COMMON INVESTMENT PHILOSOPHIES AND SHARE RESTRICTIONS OF ASSET MANAGERS

A Dissertation
Presented to the Faculty of the Graduate School
of Cornell University
in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy

by
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January 2017

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Cornell University 2017

This work addresses two important issues of investing through asset managers: similarities in the investment philosophies of low cost funds and share restrictions of hedge funds.

For the low cost funds, I create a framework to reveal their investment philosophies and study the resulting predictability of the aggregate fund trading actions. First, I develop a new methodology, which, using discrete trading observations, quantifies the fund's preferences towards available factors and their values. The approach enables us to classify quantitative factors into "Action" or "Attention" types. The latter is used to identify whether the fund is using a given factor to make trading actions or merely as a filter to concentrate its attention on a subset of stocks. I apply the model on the US mutual fund holdings data and find that 82.7% of the US mutual funds have a significant preference towards certain factor regions. For the funds which existed in the period from 1994(q1)-2014(q4), 6-month momentum is the most popular "Action" factor (used by 28% of the funds). Whereas, the most popular "Attention" factors are turnover and size (used by 43 and 36% of the funds respectively). I find that a fund's preference towards a factor value might change depending on the quantities of other factors. In particular, within different factor deciles (clusters), fund's preferences towards the same factor might be completely opposite to each other. After, I create a theoretical model, where agents follow

pre-defined investment philosophies and make trading decisions based on the changes in the underlying ”Attention” and ”Action” factors. The model is developed to work in a framework where funds have a finite-dimensional source of public information. I adjust the model for possible trading style changes and use it to predict the next quarter trades for each fund. I aggregate those predictions to test the model on the US mutual fund holdings data. I find that for stocks with large institutional holdings, the change of the holdings between adjacent quarters can be predicted with the model. The results provide evidence against the commonly accepted hypothesis which states that the mutual fund herding happens because the funds follow each other's trades. The Action |Attention model provides an alternative explanation, where an ”unintentional herding” happens because of the similarities of the trading philosophies of the low-cost funds.

For Hedge Funds, I create a framework of optimal portfolio construction that can incorporate the costs of re-balancing constraints and share restrictions. I do that by transforming a constrained portfolio construction problem into an unconstrained one by penalizing the expected returns of the underlying assets. The methodology is applied to computing the lockup premium of hedge funds in Markov-Switching and transaction cost frameworks. In contrast to the approaches I find in academia, I argue that the hedge fund lockup illiquidity should be modeled as a lost investment opportunity premium. I compute the premium for an experimental data set.

BIOGRAPHICAL SKETCH

Vardan Verdiyan was born in 1990, in a family of Armenian refugees from Azerbaijan. During his high school years Vardan was enthusiastic about competitions in mathematics, physics and computer science. Since 2005 he participated in and received awards from numerous Olympiads in those subjects. In 2009, Vardan started his undergraduate degree in mathematics at Jacobs University in Bremen, Germany. During his undergraduate studies, Vardan engaged in research projects at Max Planck Institute for Mathematics and the Institute for Pure and Applied Mathematics. He became passionate about conducting interdisciplinary research and entered the PhD program at Cornell University's Center for Applied Mathematics in 2012. Because of Vardan's interest in both behavioral and quantitative disciplines, he concentrated in Mathematical Finance. Throughout his PhD studies, Vardan also became interested in topics concerning Soft Computing, Information Science and the future of the AI in the asset management industry. Vardan practices Mindfulness Meditation and, upon graduation, hopes to fulfill his interests in public speaking, martial arts and short-films. Vardan met his best friend and now wife at Cornell. He loves spending time with his family and friends, and is looking forward to starting a new chapter in his life after completing his PhD.

To the loving memory of my grandmother Hasmik M. (1933-2016)

ACKNOWLEDGEMENTS

I owe my deepest gratitude to Robert A. Jarrow, for his generous mentorship and support. I feel very fortunate to have him as an advisor throughout my PhD studies. Robert's deep knowledge, guidance and encouragement often pushed me forward and helped to discover and pursue an exciting research topic. I am also greatly indebted to my other committee members, James Renegar and Pierre Patie, for their invaluable guidance and feedback.

I am grateful to my mentor Arthur Berd, who guided me since the start of my PhD program at Cornell. I greatly benefited from conversations with Arthur about finance and beyond. I also wish to thank Wesley Chan from Acadian Asset Management, LLC. A conversation I had with Wes motivated me to research the crowding of the mutual fund trading strategies. The internships I had at S&P, LLC, and GMO, LLC, were also very informative. I want to thank Ed Sargsyan from S&P and Anna Chetoukhina, Elliot Bernstein, Federico Berruto and Drew Spangler from GMO for their support.

This thesis would not have been possible without my family and friends. No words can capture the gratitude I have for my wife and best friend Ksenia, for her love, support and enduring patience. Ksenia was always there by my side through difficult times and late work hours. I would like to sincerely thank my mom, whose enormous devotion and love always encouraged me to strive for my dreams and boosted my confidence. I also want to thank my relatives, in particular uncle Levon and aunts Laura and Nina. I am grateful to Dmitry, Natalia and Elena for their support at my PhD defense. I wish to thank my best friend Narek Tovmasyan, for his sincere dedication. Narek always helped wherever he could and was by my side, including at my wedding and at my PhD defense. I also wish to thank Kyle Perline for various discussions about optimal control and his help in practicing my presentation. I am indebted to Marine and Kharen Musaelian, who were so kind to host me and many other students at their house. Their warmth was very comforting, particularly when I had just started my PhD, far away from my family.

I wish to acknowledge my student research assistant Gregory Stepaniouk, for his help with implementation of financial factors and various useful discussions during the summer of 2016. Chapter 4.3 of this thesis is completed in collaboration with Andrey Gushchin and Ksenia Trikoz and I want to thank both of them. I am also grateful to the Center for Applied Mathematics at Cornell University and in particular to Gennady Samorodnitsky, Alexander Vladimirsky and Karen Benavidez. Lastly, I wish to thank the Associate Director of Engineering Leadership program Werner Zorman, who introduced me to the Mindfulness Meditation. The meditation helped me to overcome one of my biggest challenges and remain concentrated and happy during even the most stressful times.

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CHAPTER 1

INTRODUCTION

I begin by informally describing my motivation to analyze the trading philosophies and the share restrictions of the investment funds. After a general discussion of the topic[1], the further organization of the thesis chapters is presented.

1.1 The Choice of the Research Topic

The question of modeling the illiquidity premium of hedge funds first came to my attention during a class project[2]. I continued exploring the idea beyond the class and became interested in analyzing asset managers in general. The research soon became quite mathematical and at some point I had to take a break from it to learn more about the stochastic optimal control theory. In addition to academic and news sources, I was fortunate to get a better understanding of the financial industry through various informal conversations with asset managers themselves. Academic and industry conferences also helped me to gain intuition in the subject.

The impression which was forming on my side was that there were two important concerns in the modern asset management industry. First, some of the trading strategies did not work as well as they did before. So, the funds were interested to know how crowded their strategies were and whether there might

[1]More formal introductions of the research topics are given in the first sections of the following chapters.

[2]I worked on the project together with Andrey Gushchin and Ksenia (Trikoz) Verdiyan. The idea to explore the aforementioned topic came through an informal conversation with Nathan Chesley from MITIMCo. Class: ORIE 5610 (Spring, 2014), Instructor: James Renegar.

have been some structural changes in the market which made famous anomalies no longer profitable. Second, funds seemed to be worried about the investor outflows and the demands for lowering the management fees. As a result, I became interested to explore the balance between the higher fees and the trading restrictions on one side and the possible similarities in the trading strategies on the other. Through various discussions with my adviser, I formulated the research topic I explore in this thesis.

1.2 Thesis Background

Financial services is a highly competitive industry which in the last decades rapidly integrated knowledge from various disciplines to generate profits. Technological advances provided affordable and fast access to financial data for the majority of the market and created the computational capacities to process it. The market became more systematic and model-driven, demanding a higher ratio of trained professionals to make successful investment choices. This, in turn, increased the number of sophisticated investment companies and the amount of money they manage. A large portion of the modern equity market is managed by investment professionals as a part of an investment fund. Different types of funds have their comparative advantages and making a choice between them is not trivial. Lower fee funds, such as mutual funds and ETFs, have transparent structure but follow simpler trading strategies which often correlate with each other. Hedge Funds, on the other hand, might have more sophisticated, high-return strategies, but could also impose liquidation constraints. The main problem of a typical institutional investor is to optimally allocate and manage its portfolio through asset managers. In such scenarios, the use of the classical portfolio construction models is often impractical, due to the incompatibility of the simplifying assumptions with the reality. Specifically, in a portfolio of Hedge Funds it is not possible to frequently re-balance assets and re-balancing weights might be associated with a certain set of rules and significant costs. On the other hand, the common trading strategies of mutual funds might also result in hidden correlations and unaccounted risks. This thesis is concerned with exploring the aforementioned two issues of investing through asset managers.

The first question we ask ourselves is **how increasing sophistication of the industry could possibly generate similarities in the trading approaches?** At the first sight, the reverse statement might sound more logical, because sophistication should also result in new methods and more diversity. I start the discussion by presenting an informal background on the topic.

Ad-Hoc Methods and Favorite Models: During the last decades the investment industry became more sophisticated, which resulted in an increased demand for trained investment professionals. The higher educational institutions proliferated the number of specialized programs to train individuals for work in the investment companies. The curriculum in those programs became systematic and the students across those programs are introduced to classical models and beliefs of how markets work. Academic consensus on the financial market itself evolved through different phases. In 1970s it was widely thought that the stock prices are "random", which was later formalized into the Efficient Market Hypothesis. However, starting from 1980s the academic literature on anomalies came into light, which concluded that markets are not completely efficient and there might be mispricings and limited arbitrage opportunities[3]. Since then,

[3]For a more detailed introduction into the subject, I refer the reader to the Chapter 1 of the Book by Qian, Hua and Sorensen [1]

various other models were proposed from purely behavioral, to mathematical or statistical ones. As the complexity of new financial models started to increase, they became harder to understand and their average acceptance rate started to slow down. Increasing mental efforts are needed to understand a new intricate concept, and to explain it to a non-sophisticated investor might be even more difficult. On the contrary, tweaking or changing an old trusted model could be more favorable.

As a result, a handful of models (e.g. CAPM, APT) and trading styles (e.g. Momentum, Value) became absolute favorites and a large ratio of the market started to use their variations. These are wide-spread in industry, are actively taught in specialized programs in the universities and some of them have been around for decades. Many of those models/strategies were discovered in the era when the trades were not automatically executed or when the computational capacities of computers were quite modest. More importantly, at the time they were discovered they were not so widely used. However, the question whether there are enough similarities in the models to make the aggregate fund actions predictable still remains open. The first chapter of this thesis is devoted to exploring the aforementioned question.

1.3 Thesis Organization Summary

Chapter 2 discusses the topic of the fund trade predictability for the low-cost funds. In the section 2.1, a formal introduction on the topic of common trading philosophies is given. After describing the input data in the section 2.2, I present the methodology of quantifying the preferences of the funds in the section 2.3.

I introduce the concept of Action and Attention factors and apply the model on the US mutual fund holdings data in the sections 2.3.5 and 2.3.6. In section 2.3.7 I analyze the overlapping signals from the different factors and introduce the "preference directionality maps". Next, in the section 2.4 I construct a theoretical framework of an agent who trades based on the changes in the Action and Attention information. Section 2.5 completes the chapter, where I present the results of fitting the Action|Attention model to the data.

Chapter 3 introduces a framework to compute the premium of a trading restriction and Chapter 4 applies it to measure the lockup premiums of the hedge funds. I model the hedge fund lockup premiums in the Markov-Switching and the Transaction-Cost frameworks in the section 4.2. The results of computing the hedge fund lockups for an index data set is presented in the section 4.3.

The conclusions of Chapters 2-4 are presented in the Chapter 5, where I also discuss the possible limitations and further improvements of the results.

CHAPTER 2

PREDICTABILITY OF FUNDS FOLLOWING COMMON INVESTMENT PHILOSOPHIES

2.1 Introduction

A large portion of the market is currently managed by investment professionals as a part of an investment fund such as a mutual fund, hedge fund or an ETF. In the US alone, in 2014 there were approximately 8000 registered mutual funds that manage around 16 trillion USD, which is a 16-fold increase compared to 1990s [2]. Fund managers often follow a certain trading philosophy and search for asset pricing anomalies using various factors. As a result, funds share beliefs on which factor values could help an equity to outperform a certain benchmark. Those beliefs might be quite general in nature, e.g. ”low P/B stocks would outperform the market” or ”stock price has a momentum but eventually reverses to the mean”. An interesting question is how closely funds follow their investment mandates and whether keeping dear to an investment philosophy makes their trading decisions predictable. For example, suppose it is possible to quantify into factors all the sources of the information which a fund uses for trading. Then, would there be most (least) ”favorite regions” of factor values where the funds is most (least) likely to buy or sell an equity? In this work, I quantify the preferences of the funds towards company-level factors and discuss how those preferences affect the aggregate fund trade predictability.

Trading philosophies of most of the funds are not kept in secret and funds usually advertise their trading philosophies in prospectuses to attract investors. That information is even mentioned in the fund names. For example, a quick textual analysis of our database shows that out of approximately 7000 US mutual fund[1] names the words "growth", "value" and "cap" were mentioned in 17%, 12% and 21% of them, respectively. There are a few popular empirical approaches of classifying a fund's trading style. For example, Grinblatt, Timan & Wermers (1995) divide the mutual funds into two categories: contrarian (buying stocks which performed poorly in the past) and momentum (buying the past winners). The authors find that 77% of the funds in their database are momentum investors. Another well-accepted approach to measure fund's trading style is based on the work of Daniel, Grinblatt, Titman & Wemers (1997). The method is to classify each stock into quintiles according to book-to-market, market capitalization and past momentum return characteristics. After, for each fund, using the fund's portfolio holdings, a style benchmark is formed. Those approaches are fundamentally different from ours, mainly because they were **designed to analyze mutual fund performance**. Investment research companies such as Morningstar® or Thomson Reuters Lipper ® also classify funds based on holdings and the classification of stocks into (usually 3) styles (see [3] and [4]). Although those approaches might be useful for understanding the exposure of a mutual fund towards different risk factors, their design does not explicitly measure fund's buying and selling (i.e. trading) preferences. In a hypothetical example, suppose we have a successful value fund which bought stocks with low price to book ratios and sold them after one year, when the price to book (P/B) ratios of the selected stocks became high. The holding measure would not quantify such a behavior as pure value trading, since the P/B ratios of stocks became high before they were liquidated and that would be counted as a "growth" signal.

[1]The funds are mainly mutual funds and ETFs. For the funds which changed their names I only use the most recent available name. More information on the database will follow.

On contrary to those approaches, I only study the trading actions made by the fund in relation to the underlying information in the market. I analyze Buy and Sell decisions separately and comparing them to each other reveals whether the factor is used within the trading philosophy of the fund. I do that by first selecting and quantifying the input information (input measure) and fund's trading actions (output measure) as to account for possible sources of noise. Later, I accumulate the signals in a way to account for fund reporting delays and to extract directionality of preference. I also introduce and later test the concept of the ″Action″ and ″Attention″ factors. I briefly describe those steps, below.

Action and Attention Factors: Due to computational limitations, it is natural to assume that fund managers pay attention only to a subset of securities and make trading actions (e.g. Buy, Sell) based on a subset of available information. Hence, if we quantify the available public information into a large number of factors, then for a given fund some of the factors are used to guide the attention towards a subset of securities whereas others might be the reason of a fund's trading decision. For example, if we have a ″small-cap momentum″ fund, then the ″size″ factor's quantity would guide the investor's attention towards a subset of assets. However, the trading decision of the fund would not be to buy the equity with the smallest ″size″, but within the small-cap funds will depend on ″momentum″ and possibly some other factors. In such a case, ″size″ would be an attention factor, whereas ″momentum″ would be an action factor. Another logical attention factor might be the quantification of security's industry.

Quantifying Trading Actions (Output Observation Measure): A Fund's Buy and Sell decisions in their pure form do not necessarily reflect the fund's trading philosophy or style. For example, relatively small Buy and Sell changes in the portfolio might be a simple risk adjustment rather than a signal of ”like” or ”dislike”. Also, a homogeneous increase (decrease) in the existing equity positions of the fund might be a result of a large inflow (outflow). I apply filters to a fund’s trading decisions to adjust for inflows/outflows, risk-adjustment noises and extract the fund’s main trading preferences. For that reason, I create 4 output observation (trading) measures. The main output measure I will use in the first part of my research is made by dividing the fund’s amount of Buy (Sell) of a given stock by the total amount of Buy (Sell) of the fund in that quarter. So, if we have a large inflow/outflow or a risk-adjustment trading observation, its importance would be reduced by the designed measure. Based on a similar logic, I create two other output measures and use them for robustness checks.

Quantifying Information (Input Observation Measure): To analyze the absolute preference towards a single source of information, I complement the 69 company-level factors from WRDS Financial Ratios suite by 11 quantitative investment signals presented in Jagadeesh, Kim, Krische & Lee (2004) and in Wermers, Yao & Zhao (2012). Many of those factors are made by combining and comparing (both within the cross-section and time-series) signals from company and market data. To make the factors for different stocks comparable to each other in current and previous time periods, I assign to a factor’s quantity a combined cross-sectional (market-cap weighted) rank. The latter quantifies a factor from a decision-maker’s perspective and helps to mitigate the systematic differences of the factor values between the industries. More details on the methodology is given in the section 2.3.3.

I make the factor rankings market-cap weighted so that there is no bias, i.e. so that the relations between the input (factors) and the output (Buy/Sell quantifications) for a "no-preference" investing are random. I test it by constructing a "no-directionality" trader from the market data and observe a completely random relation. However, when I apply the model to actual funds, I see that they have distinct Buy and Sell directionality patterns. I call those preference patterns the 1-dimensional directionality curves. To analyze the preference pattern in a larger number of funds, I systematically classify the preferences into simple curves. I apply the method on our fund holdings database to extract the funds' trading philosophies. As a result, for each fund I am able to classify the available factors into "Action", "Attention" or "Indifference" types.

After, I discuss the question of possible overlap of the input information. For example, in one-dimensional analysis it is not clear which factor's movement was the reason of the fund trading. I propose a simple method of finding preference towards one factor by clustering the trades for which all the other factors were close to each other. I find that in different factor regions, fund's directionality of preference might have different shapes. Because of that, I extend our analysis into multiple dimensions and create preferences depending from multiple factors. I call those directionality maps.

Next, I create a theoretical model where agents use action and attention factors to make trading decisions. I define decision functions for each fund and by fitting them to the data, reconstruct its expectations numerically. The fitting process is implemented in a moving window of a fixed length to account for possible trading style changes. The reason is that there might be a lot of variability on how closely a fund follows its investment mandate and trading strategies which use different trading styles (e.g. style-rotation strategy[1]) are not uncommon. Although style-rotation type of strategies gain popularity, the

[1]For a theoretical treatment of style investing see Barberis and Shleifer (2003).

non-rotating trading styles are still plentiful in the market. Besides, according to Brown, Harlow & Zhang (2015), on average the funds with lower levels of style volatility still outperform the style-volatile funds. One of the main assumptions in our model is the assumption on all the information being public. Accessing the information might still have both computational and monetary costs associated with it. But, I assume that the markets are fair and there is no ”completely private” information which certain funds are using to trade on.

2.2 Data Sources and Filtering

Before discussing the concept of directionality of preference, I start by presenting the data sources that I am using for testing the models. The reason is to be able to present some results and test robustness of our model while describing it.

Aggregating Mutual Fund Holding Data: Thomson Reuters s12 (former CDA Spectrum) is probably the most commonly used database for analyzing the holdings of the mutual funds in the academic literature. It contains both voluntary and some of the SEC mandated portfolios available through the forms N-30B-2, N-30D, N-Q, N-CSRS, N-CSR and N-Q ([5], [6]). In addition, the Thomson s12 database contains portfolio holdings information before 09/1993 which is not included in the EDGAR system ([4], [5]). Because the Thomson s12 database structure is widely used in academia, to ease reproducibility, I collect data into a database following a similar structure and terminology[2]. I combine FactSet Unadjusted Fund Holding data (1999q1-2014q4, [7]) with CRSP fund holding data (2009q1-2014q4, [8]). I also implement a textual analysis of SEC filings (https://www.sec.gov/edgar/) and compare it with the fund and security names from CRSP for (1994q1-2014q4). As a result, our data partially spans 1994-1998 and then fully covers 1999-2014 mutual fund holdings. An in-depth comparison of the CRSP, Thomson s12 databases and the SEC filings is presented in the works of Schwarz and Potter (2014).

Combining Data from Various Databases The information about the funds and the stocks themselves I extract from the CRSPSift and the CRSP databases

[2] I was not able to use the Thomson s12 database itself since Cornell University was no more subscribed to the Thomson s12 database through WRDS, hence I had no access to the recent s12 data.

([9], [8]). Compustat [10], IBES [11] and the Financial Ratios Suite by WRDS [12] are used for constructing relevant factors. I link the fund holding and the CRSP databases using fund names (which I control for changes and possible variations). I use the methodology described in MFLINKS manual [6] and Appendix A of Wermers (1999), to remove possible repetitions and mistakes. Throughout the databases, I link all the stock headers (including the Compustat GVKEY [10] headers) to PERMNO header of the CRSP. As a result, a permanent identifier of a stock in our database is PERMNO and of a fund is FUNDNO. The figure describing the process is attached, below:

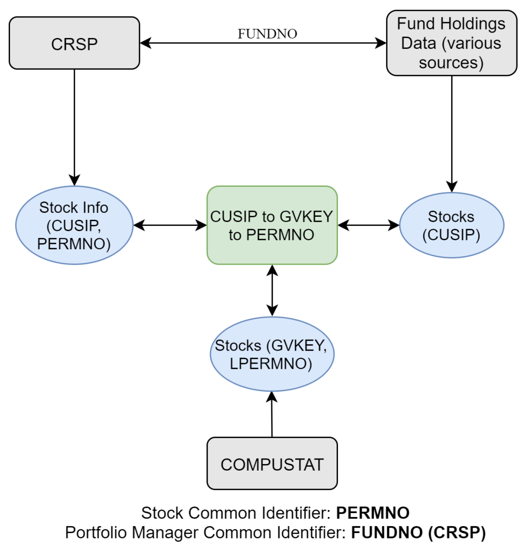


Figure 2.1: Linking Fund Holdings, CRSP and COMPUSTAT

Stale Data: Multiple file dates (FDATE) for the same report date (RDATE) of a fund often indicates repeating holding entries, so I remove them from the data and leave only the ones with the smallest FDATE. The procedure eliminates around 6 million out of the 34 million {fund, asset, holding} entries in our database. For the rest of the data, in the majority of the cases the FDATE and RDATE are in the same quarter. I remove the data for which the difference between FDATE and RDATE is more than 2 quarters.

# of Quarters	0	1	2	≥ 3
% of Data	72.07%	27.08%	0.77%	0.07 %
# of Observations	$\approx 19.7 * 10^6$	$\approx 7.4 * 10^6$	$\approx 21 * 10^4$	$\approx 2 * 10^4$

Figure 2.2: Number of quarters in between FDATE and RDATE

Share and Price Adjustment Factors from CRSP: To compare the number of the outstanding shares and prices during different time periods, I adjust them for the share distributions. CRSP ([9]) has two factors which I use: facprc and facshr. For a given stock j and months $m_1 < m_2$ we define:

$$\text{AdjS}_j(m_1, m_2) = (1 + \text{facshr}_j(m_1 + 1))...(1 + \text{facshr}_j(m_2)) \tag{2.1}$$

$$\text{AdjP}_j(m_1, m_2) = 1/((1 + \text{facprc}_j(m_1 + 1))...(1 + \text{facprc}_j(m_2))) \tag{2.2}$$

From where, we can calculate the share holdings and price changes in the following way:

$$\text{ShrChange} = \text{SHROUT}_j(m_2) - \text{SHROUT}_j(m_1) * \text{AdjS}_j(m_1, m_2) \tag{2.3}$$

$$\text{PrcChange} = \text{PRC}_j(m_2) - \text{PRC}_j(m_1) * \text{AdjP}_j(m_1, m_2) \tag{2.4}$$

Implementing Share Distribution Adjustments: To determine whether the fund bought new shares of the securities, it is important to account for share distributions. In some of the filings, holdings adjustments are already made for the stock splits, stock distributions, M&As and other corporate events. In those cases, I implement additional adjustments as described in the figure 2.3. $R(t),\ F(t)$ are the RDATE and FDATE for the quarter's holdings report and t' indicates the preceding report of the fund. First, I adjust shares from $F(t')$ to

$F(t)$ and calculate the adjusted difference between the shares. Next, I also adjust the share difference back to $R(t)$, i.e. the date closer to when the actual fund trading took place.

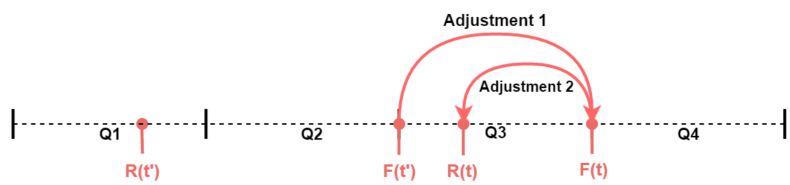


Figure 2.3: Distribution Adjustments: Example

Constructing the Matrix of Holdings and Changes: In other words, I record the share changes from the previous report's time to the time t quarter in the time $R(t)$ shares. Let us denote the adjusted share holdings variable by AHoldings and the share change by $\text{Changes}_{k,j}(t)$. Based on the aforementioned, we get that:

$$\text{AHoldings}_{k,j}(t) = \text{Holdings}_{k,j} / \text{AdjS}_j(R, F); \tag{2.5}$$

$$\text{Changes}_{k,j}(t) = \text{AHoldings}_{k,j}(t) - \text{AHoldings}_{k,j}(t') * \text{AdjS}_j(R', R). \tag{2.6}$$

Large Reporting Gaps: I also make a logical vector $\text{iRep}_k(t)$, which will indicate whether fund a k has a valid report with a RDATE in the quarter t. To reduce the noise in our data, I will calculate holding changes for a fund k at the time t only if the gap between the previous valid report date is at most 2 quarters. So, I construct a matrix $\text{iPrevRep}_k(t)$ to record the last valid report date if it is less than 3 quarters away. For the first entry of a fund, $\text{iPrevRep}_k(t)$ would be the preceding quarter. It is easy to see that the following equation holds:

$$\text{iPrevRep}_k(t) = (t-1)\text{iRep}_k(t-1) + (t-2)\text{iRep}_k(t-2)\neg\text{iRep}_k(t-1) + \tag{2.7}$$

$$+ (t-3)\text{iRep}_k(t-3)\neg\text{iRep}_k(t-1)\neg\text{iRep}_k(t-2) \tag{2.8}$$

Based on the existence of valid $iPrevRep$, I adjust the variable Changes so only the cases with a valid previous reports remain.

2.3 The Directionality of Preference

The aim of this chapter is to give a heuristic description of a model, which was designed to measure the preferences of a fund towards a source of information. The underlying data is assumed to be the set of the infrequent observations of the fund holdings, as is the quarterly Mutual Fund holdings data. I start by describing the set of available information the traders have access to in our model.

The Information Set: There exists a finite but potentially very large number d of factors $f_1, ..., f_d$, describing each asset. Those factors could include, for example, stock-specific financial ratios, macroeconomic data, analyst reviews or even a quantifications of news-sentiment. Also, the factors could be relational in the sense that a ranking of an asset based on another factor could be a factor itself. Time is continuous but the information updates at discrete instances. That is, a given stock at a time t has factor i value $v_i(t)$, where $v_i(t)$ is piece-wise constant.

Observations: For a given fund, suppose we have a finite number N of trading observations:

$$O_i = \{t_i, s_i, V_i, A_i\}, \tag{2.9}$$

where t_i is the time of the trading action of the fund towards the stock s_i, which has factor values $V_i = \{v_1^i, ..., v_d^i\}$. A_i is the dollar amount of Buy/Sell trade and represents the action of the fund towards the stock. We will call $\{t_i, s_i, V_i\}$ to be the input observations and A_i the output observations. Based on the observation data, our goal is to explore whether there is some preference in the funds towards the specific factors and their values. The observation data in its raw form could be very noisy, because an absolute quantity of a factor value without comparisons is incomplete from the decision-maker's point of view. Also, the observations do not accommodate for inflows/outflows to the fund, fire sales and external factors.

