FACE VECTORS AND HILBERT FUNCTIONS

A Dissertation
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Doctor of Philosophy

by
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This dissertation builds upon two well-known theorems: Macaulay’s theorem on the Hilbert functions of graded ideals, and the Kruskal–Katona theorem on the face vectors of simplicial complexes. There are four parts in the dissertation. The first part covers the preliminaries, and also introduces the Frankl–Füredi–Kalai theorem, which is a result in extremal graph theory concerning \( r \)-colorable \( k \)-uniform hypergraphs.

In Part II, we use the algebraic notion of Macaulay-Lex rings to show that Macaulay’s theorem, the Kruskal–Katona theorem, and the Frankl–Füredi–Kalai theorem, are in fact three special cases of one main theorem. In particular, we introduce a class of quotient rings with combinatorial significance, which we call colored quotient rings, and we characterize all possible Macaulay-Lex colored quotient rings. This gives a simultaneous generalization of the Clements–Lindström theorem and the Frankl–Füredi–Kalai theorem. As part of our characterization, we construct two new classes of Macaulay-Lex rings, thereby answering a question posed by Mermin–Peeva (2007). Using related ideas, we also show that with the exception of very special cases, the \( f \)-vectors of generalized colored simplicial complexes (resp. multicomplexes) are never characterized by “reverse-lexicographic” simplicial complexes (resp. multicomplexes).

In Part III, we look at the famous Eisenbud–Green–Harris (EGH) conjecture. It proposes an extension of Macaulay’s theorem to graded ideals containing regular sequences. We apply liaison theory to prove that the EGH conjecture is true for
a certain subclass of graded licci ideals. As an important consequence, we show that the conjecture holds for Gorenstein ideals in the three-variable case. This is the first time liaison theory appears in the context of the EGH conjecture.

In Part IV, we give complete numerical characterizations for the flag $f$-vectors of Cohen–Macaulay completely balanced complexes, and the fine $f$-vectors of generalized colored complexes. En route to these characterizations, we introduce the notion of Macaulay decomposability, which implies vertex-decomposability, and we generalize the notion of Macaulay representations. We also construct a poset $\mathcal{P}$ of such generalized Macaulay representations, and show that proving the Kruskal–Katona theorem is equivalent to finding chains in $\mathcal{P}$, while characterizing the flag $f$-vectors of (non-generalized) colored complexes is equivalent to finding Boolean sublattices in $\mathcal{P}$. A more general statement holds. Macaulay decomposability is also interesting by itself and we use it to extend results by Babson–Novik, Biermann–Van Tuyl, and Murai.
BIOGRAPHICAL SKETCH

Kai Fong Ernest Chong was born in Singapore on August 30th, 1985. Informally, he goes by the shorter name Ernest Chong. He attended high school in Singapore at Raffles Institution and Raffles Junior College.

In 2003, Ernest won the grand prize of the Singapore National Science Talent Search, which is a research-based science competition for high school students, and the Singapore equivalent of the Intel Science Talent Search held in the United States.\(^1\) As the grand winner, he was awarded $10,000 cash, and a scholarship that fully funds both his undergraduate and PhD studies. The scholarship is given by the Agency for Science, Technology and Research (A*STAR) in Singapore, and it includes a 5-year job appointment determined by A*STAR upon completion of his PhD studies. By accepting this scholarship, Ernest effectively knew how the next 17 years of his life would pan out.

From January 2004 to March 2006, Ernest served in the military, which is compulsory for all healthy Singaporean males. Subsequently, he enrolled in Cornell University in August 2006. He completed his undergraduate studies in three years with a major in mathematics, and he then returned to Singapore to work for one full year in A*STAR as part of his contractual obligation.

In August 2010, Ernest entered the mathematics PhD program at Cornell University. He will complete his PhD studies in August 2015. After graduation, he will return to Singapore for his 5-year job appointment within A*STAR. He will be looking for a new job when August 2020 approaches.

\(^1\)The main highlight of his winning research paper is a new characterization of when \(p\) and \(p + 2k\) are simultaneously prime numbers, where \(k\) is any fixed positive integer. Note however that it does not imply the twin prime conjecture (or the more general Polignac’s conjecture). This research paper also won him a gold medal in the Singapore Science and Engineering Fair, and he would have represented Singapore in Intel’s International Science and Engineering Fair, if not for the SARS outbreak in Singapore that led to the cancellation of all high school related international travels during the April–May 2003 period.
This dissertation is dedicated to my wife Amanda.
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CHAPTER 1
INTRODUCTION

This dissertation builds upon two very well-known and classic theorems in mathematics: Macaulay’s theorem in algebra, and the Kruskal–Katona theorem in combinatorics. In 1927, Macaulay [85] characterized all possible Hilbert functions of the graded ideals in polynomial rings. En route to proving this characterization, Macaulay also introduced the idea of taking initial ideals with respect to a fixed monomial order, thereby laying the foundations for Gröbner basis theory (see, e.g., [45] [122]), as well as paving the way for the development of computer algebra systems that many mathematicians use today (e.g. Macaulay2 [62]). In the 1960s, Kruskal [82] and Katona [77] independently characterized the face vectors of simplicial complexes. Their characterization, now commonly known as the Kruskal–Katona theorem, is a fundamental result in topological combinatorics and discrete geometry. It has spurred much interest in face enumeration questions for various classes of simplicial complexes, polytopes, and manifolds; see, e.g., [119] [124]. The techniques used in proving this characterization are also interesting in their own right, and they have been used successfully to resolve questions in extremal combinatorics and extremal graph theory; see, e.g., [48] [63].

There are four parts in the dissertation. The first part covers (among other things) basic concepts in combinatorics, commutative algebra, and graph theory. In particular, at the end of Part I, we introduce the Frankl–Füredi–Kalai theorem [53], which is a result in extremal graph theory concerning the minimum shadow sizes of r-colorable k-uniform hypergraphs.

In Part II, we show that Macaulay’s theorem, the Kruskal–Katona theorem, and the Frankl–Füredi–Kalai theorem, are in fact three special cases of one main
theorem (Theorem 6.5.3). This theorem uses the algebraic notion of Macaulay-Lex rings [91]. In particular, we introduce a class of quotient rings with combinatorial significance, which we call colored quotient rings, and we characterize all possible Macaulay-Lex colored quotient rings; see Chapter 6. As part of our characterization, we construct two new classes of Macaulay-Lex rings, thereby answering a question posed by Mermin–Peeva [92] in 2007.

Using related ideas, we will also show that with the exception of very special cases, the face vectors of generalized colored simplicial complexes (resp. multicomplexes) are never characterized by “reverse-lexicographic” simplicial complexes (resp. multicomplexes); see Theorem 8.2.8. This highlights the limitations of the techniques used in proving the Kruskal–Katona theorem, especially when we deal with simplicial complexes or multicomplexes with additional structure. Later in Part IV, we will avoid these limitations by introducing new techniques in face enumeration for complexes with additional color structure.

In Part III, we look at the Eisenbud–Green–Harris (EGH) conjecture [46]. This is a famous and still wide open problem in algebraic geometry and commutative algebra. If true, it would imply the generalized Cayley–Bacharach conjecture [46] in algebraic geometry, generalize the Clements–Lindström theorem [36] in combinatorics, and provide (more) evidence for the Lex-plus-powers conjecture [51] in commutative algebra. The EGH conjecture proposes an extension of Macaulay’s theorem by considering the Hilbert functions of graded ideals containing regular sequences. A numerical version of the conjecture includes the Kruskal–Katona theorem as a very special case that is known to be true (see Remark 9.3.20). In general, the EGH conjecture is notoriously difficult, and only a few cases are known to be true; see Chapter 9.2 for an exhaustive list of all currently known cases.
As our contribution to this list of known cases, we apply tools from liaison theory (a branch of algebraic geometry also sometimes known as linkage theory) to prove that the conjecture holds for a certain subclass of graded licci ideals (Theorem 10.2.2). In contrast to previously known results, this is the first time liaison theory appears in the context of the EGH conjecture. As an important consequence, we show that the EGH conjecture holds for Gorenstein ideals in the three-variable case (Theorem 10.2.5). However, far more importantly, our results suggest a completely new approach: Perhaps we should use liaison theory to tackle the EGH conjecture. For a good introduction to liaison theory, see [95].

In Part IV, we are interested in the face enumeration of generalized colored complexes. These are simplicial complexes with additional color structure defined as follows: Given an $n$-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ of positive integers, we say a simplicial complex is $\mathbf{a}$-colored if we can partition its vertex set into $n$ (non-empty disjoint) subsets $V_1, \ldots, V_n$, such that every face has at most $a_i$ vertices in $V_i$. Equivalently, we can think of this partition as a coloring of the vertices with $n$ colors, so that every face has at most $a_i$ vertices of the $i$-th color. Note that our definition of “$\mathbf{a}$-colored” extends the usual notion of colored complexes used by many authors, which coincides with our notion of $(1, \ldots, 1)$-colored complexes, i.e. every face has at most one vertex of any given color. For this reason, $\mathbf{a}$-colored complexes for arbitrary $n$-tuples $\mathbf{a}$ are called “generalized colored complexes”.

Because of the color structure, we can count the number of faces in an $\mathbf{a}$-colored complex $\Delta$ that have exactly $b_i$ vertices of the $i$-th color. Denote this number by $f_b$, where $\mathbf{b} = (b_1, \ldots, b_n)$. The array of integers $\{f_b\}_{\mathbf{b} \leq \mathbf{a}}$ is called the fine $f$-vector of $\Delta$, and it is a refinement of the face vector of $\Delta$. Although the face vectors of $(1, \ldots, 1)$-colored complexes have been numerically characterized in 1988 [53], there
has been very little progress made since then towards a numerical characterization of the more refined fine $f$-vectors.

As pointed out by Björner–Frankl–Stanley [19], part of the difficulty lies in the non-uniqueness of color-compressed $(a_1, \ldots, a_n)$-colored complexes with a given fine $f$-vector when $n > 1$. Frohmader [54] showed it is not possible to make further progress towards a numerical characterization of the fine $f$-vectors of color-compressed $(1, \ldots, 1)$-colored complexes through stronger restrictions on the color-selected subcomplexes. Walker [127] proved that the only linear inequalities satisfied by the entries of the fine $f$-vectors of $(1, \ldots, 1)$-colored complexes are the trivial ones, i.e. these entries are non-negative. On a more positive note, Walker also characterized all linear inequalities satisfied by the logarithms of the entries of these fine $f$-vectors, which yield (non-trivial) necessary numerical conditions on the fine $f$-vectors. Unfortunately, these numerical conditions do not fully characterize the fine $f$-vectors other than for the trivial cases of $(1)$-colored complexes and $(1,1)$-colored complexes. The first non-trivial case, i.e. the fine $f$-vectors of $(1,1,1)$-colored complexes, was characterized by Frohmader [55]. Already in this “simplest” non-trivial case, the characterization is complicated, and Frohmader’s proof involves the brute-force check of many cases.

One of the main results we have in Part IV is a complete numerical characterization of the fine $f$-vectors of $a$-colored complexes for arbitrary $n$-tuples $a$ (Theorem 14.2.7). This includes the case $a = (1, \ldots, 1)$, as well as the case $a = (d)$ (corresponding to the Kruskal–Katona theorem). A significant portion of Part IV is devoted to addressing the non-uniqueness issue pointed out by Björner–Frankl–Stanley [19]. Note that our (completely different) approach avoids the difficulties encountered by Walker and Frohmader. There are three key steps for getting this
numerical characterization.

For the first key step, we introduce the notion of ‘Macaulay decomposability’ in Chapter 12. All unexplained terminology used in the rest of this introduction will be defined in subsequent chapters. Macaulay decomposability implies vertex-decomposability (Proposition 12.2.6), and we show that every pure color-shifted a-balanced complex is a-Macaulay decomposable and hence vertex-decomposable (Theorem 12.3.1). This allows a geometric interpretation of ‘decomposing’ a-colored complexes into pieces we can better understand.

Recall that the Kruskal–Katona theorem is commonly stated in terms of Macaulay representations. For our second key step, we generalize these Macaulay representations. We will use a-Macaulay decomposability to define the notion of a generalized a-Macaulay representation. Our main insight is that we should think of the (usual) Macaulay representations not as sums of binomial coefficients, but as certain labeled binary trees whose labels correspond to the top components of the binomial coefficients, and whose tree structure is determined by the bottom components of the binomial coefficients. With this labeled binary tree interpretation, there is a natural way to generalize these Macaulay representations: The labels, which are integers, can be suitably replaced by n-tuples of integers, and the tree structure can be arbitrarily “complicated”. Such “general” labeled binary trees are essentially what generalized Macaulay representations are, and they correspond to sums of products of binomial coefficients. Just like how the top components of the binomial coefficients in the (usual) Macaulay representations are required to be strictly decreasing, there are also restrictions on the possible labels in generalized a-Macaulay representations. Finding suitable restrictions on these labels is the main difficulty that we will address in Chapter 13. We will also define suitable
differentials on these generalized \( a \)-Macaulay representations.

Finally, for our third key step, we define a partial order on generalized Macaulay representations (with different values of \( a \)). Thus, we can construct an infinite poset \( P \) consisting of all possible generalized Macaulay representations. What our numerical characterization (Theorem 14.2.7) says is that an array of positive integers is the fine \( f \)-vector of an \( a \)-colored complex if and only if there exists a subposet of \( P \), induced by generalized Macaulay representations corresponding to the entries in the array, with a specific poset structure: It must be anti-isomorphic to the closed interval \([0_n, a]\) contained in the poset of all \( n \)-tuples in \( \mathbb{N}^n \) ordered by \( \leq \). Consequently, we get a very satisfying conceptual description of our numerical characterization: Face vectors of simplicial complexes correspond to chains in \( P \), fine \( f \)-vectors of \( (1, \ldots, 1) \)-colored complexes correspond to Boolean sublattices in \( P \), and in general, checking if an array of integers is the fine \( f \)-vector of an \( a \)-colored complex is equivalent to checking the existence of an induced subposet in \( P \) with a certain poset structure.

Completely balanced Cohen–Macaulay complexes [116] form a special subclass of generalized colored complexes, and they include order complexes of shellable posets and barycentric subdivisions of polytopes as important examples. These are “sufficiently nice” \((1, \ldots, 1)\)-colored complexes whose fine \( f \)-vectors (equivalently, flag \( f \)-vectors) can be numerically characterized without having to go through the third key step in our characterization. In particular, checking if an array of integers is the fine \( f \)-vector of completely balanced Cohen–Macaulay complex is essentially equivalent to checking the existence of a single generalized Macaulay representation that is “compatible” with certain differentials; see Theorem 14.1.3.
The notion of Macaulay decomposability introduced in our first key step is also interesting by itself. It is a refinement of vertex-decomposability, and it can be used to extend results by Babson–Novik [8, Theorem 6.2], Biermann–Van Tuyl [13], and Murai [97, Proposition 4.2]; see Chapter 12.3. In particular, we show that the fine $f$-vector of an $a$-balanced Cohen–Macaulay complex is the fine $f$-vector of an $a$-balanced pure vertex-decomposable complex (see Corollary 12.3.6).

With the exception of Part I, which covers the preliminaries, most of the results presented in the rest of the dissertation can be found in the following papers: Most of Part II can be found in [35], slightly more than half of Part III can be found in [33], and most of Part IV can be found in [34]. Also, every chapter henceforth ends with notes containing further discussion, recent progress, related topics, open problems, conjectures, and/or historical accounts.
Part I

Preliminaries
CHAPTER 2
FACE VECTORS OF SIMPLICIAL COMPLEXES

We first begin by reviewing some basic notation and terminology that will be used throughout this dissertation. Borrowing a highly useful and popular notation in computer programming, we use the assignment operator “:=” for definitions, so that “$A := B$” means “$A$ is defined to be $B$”. Let $\mathbb{N}$ and $\mathbb{P}$ denote the non-negative integers and positive integers respectively. For convenience, let $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and $\overline{\mathbb{P}} := \mathbb{P} \cup \{\infty\}$. Throughout, we always assume that $n$ is a positive integer. We also assume all sets are countable. Let $[n]$ be the set $\{1, \ldots, n\}$, and let $[0] := \emptyset$. For $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$ in $\mathbb{Z}^n$, define $|a| := a_1 + \ldots + a_n$, and write $a \leq b$ if $a_i \leq b_i$ for all $i \in [n]$. By convention, write $a < b$ if $a \leq b$ and $a \neq b$. Also, write $a \prec b$ if $a < b$ and $|b| = |a| + 1$. For brevity, we use $0_n, 1_n, \infty_n$ to mean the $n$-tuples $(0, \ldots, 0), (1, \ldots, 1), (\infty, \ldots, \infty)$ respectively. Denote the Kronecker delta function by $\delta(i, j)$, i.e. $\delta(i, j) = 1$ if $i = j$, and $\delta(i, j) = 0$ if $i \neq j$. Define $\delta_{t,n} = (\delta(t, 1), \delta(t, 2), \ldots, \delta(t, n)) \in \mathbb{N}^n$ for each $t \in [n]$.

Let $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. The support of $a$, denoted by $\text{supp}(a)$, is the set of indices corresponding to the non-zero entries, i.e. $\text{supp}(a) := \{i \in [n] : a_i \neq 0\}$. If $a \neq 0_n$, and $i_1 < \cdots < i_m$ are all the (distinct) elements in $\text{supp}(a)$, then let $\mathbf{a} := (a_{i_1}, \ldots, a_{i_m}) \in \mathbb{P}^m$. Analogously, if $\pi = (X_1, \ldots, X_n)$ is a sequence of possibly empty subsets of a set $X$, then we also define $\text{supp}(\pi) := \{i \in [n] : X_i \neq \emptyset\}$, and we call this set the support of $\pi$. If $\text{supp}(\pi)$ is non-empty, and $i_1 < \cdots < i_m$ are all the elements in $\text{supp}(\pi)$, then define $\pi := (X_{i_1}, \ldots, X_{i_m})$. Also, for any $W \subseteq X$, define $\pi \cap W := (X_1 \cap W, \ldots, X_n \cap W)$. 

9
2.1 Simplicial complexes

A simplicial complex $\Delta$ is a collection of subsets of some given set $V$, such that $\Delta$ is closed under set inclusion (i.e. $F \in \Delta, F' \subseteq F \Rightarrow F' \in \Delta$). If $\{v\} \in \Delta$ for all $v \in V$, then we say $V$ is the vertex set of a simplicial complex of $\Delta$. For our purposes, we always assume $\Delta$ is finite and non-empty. Elements of $\Delta$ are called faces. The dimension of each $F \in \Delta$ is $\dim F = |F| - 1$, and the dimension of $\Delta$, denoted by $\dim \Delta$, is the maximum dimension of its faces. Maximal faces are called facets, 0-dimensional faces are called vertices, and 1-dimensional faces are called edges. For each $k \in \mathbb{N}$, let $\mathcal{F}_k(\Delta)$ denote the collection of all $k$-dimensional faces of $\Delta$. If all facets of $\Delta$ have the same dimension, then we say $\Delta$ is pure. Given another simplicial complex $\Delta'$ with vertex set $V'$, we say $\Delta$ and $\Delta'$ are isomorphic if there exists a bijection $\nu : V \rightarrow V'$ such that $F \in \Delta$ if and only if $\nu(F) \in \Delta'$.

Let $\Delta$ be a $(d-1)$-dimensional simplicial complex. The $f$-vector of $\Delta$ is the vector $(f_0, \ldots, f_{d-1}) \in \mathbb{N}^d$, where each $f_i := |\mathcal{F}_i(\Delta)|$ is the number of $i$-dimensional faces of $\Delta$. By default, set $f_{-1} = 1$, which corresponds to the empty face $\emptyset \in \Delta$. The $h$-vector of $\Delta$ is a vector $(h_0, \ldots, h_d) \in \mathbb{Z}^{d+1}$ defined by

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}, \quad 0 \leq i \leq d. \quad (2.1)$$

Note that the $f$-vector and $h$-vector of $\Delta$ determine each other completely, since $(2.1)$ is invertible, i.e.,

$$f_{i-1} = \sum_{j=0}^{i} \binom{d-j}{d-i} h_j, \quad 0 \leq i \leq d. \quad (2.2)$$

Equivalently, we have the relation

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1} (x - 1)^{d-i}. \quad (2.3)$$

1This is in contrast to the notion of facets of polytopes (in polytope theory), which are codimension one faces.
Given a set $X$ and an arbitrary collection $\mathcal{F} = \{F_1, \ldots, F_m\}$ of subsets of $X$, there is a (unique) minimal simplicial complex, denoted by $\langle \mathcal{F} \rangle$, that contains all elements in $\mathcal{F}$. This simplicial complex is said to be generated by $\mathcal{F}$, and a simplicial complex generated by one face, i.e. $|\mathcal{F}| = 1$, is called a simplex.

A subcomplex of $\Delta$ is a subcollection of $\Delta$ that is also a simplicial complex. The $k$-skeleton of $\Delta$ is the subcomplex $\{F \in \Delta : \dim F \leq k\}$. In particular, the 1-skeleton of $\Delta$ is called the underlying graph of $\Delta$, and can be treated as a graph without ambiguity; cf. Chapter 4.1. The deletion\(^{2}\) of a face $\sigma$ from $\Delta$ is the subcomplex $dl_{\Delta}(\sigma) := \{\tau \in \Delta : \sigma \not\subseteq \tau\}$. The link of a face $\sigma$ in $\Delta$ is the subcomplex $lk_{\Delta}(\sigma) := \{\tau \in \Delta : \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta\}$. If $v$ is a vertex of $\Delta$, then we write $dl_{\Delta}(\{v\})$ and $lk_{\Delta}(\{v\})$ simply as $dl_{\Delta}(v)$ and $lk_{\Delta}(v)$ respectively. Notice we always have $lk_{\Delta}(v) \subseteq dl_{\Delta}(v)$ for all vertices $v$ of $\Delta$.

The join of two simplicial complexes $\Delta_1$, $\Delta_2$ with disjoint vertex sets is the simplicial complex $\Delta_1 * \Delta_2 := \{\sigma_1 \cup \sigma_2 : \sigma_1 \in \Delta_1, \sigma_2 \in \Delta_2\}$. A vertex $v$ of $\Delta$ is called a conepoint if $\Delta = \langle \{v\} \rangle * dl_{\Delta}(v)$, or equivalently, if $lk_{\Delta}(v) = dl_{\Delta}(v)$.

### 2.2 Generalized colored and balanced complexes

An ordered partition of a non-empty set $X$ is a finite sequence $(X_1, \ldots, X_n)$ of non-empty, pairwise disjoint subsets of $X$ satisfying $X = X_1 \cup \cdots \cup X_n$.

**Definition 2.2.1.** Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{P}^n$. A colored complex of type $\mathbf{a}$ is a pair

\(^{2}\)We caution the reader that there is another notion of deletion used by several authors: Given a subset $U$ of the vertex set of $\Delta$, the deletion of $U$ from $\Delta$ is sometimes defined to be the subcomplex $\Delta \setminus U := \{\tau \in \Delta : \tau \cap U = \emptyset\}$. When $U$ is a face of $\Delta$, the subcomplex $\Delta \setminus U$ is more commonly known as the anti-star of $U$ in $\Delta$. Notice that $dl_{\Delta}(\sigma) = \Delta \setminus U$ when $\sigma = U = \{v\}$ consists of a single vertex, i.e. the deletions of vertices are the same in either definition. To avoid ambiguity, some authors instead call $dl_{\Delta}(\sigma)$ the face deletion of $\sigma$ in $\Delta$. However, for our purposes, the deletion of $\sigma$ from $\Delta$ will always mean the subcomplex $dl_{\Delta}(\sigma)$.  


$(\Delta, \pi)$ satisfying:

(i) $\Delta$ is a simplicial complex with a non-empty vertex set $V$;
(ii) $\pi = (V_1, \ldots, V_n)$ is an ordered partition of $V$;
(iii) $|F \cap V_i| \leq a_i$ for every $F \in \Delta$ and every $i \in [n]$.

We say a simplicial complex $\Delta$ with vertex set $V$ is colorable of type $a$ if there exists an ordered partition $\pi$ of $V$ such that $(\Delta, \pi)$ is a colored complex of type $a$. In the special case when $a = 1_n$, we say $\Delta$ is $n$-colorable. The usual notion of colored complexes\(^3\) that many authors use is equivalent to our definition of colored complexes of type $1_n$, hence our definition of colored complexes is more general. To avoid ambiguity, we sometimes say “generalized colored complexes” to mean colored complexes of arbitrary type.

Let $a = (a_1, \ldots, a_n) \in \mathbb{P}^n$. A balanced complex of type $a$ is a colored complex $(\Delta, \pi)$ of type $a$ satisfying $|a| = \dim \Delta + 1$, and a completely balanced complex is a balanced complex of type $1_n$. In contrast to the original definition given by Stanley [116], we do not assume balanced complexes to be pure. Note also that some authors use the notion “balanced” to mean what we call completely balanced. For brevity, write ‘a-colored’ and ‘a-balanced’ to mean ‘colored of type $a$’ and ‘balanced of type $a$’ respectively.

**Remark 2.2.2.** We can produce examples of balanced complexes of arbitrary type $a \in \mathbb{P}^n$ from a completely balanced complex $(\Delta, \pi)$ satisfying $\dim \Delta = |a| - 1$ as follows: If $d = |a|$ and $\pi = (V_1, \ldots, V_d)$ is the ordered partition of the vertex set $V$, then any surjective map $\phi : [d] \to [n]$ induces the ordered partition $\pi' = \phi(V_1) \cup \ldots \cup \phi(V_d)$.

\(^3\)Colored complexes are also sometimes called “color complexes” or “colorable complexes” by other authors. In most instances, they correspond (in our terminology) to simplicial complexes that are colorable of type $1_n$. There is also another class of simplicial complexes called “coloring complexes”. Despite its similar name, coloring complexes are different from colored complexes; see [68] [121].
(\(V'_1, \ldots, V'_n\)) of \(V\) such that \(v \in V_i\) implies \(v \in V'_{\phi(i)}\), and we check that \((\Delta, \pi')\) is an \(a\)-balanced complex. However, as pointed out by Swartz [123, Section 3], not all balanced complexes arise from completely balanced complexes in this manner.

Suppose \((\Delta, \pi)\) is an \(a\)-colored complex, where \(\pi = (V_1, \ldots, V_n)\) is an ordered partition of the vertex set \(V\) of \(\Delta\). For each \(b = (b_1, \ldots, b_n) \in \mathbb{N}^n\) satisfying \(b \leq a\), let \(f_b\) be the number of faces \(F\) of \(\Delta\) such that \(|F \cap V_i| = b_i\) for all \(i \in [n]\). The array of integers \(\{f_b\}_{b \leq a}\) is called the fine \(f\)-vector of \((\Delta, \pi)\), and it is a refinement of the \(f\)-vector \((f_0, \ldots, f_{\dim \Delta})\) of \(\Delta\) in the sense that
\[
 f_i = \sum_{\|b\| = i+1} f_b, \quad 0 \leq i \leq \dim \Delta. \tag{2.4}
\]
The fine \(h\)-vector of \((\Delta, \pi)\) is the array of integers \(\{h_b\}_{b \leq a}\) defined in terms of the fine \(f\)-vector by
\[
 h_b := \sum_{c \leq b} f_c \prod_{i=1}^{n} (-1)^{b_i - c_i} \left( a_i - c_i \right) \left( b_i - c_i \right), \quad 0_n \leq b \leq a. \tag{2.5}
\]
Fine \(f\)-vectors and fine \(h\)-vectors are sometimes called extended \(f\)-vectors and extended \(h\)-vectors respectively; see, e.g., [9].

**Theorem 2.2.3** ([116]). Let \((\Delta, \pi)\) be a \((d-1)\)-dimensional balanced complex of type \(a \in \mathbb{P}^n\), and let \(\{h_b\}_{b \leq a}\) be its fine \(h\)-vector. If \((h_0, \ldots, h_d)\) is the \(h\)-vector of \(\Delta\), then
\[
 h_i = \sum_{\|b\| = i} h_b. \tag{2.6}
\]

If \((\Delta, \pi)\) is completely balanced, i.e. of type \(1_d\), then we can identify each \(b = (b_1, \ldots, b_d) \in \mathbb{N}^d\) satisfying \(b \leq 1_d\) with the set \(\{i \in [d] : b_i = 1\} \subseteq [d]\). Under this identification, the corresponding arrays \(\{f_S\}_{S \subseteq [d]}\) and \(\{h_S\}_{S \subseteq [d]}\) are called the flag \(f\)-vector and flag \(h\)-vector of \((\Delta, \pi)\) respectively. In this case, (2.5) becomes
\[
 h_S = \sum_{T \subseteq S} (-1)^{|S-T|} f_S, \quad S \subseteq [d], \tag{2.7}
\]

and by Theorem 2.2.3, the flag $h$-vector is a refinement of the $h$-vector.

### 2.3 Order complexes of posets

A poset (partially ordered set) is a pair $(P, \leq)$, where $P$ is a set, and $\leq$ is a binary relation on $P$, such that for all $x, y, z \in P$, the following conditions hold:

(i) $x \leq x$ (reflexivity);
(ii) if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry); and
(iii) if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

Any binary relation $\leq$ satisfying the above three conditions (reflexivity, antisymmetry, and transitivity) is called a partial order (on $P$). By convention, write $y \geq x$ to mean $x \leq y$; write $x < y$ to mean $x \leq y$ and $x \neq y$; and write $y > x$ to mean $y \geq x$ and $y \neq x$. The poset $(P, \leq)$ is called finite (resp. infinite) if $P$ is a finite (resp. infinite) set. When the partial order is clear from the context, we denote the poset $(P, \leq)$ simply by $P$ without any ambiguity.

Given a poset $(P, \leq)$, we say two elements $x, y \in P$ are comparable if either $x \leq y$ or $y \leq x$. If any two elements in $P$ are comparable, then $(P, \leq)$ (or simply $P$) is called a linearly ordered set (or a totally ordered set), and $\leq$ is called a linear order (or a total order) on the set $P$. A well-ordering on $P$ is a linear order $\leq$ on $P$ such that every non-empty subset of $P$ has a least element. The poset $(P, \leq)$ is called a well-ordered set if $\leq$ is a well-ordering on $P$. When $P$ is finite, all linear orders are trivially well-orderings. An order ideal of $P$ is a subset $I$ of $P$ such that if $x \in P$ and $y \leq x$, then $y \in I$. If $\leq$ is a well-ordering on $P$, then the order ideal $I$ is called an initial segment of $P$ (with respect to $\leq$).
Given another poset \((Q, \preceq)\), we say \((P, \leq)\) and \((Q, \preceq)\) are isomorphic if there exists a bijection \(\phi : P \to Q\) such that \(x \leq y\) if and only if \(\phi(x) \preceq \phi(y)\), and we say \((P, \leq)\) and \((Q, \preceq)\) are anti-isomorphic if there exists a bijection \(\phi : P \to Q\) such that \(x \leq y\) if and only if \(\phi(y) \preceq \phi(x)\). Suppose \(P\) has \(n\) elements, and \(\preceq\) is another partial order on \(P\). Then we say \((P, \preceq)\) is a linear extension of \((P, \leq)\) if \(x \leq y\) implies \(x \preceq y\) for all \(x, y \in P\), and \((P, \preceq)\) is isomorphic to \(([n], \leq)\) (i.e. the poset consisting of the first \(n\) positive integers with the usual order).

A subposet\(^4\) of \((P, \leq)\) is a poset \((P', \leq')\) such that \(P'\) is a subset of \(P\), and for all \(x, y \in P'\), we have \(x \leq' y\) if and only if \(x \leq y\). Notice that \(\leq'\) is completely determined given \((P, \leq)\) and \(P'\). We say \(\leq'\) is the induced partial order on \(P'\), and we refer to the subposet \((P', \leq')\) as the subposet of \((P, \leq)\) induced by \(P'\). Again, when the context is clear, we denote the subposet \((P', \leq')\) simply by \(P'\). By abuse of notation, we write \(\leq\) in place of \(\leq'\), and we sometimes also say \(P'\) is a subposet of \(P\) with respect to \(\leq\). As a basic example, if \(x, y\) are elements in \(P\) satisfying \(x \leq y\), then the (closed) interval \([x, y] := \{z \in P : x \leq z \leq y\}\) is a subposet of \(P\) with respect to \(\leq\).

A chain of \(P\) is a subposet \(C\) of \(P\) that is a linearly ordered set. We say \(C\) is a saturated chain of \(P\) if there are no elements \(z \in P - C\) such that \(C \cup \{z\}\) is a chain of \(P\), and \(x < z < y\) for some \(x, y \in C\). Suppose \(x_0, x_1, \ldots, x_\ell (\ell \in \mathbb{N})\) are all the distinct elements in a finite chain \(C\) of \(P\), and suppose \(x_0 < x_1 < \cdots < x_\ell\). Then we call “\(x_0 < x_1 < \cdots < x_\ell\)” a chain, and we identify \(C\) with it. The integer \(\ell\) is called the length of \(C\). For all \(x, y \in P\), we write \(x < y\) and say \(y\) covers \(x\) if \(x < y\) and the subposet of \(P\) induced by \(\{x, y\}\) is a saturated chain (of length 1).

If \(P\) is finite, then the length of \(P\) is the maximum length of all the chains of \(P\).

---

\(^4\)What we have defined to be subposets are usually called induced subposets; see, e.g., [120, Chap. 3]. However, for our purposes, we will always assume all subposets are induced subposets, so we drop the word “induced”.
Furthermore, if every maximal chain of $P$ has the same length $r$, then we say $P$ is graded of rank $r$, or simply a graded poset.

Given a finite poset $P$, the collection of chains of $P$ is a simplicial complex denoted by $\Delta(P)$, and it is called the order complex of $P$. The vertex set of $\Delta(P)$ (consisting of chains) can be partitioned into $\rho = (P_1, \ldots, P_n)$, where $n - 1$ is the length of $P$ as a poset, and $x \in P_i$ if and only if $i - 1$ is the length of the longest chain $x_0 < x_1 < \cdots < x_{i-1} = x$ of $P$ with largest element $x$. By “longest”, we mean “of maximum length”. We check that the partition is still well-defined for finite posets that are not graded. Notice that $(\Delta(P), \rho)$ is a completely balanced complex (including the case when $P$ is not graded). Consequently, in view of Remark 2.2.2, we can then construct balanced complexes of arbitrary type from posets.

### 2.4 Lex, revlex, and colex orders

Let $S$ be a set. For each $k \in \mathbb{N}$, define a $k$-subset of $S$ to be a subset of $S$ with $k$ elements, and let $\binom{S}{k}$ denote the set of all $k$-subsets of $S$, which by default is empty if $|S| < k$.

**Definition 2.4.1.** If $k \in \mathbb{N}$, and $(S, \leq)$ is a well-ordered set, then the colex (colexicographic) order $\leq_{cl}$ on $\binom{S}{k}$ is defined by

$$A <_{cl} B \iff \max\{i : i \in A - B\} < \max\{j : j \in B - A\}.$$

Equivalently, if $A \Delta B$ is the symmetric difference of $A$ and $B$, then $A <_{cl} B$ if and only if $\max(A \Delta B) \in B$.

**Remark 2.4.2.** This order is sometimes known by other authors as the reverse lexicographic order (see, e.g., [119]) or the squashed order (see, e.g., [6]). For our
purposes, we always call this linear order the colex order. Part of the reason is that in contrast to the reverse lexicographic order (which has several different but widely used definitions), the colex order has a consistently used definition in the literature that coincides with Definition 2.4.1.

Let \( X = \{x_1, x_2, \ldots \} \) be a (possibly infinite) set of variables, and for each \( d \in \mathbb{N} \), let \( \mathcal{M}_X^d \) be the collection of all monomials in variables \( X \) of degree \( d \).

**Definition 2.4.3.** Let \( d \in \mathbb{N} \). If \( X \) is well-ordered by \( x_1 < x_2 < \cdots \), then the lexicographic order \( \leq_{\text{lex}} \) on \( \mathcal{M}_X^d \) induced by this given well-ordering on \( X \) is defined by

\[
x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots <_{\text{lex}} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} \cdots \iff \alpha_i < \beta_i \text{ for the largest } i \in \mathbb{P} \text{ such that } \alpha_i \neq \beta_i.
\]

(2.8)

**Remark 2.4.4.** If we fix this well-ordering \( x_1 < x_2 < \cdots \) on \( X \), and identify each squarefree monomial \( x_{i_1} \cdots x_{i_d} \in \mathcal{M}_X^d \) with the set \( \{i_1, \ldots, i_d\} \in (\mathbb{P}^d) \), then the induced lexicographic order on the subcollection of all squarefree monomials in \( \mathcal{M}_X^d \) is equivalent to the colex order defined on \( (\mathbb{P}^d) \).

Note that the lexicographic order \( \leq_{\text{lex}} \) on \( \mathcal{M}_X^d \) (as we have defined in Definition 2.4.3) depends on the given well-ordering of the variables. If we instead order the variables by \( x_1 > x_2 > \cdots \), then the lexicographic order \( \leq_{\text{lex}} \) on \( \mathcal{M}_X^d \) induced by this new linear order on \( X \) is instead given by

\[
x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots <_{\text{lex}} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} \cdots \iff \alpha_i < \beta_i \text{ for the smallest } i \in \mathbb{P} \text{ such that } \alpha_i \neq \beta_i.
\]

(2.9)

In commutative algebra, the usual convention (e.g., in [45, Section 15.2]) is to fix the linear order \( x_1 > x_2 > \cdots \) on \( X \). Thus for convenience, algebraists commonly
define the lex order on $M^d_X$ by (2.9) without specifying the well-ordering on the variables, so these variables instead inherit the order $x_1 >_{\text{lex}} x_2 >_{\text{lex}} \cdots$ (as degree one monomials).

In contrast, there seems to be no agreed-upon definition of the lex order in combinatorics (cf. [6] [48], [119]), although several highly influential texts (e.g., in [76], [119, Chap. II]) define the lex order $\leq_{\text{lex}}$ on $M^d_X$ as follows:

$$x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} \cdots <_{\text{lex}} x_1^{\beta_1}x_2^{\beta_2}x_3^{\beta_3} \cdots \iff \alpha_i > \beta_i \text{ for the smallest } i \in \mathbb{P} \text{ such that } \alpha_i \neq \beta_i.$$  

(2.10)

Notice that (2.9) and (2.10) are different. This difference arises possibly because algebraists associate ‘first’ and ‘last’ to ‘largest’ and ‘smallest’ respectively (e.g. the initial term of a polynomial is the largest term in that polynomial with respect to some monomial order), while combinatorialists instead associate ‘first’ and ‘last’ to ‘smallest’ and ‘largest’ respectively (i.e. consistent with the usual order $1 < 2 < \cdots$ on positive integers). On a similar note, some combinatorialists use our definition of the lex order in (2.8) to define the reverse lexicographic order instead.

