ON FREE ω-CONTINUOUS AND REGULAR ORDERED ALGEBRAS

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ABSTRACT. Let E be a set of inequalities between finite Σ-terms. Let $\mathcal{V}_\omega$ and $\mathcal{V}_r$ denote the varieties of all ω-continuous ordered Σ-algebras and regular ordered Σ-algebras satisfying E, respectively. We prove that the free $\mathcal{V}_r$-algebra $R(X)$ on generators X is the subalgebra of the corresponding free $\mathcal{V}_\omega$-algebra $F_\omega(X)$ determined by those elements of $F_\omega(X)$ denoted by the regular Σ-coterms. We actually establish this fact as a special case of a more general construction for families of algebras specified by syntactically restricted completeness and continuity properties. Thus our result is also applicable to ordered regular algebras of higher order.

1. ω-Continuous Algebras

Let Σ be a ranked alphabet, which will be fixed throughout. A Σ-algebra A is called *ordered* if A is partially ordered by a relation $\le$ with least element $\bot^A$ and the algebraic operations are monotone with respect to $\le$; that is, if $f \in \Sigma_n$ and $a_i, b_i \in A$ with $a_i \le b_i$ for $1 \le i \le n$, then $f^A(a_1, \ldots, a_n) \le f^A(b_1, \ldots, b_n)$. A morphism $h : A \to B$ of ordered algebras is a strict monotone map that commutes with the algebraic operations:

$$a \le b \Rightarrow h(a) \le h(b) \qquad h(\bot^A) = \bot^B \qquad h(f^A(a_1, \ldots, a_n)) = f^B(h(a_1), \ldots, h(a_n))$$

for all $a, b, a_1, \ldots, a_n \in A$ and $f \in \Sigma_n$, $n \ge 0$.

For each set X, there is a free ordered algebra $\mathsf{T}X$ freely generated by X. The elements of $\mathsf{T}X$ are represented by the finite partial Σ-terms over X; here *partial* means that some subterms may be missing, which is the same as having the empty term $\bot^{\mathsf{T}X}$ in that position. When $t \in \mathsf{T}X$ and A is an ordered algebra, t induces a function $t^A : A^X \to A$ in the usual way.

An ordered algebra A is *ω-continuous* [5, 7, 8] if it is ω-complete and the operations are ω-continuous. That is, any countable directed set (or countable chain) C has a supremum $\bigvee C$, and when $n \ge 0$, $f \in \Sigma_n$, and C_i is a countable directed set for $1 \le i \le n$, then

$$f^A(\bigvee C_1, \ldots, \bigvee C_n) = \bigvee \{f^A(x_1, \ldots, x_n) \mid x_i \in C_i,\ 1 \le i \le n\}.$$

A morphism of ω-continuous algebras is an ω-continuous ordered algebra morphism.

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For each set X, there is a free ω-continuous algebra $\mathsf{C}X$ freely generated by X. The elements of $\mathsf{C}X$ are the partial coterms over Σ (finite or infinite partial terms). When $t \in \mathsf{C}X$ and A is an ω-continuous algebra, t induces a function $t^A : A^X \to A$.

Suppose that X_ω is a fixed countably infinite set. A set of inequalities $t \sqsubseteq t'$ between terms in $\mathsf{T}X_\omega$ determines in the usual way a *variety of ordered algebras*, denoted $\mathcal{V}$, and a *variety of ω-continuous algebras*, devoted $\mathcal{V}_\omega$. An ordered algebra A belongs to $\mathcal{V}$ iff $t^A \le t'^A$ in the pointwise ordering of functions $A^{X_\omega} \to A$ whenever $t \sqsubseteq t'$ is in E. The class $\mathcal{V}_\omega$ contains all ω-continuous algebras in $\mathcal{V}$. It is known that all free algebras exist both in $\mathcal{V}$ and in $\mathcal{V}_\omega$ [3, 8]. Moreover, the free algebra in $\mathcal{V}$ freely generated by a set X can be represented as the X-generated ordered subalgebra $F(X)$ of the free ω-continuous algebra $F_\omega(X)$ in $\mathcal{V}_\omega$. Moreover, $F_\omega(X)$ is the completion of $F(X)$ by ω-ideals (see §2).