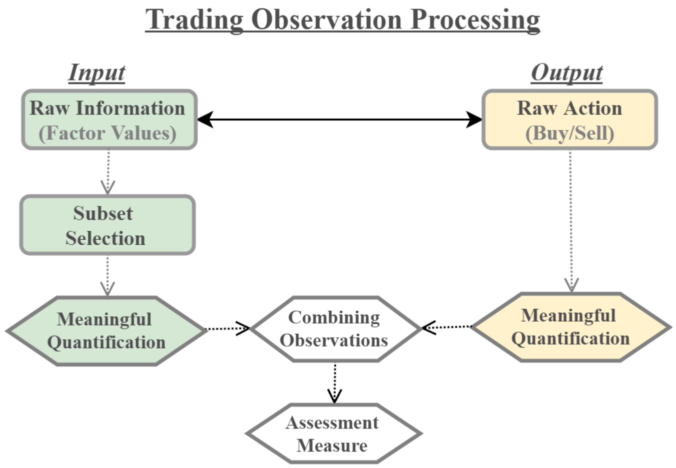


Figure 2.4: Trading Observation Modeling Steps

Thus, I design input observation measures to filter signals which are important for the funds and an output measure which will quantify the interest of the fund in a stock. The workflow of the trading observation modeling is presented in the figure 2.4. In the following part of this section, I will talk about choosing a subset of information and designing input/output measure μ_I/μ_O. But first, I will introduce and formally define the concept of directionality of preference.

2.3.1 Types of Directionality: Attention and Action

We want to explore the question whether there is any directionality in the investor's preference towards a given factor value. That is, if we have two stocks with all except one factor being identical, would the investor prefer buying stock 1 over stock 2 based on the value of that single factor? The answer probably is that it depends on the factor itself and in some cases the investor would be indifferent between the two options.

One can note that, similar to the revealed preference problems, we have input and output observations from the decision maker. However, the main difference here is that a factor value cannot be considered as a ”good”. That is, if in the preference theory, the more of a specific good we have the better it is, here the larger values might not imply more preference. So, more complex preference relations could exist (as in the figure 2.5), which are impossible to model by a simple utility function. To address the issue, I will divide the set of all of the factors into three categories: Attention, Action and Indifference.

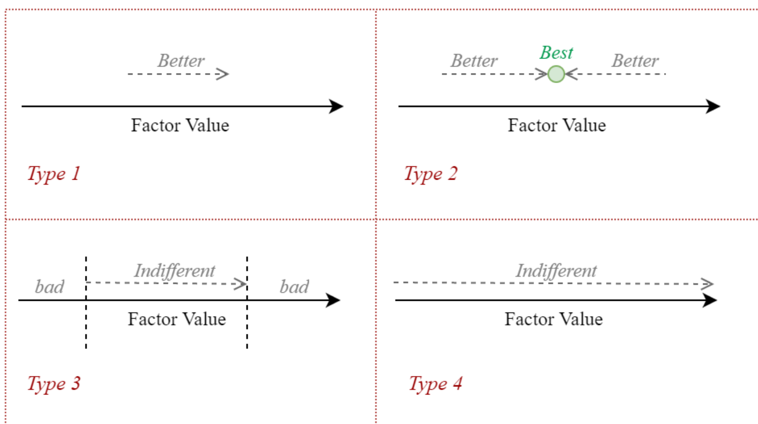


Figure 2.5: Directionality of Preference

Differentiating Between Factors (Action and Attention):

Due to computational limitations, it is natural to assume that the fund managers pay attention only to a subset of securities and make trading actions (e.g. Buy, Sell) based on a subset of the available information. Hence, if we quantify the available public information into a large number of factors, then for a given fund some of the factors are used to guide its attention towards a subset of securities, whereas others could be the triggers of a fund's trading actions. Based on the aforementioned intuition, I give the heuristic definitions of the attention, action and indifference factors.

Definition 2.3.1 (Attention Factor)**.** *For a given investor, a factor f_i is an attention factor, if there exists a fixed interval such that:*
1) the investor considers buying (selling) the asset only if the value of the factor is in that interval
2) investor is "indifferent" towards the factor's value while within the interval

Definition 2.3.2 (Action Factor)**.** *For a given investor, a factor f_i is an action factor, if:*
1) the investor's decision to Buy & Sell is affected by the value of the factor
2) there is a value of the factor which investor considers to be the best for buying the asset

Definition 2.3.3 (Indifference Factor)**.** *For a given investor, a factor f_i is an indifference factor, if the investor's trading decisions are not (in any significant way) dependent on the value of the factor*

An example of an attention factor could be the factor describing the market cap size of a stock. A fund might decide to invest only in the small caps. However, the trading decision of the fund would not be to buy the equity with the smallest "size", but to make the choices between the small cap companies based on some other factor(s). So, if the stock is in the "attention region", it could have a higher chance of being both Bought and Sold. As a result, the "attention shape" is likely to be described by opposite Buy and Sell preference curves (e.g. see the Type 3 section of the figure 2.6).

An obvious example of an action factor could be the momentum signal for certain funds. For example, if we have a ″small-cap momentum″ fund, then the ″size″ factor′s quantity would guide the investor′s attention towards a subset of assets. However, the trading decisions of the fund within the small-cap stocks will depend on the ″momentum″ factor. In such a case, the ″size″ would be an attention factor whereas the ″momentum″ would be an action factor.

Based on our definitions, we would expect the relation between the Buy/Sell signals and the stock factor values be represented by shapes similar to the ones in the figure 2.6 (green dots represent the hypothetical Buy observations and the red dots the hypothetical Sell observations).

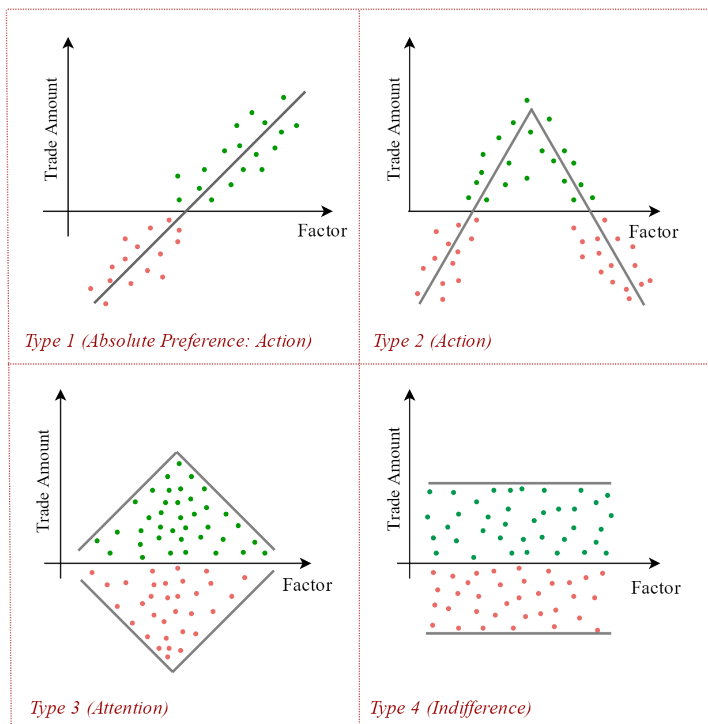


Figure 2.6: The Directionality of Preference

Buy vs. Sell signals: To understand which of the output observations reveal the investor′s trading philosophy, I discuss a few possible scenarios. A small Buy could be a sign of a re-balancing or hedging, whereas a larger Buy probably means that the investor's trading philosophy is in favor of the stock. Similarly, a "new Buy" (i.e. the case when the investor did not hold the stock in the previous quarter) could represent a "like", though a "Buy more" (i.e. increasing a position in an asset) could be a result of an inflow. However, a Sell decision might represent a larger variety of scenarios. It could be the case that for a long time the stock did not perform as the investor was expecting, and the investor's decision to liquidate its position does not reflect their core investment philosophy. So, although a Buy is often a planned action, a Sell could be a result of a forced decision. Thus, to untangle the underlying trading philosophy of a fund, I would analyze the buys and sells separately.

Standard Directionality: Based on the aforementioned, I define a few directionality patterns which we would expect the funds to have. The preference could be absolute, or there could be peaks where the fund thinks Buying/Selling is optimal. The first two simple patterns we would expect to have are I-shaped and V-shaped (when there is a peak either towards Buy or Sell). But, a trading strategy might be more complex, so we might also have behaviors with 2-peaks or 3-peaks as are the N and M-shaped patterns. For simplicity I would consider those patterns and their "smooth" versions (see 2.7).

Next, we are able to give our first definition of an investment philosophy. I define an investment philosophy to be the set of directionality graphs for Buy and Sell actions

Corollary 2.3.1 (Indistinguishable Investment Philosophies)**.** *Two funds have the same investment philosophies, if their simple directionality graphs are of the same type for all of the factors.*

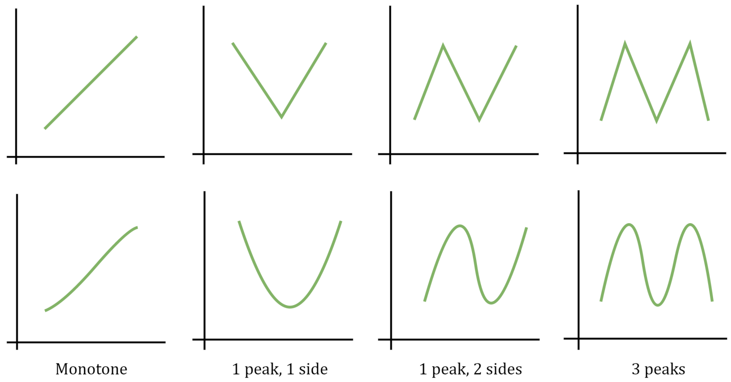


Figure 2.7: Standard Types of the Preference Shapes: IVNM

The definitions and the intuition developed in this section will be used in the next two sections to reconstruct the investment philosophies of the funds in our database.

2.3.2 Output Observation Measure

Our aim is to quantify the Buy/Sell actions of a fund in a way to compare the fund's preferences towards different stocks. However, a raw dollar quantity A_i of the amount of the stock buy/sell does not account for the fund's overall size, the inflows/outflows in a given quarter or the fund's average trading amount.

To account for large inflows/outflows or fire selling, we can divide the value A_i by the total dollar amount of the trades in a given quarter. The idea is to make the trading action values independent from the seasonality of the trading volume. I call the resulting output transformation to be the Output Measure 1.

Output Measure 1: Buy (Sell) divided by the total amount of (Buy+Sell) in a given quarter

$$\mu_O^1(A_i) = \frac{A_i}{\sum_{j|t_j=t_i} |A_j|} \tag{2.10}$$

The Output Measure 1, however, could introduce a bias in the case when in a given quarter the fund was mainly buying stocks or mainly selling them. For example, if there were many outflows in the fund and the total dollar amount of the sold stocks is large, then a significant Buy order in the same quarter would also be "unfairly" quantified to a smaller value. To resolve the aforementioned issue, I introduce two more output measures.

Output Measure 2: Buy (Sell) divided by the total amount of Buy (Sell) in a given quarter

$$\mu_O^2(A_i) = \frac{A_i}{\sum_{j|t_j=t_i} |A_j| \mathbb{I}_{A_j A_i > 0}} \tag{2.11}$$

Output Measure 3: Buy (Sell) divided by the average Buy (Sell) of the fund up to the current quarter

$$\mu_O^3(A_i) = \frac{A_i}{\sum_{j|t_j \leq t_i} |A_j| \mathbb{I}_{A_j A_i > 0}} \tag{2.12}$$

For robustness tests, we could also record the relative difference between the quantity of the Buy and the Sell action signals.

Output Measure 0:

$$\mu_O^0(A_i) = \frac{\mathbb{I}_{A_i > 0} - \mathbb{I}_{A_i < 0}}{\sum_{j|t_j = t_i} \mathbb{I}_{A_j A_i > 0}} \tag{2.13}$$

The measure I will be using for the 1-dimensional preference analysis is the Output Measure 2. This measure is quite robust, and the measures 0 and 3 output similar preference results in the 1-dimensional case. However, to be able to predict the next quarter fund trading, I also introduce a measure which is dependent on the past holding of a fund.

Output Measure 4: Buy (Sell) divided by the previous report's total amount of holding (in shares) in a given stock.

Note that this only applies to the cases when the fund bought more of the stock or sold the stock. That is, it requires that the fund held the stock in the previous quarter, else we will be dividing by zero.

2.3.3 Input Observation Measure

In this section I will discuss the question of selecting and quantifying a subset of available information for our further analysis.

Selecting an Information Subset

There are hundreds of financial factors and ratios available in the industry and it is not easy to know which ones a fund pays more attention to. Traders might have different sets of financial information they look at, but including too many factors in our model is certainly not practical. With the help of the Financial Ratio's Suite by WRDS alone we have access to 71 different firm level ratios (see [13]). Those are the most commonly used ratios in the academic research. A quick analysis by averaging (where applicable) the correlations of factors over all the stocks shows that many of those factors are significantly correlated.

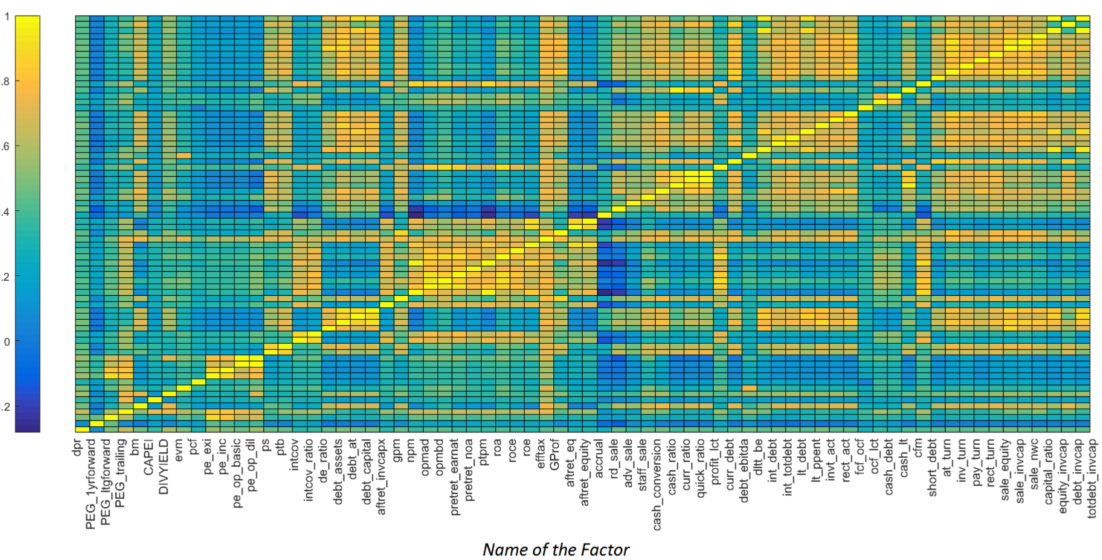


Figure 2.8: Correlations within the Factors

Since many of the factors are correlated, we don't need to consider all of them at once and can choose a subset. Also, many factors are constructed by using other factors. So, to better understand and model the financial information, first I will try to represent the financial ratios as a combination of information and its transformations.

If we fix a time and only look at one stock, then the stock usually has a parent company and is traded in the market (see figure 2.9). There is some information about the company: e.g. its profitability, debt amount, etc. The company is in an economy and also is probably a part of some industry. From the market, I assume that the only signals are the price of the stock and the transaction data. In its plain form, I assume there are no opinions and all the information is uniquely quantifiable.

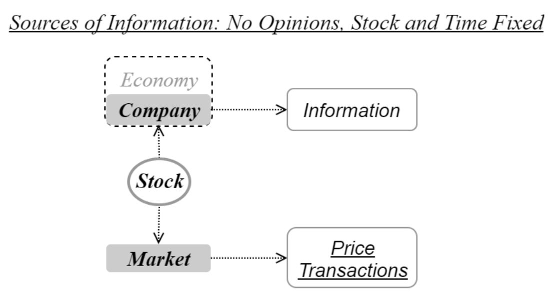


Figure 2.9: The Sources of Plain Information

Next, there are transformations such as comparisons and combinations (CO) which we can apply to our information. The comparisons could be both cross-sectional (CS) and historical (TS). Combinations in our model are assumed to be simple algebraic actions as fractions and comparisons. I assume that all information is a result of CS/TS comparisons and combinations on the plain information. For example, by using CO and CS on the price, the transaction and the market data, we can get many financial factors as described in the figures, below.

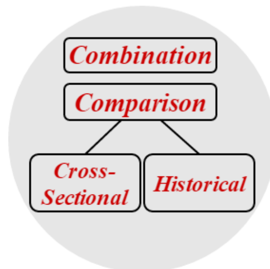


Figure 2.10: Transformations of the Information

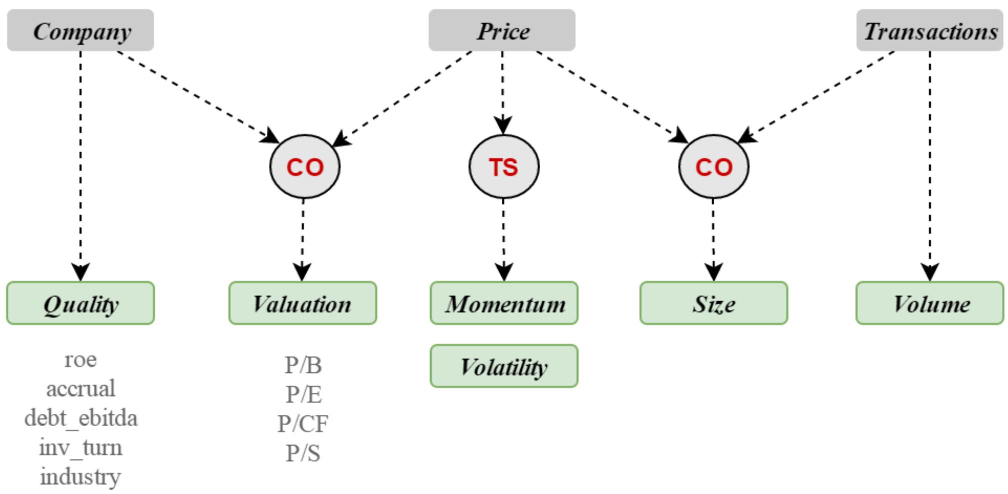


Figure 2.11: Choosing the Main Factors

In addition, we could have also performed analysis as PCA to choose combinations of those ratios which explain the most variance across all the 71 factors. Alternatively, one could cluster the factors into groups based on the inverse of the correlation value. However, to overcome possible issues of overfitting, I will choose the initial set of factors qualitatively. The WRDS factors are divided into 8 groups: Valuation, Solvency, Profitability, Financial Soundness, Liquidity, Efficiency, Capitalization and Other. In addition, we also construct factors representing: Momentum and Size indicators of a stock.

Quantifying the Information

Since I consider stocks by looking at their factor values, it is important to make the factors comparable to each other. As a result, we cannot keep the raw numerical quantity of a factor as our input component. From a decision-making perspective, I differentiate the following two classifications of a factor value: its rank in comparison to other securities (CS) and its rank in comparison to its own value in the previous time-periods (TS).

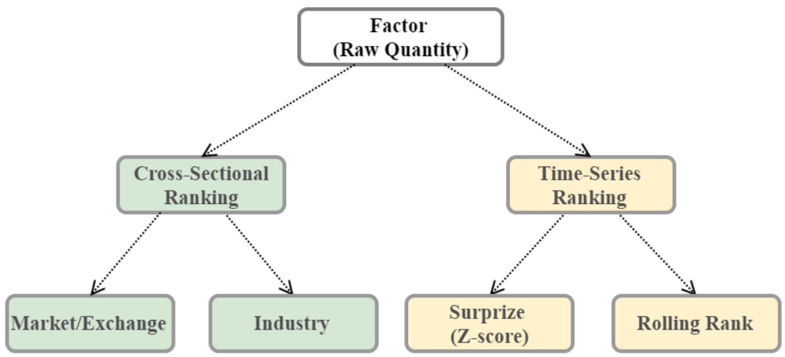


Figure 2.12: Factor Value: Decision-Maker's Perspective

Cross-Sectional ranking of stocks by a factor value is a popular way of sorting stocks and is used both in industry and in academia. An important question while making a cross-sectional ranking is choosing the set within which the ranking will be performed. Before addressing that question, I will discuss how to rank the stocks in any subset so as to not introduce a bias.

Ranking (Cap-Weighted vs. Equal-Weighted): For all of the factors, their rankings will be in the fixed interval $[1, 10]$, where 1 would represent the smallest factor value in the subset. A straightforward way of choosing a subset to rank the stocks is to choose the exchange where it is listed. I will discuss more on the topic of the subset selection, but first let us discuss the ranking process. Suppose we have a stock and have already selected a subset of stocks which we would like to compare it against. The naive way to rank the stocks' factor values, would be to sort them and give equally spaced ranks in $[1, 10]$, based on their order statistic. However, that might create a bias, since on average the market is invested in a company proportional to its market cap. As a result, companies with large caps would receive more action quantities than small cap companies for the same input factors. To resolve the issue, I will assign ranks to the stocks proportional to their market cap. I will discuss the topic in more details in the next section.

Cross-Sectional Ranking: CS

The first obvious choice to rank the factors is to perform it cross-sectionally within all the stocks under consideration. I will first perform the ranking within the stocks which have at least 10 institutional holders.

Within-Industry Cross-Sectional Ranking: CSI

Due to peculiarities of some industries, it is also important to rank the stocks within each industry. Besides, some funds only concentrate on stocks within a given industry and CSI ranking might help in such a case to measure the relative change of the factor. I use the 10 industries as presented in Kacperczyk et al. (2005). In addition, I also perform a rolling Time-Series (TS) window ranking of the stock to measure how the stock's factor performs in comparison to its past. The TS ranking might also be implicitly taken into account in the CS ranking, assuming there were no extreme market events. Another possibility is ranking in comparison to funds' portfolio holdings (CSH). Again, that ranking

[2]Note that the CS ranking here has no connection with the Characteristic Timing (CS) measure developed by DGTW (1997)

is implicitly included in the CS ranking.

Combining Rankings: CS & CSI & TS

As we discussed, the rankings themselves could be performed within all securities in the market, an exchange or even within a given fund's or style's interest set. Considering all of those options in our analysis would multiply the amount of factors by 3-5 times and is not practical. Next, I define 3 input measures, below:

Input Measure 1:

$$\mu_I^1(V) = CS(V) \tag{2.14}$$

Input Measure 2:

$$\mu_I^2(V) = 0.5 * CS(V) + 0.5 * CSI(V) \tag{2.15}$$

Input Measure 3:

$$\mu_I^3(V) = 0.5 * CS(v) + 0.3 * CSI(v) + 0.2 * TS(v) \tag{2.16}$$

For simplicity and for taking into account both the CS and the industry rankings, I will use the Measure 2 as the main one for our further analysis. I will keep the other two measures for possible robustness checks.

Initial Choice of the Factors

For the ease of presentation, I start the analysis with 8 sources of information (factors). Those are **bm** (book to market ratio), **pcf** (price to cash flow), **pe_exi** (price to earnings, diluted, excluding EI), **roe** (return on equity), **cash_ratio** (cash ratio), **debt_ebidta** (total debt over EBITDA), **size** (natural log of the market capitalization divided by 1000) and **mom** (6-month price momentum). All except the last two were retrieved from the Financial Ratios Suite by WRDS. For further description of the factors I kindly refer the reader to the WRDS manual (see [13]). The momentum factor (also the 3 and 12-month momentum) was constructed as presented in Bali, Engle and Murray (2016, [14]).

Additional factors are constructed based on the appendix of the article by Wermers et al. (2012, see [15]) which are also examined by Jegadeesh et al. (2004, see [16]). Those are **size** (natural log of the market capitalization divided by 1000), **turn** (average daily volume turnover), **sue** (standardized unexpected earnings) and **frev** (analyst forecast revision to price)[3].

In addition to the factors above, for the later sections I will also add the factors **ptb** (price to book ratio) and **divyield** (dividend yield). For the 1-dimensional factor research I will also apply our analysis to the rest ≈ 60 factors from the Financial Ratios Suite by WRDS.

[3]The construction of the factors **mom, turn, frev and sue** were implemented by my student research assistant Gregory Stepaniouk, during the summer of 2016.

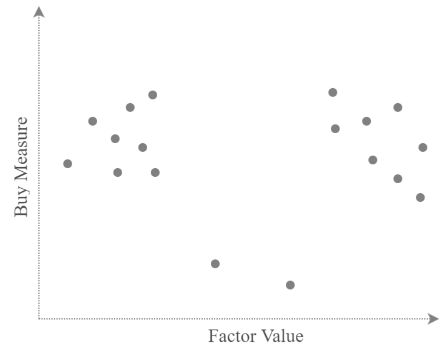


Figure 2.13: Example: Combining Observations

2.3.4 Combining Observations

After we processed the input and output observations, we need to combine them. First, note that in our model the observations of holdings are infrequent. So, we might have a few factor value changes in between two holdings observations. For example, if the holdings observations are quarterly but factors change monthly, then it is not clear due to which of the 3 month's factors the fund made the trade. Besides, we also need to account for the frequency of the observations in a given factor region. For example, if there are many Buy observations near factor value 10 and only 1, but large, Buy observation near factor value 1, then it is likely that the Buy observation at 1 is an outlier or at least not how usually the fund trades. Because of that, we need to combine the observation data in a different way.

A frequently used method for data smoothing and noise reduction is the moving average filter. However, it cannot be applied in our case since the distribution of factor values for the observations might not be uniform. For example, in an observation data as in the figure 2.13, a moving average filter with 5 entries will give large values to the infrequent small observations in the middle.

The smoothing technique I propose would solve the issues described above. Size of the action measure and the number of the actions are both important. I assume an error bound on the recorded factor values for the trades. Let us denote the bound by ϵ_f. For each observation and each factor, I combine it with the observations within ϵ_f distance from the factor. I use sum instead of average so that the number of observations also counts. I exclude the outliers on the left-most and right-most of the region, where it is not possible to put an interval with length ϵ_f. Otherwise, on the left-most and the right-most values of the factor, we would have smaller values just because of the method of construction. I discuss the possible choices of ϵ_f later in the thesis. So, for now I allow the observations to have a fixed small error bound.

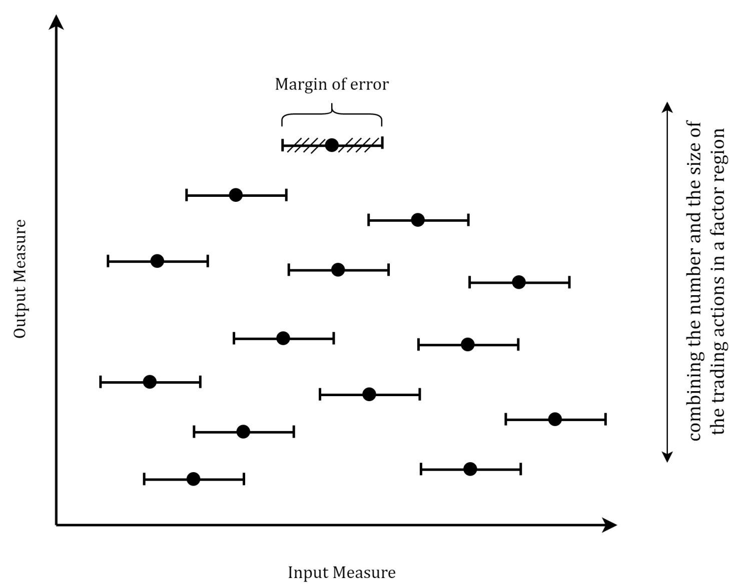


Figure 2.14: Intuition of the Data Smoothing

There are two questions under consideration here: **the likelihood of an action and the size of an action**. What we do here by summing up is combining those together and in that case we have a strong signal.

The only problem with this approach is that some of the funds would have many more observation quarters than the others, so the combined action measure would be larger for funds with more observations. One way to resolve this would be to scale the resulting value by the number of the observation quarters of each fund.

No-Directionality (ND) Trader: So, we have selected a subset of factors and designed input/output observation measures and a framework to combine those observations. Next, I go back to the question of the distribution of the actual input observation measure. In our framework, I define the No-Directionality Trader to be the one who does not have any preference towards any of the factors. The ND Trader has a large amount of money and each time period reinvests all of it randomly in the market. We can think of it as if the ND Trader buys the fraction of the market each time. In that scenario, the amount of money invested in each stock would be proportional to the market capitalization of it. Alternatively, we could have defined the mutual fund ND Trader to be the one who invests proportional to the total mutual fund holding of a stock. However, in that case the information would be biased in case if mutual funds in aggregate have some preferences towards a given factor.

Results: Directionality Curve of the ND-Trader

I check for a relation between the factor value measure and the action of the ND-Trader for 8 factors and stocks in the US markets. I chose a random subset of years: 1997-1998, 2004q1-2004q2 and 2010-2014. For those time periods, I use the market data on the factors and compute the directionality relation of the ND-Trader in different scenarios.