To avoid confusion, we always use Definition 2.4.3 for the lex order, and in particular, we will always specify a well-ordering on the variables. For example, in Part II of the dissertation, we specify the linear order $x_1 > \cdots > x_n$ for the polynomial ring $\mathbb{k}[x_1, \ldots, x_n]$, while in Part IV of the dissertation, we specify the well-ordering $x_{i,1} < x_{i,2} < \cdots$ for each $i \in [n]$, corresponding to a polynomial ring whose set of (doubly-indexed) variables is partitioned into $n$ well-ordered subsets.

What about the reverse-lexicographic order? Even if we agree on a particular definition for the lex order, there are still two possible variations for the “reverse” of the lex order. For this dissertation, we require both variations. To eliminate
any ambiguity, we call them the “revlex” order and the “degree-revlex” order, and we denote them by $\leq_{rt}$ and $\leq_{rlex}$ respectively.

**Definition 2.4.5.** Let $d \in \mathbb{N}$, suppose $X$ is well-ordered by $x_1 < x_2 < \cdots$, and let $\leq_{tex}$ be the lex order on $\mathcal{M}_X^d$ induced by this given well-ordering on $X$. Then the revlex (reverse lexicographic) order $\leq_{rt}$ on $\mathcal{M}_X^d$ induced by the given well-ordering on $X$ is defined by $m \leq_{rt} m' \iff m \geq_{tex} m'$.

**Remark 2.4.6.** Note that $x_1 < x_2 < x_3 < \cdots$ implies $x_1 >_{rt} x_2 >_{rt} x_3 >_{rt} \cdots$.

Let $\mathcal{M}_X := \bigcup_{d \in \mathbb{N}} \mathcal{M}_X^d$ be the collection of all monomials in variables $X$ of arbitrary degree, and let $\mathbb{k}[X]$ denote the polynomial ring over a field $\mathbb{k}$ on the set of variables $X$. A *monomial order* on $\mathbb{k}[X]$ is a linear order $<$ on $\mathcal{M}_X$ satisfying the following condition: If $m_1, m_2, m' \in \mathcal{M}_X$ such that $m' \neq 1$, then $m_1 > m_2$ implies $m'm_1 > m'm_2 > m_2$.

**Definition 2.4.7.** Let $m_1 = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} \cdots, m_2 = x_1^{\beta_1}x_2^{\beta_2}x_3^{\beta_3} \cdots$ be arbitrary monomials in $\mathcal{M}_X$, and suppose $X$ is well-ordered by $x_1 < x_2 < \cdots$. The *degree-revlex* order $\leq_{rlex}$ on $\mathcal{M}_X$ induced by this given well-ordering on $X$ is defined as follows: $m_1 <_{rlex} m_2$ if and only if either $\deg(m_1) < \deg(m_2)$; or $\deg(m_1) = \deg(m_2)$ and $\alpha_i > \beta_i$ for the smallest $i \in \mathbb{P}$ such that $\alpha_i \neq \beta_i$.

**Remark 2.4.8.** The degree-revlex order $\leq_{rlex}$ on $\mathcal{M}_X$ is a monomial order on $\mathbb{k}[X]$. In particular, $x_1 < x_2 < x_3 < \cdots$ implies $x_1 <_{rlex} x_2 <_{rlex} x_3 <_{rlex} \cdots$, and for monomials of the same degree, the definition of $\leq_{rlex}$ coincides with the definition of “$\leq_{tex}$” in (2.10).
2.5 Compression and shifting

Compression and shifting are two seemingly simple yet highly effective techniques in combinatorics. The idea of compression first (implicitly) appeared in 1913 [58] as a key ingredient for the first proof of Macaulay’s theorem [85] (see Chapter 3.6). The idea of (combinatorial) shifting first appeared in a paper by Erdős–Ko–Rado [49] in 1961. Since then, these two techniques have been used multiple times in various important theorems in combinatorics, and even gave birth to Sperner theory, a whole new area in combinatorics; see [48] for a good introduction to the subject. There is also an algebraic analogue of shifting known as algebraic shifting; see [76] for an excellent overview.

Later in Chapter 11, we will introduce the notion of color compression and color shifting, which are generalized (colored) extensions of compression and shifting respectively. There is also the notion of colored algebraic shifting introduced recently by Babson–Novik [8]; see Chapter 11.2. Since all results in this dissertation involving either compression or shifting are proven in the more general setting of their colored analogues, we would only need the basic definitions of compressed and shifted simplicial complexes. See, e.g., [6], [48], [63], [76], for more details and other relevant references on compression and shifting; see also Chapter 7.1.

Recall that if $\leq$ is a well-ordering on a set $\mathcal{S}$, then for each $k \in \mathbb{N}$, we can define the colex order $\leq_{c\ell}$ on $\binom{\mathcal{S}}{k}$. We then define a colex initial segment of $\binom{\mathcal{S}}{k}$ to be an order ideal of $\binom{\mathcal{S}}{k}$. We can also similarly define lex initial segments, revlex initial segments, and degree-revlex initial segments of the collection $\mathcal{M}^S_k$ of degree

---

5 Although Macaulay proved his famous Macaulay’s theorem on Hilbert functions in 1927, most mathematicians do not know that the first actual proof of this theorem can be traced back to an obscure paper by N. M. Gjunter in 1913; see Chapter 7.1 for the fascinating history on Macaulay’s theorem.
Definition 2.5.1. Let $\Delta$ be a simplicial complex with a linearly ordered vertex set $V$. Then we say $\Delta$ is compressed if $F_k(\Delta)$ is a colex initial segment of $\left(\binom{V}{k+1}\right)$ for all integers $0 \leq k \leq \dim \Delta$ (with respect to the given linear order on $V$).

Definition 2.5.2. Let $\Delta$ be a simplicial complex, and suppose its vertex set is linearly ordered by $\leq$. Then we say $\Delta$ is shifted if every $F \in \Delta$ satisfies the following property: If $u, v \in V$ such that $u < v$, $u \notin F$, and $v \in F$, then $(F \setminus \{v\}) \cup \{u\} \in \Delta$.

Remark 2.5.3. It is easy to show that every compressed simplicial complex is shifted, but not every shifted simplicial complex is compressed.

2.6 Kruskal–Katona theorem

Given positive integers $N$ and $i$, it is easy to show there exists a unique expansion

$$N = \binom{N_i}{i} + \binom{N_{i-1}}{i-1} + \cdots + \binom{N_j}{j},$$

such that $N_i > N_{i-1} > \cdots > N_j \geq j \geq 1$. (See, e.g. [63, Section 8], for a proof.) Such an expansion is called the $i$-th Macaulay representation of $N$ (or the $i$-binomial expansion of $N$). The integers $N_i, \ldots, N_j$ are sometimes called Macaulay coefficients. Since this expansion is unique for each $N \in \mathbb{P}$, we can define functions $\partial^i, \partial_i$, on $\mathbb{N}$ by

$$\partial^i(N) = \binom{N_i}{i+1} + \binom{N_{i-1}}{i} + \cdots + \binom{N_j}{j+1},$$

$$\partial_i(N) = \binom{N_i}{i-1} + \binom{N_{i-1}}{i-2} + \cdots + \binom{N_j}{j-1},$$

for all $N \in \mathbb{P}$, and $\partial^i(0) = \partial_i(0) = 0$. By convention, $\binom{n}{k} = 0$ if $k > n$. These functions allow us to numerically characterize the $f$-vectors of simplicial complexes as follows:
Theorem 2.6.1 (Kruskal–Katona theorem [107, 82, 77]). Let $f = (f_0, \ldots, f_d)$ be a $(d + 1)$-tuple of positive integers. The following are equivalent:

(i) $f$ is the $f$-vector of a $d$-dimensional simplicial complex.

(ii) $f$ is the $f$-vector of a $d$-dimensional compressed simplicial complex.

(iii) $f$ is the $f$-vector of a $d$-dimensional shifted simplicial complex.

(iv) $f_i \leq \partial^i(f_{i-1})$ for every $i \in [d]$.

(v) $\partial_{i+1}(f_i) \leq f_{i-1}$ for every $i \in [d]$.

Later in Chapter 13, we will generalize the notion of Macaulay representations, and in Theorem 14.2.8, we use these generalized Macaulay representations to give a different (numerical) formulation of the Kruskal–Katona theorem. In particular, the Kruskal–Katona theorem is equivalent to the existence of a saturated chain in the poset of generalized Macaulay representations. For the history of the Kruskal–Katona theorem and a (long) list of references for various proofs of the theorem (including some recent ones), see Chapter 7.1.

2.7 Multicomplexes and colored multicomplexes

A multicomplex $M$ on a (possibly infinite) set of variables $X$ is a collection of monomials in these variables that is closed under divisibility (i.e. $m \in M, m'|m \Rightarrow m' \in M$). Throughout, we always assume all multicomplexes $M$ are non-empty, and we do not require every $x \in X$ to be in $M$. Notice that $M$ could be infinite. A subcomplex of $M$ is a subcollection of $M$ that is also a multicomplex. For any monomial $m = \prod_{x \in X} x^{c(x)}$ in $M$, the integer $c(x)$ is called the exponent of $x$ in $m$, and the support of $m$, denoted by supp($m$), is the set $\{x \in X : c(x) \neq 0\}$. The degree of $m$ is defined by $\deg(m) := \sum_{x \in X} c(x)$. If $M$ is finite, then the maximum
degree of the monomials in $M$ is called the \textit{dimension} of $M$, otherwise $M$ is called infinite dimensional. We say $M$ is \textit{pure} if $M$ is finite and all maximal monomials in $M$ (with respect to division) have the same degree.

If $Y \subseteq X$ is a subset, then let $m_Y := \prod_{x \in Y} x^{c(x)}$, and define the subcomplex $M_Y := \{m_Y : m \in M\}$. By default, we always have $1 \in M_Y$. For each $d \in \mathbb{N}$, denote the collection of all monomials in $M$ of degree $d$ by $M^d$. The \textit{f-vector} of $M$ is $(f_0, f_1, \ldots)$, where $f_i = |M^i|$ for each $i \in \mathbb{N}$. Note that $f_0 = 1$, corresponding to $1 \in M$. If there exists some $d \in \mathbb{N}$ such that $f_i = 0$ for all $i > d$, then we identify the \textit{f-vector} of $M$ with the (finite) vector $(f_0, f_1, \ldots, f_d)$ without any ambiguity. In particular, this means we identify two vectors if they differ only in the number of trailing zeros.

Let $X = \{x_1, x_2, \ldots\}$ be a set of variables linearly ordered by $x_1 < x_2 < \cdots$. Let $M_X$ denote the collection of all monomials whose supports are contained in $X$. For every $d \in \mathbb{N}$, let $M_X^d$ denote the subcollection of all monomials in $M_X$ of degree $d$. Given any map $\phi : X \to \mathbb{P}$, let $e = (e_1, e_2, \ldots) := (\phi(x_1), \phi(x_2), \ldots)$, and let $M_X(e)$ be the collection of all monomials $x_1^{e_1}x_2^{e_2}\cdots$ such that $0 \leq c_i \leq e_i$ for each $i$. Also, for every $d \in \mathbb{N}$, let $M_X^d(e)$ be the subcollection of all monomials in $M_X(e)$ of degree $d$. In particular, if $1 = (1, 1, 1, \ldots)$, then $M_X(1)$ is the collection of all squarefree monomials whose supports are contained in $X$.

Every simplicial complex $\Delta$ with a vertex set $V \subseteq X$ corresponds bijectively to a finite multicomplex $M \subseteq M_X(1)$ via

$$\{x_{i_1}, \ldots, x_{i_t}\} \in \Delta \iff x_{i_1} \cdots x_{i_t} \in M,$$

hence multicomplexes can be considered as generalizations of simplicial complexes. Note however that the \textit{f-vector} of $\Delta$ and the \textit{f-vector} of $M$ differ by a shift in the indexing.
Analogous to the Kruskal–Katona theorem (see Chapter 2.6), there is also a numerical characterization of the $f$-vectors of multicomplexes involving Macaulay representations. It was Stanley [115] who first realized that this numerical characterization essentially follows from Macaulay’s theorem (see Chapter 3.6).

For $N, i \in \mathbb{P}$, recall that the $i$-th Macaulay representation of $N$ is

$$N = \binom{N_i}{i} + \binom{N_{i-1}}{i-1} + \cdots + \binom{N_j}{j},$$

for some integers $N_i > N_{i-1} > \cdots > N_j \geq j \geq 1$. Since the $i$-th Macaulay representation of $N$ is unique, we can define functions $\partial^{(i)}$, $\partial_{(i)}$ on $N$ by

$$\partial^{(i)}(N) = \binom{N_i + 1}{i+1} + \binom{N_{i-1} + 1}{i} + \cdots + \binom{N_j + 1}{j},$$

$$\partial_{(i)}(N) = \binom{N_i - 1}{i-1} + \binom{N_{i-1} - 1}{i-2} + \cdots + \binom{N_j - 1}{j-1},$$

for all $N \in \mathbb{P}$, and $\partial^{(i)}(0) = \partial_{(i)}(0) = 0$. By convention, let $\binom{0}{0} = 1$, and define $\binom{n}{k}$ to be 0 if $k < 0$ or $k > n$.

**Theorem 2.7.1** ([119, Thm. II.2.2]). Let $f = (f_0, f_1, f_2, \ldots)$ be a sequence of non-negative integers. The following are equivalent:

(i) $f$ is the $f$-vector of a multicomplex.

(ii) $f_0 = 1$, and $f_{i+1} \leq \partial^{(i)}(f_i)$ for all $i \in \mathbb{P}$.

(iii) $f_0 = 1$, and $\partial_{(i+1)}(f_{i+1}) \leq f_i$ for all $i \in \mathbb{P}$.

**Definition 2.7.2.** A sequence $(f_0, f_1, f_2, \ldots)$ of non-negative integers satisfying either of the equivalent conditions (ii) and (iii) in Theorem 2.7.1 is called an $O$-sequence or an $M$-sequence. If the sequence has finitely many entries, or finitely many non-zero entries, then it is sometimes called an $M$-vector. If an O-sequence ...
(or M-sequence, or M-vector) is the f-vector of a pure multicomplex, then we call it a \textit{pure O-sequence} (or pure M-sequence, or pure M-vector).

Next, we look at colored analogues of multicomplexes.

\textbf{Definition 2.7.3.} Let $a = (a_1, \ldots, a_n) \in \mathbb{P}^n$. A \textit{colored multicomplex of type $a$} is a pair $(M, \pi)$ satisfying:

(i) $M$ is a multicomplex on a non-empty set $X$;

(ii) $\pi = (X_1, \ldots, X_n)$ is an ordered partition of $X$;

(iii) $\deg(m_{X_i}) \leq a_i$ for every $m \in M$ and every $i \in [n]$.

\textbf{Remark 2.7.4.} As is the case with generalized colored complexes, we write “$a$-colored” to mean “colored of type $a$”. A \textit{colored multicomplex} is an $a$-colored multicomplex for some $a \in \mathbb{P}^n$, and we say “generalized colored multicomplexes” to mean colored multicomplexes of arbitrary type. A multicomplex $M$ on $X$ is called \textit{colorable of type $a$} if there is an ordered partition $\pi$ of $X$ such that $(M, \pi)$ is $a$-colored. In the special case when $a = 1_n$, we say $M$ is $n$-\textit{colorable}.

Given an ordered partition $\pi = (X_1, \ldots, X_n)$ of $X$, the \textit{multidegree} of a monomial $m \in M$ (with respect to $\pi$) is the vector

$$\text{Deg}_\pi(m) := (\deg(m_{X_1}), \ldots, \deg(m_{X_n})) \in \mathbb{N}^n.$$ 

For each $b \in \mathbb{N}^n$, let $f_b$ be the number of monomials $m$ in $M$ such that $\text{Deg}_\pi(m) = b$. The array of integers $\{f_b\}_{b \leq a}$ is called the \textit{fine f-vector} of $(M, \pi)$, and it is a refinement of the $f$-vector $(f_0, f_1, \ldots)$ of $M$ in the sense that

$$f_i = \sum_{|b|=i} f_b, \quad i \in \mathbb{N}.$$ 

For type $a = 1_n$, define the \textit{flag f-vector} $\{f_S\}_{S \subseteq [n]}$ of $(M, \pi)$ analogously as in the case of simplicial complexes.
Fix a set of variables $X$, and let $\pi = (X_1, \ldots, X_n)$ be an ordered partition of $X$ such that each $X_i = \{x_{i,1}, x_{i,2}, \ldots\}$ is linearly ordered by $x_{i,1} < x_{i,2} < \cdots$. For each $i \in [n]$, let $\phi_i : X_i \to \mathbb{P}$ be any map, and let $e_i = (\phi_i(x_{i,1}), \phi_i(x_{i,2}), \ldots)$. Let $M_\pi(e_1, \ldots, e_n)$ be the collection of all monomials $m$ in variables $X$ such that $m_{X_i} \in M_{X_i}(e_i)$ for every $i \in [n]$. When $e_i = (\infty, \infty, \ldots)$ for each $i \in [n]$, we simply write $M_\pi(e_1, \ldots, e_n)$ as $M_\pi$. If $M \subseteq M_\pi(e_1, \ldots, e_n)$ and $(M, \pi)$ is a colored multicomplex, then by abuse of notation, we say $(M, \pi)$ is in $M_\pi(e_1, \ldots, e_n)$. Note that it is implicitly assumed every colored multicomplex in $M_\pi(e_1, \ldots, e_n)$ has $\pi$ as its corresponding ordered partition, and the length of each sequence $e_i$ is the cardinality of $X_i$.

Suppose $\varphi : X' \to \mathbb{P}$ is a map whose domain contains $X$. We similarly define $M_X(\varphi)$ to be the collection of all monomials $m \in M_X$ such that the exponent of each $x \in (\text{supp}(m) \cap X')$ in $m$ is at most $\varphi(x)$. By abuse of notation, an $\mathbf{a}$-colored multicomplex $(M, \pi)$ is said to be in $M_X(\varphi)$ if $M \subseteq M_X(\varphi)$. For each $d \in \mathbb{N}$, define $M_X^d(\varphi) := M_X^d \cap M_X(\varphi)$. If $1 : X \to \mathbb{P}$ is the constant map defined by $x \mapsto 1$ for all $x \in X$, then $M_X(1)$ is the collection of all squarefree monomials in $M_X$. For convenience, we do not distinguish between $1 = (1, 1, 1, \ldots)$ (as a sequence of ones) and the constant map $1 : X \to \mathbb{P}$ defined by $x \mapsto 1$. Since multicomplexes can be considered as generalizations of simplicial complexes, every $\mathbf{a}$-colored complex with vertex set $V \subseteq X$ can be identified with an $\mathbf{a}$-colored multicomplex in $M_X(1)$.
2.8 Clements–Lindström theorem

The Clements–Lindström theorem is a simultaneous generalization of the Kruskal–Katona theorem (introduced in Chapter 2.6) and Macaulay’s theorem. At this moment, we have not yet stated Macaulay’s theorem in its original algebraic formulation. However, we have already seen an equivalent (combinatorial) formulation of Macaulay’s theorem (Theorem 2.7.1).

Let \( X = \{x_1, \ldots, x_n\} \) be a set of \( n \) variables. The Kruskal–Katona theorem characterizes the \( f \)-vectors of multicomplexes in \( \mathcal{M}_X(1^n) \), while an equivalent version of Macaulay’s theorem (Theorem 2.7.1) characterizes the \( f \)-vectors of multicomplexes in \( \mathcal{M}_X(\infty_n) = \mathcal{M}_X \). In comparison, the Clements–Lindström theorem generalizes these two theorems by characterizing the \( f \)-vectors of multicomplexes in \( \mathcal{M}_X(e) \) for arbitrary \( e \in \mathbb{F}^n \).

Let \( d \in \mathbb{N} \). Recall from Remark 2.4.4 that the colex order on \( {[n]\choose d} \) is equivalent to the lex order on \( \mathcal{M}_X(d) \). Note also from Definition 2.5.1 that a simplicial complex \( \Delta \) with vertex set contained in \( [n] \) is compressed if and only if \( F_k(\Delta) \) is a colex initial segment of \( {[n]\choose k+1} \) for all \( 0 \leq k \leq \dim \Delta \). Thus, by translating the statement of the Kruskal–Katona theorem (Theorem 2.6.1) into the language of multicomplexes in \( \mathcal{M}_X(1^n) \), we obtain a statement involving lex initial segments of \( \mathcal{M}_X(d) \) induced by the linear order \( x_1 < \cdots < x_n \), or equivalently, revlex initial segments of \( \mathcal{M}_X(d) \) induced by the linear order \( x_1 > \cdots > x_n \); cf. Chapter 8.2.

**Definition 2.8.1.** Let \( e = (e_1, \ldots, e_n) \in \mathbb{F}^n \), and fix a linear order on \( X \). A multicomplex \( M \) in \( \mathcal{M}_X(e) \) is called **lex-compressed** (resp. **revlex-compressed**) if \( M^d \) is a lex (resp. revlex) initial segment of \( \mathcal{M}_X(d) \) for all \( d \in \mathbb{N} \) with respect to the given linear order on \( X \). For convenience, we say \( M \) is **compressed** if it is
either lex-compressed or revlex-compressed (with respect to the given linear order on $X$).\footnote{In the math literature, compressed multicomplexes are usually either lex-compressed with respect to the linear order $x_1 < \cdots < x_n$, or revlex-compressed with respect to the linear order $x_1 > \cdots > x_n$; cf. Theorem 2.8.2.}

**Theorem 2.8.2** (Clements–Lindström theorem [36]). Let $e = (e_1, \ldots, e_n) \in \mathbb{P}^n$, and let $f = (f_0, f_1, f_2, \ldots)$ be a sequence of non-negative integers. The following are equivalent:

(i) $f$ is the $f$-vector of a multicomplex in $\mathcal{M}_X(e)$.

(ii) $f$ is the $f$-vector of a lex-compressed multicomplex in $\mathcal{M}_X(e)$ with respect to the linear order $x_1 < \cdots < x_n$.

(iii) $f$ is the $f$-vector of a revlex-compressed multicomplex in $\mathcal{M}_X(e)$ with respect to the linear order $x_1 > \cdots > x_n$.

**Notes**

There are a few definitions of simplicial complexes in the literature. The definition we gave in Chapter 2.1 is that of an abstract simplicial complex. There is another “more geometric” notion of simplicial complexes (used, e.g., in convex geometry) that builds upon the notion of geometric simplices. Given $k \in \mathbb{N}$, a geometric $k$-simplex is the convex hull of $k + 1$ affinely independent points, usually embedded in $\mathbb{R}^n$ (for some $n \geq k$). A geometric simplicial complex $\mathcal{K}$ is a collection of geometric simplices such that any face of a geometric simplex in $\mathcal{K}$ is also in $\mathcal{K}$, and the intersection of any two geometric simplices $\sigma_1, \sigma_2 \in \mathcal{K}$ is a face of both $\sigma_1$ and $\sigma_2$; see [131] for details. Some authors also distinguish the notion of geometric simplicial complexes from the topological notion of simplicial complexes as triangulations of topological spaces. The main difference is that a “topological” simplicial complex does not come equipped with a metric. For this dissertation, all simplicial complexes are assumed to be abstract, finite, and non-empty. In some instances, we want to treat a simplicial complex as a topological space, and we do so by considering “the” geometric realization of an (abstract) simplicial complex.

Face vectors are defined exactly the same as $f$-vectors, hence both notions can be used interchangeably. It is sometimes convenient to refer to the entries of $f$-vectors as either face numbers or $f$-numbers. Motivated by the Kruskal–Katona
theorem, there has been much work done finding (analogous) numerical characterizations for the $f$-vectors of various classes of simplicial complexes, polyhedral complexes, and regular CW complexes. A common strategy is to first prove that several classes of complexes share the same $f$-vectors or $h$-vectors, and then find an explicit numerical characterization for one of these classes. For example, Stanley [115] showed that the collections of $h$-vectors of Cohen–Macaulay complexes, constructible complexes and shellable complexes are each identical to the collection of $f$-vectors of multicomplexes, and he observed that Macaulay’s theorem [85] yields an explicit numerical characterization of the $f$-vectors of multicomplexes. Wegner [129] showed that the $f$-vectors of polyhedral complexes have the same numerical characterization as that of the $f$-vectors of simplicial complexes already established by the Kruskal–Katona theorem, while Nevo [99] subsequently proved a simultaneous generalization of Wegner’s theorem and Macaulay’s theorem. In fact, the crucial step in several proofs of the Kruskal–Katona theorem is to show that the $f$-vector of a simplicial complex is the $f$-vector of a compressed/shifted simplicial complex, and then give a numerical characterization for this smaller class of complexes. For a good exposition on the Kruskal–Katona theorem, see [63, Section 8]. For various proofs of the Kruskal–Katona theorem (including several recent ones), see the relevant references cited in Chapter 7.1.

In addition to the order complexes of posets that we introduced in Chapter 2.3, Coxeter complexes also form an important class of completely balanced complexes; see [4] for a good introduction to Coxeter complexes (among other things). Very recently, there has been much interest in face enumeration questions concerning completely balanced complexes. For example, in March 2015 (which is very recent relative to the time of writing this dissertation), Juhnke-Kubitzke–Murai [74] proved the balanced generalized lower bound inequality, which was a conjecture on the face numbers of completely balanced simplicial polytopes proposed by Klee–Novik [79] in September 2014. This author had the pleasure of listening to a talk by Novik on the balanced generalized lower bound inequality in November 2014.

A (simplicial) homology $d$-sphere is a triangulation of a $d$-manifold with the same homology groups as a $d$-sphere. The Dehn–Sommerville equations for homology spheres (e.g. convex simplicial polytopes; cf. [43], [64], [80], [112], [117], [119]) asserts that the $h$-vector $(h_0, \ldots, h_d)$ of a homology $(d-1)$-sphere satisfies $h_i = h_{d-i}$ for all $0 \leq i \leq d$. These equations hold more generally for Eulerian complexes [80]. (Homology spheres are important examples of Eulerian complexes; see, e.g. [120, Chap. 3], for relevant definitions and more details on Eulerian complexes and semi-Eulerian complexes.) In fact, Klee [80] proved the following analogous result for semi-Eulerian complexes$^8$: The $h$-vector $(h_0, \ldots, h_d)$ of a $(d-1)$-dimensional semi-Eulerian complex $\Delta$ satisfies

\[ h_{d-i} - h_i = (-1)^i \binom{d}{i} (\tilde{\chi}(\Delta) - \tilde{\chi}(S^{d-1})) \]  

(2.14)

$^8$Note that what are now commonly known as semi-Eulerian complexes were called Eulerian manifolds in [80].
for all $0 \leq i \leq d$, where $\widetilde{\chi}(\cdot)$ denotes the reduced Euler characteristic, and $S^{d-1}$ denotes the $(d-1)$-sphere.

Stanley [118] proved that the flag $h$-vector $\{h_S\}_{S \subseteq [d]}$ of a $(d-1)$-dimensional (completely balanced) Eulerian complex satisfies $h_S = h_{[d]\setminus S}$ for all $S \subseteq [d]$, so in particular, these generalized Dehn–Sommerville equations hold for completely balanced homology spheres. Billera–Magurn [17] observed that Stanley’s proof can be extended to the case of $a$-balanced homology spheres in a straightforward manner, thus the fine $h$-vector $\{h_b\}_{b \leq a}$ of an $a$-balanced homology sphere (for any given $a \in \mathbb{P}^n$) satisfies $h_b = h_{a-b}$ for all $0 \leq b \leq a$. We should also remark that Stanley [118, Prop. 2.2] obtained equations for the flag $h$-vectors of (completely balanced) semi-Eulerian complexes analogous to (2.14), while Swartz [123, Thm. 3.8] showed that analogous equations are satisfied by the fine $h$-vectors of $a$-balanced semi-Eulerian complexes.

Given any $a \in \mathbb{P}^n$, the generalized Dehn–Sommerville equations for $a$-balanced homology spheres are the only linear relations satisfied by the (entries of the) corresponding fine $h$-vectors (over all $a$-balanced homology spheres). The completely balanced case was proven by Bayer–Billera [9], while the general $a$-balanced case was proven by Billera–Magurn [17]. Analogous statements for (completely balanced) Eulerian complexes and $a$-balanced semi-Eulerian complexes are also true; see [9], [123].
Throughout this dissertation, \( \mathbb{k} \) is a field, and \( S := \mathbb{k}[x_1, \ldots, x_n] \) is a polynomial ring on \( n \) variables over \( \mathbb{k} \). Given any \( \mathbb{k} \)-vector space \( U \), let \( \dim_k U \) denote the dimension of \( U \) as a \( \mathbb{k} \)-vector space. All \( \mathbb{k} \)-algebras are assumed to be unital, commutative, associative and finitely generated (and so in particular they are rings), and more generally, all rings are assumed to be commutative with identity, and Noetherian.

A \( \mathbb{k} \)-algebra \( A \) is called \( \mathbb{N}^n \)-graded if \( A = \bigoplus_{a \in \mathbb{N}^n} A_a \), where each \( A_a \) is a \( \mathbb{k} \)-vector space, and \( A_a \cdot A_b \subseteq A_{a+b} \) for all \( a, b \in \mathbb{N}^n \). An \( A \)-module \( R \) is called \( \mathbb{N}^n \)-graded if \( R = \bigoplus_{a \in \mathbb{N}^n} R_a \), where each \( R_a \) is a \( \mathbb{k} \)-vector space, and \( A_a \cdot R_b \subseteq R_{a+b} \) for all \( a, b \in \mathbb{N}^n \). An element \( q \in R \) is called homogeneous (with respect to the given grading) if \( q \in R_a \) for some \( a \in \mathbb{N}^n \). If \( R_0 = \mathbb{k} \), then we say \( R \) is connected. For our purposes, we assume all \( \mathbb{N}^n \)-graded (modules over) \( \mathbb{k} \)-algebras are connected (for all \( n \in \mathbb{P} \)). We will mostly deal with \( \mathbb{N} \)-graded (modules over) \( \mathbb{k} \)-algebras, so for convenience, we say a (module over a) \( \mathbb{k} \)-algebra is graded if it is \( \mathbb{N} \)-graded.

Suppose \( R = \bigoplus_{d \in \mathbb{N}} R_d \) is an \( \mathbb{N} \)-graded \( A \)-module for some \( \mathbb{N} \)-graded \( \mathbb{k} \)-algebra \( A \). We say \( R \) is positively graded if all of the generators of \( R \) may be taken in \( \bigoplus_{d>0} R_d \), and we say \( R \) is standard graded if all of the generators of \( R \) may be taken in \( R_1 \). If \( A \) is a positively graded (connected) \( \mathbb{k} \)-algebra, then as a ring, it has a unique homogeneous maximal ideal, which we denote by \( \mathfrak{m}_A \). Usually we treat \( S = \mathbb{k}[x_1, \ldots, x_n] \) as a standard graded \( \mathbb{k} \)-algebra (i.e. \( \deg(x_i) = 1 \) for all \( i \in [n] \)), but on rare occasions (e.g. in Chapter 3.3), we may instead consider \( S \) to be \( \mathbb{N}^n \)-graded. Homogeneous polynomials in \( S \) are called forms, and we define
\( m := m_S = \langle x_1, \ldots, x_n \rangle \subseteq S \). Note that \( m \) is sometimes called the augmentation ideal or the irrelevant ideal.

### 3.1 Hilbert functions, Hilbert series, Hilbert polynomials

Suppose \( A \) is a graded \( k \)-algebra. Given a finitely generated graded \( A \)-module \( R = \bigoplus_{d \in \mathbb{N}} R_d \), the Hilbert function of \( R \) is the map \( H(R, -) : \mathbb{N} \rightarrow \mathbb{N} \) given by \( d \mapsto \dim_k R_d \), and the Hilbert series of \( R \) is the formal power series

\[
\text{Hilb}(R, t) := \sum_{d \in \mathbb{N}} H(R, d) t^d.
\]

Note in particular that \( H(R, d) < \infty \) for each \( d \in \mathbb{N} \), since both \( A \) and \( R \) are assumed to be finitely generated. (Remember that all \( k \)-algebras are assumed to be finitely generated.)

**Theorem 3.1.1** (Hilbert, Serre; see, e.g., [119, Thm. I.2.3], [23, Cor. 4.1.8]). Let \( R = \bigoplus_{d \in \mathbb{N}} R_d \) be a finitely generated standard graded \( S \)-module. Then there exists a unique integer \( d \in \mathbb{N} \), and a unique polynomial \( h(t) = h_0 + h_1 t + \cdots + h_\ell t^\ell \in \mathbb{Z}[t] \) of degree \( \ell \) with integer coefficients, such that \( h(1) \neq 0 \), and

\[
\text{Hilb}(R, t) = \frac{h(t)}{(1 - t)^d}.
\]

**Definition 3.1.2.** Let \( R \) be a finitely generated standard graded \( S \)-module, and follow the notation as in Theorem 3.1.1. Then the Krull dimension of \( R \), denoted by \( \dim R \), is the (uniquely determined) integer \( d \) in (3.1), and the \( h \)-vector of \( R \) is the vector \((h_0, h_1, \ldots, h_\ell) \in \mathbb{Z}^{\ell+1}\).

**Theorem 3.1.3** (Hilbert; see, eg., [23, Thm. 4.1.3]). Let \( R \) be a finitely generated standard graded \( S \)-module with Krull dimension \( d \). Then there exists a polynomial \( P_R(m) \in \mathbb{Q}[m] \) (in the variable \( m \)) of degree \( d - 1 \) with rational coefficients, such
that \( P_R(m) = H(R, m) \) for \( m \gg 0 \). (Note: For this theorem to hold in the case \( d = 0 \), we use the convention that the zero polynomial has degree \(-1\).)

**Definition 3.1.4.** Let \( R \) be a finitely generated standard graded \( S \)-module. Following the notation as in Theorem 3.1.3, the polynomial \( P_R(m) \) is called the **Hilbert polynomial** of \( R \).

**Remark 3.1.5.** Notice that \( \text{deg}(P_R(m)) = \dim R - 1 \).

### 3.2 Stanley–Reisner correspondence

Let \( \Delta \) be a simplicial complex with vertex set \( V = \{v_1, \ldots, v_n\} \). The **Stanley–Reisner ring** (also known as the **face ring**) of \( \Delta \) (over \( k \)) is the ring \( k[\Delta] := S/I_\Delta \), where \( I_\Delta \) is the **Stanley–Reisner ideal** of \( \Delta \) given by

\[
I_\Delta := \langle \{x_{i_1} \cdots x_{i_m} : \{v_{i_1}, \ldots, v_{i_m}\} \notin \Delta \} \rangle. 
\]

**Remark 3.2.1.** Here, we collect some remarks concerning Stanley–Reisner rings.

(i) The Stanley–Reisner ideal \( I_\Delta \) is generated by all the squarefree monomials corresponding to the non-faces of \( \Delta \).

(ii) By the definition of a vertex set, we have that \( \{v_i\} \in \Delta \) for all \( i \in [n] \). Consequently, \( I_\Delta \subseteq m^2 \).

(iii) Conversely, if \( I \subseteq m^2 \) is any ideal of \( S \) generated by squarefree monomials, then \( S/I \cong k[\Delta] \) for some simplicial complex \( \Delta \).

(iv) Thus, we get the bijective correspondence: simplicial complexes \( \leftrightarrow \) squarefree monomial ideals. In particular, if \( \Delta, \Delta' \) are simplicial complexes with the same vertex set, then \( \Delta \subseteq \Delta' \) if and only if \( I_\Delta \supseteq I_{\Delta'} \).
(v) The $h$-vector of $\Delta$ “equals” the $h$-vector of $k[\Delta]$, in the sense that two vectors that differ only in the number of trailing zeros are considered “equal”; cf (2.1) and Definition 3.1.2.

(vi) Notice that if $\Delta$ is a simplicial complex, then the Hilbert function of $k[\Delta]/\langle x_1^2, \ldots, x_n^2 \rangle$ “equals” the $f$-vector of $\Delta$. More precisely, if $\Delta$ is a $(d - 1)$-dimensional simplicial complex, and we let $R = k[\Delta]/\langle x_1^2, \ldots, x_n^2 \rangle$, then $(H(R, 1), H(R, 2), \ldots, H(R, d)) \in \mathbb{P}^d$ is the $f$-vector of $\Delta$.

### 3.3 Hilbert series of Stanley–Reisner rings

In this section, our goal is to relate the $f$-vector of a simplicial complex $\Delta$ with the Hilbert series of the corresponding Stanley–Reisner ring $k[\Delta]$. As a key step toward this goal, we will use the fine grading on $S = k[x_1, \ldots, x_n]$; this is a natural refinement of the standard grading on $S$.

Previously in Chapter 2.7, we defined the multidegree of a monomial with respect to some ordered partition of the underlying set of variables; see discussion after Remark 2.7.4. Consider the ordered partition $\pi = (\{x_1\}, \{x_2\}, \ldots, \{x_n\})$ of $\{x_1, \ldots, x_n\}$ into singletons, and define the multidegree $\text{Deg}_\pi$ on monomials in $S$ with respect to this given ordered partition $\pi$. Thus, the multidegree of a monomial $x_1^{a_1} \cdots x_n^{a_n}$ in $S$ is the vector $(a_1, \ldots, a_n) \in \mathbb{N}^n$.

For each $a \in \mathbb{N}^n$, let $S_a := \{cx^a : c \in k\}$. We check that $S = \bigoplus_{a \in \mathbb{N}^n} S_a$ is $\mathbb{N}^n$-graded by $\text{Deg}_\pi$. We can also extend this $\mathbb{N}^n$-grading to a $\mathbb{Z}^n$-grading by specifying that $S_b = 0$ for all $b \in \mathbb{Z}^n \setminus \mathbb{N}^n$. The $\mathbb{Z}^n$-grading could be useful when we want maps between multigraded $S$-modules to be homogeneous of multidegree...
0_n, in which case some of the S-modules in these maps may require shifts in the multidegrees. Whether S is treated as being ℤ^n-graded or ℤ^n-graded, we say S is finely graded, and the corresponding multigrading is called the fine grading on S. Each k-vector space S_a is called the homogeneous component of S of multidegree a.

Note that the fine grading is a refinement of the standard grading on S (i.e. \( \deg(x_i) = 1 \) for all \( i \in [n] \)) in the following sense:

\[
S_k = \bigoplus_{a \in \mathbb{N}^n, |a| = k} S_a, \quad k \in \mathbb{N}.
\]

(3.3)

Note also that ℤ^n-graded ideals of S (under this fine grading) are identically monomial ideals of S. Consequently, if I ⊆ S is a monomial ideal, then the quotient ring S/I inherits the natural ℤ^n-grading given by \( (S/I)_a = S_a/I_a \) for all \( a \in \mathbb{N}^n \). In particular, Stanley–Reisner rings are ℤ^n-graded under the fine grading.