More generally, for C a subset of an ordered algebra A, define

$$C{\downarrow} = \{c \mid \exists a \in C \;\; c \le a\}.$$

The set C is an *order ideal* if it is directed and downward closed; that is, if $a, b \in C$ implies there exists $c \in C$ such that $a, b \le c$, and if $a \le b$ and $b \in C$, then $a \in C$, i.e. $C = C{\downarrow}$. An order ideal $C \subseteq A$ is called an *ω-ideal* if it is countably generated; that is, there is a countable directed set $C_0 \subseteq C$ such that $C = C_0{\downarrow}$. The set $I_\omega(A)$ of all ω-ideals of A ordered by set inclusion $\subseteq$ is an ω-complete poset, where the supremum of an ω-ideal $\mathcal{A}$ of ω-ideals is the union $\bigcup \mathcal{A}$. We can turn $I_\omega(A)$ into an ω-continuous algebra by defining

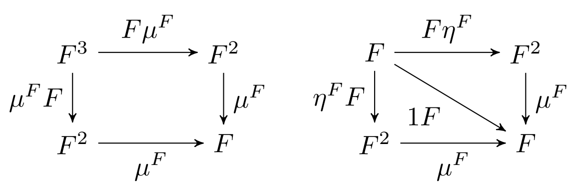
$$f^{I_\omega(A)}(C_1, \ldots, C_n) = \{f^A(a_1, \ldots, a_n) \mid a_i \in C_i,\ 1 \le i \le n\}{\downarrow},$$

the order ideal generated by the set $\{f^A(a_1, \ldots, a_n) \mid a_i \in C_i,\ 1 \le i \le n\}$. It is clear that A can be embedded in $I_\omega(A)$ by the ordered algebra morphism mapping $a \in A$ to the order ideal $\{a\}{\downarrow}$ generated by a. In fact, it is known that $I_\omega(A)$ is the *free completion* of A to an ω-continuous algebra [3, 8]. In particular, $\mathsf{C}X$ is isomorphic to $I_\omega(\mathsf{T}X)$ for all X.

2. Free Completion

Every ordered algebra can be completed to an ω-continuous algebra by the method of completion by ω-ideals. This result is well known and can be proved by standard universal-algebraic argument [3, 8]. In this section we give a more general construction that will allow us to apply this idea more widely.

Suppose we have a monad (F, μ^F, η^F) on some category C.



(2.1)

An F-*algebra* (aka Eilenberg–Moore algebra for F) is a pair (X,α), where $\alpha : FX \to X$ is the *structure map* of the algebra. It must satisfy

$$\begin{array}{ccc} F^2X & \xrightarrow{F\alpha} & FX \\ {\scriptstyle \mu^F_X}\downarrow & & \downarrow{\scriptstyle\alpha} \\ FX & \xrightarrow[\alpha]{} & X \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\eta^F_X} & FX \\ & \searrow{\scriptstyle 1_X} & \downarrow{\scriptstyle\alpha} \\ & & X \end{array} \tag{2.2}$$

An F-*algebra morphism* is a morphism of the underlying category C that commutes with the structure maps. The category of F-algebras and F-algebra morphisms is denoted F-Alg.

Examples 2.1. In our application, the underlying category C is the category of posets with $\bot$ and strict monotone maps. An F-algebra is a pair (X,α) where $\alpha : FX \to X$ is a morphism of C. For functors representing algebraic signatures (polynomial functors with an added $\bot$ or partial term algebras), this implies that the algebraic operations are monotone, but not necessarily strict.

The free ordered algebra $\mathsf{T}X$ and the free ω-continuous algebra $\mathsf{C}X$ are Eilenberg-Moore algebras for the partial term monad and partial coterm monad for the signature Σ, respectively.

Suppose further that we have another monad (D,μ^D,η^D) on C, along with a distributive law $\lambda : FD \to DF$.

$$\begin{array}{ccccc} FD^2 & \xrightarrow{F\mu^D} & FD & \xleftarrow{\mu^F D} & F^2D \\ {\scriptstyle\lambda D}\downarrow & & & & \downarrow{\scriptstyle F\lambda} \\ DFD & & \downarrow{\scriptstyle\lambda} & & FDF \\ {\scriptstyle D\lambda}\downarrow & & & & \downarrow{\scriptstyle\lambda F} \\ D^2F & \xrightarrow[\mu^D F]{} & DF & \xleftarrow[D\mu^F]{} & DF^2 \end{array} \qquad\qquad \begin{array}{ccccc} F & \xrightarrow{F\eta^D} & FD & \xleftarrow{\eta^F D} & D \\ & \searrow{\scriptstyle\eta^D F} & \downarrow{\scriptstyle\lambda} & \swarrow{\scriptstyle D\eta^F} & \\ & & DF & & \end{array} \tag{2.3}$$