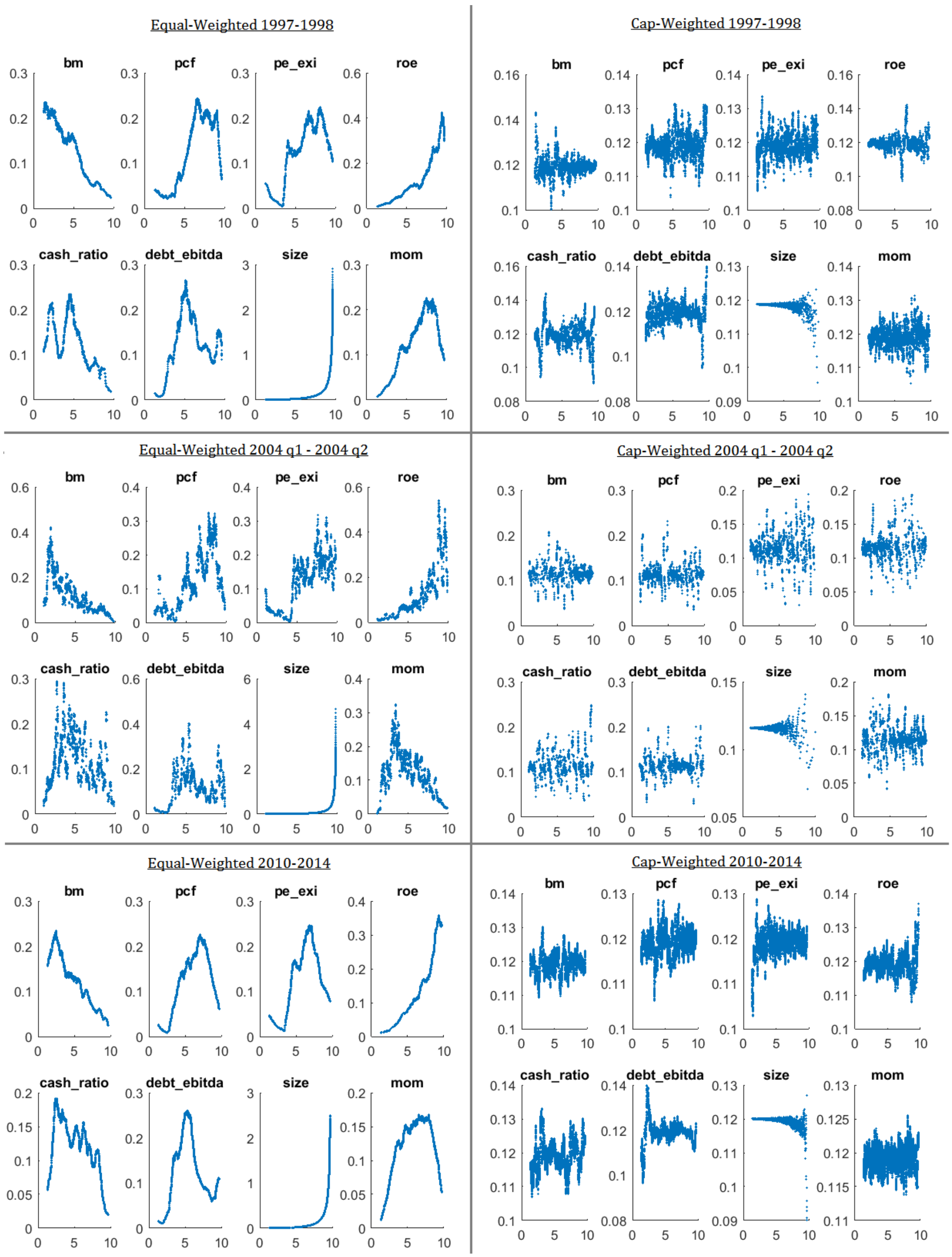
We can see in 2.15 (left side) that an Equal-Weighted ranking of the factors provides biased results for all of the factors in each of the 3 time intervals. We can see that the longer the interval the more biased the results are. However, using the cap-weighted ranking, the directionality of input/output observation is close to a random noise (right side of the figure 2.15).

In addition, I used smoothing with square root or log of the cap size for the robustness comparisons. As I have expected, neither of those make ”pure noise” for the ND-trader, which confirms the intuition of making all the rankings cap-weighted. Thus, cap weighted ranking produces no directionality for a fund with preferences that are matching with the market. Next, I test the individual mutual fund directionality versus the ND-Trader.

2.3.5 Assessing the Absolute Directionality

As we have discussed in the previous section, to avoid biased results I adopt market-cap weighted ranking. I search for directionality on a few randomly selected funds. I compare the funds in our database versus the ND-Trader (which had a random ”noise” form). The analysis is first performed towards a ran-

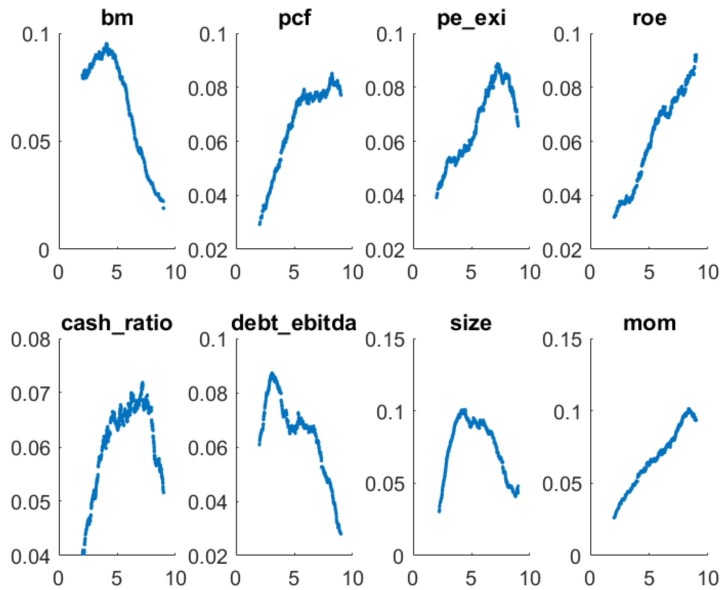
Figure 2.15: ND Trader, (left) Equal-Weighted and (right) Cap-Weighted Ranking



Horizontal Axis: $\mu_i^{eq/cap}(fac)$*, Vertical Axis: "Like/Dislike Measure" for* $\mu_o^2(Buy)$ *(Expected amount of the Buy/Sell trading fraction in an* $\epsilon_f = 0.3$ *region of a point).*

domly selected Growth (see figure 2.16) and Value fund (see figure 2.17).

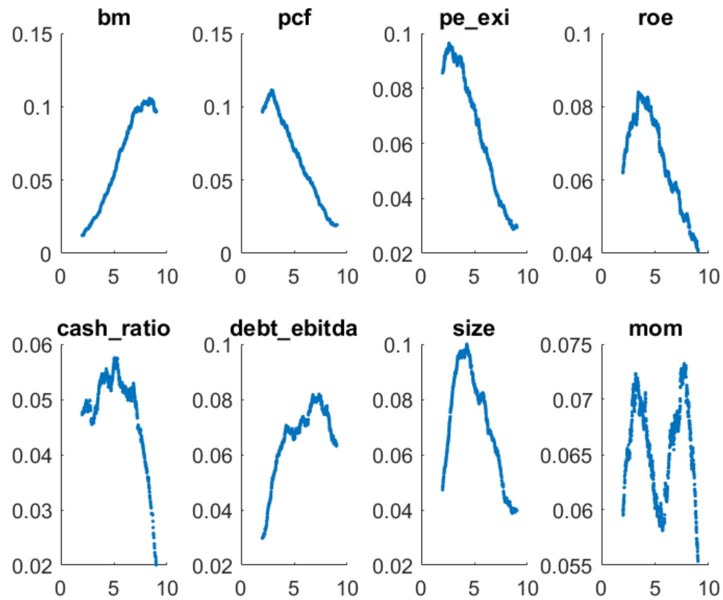
Figure 2.16: Accessor Funds: Growth Portfolio (Buy Signals)



Horizontal Axis: $\mu_i^{cap}(fac)$*, Vertical Axis: "Like/Dislike Measure" for* $\mu_o^2(Buy)$ *(Expected amount of the Buy/Sell trading fraction in an* $\epsilon_f = 0.3$ *region of a point).*

As we could see, both of the funds have "smooth-looking" directionalities towards many factors. Moreover, the growth fund seems to Buy high momentum stocks and the value fund low book to market ratio stocks. Both of those observations are in line with the funds' names. However, if we look at Buy and Sell decisions combined (see figure 2.18), we would see that many of the factors with a clean buy directionality pattern have exactly opposite sell directionality pattern. I classified those type of factors as **Attention Factors**. For the growth fund, though, we could see that the momentum factor is the only one that has both Buy and Sell directionality patterns similar which results in Buy/Sell decisions combined having a directionality. In our terminology, the momentum factor would be classified as an **Action Factor** for that fund. The Value and Income Fund, as we saw, purchases stocks which are value, but the way the

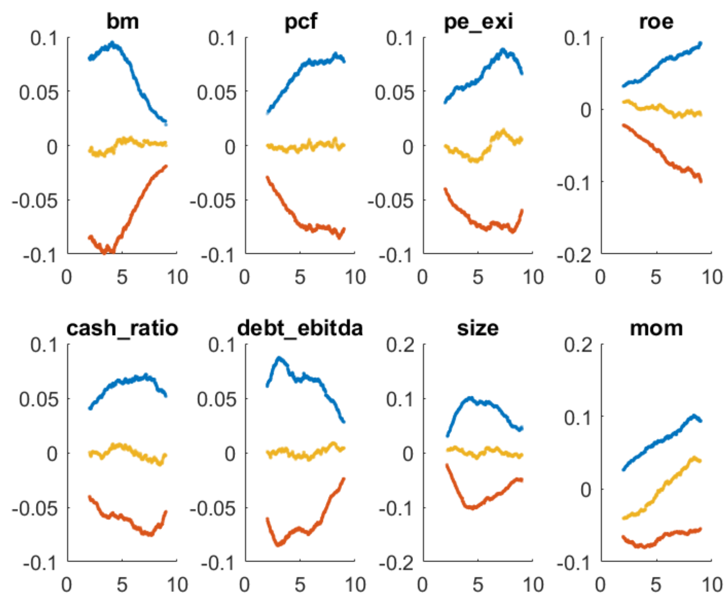
Figure 2.17: Accessor Funds: Value and Income Portfolio (Buy Signals)



Horizontal Axis: $\mu_i^{eq}(fac)$, Vertical Axis: "Like/Dislike Measure" for $\mu_o^2(Buy)$ (Expected amount of the Buy/Sell trading fraction in an $\epsilon_f = 0.3$ region of a point).

fund sells the stocks nullifies the value effect. For that fund, also all except the momentum factor are clearly attention factors. The fund's strategy could be to Buy above-average and below-average momentum stocks and sell when those return to average momentum values.

Figure 2.18: Directionality of Buy, Sell and Combined Actions



Accessor Funds: Growth Portfolio (All Signals)

2.3.6 IVNM Fitting

Our aim is to find the simplest curve that would fit the data. The *IVNM* curves have 0-3 inflection points and I would start fitting I, then V then N and then M with a fixed statistical significance. The problem with V, N and M fitting is correctly identifying the inflection points of the curve. By default, I assume that *I* is the increasing line and *V, N, M* lines are similar to their letter representations. I will denote the flipped up-down versions of *I, V, N, M* by *I′, V′, N′, M′*.

The IVNM Fitting Process

As we have discussed previously, we want to fit a curve as simple as possible. However, funds might have more complex trading strategies and V, N or M-like patterns are not necessarily an overfitting. For both Buy and Sell signals and their combination, I start by implementing a linear regression, that is fitting the

I-curve. Next, to maintain a balance between the simplicity and the goodness of fit, I define adjusted R-squared acceptance levels as follows:

- $I_{\text{adj}R^2} \geq 0.75$ (Linear Regression)
- $V_{\text{adj}R^2} \geq 0.80$ (2nd degree polynomial or 1-dimensional Gaussian)
- $N_{\text{adj}R^2} \geq 0.90$ (3rd degree polynomial)
- $M_{\text{adj}R^2} \geq 0.95$ (4th degree polynomial)

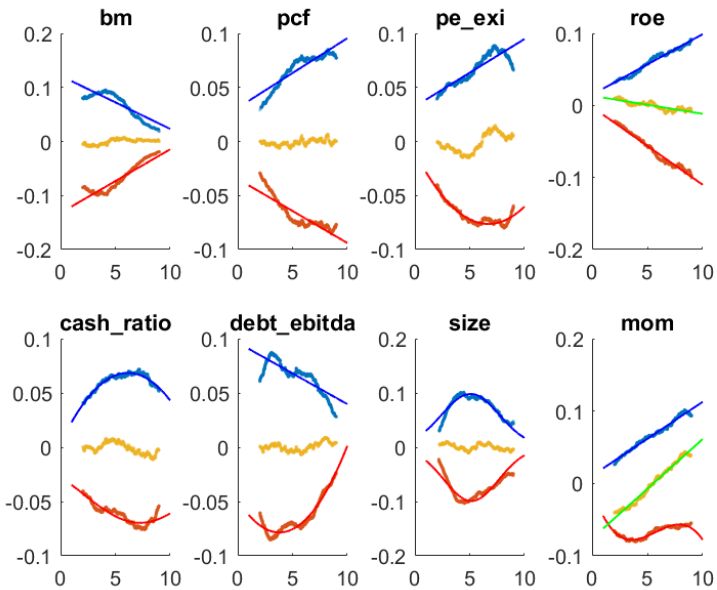
If the requirement is satisfied by a simpler curve, then I move to a more complex one only if the improvement in the adjusted R-squared value is greater than 3/4. The reason we are considering both the 2nd degree polynomial and the Gaussian function is that the latter also has a V-shape and is likely to represent the actions of an investor. In particular, if an investor is not interested in a region, then the B/S signals are probably flat there. The 1-dimensional Gaussian ($ae^{-(x-b)^2/2c^2}$) curve has exactly those properties with flat-looking tails.

Applying the IVNM fit to the fund described in the previous section, I get the figure 2.19.

Results: Buy/Sell Directionalities of the US Funds

As I have previously mentioned, I will only consider funds which have at least 50 buy/sell actions recorded. Implementing the aforementioned methodology on our fund database for years 1994-2014, I get that around the 82.7% (see figure 2.20) of the funds have a Buy or Sell directionality towards at least one out of the 16 factors.

Figure 2.19: IVNM fit of the Directionality of Buy, Sell and Combined Actions



Accessor Funds: Growth Portfolio (All Signals)

Figure 2.20: IVNM Buy/Sell Curve Fitting Statistics

Number* of the Funds which have:	
At least 50 observations	*6650*
Buy directionality towards at least 1 factor	*5500*
Sell directionality towards at least 1 factor	*5050*
B&S directionality towards at least 1 factor	*3500*

****rounded ± 25***

The N and M shapes, as we expected resulted in a very few cases of the funds. On average, the N and M directionality shapes were recorded for the 0.5% of the funds with the largest number being 3.3%. For I and V shapes the results are presented in the figure 2.21. We can note some obvious similarities of preferences from the table. For example, we see that 18% of the funds use bm factor for buying (I-shape) and 15% for selling (I′-shape), which would probably characterize it into an attention factor. The absolute favorite seems to be the turn factor with 47% Buy (I-shape) and 46% (I′-shape). However, turn is measuring the average trade volume and the relation might indicate that many funds trade stocks which are actively traded by the market. This could indicate herding behavior. Momentum factors also seem to have significant directionalities, but only in V and V′ shapes.

Percentage of the funds* which have I,I',V,V' directionality towards a given factor

	bm		pcf		divyield		ptb	
	B	S	B	S	B	S	B	S
/	18%	4%	11%	5%	4%	18%	9%	12%
\	5%	15%	8%	8%	18%	2%	16%	5%
∨	7%	1%	14%	0%	7%	0%	5%	1%
∧	1%	5%	0%	12%	0%	6%	1%	4%

	pe_exi		roe		cash_ratio		debt_ebidta	
	B	S	B	S	B	S	B	S
/	3%	4%	2%	14%	12%	0%	1%	2%
\	6%	2%	15%	3%	1%	12%	1%	1%
∨	23%	0%	8%	0%	8%	0%	4%	0%
∧	0%	18%	0%	7%	0%	7%	1%	5%

	mom3		mom6		mom9		mom12	
	B	S	B	S	B	S	B	S
/	8%	2%	10%	2%	10%	1%	11%	1%
\	4%	2%	6%	2%	7%	2%	7%	3%
∨	23%	0%	27%	0%	28%	0%	25%	0%
∧	0%	19%	0%	24%	0%	25%	0%	24%

	turn		sue		frev		size	
	B	S	B	S	B	S	B	S
/	49%	0%	3%	1%	7%	0%	0%	30%
\	0%	47%	3%	0%	1%	3%	29%	0%
∨	14%	2%	1%	0%	21%	0%	30%	1%
∧	2%	14%	0%	0%	1%	15%	1%	29%

**out of ~5500 funds in the database for whom at least 1 factor can be classified with an IVNM shape*

Figure 2.21: IV Buy/Sell Fitting Shapes by Factor

Results: Classification of Factors into Attention and Action

After performing the IVNM fitting, I search for Action and Attention patterns to classify factors for each fund. Rules for classifying a factor as "Action" for a given fund are:

- A factor has Buy and Sell directionality curves
- The direction of Buy and Sell curves are opposite to each other (e.g. $\{V, V'\}$, $\{I, I'\}$)
- The combined B & S signals: either don't have an acceptable directionality

shape, or the range of the shape is insignificant (less than 20% of the range of the B and S shapes)

The reason I add the 3rd condition is to account for the small variations of the B & S curves due to the noise. Out of approximately 5500 funds I consider, 4900 of them had at least one Action or Attention factor classification (see figure 2.22).

Number * of Funds which have at least:	
1 Action or 1 Attention factor	*4900*
1 Attention factor	*3700*
1 Action factor	*3200*
2 Attention factors	*2500*
1 Action and 1 Attention factors	*2000*
1 Action and 2 Attention factors	*1500*

rounded ± 25

Figure 2.22: Action/Attention Classification Statistics

Analogously, I define rules for classifying a factor as **Action** for a given fund

- Combined B & S signals have an acceptable directionality shape
- The range of the combined B & S directionality shapes is at least 20% of the range of the B and S curves

As we can see in the figure 2.23, the most popular Action factor is mom6 and the next 3 Action factors are mom9, mom3 and mom12. So, around a quarter of the funds in our database rely on the momentum factor while making trading decisions. As was expected, turn and size are the most common Attention factors. Size also is an Action factor for around 12% of the funds in the database, which is probably because factor size is correlated with the price of the stock and Buy/Sell decisions should certainly depend on the stock's price.

Combinations of Action and Attention Factors

Percentage of Funds* for whom a given factor is Action or Attention

Factor:	bm	pcf	divyield	ptb	pe_exi	roe	cash ratio	debt ebidta
Attention	16%	16%	15%	15%	16%	14%	12%	4%
Action	3%	3%	5%	3%	3%	3%	2%	2%

Factor:	mom3	mom6	mom9	mom12	turn	sue	frev	size
Attention	6%	10%	11%	12%	43%	0%	12%	36%
Action	24%	28%	25%	21%	7%	4%	5%	12%

*out of ~5500 funds in the database for whom at least 1 factor can be classified with an IVNM shape

Figure 2.23: Percentage of Action/Attention Classifications for a Factor

Next, I study the popular combinations of Action and Attention factors. The reason is that in the next sections I will analyze the more complex analogue of the directionality curves. For that reason, it is important to know for how many funds a pair of factors can represent Action and/or Attention.

Lastly, I implement the directionality curve analysis for the rest of the 63 factors from the WRDS suite ([13]). We can see that none of those 63 factors have outstanding Action or Attention patterns in comparison to the 16 factors I analyzed before. Hence, I will continue to use the initial 16 factors for the rest of the work.

Change of the Action/Attention Factor Popularity: In order to understand how the popularity of factors as Action or Attention have changed through the last 20 years, I divided it into four 5-year intervals and implemented the Action/Attention classification (see 2.28).

We can note that mom6 factor was not very popular in 1995-2000, but within the next 10 years gained more use as an Action factor, but again started to lose popularity during the . last 5 years[4].

[4]Note that the directionality curves are dependent on the time interval and it is hard to compare 5 and 20 year fits. Because of that, the sum of the percentages in the 5 year windows should not be the same as the percentages in the 20 year window figure.

Attention	bm	pcf	divyield	ptb	pe exi	roe	cash ratio	debt ebidta	mom3	mom6	mom9	mom12	turn	sue	frev	size
Number of the Funds for whom the specified factors are Attention																
bm	875															
pcf	333	899														
divyield	207	362	822													
ptb	623	356	215	817												
pe exi	303	454	316	301	854											
roe	362	272	198	335	306	744										
cash ratio	144	345	287	157	335	157	665									
debt ebidta	50	91	68	47	105	63	117	221								
mom3	117	141	168	114	178	156	124	33	342							
mom6	157	216	239	143	283	211	186	48	226	520						
mom9	160	280	291	162	348	257	255	63	209	366	619					
mom12	158	284	311	166	338	239	253	63	213	340	434	657				
turn	536	664	643	529	602	498	539	135	280	437	522	533	2320			
sue	8	7	5	6	6	5	4	1	5	4	5	8	9	15		
frev	281	248	224	267	293	307	191	54	142	212	262	289	481	9	648	
size	446	482	509	424	489	438	375	112	231	353	413	445	1187	9	418	1952

Figure 2.24: Attention Factors (1994-2014)

2.3.7 Overlapping Effect and the Directionality Maps

Untangling the Overlapping Signals

As we have previously described, analyzing the relation of only one factor with the actions of a fund could result in overlapping signals. In such a case we cannot be sure if the reason of the action was the given factor. Moreover, it is quite logical to expect that the modern funds use more than 1 factor in making their trading decisions. So, trading actions based on different factors would introduce a noise. The theoretical method of resolving the overlap is to consider the cases when all except one factor values are identical and then analyze the fund's trading action in relation to that one factor. In practice, of course, it is

Number of the Funds for whom the specified factors are Action																
bm	150															
pcf	31	160														
divyield	24	26	249													
ptb	50	33	23	143												
pe exi	34	47	37	34	178											
roe	33	24	18	30	31	157										
cash ratio	28	18	15	22	31	16	112									
debt ebidta	21	23	29	15	21	18	20	106								
mom3	28	23	32	32	38	36	6	12	1313							
mom6	36	33	73	39	54	50	5	10	962	1503						
mom9	37	29	83	43	57	47	9	11	778	1037	1349					
mom12	32	34	77	45	61	46	8	11	612	845	930	1131				
turn	21	24	74	15	34	33	9	11	70	108	140	134	396			
sue	17	10	18	17	12	20	5	4	74	96	103	95	25	209		
frev	14	15	13	13	11	21	2	5	99	140	152	155	26	77	262	
size	36	24	74	33	47	45	18	19	109	146	156	165	171	32	38	653
Action	**bm**	**pcf**	**divyield**	**ptb**	**pe exi**	**roe**	**cash ratio**	**debt ebidta**	**mom3**	**mom6**	**mom9**	**mom12**	**turn**	**sue**	**frev**	**size**

Figure 2.25: Action Factors (1994-2014)

very hard to find stocks with all except one value being identical.

To resolve the issue of the overlap, for each factor I compare only the cases when the other factors are ”close enough” to each other. I do that for each fund in the following way:

- We save the coordinates $(v_1, ..., v_d)$ of each stock towards which a given fund had a Buy/Sell action
- For each factor $j \in 1, ..., d$, we consider the $d - 1$ dimensional subspace of the rest of the factors
- We group the Buy/Sell observations into 4-10 clusters
- In each cluster, we analyze the directionality shape of the factor j

Number of the Funds for whom the specified factors are Attention and Action

Attention \ Action	bm	pcf	divyield	ptb	pe_exi	roe	cash ratio	debt ebidta	mom3	mom6	mom9	mom12	turn	sue	frev	size
bm		16	7	12	13	18	4	1	5	7	9	8	32	2	17	25
pcf	19		8	17	13	21	8	4	12	12	13	12	34	0	11	33
divyield	27	27		21	32	43	14	8	13	9	9	19	43	1	29	53
ptb	17	20	4		11	24	2	6	2	7	4	4	30	0	14	24
pe exi	25	20	9	13		23	12	6	8	4	8	10	34	1	22	41
roe	16	21	15	14	13		10	8	1	7	8	10	34	4	19	38
cash ratio	1	1	1	2	3	2		0	2	5	3	5	15	0	1	19
debt ebidta	1	3	0	0	2	2	1		1	0	0	2	11	0	1	15
mom3	322	354	319	323	228	191	233	70		59	123	158	693	8	220	517
mom6	383	374	321	387	227	258	223	64	47		87	131	745	8	247	562
mom9	390	312	269	375	187	253	159	55	58	42		77	623	9	219	486
mom12	350	262	194	322	143	243	114	43	43	38	26		503	5	171	391
turn	61	48	46	39	58	56	36	7	31	41	41	46		3	49	78
sue	65	41	35	58	41	43	16	9	16	26	22	30	85		37	71
frev	67	46	60	58	37	44	18	10	19	22	24	24	127	0		99
size	121	98	70	99	126	125	66	26	57	85	99	97	189	3	102	

Figure 2.26: Action & Attention Factor Pairs (1994-2014)

As an example, I performed the aforementioned steps for the fund with ID $k = 13$ of our database. I took 3 factors: size, bm and pcf for clustering and analyzed the directionality of the mom factor in each cluster. As we can see in the figure 2.30, in different clusters the fund has different directionality graphs. The combined (without clustering) directionality graph of the momentum factor is given in the figure 2.31.

The phenomenon also holds for other randomly selected funds in our database. To conclude, we saw that a fund might have different preferences towards the same factor in different factor regions. Hence, it may not be possible to aggregate the preferences of multiple funds by simply combining their 1-dimensional absolute directionality graphs.