Let \( x := (x_1, \ldots, x_n) \), and let \( t := (t_1, \ldots, t_n) \) be another n-tuple of (new) variables. For convenience, define \( x^a := x_1^{a_1} \cdots x_n^{a_n} \) and \( t^a := t_1^{a_1} \cdots t_n^{a_n} \) for every \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \). Given an ℤ^n-graded S-module \( R = \bigoplus_{a \in \mathbb{N}^n} R_a \), the multigraded Hilbert function of R is the map \( H(R, -) : \mathbb{N}^n \to \mathbb{N} \) given by \( a \mapsto \dim_k R_a \), and the multigraded Hilbert series of R is the formal power series

\[
\text{Hilb}(R, t) := \sum_{a \in \mathbb{N}^n} H(R, a)t^a.
\]

We now look at the multigraded Hilbert function of the Stanley–Reisner ring \( k[\Delta] \). Without loss of generality, suppose \( \Delta \) is a \((d - 1)\)-dimensional simplicial complex with vertex set \([n]\). Note that a monomial \( x^a \) in S is contained in the Stanley–Reisner ideal \( I_\Delta \) if and only if \( \text{supp}(a) \not\subseteq \Delta \), thus \( x^a \neq 0 \) in \( k[\Delta] \) if and only if \( \text{supp}(a) \subseteq \Delta \). These non-zero monomials in \( k[\Delta] \) form a k-basis for \( k[\Delta] \),
hence
\[
\text{Hilb}(k[\Delta], t) = \sum_{a \in \mathbb{N}^n} t^a = \sum_{F \in \Delta} \sum_{a \in \mathbb{N}^n \supseteq a \in F} t^a. \tag{3.4}
\]

Now, observe that
\[
F = \emptyset \implies \sum_{a \in \mathbb{N}^n \supseteq a \in F} t^a = 1; \quad (3.5)
\]
\[
F \neq \emptyset \implies \sum_{a \in \mathbb{N}^n \supseteq a \in F} t^a = \prod_{i \in F} \frac{t_i}{1 - t_i}; \quad (3.6)
\]
so if we use the convention that the product over an empty index set is 1, we get
\[
\text{Hilb}(k[\Delta], t) = \sum_{F \in \Delta} \prod_{i \in F} \frac{t_i}{1 - t_i}. \tag{3.7}
\]

Using (3.7), we can then explicitly determine the Hilbert series of \(k[\Delta]\) as follows.

**Theorem 3.3.1** ([119, Thm. II.1.4]). Let \(\Delta\) be a \((d - 1)\)-dimensional simplicial complex with \(f\)-vector \((f_0, \ldots, f_{d-1})\). Then the Hilbert series of \(k[\Delta]\) (as an \(\mathbb{N}\)-graded ring) is
\[
\text{Hilb}(k[\Delta], t) = \sum_{i = -1}^{d-1} f_i t^{i+1} \left(1 - t\right)^{i+1},
\]
where \(f_{-1} = 1\).

**Proof.** Observe that (3.3) says
\[
k[\Delta]_i = \bigoplus_{a \in \mathbb{N}^n \mid |a| = i} k[\Delta]_a \tag{3.9}
\]
for all \(i \in \mathbb{N}\), thus to get the Hilbert series of \(k[\Delta]\) as an \(\mathbb{N}\)-graded ring, we only need to replace every \(t_i\) in (3.7) by \(t\). This yields
\[
\text{Hilb}(k[\Delta], t) = \sum_{F \in \Delta} \prod_{j \in F} \frac{t}{1 - t} = \sum_{i = -1}^{d-1} \sum_{F \in \Delta \mid \dim F = i} \prod_{j \in F} \frac{t}{1 - t} = \sum_{i = -1}^{d-1} f_i t^{i+1} \left(1 - t\right)^{i+1}. \tag{3.10}
\]

\(\square\)
Corollary 3.3.2 ([119, Thm. II.1.4]). Let \( \Delta \) be a \((d - 1)\)-dimensional simplicial complex with f-vector \((f_0, \ldots, f_{d-1})\). Then the Hilbert function of \( \mathbb{k}[\Delta] \) (as a \( \mathbb{N} \)-graded ring) is

\[
H(\mathbb{k}[\Delta], m) = \begin{cases} 
1, & \text{if } m = 0; \\
\sum_{i=0}^{d-1} f_i \binom{m - 1}{i}, & \text{if } m > 0.
\end{cases}
\]

Proof. Consider the following identity

\[
\frac{1}{(1 - t)^{i+1}} = \sum_{j=0}^{\infty} \binom{i + j}{i} t^j;
\]

which holds for all \( i \in \mathbb{N} \). Applying (3.11) to Theorem 3.3.1, we thus get that

\[
\text{Hilb}(\mathbb{k}[\Delta], t) = \sum_{i=0}^{d-1} \frac{f_i t^{i+1}}{(1 - t)^{i+1}}
\]

\[
= 1 + f_0 t \left( \sum_{j=0}^{\infty} t^j \right) + f_1 t^2 \left( \sum_{j=0}^{\infty} \binom{j + 1}{1} t^j \right) + f_2 t^3 \left( \sum_{j=0}^{\infty} \binom{j + 2}{2} t^j \right) + \cdots
\]

\[
\vdots + f_{d-1} t^d \left( \sum_{j=0}^{\infty} \binom{j + d - 1}{d - 1} t^j \right). \tag{3.12}
\]

By definition, \( H(\mathbb{k}[\Delta], m) \) equals the coefficient of \( t^m \) in \( \text{Hilb}(\mathbb{k}[\Delta], t) \). We check that \( H(\mathbb{k}[\Delta], 0) = 1 \), while for each \( m > 0 \), we have

\[
H(\mathbb{k}[\Delta], m) = f_0 + f_1 \binom{m - 1}{1} + f_2 \binom{m - 1}{2} + \cdots + f_{d-1} \binom{m - 1}{d - 1}
\]

\[
= \sum_{i=0}^{d-1} f_i \binom{m - 1}{i}. \tag{3.13}
\]

\[
\square
\]

Corollary 3.3.3. Let \( \Delta \) be a \((d - 1)\)-dimensional simplicial complex with f-vector \((f_0, \ldots, f_{d-1})\). Then the Hilbert polynomial of \( \mathbb{k}[\Delta] \) is

\[
P_{\mathbb{k}[\Delta]}(m) = \sum_{i=0}^{d-1} f_i \binom{m - 1}{i}. \tag{3.14}
\]

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In particular, \( P_{k[\Delta]}(m) = H(k[\Delta], m) \) for all \( m \in \mathbb{N} \) if and only if \( \Delta \) has reduced Euler characteristic \( \tilde{\chi}(\Delta) = 0 \).

**Proof.** When \( m > 0 \), Corollary 3.3.2 says \( H(k[\Delta], m) = \sum_{i=0}^{d-1} f_i \binom{m-1}{i} \), which is a polynomial in \( m \), so it must coincide with the Hilbert polynomial, i.e. (3.14) holds. Evaluating this polynomial at \( m = 0 \) yields

\[
P_{k[\Delta]}(0) = \sum_{i=0}^{d-1} f_i,
\]

which equals the (non-reduced) Euler characteristic \( \chi(\Delta) \) of \( \Delta \). Since Corollary 3.3.2 says \( H(k[\Delta], 0) = 1 \), it follows that \( P_{k[\Delta]}(0) = H(k[\Delta], 0) \) if and only if \( \tilde{\chi}(\Delta) = 1 - 1 = 0 \).

**Corollary 3.3.4.** \( \dim k[\Delta] = \dim \Delta + 1 \) for all simplicial complexes \( \Delta \).

**Proof.** By Corollary 3.3.3, the Hilbert polynomial \( P_{k[\Delta]}(m) \) has degree \( \dim \Delta \), thus Theorem 3.1.3 implies \( \dim k[\Delta] = \dim \Delta + 1 \).

### 3.4 Regular sequences and depth

Recall that \( S = k[x_1, \ldots, x_n] \) and that \( \mathfrak{m} = \langle x_1, \ldots, x_n \rangle \) is the (unique) homogeneous maximal ideal of \( S \). Recall also that all rings are assumed to be commutative and Noetherian.

**Definition 3.4.1.** Let \( R \) be a ring, and let \( M \) be an \( R \)-module. A sequence \( f_1, \ldots, f_r \) of elements in \( R \) is called an \( M \)-regular sequence if

1. \( f_i \) is a non-zero-divisor in \( M/\langle f_1, \ldots, f_{i-1} \rangle M \) for all \( i \in [r] \); and
2. \( M/\langle f_1, \ldots, f_r \rangle M \neq 0 \).
Note that if $R = S$, and $\langle f_1, \ldots, f_r \rangle \subseteq \mathfrak{m}$, then condition (ii) is automatically satisfied because of Nakayama’s lemma; cf. [23, Chap. 1].

**Definition 3.4.2.** A complete intersection ideal (CI ideal) of $S$ is a (proper) ideal of $S$ generated by an $S$-regular sequence of forms.

The following result proved by Rees is a fundamental theorem about regular sequences that deserves special mention.

**Theorem 3.4.3** ([23, Prop. 1.2.14]). Let $R$ be a ring, and let $M$ be a finitely generated $R$-module. If $I \subseteq R$ is a proper ideal such that $IM \neq M$, then all maximal $M$-regular sequences contained in $I$ have the same length.

We now define the important notions of grade and depth.

**Definition 3.4.4.** Let $R$ be a ring, and let $M$ be a finitely generated $R$-module.

(i) If $I \subseteq R$ is a (proper) ideal such that $IM \neq M$, then the common length of all the maximal $M$-regular sequences contained in $I$ is called the grade of $I$ on $M$, and it is denoted by grade($I, M$). This is well-defined by Theorem 3.4.3. When $M = R$, we simply denote it by grade($I$).

(ii) If instead $I \subseteq S$ satisfies $IM = M$, then we define grade($I, M$) = $\infty$.

(iii) When $R$ is a positively graded $k$-algebra (e.g. $R = S$), then $R$ has a unique homogeneous maximal ideal $\mathfrak{m}_R$. The grade of $\mathfrak{m}_R$ on $M$ is called the depth of $M$, and it is denoted by depth($M$).

**Remark 3.4.5.** All maximal regular sequences in $\mathfrak{m}$ have length $n$, i.e. depth($S$) = $n$. In particular, $x_1, \ldots, x_n$ is an $S$-regular sequence.

**Theorem 3.4.6.** Let $M$ be a finitely generated $S$-module, and let $I \subseteq S$ be a (proper) homogeneous ideal such that $IM \neq M$. 
(i) Every permutation of an $M$-regular sequence of forms is $M$-regular.

(ii) If $f_1, \ldots, f_{i-1}, f_i, f_{i+1}, \ldots, f_r$ and $f_1, \ldots, f_{i-1}, f'_i, f_{i+1}, \ldots, f_r$ are $M$-regular sequences, then $f_1, \ldots, f_{i-1}, f'_i, f_{i+1}, \ldots, f_r$ is also an $M$-regular sequence.

(iii) If $f_1, \ldots, f_r$ is an $M$-regular sequence, then $f_1^{\alpha_1}, \ldots, f_r^{\alpha_r}$ is also an $M$-regular sequence for all $\alpha_1, \ldots, \alpha_r \in \mathbb{P}$.

(iv) If $\mathbb{k}$ is infinite, and $f_1, \ldots, f_r$ is an $M$-regular sequence of forms for some $r \leq \text{depth}(M)$, then for all integers $r < s \leq \text{depth}(M)$ and all $\alpha_{r+1}, \ldots, \alpha_s \in \mathbb{P}$, there exist forms $f_{r+1}, \ldots, f_s$ in $S$ of degrees $\alpha_{r+1}, \ldots, \alpha_s$ respectively, such that $f_1, \ldots, f_s$ is an $M$-regular sequence.

(v) If $I = \langle f_1, \ldots, f_r \rangle$ is a CI ideal of $S$ such that $\deg(f_i) = \alpha_i$ for all $i \in [r]$, then $I$ has the same Hilbert function as the ideal $\langle x_1^{\alpha_1}, \ldots, x_r^{\alpha_r} \rangle \subseteq S$. [Note: $f_1, \ldots, f_r$ need not necessarily be polynomials in the variables $x_1, \ldots, x_r$.]

Proof. For statements (i), (ii), (iii), and (iv), see [23, Prop. 1.1.6], [23, Ex. 1.1.10(a)], [23, Ex. 1.1.10(b)], and [23, Prop. 1.5.12] respectively; while for statement (v), see either [119, Chap. I.5] or [23, Ex. 4.4.14].

Remark 3.4.7. In contrast to Theorem 3.4.6(i), the permutation of an $M$-regular sequence containing non-homogeneous elements is not necessarily $M$-regular; see, e.g., [23, Ex. 1.1.13].

3.5 Cohen–Macaulay rings and complexes

Let $M$ be a finitely generated standard graded $S$-module. Earlier in Chapter 3.1, we have defined the Krull dimension of $M$ to be the order of the pole of the Hilbert series $\text{Hilb}(M, t)$ at $t = 1$ (Definition 3.1.2). There are several equivalent
definitions of Krull dimension. We now give another (equivalent) definition of
the Krull dimension for arbitrary rings and modules, then introduce the notion of
“Cohen–Macaulay” for rings, modules and simplicial complexes.

Given a ring \( R \), the \textit{spectrum} of \( R \), denoted by \( \text{Spec}(R) \), is the set\(^1\) of all prime
ideals of \( R \). Recall that an ideal \( I \subseteq R \) is \textit{prime} if \( I \neq R \), and \( xy \in I \) implies
\( x \in I \) or \( y \in I \) (for all \( x, y \in R \)). The \textit{height} of a prime ideal \( p \in \text{Spec}(R) \) is the
supremum of the lengths \( \ell \) of strictly descending chains \( p = p_0 \supseteq p_1 \supseteq \cdots \supseteq p_\ell \) of
prime ideals of \( R \). The \textit{height} of an arbitrary proper ideal \( I \not\subseteq R \) is defined by

\[
\text{height}(I) := \inf\{\text{height} \ p : p \in \text{Spec}(R), p \supseteq I\}.
\]

**Definition 3.5.1.** The \textit{Krull dimension} of a ring \( R \) is defined by

\[
\dim R := \sup\{\text{height} \ p : p \in \text{Spec}(R)\}.
\]

For a proof of the equivalence of Definition 3.5.1 and Definition 3.1.2, see, e.g.,
[7, Thm. 11.14]. Using Definition 3.5.1, we get the following result (whose easy
proof is left to the reader):

**Proposition 3.5.2.** If \( R \) is a ring, then \( \text{height} I + \dim(R/I) \leq \dim R \) for all
proper ideals \( I \not\subseteq R \).

**Remark 3.5.3.** Recall that an \textit{(integral) domain} is a non-zero ring \( R \) such that the
product of every two non-zero elements is non-zero. If \( R \) is a domain over \( k \), e.g. \( S \),
then in fact we have equality in Proposition 3.5.2, i.e. \( \text{height} I + \dim(R/I) = \dim R \);
see [45, Chap. 9].

**Proposition 3.5.4.** [23, Prop. 1.2.14] If \( R \) is a ring, then every proper ideal
\( I \not\subseteq R \) satisfies \( \text{grade}(I) \leq \text{height}(I) \).

\(^1\)Usually, \( \text{Spec}(R) \) comes equipped with the Zariski topology, but for this dissertation, we will
not be using this topology.
Definition 3.5.5. A ring $R$ is called Cohen–Macaulay if $\operatorname{grade}(I) = \operatorname{height}(I)$ for all maximal ideals $I \subsetneq R$.

Let $R$ be a ring, let $I, J$ be ideals of $R$, and let $M$ be an $R$-module. The (ideal) quotient of $I$ and $J$ (also called the colon ideal of $I$ and $J$) is the ideal

$$(I : J) := \{ r \in R : rJ \subseteq I \} \subseteq R.$$

More generally, we can define the submodule

$$(I : M J) := \{ m \in M : mJ \subseteq IM \} \subseteq M.$$

The annihilator of $M$ in $R$ is the ideal defined by

$$\operatorname{Ann}_R(M) := \{ r \in R : rm = 0 \text{ for all } m \in M \} \subseteq R.$$

Note that $\operatorname{Ann}_R(J) = (0 : J)$, and more generally, that $(I : J) = \operatorname{Ann}_R((J + I)/I)$.

Definition 3.5.6. Let $R$ be a ring. The Krull dimension of an $R$-module $M$ is defined by $\dim M := \dim(R/\operatorname{Ann}_R(M))$.

Proposition 3.5.7. [23, Prop. 1.2.12] If $A$ is a positively graded $k$-algebra, and $M$ is a finitely generated $A$-module, then $\operatorname{depth}(M) \leq \dim M$.

Definition 3.5.8. Let $A$ be a positively graded $k$-algebra (e.g. $A = S$), and let $M$ be a finitely generated $A$-module. Then we say $M$ is a Cohen–Macaulay $A$-module if $\operatorname{depth}(M) = \dim M$, and in particular, we say $A$ is a Cohen–Macaulay ring if $\operatorname{depth}(A) = \dim A$.

Definition 3.5.9. A simplicial complex $\Delta$ is called Cohen–Macaulay (over $k$) if its Stanley–Reisner ring $k[\Delta]$ is a Cohen–Macaulay ring.

Most authors (including this author) call a Cohen–Macaulay simplicial complex simply as a Cohen–Macaulay complex. For a detailed treatment of Cohen–Macaulay complexes and their significance, see [23] and [119].
3.6 Macaulay’s theorem on Hilbert functions

Let $S = \bigoplus_{d \in \mathbb{N}} S_d = \mathbb{k}[x_1, \ldots, x_n]$ be a standard graded polynomial ring on $n$ variables over a field $\mathbb{k}$. Fix the linear order $x_1 > \cdots > x_n$ on the variables. Then for each $d \in \mathbb{N}$, the lex order $\leq_{\text{lex}}$ on monomials in $S_d$ induced by $x_1 > \cdots > x_n$ (see Chapter 2.4) is given as follows.

$x_1^{\alpha_1} \cdots x_n^{\alpha_n} <_{\text{lex}} x_1^{\beta_1} \cdots x_n^{\beta_n} \iff \alpha_i < \beta_i$ for the smallest $i \in [n]$ such that $\alpha_i \neq \beta_i$.

**Definition 3.6.1.** A lex ideal of $S$ is a monomial ideal $L$ such that if $m, m'$ are monomials in $S$ satisfying $m \in L$, $\deg(m) = \deg(m')$ and $m <_{\text{lex}} m'$, then $m' \in L$.

**Remark 3.6.2.** We will take a closer look at lex ideals in Chapter 5.1.

Given positive integers $N$ and $i$, recall from Chapter 2.6 that the $i$-th Macaulay representation of $N$ is a unique expansion of $N$ as the sum of binomial coefficients

$N = \binom{N_i}{i} + \binom{N_{i-1}}{i-1} + \cdots + \binom{N_j}{j},$

such that $N_i > N_{i-1} > \cdots > N_j \geq j \geq 1$. Recall also that in Chapter 2.7 (right before Theorem 2.7.1), we defined the following functions $\partial^{(i)}$ and $\partial_{(i)}$ on $\mathbb{N}$ by:

$\partial^{(i)}(N) = \binom{N_i + 1}{i+1} + \binom{N_{i-1} + 1}{i} + \cdots + \binom{N_j + 1}{j+1},$

$\partial_{(i)}(N) = \binom{N_i - 1}{i-1} + \binom{N_{i-1} - 1}{i-2} + \cdots + \binom{N_j - 1}{j-1},$

for all $N \in \mathbb{P}$, and $\partial^{(i)}(0) = \partial_{(i)}(0) = 0$. These are well-defined functions by the uniqueness of the $i$-th Macaulay representation of $N$. By convention, $\binom{n}{k} = 0$ if $k < 0$ or $k > n$, and we define $\binom{0}{0} := 1$.

Macaulay’s theorem\footnote{Although Macaulay proved his famous Macaulay’s theorem on Hilbert functions in 1927, most mathematicians do not know that the first actual proof of this theorem can be traced back to an obscure paper by N. M. Gjunter in 1913; see Chapter 7.1 for the fascinating history on Macaulay’s theorem.} is a characterization of the Hilbert functions of graded
ideals of $S$, which can either be stated in terms of lex ideals, or in terms of Macaulay representations:

**Theorem 3.6.3** (Macaulay’s theorem [58] [85]). Let $H : \mathbb{N} \to \mathbb{N}$ be a function. The following are equivalent:

(i) $H$ is the Hilbert function of $S/I$ for some graded ideal $I \subseteq S$.
(ii) $H$ is the Hilbert function of $S/L$ for some lex ideal $L \subseteq S$.
(iii) $H(0) = 1$, and $H(i + 1) \leq \partial^{(i)}(H(i))$ for all $i \in \mathbb{P}$.
(iv) $H(0) = 1$, and $\partial_{(i+1)}(H(i + 1)) \leq H(i)$ for all $i \in \mathbb{P}$.

Equivalently, Macaulay’s theorem characterizes the $f$-vectors of multicomplexes; cf. Theorem 2.7.1. In fact, Stanley [115] proved the following:

**Theorem 3.6.4** ([115]). Let $f = (f_0, \ldots, f_d)$ be a $(d + 1)$-tuple of non-negative integers. The following are equivalent:

(i) $f$ is the $h$-vector of a $(d - 1)$-dimensional Cohen–Macaulay complex.
(ii) $f$ is the $f$-vector of a $d$-dimensional multicomplex.
(iii) $f_0 = 1$, and $f_{i+1} \leq \partial^{(i)}(f_i)$ for all $i \in [d - 1]$.
(iv) $f_0 = 1$, and $\partial_{(i+1)}(f_{i+1}) \leq f_i$ for all $i \in [d - 1]$.

**Notes**

There are several variations in terminology for some of the algebraic notions we have introduced in this chapter. For example, given a ring $R$, a finitely generated $R$-module $M$, and an ideal $I \subseteq R$, what we call the “grade of $I$ on $M$” is known to some authors (e.g. French geometers; cf. [45, Chap. 17]) as the “depth of $I$ on $M$”, and in particular, what we mean by “grade($I$)” (grade of $I$ on $R$) is written as “depth($I$)” by some other authors. This may cause confusion since the depth of $I$ as an $R$-module (as we have defined) is not the same as the grade of $I$ on $R$. Another example is the notion of height. What we call the “height” of a proper ideal $I$ is sometimes called the “codimension of $I$”, or the “rank of
Both “height” and “codimension” seem to be commonly used (cf. [45, Chap. 9]), sometimes interchangeably. However, we again caution the reader that there are two different definitions in the literature for the notion of codimension. Some authors define the codimension of $I$ to be $\dim R - \dim(R/I)$. Notice that in our terminology, we have height $I \leq \dim R - \dim(R/I)$ (Proposition 3.5.2), where equality does not necessarily hold for arbitrary rings $R$. Hence, the codimension of $I$ defined to be $\dim R - \dim(R/I)$ (for arbitrary rings $R$) is different from the codimension of $I$ defined using chains of prime ideals (i.e. what we call the height of $I$). However, for “nice” rings $R$ (e.g. when $R$ is an integral domain; see Remark 3.5.3), the two notions of codimension coincide.

The topic of Cohen–Macaulay rings and Cohen–Macaulay complexes plays a huge role in the intersection of combinatorics and commutative algebra. For excellent treatments of this topic, see [23] and [119]. One of the most celebrated appearances of Cohen–Macaulay complexes involves what was known earlier as the upper bound conjecture, and which is now commonly called the upper bound theorem; see [119, Chap II.3] for a precise statement of the conjecture/theorem. The original statement, concerning upper bounds on the face numbers of simplicial convex polytopes, was proposed by Motzkin [96] in 1957, and was proven by McMullen [86] in 1970. Victor Klee suggested an extension of the statement to all simplicial spheres, and Stanley [114] proved this extended statement. One of the key steps in Stanley’s proof is that all simplicial spheres are Cohen–Macaulay, which is a fact built upon homological results by Reisner and Munkres; see [119, Chap II.4]. In particular, the Cohen–Macaulay property for simplicial complexes is a topological property, in the sense that if $\Delta$ is a Cohen–Macaulay complex, then any simplicial complex whose geometric realization is homeomorphic to the geometric realization of $\Delta$ must also be Cohen–Macaulay.
CHAPTER 4
GRAPH THEORY

Because graph theory terminology varies widely in the literature, we shall review the basic terminology that we use. In particular, all graphs are assumed to be simple, but \( k \)-uniform hypergraphs are not necessarily simple. We allow infinite vertex sets, but the edge sets of all graphs and hypergraphs are always assumed to be finite. Additional (less frequently used) graph theory terminology encountered in Chapters 13–14 will be introduced in Chapter 13.1. We will also present the Frankl–Füredi–Kalai theorem, a result in extremal graph theory concerning the minimum shadow sizes of \( r \)-colorable \( k \)-uniform hypergraphs.

4.1 Basic Terminology

A graph is a pair \( G = (V, E) \) such that \( V \) is a set, and \( E \subseteq \binom{V}{2} \). Note that we allow \( V \) to be the empty set, and we also allow \( V \) to be an infinite set. However, we always assume that \( E \) is a finite set. The sets \( V \) and \( E \) are called the vertex set and edge set respectively, and the elements of \( V \) and \( E \) are called vertices and edges respectively. A vertex \( v \in V \) is incident to an edge \( e \in E \) if \( e = \{u, v\} \) for some vertex \( u \neq v \). The degree of \( v \in V \), denoted by \( \deg(v) \), is the number of edges incident to \( v \). We say \( u, v \in V \) are adjacent if \( \{u, v\} \in E \). A subgraph of \( G \) is a graph \( G' = (V', E') \) such that \( V' \subseteq V \) and \( E' \subseteq E \). Given a subset of vertices \( V' \subseteq V \), the subgraph \( G' = (V', E \cap \binom{V'}{2}) \) is called the subgraph of \( G \) induced by \( V' \).

For convenience, we sometimes identify graphs with finite vertex sets as simplicial complexes of dimension \( \leq 1 \).

Given \( u, v \in V \), a path from \( u \) to \( v \) is a sequence \( v_0, v_1, \ldots, v_k \) of vertices
that are all distinct except possibly \( v_0 \) and \( v_k \), such that \( v_0 = u \), \( v_k = v \), and \( \{v_0, v_1\}, \ldots, \{v_{k-1}, v_k\} \) are \( k \) distinct edges in \( E \). The length of this path is \( k \).

(Note that \( k \in \mathbb{N} \).) If \( k \geq 3 \) and \( u = v \), then we call this path a cycle. We say \( G \) is connected if there is a path from \( x \) to \( y \) for every \( x, y \in V \), and we say \( G \) is acyclic if it has no cycles. A connected acyclic graph is called a tree. Note that for all vertices \( x, y \) of a tree, there always exists a unique path from \( x \) to \( y \).

### 4.2 Shadows and colorability of \( k \)-uniform hypergraphs

A hypergraph is a pair \( G = (V, E) \), such that \( V \) is a set, and \( E \) is a finite set of non-empty subsets of \( V \). Similar to graphs, the sets \( V \) and \( E \) are called the vertex set and edge set respectively, and elements of \( V \) and \( E \) are called vertices and edges\(^1\) respectively. Note in particular that we do not require every \( v \in V \) to be contained in some edge in \( E \). We say \( G \) is non-empty if \( V \) is a non-empty set. If there exists some \( k \in \mathbb{P} \) such that \( E \subseteq \binom{V}{k} \), i.e. \( E \) is a set of \( k \)-subsets of \( V \), then we say \( G \) is \( k \)-uniform. Note that a 2-uniform hypergraph is precisely a graph. Given another hypergraph \( G' = (V', E') \), we say \( G \) and \( G' \) are isomorphic if there exists a bijection \( \nu : V \to V' \) such that \( E \in E \) if and only if \( \nu(E) \in E' \).

Recall that an ordered partition of a non-empty set \( X \) is a finite (non-empty) sequence of non-empty, pairwise disjoint subsets of \( X \) whose union equals \( X \). Let \( r \in \mathbb{P} \). A hypergraph \( G = (V, E) \) is called \( r \)-colorable if it is non-empty and there exists an ordered partition \( \pi = (V_1, \ldots, V_r) \) of \( V \) such that \( |E \cap V_i| \leq 1 \) for all \( E \in E \) and all \( i \in [r] \). Motivated by our notion of generalized colored complexes, we can analogously extend the notion of \( r \)-colorability as follows.

\(^1\)Edges of hypergraphs are also sometimes called hyperedges by some authors.
Definition 4.2.1. Let \( a = (a_1, \ldots, a_n) \in \mathbb{P}^n \). An \( a \)-colored hypergraph is a pair \((\mathcal{G}, \pi)\) satisfying:

(i) \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) is a non-empty hypergraph.
(ii) \( \pi = (\mathcal{V}_1, \ldots, \mathcal{V}_n) \) is an ordered partition of \( \mathcal{V} \);
(iii) \( |E \cap \mathcal{V}_i| \leq a_i \) for every \( E \in \mathcal{E} \) and every \( i \in [n] \).

Remark 4.2.2. A colored hypergraph is an \( a \)-colored hypergraph for some \( a \in \mathbb{P}^n \), and the \( n \)-tuple \( a \) is called its type. As is the case with generalized colored complexes/multicomplexes, we say “generalized colored hypergraphs” to mean colored hypergraphs of arbitrary type. A hypergraph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) is called colorable of type \( a \) if there is an ordered partition \( \pi \) of \( \mathcal{V} \) such that \((\mathcal{G}, \pi)\) is \( a \)-colored. In particular, the notions of “colorable of type 1, and “\( r \)-colorable” are exactly the same.

Suppose \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) is a \( k \)-uniform hypergraph for some \( k \in \mathbb{P} \). The shadow of \( \mathcal{G} \), denoted by \( \partial \mathcal{G} \), is the set of all \((k-1)\)-subsets of \( \mathcal{V} \) that are contained in some element of \( \mathcal{E} \). The shadow size of \( \mathcal{G} \), denoted by \( |\partial \mathcal{G}| \), is the number of elements (i.e. \((k-1)\)-subsets of \( \mathcal{V} \)) in \( \partial \mathcal{G} \). By default, the shadow of a 1-uniform hypergraph is the set \( \{\emptyset\} \) consisting of the empty set as its only element. See Figure 4.1 for an example of the shadow of a 3-colorable 3-uniform hypergraph.

![Diagram of a 3-uniform hypergraph](image)

This is the depiction of a 3-uniform hypergraph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} = \{v_1, \ldots, v_8\} \), and \( \mathcal{E} = \{E_1, E_2, E_3, E_4\} \).

Note that \( \mathcal{G} \) is 3-colorable via the ordered partition:
\[
\mathcal{V}_1 = \{v_1, v_4, v_7\}, \quad \mathcal{V}_2 = \{v_2, v_6, v_8\}, \quad \mathcal{V}_3 = \{v_3, v_5\}.
\]

The shadow \( \partial \mathcal{G} \) consists of 2-sets depicted by the 11 black edges, so the shadow size is \( |\partial \mathcal{G}| = 11 \).

Figure 4.1: Example of the shadow of a 3-colorable 3-uniform hypergraph.
4.3 Frankl–Füredi–Kalai theorem

Let \(k, r, N \in \mathbb{P}\), and let \(G = (\mathcal{V}, \mathcal{E})\) be an \(r\)-colorable \(k\)-uniform hypergraph with \(N\) edges. In this section, we are interested in the minimum possible shadow size of \(G\). Notice that adding new vertices to \(\mathcal{V}\) does not alter the shadow \(\partial G\), hence we may assume without loss of generality that \(\mathcal{V}\) is an infinite set.

For every \(k, r \in \mathbb{P}\), define the set

\[
P_{k,r} := \left\{ E \in \binom{\mathbb{P}}{k} : a, b \in E, a \equiv b \pmod{r} \Rightarrow a = b \right\}
\]

consisting of all \(k\)-subsets of positive integers whose elements are in distinct residue classes modulo \(r\).

**Lemma 4.3.1.** If \(k, r \in \mathbb{P}\), then every \(r\)-colorable \(k\)-uniform hypergraph with an infinite vertex set is isomorphic to some hypergraph \((\mathbb{P}, P)\) such that \(P \subseteq P_{k,r}\).

**Proof.** Let \(G = (\mathcal{V}, \mathcal{E})\) be an \(r\)-colorable \(k\)-uniform hypergraph such that \(|\mathcal{V}| = \infty\). By definition, there exists an ordered partition \(\pi = (\mathcal{V}_1, \ldots, \mathcal{V}_r)\) of \(\mathcal{V}\) such that \((G, \pi)\) is a \(1_r\)-colored \(k\)-uniform hypergraph. For each \(i \in [r]\), label and linearly order the elements in \(\mathcal{V}_i\) by \(v_{i,1} < v_{i,2} < \cdots\). Clearly \(|\mathcal{V}| = \infty\) implies there exists some \(j \in [r]\) such that \(|\mathcal{V}_j| = \infty\). Let \(v_{j,\ell}\) (for some \(\ell \in \mathbb{P}\)) be the largest element in \(\mathcal{V}_j\) (with respect to the given linear order on \(\mathcal{V}_j\)) that is contained in some edge of \(G\). Such an integer \(j\) exists since all hypergraphs are assumed to have finitely many edges.

Next, define \(\mathcal{V}_j' := \{v_{j,\ell} : t \in [\ell]\} \cup \{v_{j,t} : t \in \mathbb{P}\setminus[\ell], t \equiv j \pmod{r}\}\), and for each \(i \in [\ell]\setminus\{j\}\), define \(\mathcal{V}_i' := \mathcal{V}_i \cup \{v_{j,\ell} : t \in \mathbb{P}\setminus[\ell], t \equiv i \pmod{r}\}\). We check that \(\pi' := (\mathcal{V}_1', \ldots, \mathcal{V}_r')\) is an ordered partition of \(\mathcal{V}\). Since \(\pi\) and \(\pi'\) differ only by
elements not contained in any edge of \( \mathcal{G} \), it follows that \((\mathcal{G}, \pi')\) is also a \(1_r\)-colored \(k\)-uniform hypergraph. Now, \(|V'_i| = \infty\) for every \(i \in [r]\) by construction, thus we can relabel the elements in \(V\) so that \(V'_i = \{t \in \mathbb{P} : t \equiv i \pmod{r}\}\) for all \(i \in [r]\). Under this relabeling, \(\mathcal{G}\) becomes \((\mathbb{P}, \mathcal{P})\) for some \(\mathcal{P} \subseteq \mathcal{P}_{k,r}\).

**Theorem 4.3.2** (Frankl–Füredi–Kalai theorem [53] [83]). The minimum shadow size of an \(r\)-colorable \(k\)-uniform hypergraph \(\mathcal{G} = (\mathbb{P}, \mathcal{E})\) among all possible hypergraphs with the same number of edges is achieved when \(\mathcal{E}\) is a colex initial segment\(^2\) of \(\mathcal{P}_{k,r}\).

**Remark 4.3.3.** Theorem 4.3.2 and Lemma 4.3.1, together with the observation that we could always assume an infinite vertex set, imply a tight lower bound on the possible shadow sizes of an arbitrary \(r\)-colorable \(k\)-uniform hypergraph with a given fixed number of edges.

Let \(\Delta\) be a \((d - 1)\)-dimensional simplicial complex with vertex set \(V\). For each \(k \in [d]\), recall that \(\mathcal{F}_{k-1}(\Delta)\) denotes the collection of all \((k - 1)\)-dimensional faces of \(\Delta\). Note that \((V, \mathcal{F}_{k-1}(\Delta))\) is a \(k\)-uniform hypergraph. (In particular, recall that we assumed all simplicial complexes are finite, so \(\mathcal{F}_{k-1}(\Delta)\) is a finite subset of \(\binom{V}{k-1}\).) We could always add new vertices to this \(k\)-uniform hypergraph so that it has an infinite vertex set, and we can then identify the vertices by the positive integers. Thus, the Kruskal–Katona theorem (see Theorem 2.6.1) can be reformulated in the language of \(k\)-uniform hypergraphs as follows.

**Theorem 4.3.4** (Kruskal–Katona theorem reformulated). The minimal shadow size of a \(k\)-uniform hypergraph \(\mathcal{G} = (\mathbb{P}, \mathcal{E})\) among all possible hypergraphs with the

\(^2\)As already explained in Chapter 2.4, there are several variations of the lex order and the revlex order. Some authors would insist that the Frankl–Füredi–Kalai theorem should be stated in terms of revlex initial segments. In this case, we urge the reader to check that their definition of revlex initial segments coincides with our definition of colex initial segments for \(k\)-subsets of positive integers.
same number of edges is achieved when $E$ is a colex initial segment of $\binom{p}{k}$.

The $k$-uniform hypergraph $(V, F_{k-1}(\Delta))$ is trivially $r$-colorable for sufficiently large $r$. Note also that $\mathcal{P}_{k,r} \rightarrow \binom{p}{k}$ as $r \rightarrow \infty$, thus Theorem 4.3.4 follows from Theorem 4.3.2. Consequently, the Frankl–Füredi–Kalai theorem generalizes the Kruskal–Katona theorem.

We could also consider the Frankl–Füredi–Kalai theorem as a colored analogue of the Kruskal–Katona theorem: Observe that if $\Delta$ is an $r$-colorable simplicial complex for some $r \in \mathbb{P}$ (cf. Chapter 2.2), then $(V, F_{k-1}(\Delta))$ is an $r$-colorable $k$-uniform hypergraph for every $1 \leq k \leq \dim(\Delta)+1$. Thus, the Frankl–Füredi–Kalai theorem characterizes the $f$-vectors of $r$-colorable simplicial complexes.

Notes

The notion of “$r$-colorable” hypergraphs is frequently seen in hypergraph theory. However, some hypergraph theorists would distinguish between two different but closely related notions of $r$-colorability. What we have defined as “$r$-colorable” coincides with the notion of “strongly $r$-colorable”. There is also the notion of “weakly $r$-colorable”. A hypergraph $\mathcal{G} = (V, E)$ is called weakly $r$-colorable (or weakly $r$-choosable) if there exists an ordered partition $(V_1, \ldots, V_r)$ of $V$ such that $E \nsubseteq V_i$ for all $E \in E$ and all $i \in [r]$, or equivalently, such that none of the edges are “monochromatic”. It is clear that a strongly $r$-colorable $k$-uniform hypergraph is always weakly $r$-colorable when $k \geq 2$. For 2-uniform hypergraphs (i.e. the usual graphs), the notions of “strong $r$-colorable” and “weakly $r$-colorable coincide”, and we simply use the term “$r$-colorable graphs”.

