in the computation of the risk minimizing hedging strategy.

Table 3 presents the numerical results obtained from similar hedging experiments to the ones described in the Black-Scholes framework. The stochastic interest rates are incorporated by assuming a joint model of the real-world underlying asset dynamics and the short rate:

$$\begin{cases} \frac{dS_t}{S_t} &= (\mu - q - k\lambda)dt + \sigma_1 \, dW_{1,t} + (J-1)d\pi_t \\ dr_t &= a(\bar{r}_t - r_t)dt + \sigma_2 \, dW_{2,t} \end{cases} \tag{14}$$

Table 3: Hedging using the undelying asset in a MJD framework

	No hedge	Annually			Monthly		
	Π_T	ann	ann-r	ann-$\check{r}$	month	month-r	month-$\check{r}$
C_0	0	19.27	19.27	19.35	22.33	22.33	22.52
$\mathrm{E}(\Pi_T - G_M)$	31.52	37.01	35.75	37.17	42.72	41.19	42.87
$\mathrm{E}(\Pi_T - P_T^{\mathrm{sf}})$	31.52	0.04	-1.21	0.05	-0.10	-1.63	-0.15
$\mathrm{std}(\Pi_T - P_T^{\mathrm{sf}})$	65.36	48.83	49.60	48.78	31.38	33.58	31.50
VaR(95%)	152.31	81.16	81.28	80.61	54.06	54.73	54.53
CVaR(95%)	245.04	140.57	140.38	140.49	95.34	95.51	95.23

MJD: #scenarios = 20000, $\mu = 0.15$, $\sigma_1 = 0.2$, $\mu_J = -0.344$, $\sigma_J = 0.25$, $\lambda = 0.0916$, $S_0 = 100$
Vasicek: $r_0 = 0.05$, $\bar{r} = 0.08$, $a = 0.2$, $\sigma_2 = 0.02$

Similarly to the Black-Scholes framework, risk minimization hedging is better than no hedging and the hedging performance improves as rebalancing is more frequent. However, because of the jump risk, this improvement is not as important as in Table 1. Moreover, we notice that in the MJD framework, both annual and monthly hedging results are less sensitive to the additional interest rate risk. This is because hedging using the underlying asset is ineffective in eliminating the jump risk and thus the presence of interest risk is relatively invisible.

The computation of the hedging strategies "ann-$\check{r}$" and "month-$\check{r}$" requires the joint density function for the underlying asset and the short rate in equations (14). We have assumed, for simplicity, that the underlying asset and the short rate are independent. However, we can analyze, as in Section 3, the hedging performance on simulated paths for different values of the correlation between the Brownian motions W_1 and W_2 in (14). The numerical experiments show that, due to the relative insensitivity to the interest rate risk, the hedging performance of all the strategies remains almost unchanged for all the different correlations. This is why we are not including the numerical results.

4. Hedging using options, under interest rate risk

Due to the expansion of the option market, hedging with options has become a viable possibility. This section investigates the hedging performance of risk minimizing hedging strategies for a lookback option embedded in a guaranteed minimum benefit with ratchet feature, under interest rate risk, when the hedging instruments are options. Due to liquidity constraints, we use only short term maturity options as hedging instruments.

In our paper ([12]), we illustrate that hedging with standard options is significantly more effective than hedging with the underlying, especially under jump risk. We remark that the performance of a hedging strategy that uses options depends not only on the underlying price dynamics, but also on the option price dynamics. The paper ([12]) also investigates the modeling of volatility risk in the computation of a risk minimizing hedging strategy. While the performance of hedging with the underlying is sensitive to instantaneous volatility risk, hedging with options is sensitive to the implied volatility risk. The above mentioned paper proposes a joint model for the underlying real-world price dynamics and the stochastic at-the-money implied volatility. Computed under this model, the risk minimizing hedging strategies using options effectively reduce volatility risk.

In order to investigate the sensitivity to interest rate of a risk minimizing hedging strategy using options, we perform a similar analysis to the case in Section 3, where the hedging instruments are the underlying assets. We first compute the hedging strategy when there is no interest rate risk, in a framework where the term structure of interest rates is deterministic. The numerical results illustrating the hedging performance in this case are presented in the tables below under the label "opt". We then analyze the performance of the above strategy under a joint model for the real-world underlying price dynamics and the stochastic short rates. The results are denoted below by "opt-r". Finally, we compute the risk minimizing hedging strategy under the joint model by considering the short rate as a state variable. The hedging results in this framework are identified as "opt-$\check{r}$.

We consider a hedging portfolio rebalanced annually and consisting of 1-year maturity options and risk-free bonds. At each hedging time t_k, the options are: 3 calls with strike prices S_k, 110%S_k, 120%S_k, and 3 puts with strike prices S_k, 80%S_k, 90%S_k.