Percentage of Funds for whom a given factor is Action or Attention*

Factor:	dpr	PEG 1yrforward	PEG ltgforward	PEG trailing	CAPEI	evm	pe inc	pe op basic
Attention	13%	8%	15%	11%	21%	16%	14%	12%
Action	2%	2%	2%	2%	4%	2%	3%	3%

Factor:	ps	intcov	intcov ratio	de ratio	debt assets	debt at	debt capital	aftret invcapx
Attention	16%	7%	6%	12%	10%	4%	9%	9%
Action	2%	2%	2%	2%	2%	2%	1%	3%

Factor:	npm	opmad	opmbd	pretret earnat	pretret noa	ptpm	roa	roce
Attention	14%	13%	12%	12%	13%	14%	3%	4%
Action	2%	2%	2%	2%	3%	2%	1%	2%

Factor:	GProf	aftret eq	aftret equity	accrual	rd sale	adv sale	staff sale	cash conversion
Attention	2%	14%	15%	6%	2%	3%	3%	1%
Action	2%	3%	3%	2%	3%	3%	5%	2%

Factor:	quick ratio	profit lct	curr debt	dltt be	int debt	int totdebt	lt debt	lt ppent
Attention	17%	4%	14%	2%	12%	20%	14%	1%
Action	2%	2%	2%	2%	2%	2%	2%	2%

Factor:	rect act	fcf ocf	ocf lct	cash debt	cash lt	cfm	short debt	at turn
Attention	3%	3%	3%	3%	14%	11%	13%	8%
Action	2%	2%	2%	2%	2%	2%	2%	2%

Factor:	pay turn	rect turn	sale equity	sale invcap	sale nwc	capital ratio	equity invcap	debt invcap
Attention	12%	1%	3%	3%	10%	2%	4%	2%
Action	2%	2%	2%	2%	2%	2%	2%	1%

Factor:	pay turn	rect turn	sale equity	sale invcap	sale nwc	capital ratio	equity invcap	debt invcap
Attention	12%	1%	3%	3%	10%	2%	4%	2%
Action	2%	2%	2%	2%	2%	2%	2%	1%

Factor:	pe op dil	gpm	efftax	curr ratio	invt act	inv turn	totdebt invcap
Attention	11%	2%	9%	23%	2%	1%	3%
Action	3%	2%	2%	3%	2%	2%	2%

**out of ~5500 funds in the database for whom at least 1 factor can be classified with an IVNM shape*

Figure 2.27: Results of the IVNM A|A fitting on the remaining 63 factors (1994-2014)

Percentage of Funds for whom a given factor is Action or Attention*

2010-2014								
Factor:	**bm**	**pcf**	**divyield**	**ptb**	**pe_exi**	**roe**	**cash ratio**	**debt ebidta**
Attention	9%	12%	10%	7%	15%	7%	3%	1%
Action	4%	6%	6%	4%	6%	4%	3%	4%
Factor:	**mom3**	**mom6**	**mom9**	**mom12**	**turn**	**sue**	**frev**	**size**
Attention	2%	5%	8%	9%	23%	0%	4%	20%
Action	21%	24%	22%	21%	10%	4%	5%	26%

2005-2009								
Factor:	**bm**	**pcf**	**divyield**	**ptb**	**pe_exi**	**roe**	**cash ratio**	**debt ebidta**
Attention	8%	14%	7%	7%	7%	10%	5%	1%
Action	4%	4%	5%	3%	5%	4%	3%	3%
Factor:	**mom3**	**mom6**	**mom9**	**mom12**	**turn**	**sue**	**frev**	**size**
Attention	4%	6%	6%	7%	35%	0%	4%	23%
Action	24%	28%	25%	20%	12%	4%	6%	24%

2000-2004								
Factor:	**bm**	**pcf**	**divyield**	**ptb**	**pe_exi**	**roe**	**cash ratio**	**debt ebidta**
Attention	7%	5%	2%	7%	9%	7%	5%	2%
Action	4%	3%	4%	3%	3%	3%	2%	2%
Factor:	**mom3**	**mom6**	**mom9**	**mom12**	**turn**	**sue**	**frev**	**size**
Attention	3%	6%	8%	10%	26%	0%	6%	21%
Action	14%	18%	14%	12%	9%	4%	3%	39%

1995-2000								
Factor:	**bm**	**pcf**	**divyield**	**ptb**	**pe_exi**	**roe**	**cash ratio**	**debt ebidta**
Attention	5%	3%	2%	4%	3%	5%	7%	4%
Action	4%	2%	3%	3%	3%	2%	3%	2%
Factor:	**mom3**	**mom6**	**mom9**	**mom12**	**turn**	**sue**	**frev**	**size**
Attention	1%	1%	1%	4%	18%	0%	13%	22%
Action	6%	9%	14%	11%	10%	3%	8%	19%

**out of the funds in the database in the specified time period, s.t. at least 1 factor can be classified with an IVNM shape*

Figure 2.28: Changes in the Popularity of Factors within the Funds

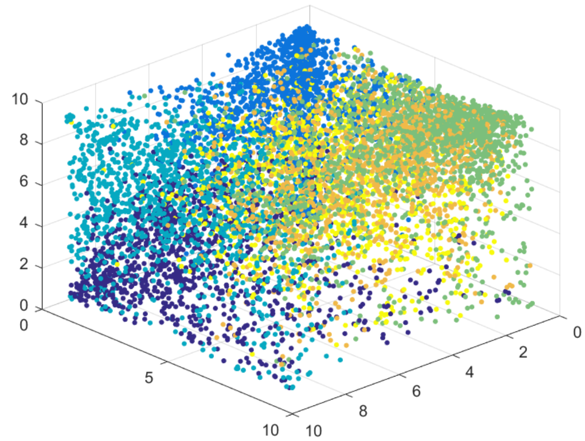


Figure 2.29: Accessor Funds, Inc: Small to Mid Cap Fund; Class A Shares

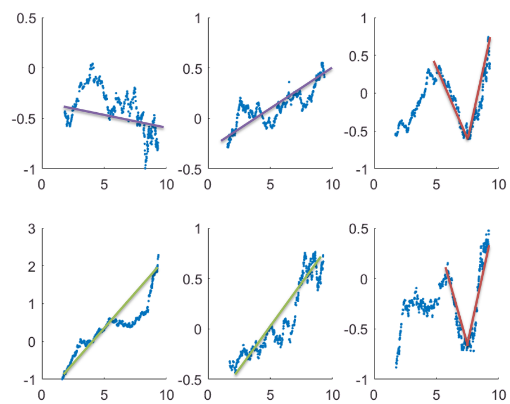


Figure 2.30: Different Directionality in Different Clusters

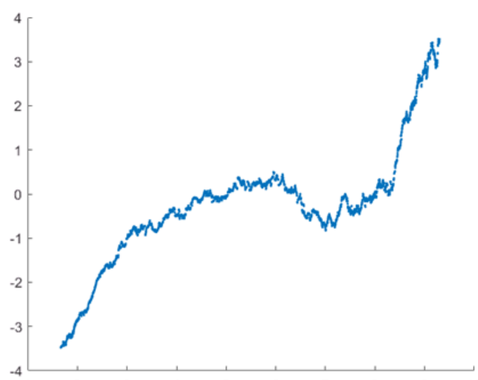


Figure 2.31: Momentum: Combined Directionality

Directionality Maps:

In theory, if we have a very large amount of observations, we could construct analogues of the directionality curves, but in multiple dimensions. In such a case, the overlapping effect will be no more relevant. However, considering too many dimensions is not practical either. The more dimensions we have, the exponentially less observations we get per a discrete lattice of a fixed length. So, the number of factors I will typically choose is less than 5. Moreover, with the 1-dimensional analysis we were able to find the absolute directionalities of the funds towards single factors. For the majority of the funds the directionality is limited to less than 5 factors.

Next, using an analogous approach as in the one-dimensional case, I construct a multi-dimensional directionality curve, which I call the **preference directionality map**. In multiple dimensions, I use the range-search algorithm (see Bentley (1979) or Robinson (1981)) for the error smoothing. The algorithm works in $O(dnlog(n))$ time and runs typically in less than a second (on a desktop PC) if applied to any single fund in our database. After applying the smoothing, I also use a triangulation-based linear interpolant to first interpolate and then extrapolate the values of the map at different points. The directionality map of the k=11 entry of our database is presented in the figure 2.32.

We could see that, although the aforementioned fund prefers low bm as its absolute directionality, as a result of multi-dimensional preference analysis, the factor mom overlaps the effect of the bm factor. Hence, we conclude once more that combining mutual funds by their general trading philosophies could introduce a bias because **funds trade differently in the different factor regions**. To resolve the issue, I will introduce a theoretical framework which will help us to

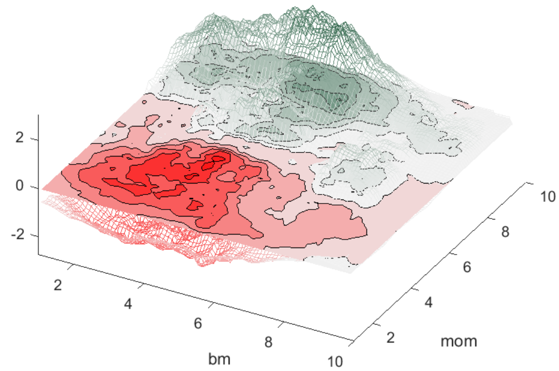


Figure 2.32: Sample Directionality Map - Accessor Funds: Growth Portfolio

understand how to combine the directionality maps of the different funds.

Further, I apply the model to reconstruct the directionality maps of two value and two growth funds. I chose the funds from BlackRock and Fidelity to represent popular US equity mutual funds (see figure: 2.33). We can see that the growth funds (the upper two maps) have completely opposite trading patterns compared to the value funds (lower two maps). Although the directionality maps from different funds look quite different, it is also easy to note the visual similarity within the value funds and the growth fund. The two value funds tend to buy high bm and low mom stocks, which they sell when the mom increases. The two growth funds, on the other hand side, buy high mom stocks and sell them when the mom becomes lower. Thus, using the directionality maps we can clearly see the similarity within the different beliefs on **how to generate profits.** Moreover, we note that two different funds under a larger company could take completely opposite bets in the equity market (counterparty).

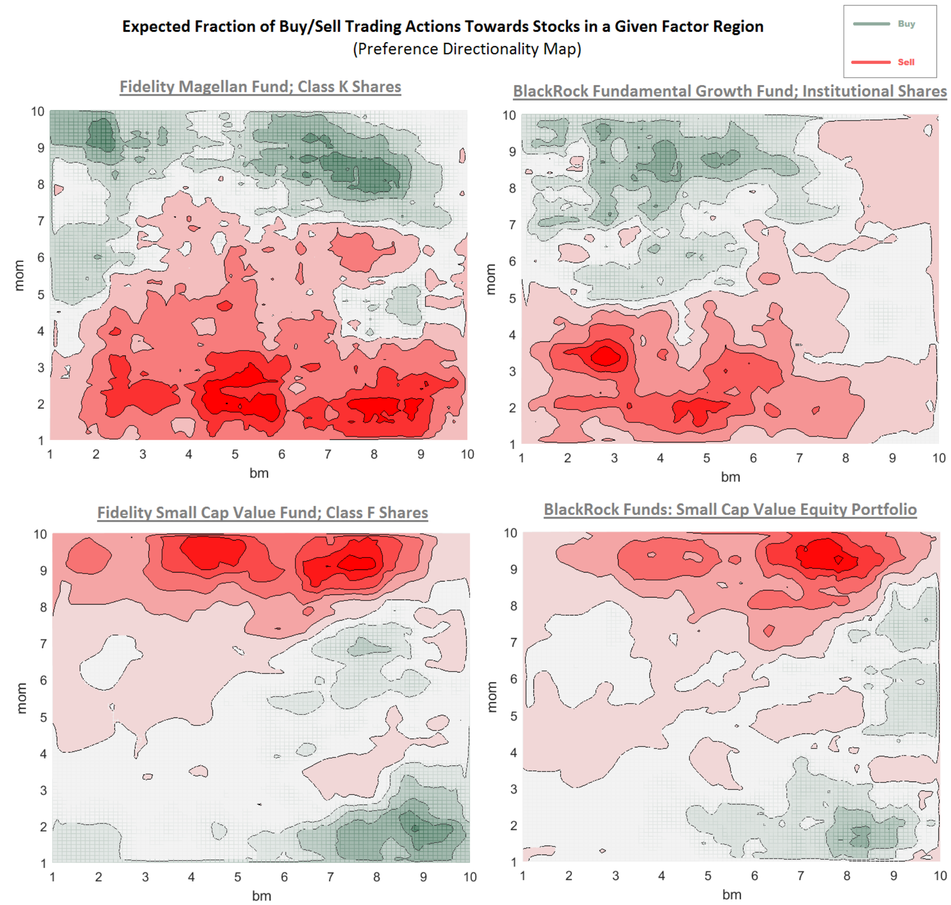


Figure 2.33: Example: Directionality Maps of Value and Growth Funds

Combining the Directionality Maps:

We have seen that there exists some persistence in the trading patterns of the mutual funds. Given the stochastic and sometimes discretionary nature of investing, it is unlikely that one could predict the exact trading decisions of an actively managed fund. However, a natural extension might be to combine the directionality maps of multiple funds. By doing so, the stochastic aspects of investing could "cancel each other out" and predicting the aggregate fund actions could become realistic. I devote the next chapter to discussing this topic.

2.4 The Action|Attention Trader

In this section, I create a framework which will allow us to combine the directionality maps of multiple funds. I start by discussing the motivation behind designing a model, which in addition to the concepts of *Attention and Action* factors, introduces a new one I call an *Interest Group*. After, I create a theoretical framework to model the trading decisions of a given fund. The framework will help us to understand better the intuition behind the model in the previous chapter and how exactly we should combine the directionality maps of the funds. Finally, I will be able to model the collective trading decisions of the US Mutual funds.

2.4.1 The Investment Process of a Fund

As we have found in the previous section, for many funds we can classify the set of factors as Attention or Action. But, apart from probabilistic reasons, the investors might also pay more attention to stocks they hold in their portfolio in comparison to stocks they never held. I will classify the investor-security relationship based on whether the investor currently holds the security, held it at some point in the past or never had the security in her portfolio. I will call the latter the **interest group** of an investor in regard to a security. The three groups of interest are represented in the figure 2.34. Note that by ”holding” I mean that the investor has a position in the security, which might be either long or short.

The issue with the interest group was not relevant in the previous chapter, because we were analyzing the past trading actions only. If we saw a Buy ob-

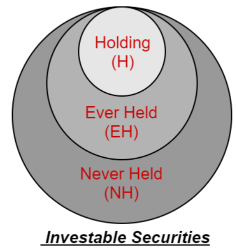


Figure 2.34: The Interest Group

servation, we already knew that the fund chose to act on the given stock. But, even for investors with the same trading philosophies, their actions towards a security might be different depending on their current portfolio holdings. In our economy traders with the same investment philosophy might still choose different assets due to their pre-existing portfolios, the subsets of assets under consideration, or purely by chance. For example, suppose there are two identical copies of the same investor: Copy 1 and Copy 2 with a long-only strategy who manage different portfolios. Assume one of them is holding the security and the other has never held it. Then, if an information event for that security arrives, the Copy 1 investor might act on it since she was keeping the stock under radar. The Copy 2 investor might not be that interested in the stock which she never held. Besides, even if she did, Copy 1 can sell the security whereas Copy 2 cannot.

For the ease of presentation, I denote the interest group of an investor towards an asset with letters H, EH or NH, where:

- H - investor currently holds the asset
- EH - investor was holding the asset in the past but does not hold it now
- NH - investor never held the asset in the past.

Next, I present a heuristic framework of an investment process (see 2.35) which

I will assume the funds follow. Here, there are *Attention* and *Action* factors for a given fund. New information on the finite number of factors is arriving to the investor continuously in time. The fund has some trading philosophy and a pre-existing portfolio.

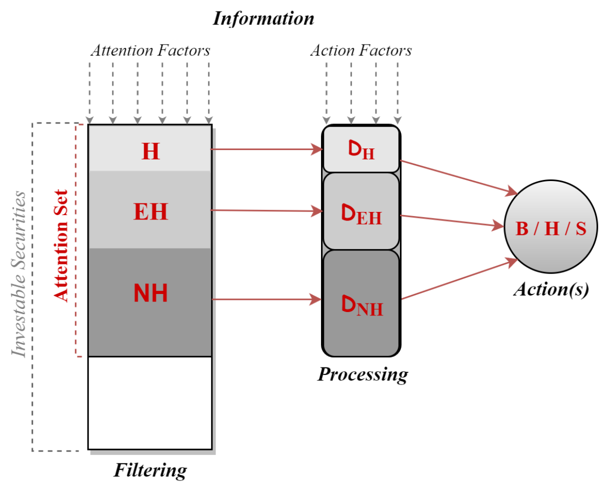


Figure 2.35: The Investment Process

Using the *Attention factors*, from each interest group the fund chooses stocks to consider. Next, out of the stocks under consideration, using *Action factors*, the fund decides which ones to act on and how much to Buy or Sell. Depending on the interest group, the fund's sensitivity towards Action factor changes might be different. Note that simple risk adjustments and re-balancing might be also considered within this model.

Based on the developed intuition, in the next section I construct a formal theoretical framework of an economy which consists of "Action|Attention" traders.

2.4.2 Theoretical Framework: Action|Attention Model

The **information set** is the same as defined in the previous section. That is, there exists a finite but potentially very large number d of factors $f = [f_1, ..., f_d]$, describing each asset and the economy. Time is continuous but the funds holdings are available at discrete times $t = [0, q, 2q, ...]$. For each $j \in [1, ..., d]$ and a fixed fund, the factor f_j belongs to only one of the 3 factor types: ***Attention, Action or Indifference***. For a given fund, I assume that the type of the factor $j \in [1, ..., d]$ does not change. However, the same factor j might be in different types for different funds. To simplify the notation, up until the section where I define the A|A economy, I choose **one of the funds in the market and fix it.**

The **market** has a finite number n of securities. For each security i the vector $V_{it} := \{v_{i,j}(t) \in \mathbb{R},\ j = 1, ..., d\} \in \mathbb{R}^d$ represents the values of factors f at time t. Also, if we drop i then V_t would represent the information of some fixed security at the time t and V would represent the factor values of a fixed security in the discussions where the time is irrelevant. We denote by $\mathcal{V}_t$ the $n \times d$ matrix of vectors V_{it} for $i = 1, ..., n$.

The Incomplete Set of Observations:

In our model the fund's decisions can only be observed discretely but the information can arrive and change continuously in time. Because of that, we cannot be sure exactly at which factor value the trading action was made (see e.g. 2.36). Later in the section we will have the tools to measure that value more accurately. Let us fix one stock, then for a time s in the period $(t, t+q)$, I define

$$\epsilon_t(s) = \frac{\max\{V_u : u \in (t,s)\} - \min\{V_u : u \in (t,s)\}}{2}, \tag{2.17}$$

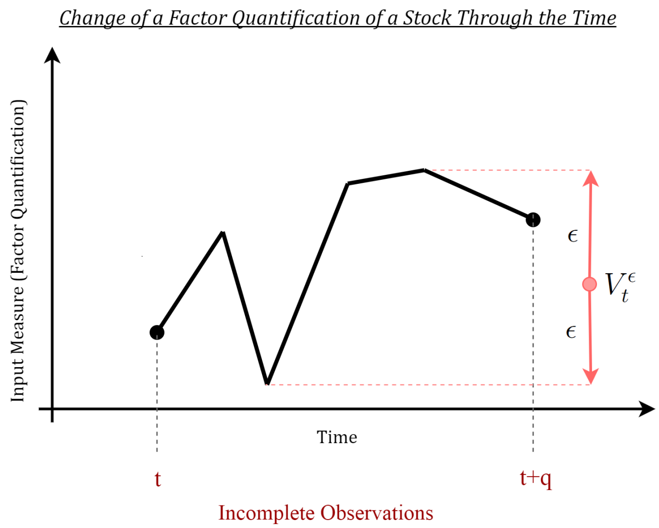


Figure 2.36: Example: Uncertainty in the Factor Value at the Time of the Trading Action

where the max/min are taken point-wise. Next, we denote

$$V_t^\epsilon(s) = \frac{\max\{V_u : u \in (t,s)\} + \min\{V_u : u \in (t,s)\}}{2} \tag{2.18}$$

and let $\epsilon_t := \epsilon_t(t+q)$, $V_t^\epsilon := V_t^\epsilon(t+q)$. I also assume that there exists an $\epsilon < \infty$ such that $\epsilon_t < \epsilon$. The idea behind such a definition, is that the trading decisions are likely to be caused by extremes in the information values in the interval. However, at this point we cannot know by which of the extremes it was caused by and thus we have to record the **range of the extremes** ϵ_t and the **average of the extremes** V_t^ϵ. We have defined $V_t^\epsilon(s)$ and $\epsilon_t(s)$ so that in the next sections we have the framework to predict the trading actions in the period $(t, t+q]$ based on the information at time $s \in (t, t+q]$. The closer we choose s to $t+q$, the better $V_t^\epsilon(s)$ and $\epsilon_t(s)$ would approximate V_t^ϵ and ϵ_t. However, if we choose s to be very close to $t+q$, the length of the prediction $(s, t+q]$ would decrease and the prediction would become less meaningful.

To simplify the presentation of the material, I assume that ϵ_t is a constant and is equal to ϵ. On practice, the values of ϵ_t are easy to compute and only complicate the notation. So, for now I assume that for each quarter the information about the securities in the market is perceived by the **"average" factor values** in the region $(t, t+q)$ denoted by V_t^{ϵ} which has a range $V_t^{\epsilon} \pm \epsilon$.

The assumptions on fund's trading which I impose are very weak and the only assumption is that funds do not explicitly copy each others' trading decisions. I will give a formal definition to such type of trading and call it the Action|Attention (A|A) Trading Process.

Description of The Trading Process:

The fund's next trading decision in regards to a security only depends on:

1. security's current and historical factor values
2. factor values of the other securities in the market
3. fund's current and previous holdings history
4. fund's investment philosophy (to be defined in the next section)

The Extended A|A Trader:

First, I introduce a one-period extended A|A model. Suppose we are at a time T, which is a multiple of q. The discrete set of information produced by the securities during the **"learning period"** $[0, 1, ..., T]$ is saved in the $T \times n \times d$ dimensional array $\mathcal{V} := [V_0, ..., V_T]$. At the time T, each fund has an initial portfolio $P_T \in \mathbb{R}^n$ and an interest group (history) vector $\mathcal{I}_T \in \{H, EH, NH\}^n$. The interest group vector is formed by recording which stocks the fund has held by the time T.

Definition 2.4.1 (The Extended A|A Process)**.** *We call* $\{D_B^*, D_S^*\}$ *an extended*

Action$|$Attention process, if D_B^*, D_S^* are $[0,\infty)^n$ valued random fields on the space $\mathbb{R}^{(T+1)\times n\times d}\times\{H,EH,NH\}^n$.

D_B^* and D_S^* represent the (time T) dollar amounts that will be invested/liquidated (if any) in each of the n assets within $(T,T+q]$. We can assume that the random fields are isotropic for simplicity. That is, the correlation between the random variables is only dependent on their spatial distance from each other. I would refer to D_B^* and D_S^* as the ***extended Buy and Sell decision processes***.

We call a fund to be Extended A|A Trader, if it has an Extended A|A trading process $\{D_B^*, D_S^*\}$ and the dollar amount which the fund invests/liquidates in the securities $1,...,n$ between $(T,T+q]$ is:

$$D_B^*(\mathcal{V}_T^\epsilon,\mathcal{V},\mathcal{I}_T) \text{ and } D_S^*(\mathcal{V}_T^\epsilon,\mathcal{V},\mathcal{I}_T). \tag{2.19}$$

Next, I discuss how to reduce the aforementioned model to make it tractable.

2.4.3 The Reduced Model: Input and Output Transformations

Making investment decisions might depend not just on the information about one given stock, but also on its comparison to the other stocks in the market or the stock's past. In other words, investing decisions are usually based on the information about more than one security. Because of that, in the extended definition of the A |A model, the decision function was dependent on all the relevant factor values in the market up to time T: $\mathcal{V} \in \mathbb{R}^{T\times n\times d}$. However, modeling the fund trading in such a manner requires a very large number of observations and is not practical. To resolve the issue, I will define measures which will reduce the problem but keep the ***relational nature*** of the trading in our model.

I introduce an Input Transformation to pre-process all the necessary comparisons and rankings, so that we are able to simplify the decision function. The aim is to make the decision function not be explicitly dependent on the whole market information from $\mathbb{R}^{T\times n\times d}$.

Input Transformation:

We call the collection of functions $\mu_{inp} = \{\mu^1_{inp}, ..., \mu^n_{inp}\}$ an Input Transformation, if for each security $i = 1, ..., n$:

$$\mu^i_{inp} : \mathbb{R}^{(T+1)\times n\times d} \times \{H, EH, NH\}^n \to [1, 10] \times \{H, EH, NH\}. \tag{2.20}$$

If we apply the μ_{inp} on $\{\mathcal{V}^{\epsilon}_T, \mathcal{V}, \mathcal{I}_T\}$, the resulting would be a "ranking" for each of the securities, coupled with information of (only) that security's interest group. There are no restrictions on the "rankings" process and it could be e.g. within the fund's portfolio, within the stock exchange or the industry. Thus, the transformation does all the necessary comparisons and helps us to **separate** the problem so that we can consider each stock ”independently”.

As we have previously discussed, the trading action observations might be noisy and do not account for inflows/outflows or risk adjustment of the fund. So, the actual trading patterns of the fund might be not in the raw dollar amounts of the trades. Thus, to further simplify the decision function, I no more assume that its output is the dollar amount of the trade. Instead, I will apply an output transformation to the trading observations to reconstruct the appropriate decision function. Later, I will apply an inverse output transformation to the decision function to track the evolution of the fund’s portfolio.

Output Transformation (Applied to Observations):

$$\mu_o : \mathbb{R}^n \times \mathbb{R}^{T\times n} \to \mathbb{R}^n. \tag{2.21}$$

Inverse Output Transformation (Applied to The Decision Functions):

$$\mu_o' : \mathbb{R}^n \to \mathbb{R}^n. \tag{2.22}$$

An example of an output transformation is dividing the time $T + q$ holding changes of a fund in a given stock by its time T holdings. An inverse output transformation in that case would be to multiply the time $T + q$ output of a decision function by the time T holdings. I assume that the $\mathcal{V} \in \mathbb{R}^{T\times n}$ (history) is implicitly incorporated in the inverse output transformation.

Information in the Reduced Framework:

Next, I assume that an appropriate Input Transformation has been already applied to the information set. So, in the reduced framework, for each security the input information has a form $[1,10]^d \times \{H, EH, NH\}$. As a result: $\mathcal{V}_t \in [1,10]^{n\times d}$, $\mathcal{V}_t^\epsilon \in [1,10]^{n\times d}$ and $V_{it} \in [1,10]^d$.

I proceed by introducing the concept of an **Investment Philosophy Type**, which will help us to reduce the dimension of the problem even more.

Definition 2.4.2 (The Investment Philosophy Type). *We call $\Pi_{\pi_1,\ldots,\pi_d}$ with $\pi_i \in \{-1,0,1\}_{i=1,..,d}$ an investment philosophy type if $\Pi_{\pi_1,\ldots,\pi_d}$ is a projection from $[1,10]^d$, taking the dimensions $i \in [1,...,d]$ for which $\pi_i \neq 0$ to $[1,10]$.*

The values of $\pi_i : \{-1,0,1\}$ represent whether the factor i is **Attention**, **Indifference** or **Action**, in that order. In some sense, it is used to reduce the dimension of the input information to remove the indifference factors. Denote $d_+ := \sum_{i=1}^d \mathbf{1}_{\pi_i>0}$ (action dimension), $d_- := \sum_{i=1}^d \mathbf{1}_{\pi_i<0}$ (attention dimension) and $d_\Pi := d_+ + d_-$. The latter would represent the dimensionality of the information set for an investor with philosophy type $\Pi_{\pi_1,\ldots,\pi_d}$ (Π for short). So, Π is a projection from $[1,10]^d$ onto $[1,10]^{d_-} \times [1,10]^{d_+}$. As a result, I am able to define a more tractable version of the **A|A process**. Next, assume the notation

$$\star_{<0} = \begin{cases} = |\star|, & \text{if } \star < 0 \\ = 0, & \text{if } \star \geq 0 \end{cases}$$

and an analogous one for $\star_{>0}$, where all the operations are done element-wise.

Definition 2.4.3 (The A|A Process). *We call $\{D, \Pi\}$ an Action|Attention process, if Π is a philosophy type and for $I \in \{H, EH, NH\}$, $D_B(\cdot, I) := D(\cdot, I)_{>0}$ and $D_S(\cdot, I) := D(\cdot, I)_{<0}$ are $[0,\infty)$ valued random fields on the space $[1,10]^{d_\Pi}$.*

I call $D_B(\cdot, I)$ and $D_S(\cdot, I)$ the **Buy and Sell decision functions.** However, the decision function has a different meaning in the reduced framework. The reason is that to have a **tractable decision function**, we need to make it dependent only on the "persistent patterns" of the fund trading. For example, the amount of Buy might be dependent on the total size of the fund. The latter changes depending on inflows and many other external factors. The amount of sell, on the other hand, is dependent on the amount of holdings of a given security in the previous quarter. So, we need an interpretation of the decision function which is independent from the aforementioned external factors. I introduce two interpretations of the decision function. I describe the first interpretation below for the Sell decision function $D_S(\cdot, I)$ and the one for $D_B(\cdot, I)$ is analogous.

Interpretation of the Decision Function:

$D_S(V_{it}^{\epsilon}, I)$ is the fraction of the fund's time t total holdings which the fund will liquidate (short) in the stock i within $(t, t+q]$, where:

- i is in the interest group $I \in \{H, EH, NH\}$ at time t
- i has a factor decomposition V_i^{ϵ} which was estimated in $(t, t+q]$ with uncertainty $\pm\epsilon$.

While computing the aforementioned fraction, I divide the time t price of the holdings changes within $(t, t+q]$ by the time t total holdings price of the fund. I do so to make the decision function independent from the realized price changes. Note that in this case the smaller positions of the fund would receive a much smaller weight. To mitigate that issue, for the case when $I = H$, I will define an alternative Decision Function.

Interpretation of the Alternative Decision Function:

We denote by $D_S^H(V_{it}^{\epsilon})$ the fraction of the fund's time t holdings of the stock i

which the fund will liquidate within $(t, t+q]$, where:

- i is in the interest group $I = H$ at time t
- i has a factor decomposition V_{it}^{ϵ} which was estimated in $(t, t+q]$ with uncertainty $\pm\epsilon$.

I assume that D_S^H and D_B^H are also $[0, \infty)$ valued random fields on $[1, 10]^{d_\Pi}$. I denote $D_S^H(V_{it}^{\epsilon}) := D_S(V_{it}^{\epsilon})$, so if the decision function has one entry it is assumed to be the alternative decision function. Note that the decision function defined in this way is only meaningful for the cases when the time t holdings of the fund in the stock i are non-zero. Otherwise, the value would be infinity. However, in a framework with no shorting a Sell requires $I = H$, so the aforementioned definition might be used.

For the ease of notation, I accumulate the decision functions $D_S(V_{it}^{\epsilon}, I_{it})$ for $i = 1, ..., n$ into a vector of the decision functions:

$$\mathcal{D}_S(\mathcal{V}_t^{\epsilon}, \mathcal{I}_t) := [D_S(V_{1t}^{\epsilon}, I_{1t}), ..., D_S(V_{nt}^{\epsilon}, I_{nt})]. \tag{2.23}$$

Independent from the interpretation of the decision function, I assume that μ_o', the inverse output measure, transforms the output of the decision function to represent the time t dollar amount of the trading action. In some case μ_o' might change depending on t or other variables and we think of it as a rule that would ease the notation, rather than a well-defined static function. Similarly, for D_B and D_S the rule μ_o' might act differently. Following our notation, the time t dollar amount of Buy/Sell/combined of the fund within $(t, t+q]$ is equal to:

$$\mu_o'(\mathcal{D}_B(\mathcal{V}_t^{\epsilon}, \mathcal{I}_t)),\ \mu_o'(\mathcal{D}_S(\mathcal{V}_t^{\epsilon}, \mathcal{I}_t)) \text{ and} \tag{2.24}$$

$$\mu_o'(\mathcal{D}(\mathcal{V}_t^{\epsilon},\mathcal{I}_t)) = \mu_o'(\mathcal{D}_B(\mathcal{V}_t^{\epsilon},\mathcal{I}_t)) - \ \mu_o'(\mathcal{D}_S(\mathcal{V}_t^{\epsilon},\mathcal{I}_t)) \tag{2.25}$$

The A|A Trader:

Now we are ready to introduce the A|A trader into our model. Assume that at the time 0, our fund has an initial portfolio $P_0 \in \mathbb{R}^n$ and an interest group (history) vector $\mathcal{I}_0 \in \{H, EH, NH\}^n$. I denote the portfolio holdings (**in number of shares**) of the fund at time $t = [0, q, 2q, ...]$ by $P_t \in \mathbb{R}^n$ and I assume that all the necessary share adjustment have been performed.