The original proof of the Frankl–Füredi–Kalai theorem in [53] is incorrect as stated. In particular, Claim 4.2(ii) in [53] is not true. See Appendix A for justification. Nevertheless, the statement of the theorem is true, and in Theorem 4.3.2, we give a (correct) non-numerical version of the Frankl–Füredi–Kalai theorem. There were three other previously known proofs for this non-numerical version (see Chapter 7.1 for details). In this dissertation, we provide another (fourth) different proof; see Chapter 6. In fact, we will give a simultaneous generalization of the Clements–Lindström theorem and the Frankl–Füredi–Kalai theorem.
On a related note, Billera–Björner [15] pointed out that the uniqueness claim in [53, Lemma 1.1] is incorrect, which makes the $\partial_k^{(\rho)}(\cdot)$ operator introduced in [53] not well-defined. Thus the numerical version of the Frankl–Füredi–Kalai theorem, as originally stated in [53], was incorrect. Nevertheless, there is an easy fix to this non-uniqueness problem, and a correct numerical version of the Frankl–Füredi–Kalai theorem is stated in [15], which includes a fix suggested by J. Eckhoff (and which agrees with what the author had in mind).
Part II

Macaulay-Lex rings and a generalization of the Clements–Lindström theorem
CHAPTER 5
MACAULAY-LEX RINGS

The study of Hilbert functions is one of the central themes in commutative algebra, and much of our understanding is enhanced by insights in combinatorics. This rich interplay between commutative algebra and combinatorics can be traced back to Macaulay’s 1927 paper [85], in which Macaulay characterized the possible Hilbert functions of graded ideals in the polynomial ring $S := \mathbb{k}[x_1, \ldots, x_n]$.

Macaulay’s key idea was that for every graded ideal of $S$, there exists a lex (lexicographic) ideal of $S$ with the same Hilbert function. Given a monomial ideal $M$ of $S$, we can similarly define lex ideals in the quotient ring $S/M$. Motivated by Macaulay’s theorem, Mermin–Peeva [91] asked if there is an analogous characterization of the Hilbert functions of graded ideals of $S/M$, thereby introducing the notion of Macaulay-Lex rings. In Chapter 5.1, we define lex ideals and briefly review some important uses of lex ideals. In Chapter 5.2, we introduce the notions of monomial spaces and revlex-segment sets, which we will use very frequently in subsequent chapters. In Chapter 5.3, we study the basic properties of Macaulay-Lex rings.

### 5.1 Lex ideals

Throughout this chapter, let $S = \bigoplus_{d \in \mathbb{N}} S_d := \mathbb{k}[x_1, \ldots, x_n]$ be a standard graded polynomial ring on $n$ variables over a field $\mathbb{k}$. Assume that the variables are linearly ordered by $x_1 > \cdots > x_n$. Some of the relevant notation involving monomials has

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1Interestingly, most mathematicians do not know that the first actual proof of Macaulay’s theorem was given by the Russian mathematician N. M. Gjunter in 1913. Gjunter was in fact an analyst! See Chapter 7.1 for the fascinating history of Macaulay’s theorem.
already been introduced in Chapter 2.7 when we encountered multicomplexes. For example, given a subset \( X \subseteq \{x_1, \ldots, x_n\} \) and a monomial \( m = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) in \( S \), we have earlier defined \( m_X := \prod_{x_i \in X} x_i^{\alpha_i} \). We now introduce more (new) notation.

If \( Y \) is a set of elements in \( S \), write \( \langle Y \rangle \) to mean the ideal of \( S \) generated by \( Y \), and write \( \text{Span}_k(Y) \) to mean the \( k \)-vector space spanned by \( Y \). For any \( k \)-vector space \( U \subseteq S \), write \( \{U\} \) to mean the set of monomials in \( U \).

Lex ideals, sometimes also known as \emph{lex-segment ideals}, are monomial ideals defined combinatorially. For each \( d \in \mathbb{N} \), recall from Chapter 2.4 that the lex order \( \preceq_{\text{lex}} \) on monomials in \( S_d \) (induced by \( x_1 > \cdots > x_n \)) is given as follows.

\[ x_1^{\alpha_1} \cdots x_n^{\alpha_n} \preceq_{\text{lex}} x_1^{\beta_1} \cdots x_n^{\beta_n} \iff \alpha_i < \beta_i \text{ for the smallest } i \in [n] \text{ such that } \alpha_i \neq \beta_i. \]

For convenience, we also recall the definition of lex ideals (Definition 3.6.1):

**Definition 5.1.1.** A \emph{lex ideal} of \( S \) is a monomial ideal \( L \) such that if \( m, m' \) are monomials in \( S \) satisfying \( m \in L \), \( \deg(m) = \deg(m') \) and \( m \preceq_{\text{lex}} m' \), then \( m' \in L \).

Lex ideals play a crucial role in Hartshorne’s proof [65] that Grothendieck’s Hilbert scheme is connected. Specifically, in each Hilbert subscheme parameterizing all graded ideals of \( S \) with a certain fixed Hilbert polynomial, there is always a sequence of deformations from any point in the subscheme to the unique point corresponding to a lex ideal with this given Hilbert polynomial. Also, every lex ideal of \( S \) has maximal Betti numbers among all graded ideals with the same Hilbert function [14, 71, 102]. Moreover, in combinatorics, Macaulay’s theorem (in which lex ideals first appeared) yields numerical characterizations for both the \( f \)-vectors of multicomplexes (see Theorem 2.7.1), and the \( h \)-vectors of Cohen–Macaulay complexes [115]; see Chapter 3.6.

Given a monomial ideal \( M \) of \( S \), we can similarly define lex ideals in the quotient
ring $S/M$. Motivated by Macaulay’s theorem, Mermin and Peeva [91] asked if there is an analogous characterization of the Hilbert functions of graded ideals of $S/M$, thereby introducing the notion of Macaulay-Lex rings. In this general context, we say $S/M$ is a Macaulay-Lex ring (or $M$ is a Macaulay-Lex ideal) if for every graded ideal of $S/M$, there exists a lex ideal of $S/M$ with the same Hilbert function.

In 1969, Clements and Lindström [36] generalized Macaulay’s theorem by showing that if $2 \leq e_1 \leq \cdots \leq e_n \leq \infty$, then $Q := S/\langle x_1^{e_1}, \ldots, x_n^{e_n} \rangle$ is Macaulay-Lex (where $x_i^\infty = 0$); cf. Chapter 2.8. Such quotient rings $Q$ are called Clements–Lindström rings, and the case $e_1 = \cdots = e_n = 2$, first proven independently by Schützenberger [107], Kruskal [82] and Katona [77] around the 1960s, is of particular interest in combinatorics (see, e.g., [63, Sec. 8]), since it yields a numerical characterization of the $f$-vectors of simplicial complexes; cf. Chapter 2.6. However, for several decades after Clements and Lindström generalized Macaulay’s theorem, these Clements–Lindström rings were the only examples of Macaulay-Lex rings known to algebraists. Later in Chapter 6, we will discuss recent progress, and in particular, we construct new classes of Macaulay-Lex rings; see Theorem 6.5.3.

5.2 Monomial spaces and revlex-segment sets

Let $d \in \mathbb{N}$, and let $\Gamma$ be a set of monomials in $S_d$. Recall from Chapter 2.4 that the revlex order $\leq_{rt}$ on $\Gamma$ is defined by $m <_{rt} m' \iff m >_{lex} m'$ for all $m, m' \in \Gamma$. In particular, $x_1 <_{rt} \cdots <_{rt} x_n$ is the reverse of the fixed linear order on the variables. Given a subset $\Gamma' \subseteq \Gamma$, we say $\Gamma'$ is a lex-segment (resp., revlex-segment)\(^2\) set in $\Gamma$

\(^2\)Lex-segment (resp. revlex-segment) sets in $\Gamma$ are identically lex (resp. revlex) initial segments, which we have defined in Chapter 2.5. The notion of ‘lex-segment’ or ‘revlex-segment’
if \( m \in \Gamma' \Rightarrow m' \in \Gamma' \) for all \( m' >_{\text{lex}} m \) (resp., \( m' >_{\text{rev}} m \)) in \( \Gamma \).

Suppose \( R = \bigoplus_{d \in \mathbb{N}} R_d := S/M \) for some monomial ideal \( M \subseteq S \). Without any ambiguity, identify the monomials in \( \{R\} \) with the monomials in \( \{S\}\setminus\{M\} \) in the natural way, so that previously defined notions (lex order, lex-segment, etc.) make sense on subsets of \( \{R_d\} \) for any \( d \in \mathbb{N} \). An \( R_d \)-monomial space is a \( k \)-vector space \( A \) spanned by some subset of the monomials in \( \{R_d\} \), and we say \( A \) is lex-segment (resp., revlex-segment) if it is spanned by a lex-segment (resp., revlex-segment) set in \( \{R_d\} \). Using the notion of lex-segment monomial spaces, we can define a lex ideal of \( R \) to be a (graded) monomial ideal \( L = \bigoplus_{d \in \mathbb{N}} L_d \) of \( R \) such that each \( L_d \) is lex-segment.

### 5.3 Basic properties of Macaulay-Lex rings

Let \( R = \bigoplus_{d \in \mathbb{N}} R_d := S/M \) for some monomial ideal \( M \subseteq S \). We remind the reader that \( R \) is a Macaulay-Lex ring (or equivalently, \( M \) is a Macaulay-Lex ideal) if for every graded ideal \( I \) of \( R \), there exists a lex ideal \( L \) of \( R \) with the same Hilbert function as \( I \). Given \( d \in \mathbb{P} \) and a subset \( \Gamma \subseteq \{R_d\} \), define

\[
\partial(\Gamma) := \left\{ m \in \{R_{d-1}\} : m \text{ divides } m' \text{ for some } m' \in \Gamma \right\}
\]

to be the lower shadow of \( \Gamma \). If \( A \) is an \( R_d \)-monomial space, then the lower shadow of \( A \), denoted by \( \partial(A) \), is the \( R_{d-1} \)-monomial space spanned by \( \partial(\{A\}) \), and the upper shadow of \( A \) is the \( R_{d+1} \)-monomial space \( R_1A \) spanned by the monomials of

is more common in commutative algebra, while the notion of ‘initial segments’ is more common in combinatorics. Note also that ‘lex-segment’ and ‘revlex-segment’, as we have used here, are adjectives. Some authors abbreviate ‘lex initial segment’ (resp. ‘revlex initial segment’) as ‘lex segment’ (resp. ‘revlex segment’) or even ‘lex-segment’ (resp. ‘revlex-segment’). This may cause confusion, since it means ‘lex-segment’ and ‘revlex-segment’ could either be adjectives or nouns, depending on the authors’ preferences.
the form \(x_im\), where \(x_i \in R_1\) and \(m \in \{A\} \).

For convenience, let \(\text{Lex}_{R_d}(A)\) and \(\text{Revlex}_{R_d}(A)\) denote the unique lex-segment and revlex-segment \(R_d\)-monomial spaces of dimension \(|A|\) respectively, where \(|A| := \dim_k(A)\) is the dimension of \(A\) as a \(k\)-vector space.

**Theorem 5.3.1.** The following are equivalent:

(i) \(R\) is a Macaulay-Lex ring.

(ii) \(|R_1(\text{Lex}_{R_d}(A))| \leq |R_1A|\) for all \(d \in \mathbb{N}\) and every \(R_d\)-monomial space \(A\).

(iii) \(\partial(\text{Revlex}_{R_{d+1}}(A)) \subseteq \text{Revlex}_{R_d}(\partial(A))\) for all \(d \in \mathbb{N}\) and every \(R_{d+1}\)-monomial space \(A\).

(iv) For all \(d \in \mathbb{N}\), the following two conditions hold:

- (a) \(|\partial(\text{Revlex}_{R_{d+1}}(A))| \leq |\partial(A)|\) for every \(R_{d+1}\)-monomial space \(A\); and
- (b) the lower shadow of every revlex-segment \(R_{d+1}\)-monomial space is revlex-segment.

**Proof.** The equivalence (i) ⇔ (ii) is straightforward (see, e.g., [111, Thm. 2.1.7]) and was first proven by Macaulay [85] for the case \(R = S\). The equivalence (i) ⇔ (iii) was first implicitly proven by Clements and Lindström [36] for Clements–Lindström rings (cf. Chapter 6.1), and a proof for arbitrary Macaulay-Lex rings was explicitly given by Shakin [109, Thm. 2.7]. Finally, Engel [48, Prop. 8.1.1] proved (iii) ⇔ (iv).

**Remark 5.3.2.** The converses to the two implications (i) ⇒ (iv)(a) and (i) ⇒ (iv)(b) in Theorem 5.3.1 are not true. For example, as observed by Shakin [109, Example 2.8], the quotient ring \(R = \mathbb{k}[x_1, x_2]/\langle x_1^3, x_1x_2^2, x_2^3 \rangle\) satisfies (iv)(a), yet this ring is not Macaulay-Lex. Also, we will later define colored quotient rings (see Definition 6.1.1) and show that all colored quotient rings satisfy (iv)(b) (see...
Proposition 6.1.2), yet not every colored quotient ring is Macaulay-Lex (see Theorem 6.5.3). In contrast, the upper shadow of every lex-segment $R_d$-monomial space (for all $d \in \mathbb{N}$) is lex-segment; see, e.g., [109, Lem. 2.1] for a proof.

Given an ideal $I$ of $S$, a lex-plus-$I$ ideal of $S$ is an ideal $L'$ that can be written as $L' = L + I$ for some lex ideal $L$ of $S$. In particular, if $P := \langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle$ for $2 \leq a_1 \leq \cdots \leq a_n \leq \infty$, then lex-plus-$P$ ideals are called lex-plus-powers ideals, and they were first introduced by Evans; see [51].

**Theorem 5.3.3.** Let $M$ be a Macaulay-Lex ideal of $S$.

(i) Every lex-plus-$M$ ideal of $S$ is Macaulay-Lex. In particular, every lex-plus-powers ideal of $S$ is a Macaulay-Lex ideal.

(ii) If $M$ is non-zero and $d > 0$ is the minimal degree of the generators of $M$, then there exists some $i \in [n]$ such that $x_1^{d-1}x_i \in M$.

(iii) If $n \geq 2$ and $d \in \mathbb{N}$, then the ideal $\langle \{m \in \mathbb{k}[x_2, \ldots, x_n] : mx_1^d \in M \} \rangle$ contained in $\mathbb{k}[x_2, \ldots, x_n]$ is Macaulay-Lex.

(iv) If $y$ is an indeterminate, then $(S[y])M$ as an ideal of the polynomial ring $S[y]$ is Macaulay-Lex with respect to the linear order $x_1 > \cdots > x_n > y$.

**Proof.** Statement (i) was independently proven by Shakin [111, Thm. 2.7.2] and Mermin-Peeva [91, Thm. 5.1]. Statements (ii) and (iii) were first proven by Shakin; see [111, Thm. 2.2.4] and [111, Prop. 2.5.2] respectively. Statement (iv) was independently proven by Shakin [111, Thm. 2.5.1] and Mermin-Peeva [91, Thm. 4.1]. □
Recall that all $k$-algebras are assumed to be unital, commutative, associative, and finitely generated, while all graded $k$-algebras are assumed to be connected, so in particular, a standard graded $k$-algebra with $n$ generators is isomorphic to a standard graded quotient ring $S/I$ for some graded ideal $I \subseteq S$. The ideal $I$ is called a presentation ideal.

Let $R$ be a standard graded $k$-algebra. What can we say about the Hilbert functions of ideals contained in $R$? If $R$ is a Macaulay-Lex ring, then we know that the ideals in $R$ can be characterized by lex ideals (in $R$). However, there are many examples of non-Macaulay-Lex rings (cf. Chapter 6). For such non-Macaulay-Lex rings, what then can we say about the Hilbert functions of ideals contained in them?

Motivated by these questions, Caviglia–Kummini [27] recently introduced (in 2013) the idea of poset embeddings of Hilbert functions as a way of classifying the collections of Hilbert functions of ideals corresponding to standard graded $k$-algebras. Specifically, let $\mathcal{I}_R$ be the poset of all graded ideals in $R$ ordered by set inclusion, and let $\mathcal{H}_R := \{H(I, -) : I \in \mathcal{I}_R\}$ be the poset of all Hilbert functions of ideals in $R$, ordered such that $H(I, -) \preceq H(J, -)$ if and only if $H(I, d) \leq H(J, d)$ for all $d \in \mathbb{N}$. Then they asked whether there exists an order-preserving poset embedding $\epsilon : \mathcal{H}_R \to \mathcal{I}_R$ such that $H(\epsilon(H(I, -))) = H(I, -)$ for all ideals $I \subseteq R$. (Recall that $H(I, -) : \mathbb{N} \to \mathbb{N}$ is the Hilbert function of $I = \bigoplus_{d \in \mathbb{N}} I_d$.) If such a poset embedding $\epsilon$ exists, then we say $\mathcal{H}_R$ admits an embedding.

Consider the map $\epsilon$ on $\mathcal{H}_R$ given by

$$H\left(\bigoplus_{d \in \mathbb{N}} I_d, -\right) \mapsto \bigoplus_{d \in \mathbb{N}} \text{Lex}_{R_d}(I_d). \quad (5.1)$$

For general $k$-algebras $R$, the image of $\epsilon$ is a $k$-vector space but not necessarily an ideal of $R$. However, notice that the image of $\epsilon$ is always an ideal of $R$ if and only if $R$ is a Macaulay-Lex ring. Note also that for any pair of lex ideals $L, L'$ in $R$, we have $L \subseteq L'$ if and only if $H(L, d) \leq H(L', d)$ for all $d \in \mathbb{N}$. (This follows from the definition of a lex ideal.) Consequently, in the language of poset embeddings of Hilbert functions, we conclude the following: If $R$ is a Macaulay-Lex ring, then $\mathcal{H}_R$ admits an embedding.

Macaulay-Lex rings are not the only $k$-algebras admitting embeddings. Thus the study of poset embeddings of Hilbert functions includes the study of Macaulay-Lex rings as a special case. For other recent results concerning poset embeddings of Hilbert functions, see [28], [30], [31].
In Chapter 5, we introduced the notion of Macaulay-Lex rings. Although basic properties of Macaulay-Lex rings are well understood (see Chapter 5.3), a complete list of all Macaulay-Lex ideals of \( S := \mathbb{k}[x_1, \ldots, x_n] \) is known only for \( n \leq 2 \), with the case \( n = 2 \) already being quite complicated [109]. As for \( n \geq 3 \), several partial results are known [1, 109, 110, 111], but otherwise, finding an explicit characterization of all possible Macaulay-Lex ideals for arbitrary \( n \) remains a wide open problem. In fact, since the 1960s, to as recent as the early 2000s, Clements–Lindström rings were the only examples of Macaulay-Lex rings known to the algebraists. In light of what was known then, Mermin–Peeva [92] posed the following fundamental problem in 2007: Find new classes of Macaulay-Lex rings.

Recently in 2010, Mermin and Murai [89] constructed a new class of Macaulay-Lex rings that they call colored squarefree rings. Their construction was inspired by the Frankl–Füredi–Kalai theorem [53], a graph-theoretic result proven in 1988 that we introduced earlier in Chapter 4.3. Already in the late 1980s, the combinatorialists knew that the Frankl–Füredi–Kalai theorem gives a numerical characterization of the \( f \)-vectors of \( 1_n \)-colored complexes, or equivalently, the \( h \)-vectors of Cohen–Macaulay completely balanced complexes [19]. (See Chapters 2.2 and 3.5 for relevant definitions.) We note that Mermin–Murai proved a refinement of the Frankl–Füredi–Kalai theorem; see [89, Remark 2.12] for details.

In this chapter, we extend Clements–Lindström rings and colored squarefree rings to a common class of quotient rings, which we call colored quotient rings. Note that the combinatorial analogues of Clements–Lindström rings and colored
squarefree rings are the “uncolored” multicomplexes and colored (simplicial) complexes respectively, both of which are special cases of generalized colored multicomplexes, cf. Chapter 2.7. In Chapter 6.1, we give the definition of colored quotient rings and study the lower shadows of revlex-segment monomial spaces arising from these rings. In Chapter 6.2, we introduce two important classes of colored quotient rings: mixed colored quotient rings, and hinged colored quotient rings. We will prove that if a colored quotient ring is Macaulay-Lex, then it must be either mixed or hinged (Theorem 6.2.3).

In Chapter 6.3, we study the “one-colored” case and prove that all colored quotient rings with “one color” are Macaulay-Lex. In contrast, colored quotient rings with “at least two colors” are in general not necessarily Macaulay-Lex. In Chapter 6.4, we introduce the notion of quasi-compression, an extension of the notion of compression introduced in Chapter 2.5 (cf. Chapter 11.1). In particular, we state a key result (Theorem 6.4.3) concerning quasi-compressed monomial spaces arising from mixed or hinged colored quotient rings. The proof of Theorem 6.4.3 requires some preparation, so we postpone it to Chapter 7.2.

Finally, in Chapter 6.5, we use quasi-compression to prove that mixed or hinged colored quotient rings are always Macaulay-Lex, thereby completely characterizing all possible Macaulay-Lex colored quotient rings (Theorem 6.5.3).

6.1 Colored quotient rings

Definition 6.1.1. Let \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{P}^n \), \( \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{P}^n \), and let \( X_\lambda := \bigsqcup_{i=1}^{n} X_i \) be a set of variables, where \( X_i = \{x_{i,1}, \ldots, x_{i,\lambda_i}\} \) for each \( i \in [n] \). Fix a linear order on \( X_\lambda \) by \( x_{i,j} > x_{i',j'} \) if \( j > j' \); or \( j = j' \) and \( i < i' \). Let
Let \( k[X_\lambda] \) be a polynomial ring on the set of variables \( X_\lambda \) over a field \( k \), graded by \( \deg(x_{i,j}) = 1 \), and let \( Q_a := \sum_{i=1}^n \langle X_i \rangle^{a_i+1} \) be a monomial ideal of \( k[X_\lambda] \), where \( \langle X_i \rangle^{\infty} = 0 \). A colored quotient ring of type \( a \) and composition \( \lambda \) is the quotient ring \( W := k[X_\lambda]/Q_a \).

A colored quotient ring is completely determined by two parameters: its type \( a \in \mathbb{P}^n \), and its composition \( \lambda \in \mathbb{P}^n \). Colored squarefree rings and Clements–Lindström rings are two important classes of colored quotient rings. Specifically, a colored squarefree ring is a colored quotient ring of type \( 1_n \) and composition \( (\lambda_1, \ldots, \lambda_n) \) satisfying \( \lambda_1 \geq \cdots \geq \lambda_n \), while a Clements–Lindström ring is a colored quotient ring of composition \( 1_n \) and type \( (a_1, \ldots, a_n) \) satisfying \( a_1 \leq \cdots \leq a_n \).

Throughout Chapters 6–8, we always reserve the notation \( W = \bigoplus_{d \in \mathbb{N}} W_d \) to mean a colored quotient ring of type \( a \in \mathbb{P}^n \) and composition \( \lambda \in \mathbb{P}^n \). Note in particular that \( S := k[x_1, \ldots, x_n] \cong k[X_1^n]/Q_{\infty} \) is a colored quotient ring of type \( a = \infty_n \) and composition \( \lambda = 1_n \).

Given an \( \mathbb{N} \)-graded quotient ring \( R = \bigoplus_{d \in \mathbb{N}} R_d \) such that \( R = S/M \) for some monomial ideal \( M \subseteq S \), recall that the lower shadow of a revlex-segment \( R_{d+1} \)-monomial space is not necessarily revlex-segment (see Remark 5.3.2). However, when we restrict to colored quotient rings, the following proposition shows that the lower shadows of revlex-segment \( W_{d+1} \)-monomial spaces are always revlex-segment.

**Proposition 6.1.2.** Let \( d \in \mathbb{N} \). If \( A \) is a revlex-segment \( W_{d+1} \)-monomial space, then \( \partial(A) \) is revlex-segment.

**Proof.** For convenience, relabel the variables in \( X_\lambda \) by \( y_1 < y_2 < y_3 < \cdots \), so that the linear order on \( X_\lambda \) is preserved, i.e. \( y_1 = x_{n,1}, y_2 = x_{n-1,1}, y_3 = x_{n-2,1}, \ldots \)
etc. Suppose $m$ is a monomial in $\partial(A)$. Then there is some $y_t$ such that $my_t$ is a monomial in $A$. Without loss of generality, choose the largest possible $t$ (i.e. the smallest possible $y_t$ in the revlex order). Next, choose an arbitrary monomial $u$ in $W_d$ such that $u >_{r\ell} m$, and write $m = y_1^{\alpha_1} \cdots y_N^{\alpha_N}$, $u = y_1^{\beta_1} \cdots y_N^{\beta_N}$, where $N = \max\{i : y_i \text{ divides } m \text{ or } y_i \text{ divides } u\}$.

To show that $\partial(A)$ is revlex-segment, it suffices to show that $u \in \partial(A)$. If there is some $i \in [t]$ such that $uy_i \neq 0$, then $y_i \geq_{r\ell} y_t$ and $u >_{r\ell} m$ together yield $uy_i >_{r\ell} my_t$, so since $A$ is revlex-segment, we get $uy_i \in A$, which implies $u \in \partial(A)$, and we are done. Thus, we can assume every $i \in [t]$ satisfies $uy_i = 0$.

Note that $my_t \neq 0$ yields $\deg(my_t) \leq |a|$, hence $\deg(u) = \deg(m) \leq |a| - 1$, and $uy_j \neq 0$ for some $j > t$. Choose the smallest possible $j$, so that $uy_i = 0$ for all $i \in [j - 1]$. Note that $j \leq n$, since otherwise $uy_i = 0$ for all $i \in [n]$, which implies $\deg(u_{X_i}) = a_i < \infty$ for all $i \in [n]$. This would force $\deg(m) = \deg(u) = |a|$ and $mx = 0$ for all $x \in X_\lambda$, which contradicts the existence of $y_t$. Also, notice that if we can show $uy_j \geq_{r\ell} my_t$, then since $A$ is revlex-segment, we get $uy_j \in A$, i.e. $u \in \partial(A)$, and we are done.

Suppose instead $uy_j <_{r\ell} my_t$. Let $\ell \in [N]$ be the largest integer such that $\alpha_\ell \neq \beta_\ell$, and note that $u >_{r\ell} m$ implies $\beta_\ell < \alpha_\ell$. By the definition of $\ell$, it follows from $uy_j <_{r\ell} my_t$ and $j > t$ that $j \geq \ell$. If $j > \ell$, then we get $\ell \in [n]$, so since $\beta_\ell < \alpha_\ell$, the definition of $\ell$ implies $\deg(u_{X_{n+1-\ell}}) < \deg(m_{X_{n+1-\ell}})$, thus $uy_\ell \neq 0$, which contradicts the minimality of $j$. Consequently, $j = \ell$. This forces $\beta_\ell = \alpha_\ell - 1$, since otherwise $\beta_\ell < \alpha_\ell - 1$ would imply the contradiction $uy_j = uy_\ell >_{r\ell} my_t$ that follows from $t < j = \ell$ and the definition of $\ell$.

Now, $uy_\ell = uy_j <_{r\ell} my_t$, and the exponents of $y_\ell$ in both $uy_\ell$ and $my_\ell$ are
equal, thus there is some $\ell' \in [\ell - 1]$ such that the exponent of $y_{\ell'}$ in $uy_\ell$ is strictly greater than the exponent of $y_{\ell'}$ in $my_t$. In particular, this means $\alpha_{\ell'} < \beta_{\ell'}$. Note that $uy_i = 0$ for all $i \in [\ell - 1]$, so it follows from $\ell \leq n$ and the definition of $W$ that $\deg(u_{X_{n+1-i}}) = a_{n+1-i} < \infty$ for all $i \in [\ell - 1]$, thus $\alpha_i \leq \beta_i$ for all $i \in [\ell - 1]$. Since $\deg(uy_\ell) = \deg(my_t)$ and $\beta_\ell = \alpha_\ell - 1$, we get $\sum_{i\in[\ell-1]} \beta_i = (\sum_{i\in[\ell-1]} \alpha_i) + 1$, so $\alpha_{\ell'} < \beta_{\ell'}$ implies $\alpha_i = \beta_i$ for all $i \in [\ell - 1] \setminus \{\ell'\}$, and $\beta_{\ell'} = \alpha_{\ell'} - 1$. Note that $\ell' \neq t$, since otherwise we would get $uy_\ell = my_t$. Thus $\deg(u_{X_{n+1-\ell'}}) < \deg(m_{X_{n+1-\ell'}})$, which implies $uy_{\ell'} \neq 0$, therefore contradicting the minimality of $j$. □

By combining with Theorem 5.3.1, we get the following useful corollary.

**Corollary 6.1.3.** $W$ is Macaulay-Lex if and only if $|\partial(\text{Revlex}_{W_{d+1}}(A))| \leq |\partial(A)|$ for all $d \in \mathbb{N}$ and every $W_{d+1}$-monomial space $A$.

### 6.2 Mixed or hinged colored quotient rings

Let $W$ be a colored quotient ring of type $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{P}^n$ and composition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{P}^n$. If there exists some integer $r$ satisfying $0 \leq r \leq n$ such that $a_i = 1$ for all $i \in [r]$, $\lambda_i = 1$ for all $i \in [n] \setminus [r]$, and $a_{r+1} \leq \cdots \leq a_n$, then we say $W$ is *mixed*. If $a_1 \leq \cdots \leq a_n$ and $\lambda_i = 1$ for all $i \neq 1$, then we say $W$ is *hinged*. Note that a Clements–Lindström ring is both mixed and hinged, while a colored squarefree ring is mixed. See Figure 6.1 for a depiction of mixed and hinged colored quotient rings.

The main result of this section is that if $W$ is Macaulay-Lex, then it must be either mixed or hinged (Theorem 6.2.3). Later in Chapter 6.5, we will prove the converse to Theorem 6.2.3, thereby obtaining a complete characterization of all
Macaulay-Lex colored quotient rings. Before we prove Theorem 6.2.3, we shall first prove a useful necessary condition for $W$ to be Macaulay-Lex.

**Proposition 6.2.1.** If $W$ is Macaulay-Lex, then $a_1 \leq \cdots \leq a_n$.

*Proof.*** Assume $n > 1$, and suppose instead that $a_t > a_{t+1}$ for some $t \in [n - 1]$. Define the lex ideal $L := \langle \{x \in X_\lambda : x > x_{t,1}\} \rangle \subseteq k[X_\lambda]$, and note that $Q_a$ is Macaulay-Lex by assumption, hence by Theorem 5.3.3(i), the ideal $M := Q_a + L$ is a Macaulay-Lex ideal of $k[X_\lambda]$. Since

$$k[X_\lambda]/M \cong k[x_{t,1}, x_{(t+1),1}, \ldots, x_{n,1}]/\langle x_{t,1}^{a_t+1}, x_{(t+1),1}^{a_{(t+1)+1}}, \ldots, x_{n,1}^{a_{n+1}+1} \rangle,$$

it follows that $J := \langle x_{t,1}^{a_t+1}, x_{(t+1),1}^{a_{(t+1)+1}}, \ldots, x_{n,1}^{a_{n+1}+1} \rangle$ is a Macaulay-Lex ideal of the polynomial ring $k[x_{t,1}, x_{(t+1),1}, \ldots, x_{n,1}]$. However, the generators of $J$ have minimal degree $\leq a_t$, which contradicts Theorem 5.3.3(ii). \hfill $\square$

**Remark 6.2.2.** Proposition 6.2.1 was previously known for the case $\lambda = 1_n$; see [111, Example 2.7.4]. By combining with the Clements–Lindström theorem [36], we conclude that a colored quotient ring of composition $1_n$ is Macaulay-Lex if and only if $a_1 \leq \cdots \leq a_n$.

**Theorem 6.2.3.** If $W$ is Macaulay-Lex, then $W$ is either mixed or hinged.
Proof. Let \( s := \max\{i \in [n] : \lambda_i \geq 2\} \). By Proposition 6.2.1, it suffices to show that if \( s \neq 1 \), then \( a_i = 1 \) for all \( i \in [s] \). Define the ideal

\[
M' := (x_{1,1}^{a_1+1}, x_{s,1}^{a_s+1}) + \langle x_{s,1}, x_{s,2} \rangle^{a_s+1} \subseteq \mathbb{k}[x_{s,2}, x_{1,1}, x_{2,1}, \ldots, x_{n,1}] \tag{6.1}
\]

Note that \( Q_a \) is Macaulay-Lex, while \( L := \langle \{x \in X_\lambda : x > x_{s,2}\} \rangle \subseteq \mathbb{k}[X_\lambda] \) is a lex ideal, so \( M := Q_a + L \) is Macaulay-Lex by Theorem 5.3.3(i). Since \( \mathbb{k}[X_\lambda]/M \cong \mathbb{k}[x_{s,2}, x_{1,1}, x_{2,1}, \ldots, x_{n,1}]/M' \), it follows that \( M' \) is Macaulay-Lex.

Next, define the ideal

\[
J = (x_{1,1}^{a_1+1}, x_{2,1}^{a_2+1}, \ldots, x_{n,1}^{a_n+1}) + \langle x_{s,1}^2 \rangle \subseteq \mathbb{k}[x_{1,1}, x_{2,1}, \ldots, x_{n,1}] \tag{6.2}
\]

Theorem 5.3.3(iii) tells us that \( J \) is Macaulay-Lex, thus \( \mathbb{k}[x_{1,1}, x_{2,1}, \ldots, x_{n,1}]/J \) is a Macaulay-Lex colored quotient ring of type \((a_1, \ldots, a_{s-1}, 1, a_{s+1}, \ldots, a_n)\). Consequently, Proposition 6.2.1 yields \( a_i = 1 \) for all \( i \in [s-1] \), so we get

\[
M' = (x_{1,1}^2, \ldots, x_{(s-1),1}^2) + \langle x_{s,1}^{a_s+1}, \ldots, x_{n,1}^{a_n+1} \rangle + \langle x_{s,1}, x_{s,2} \rangle^{a_s+1} \tag{6.3}
\]

Now, if \( s > 1 \), then \( M' \) has generators of minimal degree 2, and Theorem 5.3.3(ii) says there is some \( x \in \{x_{1,1}, \ldots, x_{n,1}\} \cup \{x_{s,2}\} \) such that \( x_{s,2}x \in M' \), thereby forcing \( x \in \{x_{s,1}, x_{s,2}\} \) and hence \( a_i = 1 \) for all \( i \in [s] \).

6.3 The one-colored case

Recall that \( W = \mathbb{k}[X_\lambda]/Q_a \), where \( a \in \mathbb{P}^n \) and \( \lambda \in \mathbb{P}^n \). For this section, suppose \( n = 1 \), i.e. \( W = \mathbb{k}[X_1]/(X_1)^{a_1+1} \). This means that \( W \) is necessarily hinged, and by Macaulay’s theorem, we know that \( W \) is Macaulay-Lex if \( a_1 = \infty \). Now, assume \( a_1 < \infty \), and note that every \( W_d \)-monomial space satisfying \( d > a_1 \) is empty. Thus
by Corollary 6.1.3, \( W \) is Macaulay-Lex if and only if \( |\partial(\text{Revlex}_{W_d}(A))| \leq |\partial(A)| \) for all \( d \in [a_1] \) and every \( W_d \)-monomial space \( A \).

Let \( T = \bigoplus_{d \in \mathbb{N}} T_d \) be the polynomial ring \( \mathbb{k}[X_1] \), equipped with the linear order \( x_{1,1} < \cdots < x_{1,\lambda_1} \) on its variables. Notice that \( T \) is Macaulay-Lex by Macaulay’s theorem. Next, fix \( d \in [a_1] \) and some \( W_d \)-monomial space \( A \). Since \( d \leq a_1 \) implies \( \{W_d\} = \{T_d\} \), it follows from Corollary 6.1.3 that

\[
|\partial(\text{Revlex}_{W_d}(A))| = |\partial(\text{Revlex}_{T_d}(A))| \leq |\partial(A)| = |\partial(A)|,
\]

therefore \( W \) is always Macaulay-Lex when \( n = 1 \).

By a similar argument, we get the following more general result.

**Proposition 6.3.1.** Let \( n \in \mathbb{P}, a, \alpha_1, \ldots, \alpha_n \in \mathbb{P}, \) and assume the variables of the polynomial ring \( S = \mathbb{k}[x_1, \ldots, x_n] \) are linearly ordered by \( x_1 > \cdots > x_n \). Then

\[
I := \langle x_1, \ldots, x_n \rangle^{a+1} + \langle x_1^{\alpha_1+1}, \ldots, x_n^{\alpha_n+1} \rangle \subseteq S
\]

is Macaulay-Lex if and only if \( \min\{a, \alpha_1\} \leq \cdots \leq \min\{a, \alpha_n\} \).

[Note: By definition, \( x_i^\infty = 0 \).]

**Proof.** Let \( R = \bigoplus_{d \in \mathbb{N}} R_d := S/I \). The case \( a = \infty \) follows from Remark 6.2.2, so assume \( a < \infty \). Since every \( R_d \)-monomial space satisfying \( d > a \) is empty, it follows from Theorem 5.3.1 that \( R \) is Macaulay-Lex if and only if

\[
\partial(\text{Revlex}_{R_d}(A)) \subseteq \text{Revlex}_{R_{d-1}}(\partial(A))
\]

(6.5)

for all \( d \in [a] \) and every \( R_d \)-monomial space \( A \).

Fix some \( d \in [a] \), and choose an arbitrary \( R_d \)-monomial space \( A \). Next, let \( \beta_i = \min\{a, \alpha_i\} \) for each \( i \in [n] \), define the ideal \( J := \langle x_1^{\beta_1+1}, \ldots, x_n^{\beta_n+1} \rangle \subseteq S \), and
let \( T = \bigoplus_{d \in \mathbb{N}} T_d := S/J \). Notice that Remark 6.2.2 (which is a consequence of the Clements–Lindström theorem) says \( T \) is Macaulay-Lex if and only if \( \beta_1 \leq \cdots \leq \beta_n \).

Since \( d \leq a \) implies \( \{ R_d \} = \{ T_d \} \), it then follows from Theorem 5.3.1 that

\[
\partial(\text{Revlex } R_d(A)) = \partial(\text{Revlex } T_d(A)) \subseteq \text{Revlex } T_{d-1}(\partial(A)) = \text{Revlex } R_{d-1}(\partial(A))
\]

if and only if \( \beta_1 \leq \cdots \leq \beta_n \), and the assertion follows from (6.5).