Table 4 shows the numerical results in a Black-Scholes framework. The joint model for the underlying asset and the short rate is given by (12).

Table 4: Hedging with options in a Black-Scholes framework

	No hedge	Option hedging		
	Π_T	opt	opt-r	opt-$\check{r}$
C_0	0	14.68	14.68	15.11
$\mathrm{E}(\Pi_T - G_M)$	24.41	28.15	27.73	29.02
$\mathrm{E}(\Pi_T - P_T^{\mathrm{sf}})$	24.41	-0.007	-0.42	0.03
$\mathrm{std}(\Pi_T - P_T^{\mathrm{sf}})$	36.83	2.33	7.94	3.17
VaR(95%)	96.92	3.09	11.68	4.68
CVaR(95%)	136.60	5.64	17.66	7.66

BS: #scenarios = 20000, $\sigma_1 = 0.2$, $\mu = 0.1$, $S_0 = 100$
Vasicek: $r_0 = 0.05$, $\bar{r} = 0.08$, $a = 0.2$, $\sigma_2 = 0.02$

Comparing the results for hedging with options in Table 4 with the results for monthly hedging with the underlying in Table 1, we remark that option hedging leads to a larger reduction in terms of standard deviation of total risk, VaR and CVaR. Even when the stochastic interest rates are not modeled in the computation of the hedging strategy ("opt-r"), option hedging is slightly better than hedging with the underlying. The average total cost for option hedging is marginally larger than the average total cost for hedging with the underlying.

We also notice that option hedging is much more sensitive to interest rate risk than hedging with the underlying. When the market interest rate is stochastic, but this is not accounted for in the computation of the hedging strategy ("opt-r"), the values of the standard deviation of the total risk, VaR, and CVaR increase more than 300% compared to the corresponding values when there is no interest rate risk ("opt"). However, when the risk minimization hedging strategy is obtained by modeling the stochastic interest rate as a state variable ("opt-$\check{r}$"), the interest rate risk is greatly reduced. As mentioned in the previous section, an important factor in achieving this interest risk reduction is the inclusion of 2 bonds in the portfolio, one bond being insufficient for maintaining the portfolio self-financing and offseting the fluctuations of the interest rate.

As in Section 3 we also investigate the sensitivity to interest rate risk in a model with jump risk, which is a more appropriate framework for the price dynamics of the underlying of a lookback option embedded in a GMDB with ratchet feature. Table 5 illustrates the performance of hedging with options in a MJD framework.

Table 5: Hedging with options in a MJD framework

	No hedge	Option hedging		
	Π_T	opt	opt-r	opt-$\check{r}$
C_0	0	19.19	19.19	19.57
$\mathrm{E}(\Pi_T - G_M)$	31.52	36.80	36.34	37.55
$\mathrm{E}(\Pi_T - P_T^{\mathrm{sf}})$	31.52	-0.01	-0.48	0.005
$\mathrm{std}(\Pi_T - P_T^{\mathrm{sf}})$	65.36	4.53	11.59	5.44
VaR(95%)	152.31	5.65	17.28	7.46
CVaR(95%)	245.04	11.24	27.18	13.07

MJD: #scenarios = 20000, $\mu = 0.15$, $\sigma_1 = 0.2$, $\mu_J = -0.344$, $\sigma_J = 0.25$, $\lambda = 0.0916$, $S_0 = 100$
Vasicek: $r_0 = 0.05$, $\bar{r} = 0.08$, $a = 0.2$, $\sigma_2 = 0.02$

Comparing the results from Tables 3 and 5, we remark that option hedging is more effective in reducing the risk than hedging with the underlying. The remark was also true in the Black-Scholes framework, however the differences between the performances of hedging with options and hedging with the underlying are larger in the MJD framework. This shows that option hedging is superior in offsetting the jump risk.

As in the case of the Black-Scholes framework, option hedging is sensitive to the interest rate risk. When the stochastic interest rates are not modeled in the computation of the hedging strategy ("opt-r"), the standard deviation of the hedging risk, VaR and CVaR are more than double compared to the corresponding values when there is no risk ("opt"). However, the hedging strategy calculated under the joint model for the underlying price dynamics and the short rates ("opt-$\check{r}$") substantially reduces the interest rate risk.

In order to simplify the computation of the option hedging strategy "opt-$\check{r}$" under the joint model of the underlying asset and the short rate, we have once again assumed in this section, for both the Black-Scholes and the MJD framework, that the underlying and the interest rate are independent. Table 6 illustrates the sensitivity of the option hedging strategy to different correlations, in the MJD framework. The results in the Black-Scholes framework show a similar trend and have been omitted.