Definition 2.4.4 (The A|A Trader)**.** *We call a fund to be an A|A Trader, if it follows an $A|A$ trading process $\{D, \Pi\}$ coupled with an inverse output transformation μ_o', such that the time t dollar amount which the fund invests/liquidates in the stocks $i = 1, ..., n$ with an interest set $\mathcal{I}_t$ between $(t, t+q]$ is equal to:*

$$\mu_o'(\mathcal{D}(\Pi(\mathcal{V}_t^{\epsilon}),\mathcal{I}_t)). \tag{2.26}$$

Denote the per-share price of the stock i at time t by the vector $M_t = [m_{t1}, ..., m_{tn}]$. Also, I denote by $R_{t_1,t_2} = \{r_1(t_1,t_2), ..., r_n(t_1,t_2)\}$ the return of the securities in the time intervals $(t_1, t_2]$ (including all the necessary adjustments). In particular, $R_t := R_{t-1,t}$ and $R_{tq} := R_{t,t+q}$.

One can note that using symbols H, EH, NH, we can represent the interest level of the fund $\mathcal{I}_t \in \{H, EH, NH\}^n$ for $t > 0$ with P_t:

$$I_{it} = [\mathrm{H}\ \mathbb{I}_{\{P_t^i \neq 0\}} + \mathrm{EH}\ \mathbb{I}_{\{P_t^i = 0,\ P_s^i \neq 0 : s < t\}} + \mathrm{NH}\ \mathbb{I}_{\{P_s^i \neq 0 : s \leq t\}}]_{\ i=1,...,n}. \tag{2.27}$$

The Evolution of the A|A's Portfolio:

Next, it is easy to see that in our framework for the fund holdings at time $t =$

$[q, 2q, ...]$ the following equation holds:

$$M_t \circ (P_{t+q} - P_t) = \mu_0'(\mathcal{D}(\Pi(\mathcal{V}_t^\epsilon), \mathcal{I}_t)), \tag{2.28}$$

where $\circ$ is the element-wise product of vectors (Hadamard product). So, if we assume that the division is element-wise for vectors/matrices, then the fund's portfolio evolves according to the equation:

$$P_{t+q} = P_t + \frac{\mu_0'(\mathcal{D}(\Pi(\mathcal{V}_t^\epsilon), \mathcal{I}_t))}{M_t}. \tag{2.29}$$

Conversely, I can represent the values of the Buy/Sell decision functions using the changes in portfolio holdings as follows:

$$\mathcal{D}_B(\mathcal{V}_t^\epsilon, \mathcal{I}_t) = \mu_o(M_t \circ (P_{t+q} - P_t))_{>0}, \tag{2.30}$$

$$\mathcal{D}_S(\mathcal{V}_t^\epsilon, \mathcal{I}_t) = \mu_o(M_t \circ (P_{t+q} - P_t))_{<0}, \tag{2.31}$$

Hence, in case of the first and the alternative interpretations of the decision function, we accordingly have:

$$\mu_o'(\mathcal{D}(\Pi(\mathcal{V}_t^\epsilon), \mathcal{I}_t) = (\mathcal{D}(\Pi(\mathcal{V}_t^\epsilon), \mathcal{I}_t) \circ M_t)\mathbf{1}P_t \tag{2.32}$$

$$\mu_o'(\mathcal{D}(\Pi(\mathcal{V}_t^\epsilon))) = \mathcal{D}(\Pi(\mathcal{V}_t^\epsilon)) \circ M_t \circ P_t, \tag{2.33}$$

where $\mathbf{1}$ is the vector of n ones.

An Ad-Hoc Method of Approximating the Future Information: For one fixed stock, note that V_t^ϵ represents the changes in the information within $(t, t+q]$ and is not known before the time $t + q$. However, if we choose $s \in (t, t + q]$, then by increasing ϵ to some ϵ', we can make sure that $[V_t^{\epsilon'}(s) - \epsilon', V_t^{\epsilon'}(s) + \epsilon']$ (element-wise) includes the sets $[V_t^\epsilon - \epsilon, V_t^\epsilon + \epsilon]$. Studying each source of information in more details, we could potentially assume a stochastic evolution function on $V_t^\epsilon(s)$ and further approximate V_t^ϵ by $V_t^{\epsilon'}(s)$. However, for simplicity, I discuss the case when s is sufficiently close to $t + q$ so that I will assume $V_t^\epsilon(s) \approx V_t^\epsilon$.

The A|A Economy

Next, I construct an economy of a finite number of A|A Traders. I start by describing the assumptions in the A|A economy. I assume that the market consists of k investors who follow an A|A process $\{D^j, \Pi^j\}$ with an initial states $(P_0^j, \mathcal{I}_0^j)$. I also assume that the decision functions D_B^j and D_S^j are independent.

The framework is the same as defined earlier. So, the information changes are exogenous and saved in the matrix $\mathcal{V}_t \in \mathbb{R}^{d\times n}$ for the time t. The cumulative shares which the funds hold in a given stock is represented by the vector $P_t^* = [P_1^*, ..., P_n^*]$. Hence, the cumulative fund holdings evolve according to

$$P_{t+q}^* = P_t^* + \frac{\sum_{j=1}^k \mu_0'(\mathcal{D}^j(\Pi^j(V_t^\epsilon), \mathcal{I}_t)}{M_t} \tag{2.34}$$

In particular, we can rewrite it as:

$$\mu_o(P_{i(t+q)}^*) = \mu_o(P_{it}^*) + \frac{\sum_{j=1}^k D^j(\Pi^j(V_{it}^\epsilon), I_t)}{M_t}, \tag{2.35}$$

which we could further expand to:

$$\mu_o(P_{i(t+q)}^*) = \mu_o(P_{it}^*) + \frac{\sum_{j=1}^k D_B^j(\Pi^j(V_{it}^\epsilon), I_t)}{M_t} - \frac{\sum_{j=1}^k D_S^j(\Pi^j(V_{it}^\epsilon), I_t)}{M_t} \tag{2.36}$$

Note that D^j are independent, which means that the trading decisions are made independently. The funds could still herd when the stock enters an "opinionated region" where funds have strong opinions to buy/sell it. So, in this model the different investment philosophies might have common factors. However, each investment philosophy has a unique set of beliefs about what kind of factor decomposition of a security has higher chances to generate profits. So, investors of different philosophies might find different decompositions more appealing.

Next, if we assume the alternative interpretation of the preference function, the cumulative demand of the market towards the asset i is equal to:

$$P^*_{i(t+q)} - P^*_{it} = \frac{\sum_{j=1}^{k} D^j(\Pi^j(V^{\epsilon}_{it}), I_t) P^j_{it}}{M_t}. \tag{2.37}$$

I assume that the demand from the institutional investors is satisfied by the remaining market participants and the process is exogenous. I will use the aforementioned framework for predicting the cumulative fund actions. But before doing so, I also introduce the preference function and discuss how to compute its expectation.

2.4.4 The Preference Function

In this section, I formally introduce the **Preference Function**. It was informally used before, to construct the directionality graphs and maps. Let $V^\epsilon \in [1, 10]^{d_\Pi}$ be an ϵ region around the factor quantity V. I will define a function which describes the fund's preferences towards the factor regions rather than towards specific stocks.

Definition 2.4.5 (The Preference Function)**.** *For a philosophy type* Π*, we call the function* $\mathcal{P}$ *a preference function, if* $\mathcal{P}_B := \mathcal{P}_{>0}$ *and* $\mathcal{P}_S := \mathcal{P}_{<0}$ *are* $\mathbb{R}^{\geq 0}$ *valued random fields on the space* $[1, 10]^{d_\Pi}$*.*

Analogous to the case with the decision functions, I assume that there exists an inverse output transformation μ_o' such that $\mu_o'(\mathcal{P}(V_{it}^\epsilon))$ represents the time t dollar price of the cumulative share changes of the fund in the factor region V_{it}^ϵ. I will give two interpretations to the preference functions, which I will apply in different scenarios. Suppose that a given fund liquidated a fraction of its position in the stock i during $(t, t+q]$. Next, assume that the stock's average of the extremes V_{it}^ϵ is in the ϵ region of a factor quantity V. Then, I call the stock i to be **liquidated in the region** V^ϵ. Next, I introduce the interpretations of the function $\mathcal{P}_S$ (the ones for $\mathcal{P}_B$ are defined in an analogous way).

Interpretation of the Preference Function:

$\mathcal{P}_S(V_t^\epsilon)$ is the fraction of the fund's sell within the region V_t^ϵ over the fund's total holdings at the time t.

Alternative Interpretation of the Preference Function:

$\mathcal{P}_S(V_t^\epsilon)$ is the fraction of the fund's sell within the region V_t^ϵ compared to the fund's total sell during $(t, t+q]$.

Note that for this case, the output transformation is the μ_O^2 defined in the previous section. In both of the cases, I take the time t prices of the securities.

The function $\mathcal{P}$ represents the Buy/Sell trading based on a factor region rather a specific stock. Both theoretically and in practice, it gives the idea on the preferences of the fund towards a factor region. In practice $\mathcal{P}_S(V_t^\epsilon)$ might be easier to fit to the data, since it is the fraction of the sell within the sell signals i.e. it is a sell **"given sell"**.

Next, I introduce some notations.

- $S_t -$ set of all the sell actions in $(t, t+q]$
- $S_t(V_t^\epsilon) -$ the subset of S_t where the average of the sell actions are within the ϵ region of the factor $V \in [1, 10]^{d_\Pi}$
- $\mathbb{S}_{tj}(V_t^\epsilon) -$ the amount of j^{th} sell action in $S_t(V_t^\epsilon)$.

Expectation of the Preference Function:

So, per the alternative interpretation, $\mathcal{P}_S(V_t^\epsilon)$ is the fraction of the total dollar amount (per all stocks) liquidated in the given factor region V_t^ϵ. Hence, we can write the expected amount of Sell in a factor region as

$$E(\mathcal{P}_S(V_t^\epsilon)) = E_S(\text{all stocks } \in V_t^\epsilon) = E_S(\text{all stocks} \in V_t^\epsilon | \text{stock was sold}). \quad (2.38)$$

which we can approximate by

$$E(\mathcal{P}(V_t^\epsilon)) \approx \sum_{j=1}^{\#S_t(V_t^\epsilon)} \mu_o(\mathbb{S}_{tj}(V_t^\epsilon)) \quad (2.39)$$

Note that the equation above has the same form as the error-smoothing procedure of the previous section. So, we can see that (for the output measure μ_O^2), the directionality graphs and maps represented the expected fraction of the Buy/Sell action in a given factor region.

2.4.5 Expectation of the Decision Function

In this section, I discuss how to interpret and reconstruct the expectations of the decision functions from a discrete set of observations.

Definition 2.4.6 (Expectation of the Decision Function)**.** *We denote by m_B (m_S) the expectations of the decision function D_B (D_S).*

Note that m_B and m_S are deterministic functions defined on $[1, 10]^{d_\Pi}$. First, for the ease of presentation, I assume that Π has been applied to the input data V and I suppress writing V as an argument. In particular, I assume that $d_\Pi = d$ for this section.

Interpretation of the Expectations:

By the definition, $\mu'_o(D_S(V_t^\epsilon, I))$ is the dollar amount of a liquidation per 1 given stock within the interest group I, if the stock was in the region V_t^ϵ during $(t, t+q]$. Thus, its mean is the expected amount of Sell per 1 stock with parameters $\{V_t^\epsilon, I\}$. We can informally represent the aforementioned expectation as:

$$E_S(1 \text{ stock } \in V_t^\epsilon) = E_S(1 \text{ stock} \in V_t^\epsilon | \text{stock was sold}) P(\text{stock was sold}). \quad (2.40)$$

It is easy to see that the expectation depends on the probability that the fund will liquidate a given stock. That probability varies significantly in between the interest groups H, EH and NH. Moreover, for an environment with no shorting, that probability is obviously 0 for the Sell action in EH and NH. Because of that, I will discuss and fit the decision functions in the different interest groups, separately. After introducing some definitions, I will first consider the case when the interest group is H.

Generating the Trading Observations:

We generate trading observations from the evolution of the portfolio holding process and introduce new definitions. To keep the notation simple, I assume that all the stock shares are adjusted for share distributions. The process of adjusting for share distributions on practice is discussed in the previous chapter. Let us denote the **changes of the holdings** of a fund from t to $t+q$ by $C_t := P_{t+q} - P_t$. As a result, indexes $i \in [1, ..., n]$ such that $C_t^i > 0$ would indicate a Buy action, $C_t^i < 0$ would indicate a Sell action and cases when $C_t^i = 0$ would indicate a Hold decision (no change) for the stock i in the fund's portfolio.

Next, I introduce some definitions for Buy (B), Sell (S) and Hold (H) actions within the interest groups I={H, EH, NH}. For simplicity, I will write them only for S and the definitions for B and H are analogous.

- N_t − the set of stocks held by the fund at time t.
- $N_t(V_t^\epsilon)$ − the subset of stocks in N_t which are in the region V_t^ϵ at the time $t+q$.
- S_t^I − set of all the sell actions in $(t, t+q]$ within the interest group $I \in \{H, EH, NH\}$
- $S_t^I(V_t^\epsilon)$ − the subset of S_t^I, such that the average of the factor quantifications of the stock is within the ϵ region of the factor $V \in [1, 10]^{d_\Pi}$
- $\mathbb{S}_{tj}^I(V_t^\epsilon)$ − the amount of j^{th} sell action in $S_t^I(V_t^\epsilon)$.

Estimating the Expectations of the Decision Function

The Interest Group H:

We don't have access to the shorting data in our databases. As a result, the sell decisions can act only within the interest group H. For the case when the interest group I is H, I omit writing the superscript, e.g. $S_t^H = S_t$.

Next, note that we can re-write the equation 2.40 as:

$$E(D_S(V_t^\epsilon, I)) = E(D_S(V_t^\epsilon, I)|V_t^\epsilon \in S_t^I(V_t^\epsilon))P(V_t^\epsilon \in S_t^I(V_t^\epsilon)), \tag{2.41}$$

which for the interest group I=H we can further simplify to

$$E(D_S(V_t^\epsilon)) = E(D_S(V_t^\epsilon)|V_t^\epsilon \in S_t(V_t^\epsilon))P(V_t^\epsilon \in S_t(V_t^\epsilon)). \tag{2.42}$$

Thus, for a fixed time t we can estimate the mean function at sample points V_{it}^ϵ where $i = 1, 2, ..., \#S_t$ by:

$$\widehat{m}_S[V_{it}^\epsilon] \approx \frac{\sum_{j=1}^{\#S_t(V_{it}^\epsilon)} \mu_o(\mathbb{S}_{tj}(V_{it}^\epsilon))}{\#S_t(V^\epsilon)} \frac{\#S_t(V_{it}^\epsilon)}{\#N_t(V_{it}^\epsilon)} \tag{2.43}$$

Which we can re-write as:

$$\widehat{m}_S[V_{it}^\epsilon] \approx \frac{\sum_{j=1}^{\#S_t(V_{it}^\epsilon)} \mu_o(\mathbb{S}_{tj}(V_{it}^\epsilon))}{\#N_t(V_{it}^\epsilon)} \tag{2.44}$$

To give a better understanding of the process, see the figure 2.37, below.

We can see that the expected amount of sell per 1 stock in a factor region is in fact equal to the total sell in the region, divided by the combined number of B/S/H observations.

Combining the Observations from Multiple Quarters

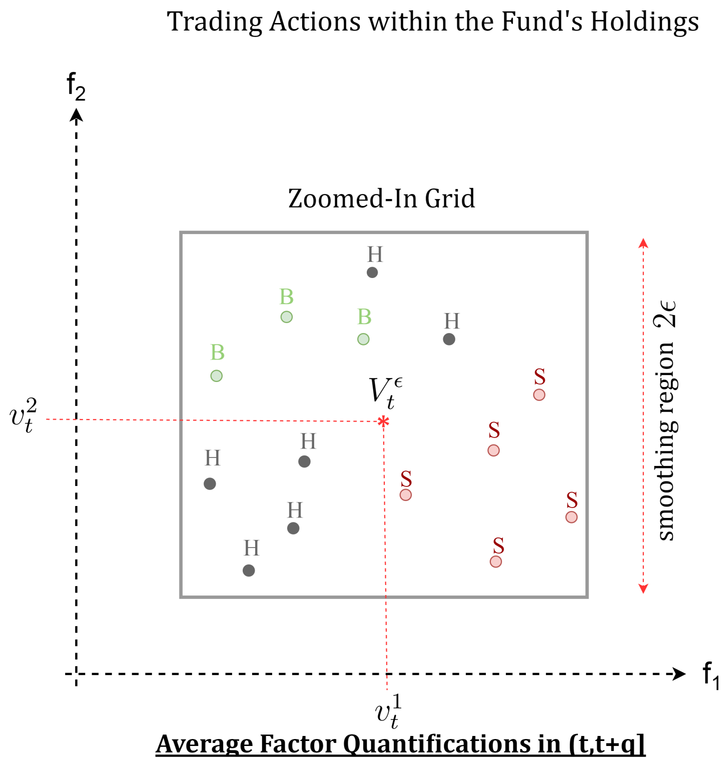


Figure 2.37: Hypothetical Example - Trading Actions in a Factor Region

Our aim is to reconstruct the functions m_S and m_B. It might be unrealistic to assume that a given analytic function represents those expected values. So, I will reconstruct the expectations of the decision functions numerically. To do so, I will compute the values of m_S and m_B on a finite grid and then extrapolate at select points of interest, as needed. Note that if the number of the dimensions d_Π is large, and if the length of the smoothing region is $\#epsilon$, then the space of the observations would be divided into roughly $(9/\epsilon)^{d_\Pi}$ number of parts. As a result, if we assume that the observations are uniformly distributed in d^Π, then we can expect to have $\frac{\text{\# of obsevations}}{(9/\epsilon)^{d_\Pi}}$ points in each ϵ region. Thus, the more dimensions we have, the exponentially less observation we will get for an ϵ region. So, having too many dimensions in d_Π would result in few to no observations in each region and will make the fitted expectations volatile. To resolve the aforementioned issue, on practice we would keep the number of the dimensions in each of the trading philosophy type small (1-3 factors).

But, even for 2 factors and an $\epsilon = 0.9$, the average number of points for each ϵ-region is $\frac{\text{\# of obsevations}}{100}$. The average number of trading observations per quarter per fund typically varies between **90 and 120**. So, if the trading observations were distributed uniformly, for an $\epsilon = 0.9$ region we would have around 1 observation. Fortunately, the distribution of the observations is not uniform and on practice the aforementioned example would produce around $10 - 30$ observation per non-empty ϵ region. However, $10 - 30$ observation per region might still not be enough to find a reliable estimate of an expectation.

To resolve the issue, I combine observations from T quarters at once. Taking T to be too large might result in neglecting the changes of a fund's trading style. I assume that it takes more than 5 years for a fund to completely change it's investment philosophy. We take T to be a multiple of q, typically $T \in [1{*}4q, 5{*}4q]$ and T will represent the length of the moving window for estimating the first moment of the decision functions.

Next, I join observations from T quarters: $q_s, ..., q_e$ and approximate the $\widehat{m}_S$ at the observation points based on the formula, below:

$$\widehat{m}_S(V_{it}^{\epsilon}) \approx \frac{\sum_{j=1}^{\#S_T^*(V_{it}^{\epsilon})} \mu_o(S_{jt}^*(V_{it}^{\epsilon}))}{\#N_T^*(V_{it}^{\epsilon})}, \tag{2.45}$$

where $N_T^* = \cup_{t\in[q_s,q_e]} N_t$ and $S_T^* = \cup_{t\in[q_s,q_e]} S_t$. I will assume the aforementioned approximation for all except the case when $d_{\Pi} = 1$. The intuition behind such an approximation comes from the interpretation of the expected value we are trying to approximate. Since the desired expectation is the expected amount of sell per 1 stock in a factor region, we can combine multiple quarters, because it will mean combining all observations B/S/H at the same time. Hence, the probability of a sell per given stock and the average amount of sell per given stock should not change considerably. To graphically explain the intuition, I present a hypothetical example in the figure 2.38, below. The presented scenario is the case when the trading actions of the fund during two consecutive factors were similar (I combined the 2.37 with its slightly shifted version).

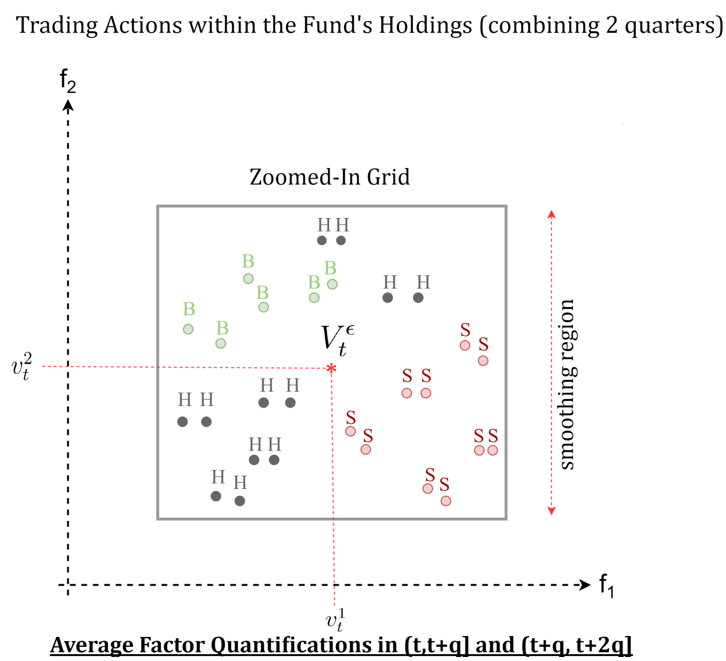


Figure 2.38: Hypothetical Example - Combining Trading Actions from Two Quarters

An alternative way to compute the $\widehat{m}_S$ is to estimate the $\widehat{m}_S$ for multiple quarters and then average the estimate. The approximation in that case would be:

$$\widehat{m}_S(V_{it}^\epsilon) \approx \frac{1}{T} \sum_{t \in [q_s, q_e]} \frac{\sum_{j=1}^{\#S_t(V_{it}^\epsilon)} \mu_o(S_{tj}(V_{it}^\epsilon))}{\#N_t(V_{it}^\epsilon)} \tag{2.46}$$

However, the alternative approximation is susceptible to the issue of the rare observations (described earlier)for $d_\Pi \geq 2$. Because of that I will assume the alternative approximation only for the case $d_\Pi = 1$.

The formulas and approximation methods for Buy|Hold (B|H) are analogous to the S case.

Constructing a Gridded Interpolant

Next, I compute the approximation of $\widehat{m}_S$ and $\widehat{m}_B$ on the vertices of a finite grid on $[1, 10]^{d_\Pi}$ with the grid-length g . I denote the vertices of the grid by V^ϵ_{kt} for $k = 1, ..., (1 + 9/g)^{d_\Pi}$. Note that independent of how small we choose the grid-length, the number of observations in a region is still dependent on the size of ϵ. Next, I plug in the value V^ϵ_{kt} in the equation 2.45 to approximate the expectations from the trading observations on the grid (I require 20 observations for each factor region, otherwise I do not save its value).

An important step in computing efficiently the expectations of the decision function is to find the values $\#N_t(V^\epsilon)$ and determine the set $S_t(V^\epsilon)$. To do so, for each trading observation and/or a grid-vertex we would need to find all other trading observations within its ϵ neighborhood. I use the range-search algorithm (see Bentley (1979), Robinson (1981)) to efficiently compute the aforementioned values.

The Interest Groups EH and NH:

I begin by describing the motivation behind using a different approach of retrieving trading signals within the interest groups EH and NH. For a given fund, the interest group H is typically small (tens to hundreds of stocks). Besides, a fund tends to be more active within the interest group H and in a given quarter we typically see trading actions towards at least 10% of H. For the cases when the set EH is small (less than 10 times H), I use the same methodology as was described for the interest group H.

However, usually the number of the stocks in the interest groups EH and NH is considerably larger than in the interest group H (see the section 2.4.5). So, a B or S action is expected towards a very small percentage of the $\{EH, NH\}$ securities. As a result, it might be impossible to accurately pinpoint which stocks would be bought or sold, particularly within the set NH. Alternatively, note that the value $P(\text{stock was sold})$ can be very close to 0 for all of the stocks. The latter will results in volatile predictions if I use the methodology of the interest group H (see equation 2.40).

The interest group $\{EH, NH\}$ has all the stocks which are not currently in the fund's portfolio. In the majority of the funds with more than 5 years of holdings history, that set covers the factor values pretty much uniformly, as shown below in the figure 2.39.

Fortunately, the estimation of the preference function was independent from the choice of the interest groups. Using the first interpretation of the preference function, I estimate the expected fraction of B/S (in comparison to fund's previous quarter holdings) in a given factor region. Next, I consider only "opin-

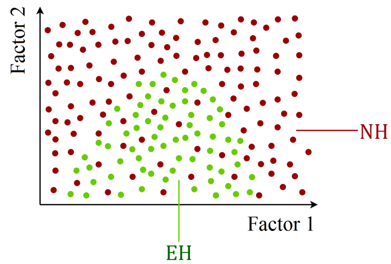


Figure 2.39: Hypothetical Representation of the EH and NH Interest Groups

ionated" regions, that is the ones where the value of the expectation of the actions is comparatively large (more than 10% of the trading actions fall within the region).

I further describe the approach on a hypothetical example in the figure 2.40, below. Suppose that based on the 3 years of observations, we found a factor region from where the fund tends to Buy a large portion of its stocks (left side of the figure). I assume that the fund is "opinionated" towards that factor region and based on its investment philosophy and/or experience expects stocks from that region to outperform. Next, at a time period $(T, T+q]$ I observe the stocks which enter that factor region (right side of the figure). We can see that there are stocks from EH as well as NH in it. Note that, using the **preference function**, we have that the expected fraction of Buy in the given factor region is: $E(\mathcal{P}_B(V_t^\epsilon))$, which I estimated independently from the choice of the interest groups. Thus, if the number of the stocks which are in the factor region during the $(T, T+q]$ is small (less than 20 stocks), then assuming each of the stocks is chosen with the same probability, I compute the **expected Buy fraction** of a fund towards a stock in the EH/NH region.

Note that opposite to the case of the interest group H, larger dimension of the

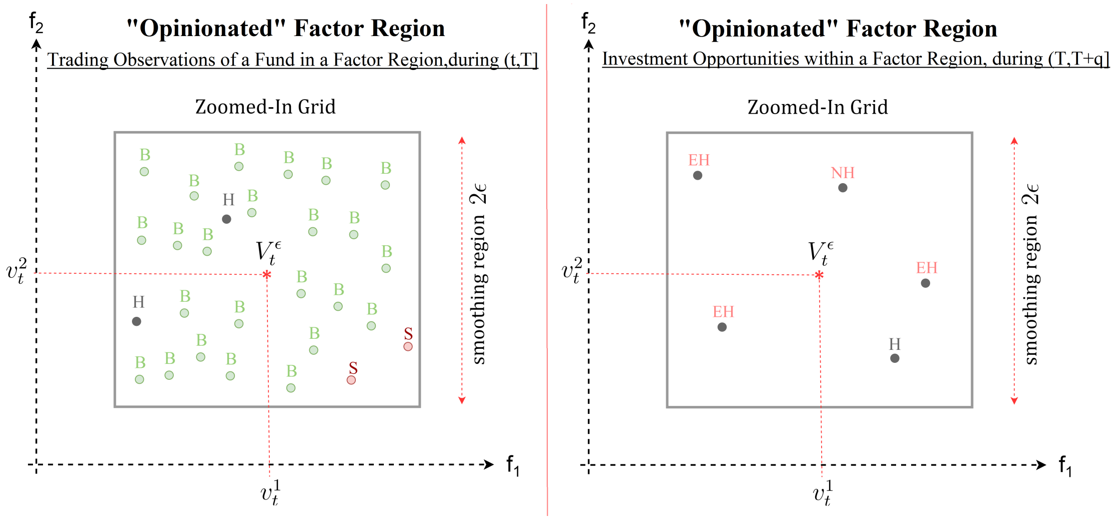


Figure 2.40: Hypothetical Example - Retrieving Signals from an "Opinionated" Factor Region

input information d_{Π} might help to decrease the number of the stocks in a factor region. Thus, we would have a larger chance of pinpointing a stock which a fund is likely to buy from the region EH/NH ("bless of dimensionality"). I combine those signals to predict the cumulative fund holding changes. The details of the implementation are described in the next section.