**Remark 6.3.2.** Although we will use the fact that \( W \) is Macaulay-Lex for \( n = 1 \) when we characterize all possible Macaulay-Lex colored quotient rings (see Chapter 6.5), we will not use Proposition 6.3.1 until in Chapter 8.2 (see proof of Theorem 8.2.7). Thus, when we prove our characterization of all Macaulay-Lex colored quotient rings (Theorem 6.5.3), we actually prove the Clements–Lindström theorem (instead of using it).

### 6.4 Quasi-compression

Suppose \( n > 1 \). For each \( t \in [n] \), let \( \widehat{S}^{(t)} := k[X_\lambda - X_t] \) be a polynomial ring on the set of variables \( X_\lambda - X_t \) over a field \( k \), and fix the linear order on \( X_\lambda - X_t \) induced by the order on \( X_\lambda \) given in Definition 6.1.1. Define the colored quotient ring \( \widehat{W}^{(t)} = \bigoplus_{d \in \mathbb{N}} \widehat{W}_d^{(t)} := \widehat{S}^{(t)}/\widehat{Q}_a^{(t)} \), where \( \widehat{Q}_a^{(t)} := \sum_{i \in [n] \setminus \{ t \}} (X_t)^{a_i+1} \) is an ideal of \( \widehat{S}^{(t)} \). Note that if \( W \) is mixed (resp., hinged), then \( \widehat{W}^{(t)} \) is mixed (resp., hinged).

Let \( A \) be a \( W_d \)-monomial space for some \( d \in \mathbb{N} \). Define

\[
\| A \|_{(a, \lambda)} := \sum_{m \in \{ A \}} \| m \|_{(a, \lambda)}, \tag{6.7}
\]

where \( \| m \|_{(a, \lambda)} := |\{ m' \in \{ W_{\deg(m)} \} : m' \geq_{r\ell} m \}| \) for each \( m \in \{ W \} \). If \( q \) is a
monomial in $W$ such that $\text{supp}(q) \subseteq X_t$ for some $t \in [n]$, then define

$$A[t; q] := \text{Span}_k \{ m \in \{ A \} : m_{X_t} = q \} \quad (6.8)$$

$$A[t; q]/q := \text{Span}_k \{ \frac{m}{q} : m \in \{ A[t; q] \} \}. \quad (6.9)$$

Also, given any $\hat{W}^{(t)}_{d-\deg(q)}$-monomial space $A'$, define

$$q \cdot A' := \text{Span}_k \{ q \cdot m : m \in \{ A' \} \}. \quad (6.10)$$

In particular, $A[t; q] = q \cdot (A[t; q]/q)$, and we can write $\{ A \} = \bigsqcup_q \{ A[t; q] \}$, where the disjoint union is over all monomials $q$ in $W$ satisfying $\text{supp}(q) \subseteq X_t$ for some fixed $t \in [n]$.

**Definition 6.4.1.** For every $t \in [n]$, define the operation $C_t$ on $W_d$-monomial spaces as follows: If $n = 1$, then $C_1(A) := A$, while if $n > 1$, then

$$C_t(A) := \bigsqcup_{q \in \{ W \} \text{ s.t. } q_{X_t} = q} \text{Relex}_{\hat{W}^{(t)}_{d-\deg(q)}}(A[t; q]/q)$$

$$= \bigsqcup_{q \in \{ W \} \text{ s.t. } q_{X_t} = q} \text{Span}_k \left\{ qm : m \in \{ \text{Relex}_{\hat{W}^{(t)}_{d-\deg(q)}}(A[t; q]/q) \} \right\}. \quad (6.11)$$

It is clear by definition that $C_t(A)$ is a $W_d$-monomial space satisfying $|C_t(A)| = |A|$ and $\|C_t(A)\|_{(a, \lambda)} \leq \|A\|_{(a, \lambda)}$. Assuming $n > 1$, we say $A$ is $(X_\lambda - X_t)$-compressed if $C_t(A) = A$, and we say $A$ is quasi-compressed if $A$ is $(X_\lambda - X_t)$-compressed for all $t \in [n]$. This notion of ‘$(X_\lambda - X_t)$-compressed’ agrees with the notion of $A$-compression introduced by Mermin [88] (for $A = X_\lambda - X_t$). Also, the notion of quasi-compression coincides with the usual notion of compression in the case $\lambda = 1_n$ (and $n > 1$); cf. [91]. However, a quasi-compressed $W_d$-monomial space is in general not necessarily compressed. As for the special case $n = 1$, since $C_1(A) = A$ by definition, we then define $A$ to be always quasi-compressed.
Lemma 6.4.2. Let \( d \in \mathbb{N} \). For every \( W_d \)-monomial space \( A \), there exists a finite sequence \( t_1, \ldots, t_r \) of integers in \([n]\) such that \( C_{t_r}(C_{t_{r-1}}(\cdots C_1(A))) \) is quasi-compressed. Furthermore, we can choose \( t_1, \ldots, t_r \) so that

\[
\|C_{t_{i+1}}(C_{t_i}(\cdots C_1(A)))\|_{(a, \lambda)} < \|C_{t_i}(\cdots C_1(A))\|_{(a, \lambda)} \tag{6.11}
\]

for every pair of consecutive terms \( t_i, t_{i+1} \).

Proof. This is trivial, since \( \|C_t(A)\|_{(a, \lambda)} \leq \|A\|_{(a, \lambda)} \) for every \( t \in [n] \), with equality holding if and only if \( C_t(A) = A \). \( \square \)

To prove that a given colored quotient ring \( W \) is Macaulay-Lex, Corollary 6.1.3 tells us it suffices to show that for any \( W_d \)-monomial space \( A \), there exists a finite sequence \( A_0, A_1, \ldots, A_N \) of \( W_d \)-monomial spaces starting with \( A_0 = A \) and ending with \( A_N = \text{Revlex}_{W_d}(A) \), such that \( |\partial(A_i)| \leq |\partial(A_{i-1})| \) for all \( i \in [N] \). By first defining the term \( A_0 = A \), we construct such a sequence algorithmically as follows:

Step \( i \): \( A_{i-1} \) is already defined; \( A_i \) is not yet defined.

if \( A_{i-1} \) is not quasi-compressed then
  Choose some \( t \in [n] \) such that \( \|A_{i-1}\|_{(a, \lambda)} > \|C_t(A_{i-1})\|_{(a, \lambda)} \).
  Set \( A_i = C_t(A_{i-1}) \).
else (i.e. \( A_{i-1} \) is quasi-compressed)
  if there exists a \( W_d \)-monomial space \( A' \) such that \( |A'| = |A_{i-1}| \), \( \|A'\|_{(a, \lambda)} < \|A_{i-1}\|_{(a, \lambda)} \), and \( |\partial(A')| \leq |\partial(A_{i-1})| \) then
    Choose any such \( A' \) and set \( A_i = A' \).
  else
    Terminate algorithm.
end if

end if

This algorithm is well-defined by Lemma 6.4.2. Also, note that

\[
\|A_0\|_{(a, \lambda)} > \|A_1\|_{(a, \lambda)} > \ldots > \|A_N\|_{(a, \lambda)} \tag{6.12}
\]

by construction. To guarantee \( A_N = \text{Revlex}_{W_d}(A) \) and \( |\partial(A_i)| \leq |\partial(A_{i-1})| \) for all \( i \in [N] \), we need to show the following:
(i) Every $W_d$-monomial space $A$ satisfies $|\partial(C_t(A))| \leq |\partial(A)|$ for all $t \in [n]$.

(ii) If $A$ is a quasi-compressed $W_d$-monomial space, then either $A$ is revlex-segment, or there exists some $W_d$-monomial space $A'$ such that $|A'| = |A|$, $\|A'\|_{(a, \lambda)} < \|A\|_{(a, \lambda)}$, and $|\partial(A')| \leq |\partial(A)|$.

In general, we can find pairs $(a, \lambda)$ such that at least one of these two statements does not hold. This is expected, since Theorem 6.2.3 says $W$ is Macaulay-Lex only if $W$ is either mixed or hinged. In the case when $W$ is either mixed or hinged, we have the following key result:

**Theorem 6.4.3.** Let $d \in \mathbb{P}$, $n > 1$, $a \in \mathbb{P}^n$, and suppose $W$ is either mixed or hinged. If $A$ is a non-empty quasi-compressed $W_d$-monomial space that is not revlex-segment, then there exists some $W_d$-monomial space $A'$ satisfying $|A'| = |A|$, $\|A'\|_{(a, \lambda)} < \|A\|_{(a, \lambda)}$, and $|\partial(A')| \leq |\partial(A)|$.

Remarkably, the condition that $W$ is either mixed or hinged is not just necessary, but crucial as well. Our proof of Theorem 6.4.3 is purely combinatorial and uses the combinatorial structure of mixed and hinged colored quotient rings in a fundamental way. This proof requires some preparation, so we postpone it to Chapter 7.2. Note that Theorem 6.4.3, together with the preceding discussion, implies the following result.

**Corollary 6.4.4.** Let $n > 1$, $a \in \mathbb{P}^n$, and suppose $W$ is either mixed or hinged. If $|\partial(C_t(A))| \leq |\partial(A)|$ for all $t \in [n]$, $d \in \mathbb{P}$, and every $W_d$-monomial space $A$, then $W$ is Macaulay-Lex.
6.5 Macaulay-Lex colored quotient rings

In this section, we give an explicit characterization of all Macaulay-Lex colored quotient rings: A colored quotient ring is Macaulay-Lex if and only if it is either mixed or hinged (see Theorem 6.5.3). We begin with a key result required in proving this characterization.

**Theorem 6.5.1.** Let $n, d \in \mathbb{P}$, $a \in \mathbb{P}^n$, suppose $W$ is either mixed or hinged, and let $A$ be a $W_d$-monomial space. Then the following two statements hold:

(i) $|\partial(\text{Revlex}_{W_d}(A))| \leq |\partial(A)|$.

(ii) $|\partial(C_t(A))| \leq |\partial(A)|$ for all $t \in [n]$.

**Proof.** We will prove (i) and (ii) simultaneously by induction on $n$. For the base case $n = 1$, (i) was proven in Chapter 6.3, while (ii) is trivially true. Assume $n > 1$, fix some $t \in [n]$, and let $\hat{A} := \partial(A)$. For brevity, let $\Gamma$ be the set of all monomials $q$ in $W$ satisfying $\text{supp}(q) \subseteq X_t$. Next, define $B := \text{Span}_k(W_d \setminus \{C_t(A)\})$ and $\hat{B} := \text{Span}_k(W_{d-1} \setminus \{C_t(\hat{A})\})$.

We claim that $xq \cdot \hat{D} \subseteq B$ for all $x \in X_\lambda$. If this claim is true, then $\partial(C_t(A)) \subseteq C_t(\hat{A})$, which implies $|\partial(C_t(A))| \leq |C_t(\hat{A})| = |\hat{A}| = |\partial(A)|$, i.e. (ii) is true, thus (i) follows from Corollary 6.4.4, and we are done. For each $q \in \Gamma$ of degree $r$, define

$$D[q] := \text{Span}_k \left( \{\hat{W}_{d-r}^{(t)}\} - \{\text{Revlex}_{\hat{W}_{d-r}^{(t)}}(A[t;q]/q)\} \right);$$

$$\hat{D}[q] := \text{Span}_k \left( \{\hat{W}_{d-r-1}^{(t)}\} - \{\text{Revlex}_{\hat{W}_{d-r-1}^{(t)}}(A[t;q]/q)\} \right);$$

$$I[q] := \text{Span}_k \left( \bigcup_{j \geq d-r-1} \{\hat{W}_{j}^{(t)}\} - (A[t;q]/q) \cup (\hat{A}[t;q]/q) \right).$$

Note that $B = \bigoplus_{q \in \Gamma} (q \cdot D[q])$ and $\hat{B} = \bigoplus_{q \in \Gamma} (q \cdot \hat{D}[q])$. Consequently, to prove the claim, it suffices to show that $xq \cdot \hat{D}[q] \subseteq B$ for all $x \in X_\lambda$ and all $q \in \Gamma$. 

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Choose any $q \in \Gamma$, let $r = \deg(q)$, and note that $I^{[q]} = \bigoplus_{j \in \mathbb{N}} I^{[q]}_j$ is a graded monomial ideal of $\hat{W}^{(t)}$. By induction hypothesis, $|\partial(\text{Revlex} \, \hat{W}^{(t)}_j(A'))| \leq |\partial(A')|$ for all $j \in \mathbb{P}$ and every $\hat{W}^{(t)}_j$-monomial space $A'$, so $\hat{W}^{(t)}$ is Macaulay-Lex by Corollary 6.1.3, i.e. there is a lex ideal $L^{[q]} = \bigoplus_{j \in \mathbb{N}} L^{[q]}_j$ in $\hat{W}^{(t)}$ with the same Hilbert function as $I^{[q]}$. We check that $L^{[q]}_d = \hat{D}^{[q]}_d$ and $L^{[q]}_d = D^{[q]}$ by definition, thus if $x \in X_{\lambda} - X_t$, then $xq \cdot \hat{D}^{[q]} \subseteq q \cdot D^{[q]} \subseteq B$. Suppose instead $x \in X_t$, and assume without loss of generality that $xq \neq 0$. Since $I := \bigoplus_{q \in \Gamma}(q \cdot I^{[q]})$ is a monomial ideal of $W$, we have $x(q \cdot I^{[q]}) \subseteq xq \cdot I^{[xq]}$, thus $I^{[q]} \subseteq I^{[xq]}$, and in particular, $|\hat{D}^{[q]}| = |I^{[xq]}_d| \leq |I^{[xq]}_d| = |D^{[xq]}|$. Now, $\hat{D}^{[q]}$ and $D^{[xq]}$ are lex-segment $\hat{W}^{(t)}_d$-monomial spaces by construction, so $\hat{D}^{[q]} \subseteq D^{[xq]}$, therefore $xq \cdot \hat{D}^{[q]} \subseteq xq \cdot D^{[xq]} \subseteq B$. 

The author thanks an anonymous referee for suggesting the current proof of Theorem 6.5.1, which is significantly shorter than the original proof. For completeness, we shall also present the original proof. Notice that the original proof (Proposition 6.5.2) is “more combinatorial” and could potentially be adapted in proving other non-algebraic “Kruskal–Katona-type” analogues.

Suppose $A$ is a $W_d$-monomial space for some $d \in \mathbb{P}$. For each monomial $q$ in $W$ satisfying $\text{supp}(q) \subseteq X_t$ for some $t \in [n]$, define $\partial_{[t; q]}(A) := (\partial(A[t; q]))[t; q]$ and $\widetilde{\partial}_{[t; q]}(A) := \partial(A[t; q]) - \partial_{[t; q]}(A)$. For each $s \in [a_t]$, define

$$U_s := \left( \bigoplus_{p \in \mathbb{N}} \partial_{[t; p]} \left( p \cdot \text{Revlex} \, \hat{W}^{(t)}_{d-s+1} \left( A[t; p]/p \right) \right) \right) + \sum_{q \in \mathbb{N}} \widetilde{\partial}_{[t; q]} \left( q \cdot \text{Revlex} \, \hat{W}^{(t)}_{d-s} \left( A[t; q]/q \right) \right),$$

$$V_s := \left( \bigoplus_{p \in \mathbb{N}} \partial_{[t; p]}(A[t; p]) \right) + \sum_{q \in \mathbb{N}} \widetilde{\partial}_{[t; q]}(A[t; q]).$$

**Proposition 6.5.2.** Let $n, d \in \mathbb{P}$, $a \in \mathbb{P}^n$, suppose $W$ is either mixed or hinged, and let $A$ be a $W_d$-monomial space. Then the following four statements hold:

1. $\sum_{s \in [a_t]} U_s = \sum_{s \in [a_t]} V_s$.
2. $\sum_{s \in [a_t]} U_s = \sum_{s \in [a_t]} V_s$.
3. $\sum_{s \in [a_t]} U_s = \sum_{s \in [a_t]} V_s$.
4. $\sum_{s \in [a_t]} U_s = \sum_{s \in [a_t]} V_s$.
(i) $|\partial(\text{Revlex}_{W_d}(A))| \leq |\partial(A)|$.

(ii) $|\partial(C_t(A))| \leq |\partial(A)|$ for all $t \in [n]$.

(iii) $|\partial_{[t]q}(q \cdot \text{Revlex}_{\hat{W}_{d-r}^{(t)}}(A[t; q]/q))| \leq |\partial_{[t]q}(A[t; q])|$ for all $t \in [n]$ and every $q \in \{k[X_t]\}$ of degree $r$.

(iv) $|U_s| \leq |V_s|$ for all $t \in [n]$, $s \in [a_t]$.

Proof. We will prove these four statements simultaneously by induction on $n$. For the base case $n = 1$, (ii), (iii) and (iv) are trivially true, while (i) is an easy consequence of Macaulay’s theorem (see Chapter 6.3). Assume $n > 1$. The $W_{d-1}$-monomial spaces $\partial(C_t(A))$ and $\partial(A)$ can be decomposed as

$$\partial(C_t(A)) = \left( \bigoplus_{q \in \{k[X_t]\}, \deg(q) = a_t} \partial_{[t]q}(q \cdot \text{Revlex}_{\hat{W}_{d-r}^{(t)}}(A[t; q]/q)) \right) \oplus \left( \bigoplus_{s \in [a_t]} U_s \right),$$

$$\partial(A) = \left( \bigoplus_{q \in \{k[X_t]\}, \deg(q) = a_t} \partial_{[t]q}(A[t; q]) \right) \oplus \left( \bigoplus_{s \in [a_t]} V_s \right),$$

hence (ii) follows from (iii) and (iv) by taking dimensions, which then implies (i) by Corollary 6.4.4. Consequently, we only need to prove (iii) and (iv).

First, we prove (iii). Choose any $q \in \{k[X_t]\}$ of degree $r$, and assume without loss of generality that $A[t; q] \neq (0)$. Note that $|\partial(A[t; q]/q)| = |\partial_{[t]q}(A[t; q])|$, so by induction hypothesis,

$$|\partial_{[t]q}(q \cdot \text{Revlex}_{\hat{W}_{d-r}^{(t)}}(A[t; q]/q))| = |\partial(\text{Revlex}_{\hat{W}_{d-r}^{(t)}}(A[t; q]/q))| \leq |\partial(A[t; q]/q)| = |\partial_{[t]q}(A[t; q])|.$$  

Next, we prove (iv). Fix some $s \in [a_t]$. For all monomials $p, q$ in $k[X_t]$ satisfying
\[
\deg(p) = s - 1, \quad \deg(q) = s, \text{ write}
\]
\[
T_{[s]}^{(p)} := \partial_{[t;p]} (p \cdot \text{Revlex}_W^{(t)} (A[t; p]/p)),
\]
\[
U_{[s]}^{(p,q)} := (\partial_{[t;q]} (q \cdot \text{Revlex}_W^{(t)} (A[t; q]/q))) \cap W_{d-1}[t; p],
\]
\[
V_{[s]}^{(p,q)} := (\partial_{[t;q]} (A[t; q])) \cap W_{d-1}[t; p],
\]

By the definition of \( \partial_{[t;q]} \),
\[
\sum_{q \in \{ k[X_i] \}} \sum_{\deg(q) = s} \partial_{[t;q]} (A[t; q]) = \bigoplus_{p \in \{ k[X_i] \}} \sum_{\deg(p) = s-1} \left[ \sum_{q \in \{ k[X_i] \}} \partial_{[t;p]} (A[t; p]) \right] + \left[ \sum_{q \in \{ k[X_i] \}} V_{[s]}^{(p,q)} \right],
\]

hence
\[
V_{[s]} = \bigoplus_{p \in \{ k[X_i] \}} \left[ T_{[s]}^{(p)} + \sum_{q \in \{ k[X_i] \}} V_{[s]}^{(p,q)} \right].
\]

Similarly, we have
\[
U_{[s]} = \bigoplus_{p \in \{ k[X_i] \}} \left[ T_{[s]}^{(p)} + \sum_{q \in \{ k[X_i] \}} U_{[s]}^{(p,q)} \right],
\]

so to prove (iv), it suffices to prove that
\[
\left| T_{[s]}^{(p)} + \sum_{q \in \{ k[X_i] \}} U_{[s]}^{(p,q)} \right| \leq \left| \partial_{[t;p]} (A[t; p]) \right| + \left| \sum_{q \in \{ k[X_i] \}} V_{[s]}^{(p,q)} \right|
\]

for every monomial \( p \) in \( k[X_i] \) of degree \( s - 1 \).

For all \( p, q \in \{ k[X_i] \} \) such that \( p \) divides \( q \), \( \deg(p) = s - 1 \) and \( \deg(q) = s \), let \( \hat{T}_{[s]}^{(p)} \), \( \hat{U}_{[s]}^{(p,q)} \), \( \hat{V}_{[s]}^{(p,q)} \) be the \( \hat{W}_{d-s}^{(t)} \)-monomial spaces \( \text{Span}_k \{ m/p : m \in T_{[s]}^{(p)} \} \), \( \text{Span}_k \{ m/p : m \in U_{[s]}^{(p,q)} \} \), \( \text{Span}_k \{ m/p : m \in V_{[s]}^{(p,q)} \} \) respectively. By Proposition 6.1.2 and the definition of the revlex order, \( \hat{U}_{[s]}^{(p,q)} \) is revlex-segment of dimension \( |U_{[s]}^{(p,q)}| \). Since the union of a collection of revlex-segment subsets in \( \{ \hat{W}_{d-s}^{(t)} \} \) is
revlex-segment, it then follows that for each \( p \in \{ \kx \} \) of degree \( s - 1 \),

\[
\sum_{q \in \{ \kx \} \atop \deg(q) = s} U^{(p,q)}_{[s]} = U^{(p,q_{\text{max}})}_{[s]}, \tag{6.20}
\]

where \( q_{\text{max}} \) is any monomial in \( \{ q \in \{ \kx \} : p \text{ divides } q \} \) such that \( |U^{(p,q)}_{[s]}| \) is maximal. Note also that Proposition 6.1.2 says

\[
\hat{T}^{(p)}_{[s]} = \partial \left( \text{Revlex}_{\hat{W}_{d-s+1}} \left( \text{A}[t;p]/p \right) \right), \tag{6.21}
\]

is revlex-segment, so \( \hat{T}^{(p)}_{[s]} + \hat{U}^{(p,q_{\text{max}})}_{[s]} \) is a revlex-segment \( W_{d-s} \)-monomial space, and

\[
\left| T^{(p)}_{[s]} + \sum_{q \in \{ \kx \} \atop \deg(q) = s} U^{(p,q)}_{[s]} \right| = \left| \hat{T}^{(p)}_{[s]} + \hat{U}^{(p,q_{\text{max}})}_{[s]} \right| = \max \left\{ |T^{(p)}_{[s]}|, \left( \max \left\{ |U^{(p,q)}_{[s]}| : q \in \{ \kx \} \text{, } p \text{ divides } q \right\} \right) \right\}. \tag{6.22}
\]

Now, we have already proven that (iii) holds, so \( |T^{(p)}_{[s]}| \leq |\partial_{(t;p)}(\text{A}[t;p])| \). Consequently, by (6.19) and (6.22), it suffices to show that \( |U^{(p,q)}_{[s]}| \leq |V^{(p,q)}_{[s]}| \) for every \( p, q \in \{ \kx \} \) satisfying \( \deg(p) = s - 1 \) and \( \deg(q) = s \). Clearly

\[
|q \cdot \text{Revlex}_{\hat{W}_{d-s}}(\text{A}[t;q]/q)| = |\text{A}[t;q]| \tag{6.23}
\]

for each \( q \in \{ \kx \} \), so \( |U^{(p,q)}_{[s]}| = |V^{(p,q)}_{[s]}| \) for all possible \( p, q \), i.e. (iv) holds. \( \square \)

Finally, we characterize all Macaulay-Lex colored quotient rings:

**Theorem 6.5.3.** Let \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{P}^n \) and \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{P}^n \). A colored quotient ring of type \( \mathbf{a} \) and composition \( \lambda \) is Macaulay-Lex if and only if (at least) one of the following conditions hold:

(i) \( \mathbf{a} = (1, \ldots, 1, a_{r+1}, \ldots, a_n), \lambda = (\lambda_1, \ldots, \lambda_r, 1, \ldots, 1) \) and \( a_{r+1} \leq \cdots \leq a_n \) for some integer \( r \) satisfying \( 0 \leq r \leq n \).

(ii) \( a_1 \leq \cdots \leq a_n \) and \( \lambda_i = 1 \) for all \( i \neq 1 \).
Remark 6.5.4. Colored quotient rings satisfying condition (i) are called mixed, while colored quotient ring satisfying condition (ii) are called hinged.

Proof. In view of Corollary 6.1.3, Theorem 6.2.3 and Theorem 6.5.1, we are left to show that if \( n > 1 \), \( \mathbf{a} \in \mathbb{P}^n \) satisfies \( a_i = \infty \) for some \( i \in [n] \), and \( W \) is either mixed or hinged, then \( W \) is Macaulay-Lex. (Note that the case \( n = 1 \), \( \mathbf{a} = (\infty) \) is identically Macaulay’s theorem.) Given such \( \mathbf{a} \in \mathbb{P}^n \) and \( W \), fix some \( d \in \mathbb{P} \) and define \( \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{P}^n \) by \( b_i = \min\{a_i, d + 1\} \) for each \( i \in [n] \). Since \( W \) is either mixed or hinged, the colored quotient ring \( \mathbb{k}[X_{\lambda}]/Q_{\mathbf{b}} \) is also either mixed or hinged, so we know \( \mathbb{k}[X_{\lambda}]/Q_{\mathbf{b}} \) is Macaulay-Lex. Since \( \{W_d\} = \{(\mathbb{k}[X_{\lambda}]/Q_{\mathbf{b}})_d\} \), Corollary 6.1.3 yields \( |\partial(\text{Revlex}_{W_d}(A))| \leq |\partial(A)| \) for every \( W_d \)-monomial space \( A \). This is true for all \( d \in \mathbb{P} \), therefore \( W \) is Macaulay-Lex by Corollary 6.1.3.

Notes

In 1966, Hartshorne [65] proved that Grothendieck’s Hilbert scheme is connected. More recently, Murai–Peeva [98] proved in 2012 the following analogue of Hartshorne’s theorem: The Hilbert scheme that parametrizes all graded ideals of a Clements–Lindström ring with a certain fixed Hilbert function is connected. Hartshorne’s theorem corresponds to the Macaulay-Lex ring \( S \), while Murai–Peeva’s theorem corresponds to another Macaulay-Lex ring, i.e. the Clements–Lindström ring. Although the two proofs use different constructions of paths on the corresponding Hilbert subschemes (to show connectedness), lex ideals still play a common role. In both proofs, it is shown that there is always a path from any point in the subscheme to the unique point corresponding to a lex ideal. This suggests the following very natural question:

Question 6.5.5. Let \( R \) be a ring, and let \( \mathcal{H}_R(\omega) \) be the Hilbert scheme that parametrizes all graded ideals of \( R \) with the given Hilbert function \( \omega \). Is \( \mathcal{H}_R(\omega) \) always connected whenever \( R \) is a Macaulay-Lex ring?

On a related note, we pose the following conjecture:

Conjecture 6.5.6. Let \( W \) be a colored quotient ring that is either mixed or hinged, and let \( \mathcal{H}_W(\omega) \) be the Hilbert scheme that parametrizes all graded ideals of \( W \) with the given Hilbert function \( \omega \). Then \( \mathcal{H}_W(\omega) \) is connected.
In the previous chapter, we stated without proof a key result (Theorem 6.4.3) concerning quasi-compressed monomial spaces arising from mixed or hinged colored quotient rings, and we used this result to characterize all Macaulay-Lex colored quotient rings (Theorem 6.5.3).

In this chapter, our purpose is twofold. In Chapter 7.1, we give a historical overview of the Clements–Lindström theorem (introduced in Chapter 2.8) and the Frankl–Füredi–Kalai theorem (introduced in Chapter 4.3), as well as other related theorems. In particular, we explain how Theorem 6.5.3 gives a simultaneous generalization of both the Clements–Lindström theorem and the Frankl–Füredi–Kalai theorem. In Chapter 7.2, we will prove Theorem 6.4.3.

### 7.1 Historical overview

The original version of the Clements–Lindström theorem [36] was given in terms of collections of $n$-tuples in $\mathbb{N}^n$ arranged in the lex order. We could always treat these $n$-tuples as the exponents of monomials in variables $x_1, \ldots, x_n$, and then identify collections of monomials with the corresponding $k$-vector spaces spanned by these collections. By doing so, this original version becomes, in our terminology, the following statement: If $W$ is a Clements–Lindström ring, then $\partial(\text{Revlex } W_{d+1}(A)) \subseteq \text{Revlex } W_d(\partial(A))$ for all $d \in \mathbb{N}$ and every $W_{d+1}$-monomial space $A$. In view of Theorem 5.3.1, the Clements–Lindström theorem is thus equivalent to the algebraic statement that all Clements–Lindström rings are Macaulay-Lex. For alternative
proofs of the Clements–Lindström theorem, see, e.g., [22] [48, Thm. 8.1.1] [92].

Before the notion of Macaulay-Lex rings was introduced in 2006 [91], and even before the equivalence \((i) \iff (iii)\) in Theorem 5.3.1 was explicitly proven in 2001 [109], it was already well-known to the commutative algebraists (at least since the 1980-90s, e.g. [46]) that the Clements–Lindström theorem yields an analogue of Macaulay’s theorem for Clements–Lindström rings. This is not unexpected, since one of the main underlying motivations for the Clements–Lindström theorem was to generalize Macaulay’s theorem [85].

Macaulay’s theorem [85], which we introduced in Chapter 3.6, is a well-known and fundamental result in algebra that was proven in 1927. In this 1927 paper, Macaulay not only characterized the Hilbert functions of homogeneous ideals of polynomial rings, but he also introduced the idea of taking initial ideals of homogeneous ideals (with respect to the lex order), and he showed that Hilbert functions remain invariant when taking initial ideals. Thus, Macaulay is usually credited for laying the basic foundations of Gröbner basis theory, although we note that the earliest use of Gröbner bases was by Gordan [60] in 1900; see [45, Chap. 15.6] for a historical overview of Gröbner basis theory.

In 1929, Sperner [113] gave a shorter proof of Macaulay’s theorem. Interestingly, there is an obscure paper [58], published in 1913 by a Russian mathematician N. M. Gjunter, containing the first proof of Macaulay’s theorem. Gjunter’s proof differs from the proofs by Macaulay and Sperner. In fact, Gjunter introduced the idea of taking initial ideals in [58], and subsequently in 1925, he also proved what is today called Buchberger’s criterion [59]. (Note that Buchberger proved his famous Buchberger’s criterion in his PhD thesis in 1965; see [45, Chap. 15.6].) However, Gjunter’s contributions were completely overlooked until very recently [105], and
even so, they are still probably unknown to most algebraists today. One possible explanation suggested in [105] is that Gjunter was an analyst, and his algebraic results were motivated by problems in analysis. In particular, Gjunter’s papers were categorized under analysis. See [105] for an overview of Gjunter’s contributions.

The Clements–Lindström theorem also generalizes the famous Kruskal–Katona theorem in combinatorics, which we introduced in Chapter 2.6. Note that the Kruskal–Katona theorem is equivalent to the algebraic statement that every colored quotient ring of type $1_n$ and composition $1_n$ is Macaulay-Lex. This theorem also has an interesting history. Kruskal [82] and Katona [77] independently proved this theorem in 1963 and 1968 respectively (with different proofs), unaware of each other’s result. Since then, many different and simpler proofs of the theorem have surfaced (e.g. [5, 22, 41, 42, 44, 52, 70, 78, 84, 88, 101, 125]), the shortest of which is by Daykin [41] and is only 1.5 pages long! In comparison, the original proofs by Kruskal and Katona were each over 25 pages long.

Around the 1980-90s, some combinatorialists realized that the “Kruskal–Katona theorem” was actually first proven by Schützenberger [107] in 1959 (cf. [119, Chap. II.2]). Schützenberger’s result went unnoticed, possibly because it was published in a somewhat obscure tech report dealing with information theory. In this tech report, Schützenberger noted that his result implied a numerical characterization of the $f$-vectors of simplicial complexes, and he also remarked that it was unlikely his result was new (when in fact it was). In light of this, the theorem should accurately be called the Schützenberger–Kruskal–Katona theorem. Unfortunately, by the 1980-90s, the name “Kruskal–Katona theorem” was already popular, and it continues to be used today.

The Frankl–Füredi–Kalai theorem [53], which we introduced in Chapter 4.3, is
• Characterizes the Hilbert functions of graded ideals of $\mathbb{k}[x_1, \ldots, x_n]$.
• Originally formulated in algebraic language.
• Not well-known: Earlier proven by Gjunter (1913).
• Observed by Stanley (1977) to be equivalent to a characterization of the $f$-vectors of multicomplexes.

• Simultaneous generalization of Macaulay’s theorem and the Kruskal-Katona theorem.
• Originally formulated in set-theoretic language.

• A colored analogue of Kruskal-Katona theorem.
• Characterizes the $f$-vectors of $r$-colorable complexes.
• Originally formulated in graph-theoretic language.

Figure 7.1: Brief historical overview of the Clements–Lindström theorem, the Frankl–Füredi–Kalai theorem, and other related theorems.
a colored analogue of the Kruskal–Katona theorem, and hence sometimes called the “colored Kruskal–Katona theorem”. It was originally formulated in terms of the minimum shadow sizes of $r$-colorable $k$-uniform hypergraphs, so it can also be considered as a graph-theoretic analogue of the Kruskal–Katona theorem. Note that if the vertices of a $k$-uniform hypergraph are treated as variables of a polynomial ring, then the edges in this hypergraph can be identified with degree $k$ monomials. Using our terminology, the Frankl–Füredi–Kalai theorem (as originally stated) is thus equivalent to the algebraic statement that every colored quotient ring of type $1_n$ and composition $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying $\lambda_1 = \cdots = \lambda_n$ is Macaulay-Lex. In particular, this includes the case $\lambda = 1_n$, so the Frankl–Füredi–Kalai theorem generalizes the Kruskal–Katona theorem; cf. discussion after Theorem 4.3.4.

Note that Claim 4.2(ii) in [53] is not true, hence the original proof of the Frankl–Füredi–Kalai theorem in [53] is incorrect as stated. However, London [83] gave a different (and correct) proof, Engel [48, Chap. 8] gave a proof using properties of Macaulay posets, while Mermin–Murai [89] gave a proof in the language of monomial ideals, so the statement of the theorem is still true. In particular, after translating what they proved into our terminology, we get the following: London proved that every colored quotient ring of type $1_n$ and composition $(\lambda_1, \ldots, \lambda_n)$ satisfying $\lambda_1 \leq \cdots \leq \lambda_n \leq \lambda_1 + 1$ is Macaulay-Lex, while both Engel and Mermin–Murai proved that every colored squarefree ring (i.e. a colored quotient ring of type $1_n$ and composition $(\lambda_1, \ldots, \lambda_n)$ satisfying $\lambda_1 \geq \cdots \geq \lambda_n$) is Macaulay-Lex. All three proofs include the case $\lambda_1 = \cdots = \lambda_n$ corresponding to the original statement of the Frankl–Füredi–Kalai theorem. In fact, Theorem 6.5.3 says that a colored quotient ring of type $1_n$ is Macaulay-Lex for arbitrary composition $\lambda \in \mathbb{P}^n$.

On a related note, it was pointed out by Billera–Björner [15] that the uniqueness
claim in [53, Lemma 1.1] is incorrect, which makes the $\partial_k^{(r)}(\cdot)$ operator introduced in [53] not well-defined. The numerical version of the Frankl–Füredi–Kalai theorem stated in [15] includes a fix suggested by J. Eckhoff. To the best of the author’s knowledge, all applications of the Frankl–Füredi–Kalai theorem use the non-numerical version, i.e. that every colored quotient ring of type $1_n$ and composition $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying $\lambda_1 = \cdots = \lambda_n$ is Macaulay-Lex, or equivalently, that the $f$-vectors of $r$-colorable complexes can be characterized by the $f$-vectors of certain “revlex” $r$-colorable complexes; cf. Chapter 8.

Notice that both Clements–Lindström rings and colored squarefree rings are mixed colored quotient rings. Theorem 6.5.3 says that a colored quotient ring is Macaulay-Lex if and only if it is either mixed or hinged, so in particular, every mixed colored quotient ring is Macaulay-Lex. Consequently, Theorem 6.5.3 gives a simultaneous generalization of the Clements–Lindström theorem and the Frankl–Füredi–Kalai theorem, and it also includes Macaulay’s theorem and the Kruskal–Katona theorem as special cases. In particular, mixed colored quotient rings form a new class of Macaulay-Lex rings that interpolate between Clements–Lindström rings and colored squarefree rings, and these quotient rings correspond combinatorially to joins of $1_r$-colored complexes with “uncolored” multicomplexes. At this moment, the author is not aware of any “natural” combinatorial significance for the Macaulay-Lex property of hinged colored quotient rings.

As we have mentioned earlier, colored quotient rings of type $1_n$ are Macaulay-Lex for all compositions $\lambda \in \mathbb{P}^n$, hence Theorem 6.5.3 actually extends the original statement of the Frankl–Füredi–Kalai theorem, as well as the refinements of the Frankl–Füredi–Kalai theorem proven by London, Engel, and Mermin–Murai. In contrast, the condition “$a_1 \leq \cdots \leq a_n$” assumed in the Clements–Lindström the-
orem is necessary, i.e. a colored quotient ring of type \((a_1, \ldots, a_n)\) and composition \(1_n\) is Macaulay-Lex if and only if \(a_1 \leq \cdots \leq a_n\); see Remark 6.2.2.

### 7.2 Simultaneous generalization

Throughout this section, let \(n > 1\), and let \(W = k[X_\lambda]/Q_a\) be a colored quotient ring of type \(a \in \mathbb{P}^n\) (note: \(a_i \neq \infty\)) and composition \(\lambda \in \mathbb{P}^n\). Assume that \(W\) is either mixed or hinged. Recall that \(W\) is mixed if there is some integer \(r\) satisfying \(0 \leq r \leq n\) such that \(a_i = 1\) for all \(i \in [r]\), \(\lambda_i = 1\) for all \(i \in [n] \setminus [r]\), and \(a_{r+1} \leq \cdots \leq a_n\), while \(W\) is hinged if \(a_1 \leq \cdots \leq a_n\) and \(\lambda_i = 1\) for all \(i \neq 1\).

Recall also that for any \(k\)-vector space \(U\) spanned by monomials, we write \(\{U\}\) to denote the set of monomials contained in \(U\).