Table 6: Annually hedging with options for different correlations

		Correlation				
		-0.4	-0.2	0	0.2	0.4
Initial	opt-r	19.19	19.19	19.19	19.19	19.19
	opt-$\check{r}$	19.57	19.57	19.57	19.57	19.57
$\mathrm{E}(\Pi_T - G_M)$	opt-r	37.40	36.91	36.30	35.52	35.04
	opt-$\check{r}$	38.48	38.04	37.73	37.10	36.68
$\mathrm{E}(\Pi_T - P_T^{\mathrm{sf}})$	opt-r	0.58	0.09	-0.52	-1.29	-1.77
	opt-$\check{r}$	0.93	0.49	0.18	-0.44	-0.86
$\mathrm{std}(\Pi_T - P_T^{\mathrm{sf}})$	opt-r	12.54	12.13	11.81	12.09	12.18
	opt-$\check{r}$	5.80	5.47	5.53	5.59	5.55
VaR(95%)	opt-r	20.37	19.51	16.84	15.48	14.65
	opt-$\check{r}$	9.39	8.21	7.92	6.84	6.59
CVaR(95%)	opt-r	33.99	31.27	27.14	24.23	22.13
	opt-$\check{r}$	14.74	13.72	13.97	13.03	12.80

MJD: #scenarios = 20000, $\mu = 0.15$, $\sigma_1 = 0.2$, $\mu_J = -0.344$, $\sigma_J = 0.25$, $\lambda = 0.0916$, $S_0 = 100$
Vasicek: $r_0 = 0.05$, $\bar{r} = 0.08$, $a = 0.2$, $\sigma_2 = 0.02$

We remark that the hedging results for strategy “opt” have been omitted from Table 6, since, as mentioned before, its performance does not depend on the correlation. For comparison, the hedging results for this strategy, on the simulated paths for different values of the correlation, are very similar to the results in column “opt” of Table 5.

Comparing the hedging results for strategy “opt-r” in Table 6 with the results in Table 2, it can be observed that hedging with options under a MJD model is more sensitive to the correlation between the changes in the account values and interest rates. However, by modeling the interest rate changes in the hedging computation, strategy “opt-$\check{r}$” leads to significant risk reduction for all correlations. In addition, the hedging performance of strategy “opt-$\check{r}$”, is approximately the same for different correlations.

In contrast with hedging using the underlying, the option hedging strategy “opt-$\check{r}$” tends to slightly underhedge the liability for positive correlations and to slightly overhedge it when the correlation is negative.

5. Conclusions

The hedging of the options embedded in guaranteed minimum benefits of variable annuities is a difficult problem due to the sensitivity of these benefits to the tail distributions of the underlying accounts and their long maturity. The popular Black-Scholes model is not adequate for calibrating the fat tails of the equity returns. Moreover, delta hedging assumes continuous rebalancing of the hedging portfolio. A more appropriate model for the underlying, such as Merton’s jump diffusion model or discretely rebalancing the hedging portfolio, leads to incomplete markets. In an incomplete market framework, a risk minimization criterion is usually used to compute a hedging strategy. A risk minimization hedging strategy is not only optimal with respect to the particular criterion, but it is also computed under the real-world price dynamics, in contrast to a delta hedging strategy which is computed in a risk-neutral framework. This is important, since the performance of a hedging strategy has to be analyzed with respect to the real-world dynamics. In addition, when hedging the embedded options, due to their long maturities, it is crucial to investigate the interest rate risk.

We compute local risk minimizing hedging strategies under a joint model for the real-world underlying price dynamics and the short rate. We analyze the hedging performance under both equity and interest rate risks. The hedging instruments considered are either the underlying assets or liquid options.

Hedging with options leads to a considerably better performance than annual or monthly hedging, especially under jump risk. Since option hedging effectively eliminates equity risks, including jump risk, the sensitivity of the hedging performance to interest rate risk becomes significant and can be more visibly observed. Computing the hedging strategies by ignoring the stochastic interest rate risk may lead to sufficiently larger hedging errors, possibly losing the advantage of option hedging over hedging with the underlying. It is possible to reduce the interest rate risk by modeling the stochastic interest rates in the computation of the hedging strategies. In fact the hedging errors for the strategies computed under the joint model of the underlying asset and the short rates are close in values to the hedging errors under no interest risk.

For simplicity, when computing the local risk minimizing hedging strategies under the joint model we assume the independence of the underlying and the interest rates. We analyze, however, the sensitivity of the hedging strategies to different correlations. The numerical results illustrate that the relative risk reduction achieved by the hedging strategy which takes into account the stochastic interest rates is approximately the same for the different values of the correlation.

In addition to the jump risk and interest rate risk, hedging the embedded options in GMDB is susceptible to other risk factors such as volatility risk, mortality risk, lapsation risk or basis risk. We have investigated the modeling of volatility risk in a previous paper ([12]). We are currently analyzing the sensitivity of the hedging performance to basis risk.

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