One can show that D lifts to a monad $(\widehat{D},\mu^D,\eta^D)$ on F-Alg with

$$\widehat{D}(X,\alpha) = (DX, D\alpha\circ\lambda_X) \qquad\qquad \widehat{D}h = Dh.$$

The monad operations μ^D and η^D of $\widehat{D}$ are the same as those of D. The monad laws (2.1) hold for $\widehat{D}$ because they are the same as those for D. We must also show that the Eilenberg-Moore properties (2.2) hold and that μ^D_X, η^D_X are F-Alg morphisms.

$$\begin{array}{ccc} F^2DX & \xrightarrow{F(D\alpha\circ\lambda_X)} & FDX \\ {\scriptstyle \mu^F_{DX}}\downarrow & & \downarrow{\scriptstyle D\alpha\circ\lambda_X} \\ FDX & \xrightarrow[D\alpha\circ\lambda_X]{} & DX \end{array} \qquad\qquad \begin{array}{ccc} DX & \xrightarrow{\eta^F_{DX}} & FDX \\ & \searrow{\scriptstyle 1_{DX}} & \downarrow{\scriptstyle D\alpha\circ\lambda_X} \\ & & DX \end{array}$$

$$\begin{array}{ccc} FD^2X & \xrightarrow{F\mu^D_X} & FDX \\ {\scriptstyle D(D\alpha\circ\lambda_X)\circ\lambda_{DX}}\downarrow & & \downarrow{\scriptstyle D\alpha\circ\lambda_X} \\ D^2X & \xrightarrow[\mu^D_X]{} & DX \end{array} \qquad\qquad \begin{array}{ccc} FX & \xrightarrow{F\eta^D_X} & FDX \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle D\alpha\circ\lambda_X} \\ X & \xrightarrow[\eta^D_X]{} & DX \end{array}$$

These results follow from the monad properties (2.1) for F and D, the Eilenberg-Moore properties (2.2) for (X,α), and the distributive laws (2.3) by straightforward arrow chasing.

Now suppose (X,β) is a D-algebra. Under what conditions can we sensibly augment an F-algebra (X,α) with the new operation β? A sufficient condition is when $((X,\alpha),\beta)$ forms a $\widehat{D}$-algebra; that is, when β is an F-Alg morphism $\beta : \widehat{D}(X,\alpha) \to (X,\alpha)$.

$$\begin{array}{ccc} FDX & \xrightarrow{F\beta} & FX \\ {\scriptstyle D\alpha\circ\lambda_X}\downarrow & & \downarrow{\scriptstyle \alpha} \\ DX & \xrightarrow[\beta]{} & X \end{array} \tag{2.4}$$

This gives an $F{+}D$-algebra $(X,\alpha,\beta) = ((X,\alpha),\beta)$ is in which D distributes over F. Moreover, this is also a $\widehat{D}$-algebra, since the conditions (2.2) are the same.

Now, as is well known, because we have a monad $(\widehat{D},\mu^D,\eta^D)$, the functor $(X,\alpha) \mapsto (DX, D\alpha\circ\lambda_X, \mu^D_X)$ is left adjoint to the forgetful functor $(X,\alpha,\beta) \mapsto (X,\alpha)$. The algebra $(DX, D\alpha\circ\lambda_X, \mu^D_X)$ is the free completion of (X,α). The unit of the adjunction $\eta^D_X : (X,\alpha) \to (DX, D\alpha\circ\lambda_X)$ embeds the original F-algebra in its completion. The counit $(DX, D\alpha\circ\lambda_X, \mu^D_X) \to (X,\alpha,\beta)$ is β.