Let \(d \in \mathbb{P}\), and let \(A\) be a non-empty quasi-compressed \(W_d\)-monomial space that is not revlex-segment. The purpose of this section is to prove Theorem 6.4.3, i.e. we will prove the existence of some \(W_d\)-monomial space \(A'\) satisfying \(|A'| = |A|\), \(\|A'\|_{(a,\lambda)} < \|A\|_{(a,\lambda)}\), and \(|\partial(A')| \leq |\partial(A)|\).

Note that such a \(W_d\)-monomial space \(A'\) does not necessarily exist if \(W\) is neither mixed nor hinged. Furthermore, as we have explained in Chapter 7.1, the special case of Theorem 6.4.3 when \(W\) is mixed would already simultaneously imply both the Clements–Lindström theorem [36] and the Frankl–Füredi–Kalai theorem [53]. Consequently, any proof of Theorem 6.4.3 must necessarily contain the “difficult” parts of any proofs of these two classic theorems. A quick look at the existing proofs of these two theorems would convince the reader that a long list of lemmas is expected, so it should not come as a surprise that our proof of
Theorem 6.4.3 is indeed quite long and is accompanied by eight lemmas.

We first introduce some auxiliary notation. For each $x \in X_\lambda$, let $\tau(x)$ be the unique integer in $[n]$ such that $x \in X_{\tau(x)}$, and let $\hat{\tau}(x)$ be the unique integer in $[\lambda_{\tau(x)}]$ such that $x = x_{\tau(x), \hat{\tau}(x)}$. If $X' \subseteq X_\lambda$, then let $\max_{\leq \ell} X'$ be the largest element of $X'$ in the revlex order, and define $\min_{\leq \ell} X'$ analogously. Given a monomial $m$ in $W$, define

$$\sigma(m) := \{\tau(x) : x \in \text{supp}(m)\} \subseteq [n].$$

Whenever we say $m$ is the largest (or smallest) monomial satisfying certain conditions, it is always with respect to the revlex order. If $\deg(m) = d$ and $m$ is not the smallest monomial in $W_d$, then there is some monomial $m^\dagger$ in $W_d$ such that $m$ covers $m^\dagger$ in the revlex order, i.e. there does not exist any $m' \in \{W_d\}$ satisfying $m >_{\ell} m' >_{\ell} m^\dagger$. Such a monomial $m^\dagger$ is uniquely determined by $m$, and we reserve the superscript ‘‘$\dagger$’’ for this cover relation.

Let $p$ be the smallest monomial in $W_d$ satisfying $p \not\in A$ and $p^\dagger \in A$. For each $x \in X_\lambda$, let $\rho(x)$ be the largest divisor of $p$ whose support is contained in $\{x' \in X_\lambda : x' >_{\ell} x\}$. Let $y_0 := \max_{\leq \ell}(\text{supp}(p))$, and inductively define $y_i = y_{i-1}^\dagger$ for $i \geq 1$. Let $j$ be the unique integer in $[d]$ such that $y_{j-1}^{a_{\tau(y_{j-1})}}$ divides $p$ for every $i \in [j-1]$ (if any), and $y_j^{a_{\tau(y_j)}}$ does not divide $p$. Define

$$z := \max_{\leq \ell} \left\{x \in X_\lambda : x \leq_{\ell} y_j, \frac{p}{\rho(x)} \cdot x \neq 0\right\},$$

let $p_0 := \rho(z)$, and observe that $z$ is well-defined since $p$ is not the smallest monomial in $W_d$. In fact, the definition of the revlex order tells us that $p^\dagger$ is the largest monomial in $W_d$ divisible by $\frac{p}{p_0} \cdot z$. This means we can factorize $p^\dagger = \frac{p}{p_0} \cdot z \cdot \tilde{p}_0$, where $\tilde{p}_0$ is the largest monomial of $W$ of degree $\deg(p_0) - 1$ such that $\frac{p}{p_0} \cdot z \cdot \tilde{p}_0 \neq 0$. For any variables $y, y' \in X_\lambda$ that divide $\tilde{p}_0$, the maximality of $\tilde{p}_0$ tells us that
\[\tau(y) = \tau(y')\] if and only if \(y = y'\), thus \(\text{supp}(\tilde{p}_0) = \{x_{i,1} : i \in \sigma(\tilde{p}_0)\}\).

Suppose \(m \in \{W_d\} \setminus \{A\}\) such that \(m \geq_{\text{rev}} p\). If \(\partial(\{m\}) \subseteq \{\partial(A)\}\), then by letting \(A' := \text{Span}_k((\{A\} \setminus \{p^1\}) \cup \{m\})\), we get \(|\partial(A)| \geq |\partial(A')|\), and we are done. Thus for each such \(m\), assume henceforth that \(\partial(\{m\}) \not\subseteq \{\partial(A)\}\), and define the (non-empty) set \(\Gamma_m := \{x \in \text{supp}(m) : \frac{m}{x} \not\in \{\partial(A)\}\}\).

Next, let \(p_{\text{min}}\) be the smallest monomial in \(A\), and let \(q\) be any monomial in \(W_d\) satisfying \(p_{\text{min}} \leq_{\text{rev}} q \leq_{\text{rev}} p^1\). The minimality of \(p\) implies \(q \in A\), so since \(A\) is quasi-compressed and \(n > 1\), the fact that \(p \not\in A\) forces \(p_{X_t} \neq q_{X_t}\) for all \(t \in \{n\}\). In particular, when \(q = p^1\), the factorization \(p^1 = \frac{p}{p_0} \cdot z \cdot \tilde{p}_0\) implies \(\sigma(p_0) \cup \sigma(z) \cup \sigma(\tilde{p}_0) = [n]\).

**Lemma 7.2.1.** \(\{i \in [n] : a_i = 1\} \subseteq \sigma(p_0) \cup \sigma(z)\).

**Proof.** Suppose not, then since \(\sigma(p_0) \cup \sigma(z) \cup \sigma(\tilde{p}_0) = [n]\), there exists some \(t \in [n]\) such that \(a_t = 1\) and \(t \in \sigma(\tilde{p}_0) \cap ([n] \setminus (\sigma(p_0) \cup \sigma(z)))\). Note that \(\text{supp}(\tilde{p}_0) = \{x_{i,1} : i \in \sigma(\tilde{p}_0)\}\) implies \(x_{t,1}\) divides \(\tilde{p}_0\), so the factorization \(p^1 = \frac{p}{p_0} \cdot z \cdot \tilde{p}_0\) gives us \(\frac{p}{p_0} \cdot x_{t,1} \neq 0\). Choose some \(y \in \Gamma_p\). Since \(t \not\in \sigma(p_0)\), we thus get \(\frac{p}{y} \cdot x_{t,1} \neq 0\), hence \(\frac{p}{y} \cdot x_{t,1} \not\in A\) by the definition of \(\Gamma_p\).

Next, write \(m := \frac{p}{y} \cdot x_{t,1}\). We check that \(\min_{\leq_{\text{rev}}} \{y, z\}\) is the smallest variable whose exponents in \(p^1 = \frac{p}{p_0} \cdot z \cdot \tilde{p}_0\) and \(m\) are different, so the definition of the revlex order yields \(p^1 <_{\text{rev}} m\). Now, \(a_t = 1\) implies \(m_{X_t} = (p^1)_{X_t} = x_{t,1}\), so since \(m \not\in A\), we get \(C_t(A) \neq A\), which contradicts the assumption that \(A\) is quasi-compressed. \(\square\)

**Lemma 7.2.2.** Let \(m, m'\) be monomials in \(W_d\) such that \(m' = \frac{m}{x} \cdot x'\) for some \(x \in \text{supp}(m)\) and \(x' \in X_\lambda\) satisfying \(x' \geq_{\text{rev}} x\). If \(n > 2\) and \(m \in A\), then \(m' \in A\).
Proof. If \( n > 2 \), then we can find some \( t \in [n] \), distinct from \( \tau(x) \) and \( \tau(x') \), such that \( m_{X_t} = m'_{X_t} \). Since \( x' \geq_{r_t} x \) implies \( m' \geq_{r_t} m \), it then follows from \( C_\ell(A) = A \) that \( m \in A \) implies \( m' \in A \). \( \square \)

**Lemma 7.2.3.** If \( n > 2 \) and \( a_n \neq 1 \), then \( \frac{p}{p_0} \cdot z \cdot x_{n,1} \) divides every \( q \in \{ W_d \} \) satisfying \( p_{\min} \leq_{r_t} q \leq_{r_t} p^\dagger \).

Proof. Let \( p'_0 \) be the smallest monomial in \( W \) of degree \( \deg(p_0) - 1 \), whose support is contained in \( \rho(z) \), such that \( \frac{p}{p_0} \cdot z \cdot p'_0 \neq 0 \). Clearly such a monomial \( p'_0 \) exists, since \( \pi_0 \) is the largest monomial satisfying the same properties. We first show that \( \frac{p}{p_0} \cdot z \cdot p'_0 \notin A \). Note that

\[
\frac{p}{p_0} \cdot z \cdot p'_0 \leq_{r_t} \frac{p}{p_0} \cdot z \cdot \pi_0 = p^\dagger <_{r_t} p = \frac{p}{p_0} \cdot p_0,
\]

hence \( z \cdot p'_0 <_{r_t} p_0 \). Let \( v_1 <_{r_t} v_2 \leq_{r_t} \cdots \leq_{r_t} v_{\deg(p_0)} = y_0 \) be the \( \deg(p_0) \) uniquely determined variables such that \( p_0 = v_1v_2 \cdots v_{\deg(p_0)} \). Similarly, let \( z = v'_1 <_{r_t} v'_2 \leq_{r_t} \cdots \leq_{r_t} v'_{\deg(p_0)} \) be the \( \deg(p_0) \) uniquely determined variables such that \( z \cdot p'_0 = v'_1v'_2 \cdots v'_{\deg(p_0)} \). The minimality of \( p'_0 \) implies \( v'_i \leq v_i \) for each \( i \in [\deg(p_0)] \), so since \( p \notin A \), the repeated use of Lemma 7.2.2 yields \( \frac{p}{p_0} \cdot z \cdot p'_0 \notin A \) as claimed.

Consequently, the minimality of \( p_{\min} \) implies \( p_{\min} >_{r_t} \frac{p}{p_0} \cdot z \cdot p'_0 \), so in particular, \( \frac{p}{p_0} \cdot z \) divides every monomial \( q \) in \( W_d \) satisfying \( p_{\min} \leq_{r_t} q \leq_{r_t} p^\dagger \), and we can factorize each such \( q \) by \( q = \frac{p}{p_0} \cdot z \cdot \tilde{q} \), where \( \tilde{q} \) is a monomial in \( W \) of degree \( \deg(p_0) - 1 \) satisfying \( \text{supp}(\tilde{q}) \subseteq \{ x \in X_\lambda : x >_{r_t} z \} \).

Suppose we can choose some \( q \) that is not divisible by \( x_{n,1} \). Since \( a_n \neq 1 \) implies \( \lambda_n = 1 \), we get \( q_{X_n} = 1 \), thus \( p_{X_n} \neq 1 \), i.e. \( x_{n,1} \) divides \( p \). Let \( b = \deg(p_{X_n}) \) (i.e. \( p_{X_n} = x_{n,1}^b = y_0^b \)), and note that \( b \leq \deg(p_0) \). Let \( q' \) be the largest degree \( b \) divisor of \( q \). If \( b < \deg(p_0) \), then \( q' \) divides \( \tilde{q} \), hence \( \frac{p}{p_0} \cdot z \cdot \tilde{q} \cdot x_{n,1}^b \) is a monomial in \( W_d \)

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satisfying \( p_{\text{min}} \leq r \ell \frac{p}{p_0} \cdot z \cdot \frac{q}{q} \cdot x_{n,1}^b < r \ell p \), thus \( \frac{p}{p_0} \cdot z \cdot \frac{q}{q} \cdot x_{n,1}^b \in A \), which gives the contradiction \( (\frac{p}{p_0} \cdot z \cdot \frac{q}{q} \cdot x_{n,1}^b) X_n = pX_n = x_{n,1}^b \).

Therefore, the equality \( b = \deg(p_0) \) must hold, i.e. \( p_0 = x_{n,1}^b \), which forces \( \tilde{p}_0 = x_{n,1}^{b-1} \). However, the fact that \( \sigma(p_0) \cup \sigma(z) \cup \sigma(\tilde{p}_0) = [n] \) would contradict the assumption \( n > 2 \).

**Lemma 7.2.4.** If \( p \neq p_0 \), then \( W \) is hinged, \( a_1 \neq 1 \), and \( \tau(z) = 1 \).

**Proof.** Suppose there exists some \( y \in \text{supp}(\frac{p}{p_0}) \). The definition of \( p_0 \) tells us that \( z \) is strictly smaller than every variable in \( \text{supp}(p_0) \) or \( \text{supp}(\tilde{p}_0) \), so it follows from \( \sigma(p_0) \cup \sigma(z) \cup \sigma(\tilde{p}_0) = [n] \) that \( y \leq r \ell z \leq r \ell x_{1,1} \). If \( a_1 = 1 \), then whether \( W \) is mixed or hinged, it follows from \( y \leq r \ell x_{1,1} \) that \( a_{\tau(y)} = 1 \), which forces the contradiction \( px_{\tau(y)} = (p^\dagger)x_{\tau(y)} = y \). Thus \( a_1 \neq 1 \), therefore \( W \) is hinged and \( \tau(z) = 1 \).

**Lemma 7.2.5.** If \( a_n \neq 1 \) and \( x_{n,1}^{a_n} \) divides \( p \), then \( p_0 = x_{n,1}^{a_n} \), \( z = x_{n-1,1} \), and \( n = 2 \).

**Proof.** Clearly if \( x_{n,1}^{a_n} \) divides \( p \), then \( x_{n,1}^{a_n} \) divides \( p_0 \). Furthermore, if \( \deg(p_0) > a_n \), then \( \deg(\tilde{p}_0) \geq a_n \), which yields the contradiction \( px_n = (p^\dagger)x_n = x_{n,1}^{a_n} \), where the last equality follows from the fact that \( a_n \neq 1 \) implies \( \lambda_n = 1 \). Thus \( p_0 = x_{n,1}^{a_n} \).

Next, we show that \( z = x_{n-1,1} \). Suppose instead \( z < r \ell x_{n-1,1} \), then by the maximality of \( z \), we get \( \deg(p_{X_{n-1}}) = a_{n-1} \), so the definition of \( j \) forces \( \lambda_{n-1} > 1 \). Also, since \( n - 1 \not\in \sigma(p_0) \) implies \( \frac{p}{p_0} \neq 1 \), Lemma 7.2.4 yields \( W \) is hinged, thus \( n = 2 \) in this case, and by the maximality of \( z \), the largest variable in \( \text{supp}(p_{X_{n-1}}) = \text{supp}(p_{X_1}) \) must divide \( p_0 \), which is a contradiction. Therefore \( z = x_{n-1,1} \), and so \( \tilde{p}_0 = x_{n,1}^{a_n-1} \). Finally, since \( \sigma(p_0) \cup \sigma(z) \cup \sigma(\tilde{p}_0) = [n] \), we get \( n = 2 \).

**Lemma 7.2.6.** If \( y \in \Gamma_p \), then either \( \tau(y) = \tau(z) \) or \( a_{\tau(y)} > 1 \).
Proof. Assume there is some \( y \in \Gamma_p \) such that \( \tau(y) \neq \tau(z) \) and \( a_{\tau(y)} = 1 \). Note that \( \frac{p}{y} \cdot x_{\tau(y),1} \) is a monomial in \( \{ W_d \} \setminus \{ A \} \) that satisfies \( \frac{p}{y} \cdot x_{\tau(y),1} \geq_{r\ell} p^\dagger \). This implies \( \tau(y) \notin \sigma(\tilde{p}_0) \), since otherwise \( a_{\tau(y)} = 1 \) forces \( \left( \frac{p}{y} \cdot x_{\tau(y),1} \right)_{x_{\tau(y)}} = (p^\dagger)_{x_{\tau(y)}} = x_{\tau(y),1} \), which contradicts the assumption that \( A \) is quasi-compressed.

Next, we show that \( \{ i \in [n] : a_i = 1 \} \neq [n] \). Suppose not, then by Lemma 7.2.1, either (i): \( \sigma(p_0) = [n], \tau(z) \in \sigma(p_0), \) and \( \deg(p_0) = n \); or (ii): \( \sigma(p_0) = [n] \setminus \sigma(z), \tau(z) \notin \sigma(p_0), \) and \( \deg(p_0) = n - 1 \). Note that \( p = p_0 \) for both cases. In case (i), the definition of \( \tilde{p}_0 \) yields \( \deg(\tilde{p}_0) = n - 1 \) and \( \tau(z) \notin \sigma(\tilde{p}_0) \), so it follows from \( \tau(y) \notin \sigma(\tilde{p}_0) \) that \( \tau(y) = \tau(z) \), which is a contradiction. In case (ii), note that if \( \hat{\tau}(z) = 1 \), then \( z = x_{1,1} \) and \( p = x_{2,1} \ldots x_{n,1} \), which contradict the assumptions that \( px_n \neq (p^\dagger)x_n \) and \( \partial(\{ p \}) \notin \{ \partial(A) \} \) when \( n > 2 \) and \( n = 2 \) respectively. Hence \( \hat{\tau}(z) > 1 \), and \( \frac{p}{y} \cdot x_{\tau(z),1} \) is a monomial in \( \{ W_d \} \setminus \{ A \} \) that satisfies \( \frac{p}{y} \cdot x_{\tau(z),1} \geq_{r\ell} p^\dagger \). Since \( \tau(y) \notin \sigma(\tilde{p}_0) \), we then get the contradiction \( \left( \frac{p}{y} \cdot x_{\tau(z),1} \right)_{x_{\tau(y)}} = (p^\dagger)_{x_{\tau(y)}} = 1 \).

Thus, \( \{ i \in [n] : a_i = 1 \} \neq [n] \) as claimed, and in particular, whether \( W \) is mixed or hinged, we must have \( a_n \neq 1 \) and \( \lambda_n = 1 \). It is easy to see that \( x_{n,1}^{a_n} \) divides \( p \), since otherwise \( \frac{p}{y} \cdot x_{n,1} \) is a monomial in \( \{ W_d \} \setminus \{ A \} \) that satisfies \( \frac{p}{y} \cdot x_{n,1} \geq_{r\ell} p^\dagger \), which gives the contradiction \( \left( \frac{p}{y} \cdot x_{n,1} \right)_{x_{\tau(y)}} = (p^\dagger)_{x_{\tau(y)}} = 1 \). Consequently, Lemma 7.2.5 yields \( p_0 = x_{n,1}^{a_n}, z = x_{n-1,1}, \) and \( n = 2 \), so the assumption \( a_{\tau(y)} = 1 \) forces \( a_{n-1} = a_1 = 1, \tau(y) = 1, \) and \( y \in \supp(\frac{p}{p_0}) \). Yet, the existence of \( y \) implies \( p = p_0 \), which contradicts Lemma 7.2.4.

\( \square \)

Lemma 7.2.7. Suppose \( \{ i \in [n] : a_i = 1 \} = [n] \), and suppose \( \Gamma_p \) contains a unique element \( z' \). If \( m \) is a monomial in \( W_d \) that is divisible by \( \frac{p}{z'} \) and satisfies \( m \geq_{r\ell} p \), then \( |\partial(\{ m \}) - \partial(\{ A \})| = 1 \).

Proof. From the assumption \( \{ i \in [n] : a_i = 1 \} = [n] \), Lemma 7.2.6 tells us \( \tau(z') = \frac{p}{z'} \).
\[ \tau(z), \] while Lemma 7.2.4 gives us \( p = p_0 \), hence \( \tau(z') \in \sigma(p_0) \), and Lemma 7.2.1 yields \( \sigma(p) = \sigma(p_0) = [n] \), so we must have \( d = n \). Write \( p' = \frac{p}{\tau} \) and fix some monomial \( m = p' \cdot y' \) in \( W_d \), where \( y' \in X_\Lambda \) satisfies \( y' \geq_{\tau \ell} z' \). Note that \( d = n \) implies \( \tau(y') = \tau(z') \). The case \( \text{supp}(m) = \{y'\} \) is trivial, so assume not, and fix some \( y \in \text{supp}(m) \) such that \( y \neq y' \). In particular, \( y \neq y' \) implies \( \tau(y) \neq \tau(y') \).

Clearly \( y \in \text{supp}(p') \), so since \( a_{\tau(z')} = 1 \) means \( z' \) does not divide \( p' \), we get \( y \neq z' \), hence the uniqueness of \( z' \) yields \( \frac{p}{y} \in \partial(A) \). Consequently, the fact that \( A \) is quasi-compressed implies \( \tilde{p} := \frac{p}{y} \cdot x_{\tau(y),1} \in \{A\} \).

To prove this lemma, we have to show that \( \frac{m}{y} \in \partial(A) \). Note that \( \frac{m}{y} = \frac{y' \cdot y}{x_{\tau(y),1}} = \frac{p}{x_{y}} \cdot y' \) (where \( \frac{p}{x_{y}} \) is a monomial), thus \( \frac{m}{y} \cdot x_{\tau(y),1} = \frac{p}{x_{y}} \cdot y' \in \{W_d\} \). Now since \( \tilde{p} \in \{A\} \), \( y' \geq_{\tau \ell} z' \), \( \tau(y') = \tau(z') \), and since \( A \) is quasi-compressed, we get \( \frac{m}{y} \cdot x_{\tau(y),1} \in \{A\} \), i.e. \( \frac{m}{y} \in \partial(A) \).

**Lemma 7.2.8.** If \( n > 2 \) and \( a_n \neq 1 \), then there exists a monomial \( \tilde{m} \in W_d \) not contained in \( A \), such that \( \tilde{m} \geq_{\tau \ell} p \) and \( \Gamma_{\tilde{m}} = \{x_{n,1}\} \).

**Proof.** Fix some \( y \in \Gamma_p \). Since \( n > 2 \), \( a_n \neq 1 \) and hence \( \lambda_n = 1 \), it follows from Lemma 7.2.5 that \( x_{n,1}^{a_n} \) does not divide \( p \), thus \( \frac{p}{y} \cdot x_{n,1} \) and \( m_0 := \frac{p}{y_0} \cdot x_{n,1} \) are monomials in \( W_d \). The definition of \( \Gamma_p \) implies \( \frac{p}{y} \cdot x_{n,1} \not\in A \), so Lemma 7.2.2 yields \( m_0 \not\in A \), which in particular means \( x_{n,1} \in \Gamma_{m_0} \). Write \( b_0 := \deg((m_0)_{x_n}) \), \( b := \min\{a_n, \deg(p_0) - 1\} \), and note that \( (p^\dagger)_{x_n} = x_{n,1}^b \) by construction, so \( (m_0)_{x_n} \neq (p^\dagger)_{x_n} \) implies \( b_0 \neq b \).

If \( y_0 = x_{n,1} \), then \( m_0 = p \), which means \( x_{n,1}^{b_0} \) divides \( p_0 \), hence we get that \( b_0 \leq \min\{a_n, \deg(p_0)\} \). Also, we either have \( \deg(p_0) - 1 \geq a_n \), or \( b_0 < \deg(p_0) \), since otherwise we would get \( p_0 = x_{n,1}^{b_0} \), which forces \( \tilde{p}_0 = x_{n,1}^{b_0 - 1} \), and the fact that \( \sigma(p_0) \cup \sigma(z) \cup \sigma(\tilde{p}_0) = [n] \) would then contradict the assumption \( n > 2 \). Thus, in

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either case, it follows from \( b_0 \neq b \) that \( 1 \leq b_0 < b \). If instead \( y_0 \neq x_{n,1} \), then since Lemma 7.2.3 yields \( b \neq 0 \), we get \( 1 = b_0 < b \). Consequently, whether \( y_0 = x_{n,1} \) or \( y_0 \neq x_{n,1} \), we always have \( 1 \leq b_0 < b \).

Now if \( \Gamma_{m_0} = \{ x_{n,1} \} \), then \( \tilde{m} = m_0 \) is our desired monomial. If not, then let \( v_0 \) be the largest variable in \( \Gamma_{m_0} \setminus \{ x_{n,1} \} \) and define \( m_1 := \frac{m_0}{v_0} \cdot x_{n,1} \), which is a monomial in \( W_d \) with \( (m_1)_{X_n} = x^{b_0+1}_{n,1} \). Since \( m_0 \notin A \), Lemma 7.2.2 tells us that \( m_1 \notin A \), so it follows from \( m_1 >_{rt} m_0 >_{rt} p^\dagger \) that \( (m_1)_{X_n} \neq (p^\dagger)_{X_n} \), i.e. \( b_0 + 1 < b \).

In general, if \( \Gamma_{m_i} \neq \{ x_{n,1} \} \) and \( b_0 + i < b \) for some \( i \), then we let \( v_i \) be the largest variable in \( \Gamma_{m_i} \setminus \{ x_{n,1} \} \) (which is well-defined since \( \Gamma_{m_i} \) is non-empty) and define \( m_{i+1} := \frac{m_i}{v_i} \cdot x_{n,1} \), which is a monomial in \( W_d \) with \( (m_{i+1})_{X_n} = x^{b_0+i+1}_{n,1} \). The same argument as before gives \( m_{i+1} \notin A \) and \( b_0 + i + 1 < b \). Repeat this process until we get some integer \( i' < b - b_0 \) such that \( \Gamma_{m_{i'}} = \{ x_{n,1} \} \). Then \( \tilde{m} = m_{i'} \) is our desired monomial.

\[ \square \]

**Proof of Theorem 6.4.3.** Consider the three cases (i): \( \{ i \in [n] : a_i = 1 \} = [n] \); (ii): \( \{ i \in [n] : a_i = 1 \} \neq [n], n > 2 \); and (iii): \( \{ i \in [n] : a_i = 1 \} \neq [n], n = 2 \).

**Case (i):** Suppose \( \{ i \in [n] : a_i = 1 \} = [n] \), then Lemma 7.2.6 says \( \Gamma_p \) contains a unique element \( z' \), which satisfies \( \tau(z') = \tau(z) \). Lemma 7.2.4 implies \( p = p_0 \), thus \( \tau(z') \in \sigma(p_0) \), and Lemma 7.2.1 gives \( \sigma(p) = \sigma(p_0) = [n] \), which implies \( d = n \). Let \( \tilde{y} := \max_{\leq rt} (\text{supp}(p_{\text{min}})) \) and \( \tilde{q} := \frac{p_{\text{min}}}{\tilde{y}} \in \partial(\{A\}) \). Note that \( z = \min_{\leq rt} (\text{supp}(p^\dagger)) \), so since \( p_{\text{min}} \leq_{rt} p^\dagger \), we get \( \min_{\leq rt} (\text{supp}(\tilde{q})) \leq_{rt} z \), thus \( \tilde{q} \neq \frac{p}{z} \). Define the sets of monomials

\[ \mathcal{A}' := \{ m \in \{ W_d \} : m \geq_{rt} p, \frac{p}{z} \text{ divides } m \}, \]

\[ \mathcal{A}'' := \{ m \in \{ A \} : \tilde{q} \text{ divides } m \}. \]
For each variable \( x \in X_\lambda \), recall that \( \hat{\tau}(x) \) is the unique integer in \([\lambda_{\tau(x)}]\) satisfying \( x = x_{\tau(x)}, \hat{\tau}(x) \). Since \( d = n \) and \( A \) is quasi-compressed, we get \( |A'| = \hat{\tau}(z') \) and \( |A''| = \hat{\tau} (\tilde{y}) \).

Next, we show \( |A'| \geq |A''| \). Suppose instead \( \hat{\tau}(z') < \hat{\tau} (\tilde{y}) \). Since \( p = p_0 \) and \( a_{\tau(z')} = 1 \), the maximality of \( z \) gives \( \hat{\tau}(z') = \hat{\tau}(z) - 1 \), so \( \hat{\tau}(z) \leq \hat{\tau}(\tilde{y}) \). If \( \tau(\tilde{y}) \neq \tau(z) \), then \( d = n \) implies \( \min_{\leq r \ell} (\text{supp}(p_{\text{min}})) \leq r \ell \ z \), with equality holding if and only if \( \sigma(p_{\text{min}}) = [n] \) and \( \hat{\tau}(x) = \hat{\tau}(z) \) for all \( x \in \text{supp}(p_{\text{min}}) \). If \( \tau(\tilde{y}) = \tau(z) \), then \( \tilde{y} \leq r \ell \ z \), and \( d = n \geq 2 \) implies \( \min_{\leq r \ell} (\text{supp}(p_{\text{min}})) < r \ell \max_{\leq r \ell} (\text{supp}(p_{\text{min}})) \leq r \ell \ z \). In either case, \( p_{\text{min}} \leq r \ell \frac{p}{z} \cdot z < r \ell \ p \), hence \( \frac{p}{z} \cdot z \in A \), which yields the contradiction \( (\frac{p}{z} \cdot z)_{X_i} = p_{X_i} \) for all \( t \in [n] \{ \tau(z) \} \).

Now let \( A := (\{A\} \cup A')\setminus A'' \), and note that \( |A| = |A| + |A'| - |A''| \geq |A| \). Let \( A' \) be the \( W_\ell \)-monomial space spanned by the \( |A| \) largest monomials in \( A \), and observe that \( ||A'||_{(a,\lambda)} < ||A||_{(a,\lambda)} \) by construction. Finally, Lemma 7.2.7 tells us that \( \partial(A') - \partial(\{A\}) = \{ \frac{p}{z} \} \), so since \( \tilde{q} \neq \frac{p}{z} \), we get

\[
|\partial(A')| \leq |\partial(A)| + 1 - 1 = |\partial(A)|,
\]

and we are done with this case.

Case (ii): Suppose instead \( \{i \in [n] : a_i = 1\} \neq [n] \) and \( n > 2 \). Note that \( a_n \neq 1 \), so Lemma 7.2.8 tells us there exists a monomial \( \tilde{m} \) in \( W_\ell \) not contained in \( A \), such that \( \tilde{m} \geq r \ell \ p \) and \( \Gamma_{\tilde{m}} = \{x_{n,1}\} \). This means \( x_{n,1} \) divides \( \tilde{m} \), and \( \partial(\{A\} \cup \{\tilde{m}\}) = (\partial(\{A\})) \cup \{\frac{\tilde{m}}{x_{n,1}}\} \). Lemma 7.2.3 says \( x_{n,1} \) divides \( p_{\text{min}} \), so the minimality of \( p_{\text{min}} \) yields \( \frac{p_{\text{min}}}{x_{n,1}} \notin \partial(\{A\}\setminus \{p_{\text{min}}\}) \). Furthermore, Lemma 7.2.3 also tells us \( \frac{p}{p_0} \cdot z \) divides \( p_{\text{min}} \), so it follows from \( \tilde{m} \geq r \ell \ p = \frac{p}{p_0} \cdot p_0 \) that the exponents of \( z \) in \( p_{\text{min}} \) and \( \tilde{m} \) are different. Consequently, \( \frac{p_{\text{min}}}{x_{n,1}} \neq \frac{\tilde{m}}{x_{n,1}} \), thus

\[
\partial((\{A\}\setminus \{p_{\text{min}}\}) \cup \{\tilde{m}\}) \subseteq \left( (\partial(\{A\}) \setminus \{\frac{p_{\text{min}}}{x_{n,1}}\} \right) \cup \{\frac{\tilde{m}}{x_{n,1}}\}.
\]
Therefore, if $A' := \text{Span}_k ((\{A\}\backslash\{p_{\text{min}}\}) \cup \{\tilde{m}\})$, then we have $|A'| = |A|$, $\|A'\|_{(a,\lambda)} < \|A\|_{(a,\lambda)}$, and $|\partial(A')| \leq |\partial(A)|$.

Case (iii): Finally, consider the case $\{i \in [n] : a_i = 1\} \neq [n]$ and $n = 2$. Equivalently, we have the conditions $a_1 \leq a_2$, $a_2 > 1$, and $\lambda_2 = 1$. It is an easy consequence of Macaulay's theorem that $Q_{(a_1)} = (X_1)^{a_1+1}$ is a Macaulay-Lex ideal of $k[X_1]$ with respect to the linear order $x_{1,1} < \cdots < x_{1,\lambda_1}$ on its variables (see Chapter 6.3), hence the colored quotient ring $k[X_\lambda]/Q_{(a_1,\infty)}$ is Macaulay-Lex by Theorem 5.3.3(iv). If $d \leq a_2$, then $W_d = (k[X_\lambda]/Q_{(a_1,\infty)})d$, which implies

$$\text{Revlex}_{W_d}(A) = \text{Revlex} (k[X_\lambda]/Q_{(a_1,\infty)})_d(A),$$

so Corollary 6.1.3 yields $|\partial(\text{Revlex}_{W_d}(A))| \leq |\partial(A)|$, and we are done by letting $A' = \text{Revlex}_{W_d}(A)$.

Assuming $a_2 < d \leq a_1 + a_2$, it follows from $a_1 \leq a_2$ that $x_{2,1}$ divides every monomial in $W_d$, and we note that $p_{\text{min}}$ is the largest monomial in $W_d$ divisible by $\frac{p_{\text{min}}}{x_{2,1}}$. Consequently, if we can find a monomial $\tilde{m} \in \{W_d\}\backslash\{A\}$ such that $\tilde{m} \geq_{rl} p$ and $|\Gamma_{\tilde{m}}| = 1$, then $A' := \text{Span}_k ((\{A\}\backslash\{\tilde{m}\})\backslash\{p_{\text{min}}\})$ satisfies $|A'| = |A|$, $\|A'\|_{(a,\lambda)} < \|A\|_{(a,\lambda)}$, and

$$|\partial(A')| \leq |\partial(A)| + |\Gamma_{\tilde{m}}| - \left|\left\{\frac{p_{\text{min}}}{x_{2,1}}\right\}\right| = |\partial(A)|,$$

and we would be done.

We will show that such a monomial $\tilde{m}$ exists. For each $m \in \{W_d\}\backslash\{A\}$ such that $m \geq_{rl} p$, let $m_{\text{max}}$ denote the largest monomial in $\{W_d\}\backslash\{A\}$ satisfying $(m_{\text{max}})_{X_2} = m_{X_2}$. Note that $\frac{m_{\text{max}}}{x} \cdot x_{1,1} \geq_{rl} m_{\text{max}}$ for each $x \in \text{supp}(m_{\text{max}}) \cap X_1$, with equality holding if and only if $x = x_{1,1}$, thus the maximality of $m_{\text{max}}$ implies $\Gamma_{m_{\text{max}}} \cap X_1 \subseteq \{x_{1,1}\}$, i.e. $\Gamma_{m_{\text{max}}} \subseteq \{x_{1,1}, x_{2,1}\}$. If $x \in \{x_{1,1}, x_{2,1}\}$, then since $\Gamma_{m_{\text{max}}} =$
assumed to be non-empty, we get $\Gamma_{m_{\text{max}}} = \{x_{2,1}\}$, so we can let $\tilde{m} = m_{\text{max}}$ and we are done. Consequently, we assume henceforth that $x_{1,1} \in \Gamma_{m_{\text{max}}}$ for all monomials $m \in \{W_d\} \setminus \{A\}$ satisfying $m \geq r_{\ell}$. Let $p^*$ denote the largest monomial in $\{W_d\} \setminus \{A\}$. Among all pairs $(q, q^\dagger)$ of consecutive monomials in $W_d$ such that $q \not\in A$ and $q^\dagger \in A$, choose a pair for which $\deg(q_{X_1})$ is minimized. Clearly $p^* = p^*_{\text{max}}$, and note that $x_{1,1}$ is, by assumption, contained in each of $\Gamma_{p^*}$ and $\Gamma_{q_{\text{max}}}$. In particular, $x_{1,1} \in \Gamma_{p^*}$ implies $x_{2,1}^{a_2}$ divides $p^*$, since otherwise we would have $\frac{p^*}{x_{1,1}^i} \cdot x_{2,1} \in \{W_d\} \setminus \{A\}$ by the definition of $\Gamma_{p^*}$, and the fact that $\frac{p^*}{x_{1,1}^i} \cdot x_{2,1} > r_{\ell} p^*$ would then contradict the maximality of $p^*$.

Next, we show that $\deg(q_{X_1}) < a_1$. Suppose instead $\deg(q_{X_1}) = a_1$. For every $0 \leq i \leq a_1 + a_2 - d$, define $m_i := \frac{p^*}{x_{2,1}^i} \cdot x_{1,1}^i$, which is a monomial in $W_d$ since the fact that $x_{2,1}^{a_2}$ divides $p^*$ implies $\deg((m_i)_{X_1}) = d - a_2 + i$. Observe that

$$m_0 > r_{\ell} m_1 > r_{\ell} \cdots > r_{\ell} m_{a_1 + a_2 - d} \quad (7.6)$$

are $(a_1 + a_2 - d + 1)$ consecutive monomials in $W_d$, so since $p^* \geq r_{\ell} q_{\text{max}}$ and $\deg((m_{a_1 + a_2 - d})_{X_1}) = \deg((q_{\text{max}})_{X_1}) = a_1$ by assumption, our choice of $(q, q^\dagger)$ (where $\deg(q_{X_1})$ is minimized) implies that these $(a_1 + a_2 - d + 1)$ consecutive monomials are all not in $A$. However, $q_{\text{max}} > r_{\ell} p^\dagger$, and there exists some $0 \leq i' \leq a_1 + a_2 - d$ such that $(m_{i'})_{X_2} = (p^\dagger)_{X_2}$, which contradicts the assumption that $A$ is quasi-compressed. Thus, $\deg(q_{X_1}) < a_1$ as claimed.

Consequently, we have $\deg((q_{\text{max}})_{X_1}) < a_1$, which implies $q^\dagger = \frac{q}{x_{2,1}} \cdot x_{1,1}$ and $(q_{\text{max}})^\dagger \equiv \frac{q_{\text{max}}}{x_{2,1}} \cdot x_{1,1}$, so $((q_{\text{max}})^\dagger)_{X_2} = (q^\dagger)_{X_2}$. Now, $q^\dagger \in A$, and $A$ is quasi-compressed by assumption, hence $(q_{\text{max}})^\dagger \in A$, thus $\frac{q_{\text{max}}}{x_{2,1}} \in \partial(A)$ and $x_{2,1} \not\in \Gamma_{q_{\text{max}}}$, i.e. $\Gamma_{q_{\text{max}}} = \{x_{1,1}\}$, therefore we can let $\tilde{m} = q_{\text{max}}$. □
Notes

Theorem 6.5.3 gives a simultaneous generalization of the Clements–Lindström theorem and the Frankl–Füredi–Kalai theorem, so it includes the Kruskal–Katona theorem and Macaulay’s theorem as special cases too. Although stated in terms of the algebraic notion of Macaulay-Lex rings, there is nothing inherently algebraic about our actual proof for the Macaulay-Lex property of mixed and hinged colored quotient rings. This is evident in Chapter 7.2, where we proved Theorem 6.4.3 in a “very combinatorial” manner. We could have given an equivalent formulation of Theorem 6.5.3 in set theoretic language.