Additional Notes on Constructing the Sets H/EH/NH

This sub-section provides some additional details on the implementation of the H/EH/NH sets for the Action |Attention model. It can be skipped by readers not interested in the specifics of the implementation.

Action Indicators (Buy/Sell/Hold): For each pair of {fund: k, security: j} we need to find whether the action of the fund towards the security was Buy/Buy More, Sell/ Sell All or Hold. Finding this might be tricky since the variable SHARES might change its value only because of the share distributions. However, I have already accounted for all such cases.

We are defining the following variables:

- iB - indicator that the action of the fund towards the asset was Buy
- iS - indicator that the action of the fund towards the asset was Sell
- iH - indicator that the action of the fund towards the asset was Hold (non-zero weight)

Attention Indicators (H/EH/NH): For each pair {fund: k, security: j}, we need to track the attention of the fund towards the stock. For that reason, I create the following 3 indicators.

- $H_{k,j}(t)$ - indicates whether at time $t-1$ the fund k was holding the stock j
- $EH_{k,j}(t)$ - indicates whether at times $1,...,t-2$ but not at time $t-1$ the fund k was holding the stock j
- $NH_{k,j}(t)$ - indicates whether at times $1,...,t-1$ fund k did not hold the stock j

The amount of non-zero inputs in those three sets across all times, funds and stocks are as follows: $H \approx 25$ million, $EH \approx 120$ million and $NH \approx 13$ billion. The reason that NH is so large might be that some small stocks are held only by a few funds and for the rest of the funds, for all the quarters their value will be NH. To resolve the issue with disproportionate NH set I will restrict the calculation of NH within an investment philosophy.

Next, I modify the indicators to adjust for the cases when the fund no more exists. If that happens, then there is no need to continue holding it in the EH attention group. I do that by excluding funds which did not report at least once in the next 3 quarters. In that case there are around 90 million valid entries in EH and with a similar procedure ≈ 5 billion valid entries in NH.

Action|Attention Events: So, in our model there are 7 variations of the action/attention events. Those are represented in the chart below.

Action \Attention	Hold	Ever Held	Never Held
Buy	B|H	B|EH	B|NH
Sell	S|H	S|NH	S|EH
Hold	H|H	*	*

Figure 2.41: The Types of the Action|Attention Events

For ease of implementation, I will record those into variables B|H, B|EH, B|NH, S|H, S|EH, S|NH, H|H as a structure and each of them would have information on three variables: index, amount (of trade) and count (of number of shares traded).

2.5 Predicting the Aggregate Fund Actions

As we have previously noted, we do not have access to the fund shorting data. So, I further fit the implementation of the A|A model to a framework with no shorting. In such a setting, the trading decisions of a fund in the period $(t, t+q]$ are to Sell from H and to Buy from H/EH/NH, as described in the Figure 2.42, below.

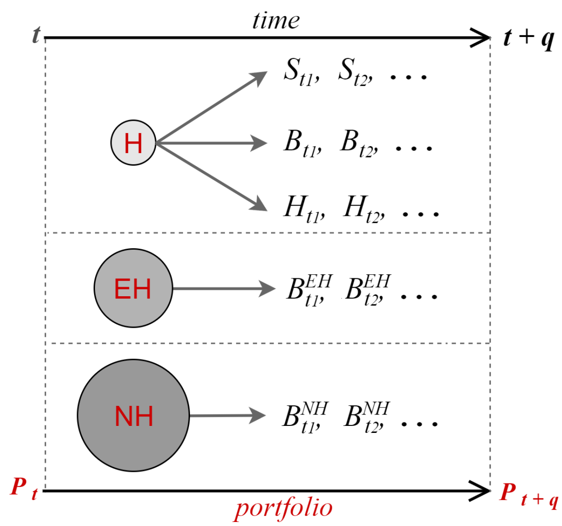


Figure 2.42: Trading Actions in $(t, t+q]$

I divide the process of predicting the fund actions into three steps: **estimation**, **prediction** and **testing**. To incorporate the possible changes in the fund's trading philosophy, I fit the values of their decision functions by a moving window of a fixed length $T = 3 * 4$ **quarters (3 years)**. I define $q_e = q_s + T$ and will test the model at the time $Q := q_e + q$. The exact steps of the estimation process are described below.

Step 1: Reconstructing the Investment Philosophy Type Π^j for each Fund

- Using the observation of the fund trades in $[q_s, q_e]$, for each fund fit an IVNM curve and save the adjusted R^2 values
- Based on the IVNM analysis in $[q_s, q_e]$, classify the 16 factors into Action and/or Attention types (if possible)
- For each fund, choose 1 Action and 1 Attention factor by choosing the ones with the largest adjusted R^2 values
- If there are no Action factors, then choose the 2 Attention factors with the largest adjusted R^2 values
- If less than 2 Attention factors are available for the fund, merge the trading observations of the fund to the category ”other”
- Perform the aforementioned analysis for the merged fund category ”other”. If there are still less than 2 Attention factors for the category, no further predictions will be made for the category

The best Action and Attention factor indexes are saved in Π^j for each fund j. So, using in the first step I constructed the **investment philosophy types** for each of the qualified [5] funds. In the next step, I approximate the expectations of the decision functions.

Step 2: Approximating m_B and m_S for the Interest Group H

- Construct a grid on $[1, 10]^2$ with a grid-length $g = 0.05$
- For each grid point, approximate the values $m_B^j(\star)$ and $m_S^j(\star)$ for the interest set H using the equation 2.45

[5]The funds for whom the 1-dimensional analysis produced at least 1 Action & 1 Attention or 2 Attention factors

- Observe the values of $V_t^{\epsilon'}$ for the interval $[q_e, Q-1)$. Save it as V_t^{ϵ} but choose $\epsilon = 2\epsilon'$ [6]

Thus, at the time $Q-1$, using the information on $P^j_{q_e}$, the expected amount of Sell and (Buy|H) during $[q_e, Q)$ is:

$$\mu_o(\widehat{m_S}(P^j_{q_e})) \text{ and } \mu_o(\widehat{m_B}(P^j_{q_e})) \tag{2.47}$$

Step 3: Combining with the Signals in EH and NH

- Approximate $E(\mathcal{P}^j(\star))$ on each point of the previously constructed grid and take $\epsilon = 0.9$
- For the cases when in an ϵ-region the fraction of the total Buy is at least 10% ("opinionated" region), for each EH/NH stock, I increase the prediction of Buy by the fraction of $E(\mathcal{P}^j(\star))$ over the total number of stocks in the region (i.e. the expected amount of Buy per stock) [7]
- Denote the resulting prediction by $\widehat{m}_B(P^j_{iq_e}, \{EH, NH\})$

Since we are computing the expectations of the decision function, predicting an instance of a buy or a sell of one stock by one fund is quite unlikely. However, combining funds and/or combining stocks (with the help of the law of large numbers) might have a predictive power. Potentially, the model can make predictions in the following 3 cases:

[6] I save the average factor values of the two months as V^{ϵ}, but to incorporate a larger range of variability I multiply the ϵ' by two.

[7] Note that if the Buy observations were spread uniformly across $[1, 10]^2$, then for each ϵ-region would contain approximately 1% of all Buy observations

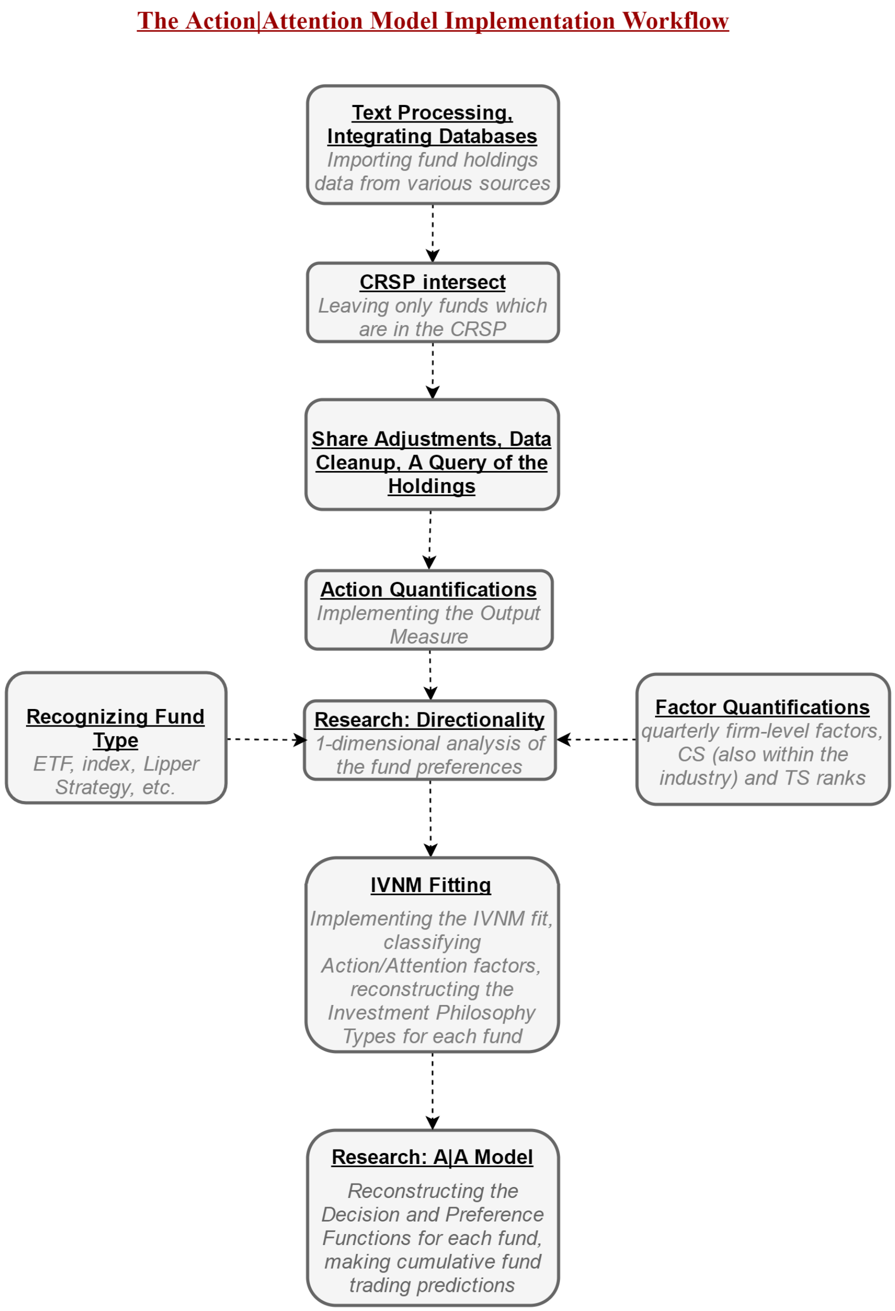


Figure 2.43: The Implementation Workflow

- 1 fund – prediction of cumulative B/S/H
- 1 stock – prediction of cumulative B/S from a subset of funds
- subset of stocks – prediction of cumulative B/S from a subset of funds.

However, I will only assess the predictive power of our model in the case of the cumulative fund Sell actions towards each stock. The implementation of the model is performed in Matlab and the general workflow of the implementation is presented in the figure 2.43.

Predicting the Next Quarter Cumulative Fund Trades for a Stock:

For a given stock i, if we denote $p_i^j := P_{iq_e}^j$, then using the Step 2, the estimated cumulative fund Sell equal to:

$$pred_{iS}(Q) = \sum_{j=1}^{k} \mu_o(\widehat{m}_S(p_i^j)). \tag{2.48}$$

Similarly, combining the Steps 2 and 3, the estimated cumulative Buy is equal to:

$$pred_{iB}(Q) = \sum_{j=1}^{k} (\mu_o(\widehat{m}_B(p_i^j)) + \widehat{m}_B(p_i^j, \{EH, NH\})). \tag{2.49}$$

Discussing the Accuracy of Predictions:

Next, I briefly present and discuss the accuracy of the predictions of the model in case of Sell actions. I compute the realized cumulative fund sell during $(q_e, Q]$ and for the stock i denote it by $real_{iS}(Q)$. Motivated by the findings of Sias (2004), that institutional demand for a stock is positively correlated in adjacent quarters, I also compute the previous quarter institutional sell of a stock and denote it by $prev_{iS}(Q)$. Besides, I record the mean of the expanding window of the realized cumulative fund sell in the variable $mean_{iS}(Q)$. Because the amount of the institutional holdings and sell from one stock to another might change significantly, I further analyze the fraction of the prediction over the actual value, as described below:

$$\text{fpred}_{iS}(Q) = \frac{pred_{iS}(Q)}{real_{iS}(Q)},\ \text{fprev}_{iS}(Q) = \frac{prev_{iS}(Q)}{real3_{iS}(Q)},\ \text{fmean}_{iS}(Q) = \frac{mean_{iS}(Q)}{real_{iS}(Q)} \tag{2.50}$$

Also, I limit the analysis to the stocks which have at least 100 funds holding them at the time of prediction. I further winsorize the prediction $pred_{iS}(Q)$ by $10 * mean_{iS}(Q)$, in order to control for the outlier predictions. I run simulations for $Q = 2003$(q1),...,2014(q2) and compute the values fpred (A|A model), where possible. Within the subset of stocks, for which there was a prediction of the A|A model , I also compute the fprev (previous quarter's cumulative institutional sell) and fmean (mean). The results are presented in the figure. 2.44. We can see that the numbers of predictions within a range of accuracy has a seasonal character. The underlying reason is that some of the funds in our database report semi-annually. We can see that in almost all of the cases the predictions of the A|A model are approximately 2-3 times better than the ones produced by the other two predictions.

For a randomly selected quarter, I also plot the histogram of the prediction ratio (see figure 2.45). It is easy to note that the A|A model tends to slightly over-estimate the predictions, whereas, the previous quarter sell (in the given case) under-estimates the cumulative fund Sell. The possible reason is that the A|A is more likely to make a prediction towards a stock which moves to an "opinionated" region and I only record the stocks for which A|A did make a prediction. On the other hand, the previous quarter's institutional sale does not contain such information and hence its prediction would be the same as when the stock

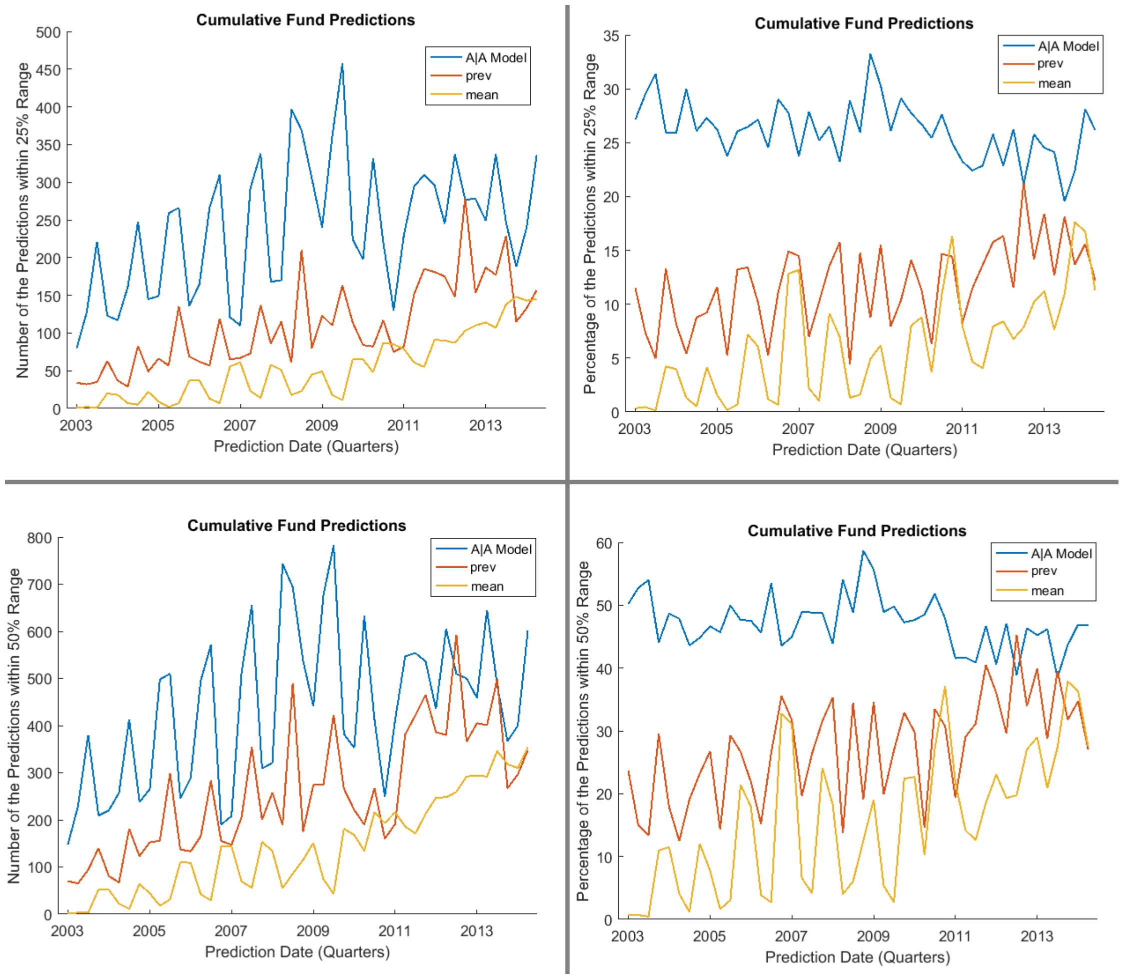


Figure 2.44: Prediction Results for $Q = 2003$ (q1),...,2014 (q2)

was out of the ”opinionated” region, which explains the lower than actual cumulative trade.

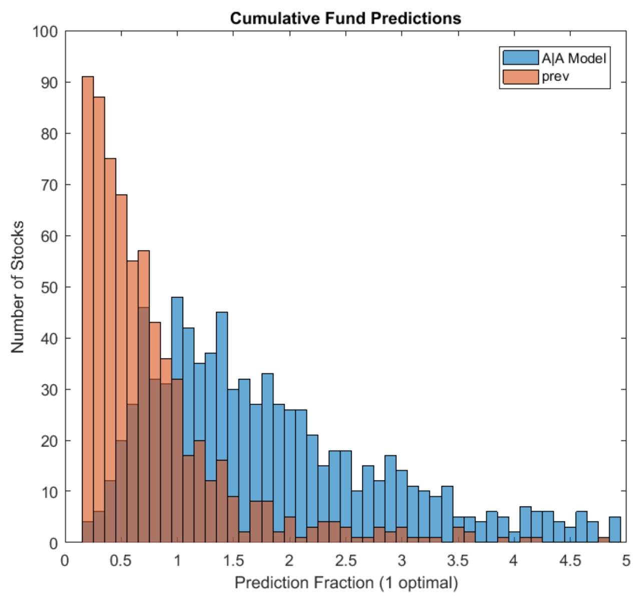


Figure 2.45: Histogram of the Prediction Accuracy for $Q = 2013$ (q4)

CHAPTER 3

GENERAL CONSTRAINT REDUCTION FRAMEWORK

3.1 Introduction

Some studies have been conducted to empirically measure the illiquidity premium associated with liquidation restrictions of hedge funds (see [17], [18]). In case of transaction costs, a theoretical definition of illiquidity premium was given by Constantinides (1986) [19]. But, there is no commonly accepted way in academia to define an illiquidity premium that an investor should be given because of additional restrictions. In this section, a general formulation of a constraint premium will be given. The approach was initially designed to address the problem of defining and efficiently calculating lockup premiums of institutional investors. However, this approach can also be applied to a great range of terminal utility maximization problems with a finite number of securities and no market impact.

First, the general portfolio construction problem will be defined. The definitions are be presented in the most general, abstract setting to expose their underlying economic reasoning. Next, I will add constraints to the model and discuss how to transform it back to the un-constrained setting by penalizing the expected returns of the underlying assets. Based on the gained intuition, I will define a premium the investor should be given to be indifferent between the constrained and un-constrained models. The derived definitions and methodology will be used in the next chapter to measure the lockup premium of hedge funds.

We are given a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{P})$ where T can also be infinite.

Exogenous Process: $\mathcal{S}$

An investor has access to a finite number of securities:

$$\mathcal{S} = \left\{ S_t^i \,|\, S_t^i : \Omega \to \mathbb{R}, \text{ adapted to } \mathcal{F}_t, \; E(S_t^i) = \mu_t^i < \infty, \; i = 1, ..., n, \; t \in [0, T] \right\}. \tag{3.1}$$

Some or all of the securities might be risky and at this point I do not assume any particular evolution on S_t^i except that it has a finite expectation. In other words, $\mathcal{S}$ represents the stochastic part of our model over which the investor has no control. I denote the column vector $\{S_t^i, \; i = 1, ..., n\}$ by S_t and the row vector of $\{\mu_t^i, \; i = 1, ..., n\}$ by μ_t.

Control Process: $\mathcal{A}$

The investor controls the weights of each asset S_t^i in her portfolio with a control process α_t^i, $\sum_{i=1}^n \alpha_t^i = 1$. At each time t, the weights $\alpha_t = \{\alpha_t^1, ..., \alpha_t^n\}$ are in a region $\mathcal{A}_t$ which represents the set of actions the investor can take to re-balance her portfolio up to time t and for convenience I will drop T in $\mathcal{A}_T$.

Cost Function: $\mathcal{C} : \mathcal{A} \times \mathcal{C}$

Suppose, there is also a cost function $\mathcal{C}$ defined on $\mathcal{S} \times \mathcal{A}$ that results from the actions of the investor while re-balancing her portfolio. The cost function must be non-negative.

Utility Functional: $\mathcal{U}$

Denote by U the utility functional of the investor. The functional can be a utility function, an integral of instantaneous utility function and vary through time, but I define here it in an abstract sense. The only requirement is that the function is non-decreasing depending on the cost-adjusted profit of the investor. I also assume that the functional U includes the adjustment for the discount rate when applicable.

Constant Initial Values: $\mathcal{I}$

I store all the constant initial values of the model at time t in $\mathcal{I}_t$. For example, the values S_0 and α_0 in $\mathcal{I}_0 := \mathcal{I}$ will represent the initial values of the assets and the initial portfolio allocation weights.

Optimal Portfolio Allocation Problem: Classical Framework

Given the model $\mathbb{M}$ consisting of the framework $\mathcal{M} = \{\mathcal{S}, \mathcal{A}, \mathcal{C}, U\}$ and the initial values $\mathcal{I}$, the problem of the investor at time 0 is to find an optimal admissible control $\alpha \in \mathcal{A}$ in order to maximize:

$$J_{\mathcal{M}}(\mathcal{I}, \alpha) = E(U(S_T\alpha_T - \mathcal{C}(\alpha_T, S_T))), \tag{3.2}$$

We denote the optimal terminal value function by:

$$V_{\mathcal{M}}(\mathcal{I}) = \sup_{\alpha \in \mathcal{A}} E(U(S_T\alpha_T - \mathcal{C}(\alpha_T, S_T))), \tag{3.3}$$

and for a given model $\mathbb{M}$ consisting of the framework $\mathcal{M}$ and initial values $\mathcal{I}$, we denote:

$$V_{\mathbb{M}} = V_{\mathcal{M}}(\mathcal{I}). \tag{3.4}$$

For example, if the cost functional $\mathcal{C} \equiv 0$, the problem can be reduced to the standard Stochastic Control framework that is discussed in e.g. Pham (2009). The case $\mathcal{C} \equiv 0$ represents the problem of optimal portfolio construction with no transaction costs, no consumption and can incorporate models of Merton (1969) if the assets follow a Geometrical Brownian Motion, or the model described in Elliott and Siu (2009), in case the stochastic process follows a Markovian Regime-Switching Model. On the other hand, if the control set $\mathcal{A}$ in any given interval $[0, t)$ consists of finite number of points, our framework reduces to classical Impulse Control problem as in Oksendal and Sulem (2008), Carmona and Ludkovski (2007), Baccarin and Marazzina (2014) and can be used to model the problem with transaction costs as in Korn (1998), Morton and Pliska (1993), Constantinindes (1986).

In the classical literature (e.g. Shreve and Soner (1994), Constantinindes (1986)), to solve the optimal portfolio allocation problem, the value function $V_{\mathcal{M}}$ is usually described as a function of the initial values $\mathcal{I}_t$ at time t. The latter is done in order to reduce the time-range of the problem and apply the dynamic programming principle. However, I will take a different approach and consider the value function from an alternative perspective.

3.2 The Constrained Model

Define models $\mathbb{M}_i(p) = \{\mathcal{M}_i(p), \mathcal{I}\}$, $i = 1, ..., n$ as versions of the model $\mathbb{M} = \{\mathcal{M}, \mathcal{I}\}$ where the only difference is that the expectation of the i^{th} asset is multiplied by a scalar $p \in (0, \infty)$ and is equal to $p\mu_t^i$. Similarly, we define models $\mathbb{M}(p) = \{\mathcal{M}(p), \mathcal{I}\}$ as versions of the model $\mathbb{M} = \{\mathcal{M}, \mathcal{I}\}$ where the only difference is that the expectation of all the asset is scaled by a parameter p.

I consider the value functions:

$$V_{\mathbb{M}_i}(p) := V_{\mathcal{M}_i(p)}(\mathcal{I}) \tag{3.5}$$

$$V_{\mathbb{M}}(p) := V_{\mathcal{M}(p)}(\mathcal{I}) \tag{3.6}$$

I will use those notations interchangeably in cases when the definition/properties hold for both of the cases. For simplicity, lets for the moment assume that μ_t^i is constant. Intuitively, we are interested in answering the question of what will happen to the optimal investor's terminal wealth if we keep everything the same except increase and/or decrease the expected return of one of its assets. Of course, in real-world scenario we would expect the function $V_{\mathbb{M}_i}(p)$ to be non-decreasing. In the mathematical framework, assuming existence and uniqueness of the value function and some regularity, by keeping in mind that $\mathcal{U}$ is non-decreasing, one could also expect the function $V_{\mathbb{M}}(\mu)$ to be non-decreasing.

Definition 3.2.1 (Constrained Model)**.** *We call a model* $\mathbb{M}' = \{\{\mathcal{S}, \mathcal{A}', \mathcal{C}', \mathcal{U}\}, \mathcal{I}\}$ *a constrained version of model* $\mathbb{M} = \{\{\mathcal{S}, \mathcal{A}, \mathcal{C}, \mathcal{U}\}, \mathcal{I}\}$ *and denote by* $\mathbb{M} \succ \mathbb{M}'$ *if the following two conditions hold:*

1) Action Constraint: $\mathcal{A}' \subseteq \mathcal{A}$

2) Cost Constraint: $\mathcal{C}' \geq \mathcal{C}$

In such a case we call the $\mathcal{A}'$ *and* $\mathcal{C}'$ *to be the additional constraints imposed on* $\mathbb{M}$*.*

As we can see, the constrained model is the one where the investor has to pay larger (or equal) fees for the same portfolio re-balancing operations and for which the admissible set of re-balancing decisions of the investor is smaller (or equal).

Corollary 3.2.1. *If $\mathbb{M}'$ is a constrained version of the model $\mathbb{M}$, assuming the existence and uniqueness of the functions $V_{\mathbb{M}}(p)$ and $V_{\mathbb{M}'}(p)$, the following relation holds:*

$$V_{\mathbb{M}}(p) \geq V_{\mathbb{M}'}(p) \tag{3.7}$$

Proof. The proof immediately follows by combining the property of supremum with the fact that $\mathcal{U}$ is non-decreasing: for any control $\alpha \in \mathcal{A}'$, $\alpha \in \mathcal{A}$ and $E(U(S_T\alpha_T - \mathcal{C}'(\alpha_T, S_T))) \leq E(U(S_T\alpha_T - \mathcal{C}(\alpha_T, S_T)))$. □

Definition 3.2.2 (Cost of the Constraints)**.** *We call the quantity $C_{\{\mathcal{A}',\mathcal{C}'\}} := V_{\mathbb{M}} - V_{\mathbb{M}'}$ the cost that the investor in model $\mathbb{M}$ carries because of the additional constraints $\mathcal{A}'$ and $\mathcal{C}'$.*

As we see from the corollary, the cost of a constraint for an investor is always nonnegative. If we suppose that $\mathcal{A} = \mathcal{A}'$ and $\mathcal{C} \equiv 0$ then our cost of the constraints definition reduces to the one given by Constantinindes (1986), where he defined the liquidity premium of a transaction cost to be the maximum expected return an investor is willing to exchange for zero transaction costs. In our analysis, though, I will mainly consider the case when $\mathcal{A}' \subset \mathcal{A}$ and $\mathcal{C} = \mathcal{C}'$. Next, suppose that the constraints $\mathcal{A}'$ and $\mathcal{C}'$ are only applied to the trading of the security i.

Definition 3.2.3 (The Constraint Premium)**.** *For a given model $\mathbb{M}$ and its constrained version $\mathbb{M}'$, if there exists a scalar p such that*

$$V_{\mathbb{M}}(p) = V_{\mathbb{M}'}, \tag{3.8}$$

then I call the quantity $\mu(1-p)$ the liquidity premium of the constraints $\mathcal{A}'$ and $\mathcal{C}'$.