Examples 2.2. Let C be the category of posets with $\bot$ and strict monotone maps. Consider the monad (D,μ^D,η^D) such that DX is the set of all countably generated order ideals ordered by set inclusion.

$$DX = \{A{\downarrow} \mid A \subseteq X,\ A \text{ is a countable directed set}\}.$$

For $\mathcal{A} \subseteq DX$ a countable directed set of order ideals of X, and for $x \in X$, let

$$\mu^D_X(\mathcal{A}{\downarrow}) = (\bigcup\mathcal{A}){\downarrow} \qquad\qquad \eta^D_X(a) = \{a\}{\downarrow}\,.$$

The distributive law $\lambda_X : FDX \to DFX$ takes a term or coterm over a countable collection of order ideals to an order ideal of terms or coterms over X of the same shape. For example, for $f \in \Sigma_n$,

$$\lambda_X : f(A_1,\ldots,A_n) \ \mapsto\ \{f(a_1,\ldots,a_n) \mid a_i \in A_i,\ 1 \le i \le n\}{\downarrow}\,.$$

If (X,α) is an ordered Σ-algebra, applying $D\alpha$ to the right-hand side applies α to every element of that set, which is the same as interpreting the term in the F-algebra (X,α). For example,

$$D\alpha\circ\lambda_X : f(A_1,\ldots,A_n) \ \mapsto\ \{f^{(X,\alpha)}(a_1,\ldots,a_n) \mid a_i \in A_i,\ 1 \le i \le n\}{\downarrow}\,.$$

This is the algebra $\widehat{D}(X,\alpha) = (DX, D\alpha\circ\lambda_X)$.

An Eilenberg-Moore algebra (X,β) for the monad D is simply an ω-complete poset, the operation $\beta : DX \to X$ giving the supremum of a countably generated directed set. The property (2.4) says exactly that the algebraic operations commute with the supremum operator; that is, the algebraic operations are ω-continuous.

The $\widehat{D}$-algebra $(DX, D\alpha\circ\lambda_X, \mu^D_X)$, the free extension of (X,α) to an $F+D$-algebra, is the completion by countably generated order ideals. The unit of the adjunction $\eta^D_X : (X,\alpha) \to (DX, D\alpha\circ\lambda_X, \mu^D_X)$ embeds the original F-algebra in its ideal completion. The counit $(DX, D\alpha\circ\lambda_X, \mu^D_X) \to (X,\alpha,\beta)$ is the supremum operator β.

3. Quasi-Regular Families

In this section we study limited completeness conditions in which not all suprema need exist, but only those of a certain syntactically restricted form.

For each set X, let ΔX denote an ordered subalgebra of $\mathsf{C}X$ containing X. The set ΔX is a set of partial coterms closed under the algebraic operations Σ. Note that each finite term in $\mathsf{T}X$ is in ΔX, thus $\mathsf{T}X \subseteq \Delta X \subseteq \mathsf{C}X$.

We further assume for all X and Y and for all morphisms $h : \mathsf{C}X \to \mathsf{C}Y$ of ω-continuous algebras with $h(X) \subseteq \Delta Y$ that $h(\Delta X) \subseteq \Delta Y$. A family of term algebras $(\Delta X)_X$ satisfying this property is called a *quasi-regular family.*

Under these conditions, Δ determines a submonad $(\Delta, \mu^\Delta, \eta^\Delta)$ of the coterm monad. The monad operations are the same, except with suitably restricted domain. Thus $\eta^\Delta_X : X \to \Delta X$ makes a singleton term x out of an element $x \in X$, and $\mu^\Delta_X : \Delta^2 X \to \Delta X$ takes a term of terms over X and collapses it to a term over X.

Examples 3.1. Two examples of quasi-regular families are $\Delta X = \mathsf{C}X$, the set of all partial coterms over X, and $\Delta X = \mathsf{T}X$, the set of all partial terms over X. These are the maximum and minimum quasi-regular families, respectively, for a given signature Σ. The regular or algebraic trees [5, 8], or more generally, the set of regular (or rational) trees of order n [6] also form quasi-regular families. Also, the union of the n-regular trees over X for all $n \geq 0$ is a quasi-regular family.

Lemma 3.2. *For any set X, the set ΔX is uniquely determined by ΔX_ω.*

Proof. Let $h : X \to Y$ be an injection. Then h extends uniquely to a morphism $\widehat{h} = \mu_Y \circ \Delta h : \Delta X \to \Delta Y$ of ω-continuous algebras. We claim that

$$\Delta Y = \{\widehat{h}(t) \mid X \subseteq X_\omega,\ h : X \to Y \text{ is an injection},\ t \in \Delta X\}.$$