Recall from Theorem 5.3.1 that \( R = \bigoplus_{d \in \mathbb{N}} R_d \) is a Macaulay-Lex ring if and only if
\[
\partial(\text{Revlex}_{R_{d+1}}(A)) \subseteq \text{Revlex}_{R_d}(\partial(A))
\]
for all \( d \in \mathbb{N} \) and every \( R_{d+1} \)-monomial space \( A \). Inherent in the definition of Macaulay-Lex rings is the notion of the lex order for upper shadows, or equivalently, the notion of the revlex order for lower shadows. What about other monomial orders? More generally, if we are working with collections of \( k \)-subsets (of a given set) instead of ideals and monomial spaces, what if we replace the lex order (or revlex order) by an arbitrary linear order? Is there an analogous notion of Macaulay-“Lex” rings with respect to other linear orders?

The answer is yes. There is the notion of “Macaulay posets”. Macaulay posets, which were first introduced and systematically studied by Engel [48] in the context of Sperner theory (an area within extremal combinatorics), can be thought of as the essential combinatorial structure for which Kruskal–Katona-type theorems can be proven. Before defining these Macaulay posets, we give some additional poset terminology.

A function \( \rho : P \to \mathbb{N} \) on a poset \((P, \leq)\) is called a rank function if \( \rho(u) = 0 \) for some (not necessarily unique) minimal element \( u \in P \), and \( \rho(y) = \rho(x) + 1 \) whenever \( x \prec y \). Note that rank functions on posets do not always exist, but if there is a rank function \( \rho \) on a poset \( P \), then we say \( P \) is a ranked poset. Given \( \rho \), the rank of each element \( x \in P \) is the integer \( \rho(x) \), and if for every \( d \in \mathbb{N} \), there are only finitely many elements of rank \( d \), then we say \( P \) is a finitely ranked poset. Notice that every graded poset is ranked, but not every ranked poset is graded. (For example, the pairs of integers \((0,0), (1,0), (2,0), (2,-1)\) ordered by \( \leq \) form a poset that admits the rank function \( \rho((x,y)) = x + y \), but this ranked poset is not graded.)

Let \((P, \leq)\) be a finitely ranked poset with rank function \( \rho : P \to \mathbb{N} \), and for each \( d \in \mathbb{N} \), let \( P_d \) denote the subposet of \( P \) induced by the elements in \( P \) of rank \( d \). Given \( d \in \mathbb{P} \) and a subset \( A \subseteq P_d \), the lower shadow of \( A \) is defined by
\[
\partial(A) := \{ a \in P_{d-1} : a < b \text{ for some } b \in A \}.
\]

\[1\]There is another class of poset with a very similar name called Cohen–Macaulay posets. Macaulay posets and Cohen–Macaulay posets are completely different posets studied in different contexts.
For any well-ordering $\preceq$ on $P$ (not necessarily a linear extension of $\leq$), let $C_{\preceq}(A)$ be the initial segment of $P_d$ (with respect to $\preceq$) of size $|A|$. Note that $C_{\preceq}(A)$ is well-defined, since $P$ is assumed to be finitely ranked.

A finitely ranked poset $(P, \leq)$ is called a Macaulay poset if there exists a well-ordering $\preceq$ on $P$, which we call the Macaulay order, such that

$$\partial(C_{\preceq}(A)) \subseteq C_{\preceq}(\partial(A))$$

for all $d \in P$ all subsets $A \subseteq P_d$. Note that $\preceq$ is not necessarily a linear extension of $\leq$. The triple $(P, \leq, \preceq)$ is called a Macaulay structure. For more information on Macaulay posets, see, e.g., [48, Chap. 8], [10], [11], [12].

Thus, for every Macaulay-Lex ring $R$, there is an underlying Macaulay structure on the monomials in $R$. As an example, the Kruskal–Katona theorem, which can be formulated as the statement that $S/\langle x_1^2, \ldots, x_n^2 \rangle$ is a Macaulay-Lex ring, can also be equivalently formulated as the statement that $(2^{[n]}, \subseteq, \leq_{\text{lex}})$ is a Macaulay structure, where $2^{[n]}$ denotes the set of all subsets of $[n]$. More succinctly, the Kruskal–Katona theorem says that the Boolean lattice is a Macaulay poset.
CHAPTER 8

F-VECTORS OF GENERALIZED COLORED MULTICOMPLEXES

This chapter deals with the \( f \)-vectors of generalized colored multicomplexes with arbitrarily prescribed maximum possible degrees of its variables. See Chapter 2.7 for basic definitions and terminology. In Chapter 8.1, we introduce what we call “truncations” of colored quotient rings, and we construct a class of non-Macaulay-Lex truncations of colored quotient rings. In Chapter 8.2, we study “reverse-lexicographic” generalized colored multicomplexes. Recall that the Frankl–Füredi–Kalai theorem [53] says for every \( 1_n \)-colored complex, there exists a “reverse-lexicographic” \( 1_n \)-colored complex with the same \( f \)-vector. As the main result of Chapter 8.2, we show that an analogous statement does not hold for \( a \)-colored complexes or \( a \)-colored multicomplexes when \( n > 1 \) and \( a \neq 1_n \) (Theorem 8.2.8).

8.1 Truncations of colored quotient rings

Throughout this section, \( W = k[X_\lambda]/Q_a \) is always a colored quotient ring of type \( a = (a_1, \ldots, a_n) \in \mathbb{P}^n \) and composition \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{P}^n \). Given a map \( \phi : X_\lambda \to \mathbb{P} \), define the graded ring

\[
W(\phi) = \bigoplus_{d \in \mathbb{N}} W_d(\phi) := k[X_\lambda]/(Q_a + T^\phi),
\]

where \( T^\phi := \langle \{ x^{\phi(x)+1} : x \in X_\lambda \} \rangle \) is an ideal contained in \( k[X_\lambda] \). We say \( W(\phi) \) is the truncation of \( W \) induced by \( \phi \).

For the following proposition, recall that \( \delta_{n,n} = (0, \ldots, 0, 1) \).

**Proposition 8.1.1.** Let \( n, d > 1 \), \( \lambda \in \mathbb{P}^n \) and \( a \in \mathbb{P}^n \), such that \( \lambda \geq 2 \cdot 1_n + \delta_{n,n} \), \( a \not\geq d \cdot 1_n \), \( a \neq 1_n \), and \( |a| > d \). Suppose \( \phi : X_\lambda \to \mathbb{P} \) is a map satisfying
Let \( \hat{s} x \geq t \) for every \( t \in [n] \). Then there exists some \( W_d(\phi) \)-monomial space \( A \) such that \( |\partial(A)| < |\partial(\text{Revlex}_{W_d(\phi)}(A))| \).

Proof. We will explicitly construct such a \( W_d(\phi) \)-monomial space \( A \). First, we fix some notation. For all monomials \( m, m' \) in \( W(\phi) \), write \( m \nmid m' \) to mean that \( m \) does not divide \( m' \). Whenever we say \( m \) (resp. \( x \in X_\lambda \)) is the largest or smallest monomial (resp. variable) satisfying certain conditions, it is always with respect to the revlex order (induced by the linear order on \( X_\lambda \) as given in Definition 6.1.1).

Given any \( s \in \mathbb{N} \), let \( \bar{s} \) denote the unique integer in \( [n] \) such that \( s \equiv \bar{s} \pmod{n} \).

For each \( x \in X_\lambda \), write \( x^\dagger \) to mean the largest variable in \( X_\lambda \) that is smaller than \( x \). For each \( t \in [n] \), define \( \hat{X}_t := \{ m \in M_{X_{\lambda}}(\phi) : x_{t,1} \nmid m \} \), and let \( \hat{m}_t \) be the largest monomial in \( \hat{X}_t \), which is well-defined as \( \sum_{x \in X_{t,1}} \phi(x) \geq a_t \) implies \( \hat{X}_t \) is non-empty.

Since \( a \nmid d \cdot 1_n \) and \( a \neq 1_n \), there exists some \( s \in [n] \) such that \( a_s \leq d - 1 \) and \( a \frac{s-1}{s} \geq 2 \). If \( s > 1 \), then define

\[
\hat{Y} := \left\{ m \in M_{X_{\lambda}}^{a_{s-1}}(\phi) : x_{s-1,1} \nmid m \text{ and } x_{s-1,2}^{\min\{\phi(x_{s-1,2}),a_{s-1}\}} \nmid m \right\},
\]

while if \( s = 1 \), then define

\[
\hat{Y} := \left\{ m \in M_{X_{\lambda}}^{a_{n-2}}(\phi) : x_{n,1} \nmid m \text{ and } y_{\min\{\phi(y),a_n\}} \nmid m \text{ for each } y \in \{x_{n,2},x_{n,3}\} \right\}.
\]

In either case, \( \hat{Y} \) is non-empty, since \( \sum_{x \in X_{\lambda} \setminus \{x_{s-1,1}\}} \phi(x) \geq a \frac{s-1}{s} \). In particular, if \( a \frac{s-1}{s} = 2 \), then \( \hat{Y} = \{1\} \). Let \( \bar{p} \) be the largest monomial in \( \hat{Y} \), let \( \Gamma := [n] \setminus \{s,\bar{s}-1\} \), and let \( \bar{m} := \bar{p} \cdot \prod_{t \in \Gamma} \hat{m}_t \), which by construction is a monomial in \( W(\phi) \) and has degree \( |a| - a_s - 2 \geq d - 1 - a_s \). Also, let \( m' \) be the smallest monomial of degree \( d - 1 - a_s \) in \( W(\phi) \) that divides \( \bar{m} \), let \( q := m' \bar{m}_s \), and let \( \bar{q} := \frac{q(x_{s,2})}{x_{s,2}} \). In particular, \( \lambda \geq 2 \cdot 1_n + \delta_{n,n} \) implies \( x_{s,2}^\dagger \in X_{\bar{s}-1} \), while \( x_{s,2} \) divides \( \bar{m}_s \) by construction, so \( q \) and \( \bar{q} \) are monomials in \( W_{d-1}(\phi) \), with \( \bar{q} <_r q \).
Next, let \( \bar{x} \) be the smallest variable in \( \{ x \in X_\lambda : x >_r t x_{s,2} \} \) such that \( q\bar{x} \) is a monomial in \( W_d(\phi) \). Define the following sets:

\[
\tilde{A} := \{ m \in \{ W_d(\phi) \} : m \geq_r \tilde{q}\bar{x} \}, \\
A_q := \{ m \in \tilde{A} : q \text{ divides } m \}, \\
A_{\tilde{q}} := \{ m \in \tilde{A} : \tilde{q} \text{ divides } m \}.
\]

Note that \( \tilde{A} \setminus A_{\tilde{q}} \) is a revlex-segment set in \( \{ W_d(\phi) \} \), while \( A_q \cap A_{\tilde{q}} = \emptyset \), since \( qx_{s,2} = \tilde{q}x_{s,2} \) is not contained in \( \tilde{A} \). Also, \( qX_t = \tilde{q}X_t \) for every \( t \in \Gamma \), thus for each \( x \in \bigcup_{t \in \Gamma} X_t \), we have \( qx \in \{ W_d(\phi) \} \) if and only if \( \tilde{q}x \in \{ W_d(\phi) \} \). The definition of \( \hat{p} \) implies \( qx_{s,1}, \tilde{q}x_{s,1}, qx_{s-1,1}, \tilde{q}x_{s-1,1} \) are monomials in \( W_d(\phi) \), while the definition of \( \tilde{m}_s \) implies \( qx_{s,1} \notin \{ W_d(\phi) \} \) and \( \tilde{q}x_{s,1} \in \{ W_d(\phi) \} \). Consequently,

\[
|A_q| = |A_{\tilde{q}}| - 1 < |A_{\tilde{q}}|.
\]

Now, define \( \mathcal{A} := \tilde{A} \setminus A_q \), and let \( x_0 \) be the (unique) variable in \( X_\lambda \) such that \( \tilde{q}x_0 \) is the largest monomial in \( A_{\tilde{q}} \). The set \( \mathcal{I} := (\tilde{A} \setminus A_{\tilde{q}}) \cup \{ \tilde{q}x_0 \} \) is revlex-segment in \( \{ W_d(\phi) \} \), and \( |\mathcal{I}| = |\mathcal{A}| \) implies \( \text{Span}_k(\mathcal{I}) = \text{Revlex}_{W_d(\phi)}(\text{Span}_k(\mathcal{A})) \).

Furthermore, for any \( x, x' \in X_\lambda \) such that \( \tilde{q}x \in \tilde{A} \), \( x' \neq x \), and \( x' \) divides \( \tilde{q} \), we claim that \( \frac{\tilde{q}x}{x'} \in \partial(\tilde{A} \setminus A_{\tilde{q}}) \). Indeed, let \( t' \) be the unique integer in \( [n] \) satisfying \( x' \in X_{t'} \), and note that \( x' \neq x_{t',1} \) by the construction of \( \tilde{q} \), so \( \frac{\tilde{q}x}{x'} \cdot x_{t',1} \in \tilde{A} \setminus A_{\tilde{q}} \), which means \( \frac{\tilde{q}x}{x'} \in \partial(\tilde{A} \setminus A_{\tilde{q}}) \) as claimed. This implies \( \partial(\mathcal{I}) = \partial(\tilde{A}) \). Finally, since \( \partial(\mathcal{A}) \subseteq \partial(\tilde{A}) \setminus \{ q \} \), we conclude that

\[
|\partial(\text{Span}_k(\mathcal{A}))| < |\partial(\tilde{A})| = |\partial(\mathcal{I})| = |\partial(\text{Revlex}_{W_d(\phi)}(\text{Span}_k(\mathcal{A})))|,
\]

therefore the assertion follows by letting \( A = \text{Span}_k(\mathcal{A}) \). \( \square \)

**Remark 8.1.2.** In the proof of Proposition 8.1.1, the only time we used the condition \( \lambda \geq 2 \cdot 1_n + \delta_{n,n} \) was in the case when \( s = 1 \), i.e. when every \( t \in [n-1] \).
satisfying $a_{t+1} \leq d - 1$ also satisfies $a_t = 1$. In all other cases, i.e. when there is some $t \in [n - 1]$ such that $a_{t+1} \leq d - 1$ and $a_t \geq 2$, we only need the weaker condition that $\lambda \geq 2 \cdot 1_n$.

**Corollary 8.1.3.** Let $n > 1$, $a \in \mathbb{P}^n$, $\lambda \in \mathbb{P}^n$ such that $\lambda \geq 2 \cdot 1_n + \delta_{n,n}$ and $a \neq 1_n$. If $\phi : X_\lambda \to \mathbb{P}$ is a map such that $\sum_{x \in X_\lambda \setminus \{x_{t,1}\}} \phi(x) \geq a_t$ for every $t \in [n]$, then $W(\phi)$ is not Macaulay-Lex.

**Proof.** Setting $d = \min\{a_i + 1 : i \in [n]\}$, we clearly have $a \not\geq d \cdot 1_n$, while the condition $a \neq 1_n$ implies $|a| > d$. The assertion then follows from Proposition 8.1.1 and Theorem 5.3.1.

\[\square\]

### 8.2 Compressed and revlex multicomplexes

Suppose $X$ is a non-empty (possibly infinite) well-ordered set. Recall that in Chapter 2.4, we defined the revlex order $\leq_{rl}$ on $\mathcal{M}_X^d$ for arbitrary (possibly infinite) sets $X$. Thus, the notion of a revlex-segment set $A$ in $\mathcal{M}_X^d$, which we have defined in Chapter 5.2 for finite $X$, makes sense even when $X$ is infinite, provided we assume that $A$ is finite.

Given a non-empty subcollection $\mathcal{M} \subseteq \mathcal{M}_X$, let $\mathcal{M}^d := \mathcal{M} \cap \mathcal{M}_X^d$, and define a revlex-segment set in $\mathcal{M}^d$ to be a finite subset $\mathcal{A} \subseteq \mathcal{M}^d$ such that if $m \in \mathcal{A}$, $m' \in \mathcal{M}^d$ and $m' >_{rl} m$, then $m' \in \mathcal{A}$. For technical reasons, we will restrict the notion of sets being “revlex-segment” to only finite sets.

**Definition 8.2.1.** Let $\mathcal{M} \subseteq \mathcal{M}_X$ be any non-empty subcollection of monomials. A multicomplex $M$ on $X$ is called a revlex-compressed multicomplex in $\mathcal{M}$ if $M$ is
finite, and $M^d$ is a revlex-segment set in $\mathcal{M}^d$ for every $d \in \mathbb{N}$. When $\mathcal{M}$ is not specified, we assume $\mathcal{M} = \mathcal{M}_X$.

For any multicomplex $M$ on $X$, the $k$-vector space $\text{Span}_k(\mathcal{M}_X \setminus M)$ forms a (graded) monomial ideal of the standard graded polynomial ring $k[X]$, and we denote this monomial ideal by $I^M = \bigoplus_{d \in \mathbb{N}} I^M_d$. Note that the map $M \mapsto I^M$ gives a bijection between multicomplexes on $X$ and monomial ideals of $k[X]$, and the corresponding inverse map is $I \mapsto \{k[X]/I\}$.

**Lemma 8.2.2.** Let $M$ be a multicomplex on $X$. The following are equivalent:

(i) For every multicomplex $M' \subseteq M$, there exists a revlex-compressed multicomplex $M''$ in $M$ with the same $f$-vector as $M'$.

(ii) $I^M$ is a Macaulay-Lex ideal of the polynomial ring $k[X]$.

**Proof.** For every graded ideal $I$ of $R := k[X]/I^M$, let $\text{in}(I)$ be its initial ideal with respect to some monomial order (see [45, Chap. 15]). Hilbert functions are additive on exact sequences, so by [45, Thm. 15.26], $R$ is Macaulay-Lex if and only if for every graded ideal $I$ of $R$, there exists a lex ideal $L$ of $R$ such that $R/\text{in}(I)$ and $R/L$ have the same Hilbert function, or equivalently, $\{R\} \setminus \{\text{in}(I)\}$ and $\{R\} \setminus \{L\}$ are multicomplexes with the same $f$-vectors. By definition, $L$ is a lex ideal of $R$ if and only if $\{R\} \setminus \{L\}$ is a revlex-compressed multicomplex in $\{R\}$. Now, for every multicomplex $M' \subseteq M$, the image of $I^{M'}$ under the natural quotient map $k[X] \to R$ is a monomial ideal of $R$, thus the assertion follows. \[\square\]

For the rest of this section, let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{P}^n$ (so in particular $a_i \neq \infty$ for all $i \in [n]$), let $\pi = (X_1, \ldots, X_n)$ be an ordered partition of a non-empty set $X$, and let $\phi : X \to \mathbb{P}$ be a map. For each $t \in [n]$, label the elements in $X_t$ by
$X_t = \{x_{t,1}, x_{t,2}, \ldots \}$, and fix a linear order on $X$ by $x_{i,j} > x_{i',j'}$ if $j > j'$; or $j = j'$ and $i < i'$ (cf. Definition 6.1.1). We check that this linear order on $X$ is a well-ordering. (To show it is a well-ordering, we used the fact that $\pi = (X_1, \ldots, X_n)$ is an ordered partition of $X$ into a finite number of subsets.) Define

$$M_{a,\pi} := \{m \in M_X : \deg(m_{X_t}) \leq a_t \text{ for all } t \in [n]\},$$

and let $M_{a,\pi}(\phi) := M_{a,\pi} \cap M_X(\phi)$. Note that $(M, \pi)$ is an $a$-colored multicomplex in $M_X(\phi)$ if and only if $M$ is a multicomplex in $M_{a,\pi}(\phi)$.

**Definition 8.2.3.** An $a$-colored multicomplex $(M, \pi)$ in $M_X(\phi)$ is called revlex (reverse-lexicographic) if $M$ is a revlex-compressed multicomplex in $M_{a,\pi}(\phi)$ with respect to this given linear order on $X$. In particular, recall that our definition of the revlex order yields $x_{i,j} > x_{i',j'} \iff x_{i,j} <_{rt} x_{i',j'}$.

**Example 8.2.4.** Let $a = (1, 2, 1)$ and $X = \{1, 2, 3\} \times \mathbb{P}$. For each $i \in \{1, 2, 3\}$, define $X_i = \{(i, j) : j \in \mathbb{P}\}$ and identify each pair $(i, j)$ with the variable $x_{i,j}$, so that $\pi = (X_1, X_2, X_3)$ is an ordered partition of $X$. Fix a linear order on $X$ by $x_{i,j} > x_{i',j'}$ if $j < j'$; or $j = j'$ and $i > i'$. Next, observe that

$$x_{3,1}x_{2,1} >_{rt} x_{3,1}x_{1,1} >_{rt} x_{2,1}x_{1,1} >_{rt} x_{2,1}x_{3,2} >_{rt} x_{1,1}x_{3,2} >_{rt} x_{3,1}x_{2,2} >_{rt} \cdots,$$

are the largest few quadratic monomials in $M_{a,\pi}(1)$, hence if we let $M$ denote the collection of the largest five quadratic monomials (boldfaced above) together with their divisors, then $(M, \pi)$ is a revlex $a$-colored multicomplex in $M_X(1)$.

**Proposition 8.2.5.** The following are equivalent:

(i) For every $a$-colored multicomplex $(M, \pi)$ in $M_X(\phi)$, there exists a revlex $a$-colored multicomplex $(M', \pi)$ in $M_X(\phi)$ such that $M$ and $M'$ have the same $f$-vector.
(ii) For every $\lambda \in \mathbb{P}^n$ satisfying $\lambda \leq (|X_1|, \ldots, |X_n|)$, the truncation of the colored quotient ring $k[X_\lambda]/Q_\lambda$ induced by $\phi$ is Macaulay-Lex.

Proof. Let $|\pi| := (|X_1|, \ldots, |X_n|) \in \mathbb{P}^n$, and define the set $X_i^r := \{x_{i,t} : i \in [r]\} \subseteq X_t$ for every $r \in \mathbb{P}$, $t \in [n]$. For each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{P}^n$ satisfying $\lambda \leq |\pi|$, consider $X_\lambda$ as a subposet of $X$, let $\pi_\lambda := (X_1^{[\lambda_1]}, \ldots, X_n^{[\lambda_n]})$ be an ordered partition of $X_\lambda$, and let $\phi_\lambda : X_\lambda \to \mathbb{P}$ be the restriction of $\phi$ to $X_\lambda$ as its domain.

Suppose $(M, \pi)$ is an a-colored multicomplex in $M_X(\phi)$, i.e. $M$ is a multicomplex in $M_{a,\pi}(\phi)$. Choose any $\lambda \in \mathbb{P}^n$ such that $\lambda \leq |\pi|$ and $M \subseteq M_{X_\lambda}$, which is possible since $a \in \mathbb{P}^n$ implies $M$ is finite. Every revlex-compressed multicomplex $M'$ in $M_{a,\pi}(\phi)$ with the same $f$-vector as $M$ is necessarily contained in $M_{a,\pi}(\phi_\lambda)$, hence $(M', \pi)$ is a revlex a-colored multicomplex in $M_{X_\lambda}(\phi_\lambda) \subseteq M_X(\phi)$ for each such revlex-compressed multicomplex $M'$. Since $M_{X_\lambda}(\phi_\lambda)$ is a multicomplex on $X_\lambda$, Lemma 8.2.2 tells us that for every $\lambda \in \mathbb{P}^n$ satisfying $\lambda \leq |\pi|$, the following are equivalent:

- For every a-colored multicomplex in $M_X(\phi) \cap M_{X_\lambda}$, there exists a revlex a-colored multicomplex $(M', \pi)$ in $M_X(\phi)$ such that $M$ and $M'$ have the same $f$-vector.

- The ideal

$$I^{M_{X_\lambda}(\phi_\lambda)} = \left( \sum_{i \in [n]} \langle X_i^{[\lambda_i]} \rangle^{a_i+1} \right) + \langle \{x^{\phi_\lambda(x)+1} : x \in X_\lambda\} \rangle$$

is a Macaulay-Lex ideal of $k[X_\lambda]$.

The assertion then follows by considering all possible $\lambda \in \mathbb{P}^n$ satisfying $\lambda \leq |\pi|$.

\[ \square \]

**Theorem 8.2.6.** For every $1_n$-colored multicomplex $(M, \pi)$, there exists a revlex $1_n$-colored multicomplex $(M', \pi)$ such that $M$ and $M'$ have the same $f$-vector.
Proof. This follows from Proposition 8.2.5 and the case $a_1 = 1_n$ of Theorem 6.5.3.

\[\square\]

**Theorem 8.2.7.** Suppose $\pi$ and $\phi$ satisfy $(|X_1|, \ldots, |X_n|) \geq 2 \cdot 1_n + \delta_{n,n}$, and suppose $\sum_{x \in X_1 \setminus \{x_{t,1}\}} \phi(x) \geq a_t$ for all $t \in [n]$. Then the following are equivalent:

(i) For every $a$-colored multicomplex $(M, \pi)$ in $M_X(\phi)$, there exists a revlex $a$-colored multicomplex $(M', \pi)$ in $M_X(\phi)$ such that $M$ and $M'$ have the same $f$-vector.

(ii) Either $n = 1$ and $x > x'$ for all $x, x' \in X$ satisfying $\min \{a_1, \phi(x)\} < \min \{a_1, \phi(x')\}$ (if any); or $a = 1_n$.

**Proof.** If $n > 1$ and statement (i) holds, then $a = 1_n$ by Corollary 8.1.3 and Proposition 8.2.5. Conversely, if $n > 1$ and $a = 1_n$, then statement (i) holds by Theorem 8.2.6. As for the case $n = 1$, i.e. $a = (a_1) \in \mathbb{P}$ and $\pi = (X)$, Proposition 8.2.5 and Proposition 6.3.1 together imply that statement (i) holds if and only if $\min \{a_1, \phi(x_{1,1})\} \geq \cdots \geq \min \{a_1, \phi(x_{1,\lambda})\}$ for every $\lambda \in \mathbb{P}$ satisfying $\lambda \leq |X|$, or equivalently, $x > x'$ for all $x, x' \in X$ satisfying $\min \{a_1, \phi(x)\} < \min \{a_1, \phi(x')\}$ (if any).

Finally, we prove the main result of this chapter.

**Theorem 8.2.8.** Let $a = (a_1, \ldots, a_n)$ be an $n$-tuple of positive integers. The following are equivalent:

(i) For every $a$-colored complex (resp., multicomplex), there exists a revlex $a$-colored complex (resp., multicomplex) with the same $f$-vector.

(ii) Either $n = 1$, or $a = (1, \ldots, 1)$.  

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Proof. Let \((M, \pi)\) be an \(a\)-colored multicomplex in \(\mathcal{M}_X(\phi)\). Since we do not require every \(x \in X\) to be in \(M\), we may assume without loss of generality that \((|X_1|, \ldots, |X_n|) = \infty_n\) by adding new variables if necessary. Consequently, the map \(\phi\) trivially satisfies \(\sum_{x \in X_1 \setminus \{x_t, 1\}} \phi(x) = \infty \geq a_t\) for every \(t \in [n]\). Therefore, by considering (i): \(\phi = 1\), where \(1 : X \to \mathbb{P}\) is the constant map defined by \(x \mapsto 1\) for all \(x \in X\); and (ii): \(\phi : X \to \mathbb{P}\) a constant map defined by \(x \mapsto \infty\) for all \(x \in X\), the assertion follows from Theorem 8.2.7. \(\square\

Notes

There is an interesting backstory to Theorem 8.2.8. In January 2012, the author already had a proof of Theorem 8.2.7 (and hence a proof of Theorem 8.2.8), but the result was cast aside and ignored for more than a year after that. His original intention earlier in Fall 2011 was to investigate whether Frankl–Füredi–Kalai’s characterization of the \(f\)-vectors of \(1_n\)-colored complexes could be extended to give a characterization of the \(f\)-vectors of \(a\)-colored complexes for arbitrary \(a \in \mathbb{P}^n\). During that time (in Fall 2011), the author had already realized that the original proof of the Frankl–Füredi–Kalai theorem [53] was incorrect (by the reasons stated in Appendix A). He had subsequently checked that other known proofs of the theorem are correct. He also tried, with much difficulty, to extend the ideas in these correct proofs to the case of \(a\)-colored complexes.

Around some time in late 2011, his advisor suggested the possibility that maybe it was so difficult to extend those ideas, simply because it was actually impossible. Working on the suggestion to look for counterexamples, the author tried small cases and realized that all of them “failed spectacularly”. What he was trying to prove for months was simply not true! These counterexamples all had some kind of extremal property, and they were not obvious, but once he started looking for extremal cases that might fail, they always did fail. Thus, from this perspective, it is actually surprising that the Frankl–Füredi–Kalai theorem is true. The only cases \(n = 1\) and \(a = 1_n\) that do not fail correspond to the Kruskal–Katona theorem and the Frankl–Füredi–Kalai theorem respectively. For all the remaining cases that did fail, they led to the current proof of Theorem 8.2.7. Since the author’s motivation was to characterize the \(f\)-vectors of \(a\)-colored complexes, this “negative” result was cast aside.

In summer 2013, after having successfully characterized the fine \(f\)-vectors of \(a\)-colored complexes by completely different ideas (see Part IV), the author went back to this “negative” result. He had this nagging feeling that the condition
“(|X_1|, \ldots, |X_n|) \geq 2 \cdot 1_n + \delta_{n,n}” in the statement of Theorem 8.2.7 was somewhat artificial. He succeeded in incorporating the Clements–Lindström theorem, and what resulted has been given in Part II of this dissertation (i.e. this part).
Part III

An application of liaison theory
to the Eisenbud–Green–Harris conjecture
CHAPTER 9
THE EISENBUD–GREEN–HARRIS CONJECTURE

The Eisenbud–Green–Harris (EGH) conjecture [46] is a famous open problem in commutative algebra and algebraic geometry. It concerns the Hilbert functions of graded ideals containing regular sequences. If true, it would imply the generalized Cayley–Bacharach conjecture [46] in algebraic geometry, generalize the Clements–Lindström theorem [36] in combinatorics (see Chapter 2.8), and provide evidence for the Lex-plus-powers conjecture [51] in commutative algebra.

Recall that in Chapter 7, we have a simultaneous generalization of the Clements–Lindström theorem and the Frankl–Füredi–Kalai theorem using the algebraic notion of Macaulay-Lex rings (see also Chapter 5). The EGH conjecture instead proposes a different generalization of the Clements–Lindström theorem. The main motivation behind this conjecture is geometric in nature and involves the well-known Cayley–Bacharach theorem in algebraic geometry. In Chapter 9.1, we will look at various aspects of the Cayley–Bacharach theorem, including its rich history, and known generalizations of the theorem. In particular, several special cases of the Cayley–Bacharach theorem are classic results in geometry, the earliest of which dates back to around 300+ AD.

The EGH conjecture is still wide open. In Chapter 9.2, we will state the EGH conjecture and give an exhaustive list of all the known cases for which this conjecture holds. In Chapter 9.3, we give other versions of the EGH conjecture. Later in Chapter 10, we will apply ideas from liaison theory (a branch of algebraic geometry) to prove that the EGH conjecture is true for a new class of ideals, which in particular includes Gorenstein ideals in the three-variable case.
9.1 Cayley–Bacharach theorem and generalizations

Note: In this section, we shall use $\mathbb{P}^n$ (instead of the more commonly used notation $\mathbb{P}^n$ found in the literature) to denote projective $n$-space. For us, $\mathbb{P}^n$ would always mean $n$-tuples of positive integers.

The Cayley–Bacharach theorem is a classic result in algebraic geometry. Before the birth of modern algebraic geometry in the 20th century, there were already several different versions of the Cayley–Bacharach theorem. What mathematicians today call the “classic version” of the Cayley–Bacharach theorem could mean any of these several versions. For our purposes, we choose the following version:

**Theorem 9.1.1** ("Classic" Cayley–Bacharach theorem, 1837). Let $C_1, C_2 \subseteq \mathbb{P}^2$ be two cubic curves that intersect at 9 distinct points $p_1, \ldots, p_9$. If $C \subseteq \mathbb{P}^2$ is a cubic curve (on the same projective plane) containing $p_1, \ldots, p_8$, then $C$ must contain the point $p_9$ as well.

Special (degenerate) cases were already known much earlier. *Pascal’s theorem*, proven in 1640, states that the opposite sides of a hexagon inscribed in a conic meet at three collinear points. This follows from our “classic” version of the Cayley–Bacharach theorem by taking each of the cubic curves to be the union of a conic and a line. Even earlier, Pappus of Alexandria proved his famous *Pappus’s theorem* (also known as *Pappus’s hexagon theorem*) around 300+ AD, and it is equivalent to Pascal’s theorem by taking pairs of lines as conics.

It is also interesting to note that it was actually Chasles in 1837 (and not Cayley and Bacharach) who first proved this “classic” version of the theorem. In 1843, Cayley published a note that claimed an extension of Chasles’s theorem
(Theorem 9.1.1) to the case of curves of higher (not necessarily equal) degrees. However, there were gaps in his claim. First of all, he assumed without further comment a subtle fact that, in today’s language, is equivalent to the statement that $k[x_1, x_2, x_3]$ is Cohen–Macaulay. This fact was not proven until in 1873 by Max Noether. The second (and more serious) gap is that Cayley’s assertion as he had stated was false, but this gap went unnoticed for years (possibly decades). In 1881, Bacharach proposed and proved a modified version of Cayley’s assertion, which had additional hypotheses imposed. Bacharach became famous because of this, and the theorem henceforth became widely known as the Cayley–Bacharach theorem. See [47] for an account of the rich history behind this theorem.

With the infusion of techniques in commutative algebra, as well as with the introduction of schemes, algebraic geometry changed dramatically, and so did the statement of the Cayley–Bacharach theorem. This theorem was reinterpreted and generalized multiple times, and we now state one of the several versions of the “modern” Cayley–Bacharach theorem, which can be found in [47]:

**Theorem 9.1.2 (“Modern” Cayley–Bacharach theorem).** Let $\Gamma \subseteq \mathbb{P}^r$ be a reduced complete intersection of hypersurfaces $X_1, \ldots, X_r$ of degrees $d_1, \ldots, d_r$ respectively. Then any hypersurface of degree $(\sum_{i \in [r]} d_i) - r - 1$ containing a closed subscheme of $\Gamma$ of degree $(\prod_{i \in [r]} d_i) - 1$ must contain $\Gamma$.

Motivated by more recent developments concerning the Cayley–Bacharach theorem [40], Eisenbud–Green–Harris [46] proposed in 1993 the following generalization of the Cayley–Bacharach theorem:

**Conjecture 9.1.3 (Generalized Cayley–Bacharach conjecture).** Let $\Gamma \subseteq \mathbb{P}^r$ be a complete intersection of $r$ quadrics. Then any hypersurface of degree $k$ containing a closed subscheme of $\Gamma$ of degree $> 2^r - 2^{r-k}$ must contain $\Gamma$. 

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There are several known cases of this conjecture. The “modern” Cayley–Bacharach theorem (Theorem 9.1.2) implies that the conjecture is true for the case $k = r - 1$, while Bézout’s theorem says the conjecture holds when $k = 1$. Also, Eisenbud–Green–Harris in their same paper [46] proved the cases when $r \leq 6$ for all $k$.

Motivated by this generalized Cayley–Bacharach conjecture, Eisenbud–Green–Haris proposed an even stronger conjecture, which is now commonly known as the “Eisenbud–Green–Harris conjecture”, and usually abbreviated by several authors as the “EGH conjecture”. We postpone giving a precise statement of the EGH conjecture to Chapter 9.2, and instead end this section with several remarks.

The generalized Cayley–Bacharach conjecture as we have given in Conjecture 9.1.3 is a geometric assertion. In contrast, the EGH conjecture is a purely algebraic assertion and concerns the Hilbert functions of graded ideals containing regular sequences. This EGH conjecture was also in part motivated by Macaulay’s theorem (see Chapter 3.6), which concerns the Hilbert functions of arbitrary graded ideals. As we will later see (Remark 9.3.20), the Kruskal–Katona is also a special case of a numerical version of the EGH conjecture. At the time of writing, the EGH conjecture is still wide open!

9.2 The EGH conjecture and known cases

Before we state the EGH conjecture, we remark that there are several statements in the literature that are called “the EGH conjecture”. Some of them seem very different. Yet remarkably, we will see later in Chapter 9.3 that these variations
are in fact equivalent formulations of one common conjecture. When we say two formulations of a conjecture are equivalent, we mean that the first formulation is true if and only if the second formulation is true. We caution the reader that a proof for a special case in a given formulation (e.g. for a given class of ideals) does not necessarily translate to a proof for the analogous case (e.g. the same given class of ideals) in another equivalent formulation; see, e.g., Remark 9.3.6.

Nevertheless, we should think of these equivalent formulations as different versions of the (same) EGH conjecture. Depending on one’s approach, some versions are “easier to work with”. There is also a closely related conjecture in commutative algebra called the Lex-plus-powers (LPP) conjecture, which concerns bounds on graded Betti numbers. If the LPP conjecture is true, then it would imply the EGH conjecture [106]. It is not known whether the EGH conjecture and the LPP conjecture are equivalent; cf. [51, Sec. 6]. For a good overview of the equivalences of various “EGH conjectures” in the context of the LPP conjecture, see [51].

For the rest of this chapter, let $S := \mathbb{k}[x_1, \ldots, x_n]$ be a standard graded polynomial ring on $n$ variables over a field $\mathbb{k}$. Since passing to an infinite field extension of $\mathbb{k}$ preserves Hilbert functions, we shall assume that $\mathbb{k}$ is an infinite field. For any minimal set of generators of a given homogeneous or graded ideal (whether contained in $S$ or otherwise), we always assume the generators are homogeneous. Recall that if $\mathcal{A} = \{p_1, \ldots, p_r\}$ is a collection of forms in $S$, then we write $\langle \mathcal{A} \rangle$ or $\langle p_1, \ldots, p_r \rangle$ to mean the graded ideal of $S$ generated by $\mathcal{A}$. For any proper ideal $I \subsetneq S$, recall that grade($I$) denotes the common length of all maximal $S$-regular sequences contained in $I$.

In this section, we will work with the following version of the EGH conjecture.
**Conjecture 9.2.1** (Eisenbud–Green–Harris conjecture). Let $2 \leq e_1 \leq \cdots \leq e_n$ be integers. If $I \subset S$ is a graded ideal containing an $S$-regular sequence $f_1, \ldots, f_n$ of forms of degrees $e_1, \ldots, e_n$ respectively, then there exists a graded ideal $J \subset S$ containing $x_1^{e_1}, \ldots, x_n^{e_n}$, such that $I$ and $J$ have the same Hilbert function.