Note that the constraint premium is the adjustment that should be made to the average returns of the securities, so that the investor with a given utility is indifferent between the model $\mathbb{M}$ with returns $\mu - p$ and the constrained model $\mathbb{M}'$ with real-world returns. We can think of it as $\mathcal{A}'$ and $\mathcal{C}'$ transform the unconstrained model to the constrained one and by penalizing the expected returns I transform it back, as presented in the equation, below:

$$\{\mathbb{M},\ \mu\} \rightarrow_{\mathcal{A}',\mathcal{C}'} \rightarrow \{\mathbb{M}',\ \mu\} \rightarrow_{\mu'=p\mu} \rightarrow \{\mathbb{M},\ \mu'\} \tag{3.9}$$

This framework enables us to compute premiums of different constraints and combine all that in the initial unconstrained model. However, one can note that it is possible to define a constrained premium if:

$$\exists p \text{ s.t. } V_{\mathbb{M}}(p) = V_{\mathbb{M}'}(1), \text{ from where we get that } \mu' = p\mu. \tag{3.10}$$

I will address the existence of such p in some specific cases when $V_{\mathbb{M}'}(p)$ can be proven to be increasing.

Example: Merton Portfolio Construction Model

To see what type of shape to expect from the function $V_{\mathbb{M}}(p)$ I simulated it in case of Merton Portfolio for an investor with CRRA utility function with elasticity $\gamma = 0.5$, parameters $\sigma = 0.2$, $r = 0.02$, $T = 4$ and the time 0 investment of 1.

In this example I assume no consumption and that there exist one risky and one riskless asset defined by the following two equations:

$$dS_t^0 = rS_t^0 dt, \tag{3.11}$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{3.12}$$

The wealth process is given by:

$$dX_t = \frac{X_t \alpha_t}{S_t} dS_t + \frac{X_t(1-\alpha_t)}{S_t^0} dS_t^0, \tag{3.13}$$

and the Value function is defined by:

$$V(t,x) = \sup_{\alpha \in A} E[U(X_T^{t,x}).] \tag{3.14}$$

For the CRRA(0.5) utility function: $U(X) = X^{0.5}/0.5$, the explicit solution and the optimal wealth process (see Pham (2009)) is:

$$dX_t = X_t(\alpha' \mu + (1-\alpha')r)dt + X_t \alpha' \sigma dW_t, \tag{3.15}$$

$$\alpha' = 2\frac{(\mu - r)^2}{\sigma^2}, \tag{3.16}$$

and the terminal utility is:

$$V(0,1) = 2\exp(\rho T), \text{ where } \rho = \frac{(\mu - r)^2}{2\sigma^2} + \frac{r}{2}. \tag{3.17}$$

From where I generate the terminal utility function for different scaling parameters p and plot it in 3.1, below.

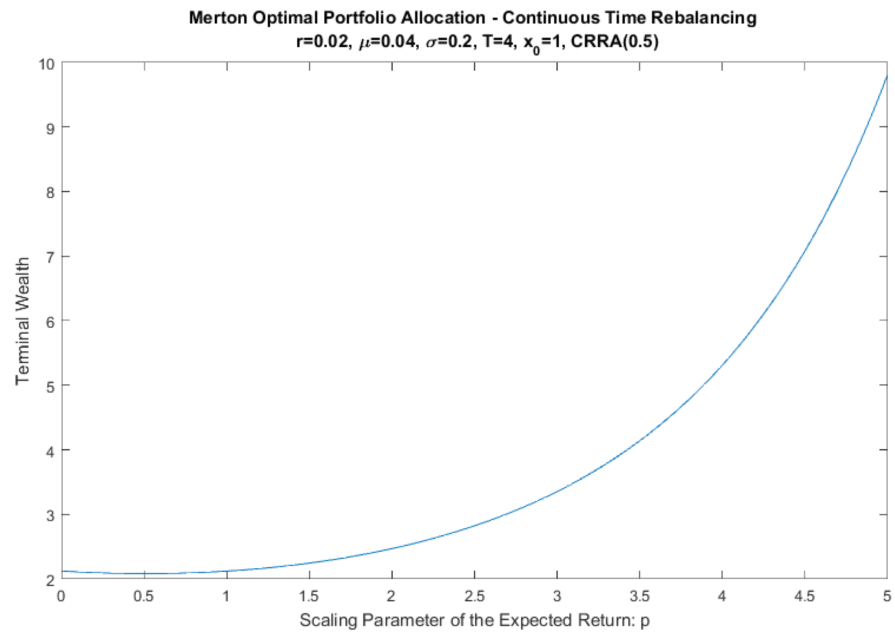


Figure 3.1: No-Transaction Cost Merton Model: $V_{\mathbb{M}}(p)$

As we can see from the figure, the terminal value increases exponentially as p increases. So, for this model I will be able to apply the definitions of the premium of a constraint.

In case that approximating such a p is numerically unfeasible, I will use an alternative definition of the constraint premium from the point of view of average returns. Note that if $V_{\mathbb{M}}$ and $V_{\mathbb{M}'}$ can be uniquely determined, then there exist average rate of returns α, α' such that $e^{\alpha T} = V_{\mathbb{M}}$ and $e^{\alpha' T} = V_{\mathbb{M}'}$. Since $V_{\mathbb{M}'} \leq V_{\mathbb{M}}$, we will also have $\alpha - \alpha' = \frac{\log V_{\mathbb{M}} - \log V_{\mathbb{M}'}}{T} \geq 0$ and the constraints $\mathcal{A}, \mathcal{C}$ are the only reason of the decrease of the relative rate of return. Hence, we can use the following definition:

Definition 3.2.4 (The Constraint Premium Rate)**.** *We call the quantity* $\frac{\log V_{\mathbb{M}} - \log V_{\mathbb{M}'}}{T}$

the Constraint Premium Rate that the investor in model $\mathbb{M}$ carries because of the additional constraints $\mathcal{A}'$ and $\mathcal{C}'$.

As we will see in the next section, exploring the functions $V_{\mathbb{M}}(p)$ and $V_{\mathbb{M}'}(p)$ might be useful in finding analogues of continuous time policies in discrete-time setting. Suppose the only restriction of M' is that the investor can re-balance her portfolio on pre-determined dates. Unfortunately, the explicit solutions of the continuous-time Optimal Portfolio Construction problems are not necessarily optimal if applied discretely. The reason is that the continuous time solutions were designed in a setting where continuous re-balancing was possible. However, if we can find $V_{\mathbb{M}}(p)$ explicitly and $V_{\mathbb{M}'}(p)$ through a comparatively slow numerical iteration, we could compute $V_{\mathbb{M}'}(p)$ for a few points and use the inverse relation with $V_{\mathbb{M}}(p)$ in order to approximate the values of $V_{\mathbb{M}'}(p)$ on some closed interval. The methodology will be applied to the Markov-Switching framework and described in more details in the next section.

CHAPTER 4

LOCKUP AS A LOST INVESTMENT OPPORTUNITY

4.1 Introduction

Money managers typically have agreements with their investors specifying a fee structure and redemption rules of the investments. The redemption rules usually include share restrictions such as lockup and notice periods. A lockup indicates the time period during which the investor cannot withdraw any money from the money manager. A notice period is the amount of the time the investor has to provide before withdrawing her money. Share restrictions are specific to each agreement and might also include conditions on the fraction of the initial investment that can be withdrawn at once.

Lockup restrictions are particularly useful for money managers that invest in more illiquid assets (see for example [20]). Aragon (2007) [17] argues that the share restrictions allow funds to manage illiquid assets more efficiently. Studies by Bali, Gockan and Liang (2007) [21] , Ackerman, McEnally and Ravenscraft (1999) [22] and Liang(1999) [23] suggest that the hedge funds that have lockup periods have higher expected returns than the ones that don't. But, studies conducted by Aragon (2007) also show that, after controlling for share restrictions, on average the alpha of all hedge funds is either negative or insignificant. Hence, one can argue that the expected returns of the money managers with greater share restrictions are significantly higher because of the illiquidity risk the share restrictions bear. But, the classic optimal portfolio allocation models do not account for such an illiquidity risk and might be tricked by the resulting high expected returns.

Literature Review:

There are not many studies in academia that address the problem of estimating the lockup illiquidity premium of Hedge Funds. After an extensive academic research on the topic, I found only three not purely empirical papers that address the question of illiquidity premiums in Hedge Funds. Ang and Bollen (2010) [24] model lockup premium by considering it as an option exercised when the investor's own valuation of a share of ownership in the Hedge Fund falls below the fund's reported NAV. The model assumes that the investor is sophisticated enough to take into account the probability of fund failure, liquidation costs and the impact of the future exercise decisions. That approach might be applicable in cases when the Hedge Funds invest in assets that are very illiquid or for which market prices are not available and Hedge Fund's NAV report might be biased. The authors also take into account fund failures which they are calculating by looking at the time when Hedge Funds stopped reporting their performance. However, Hedge Funds stopping reporting of their NAVs might be connected with many other reasons, one of them being investor fund outflows. Also, as noted by Derman et al. (2008) [25], dead funds are not of a large concern for investors with lockups, since if the Hedge Fund is dead, then the investors will be paid independently from the fact whether there was a lockup or not. Ang and Bollen (2010) do not consider the more complex question of calculating the lockup premium in a context of a full-scale asset allocation problem. The reason they state is that for small institutional investors, because of the minimum investment requirements in Hedge Funds, the decision is to typically remain invested in a given Hedge Fund or fully liquidate the position and withdraw the capital as cash. If we consider this in the framework of optimal asset allocation, while also taking into account the constraint on the minimum investment requirements, the approach of Ang and Bollen (2010) can be treated as modeling the special case when an investor is choosing between holding **the Hedge Fund or a risk-free asset**. From the financial perspective one can treat it as an investor looking at the absolute performance of a Hedge Fund while making a liquidation decision. The latter is a special case of our approach, when we assume that apart from a given Hedge Fund, the investor *has only access to a risk-free asset.*

In our opinion, the assumption that the institutional investor can calculate the NAV of an illiquid asset more accurately than the Hedge Fund manager and can spot the misreporting might not be realistic. It assumes a high level of sophistication from the investor's side, information about exactly what type of assets the Hedge Fund holds and expertise in those assets to the level that will enable the investor to price them more realistically. Besides, for the majority of the assets the prices are available publicly. But, even in such a case the Hedge Fund would not necessarily reveal its holdings to the investor so that the latter can re-calculate the NAV. Hence, I consider the question of the NAV misreporting as irrelevant in the calculation of the lockup premium.

A different approach to model lockups was proposed by Derman, Park and Whitt (2008), who estimated the lockup by comparing a given Hedge Fund with an extended lockup with a Hedge Fund with one year lockup. They look only at similar funds in the same strategy category and use a discrete-time 3-state process for their estimation. The authors note that their definition of illiquidity does not account for other investment opportunities that the investor lost because of the lockup. A generalization of the model of Derman et al. (2008) is done by Park and Whitt (2013), where the authors consider a similar 3-state

Markov Process but in continuous time setting. In that paper as well the authors note that they did not fully account for lost investment opportunities that resulted because of the lockup premium. However, the authors of those two papers define and use the idea of good and bad state in relation to the average return rate for Hedge Funds in a given strategy category. Again, if we consider this in the framework of asset allocation, we can treat is as an investor choosing between a given Hedge Fund and comparing it with the benchmark. So, what these approaches actually do is mainly taking into account a subset of lost investment opportunities, those *within a given category of funds and in a special case*.

The Proposed Approach:

It is reasonable to assume that the Hedge Fund portfolio managers are more knowledgeable in the assets in which they invest, compared to the institutional investor. So, assuming there is no conflict of interest, portfolio managers can make better predictions on market timing and can partially liquidate institutional investor's portfolio when there is a significant downside risk or not many good opportunities in the market. Moreover, lockups might protect against fire-selling behavior of non-sophisticated investors and as noted by Aragon (2007) might help funds to manage their illiquid portfolios more efficiently. As a result, I argue that the lockup premium should only be measured as a premium for the lost investment opportunities of the institutional investor.

Apart from returns relative to the benchmark or relative to the similar classes of funds, investors also look at the absolute returns of the fund and the relative returns of the fund in connection with the other fund classes. For example, if the Hedge Funds which are invested in Emerging Markets are projected not to perform well compared to the Hedge Funds or Mutual Funds in the US equities, then the institutional investor might decide to re-allocate her assets from the Emerging Markets. So, the main risk that an investor with a lockup faces is the inability to re-balance her portfolio by reinvesting the money: **in a fund of a same class, in a fund from a different class or in a riskless asset.**

Based on the aforementioned intuition, I construct models to measure the illiquidity premium of the hedge fund lockups. I model the lockups in the context of an optimal portfolio construction problem. To be able to measure the illiquidity premium independent from the choice of a specific portfolio construction approach, I model it for the optimal investor. The latter is done using the constraint reduction framework from the previous section.

Thus, I assume that the institutional investor makes a decision to invest the part of her money in a given Hedge Fund with some fixed lockup period. She also has access to a riskless asset, Hedge Funds of a similar class and the general market that is available to institutional investors. The investor has some utility function and tries to optimize her utility over a finite period of time $[0, T]$.

First, I model the hedge fund lockups in the framework of proportional transaction costs. After, I consider the problem in the Markov-Switching case. Due to the restrictions to access the Hedge Fund returns data, I am not able to implement the proposed models. However, we also construct a simplified framework to model the cost of lockups and by using the data on hedge fund indexes, I approximate lockup premium values for different type of funds. Also, I demonstrate the dependence of the lockup premium on the chosen portfolio construction model by examining it for 4 optimal portfolio construction cases.

4.2 Lockup Premium: Optimal Investor

We are given a complete filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{P})$ that satisfies the usual hypothesis as presented in Protter [26]. T represents the investment horizon of the institutional investor and can be any positive real number, but for convenience I assume that T is the amount of time in months. Assume that an institutional investor invests $H_1(0) > 0$ dollars in a given Hedge Fund that has lockup restrictions of $L < T$ months and let $k := T/L$ be a positive integer. In case of lockups the investor is allowed to re-balance her portfolio at times $L, 2L, ..., kL$

Suppose that the institutional investor has access to one riskless and 3 risky assets:

- *Riskless Asset:* $H_0(t)$ evolves according to $H_0(t) = h_0 e^{rt}$
- *Hedge Fund:* $H_1(t)$ represents the evolution of the NAV of the given Hedge Fund
- *Hedge Fund Style:* $H_2(t)$ represents the evolution of the NAV of a hedge fund style that H_1 is in. That is, $H_2(t)$ represents the overall evolution (benchmark) of a given Hedge Fund style
- *Market:* $H_3(t)$ represents the average NAV evolution of all the assets available to the investor in the economy: that is the average of NAV's over all Mutual and Hedge Funds available to the institutional investor.

The risky assets $H_i(t)$ $t \in [0, T]$, $i = 1, 2, 3$ are adapted to $(\mathcal{F}_t)_{t\in[0,T]}$ and evolve according to a Geometric Brownian Motion:

$$\frac{dH_i(t)}{H_i(t)} = \mu_i dt + \sigma_i dW_i(t) \tag{4.1}$$

Also, suppose that μ_i's are already adjusted for the fixed and incentive fees of the HF, HFs style or the HFs/MFs market. Note that I also assume that $H_2(0) = 0$ and $H_3(0) = 0$, that is, at time 0 the assets 2 and 3 play a role of investment opportunities that the investor did not use, yet. $W_i(t)$ can be correlated and we denote the correlation matrix of W_i by $(\rho_{i,j})$. The correlation of the HF with other HFs of that style is probably higher and the volatility of a given hedge fund might also be higher than the volatility of the HF style index. So, we expect $\rho_{1,j}$ and $\sigma(j)$ to decrease as j increases. Also, we assume $\mu_i > r$, that is net of fees MFs/HFs perform better than the riskless asset.

I assume that there are proportional transaction costs and adjust the model proposed by Liu and Lowenstein (2002) for our purposes. That is, suppose that $B_i(t)$ and $S_i(t)$ are $\mathcal{F}_t$ adapted processes and represent the total amount of money spent on buying/selling the asset i by time t. I assume that those processes are non-decreasing, right continuous and start at 0. I will discuss two cases separately, in case of no lockups, the values of $B_i(t)$ and $S_i(t)$ will be defined on $[0, T]$ and in case of lockups they will be defined on $t = L, 2L, ..., kL$, which will be the discrete version of our continuous time model.

Hedge funds also impose notice periods which have a typical length of a few months. As I have discussed in the previous section, lockups disadvantage institutional investors because of the lost investment opportunities rather than market-timing and fire-liquidation constraints. For an efficient liquidation of a large portfolio, the liquidation orders should be spread days or weeks apart. A fast liquidation of a large position might be costly on the investor's side and waiting period of a few months is a reasonable time-frame, which is also negligible in comparison with the long-horizon of institutional investors. As a result, we think that explicitly modeling costs of notice periods in our model is an unnecessary complication. However, I will model the notice periods as a constraint on rebalancing in our model. It is realistic to assume that over a long horizon investor's average return cannot exceed 50% per year. Hence, the upper bound on investor's terminal return is equal to $e^{0.5T}$. Now for $K = \frac{1}{N}e^{0.5T}ds$, note that $\int_t^{t+N} K = e^{0.5T}$. So, I restrict the set of admissible policies to the form:

$$B_i(t) = \int_0^t b_i(u)du \text{ and } S_i(t) = \int_0^t s_i(u)du, \text{ where } b_i(u) < K,\ s_i(u) < K. \quad (4.2)$$

Also, let $1 > \lambda_i > 0$ be the proportional transaction cost that the investor carries while buying and selling the asset i. Hence, the money the investor has in asset i will evolve according to the following equation:

$$\frac{dM_i(t)}{M_i(t)} = \mu_i dt + \sigma_i dW_i(t) + dB_i(t) - dS_i(t) \quad (4.3)$$

and the money in the riskless asset would hence be equal to:

$$dM_0(t) = rM_0(t)dt - \sum_{i=1}^{3}(1+\lambda_i)dB_i(t) + \sum_{i=1}^{3}(1-\lambda_i)dS_i(t). \quad (4.4)$$

Hence, the investor's net wealth at time t i.e. the wealth if all the positions were immediately liquidated, is equal to:

$$M(t) = M_0(t) + \sum_{i=1}^{3} M_i(t)(\mathbb{I}_{M_i(t)>0}(1-\lambda_i) - \mathbb{I}_{M_i(t)<0}(1+\lambda_i)). \quad (4.5)$$

Denote by $\mathbb{S}$ the set $\{m_0, m_1, m_2, m_3\} \in \mathbb{R}^4$ where $m_0(t)+\sum_{i=1}^{3} m_i(t)(\mathbb{I}_{m_i>0}(1-\lambda_i) + \mathbb{I}_{m_i<0}(1+\lambda_i)) > 0$. We will call $\mathbb{S}$ the solvency region, that is when the investor's net wealth is positive. We call a buy/sell strategy $B_i(t), S_i(t)$ admissible from time s, if $M_s = m > 0$ and $\{M_0(t), ..., M_3(t)\} \in \mathbb{S}$ and denote the set of all such strategies by $\mathcal{A}(s, w)$. Hence, our institutional investor's problem is to maximize the quantity:

$$\sup_{S_i, B_i \in \mathcal{A}(0, H_0(0))} E_0[e^{-rT} U(M_T)], \tag{4.6}$$

from where we can define the value function:

$$V(t, m) = \sup_{\{B_i, S_i\} \in \mathcal{A}(t,m)} E_t[e^{-rT} U(M_T) | M_t = m]. \tag{4.7}$$

<u>Optimal Value Function without a Lockup Constraint</u>

In the continuous-time case for CRRA utility function: $U(m) = \frac{m^\gamma}{\gamma}$ for $\gamma > 0$ and $U(m) = \log(m)$ for $\gamma = 0$, Shreve and Soner (1994) showed that the value function satisfies the HJB equation:

$$\max\left\{\frac{\partial V}{\partial t} + \mathcal{L}_V, \max_{i=1,2,3} \mathcal{S}_{0i} V, \max_{i=1,2,3} \mathcal{B}_{0i} V\right\} = 0, \tag{4.8}$$

on the solvency region for $t \in [0, T)$, and terminal function:

$$V(T, m) = U(m_0 + \sum_{i=1}^{3} m_i(\mathbb{I}_{m_i>0}(1-\lambda_i) + \mathbb{I}_{m_i<0}(1+\lambda_i)) > 0). \tag{4.9}$$

where,

$$\mathcal{L}_V = \frac{1}{2} \sum_{3 \geq i,j \geq 1} \rho_{ij} \sigma_i \sigma_j m_i m_j \frac{\partial^2 V}{\partial m_i \partial m_j} + \sum_{i=1}^{3} \frac{\partial V}{\partial m_i} + r m_0 \frac{\partial V}{\partial m_0} - rV \tag{4.10}$$

$$\mathcal{S}_{0i}V = -(1+\lambda_i)\frac{\partial V}{\partial m_0} + \frac{\partial V}{\partial m_i},\ \mathcal{B}_{0i}V = (1-\lambda_i)\frac{\partial V}{\partial m_0} - \frac{\partial V}{\partial m_i} \tag{4.11}$$

These type of problems can be solved numerically using Penalty Method (Karatzas, Shreve (1984)) or also formulated into impulse control problem and solved using Monte-Carlo regressions (Carmona, Ludkovski (2007)). I will use penalty method as proposed by Dai and Zhong (2010). Applying the method of Dai and Zhong (2010) and keeping in mind the restrictions on $\mathcal{B}_i(t)$, $S_i(t)$ we get that the initial problem is equivalent to finding:

$$-\frac{\partial W}{\partial t} - \mathcal{L}W = K\sum_{i=1}^{3}[(\mathcal{B}_iW)^+ + (\mathcal{S}W)^+] \tag{4.12}$$

$$W(y,T) = \log(1-\sum_{i=1}^{3}\lambda_i(y_i^+ + y_i^-)) \tag{4.13}$$

where the latter is an approximation of the original problem restricted to admissible policies of the form $B_i(t) = \int_0^t b_i(u)du$ and $S_i(u) = \int_0^t s_i(u)du$.

This problem can be solved by using finite difference discretization of a truncated domain in $\mathbb{R}^3$. I use upwind scheme for the first order partial derivatives and a scheme originally proposed by Oksendal and Sulem (2005) for 2nd order derivatives.

Optimal Value Function with a Lockup Constraint

In the discrete-time setting, the multi-asset portfolio allocation model with proportional transaction costs can be solved by approximate dynamic programming (ADP) methods. Cai, Judd and Xu (2013) demonstrated tractable numerical methods using ADP to solve the problem for up to 6 risky assets. Shen and Wang (2015), proposed an even faster method that is based on theoretical findings of the shape of the non-trading region. The authors claim that their numerical methods based on the Fast Löwner-John Ellipsoid Approximation significantly cut the computational costs. The equations in the discrete time setting are analogous to the continuous time case, with the only difference that t spans the discrete points $0, L, ..., kL$.

ADP as in Cai, Judd and Xu (2013) will suffice our case and I shortly describe the ADP approach, below:

- constructing grid points using Sobol low-discrepancy numbers
- approximate the value functions with Chebyshev Polynomials (spectral method)
- use Gauss-Hermite quadrature for efficient calculation of conditional expectations
- iteratively-maximize the value function

After computing the value functions, I use the definition of the constraint premium of the previous section to implement the lockup premium for our case. The implementations were done in Matlab. Unfortunately, at the time of this research, to the best of our knowledge Cornell University was not subscribed to any Hedge Fund databases (such as TASS or Morningstar). Because of that, I was not able to apply our theoretical model on the hedge fund return data.

Lockup Premium: Markov-Switching Framework

In this section, I discuss how to model the lockup premium in the Markov-Switching framework. The Markov-Switching framework was chosen because it best addresses the issue of the structural changes in the expected returns of the funds. For example, suppose that after monitoring a given hedge fund class, the institutional investor notes that the expected returns changed to a different state and the whole hedge fund style under-performs. Those type of return evolutions are easily captured in the Markov-Switching framework. Besides, the investment horizon of an institutional investor is typically quite long and the Markov-Switching framework is suitable for long-horizon modeling, too (see e.g. Hardy (2001), Freeland, Hardy and Till (2009), Hamilton (2010), Bulla (2011)).

As in the previous section, we have 3 risky assets which represent a given hedge fund, the hedge fund style and the market. In addition, we have 1 riskless asset and a 4-state Markov-Chain describing the changes of the underlying expected returns of the risky assets. To model the lockup premium in this situation, I adjust the setup and use the results obtained by Canakoglu and Ozekici (2009, 2012).

The evolution of the risky assets in the given framework are modeled by the following equation:

$$dH_k(t) = \mu_k(i)H_k(t)dt + H_k(t)\sum_{j=1}^{3}\sigma_{kj}dW_j(t), \tag{4.14}$$

$i = 1, .., 4$, $k = 1, .., 3$.

From where, the wealth evolves according to:

$$dX_t^u = \sum_{k=1}^{m} u_k(t)\frac{dH_k(t)}{H_k(t)} + (X_t^u - \sum_{k=1}^{m} u_k(t))\frac{dB(t)}{B(t)}. \tag{4.15}$$

In this case the Value functions $V(x,t)$ is:

$$V(x,t) = \sup_{u \in \mathcal{A}} E_1[U(Y_T, X_T^u)] \tag{4.16}$$

For the HARA utility function, Canakoglu and Ozekici (2012) provide explicit solutions for the continuous-time case. For the discrete-time case a numerical solution is provided in Canakoglu and Ozekici (2009). Combining those results with the methodology developed in the previous chapter, I compute the illiquidity premium. I use the definition 3.8 of the premium of a constraint. For simplicity, let us assume that we have an investor with an exponential utility function $U(x) = 1 - e^{-x}$. Then, the expectation of the optimal wealth in unconstrained (Continuous-Time) and Constrained (Discrete-Time) cases are:

Continuous-Time Case:

$$E_1(X_T^*) = x_0 e^{rT} + m_e(p), \text{ where } m_e = p^2 m_e(1) \tag{4.17}$$

Discrete-Time Case:

$$E_1[X_T^*] = x_0 r^T + m_e(p), \text{ where } m_e(p) = p^2 m_e(1) \tag{4.18}$$

The dependence on p is not explicit in the terminal value function: $V(x_0, 0) = 1 - E_1[e^{-X_T^*}]$, so I estimate it using Monte-Carlo simulations and Taylor Expansion. The implementation is done in Matlab (running time ≈ 2 minutes) for a sample data set built based on the appendix in the Canakoglu and Ozekici (2010). In the figure 4.1 we can see that the expected optimal utility increases in both constrained and un-constrained frameworks. The latter enables us to compute the constraint premium using the definition 3.8. Because I used a "random" data set, the results don't represent an actual value of a lockup premium.

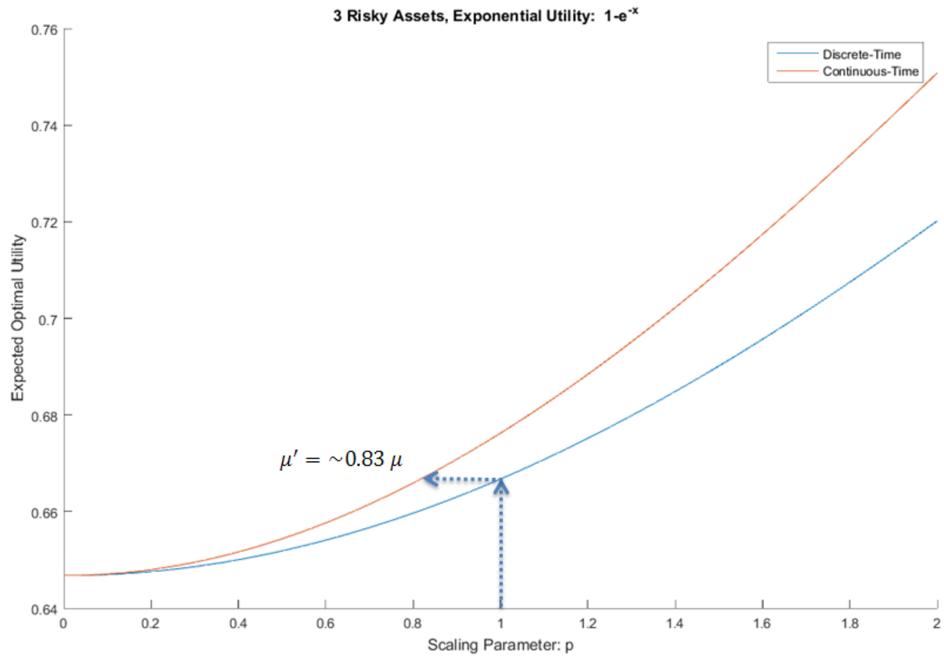


Figure 4.1: Application of the Regime-Switching Model

Number of Markov-Chain States: 4, Number of Risky Assets: 3, Investment Horizon: 4 years, Length of the Lockups in Years: 2, Wealth at Time 0: 1, Mean of the Returns in Different States: $\mu = [1.05\ 1.04\ 1.03;\ 1.04\ 1.031.05;\ 1.03\ 1.04\ 1.05;\ 1.05\ 1.03\ 1.04]$, Covariance Matrix: $\sigma = [2.425\ 1.809\ 0.607; 1.809\ 5.990\ 0.684;\ 0.607\ 0.684\ 1.893]$, The Markov-Switching Matrix:

$Q = [0.6\ 0.3\ 0.05\ 0.05;\ 0.05\ 0.6\ 0.3\ 0.05;\ 0.05\ 0.3\ 0.6\ 0.05;\ 0.05\ 0.05\ 0.3\ 0.6]$.