The reverse inclusion holds by our assumption $\widehat{h}(\Delta X) \subseteq \Delta Y$. For the forward inclusion, suppose $s \in \Delta Y$. Then $s \in \Delta Y'$ for some finite or countable subset $Y' \subseteq Y$, as there are at most countably many subterms of s. Let $X \subseteq X_\omega$ be of the same cardinality as Y' and let $h : X \to Y'$ be a bijection. Then $\widehat{h}^{-1}(s) \in \Delta X_\omega$, $h : X \to Y$ is an injection, and $s = \widehat{h}(\widehat{h}^{-1}(s))$. ∎

Definition 3.3. For terms $t_1, t_2 \in \mathsf{C}X$, define $t_1 \ll t_2$ if t_1 is finite, but agrees as a labeled tree with t_2 wherever it is defined. That is, $t_1 \in \mathsf{T}X$ and t_1 can be obtained from t_2 by deleting subterms. Clearly $t_1 \leq t_2$ whenever $t_1 \ll t_2$.

Let A be an ordered Σ-algebra. The structure map $\beta : \mathsf{T}A \to A$ interprets finite terms as elements of A. A set $B \subseteq A$ is called a Δ-*set* if there is a term $t \in \Delta A$ such that $B = \{\beta(s) \mid s \ll t\}$. The set $\{s \mid s \ll t\}$ is a countable directed subset of $\mathsf{T}A$, and as β preserves order, B is a countable directed subset of A.

We say that an ordered algebra A is a Δ-*regular algebra* if the suprema of all Δ-sets exist and the algebraic operations preserve the suprema of Δ-sets.

We will show that any Δ-regular algebra can be extended to an Eilenberg-Moore algebra for the monad Δ. Note that ΔX itself is a Δ-regular algebra. This is due to the fact that Δ is a monad.

Theorem 3.4. *Let A be a Δ-regular algebra with structure map $\beta : \mathsf{T}A \to A$. Extend β to domain ΔA by defining*

$$\beta(t) = \bigvee\{\beta(s) \mid s \ll t\}.$$

The suprema of Δ-sets exist by assumption. Then A with the extended structure map β is an Eilenberg-Moore algebra for the monad Δ.

Proof. We must show that the Eilenberg-Moore properties (2.2) hold: $\beta(a) = a$, and for any indexed set $(s_i \mid i \in I)$ of elements of ΔA and $t(s_i \mid i \in I) \in \Delta^2 A$,

$$\beta(t(s_i \mid i \in I)) = \beta(t(\beta(s_i) \mid i \in I)). \tag{3.1}$$

The first property holds since (A, β) is an Eilenberg-Moore algebra for the term monad T. Property (3.1) holds for finite t and finite s_i for the same reason.

More generally, by our assumption that the algebraic operators preserve suprema of Δ-sets, it follows by induction that for finite $t \in \mathsf{T}\{x_1, \ldots, x_n\}$ and $s_i \in \Delta A$, $1 \le i \le n$,

$$\beta(t(\beta(s_1), \ldots, \beta(s_n)) = \bigvee_{\substack{s_i' \ll s_i \\ 1 \le i \le n}} \beta(t(\beta(s_1'), \ldots, \beta(s_n'))). \tag{3.2}$$

Now to show (3.1) in the general case,

$$\begin{aligned}
\beta(t(s_i \mid i \in I)) &= \bigvee_{t' \ll t(s_i \mid i \in I)} \beta(t') && (3.3)\\
&= \bigvee_{t' \ll t} \bigvee_{\substack{s_i' \ll s_i \\ i \in I}} \beta(t'(s_i' \mid i \in I)) && (3.4)\\
&= \bigvee_{t' \ll t} \bigvee_{\substack{s_i' \ll s_i \\ i \in I}} \beta(t'(\beta(s_i') \mid i \in I)) && (3.5)\\
&= \bigvee_{t' \ll t} \beta(t'(\beta(s_i) \mid i \in I)) && (3.6)\\
&= \beta(t(\beta(s_i) \mid i \in I)). && (3.7)
\end{aligned}$$

Step (3.3) is by definition of β. Step (3.4) is by definition of $\ll$. Step (3.5) is by property (3.1) for finite terms. Step (3.6) is by property (3.2). Finally, step (3.7) is by definition of β. □

We have shown that every Δ-regular algebra is a Δ-algebra, that is, an Eilenberg-Moore algebra for the monad Δ. Henceforth, we drop the "regular" and just call them Δ-algebras.

The value of β is uniquely determined on all elements of ΔA, thus each ordered algebra with $\perp$ can be turned into a Δ-algebra in at most one way. Moreover, every Δ-algebra is determined by the interpretation of the terms in $\Delta(X_\omega)$.

A *morphism* of Δ-algebras is a morphism on the carriers as ordered sets with $\perp$ that commutes with the structure maps. It follows that any morphism of Δ-algebras preserves suprema of Δ-sets, as we now show.

Theorem 3.5. *Let (A, α) and (B, β) be Δ-algebras and $h : A \to B$ a morphism. For any Δ-set $D \subseteq A$, its image $\{h(a) \mid a \in D\} \subseteq B$ is a Δ-set in B, and*

$$h(\bigvee D) = \bigvee\{h(a) \mid a \in D\}.$$

Proof. Let $t \in \Delta A$ such that $D = \{\alpha(s) \mid s \ll t\}$. We must show that $\{h(\alpha(s)) \mid s \ll t\}$ is a Δ-set in B and

$$h(\alpha(t)) = \bigvee\{h(\alpha(s)) \mid s \ll t\}.$$

Since h commutes with the structure maps,

$$\{h(\alpha(s)) \mid s \ll t\} = \{\beta(\Delta h(s)) \mid s \ll t\} = \{\beta(u) \mid u \ll \Delta h(t)\},$$

and this is a Δ-set in B. Moreover,

$$h(\alpha(t)) = \beta(\Delta h(t)) = \bigvee\{\beta(u) \mid u \ll \Delta h(t)\} = \bigvee\{h(\alpha(s)) \mid s \ll t\}.$$

□

Examples 3.6. If $\Delta = \mathsf{T}$, then a Δ-algebra is just an ordered algebra. If ΔX is the collection of all regular trees over X, then a Δ-algebra is an ordered regular algebra [10, 11]. If ΔX is the set of all regular trees of order n, $n \geq 0$, then a Δ-algebra is an n-regular (or n-rational) algebra of [6].

We have defined Δ-algebras A in terms of suprema of Δ-sets in A. However, one could generalize the notion of Δ-set to include any set of terms with a supremum in ΔA.

Suppose that (A, α) is a Δ-algebra. A subset $E \subseteq \Delta A$ is said to be *consistent* if any two elements of E, considered as labeled trees, agree wherever both are defined. That is, if $t_1, t_2 \in E$, and t_1 and t_2 both have a node at position x in the tree, then the element of Σ labeling t_1 at x is the same as the element of Σ labeling t_2 at x. Any consistent set of terms has a unique supremum.

We say that a set $D \subseteq A$ is an *extended* Δ*-set* if there is a consistent set $E \subseteq \Delta A$ such that $\bigvee E \in \Delta A$ and $D = \{\alpha(t) \mid t \in E\}$.

We state the following theorem without proof, but it is not difficult to prove using the same technique as in Theorems 3.4 and 3.5.

Theorem 3.7. *The algebraic operations of any* Δ*-algebra and all* Δ*-algebra morphisms preserve suprema of extended* Δ*-sets.*

As every Δ-set is an extended Δ-set, we can replace the definition of Δ-regular algebra by the stronger property that suprema of all extended Δ-sets exist and are preserved by the algebraic operations.

4. Main Result

We can introduce varieties of Δ-algebras defined by a set of inequalities between finite partial terms in $\mathsf{T}X_\omega$ in the expected way.

Suppose that $\mathcal{V}_\omega$ is a variety of ω-continuous algebras defined by a set of inequalities between finite partial terms. Let $\mathcal{V}_r$ denote the variety of all Δ-algebras defined by the same set of inequalities. Let $F_r(X)$ denote the free ordered algebra on generators X in $\mathcal{V}_r$. Let $F_\omega(X)$ denote the free ω-continuous algebra on generators X in $\mathcal{V}_\omega$ (see [8]). Let $R(X)$ denote the ordered Δ-subalgebra of $F_\omega(X)$ generated by X. The elements of $R(X)$ are all elements of $F_\omega(X)$ denoted by the terms in ΔX:

$$R(X) = \{t^{F_\omega(X)} \mid t \in \Delta X\}.$$

Theorem 4.1. $R(X)$ *is the free* Δ*-algebra in* $\mathcal{V}_r$ *on generators* X*. It is isomorphic to the completion of* $F_r(X)$ *by* Δ*-ideals. (A* Δ-ideal *is simply the down-closure* $B{\downarrow}$ *of a* Δ*-set* B*.)*

Proof. Since $R(X)$ embeds in $F_\omega(X)$, we have that $R(X) \in \mathcal{V}_r$ by Birkhoff's theorem.