However, the results demonstrate the possibility to numerically compute the lockup premium in the proposed Markov-Switching framework.

4.3 Lockup Premium: Simplified Framework

Based on the work by Gushchin, Trikoz and Verdiyan (2014)

In this section we will introduce a simplified framework to measure the illiquidity premium of the hedge fund lockups. The motivation is to simplify the model in a way to be able to implement it on the type of the data which we have access to, which is the data on different hedge fund index returns.

Throughout the chapter, we will use the phrases "illiquidity premium induced by share restrictions" and "illiquidity premium", interchangeably. Our definition of the illiquidity premium enables us to compare it with the yearly return rates of different asset classes. This section also models portfolios similar to those of large endowments, such as Yale and MIT, which outsource most of their investing to outside managers. We measure the lockup illiquidity premium an endowment receives for taking on illiquidity risk through money managers in asset classes such as Hedge Funds (Absolute Return), Private Equity and Foreign Equity. To compute the illiquidity premium in the cases above, we add more structure and assumptions on the dynamics of the asset movements.

We are given a complete filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{P})$ that satisfies the usual hypothesis as presented in Protter [26]. T can be any positive real number, but for convenience we assume that T is the amount of time in months. Denote by r_t the default-free spot rate of interest that is adapted to $\mathcal{F}_t$ and is integrable. We consider a long-horizon investor with a utility function $U(\cdot)$ that has an access to a riskless asset described by a process $X_0(t) = e^{\int_0^t r_s ds}$ and n money managers investing in different asset classes. The investor has a fixed amount of money V at time 0 and wants to construct a portfolio P to maximize its expected utility at time T. We denote by $X_i(t)$ the investment return process of the money manager i and assume that the measurable index set $I_i \in [0, T]$ includes all the times at which it is possible to put a withdrawal notification. We assume that $X_i(t)$ are semimartingales adapted to $\mathcal{F}_t$. Let us denote the constant length of the notice period of the money manager i by $N_i \geq 0$. Money managers investing in more illiquid asset classes usually have longer lockup periods. Note that in our model we can also have money managers with no share restrictions. Those will be the ones for whom $N_i = 0$ and $I_i = [0, T]$. We will say those money managers are liquid.

The standard notion of self-financing trading strategies is not applicable in this framework, since we cannot short money managers. Continuous re-balancing of the portfolio is also not possible due to notice periods and share restrictions. Even in case of liquid money managers, frequent inflow/outflow of money is not realistic since it might not allow them to maintain a specific trading strategy. So, we can expect the re-balancing to be infrequent even when all the chosen money managers are liquid. Motivated by the aforementioned, we describe the actions of our long-horizon investor at times $(0, T]$ by a finite number of decisions to re-balance the portfolio. Our aim is to find a definition for share restrictions illiquidity premium, so that it does not depend on the availability of the money managers to an investor. Because of it, we make a simplifying assumption that if the investor decides to withdraw the money from the manager i, he will put that money in the riskless asset and hold it there until time T. Next, we define a notion of a restricted stopping time.

Definition 4.3.1. *Given the measurable index set $I \subseteq [0, T]$, we denote by $\mathcal{R}[I]$ the set of all random variables τ taking values in I such that $\{\tau \leq t\} \in \mathcal{F}_t$. We call elements of $\mathcal{R}[I]$ restricted stopping times.*

Next, after denoting by $\mathcal{R}$ the space $(\mathcal{R}[I_1], ..., \mathcal{R}[I_n])$, we can define the optimal investor problem:

The Investor Problem: At time 0, find the optimal allocation of money V into n money managers and a riskless asset: V_0,...,V_n (where $V_i \geq 0$ for $i \geq 1$ and $V_0 + ... + V_n = V$), so that to maximize the value of

$$\sup_{\tau \in \mathcal{R}} \mathbb{E}^{\mathcal{P}} \left[U \left(V_0 e^{\int_0^T r_s ds} + \sum_{i=1}^{n} V_i X_i(\tau_i + N_i) e^{\int_{\tau_i+N_i}^T r_s ds} \right) |\mathcal{F}_0 \right] \tag{4.19}$$

Using our notations, the expected cost an investor carries because of the share restrictions of the money manager i is:

$$C_i = V_i \left(\sup_{\tau_i \in \mathcal{R}[0,1]} \mathbb{E}^{\mathcal{P}}[X_i(\tau_i + N_i) e^{\int_{\tau_i+N_i}^T r_s ds}] - \sup_{\tau_i \in \mathcal{R}[I_i]} \mathbb{E}^{\mathcal{P}}[X_i(\tau_i + N_i) e^{\int_{\tau_i+N_i}^T r_s ds}] \right) \tag{4.20}$$

In choosing a way to define share restrictions illiquidity premium in our theoretical framework we were motivated by the following intuition. We should be able to compare it with the yearly interest rates and the yearly returns of other assets. So, we should scale appropriately for the starting investment amount and for the time to the end of our investment period T. As a result, we define the predicted yearly illiquidity premium of money manager i to be:

$$p_i = \left(1 + \frac{C_i}{V_i} \right)^{12/T} - 1. \tag{4.21}$$

Note that the previous equation can be rewritten as:

$$p_i = \left(1 + \sup_{\tau_i \in \mathcal{R}[0,1]} \mathbb{E}^{\mathcal{P}}[X_i(\tau_i + N_i) e^{\int_{\tau_i+N_i}^{T} r_s ds}] - \sup_{\tau_i \in \mathcal{R}[I_i]} \mathbb{E}^{\mathcal{P}}[X_i(\tau_i + N_i) e^{\int_{\tau_i+N_i}^{T} r_s ds}]\right)^{12/T} - 1. \tag{4.22}$$

As we can see, the illiquidity premium does not depend on the initial portfolio allocation choice V_i. Besides, $(1 + p_i)^{\frac{T}{12}} - 1 = \frac{C_i}{V_i}$ represents the expected fraction of loss for each invested dollar. Hence, our definition is indeed comparable with yearly returns and interest rates. We call the illiquidity premium p_i "predicted" since we can compute it for the time-period $[0, T]$, standing at time 0. Also note that the illiquidity premium defined above is from a perspective of an optimal investor and can be used as an absolute measure of illiquidity.

Notes on Data Collection:

We use as our input data indices and composites that the endowments themselves use to track their performance. These would serve as inputs to our model that would mimic the actual endowment allocations. The selection of the indices was based on analysis of asset allocations of Yale, Harvard and MIT and their respective benchmarks for hedge funds, private equity and real estate pools. These are published in the endowment updates online [27]. Once the index selection was finalized, the data was downloaded. For publicly managed data, the indices are readily available through Bloomberg terminal. In cases where the endowment used a custom in-house benchmark, an approximation of that benchmark was made with an available index. For private equity and real estate, we used Cambridge Associates return aggregates published on their website.

Below is the list of indices that were used for each asset class: Domestic Equity

(RAY Index: RUSSELL 3000 Index), Foreign Equity (SPBMUMUT Index: S&P Developed Ex-U.S. BMI MidCap TR), Emerging Markets Equity (SCRTEM Index: S&P Emerging BMI), Absolute Return (HEDGNAV Index: Credit Suisse Hedge Fund Index (Tremont Composite)), Real Estate (Cambridge Composite: Real Estate Index [28]), US Private Equity (Cambridge Composite: U.S. Private Equity Index [29]), Fixed Income (3 year treasury yields from the U.S. Department of the Treasury [30]), Cash (Yearly LIBOR rate [31]).

Cambridge Associates PE Composite returns are quarterly returns. Actual Private Equity returns don't behave like any Private Equity index and in fact, are closer related to a hedge fund index for Absolute Return. Further research backs up this conclusion: the nature of the private equity asset class prevents a construction of a replicable benchmark. Towers Watson reports that the "the lack of a readily available universe of transactions and assets makes it challenging to construct a replicable index. As such, there is no recognized index that captures the entire opportunity set available to private equity managers. [32]" The timing of the cash flows in private equity is unpredictable and often, the private equity universe cannot be defined.

Implementation:

In order to run simulations and compute the desired illiquidity premium, we need to assume some dynamics on the underlying processes. Money managers usually can be grouped by types of strategies and the asset classes they invest in. There are also some indices that track average performance statistics of different categories of money managers. A widely used approach in academia and in industry is to model price processes of equities and indices by a Geometric Brownian Motion. Our framework allows us to use more advanced dynamics such as the double-exponential jump diffusions (see [33], [34]), time-changed Brownian motions (see [35]) or affine jump-diffusion models (see [36]). However, to ease computational tractability we will also assume that our risky assets follow a Geometric Brownian motion. Parameters such as the correlation matrix, mean returns and volatility are found from the historical data and will be discussed later.

Let the values $\mu_0, ..., \mu_n$ be the per-period returns of our assets. So, in the continuous time framework our riskless asset will grow by the formula:

$$X_0(t) = \exp((t)ln(1 + \mu_0)) \tag{4.23}$$

Suppose, the assets in our portfolio P are described by the following SDE

$$dX_i(t) = ln(1 + \mu_i)X_i(t)dt + X_i(t)\sum_{j=1}^{n} \sigma_{i,j} dW_j(t). \tag{4.24}$$

where $i = 1, .., n$ and we assume that the $n \times n$ matrix σ with entries $\sigma_{i,j}$ in invertible.

The latter evolution in the literature is described as a special case of the Multidimensional Market Model ([37], p.226). We will modify it to fit our theoretical framework. It is easy to see that (4.24) can be rewritten as:

$$dX(t) = \mu X(t)dt + \mathbb{I}(X(t))\sigma dW(t) \tag{4.25}$$

where $X(t)$, μ, $dW(t)$ are the vectors of $X_i(t)$, $ln(1+\mu_i)$, $dW_i(t)$, appropriately, σ is the matrix $\sigma_{i,j}$ and $\mathbb{I}(X(t))$ is a matrix with vector $X(t)$ on its diagonal. The notice periods are usually standard and much shorter than the lockup periods, so for our particular example we will assume $N_i = 0$.

In the described framework, the predicted illiquidity premium for money manager i is:

$$p_i = \left(1 + \sup_{\tau_i \in \mathcal{R}[0,1]} \mathbb{E}^{\mathcal{P}}[X_i(\tau_i) e^{\int_{\tau_i}^{T} r_s ds}] - \sup_{\tau_i \in \mathcal{R}[I_i]} \mathbb{E}^{\mathcal{P}}[X_i(\tau_i) e^{\int_{\tau_i}^{T} r_s ds}]\right)^{12/T} - 1. \quad (4.26)$$

Results: For implementation, we first determined the historical returns, correlations and volatilities of our assets given the data for years 2001-2013. Next, with those inputs we simulated correlated Geometric Brownian Motions for the next 6 years. For the implementation of the Geometric Brownian Motions we heavily used the Econometrics and Financial Toolboxes of Matlab. We implemented Monte Carlo simulations to find the values of the predicted illiquidity premiums for different asset classes. To simulate restricted/unrestricted stopping times, we created various rules when to sell the asset. Those rules comprised of sets of upper and lower barriers. For example, if the asset price goes over a high enough barrier then the model sells the asset, assuming that it is an optimally high value and the asset values will probably decrease in the future. We calculate the predicted illiquidity costs for investing with money managers for 6 years starting 12/2013.

Our computed (yearly) illiquidity premiums for money managers investing in the following classes of assets are: Domestic Equity 1.47% (2 year lockup), Foreign Equity 2.42% (2 year lockup), Fixed Income 2.69% (2 year lockup), Absolute Return 2.82% (2 year lockup) and Private Equity 1.87% (6 year lockup). For our further analysis we will pick only the last two assets. For around 20,000

Monte Carlo simulations, our illiquidity premium values were converging with accuracy 0.0002 and for 40,000 Monte Carlo simulations the outputs were converging with accuracy 0.00002. Performing 100,000 simulations the predicted illiquidity premium values for the Hedge Funds with illiquidity period 2 years were: 2.45% annually, for Private Equity with illiquidity period 6 years: 1.85% annually and for the period 2 years: 2.58% annually.

Correlation and Expected Return Analysis

Figure 4.2 shows the average of the correlation matrix that was used in our simulations. The scale goes from dark red to white to green to represent positive correlation, no correlation and inverse correlation, respectively. As expected, we see high correlation among hedge fund returns and little correlation among cash, fixed income asset classes. Private Equity is somewhat equally correlated with others, while Real Estate is the least correlated with others among illiquid asset classes.

Correlation Matrix for Whole Period: 2002-2014

	Domestic Equity	Foreign Equity	Emerging Markets Equity	Absolute Return	Fixed Income (3 yr)	LIBOR (1 year)	US Private Equity	Real Estate
Domestic Equity	1.00	0.55	0.55	0.47	0.00	-0.03	0.29	0.12
Foreign Equity	0.55	1.00	0.54	0.48	0.00	-0.02	0.24	0.10
Emerging Markets Equity	0.55	0.54	1.00	0.51	0.00	-0.02	0.27	0.12
Absolute Return	0.47	0.48	0.51	1.00	0.02	-0.01	0.31	0.19
Fixed Income (3 yr)	0.00	0.00	0.00	0.02	1.00	0.11	0.03	0.02
LIBOR (1 year)	-0.03	-0.02	-0.02	-0.01	0.11	1.00	0.00	-0.02
US Private Equity	0.29	0.24	0.27	0.31	0.03	0.00	1.00	0.28
Real Estate	0.12	0.10	0.12	0.19	0.02	-0.02	0.28	1.00

Figure 4.2: Average Correlation Matrix for 2002-2014

For comparison, let us look on the correlation matrix from 2008-2010. The matrix is shown in Figure 4.3. It's largely the same as the full period correlation matrix. However, on average, all asset classes are slightly more correlated among each other and cash is more negatively correlated with all asset classes.

This confirms that the asset classes tend to correlate more during crisis than during normal markets. This knowledge may be useful to an investor who is trying to hedge their risk in the future.

Correlation Matrix for Financial Crisis: 2008-2010

	Domestic Equity	Foreign Equity	Emerging Markets Equity	Absolute Return	Fixed Income (3 yr)	LIBOR (1 year)	US Private Equity	Real Estate
Domestic Equity	1.00	0.58	0.56	0.48	-0.05	-0.12	0.29	0.18
Foreign Equity	0.58	1.00	0.57	0.48	-0.06	-0.11	0.24	0.14
Emerging Markets Equity	0.56	0.57	1.00	0.54	-0.04	-0.11	0.29	0.17
Absolute Return	0.48	0.48	0.54	1.00	-0.01	-0.11	0.42	0.29
Fixed Income (3 yr)	-0.05	-0.06	-0.04	-0.01	1.00	0.28	0.00	-0.06
LIBOR (1 year)	-0.12	-0.11	-0.11	-0.11	0.28	1.00	-0.11	-0.14
US Private Equity	0.29	0.24	0.29	0.42	0.00	-0.11	1.00	0.43
Real Estate	0.18	0.14	0.17	0.29	-0.06	-0.14	0.43	1.00

Figure 4.3: Average Correlation Matrix for 2008-2010

	Whole Period: 2002-2014				Financial Crisis: 2008- 2010		
	Mean	**Stdev**	**Corr**		**Mean**	**Stdev**	**Corr**
Domestic Equity	0.005	0.045	1.000		-0.006	0.071	1.000
Foreign Equity	0.011	0.060	0.552		-0.004	0.110	0.580
Emerging Markets Equity	0.010	0.066	0.552		-0.003	0.107	0.561
Absolute Return	0.006	0.016	0.466		-0.001	0.027	0.477
Fixed Income (3 yr)	0.002	0.001	-0.004		0.002	0.000	-0.047
LIBOR (1 year)	0.002	0.001	-0.028		0.002	0.001	-0.116
US Private Equity	0.011	0.017	0.290		-0.004	0.025	0.288
Real Estate	0.006	0.021	0.124		-0.029	0.025	0.183

Figure 4.4: Average Mean and Standard Deviation of Each Asset Class During Two Periods

In addition, we look at the expected return and standard deviation of each asset class during the full period and financial crisis. Many studies indicate that during financial crises or times of high volatility, correlations between assets generally increase (see for example [38], [39], [40] and [41]). To address that issue, we assess the sensitivity of our illiquidity premium depending on the correlations and mean returns of our assets.

Next, we see how sensitive our computed illiquidity premiums are while scaling the underlying correlation matrix. Except the entries on the diagonal, we scale the correlation entries and make them artificially smaller or higher. A problem one might encounter in scaling the values of correlation matrix is that after the changes it might not be positive-semidefinite anymore. To solve the latter problem we are using the Alternating Projections Algorithm to find the nearest positive semidefinite matrix as described in Higham (2002) [42].

The results (see Figure 4.5) indicate that when all the asset classes (including the interest rates) become more (less) correlated, the values of our computed illiquidity premiums do not change much. So, our model is quite robust. But, the graphs also indicate that while the correlations increase, the illiquidity premium also has a positive dynamics. During crises the returns of all asset classes (except the riskless one) also decrease. We analyzed how the change in the expected returns affect the illiquidity premium (see Figure 4.6).

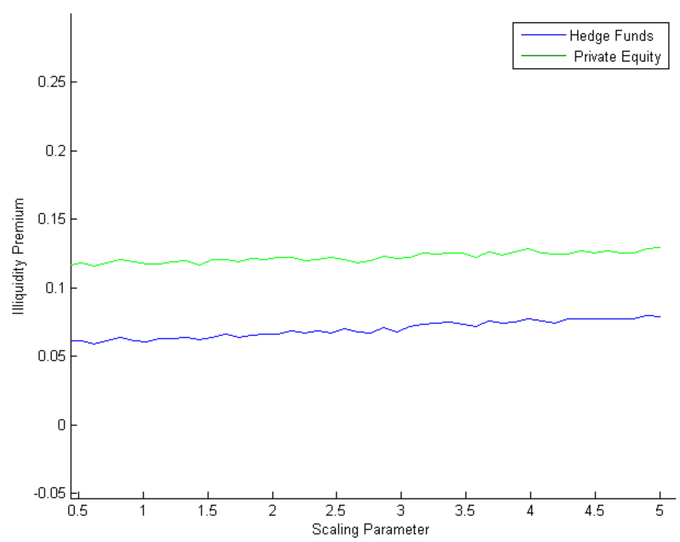


Figure 4.5: Scaling the Correlation Matrix

One can see that there is a positive correlation between the expected returns and the illiquidity premium. When we scale the expected returns up, the illiquidity premium also increases. For the Hedge Fund class in particular,

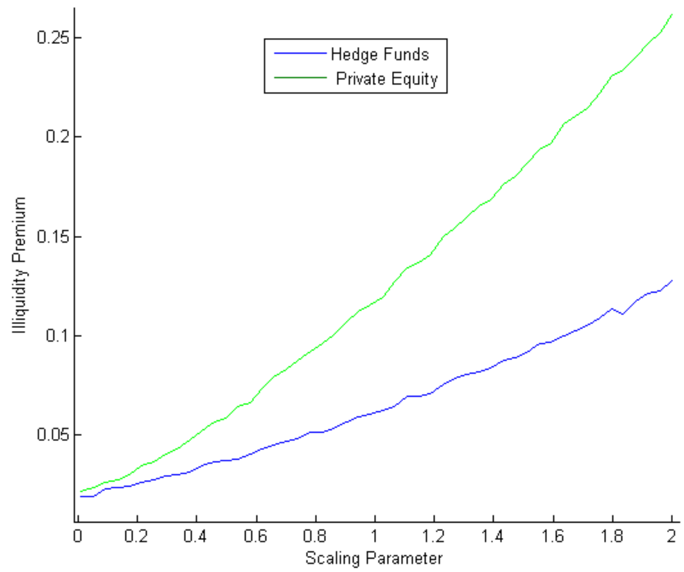


Figure 4.6: Scaling the Mean of the Expected Returns

this result confirms the findings of Bali, Gockan and Liang (2007) [21] , Ackerman, McEnally and Ravenscraft (1999) [22] and Liang(1999) [23], that the Hedge Funds with more illiquidity (longer lockup restrictions) have higher expected returns.

CHAPTER 5

CONCLUSIONS

In the first part of this work, I introduced theoretical definitions to describe two type of factors: **Attention** (ones that guide investor's attention towards a subset of stocks) and **Action** (ones that could be the triggers of the trading actions). After creating a framework to quantify various input and output information, I presented a model which enabled us to combine the Buy/Sell trading signals of a given fund. The implementation of the model on the US Mutual Fund data confirmed the existence of the previously hypothesized Attention/Action factors. I found that factors such as **turn** or **size** are often used to direct the investor's attention towards a subset of stocks. However, other factors such as 3 or 6 month momentum, throughout the last 25 years, remain the main reason of a trading action of a fund. Surprisingly, the factors **bm, pcf or ptb**, although being widely used by the Value Investors, are used only by 3% of the funds as triggers for their trading actions. Using cluster analysis, we saw that a fund's preference towards one factor might vary depending on the values of the other factors. By extending the methodology of finding the trading preferences of a fund towards 1 factor, I introduced the "Preference Directionality Map". The latter is used to find the preference of the fund's Buy/Sell trading actions depending on 2 or more factors. The implementation of the directionality map on four sample US Mutual Funds demonstrated the similar patterns of preference within the same investment philosophy.

The described methodology of determining the trading preferences of the funds is fundamentally different from the commonly accepted empirical approaches of finding fund's trading style based on the fund holdings. This is because the holdings-based trading style measures by Grinblatt et al. (1995) and Daniel et al. (1997) were designed to address the question of the fund performance rather than finding the Buy/Sell preferences in relation to various sources of information. To the best of my knowledge, the described approach is also fundamentally different from the other popular (empirical) models in academia which extract mutual fund preferences (e.g. those by Falkenstein (1996)[1] or Bushee (1998, 2001) [2]).

To better understand how to combine the "preference directionality maps" of multiple funds, I introduced the Action|Attention (A|A) model. Exploring the A|A model, we saw that the previously constructed "preference directionality maps" were representing the expected amounts of a trading action in a given factor region. After introducing the **Preference** and the **Decision** functions, I was able to use the A|A model to predict the aggregate mutual fund sell actions for 20-50% (depending on the chosen range of accuracy) of stocks[3]. In the majority of the cases, the A|A model performed 2-3 times better than the fund predictions made by the time-series predictors. The previous quarter sell was the second best predictor. The latter confirms the finding of Sias (2004), that the investor's aggregate demand for a stock is similar in the adjacent quarters.

An important conclusion of this work is that the aggregate fund trading and "herding" does not require funds to follow each other's trading actions. Note that the main assumption in the Action|Attention model actually was that the funds do not copy each other's trading decisions[4]. However, fitting the model

[1]The paper documents fund preferences towards stocks with low transaction costs by analyzing the mutual fund holdings for the year 1991 and 1992.

[2]Bushee (1998, 2001) classifies the US funds using empirical analysis as either dedicated, transient or quasi-index fund.

[3]Within the stocks for which the A|A model produced a sell prediction.

[4]The reason of introducing such an assumption was to allow us to use the law of large numbers to predict aggregate fund actions.

to the data produced solid predictions for the aggregate fund trades which were better [5] than the ones produced by the previous quarter predictor. In contrast to the conclusions of Sias (2004), we saw that a model where funds do not intentionally follow each other's trading actions can fit and predict the aggregate US Mutual fund sales. Thus, the Action|Attention model provides important evidence against the commonly used hypothesis, which states that the reason why institutions herd is because they follow each other's trading actions (see models by Bikhchandani, Hirshleifer, and Welch (1992) or Scharfstein and Stein (1990)). The alternative explanation of the aggregate fund trading patterns provided by the A|A model is that the herding is "unintentional" because the funds follow similar trading philosophies[6]. So, when a stock enters a region towards which many funds have strong one-sided (i.e. either Buy or Sell) preferences [7], the probability of a one-sided trading action increases and "herding" could happen.

An important observation is that we do not have access to the complete input information which a fund manager actually uses for trading. For example, if a fund made a purchase based on a meeting of a company's CEO with the shareholders, that set of the information would not be included in our dataset. So, our analysis would be about the relation of the public information with the fund's trading decisions. That public information might still not be free to access. But, the main assumption of our model is that any fund, given the will to spend the necessary resources, can access an information set, fairly. If a fund trades

[5] That is producing a larger percentage of predictions which are within the 25% and 50% range of the actual aggregate fund sell actions. For example, for the 50% range of accuracy, in the 96% of the cases the A|A predictor's success rate was higher (for 34% of the cases at least twice higher) than the one produced by the previous quarter predictor.

[6] That is, the funds have similar beliefs about stocks with which factor values could outperform the market.

[7] I call them the "opinionated" regions.

based on private or purely discretionary reasons, I expect the relation not to be explainable by my analysis. Fortunately, with the introduction of the machine learning tools, the amount of ”non-quantifiable” information decreases. Thus, I hypothesize that the predictability of the low-cost fund trades would continue to increase.

There are various possible improvements and further research options, which remained outside of the scope of this work. For example, based on the initial IVNM analysis, it is possible to increase the accuracy of the IVNM predictions by taking instead of V^{ϵ}, the maximum or the minimum values of the factors produced in a quarter. Another improvement could be to increase the accuracy of the trading observations, by reverse-engineering the mutual fund holdings from their daily return data.

Note that the aggregate fund predictions were targeted towards individual stocks. A further analysis of stocks might show which stocks are more predictable and one could combine stocks in portfolios to decrease the variance of the prediction. Lastly, the aggregate institutional trades are correlated with the short-term stock price movements (see e.g. Wermers (1999) or Dasgupta, Prat and Verardo (2011)). An interesting question would be to further analyze the correlation of the predictions of the A|A model with short-term stock price movements.

In the constraint reduction framework, I developed a theory to correctly measure the costs of the portfolio constraints. My motivation was to understand how to model the Hedge Fund lockups by treating them as re-balancing restrictions. We saw that the value of the expected terminal utility function is naturally smaller in the constrained setting. So, I proposed to penalize the expected returns of the assets in the un-constrained model and, as a result, lower the resulting terminal utility's expected value. When it is possible to lower that value to the level of the constrained model, we are able to define the premium of a constraint. The approach was described in a general theoretical setting and I demonstrated its use on a classical Merton Portfolio Construction example.

I argued that the hedge fund lockups should be modeled as a lost investment opportunity premium. The investment opportunities of the institutional investors typically consist of investing through the asset managers. I modeled lockups as an inability to rebalance investor's portfolio into: a riskless asset, another hedge fund class or another mutual fund. The lockup premium was constructed in the transaction cost and Markov-Switching frameworks. Generally, implementing optimal portfolio construction model with transaction costs might not be practical (Korn, 2004). However, we saw that by using the recent advances in the approximate dynamic programming (Cai, Judd and Xu, 2013), it is possible to calculate the hedge fund lockups in a transaction cost setting. I also modeled the hedge fund lockups in a Markov-Switching framework, which I argued to be a better representation of the Hedge Fund return process. Due to the limitations on the data access, I only implemented the latter model on a "random" data set. However, it demonstrated the possibility to numerically compute the lockup premium in the Markov-Switching framework. I conclude that the modern numerical tools enable us to model hedge fund lockups in a realistic setting.

Next, we simplified the lockup premium framework in order to apply it to the data on hedge fund indexes (the one we had access to). We fitted a simplified theoretical model to historic data and simulated asset price paths for the next

6 years. This model produced lockup restriction illiquidity premium values for different asset classes. Those values were typically between $\approx 1.5-2.5\%$, yearly. Thus, in some cases the implicit lockup premium is more than the standard 2% hedge fund management fee. So, we conclude that the hedge fund lockup premium values are significant and need to be considered by a sophisticated institutional investor to choose between hedge funds and lower cost funds.

Scaling the input correlation matrix indicates that our model is robust with respect to collective changes of the correlations of the assets. Our results also indicate that the the higher the expected returns of the asset, the more illiquidity premium it gets because of the share restrictions. For specific case of hedge funds, our results support the findings of Bali, Gockan and Liang (2007) [21] , Ackerman, McEnally and Ravenscraft (1999) [22] and Liang(1999) [23].

An obvious extension of this work would be to apply the models to actual Hedge Fund return data and compute the lockup premiums. Note that the proposed approach measured the illiquidity premium of the hedge fund lockups independent from the portfolio construction model of the institutional investor. So, the approach is mainly limited to the cases when the aim is to estimate an absolute measure of the illiquidity premium. If the optimal portfolio construction strategy of the institutional investor is known, the lockup premium should be calculated based on the that strategy. The reason is that the illiquidity premium heavily depends on how efficient the portfolio construction model of the investor is (Gushchin, Trikoz and Verdiyan (2014)). In particular, the hedge fund lockups could even be beneficial by protecting an inexperienced investor from market timing mistakes and fire sales.

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