Suppose that A is an ordered Δ-regular algebra and $h : X \to A$. Consider the completion $I_\omega(A)$ of A by ω-ideals, which is an ω-continuous algebra in $\mathcal{V}_\omega$ [3, 8]. The function $\eta_A^D : a \mapsto \{a\}{\downarrow}$ embeds A in $I_\omega(A)$, so we can view h as a function $X \to I_\omega(A)$. Since $I_\omega(A) \in \mathcal{V}_\omega$, there is a unique extension of h to a morphism $\widehat{h} : F_\omega(X) \to I_\omega(A)$ of ω-continuous algebras. The restriction of $\widehat{h}$ to $R(X)$ is an ordered Δ-regular algebra morphism

$R(X) \to I_\omega(A)$. Let $R(A)$ denote the image of $R(X)$ under $\widehat{h}$. Then $R(A)$ is a Δ-regular subalgebra of $I_\omega(A)$ generated by A, and $R(A)$ is in $\mathcal{V}_r$ because it is a homomorphic image of $R(X)$, which is in $\mathcal{V}_r$.

Finally, let $\bigvee$ be the supremum operator on $R(A)$. The elements of $R(A)$ are ω-ideals generated by Δ-sets, therefore the suprema exist. The map $\bigvee$, being a component of the counit of the completion construction, is a morphism of Δ-algebras; that is, a strict monotone function that preserves suprema of Δ-sets and commutes with the algebraic operations.

Now the restriction of $\widehat{h}$ composed with $\bigvee$ is the required unique extension of h to a morphism $R(X) \to A$ of Δ-algebras.

We can also construct $R(X)$ directly as the free completion of $F_r(X)$ by Δ-ideals. Here we use the general construction of §2. Let $F = \mathsf{T}$, the finite term monad for the signature Σ and D the monad of Δ-ideals. The algebra $F_r(X)$ is an F-algebra. We must check that the monad D satisfies the condition (2.4); but this is almost automatic, as D is a submonad of the monad of Example 2.2.

The free completion is $(D(F_r(X)), D\alpha \circ \lambda_{F_r(X)}, \mu^D_{F_r(X)})$, where α is the structure map of $F_r(X)$. It is the free Δ-algebra on generators X. As $R(X)$ is also a Δ-algebra, there is unique Δ-algebra morphism g from $D(F_r(X))$ to $R(X)$ mapping $x \in X$ to x. Since both algebras are generated by X, g is surjective. Similarly, since $D(F_r(X))$ is a Δ-algebra, as noted above, there is a Δ-algebra morphism h from $R(X)$ to $D(F_r(X))$ mapping $x \in X$ to x. The composition $h \circ g$ is the unique morphism on $D(F_r(X))$ extending the identity on X, thus must be the identity morphism. Thus $R(X)$ and $D(F_r(X))$ are isomorphic. □

Examples 4.2. The free ω-continuous semiring on generators X is the semiring of power series $\mathbb{N}_\infty\langle\!\langle X^*\rangle\!\rangle$ with a countable support having coefficients in $\mathbb{N}_\infty$ obtained from the semiring of natural numbers by adding a point at infinity. It follows that the free regular semiring over X is the semiring $\mathbb{N}_\infty^{\mathrm{alg}}\langle\!\langle X^*\rangle\!\rangle$ of all algebraic elements of $\mathbb{N}_\infty\langle\!\langle X^*\rangle\!\rangle$.

The free ω-continuous idempotent semiring on X is the semiring of all at most countably infinite languages in X^*. The ω-continuous idempotent semirings are the same as the closed semirings used in the study of shortest-path algorithms [2] (modulo minor corrections to the axiomatization). The free "1-regular" idempotent semiring over X is the semiring of OI-macro languages over X.

The free star-continuous Kleene algebra on generators X is the family of regular subsets of X^* [9]. This is the free Δ-algebra for Δ the set of regular coterms defined by finite systems of linear affine equations in the variety of idempotent semirings. The family of context-free languages is the free Δ-algebra for Δ the set of regular coterms defined by finite systems of algebraic equations in the variety of idempotent semirings. The free ω-continuous idempotent semiring is the completion of the free Kleene algebra by countably generated star-ideals [9].

A final example is given by the category of iteration theories [4], which is isomorphic to the category of Eilenberg-Moore algebras for the rational-tree monad on the category of signatures [1].

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