**Example 9.2.2.** Consider the ideal $I := \langle x^3, xy, y^3 \rangle \subseteq \mathbb{k}[x, y]$. It contains the $\mathbb{k}[x, y]$-regular sequence $xy, x^3 + y^3$ of forms of degrees 2, 3 respectively. We check that $J := \langle x^2, y^3, xy^2 \rangle \subseteq \mathbb{k}[x, y]$ (which in particular contains $x^2$ and $y^3$) has the same Hilbert function as $I$. In fact, $J$ is the unique monomial ideal in $\mathbb{k}[x, y]$ containing $x^2, y^3$ that has the same Hilbert function as $I$. Notice also that if we fix the linear order $x > y$ on the variables, then $J$ is a lex-plus-$\langle x^2, y^3 \rangle$ ideal with respect to the induced lex order.

**Remark 9.2.3.** Here, we collect some remarks concerning Conjecture 9.2.1.

(i) The sequence $f_1, \ldots, f_n$ is not only a maximal $S$-regular sequence in $I$, but also a maximal $S$-regular sequence in the maximal homogeneous ideal $m \subseteq S$. Later in Chapter 9.3, we will look at another version of the EGH conjecture involving arbitrary $S$-regular sequences that are not necessarily maximal; see Conjecture 9.3.3.

(ii) The special case when $e_i = 2$ for all $i \in [n]$ suffices in proving the generalized Cayley–Bacharach conjecture [46] (Conjecture 9.1.3); cf. Chapter 9.1.

(iii) We can assume that $J$ is a monomial ideal: If there exists a graded proper ideal $J \subset S$ containing $x_1^{e_1}, \ldots, x_n^{e_n}$ with the same Hilbert function as $I$, then we could always replace $J$ by an initial ideal $\text{in}_<(J)$ (with respect to any monomial order $<$). Clearly $\text{in}_<(J)$ is a monomial ideal containing $x_1^{e_1}, \ldots, x_n^{e_n}$ with the same Hilbert function as $I$.

(iv) Example 9.2.2 shows that even if $I$ is a monomial ideal, the corresponding (monomial) ideal $J$ could be a completely different ideal. Nevertheless, the
following proposition tells us that for certain monomial ideals $I$, we can choose $J$ to be the ideal $I$ itself.

**Proposition 9.2.4.** Let $k \in \mathbb{P}$, and let $I \subseteq S$ be a proper ideal generated by monomials of degrees at most $k$. If $\text{grade}(I) = n$, then $(x_1^k, \ldots, x_n^k) \subseteq I$.

**Proof.** Suppose $x_j^k \not\in I$ for some $j \in [n]$. Since $I$ is generated by monomials of degrees $\leq k$, it follows that $x_j^t \not\in I$ for all $t \geq k$, thus $\dim_k(S/I) \neq \infty$, which in particular implies $S/I$ has non-zero Krull dimension [45, Cor. 9.1]. However, since $S$ is a domain over $k$, Remark 3.5.3 and Proposition 3.5.4 together imply that

$$\text{grade}(I) \leq \text{height}(I) = \dim S - \dim(S/I) \leq n - 1,$$

(9.1)

which is a contradiction. \qed

**Corollary 9.2.5.** Conjecture 9.2.1 holds if $I$ is an ideal generated by monomials of degrees $\leq e_1$. In particular, Conjecture 9.2.1 holds if $I$ is a quadratic monomial ideal.

**Proof.** If $I$ is an ideal generated by monomials of degrees $\leq e_1$, then Proposition 9.2.4 implies $I$ contains $x_1^{e_1}, \ldots, x_n^{e_1}$ and hence must also contain $x_1^{e_i}, \ldots, x_n^{e_i}$. The second assertion follows by setting $e_i = 2$ for all $i \in [n]$. \qed

We now list all currently known cases for which the EGH conjecture is true. First of all, the cases $n = 1$ and $(f_1, \ldots, f_n) = (x_1^{e_1}, \ldots, x_n^{e_n})$ are trivially true. Richert [106] showed the case $n = 2$, but already the case $n = 3$ is in general open. When $I$ is a complete intersection (CI) ideal, i.e. $I = \langle f_1, \ldots, f_n \rangle$, it is a routine exercise to show that the conjecture is true; see Theorem 3.4.6(v). Francisco [50]
proved the case when $I$ is an almost CI ideal.$^1$

The conjecture is also true when $e_{j+1} > \sum_{i=1}^{j}(e_i - 1)$ for all $1 \leq j < n$: Cooper [38] proved this for $n = 3$, while around the same time, Caviglia and Maclagan [29] independently proved this for all $n$. Cooper [39] subsequently showed that the conjecture holds for all triples $(e_1, e_2, e_3)$ satisfying $e_1 \leq 3$ by considering the remaining triples not covered by these inequalities.

For the special case when $e_i = 2$ for all $1 \leq i \leq n$, Richert [106] claimed without proof that the conjecture holds for $n \leq 5$, Chen [32] gave a proof for $n \leq 4$, Herzog–Popescu [69] proved that the conjecture holds if $\mathbb{k}$ has characteristic zero and $I$ is minimally generated by generic quadratic forms, while Gasharov [57] showed that Herzog–Popescu’s result is still true when $\mathbb{k}$ is replaced by a field of arbitrary characteristic.

Very recently, Abedelfatah [3] showed the case when every $f_i$ is a product of linear forms, Caviglia–Constatinescu–Varbaro [25] proved the case when $I$ is a quadratic monomial ideal$^2$, and Abedelfatah [2] proved that the conjecture is true when $I$ is an arbitrary monomial ideal. Later in Chapter 10, we give a new class of ideals for which the EGH conjecture is true.

Notice that all known cases of the EGH conjecture can be categorized into the following four different approaches:

(i) Bound the value of $n$.

$^1$An ideal $I \subseteq S$ is called perfect if grade($I$) equals the projective dimension of $S/I$. An almost complete intersection ideal of $S$ is a perfect ideal $I$ that is minimally generated by grade($I$) + 1 elements. Every almost CI ideal $I \subseteq S$ can be written as $I = \langle f_1, \ldots, f_r, g \rangle$, where $f_1, \ldots, f_r$ is a maximal $S$-regular sequence contained in $I$, and $g$ is another element not in $\langle f_1, \ldots, f_r \rangle$.

$^2$Caviglia–Constatinescu–Varbaro proved the more general statement that the EGH conjecture involving arbitrary (not necessarily maximal) $S$-regular sequences (i.e. Conjecture 9.3.3) is true for quadratic monomial ideals. In the maximal $S$-regular sequences version of the EGH conjecture (i.e. Conjecture 9.2.1), this is trivially true; see Corollary 9.2.5.
(ii) Restrict the values of $e_1, \ldots, e_n$.

(iii) Impose conditions on the $S$-regular sequence $f_1, \ldots, f_n$.

(iv) Restrict $I$ to a certain class of ideals.

<table>
<thead>
<tr>
<th>Bound value of $n$</th>
<th>Impose conditions on regular sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>• $n = 1$ (trivial)</td>
<td>• $(f_1, \ldots, f_n) = (x_1^{e_1}, \ldots, x_n^{e_n})$ (trivial)</td>
</tr>
<tr>
<td>▶ $n = 2$ (Richert, 2004)</td>
<td>▶ Each $f_i$ a product of linear forms</td>
</tr>
<tr>
<td></td>
<td>(Abedelfatah, 2015)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Restrict values of $e_1, \ldots, e_n$</th>
<th>Restrict $I$ to certain classes of ideals</th>
</tr>
</thead>
<tbody>
<tr>
<td>◀ $e_{j+1} &gt; \sum_{i=1}^{j}(e_i - 1)$</td>
<td>• CI ideals (textbook exercise)</td>
</tr>
<tr>
<td>(Cooper, 2005)</td>
<td>▶ Ideals minimally generated by generic quadratic forms</td>
</tr>
<tr>
<td>◀ $n = 3$ and $e_1 \leq 3$</td>
<td>(Gasharov, 1999)</td>
</tr>
<tr>
<td>(Cooper, 2012)</td>
<td>▶ Almost CI ideals (Francisco, 2004)</td>
</tr>
<tr>
<td>For $(\alpha_1, \ldots, \alpha_n) = (2, \ldots, 2)$:</td>
<td>▶ Monomial ideals</td>
</tr>
<tr>
<td>◀ $n \leq 4$ (Chen, 2012)</td>
<td>(Caviglia–Constatinescu–Varbaro, 2014)</td>
</tr>
<tr>
<td>◆ $n \leq 5$? (Richert)</td>
<td>(Abedelfatah, 2013 preprint)</td>
</tr>
<tr>
<td></td>
<td>• Sequentially bounded licci ideals</td>
</tr>
<tr>
<td></td>
<td>(Chong, 2013 preprint; see Chapter 10)</td>
</tr>
</tbody>
</table>

Note: ◀ uses combinatorial structure of lex-plus-powers ideals in proof.

Figure 9.1: Summary of all known cases of the EGH conjecture.

In Figure 9.1, we give a summary of all known cases of the EGH conjecture categorized by these four approaches. In particular, if we look into the details of the proofs of all these known cases, we would notice that with the exception of the “trivial” cases and our new case presented in Chapter 10, all of the other known cases are proven using the combinatorial structure of lex-plus-powers ideals. In contrast, we do not use lex-plus-powers ideals and instead use liaison theory as the main tool when proving our new case of the EGH conjecture.
9.3 Variations of the EGH conjecture

In this section, we look at several variations of the EGH conjecture. Our goal is to show that these variations are all equivalent. For convenience, we shall label these variations by different acronyms. First, we recall the version of the EGH conjecture introduced in the previous section:

**Conjecture 9.2.1** (EGH\(_{e,n}\)). Let \(e = (e_1, \ldots, e_n)\) be an \(n\)-tuple of integers such that \(2 \leq e_1 \leq \cdots \leq e_n\). If \(I \subset S\) is a graded ideal containing an \(S\)-regular sequence \(f_1, \ldots, f_n\) of forms of degrees \(e_1, \ldots, e_n\) respectively, then there exists a graded ideal \(J \subset S\) containing \(x_1^{e_1}, \ldots, x_n^{e_n}\), such that \(I\) and \(J\) have the same Hilbert function.

Given a graded ideal \(I \subset S\), we say \(I\) minimally contains an \((a_1, \ldots, a_r)\)-regular sequence of forms if \(\text{grade}(I) = r\), the minimum degree of the forms in \(I\) is \(a_1\), and for each \(2 \leq i \leq r\), the integer \(a_i\) is the smallest degree such that \(I\) contains an \(S\)-regular sequence \(f_1, \ldots, f_i\) of forms of degrees \(a_1, \ldots, a_i\) respectively. Note that such an \(S\)-regular sequence \(f_1, \ldots, f_r\) of forms with minimal degrees always exists. Note also that \(a_1 \leq \cdots \leq a_r\) by Theorem 3.4.6(i).

**Conjecture 9.3.1** (EGH\(_{\text{min deg} e,n}\)). Let \(e = (e_1, \ldots, e_n)\) be an \(n\)-tuple of integers such that \(2 \leq e_1 \leq \cdots \leq e_n\). If \(I \subset S\) is a graded ideal that minimally contains an \(e\)-regular sequence of forms, then there exists a graded ideal \(J \subset S\) containing \(x_1^{e_1}, \ldots, x_n^{e_n}\), such that \(I\) and \(J\) have the same Hilbert function.

**Proposition 9.3.2.** Conjecture 9.2.1 is true if and only if Conjecture 9.3.1 is true. In particular, given \(e = (e_1, \ldots, e_n)\) \(\in \mathbb{P}^n\) such that \(2 \leq e_1 \leq \cdots \leq e_n\), and given a graded ideal \(I \subset S\) that minimally contains an \(e\)-regular sequence of forms, then EGH\(_{\text{min deg} e,n}\) is true for \(I\) if and only if EGH\(_{e',n}\) is true for \(I\) and all \(e' \in \mathbb{P}^n\) satisfying \(e' \geq e\).
Proof. If $\text{EGH}_{e,n}^{\text{min deg}}$ is true for $I$, then there is a graded ideal $J \subset S$ containing $x_{e_1}^{e_1}, \ldots, x_{e_n}^{e_n}$ with the same Hilbert function as $I$. Choose any $e' = (e'_1, \ldots, e'_n) \in \mathbb{P}^n$ such that $e' \geq e$. Since $J$ is an ideal, it must also contain $x_{e_1}^{e'_1}, \ldots, x_{e_n}^{e'_n}$, thus we conclude that $\text{EGH}_{e',n}$ is true for $I$ and all $e' \in \mathbb{P}^n$ satisfying $e' \geq e$. Conversely, if $\text{EGH}_{e',n}$ is true for $I$ and all $e' \in \mathbb{P}^n$ satisfying $e' \geq e$, then in particular $\text{EGH}_{e,n}$ (i.e. $e' = e$) is true for $I$, therefore $\text{EGH}_{e,n}^{\text{min deg}}$ is true for $I$. \qed

The following versions of the EGH conjecture allows for non-maximal $S$-regular sequences.

**Conjecture 9.3.3 (EGH$_{n,e,r}$).** Let $r \in [n]$, let $e = (e_1, \ldots, e_r)$ be an $r$-tuple of integers satisfying $2 \leq e_1 \leq \cdots \leq e_r$, and let $f_1, \ldots, f_r$ be an $S$-regular sequence of forms of degrees $e_1, \ldots, e_r$ respectively. If $I \subset S$ is a graded ideal containing $f_1, \ldots, f_r$, then there exists a graded ideal $J \subset S$ containing $x_{e_1}^{e_1}, \ldots, x_{e_r}^{e_r}$, such that $I$ and $J$ have the same Hilbert function.

**Conjecture 9.3.4 (EGH$_{n,e,r}^{\text{min deg}}$).** Let $r \in [n]$, let $e = (e_1, \ldots, e_r)$ be an $r$-tuple of integers satisfying $2 \leq e_1 \leq \cdots \leq e_r$, and let $f_1, \ldots, f_r$ be an $S$-regular sequence of forms of degrees $e_1, \ldots, e_r$ respectively. If $I \subset S$ is a graded ideal that minimally contains an $e$-regular sequence of forms, then there exists a graded ideal $J \subset S$ containing $x_{e_1}^{e_1}, \ldots, x_{e_r}^{e_r}$, such that $I$ and $J$ have the same Hilbert function.

Using the same argument as in Proposition 9.3.2, we check that Conjecture 9.3.3 and Conjecture 9.3.4 are equivalent. The reader may suspect that Conjecture 9.3.3 is stronger than Conjecture 9.2.1, since the special case $\text{EGH}_{n,e,n}$ is identical to $\text{EGH}_{e,n}$. Yet remarkably, Caviglia and Maclagan [29] showed that both conjectures are equivalent:

**Theorem 9.3.5 ([29]).** If $r \in [n]$, $e = (e_1, \ldots, e_r) \in \mathbb{P}^r$ satisfies $2 \leq e_1 \leq \cdots \leq e_r$, 

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and \( EGH_{e',n} \) is true for all \( e' = (e'_1, \ldots, e'_n) \in \mathbb{P}^n \) such that \( 2 \leq e'_1 \leq \cdots \leq e'_n \) and \( e'_i = e_i \) for each \( i \in [r] \), then \( EGH_{n,e,r} \) is true. Conversely, if \( EGH_{e,r} \) holds for some \( r \in \mathbb{P} \) and some \( e = (e_1, \ldots, e_r) \in \mathbb{P}^r \) satisfying \( 2 \leq e_1 \leq \cdots \leq e_r \), then \( EGH_{n',e,r} \) holds for all integers \( n' \geq r \).

**Remark 9.3.6.** Although Conjecture 9.3.3 is true if and only if Conjecture 9.2.1 is true, we caution the reader that proofs for specific cases of Conjecture 9.2.1 do not necessarily translate to proofs for the analogous cases of Conjecture 9.3.3. For example, we earlier gave an easy proof (see Corollary 9.2.5) that Conjecture 9.2.1 is true when \( I \) is a quadratic monomial ideal. Our proof uses the assumption that \( \text{grade}(I) = n \), and hence does not translate to a proof of Conjecture 9.3.3 for quadratic monomial ideals. In fact, it was only very recently proven by Caviglia–Constatinescu–Varbaro [26] that Conjecture 9.3.3 is true for quadratic monomial ideals. Subsequently, Abedelfatah [2] proved that Conjecture 9.3.3 holds for arbitrary monomial ideals.

For the rest of this section, fix the linear order \( x_1 > \cdots > x_n \) on the variables of \( S \). This linear order induces a lex order on the monomials in \( S \) (see Chapter 2.4). Recall that for any ideal \( I \subseteq S \), a lex-plus-\( I \) ideal of \( S \) is an ideal \( L' \) that can be written as \( L' = L + I \) for some lex ideal \( L \) of \( S \). If \( P := \langle x_1^{e_1}, \ldots, x_n^{e_n} \rangle \) for some positive integers \( e_1, \ldots, e_n \), then a lex-plus-\( P \) ideal is called a lex-plus-powers ideal.\(^3\) Using the notion of lex-plus-powers ideals, the Clements–Lindström theorem [36] (see Chapter 2.8) can be equivalently stated as follows:

**Theorem 9.3.7.** Let \( r \in [n] \), and let \( 2 \leq e_1 \leq \cdots \leq e_r \) be integers. If \( I \subseteq S \) is a graded ideal containing \( x_1^{e_1}, \ldots, x_r^{e_r} \), then there exists a lex-plus-\( \langle x_1^{e_1}, \ldots, x_n^{e_n} \rangle \) ideal.

\(^3\)Contrary to the definition of lex-plus-powers ideals used by Francisco–Richert [51] in the context of the LPP conjecture, we do not require the powers \( x_1^{e_1}, \ldots, x_n^{e_n} \) to be minimal generators of the lex-plus-powers ideal.
L, such that I and L have the same Hilbert function.

Consequently, by incorporating this formulation of the Clements–Lindström theorem, we have the following equivalent versions of the EGH conjecture.

**Conjecture 9.3.8 (EGH\textsuperscript{lp}\textsubscript{pp},e\textsubscript{n})**. Let \( e = (e_1, \ldots, e_n) \) be an \( n \)-tuple of integers such that \( 2 \leq e_1 \leq \cdots \leq e_n \). If \( I \subsetneq S \) is a graded ideal containing an \( S \)-regular sequence \( f_1, \ldots, f_n \) of forms of degrees \( e_1, \ldots, e_n \) respectively, then there exists a \( \text{lex-plus-}(x_1^{e_1}, \ldots, x_n^{e_n}) \) ideal with the same Hilbert function as \( I \).

**Conjecture 9.3.9 (EGH\textsuperscript{lp}\textsubscript{pp} min deg,e\textsubscript{n})**. Let \( e = (e_1, \ldots, e_n) \) be an \( n \)-tuple of integers such that \( 2 \leq e_1 \leq \cdots \leq e_n \). If \( I \subsetneq S \) is a graded ideal that minimally contains an \( e \)-regular sequence of forms, then there exists a \( \text{lex-plus-}(x_1^{e_1}, \ldots, x_n^{e_n}) \) ideal with the same Hilbert function as \( I \).

**Conjecture 9.3.10 (EGH\textsuperscript{lp}\textsubscript{pp},e\textsubscript{n},e\textsubscript{r})**. Let \( r \in [n] \), let \( e = (e_1, \ldots, e_r) \) be an \( r \)-tuple of integers satisfying \( 2 \leq e_1 \leq \cdots \leq e_r \), and let \( f_1, \ldots, f_r \) be an \( S \)-regular sequence of forms of degrees \( e_1, \ldots, e_r \) respectively. If \( I \subsetneq S \) is a graded ideal containing \( f_1, \ldots, f_r \), then there exists a \( \text{lex-plus-}(x_1^{e_1}, \ldots, x_n^{e_n}) \) ideal with the same Hilbert function as \( I \).

**Conjecture 9.3.11 (EGH\textsuperscript{lp}\textsubscript{pp} min deg,e\textsubscript{n},e\textsubscript{r})**. Let \( r \in [n] \), and let \( e = (e_1, \ldots, e_r) \) be an \( r \)-tuple of integers such that \( 2 \leq e_1 \leq \cdots \leq e_r \). If \( I \subsetneq S \) is a graded ideal that minimally contains an \( e \)-regular sequence of forms, then there exists a \( \text{lex-plus-}(x_1^{e_1}, \ldots, x_r^{e_r}) \) ideal with the same Hilbert function as \( I \).

There are also “stepwise” versions of the EGH conjecture. Before stating them, we shall prove the following useful lemma.

**Lemma 9.3.12**. Let \( r \in [n] \), let \( 2 \leq e_1 \leq \cdots \leq e_r \) be integers, and suppose \( I \subsetneq S \) is a graded ideal containing an \( S \)-regular sequence \( f_1, \ldots, f_r \) of forms of degrees...
e_1, \ldots, e_r respectively. Then for a fixed \( t \in \mathbb{N} \), there exists a lex-plus-\( \langle x_1^{e_1}, \ldots, x_r^{e_r} \rangle \) ideal \( L \subseteq S \) such that \( H(I, t) = H(L, t) \), or equivalently, \( H(S/I, t) = H(S/L, t) \).

Proof. Let \( P := \langle x_1^{e_1}, \ldots, x_r^{e_r} \rangle \) and \( F := \langle f_1, \ldots, f_r \rangle \) be graded CI ideals in \( S \). Consider the following short exact sequence of graded \( S \)-modules:

\[
0 \longrightarrow F \longrightarrow I \longrightarrow I/F \longrightarrow 0 \tag{9.2}
\]

By the additivity of Hilbert functions, it follows from Theorem 3.4.6(v) that

\[
H(I, d) = H(F, d) + H(I/F, d) = H(P, d) + H(I/F, d) \geq H(P, d) \tag{9.3}
\]

for all \( d \in \mathbb{N} \). Consequently, by letting \( A \) be the first \((H(I, t) - H(P, t))\) monomials in \( S_t \) (with respect to the lex order) that are not in \( P_t \), we get that \( L := \langle A \rangle + P \) is a lex-plus-\( P \) ideal satisfying \( H(I, t) = H(L, t) \). A similar argument using the short exact sequences \( 0 \to I \to S \to S/I \to 0 \) and \( 0 \to L \to S \to S/L \to 0 \), and the additivity of Hilbert functions, shows that \( H(I, t) = H(L, t) \) if and only if \( H(S/I, t) = H(S/L, t) \).

Remark 9.3.13. Given an ideal \( I \) satisfying the hypotheses of the EGH conjecture, Lemma 9.3.12 says that for any fixed \( t \in \mathbb{N} \), there is a lex-plus-powers ideal \( L \) (dependent on \( t \)) such that \( H(I, t) = H(L, t) \). The EGH conjecture asserts that there is a common lex-plus-powers ideal \( L' \) such that \( H(I, d) = H(L', d) \) for all \( d \in \mathbb{N} \), not just for a particular \( d \); see Conjectures 9.3.8–9.3.11.

Given graded ideals \( P \) and \( L = \bigoplus_{d \in \mathbb{N}} L_d \) in \( S \), and given any \( t \in \mathbb{N} \), define the ideal \( L_{(t)}^{[P]} := (\langle L_t \rangle + P) \subseteq S \) generated by \( L_t \) and \( P \). Notice that if \( P = \langle x_1^{e_1}, \ldots, x_r^{e_r} \rangle \) for some integers \( 2 \leq e_1 \leq \cdots \leq e_r \), and \( L \) is a lex-plus-\( P \) ideal, then \( L_{(t)}^{[P]} \) is also a lex-plus-\( P \) ideal (cf. Remark 5.3.2 and Theorem 9.3.7), and it is uniquely determined by \( H(L, t) \) and \( P \). Also, it satisfies \( H(S/L, t) = H(S/L_{(t)}^{[P]}, t) \) and \( H(S/L, t + 1) \leq H(S/L_{(t)}^{[P]}, t + 1) \).
**Conjecture 9.3.14** (EGH\textsubscript{step}\textsubscript{d}). Let \( d \in \mathbb{N} \), let \( \mathbf{e} = (e_1, \ldots, e_n) \) be an \( n \)-tuple of integers such that \( 2 \leq e_1 \leq \cdots \leq e_n \), and define \( P := \langle x_1^{e_1}, \ldots, x_n^{e_n} \rangle \subseteq S \). Suppose \( I \subsetneq S \) is a graded ideal that contains an \( S \)-regular sequence \( f_1, \ldots, f_n \) of forms of degrees \( e_1, \ldots, e_n \) respectively, and let \( L \) be a lex-plus-\( P \) ideal such that \( H(S/I, d) = H(S/L, d) \) (which exists by Lemma 9.3.12). Then

\[
H(S/I, d + 1) \leq H(S/L^{[P]}_{(d)}, d + 1).
\]

\[\text{(9.4)}\]

**Proposition 9.3.15.** Conjecture 9.3.14 is true if and only if Conjecture 9.3.8 is true. In particular, given \( e = (e_1, \ldots, e_n) \in \mathbb{P}^n \) such that \( 2 \leq e_1 \leq \cdots \leq e_n \), and given a graded ideal \( I \subsetneq S \) that contains an \( S \)-regular sequence \( f_1, \ldots, f_n \) of forms of degrees \( e_1, \ldots, e_n \) respectively, then EGH\textsubscript{step}\textsubscript{d} is true for \( I \) if and only if EGH\textsubscript{step} is true for \( I \) and all \( d \in \mathbb{N} \).

**Proof.** Fix the integers \( 2 \leq e_1 \leq \cdots \leq e_n \), fix a graded ideal \( I \subsetneq S \) that contains an \( S \)-regular sequence \( f_1, \ldots, f_n \) of forms of degrees \( e_1, \ldots, e_n \) respectively, and define \( P := \langle x_1^{e_1}, \ldots, x_n^{e_n} \rangle \subseteq S \). For each \( d \in \mathbb{N} \), choose a (graded) lex-plus-\( P \) ideal \( L^{(d)} = \bigoplus_{i \in \mathbb{N}} L_i^{(d)} \) such that \( H(S/I, d) = H(S/L^{(d)}, d) \), which exists by Lemma 9.3.12. Independent of our choices of the ideals \( L^{(0)}_0, L^{(1)}_1, L^{(2)}_2, \ldots \), we note that the homogeneous components \( L^{(0)}_0, L^{(1)}_1, L^{(2)}_2, \ldots \) are uniquely determined by \( I \). For each \( d \in \mathbb{N} \), define the graded lex-plus-\( P \) ideal

\[
K^{(d)} = \bigoplus_{i \in \mathbb{N}} K_i^{(d)} := (L^{(d)})^{[P]}_{(d)} = (L^{(d)}_{d}) + P,
\]

and observe that \( H(S/I, d + 1) \leq H(S/(K^{(d)}), d + 1) \) if and only if

\[
\dim_k(I_{d+1}) = H(I, d + 1) \geq H(K^{(d)}, d + 1) = \dim_k(S_1K^{(d)}_{d}).
\]

\[\text{(9.5)}\]

Suppose EGH\textsubscript{step}\textsubscript{d} is true for \( I \) and all \( d \in \mathbb{N} \). This means (9.5) holds for every \( d \in \mathbb{N} \). Note that \( K^{(d)}_{d} \) is a lex-segment \( S_d \)-monomial space by definition (see
Chapter 5.2), and recall that the upper shadow of a lex-segment $S_d$-monomial space is lex-segment (see Remark 5.3.2). Consequently, $K' := \bigoplus_{d \in \mathbb{N}} K'_d$ is an ideal of $S$. We check that $H(K', d) = H(I, d)$ for all $d \in \mathbb{N}$, and $K'$ is a lex-plus-$P$ ideal by construction, thus $\text{EGH}^{\text{pp}}_{e,n}$ is true for $I$.

Conversely, suppose instead that $\text{EGH}^{\text{pp}}_{e,n}$ is true for $I$, i.e. there exists a lex-plus-$P$ ideal $L' = \bigoplus_{d \in \mathbb{N}} L'_d$ with the same Hilbert function as $I$. Since $L'$ is in particular an ideal, it follows that $S_1L'_d \subseteq L'_{d+1}$ for all $d \in \mathbb{N}$. Given any $d \in \mathbb{N}$, note that $L'_d$ and $L^{(d)}_d$ are lex-segment $S_d$-monomial spaces whose dimensions as $k$-vector spaces both equal $H(I, d)$, thus $L'_d = L^{(d)}_d$, which implies

$$\dim_k(S_1L^{(d)}_d) = \dim_k(S_1L'_d) \leq \dim_k(L'_{d+1}) = \dim_k(I_{d+1}) = H(I, d + 1). \quad (9.6)$$

Observe also that

$$H(K^{(d)}, d + 1) = \dim_k(K^{(d)}_{d+1}) = \dim_k(S_1K^{(d)}_d) = \dim_k(S_1L^{(d)}_d) \quad (9.7)$$

by the definition of $K^{(d)}$. By combining (9.6) and (9.7), we get that $H(K^{(d)}, d+1) \leq H(I, d + 1)$ for all $d \in \mathbb{N}$, or equivalently, $H(S/I, d + 1) \leq H(S/(K^{(d)}), d + 1)$ for all $d \in \mathbb{N}$. Finally, since our choice of $L^{(d)}$ was arbitrary, we conclude that $\text{EGH}^{\text{step} : d}_{e,n}$ is true for $I$ and all $d \in \mathbb{N}$. 

Using the same argument as in the proof of Proposition 9.3.15, we also get the following equivalent versions of the EGH conjecture:

**Conjecture 9.3.16 (EGH$_{e,n}^{\text{min deg step} : d}$).** Let $d \in \mathbb{N}$, let $e = (e_1, \ldots, e_n)$ be an $n$-tuple of integers such that $2 \leq e_1 \leq \cdots \leq e_n$, and define $P := \langle x_1^{e_1}, \ldots, x_n^{e_n} \rangle \subseteq S$. Suppose $I \subset S$ is a graded ideal that minimally contains an $e$-regular sequence of forms, and let $L$ be a lex-plus-$P$ ideal such that $H(S/I, d) = H(S/L, d)$ (which exists by Lemma 9.3.12). Then $H(S/I, d + 1) \leq H(S/L^{[P]}_d, d + 1)$. 

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Conjecture 9.3.17 (EGH\textsubscript{step\textsuperscript{d}}\textsubscript{n,e,r}). Let \( r \in [n], d \in \mathbb{N}, \) let \( e = (e_1, \ldots, e_r) \) be an \( r \)-tuple of integers satisfying \( 2 \leq e_1 \leq \cdots \leq e_r \), and define \( P := \langle x_1^{e_1}, \ldots, x_r^{e_r} \rangle \subseteq S \).

Suppose \( I \subsetneq S \) is a graded ideal that contains an \( S \)-regular sequence \( f_1, \ldots, f_r \) of forms of degrees \( e_1, \ldots, e_r \) respectively, and let \( L \) be a lex-plus-\( P \) ideal such that \( H(S/I, d) = H(S/L, d) \) (which exists by Lemma 9.3.12). Then

\[
H(S/I, d + 1) \leq H(S/L^{[P]}(d), d + 1).
\] (9.8)

Conjecture 9.3.18 (EGH\textsubscript{min deg step\textsuperscript{d}}\textsubscript{n,e,r}). Let \( r \in [n], d \in \mathbb{N}, \) let \( e = (e_1, \ldots, e_r) \) be an \( r \)-tuple of integers satisfying \( 2 \leq e_1 \leq \cdots \leq e_r \), and define \( P := \langle x_1^{e_1}, \ldots, x_r^{e_r} \rangle \subseteq S \).

Suppose \( I \subsetneq S \) is a graded ideal that minimally contains an \( e \)-regular sequence of forms, and let \( L \) be a lex-plus-\( P \) ideal such that \( H(S/I, d) = H(S/L, d) \) (which exists by Lemma 9.3.12). Then \( H(S/I, d + 1) \leq H(S/L^{[P]}(d), d + 1) \).

Finally, we look at the numerical version of an important special case of the EGH conjecture: Conjecture 9.3.14 (EGH\textsubscript{step\textsuperscript{d}}\textsubscript{n,e}) in the case when \( e = (2, \ldots, 2) \).

Given \( N, i \in \mathbb{P} \), recall that the \( i \)-th Macaulay representation of \( N \) is

\[
N = \binom{N_i}{i} + \binom{N_{i-1}}{i-1} + \cdots + \binom{N_j}{j},
\] (9.9)

for some integers \( N_i > N_{i-1} > \cdots > N_j \geq j \geq 1 \). Such an expansion is unique given \( N \) and \( i \) (see Chapter 2.6). Recall also that

\[
\partial^i(N) = \binom{N_i}{i + 1} + \binom{N_{i-1}}{i} + \cdots + \binom{N_j}{j + 1},
\]

\[
\partial_i(N) = \binom{N_i}{i - 1} + \binom{N_{i-1}}{i - 2} + \cdots + \binom{N_j}{j - 1},
\]

for all \( N \in \mathbb{P} \), and \( \partial^i(0) = \partial_i(0) = 0 \).

Conjecture 9.3.19 (Quadrics case of the EGH conjecture (numerical version)).

Let \( I \subsetneq S \) be a graded ideal that contains an \( S \)-regular sequence \( f_1, \ldots, f_n \) of quadratic forms. Then all of the following (equivalent) conditions hold:
(i) $H(S/I, t + 1) \leq \partial^t(H(S/I, t))$ for all $t \in \mathbb{N}$.

(ii) $\partial_{t+1}(H(S/I, t + 1)) \leq H(S/I, t)$ for all $t \in \mathbb{N}$.

(iii) $(H(S/I, 1), H(S/I, 2), \ldots, H(S/I, d))$ is the $f$-vector of a $(d+1)$-dimensional simplicial complex, where $d$ is the largest integer such that $H(S/I, d) \neq 0$.

By Remark 9.2.3(ii), this special case of the EGH conjecture suffices in proving the generalized Cayley–Bacharach conjecture (Conjecture 9.1.3). In fact, this was the original version of the conjecture as stated in Eisenbud–Green-Harris’s paper [46], although only condition (i) in the list of equivalent conditions in Conjecture 9.3.19 was given.

**Remark 9.3.20.** The special case of Conjecture 9.3.19 when $(f_1, \ldots, f_n) = (x_1^2, \ldots, x_n^2)$ is precisely the Kruskal–Katona theorem (see Chapter 2.6) and is equivalent to the algebraic statement that $S/\langle x_1^2, \ldots, x_n^2 \rangle$ is a Macaulay-Lex ring (see Chapter 5).

**Notes**

Given a standard graded $k$-algebra $A$ and a finitely generated standard graded $A$-module $M$, the $(i, j)$-th graded Betti number of $M$ over $A$ is defined by

$$
\beta_{i,j}^A(M) := \dim_k(\Tor_i^A(M, k)_j) = \dim_k(\Ext_i^A(M, k)_j)
$$

for all $i, j \in \mathbb{N}$. The index $i$ is called the *homological degree*, while the index $j$ is called the *inner degree*.

The LPP conjecture was proposed by Charalambous and Evans [106], and in its general form, it says the following: Let $r \in [n]$, let $2 \leq e_1 \leq \cdots \leq e_r$ be integers, and let $f_1, \ldots, f_r$ be an $S$-regular sequence of forms of degrees $e_1, \ldots, e_r$ respectively. If $I \subset S$ is a graded ideal containing $f_1, \ldots, f_r$, then there exists a lex-plus-$\langle x_1^{e_1}, \ldots, x_r^{e_r} \rangle$ ideal $L \subset S$ such that $\beta_{i,j}^S(I) \leq \beta_{i,j}^S(L)$ for all $i, j \in \mathbb{N}$.

One of the most important progress in the LPP conjecture is a result by Mermin–Murai [90], which says that the LPP conjecture is true when $f_1, \ldots, f_r$ are monomials. For more information regarding the LPP conjecture, see [106]. See also the very recent work by Caviglia–Sbarra [31].
It may be surprising to the reader that the EGH conjecture for maximal $S$-regular sequences (i.e. Conjecture 9.2.1) has no “trivial” proof for the case of monomial ideals. After all, if $\text{grade}(I) = n$, then $S/I$ is Artinian.

To see the subtle difficulty, look at Example 9.2.2. Part of the difficulty is highlighted in Remark 9.2.3(iv) (i.e. $J$ could be a completely different ideal from $I$), although Proposition 9.2.4 and Corollary 9.2.5 address the question of when we can choose $J$ to be $I$ itself. Despite an apparent lack of a “trivial” proof, the EGH conjecture is still true for monomial ideals. Abedelfatah [2] actually proved the more general statement that the EGH conjecture involving arbitrary (not necessarily maximal) $S$-regular sequences (i.e. Conjecture 9.3.3) is true for monomial ideals. He also claimed that the version of the EGH conjecture with maximal $S$-regular sequences (i.e. Conjecture 9.2.1) can be proven directly for monomial ideals without using his general proof, but he gave an erroneous sketch of a direct proof [2, Remark 3.6]. In particular, Example 9.2.2 is a counter-example to his assertion that if $I$ is a monomial ideal, then the ideal $J$ in Conjecture 9.2.1 can be chosen to be $I$. Nevertheless, the EGH conjecture (whether Conjecture 9.2.1 or Conjecture 9.3.3) is true for monomial ideals by his general proof.
Liaison theory (also sometimes called linkage theory) is a branch of algebraic geometry. In its usual formulation, it involves the study of subschemes of a fixed codimension in projective space that are related (linked) by an equivalence relation called liaison. There is a lot of rich geometry involved in liaison theory. For an excellent introduction to the subject, see [95].

The goal of this chapter is to apply liaison theory to the EGH conjecture (introduced in the previous chapter). In contrast to previously known results on the EGH conjecture, we use liaison theory as our main tool. In fact, this is the first time liaison theory has appeared in the context of the EGH conjecture. We begin in Chapter 10.1 by reviewing the basic terminology and relevant results in liaison theory.

Licci ideals are homogeneous ideals in the liaison class of a complete intersection ideal, and projective schemes defined by licci ideals are fundamental objects studied in liaison theory. In Chapter 10.2, we introduce a new subclass of licci ideals, which we call ‘sequentially bounded’, and we prove that the EGH conjecture holds for every sequentially bounded licci ideal that ‘admits a minimal first link’ (Theorem 10.2.2); see Definition 10.2.1 for precise definitions. As an important consequence, we show that the EGH conjecture holds for Gorenstein ideals in the case of three variables (Theorem 10.2.5).

Finally, in Chapter 10.3, we prove analogues of Theorem 10.2.2 and Theorem 10.2.5 for another equivalent version of the EGH conjecture that involves arbitrary S-regular sequences (i.e. not necessarily maximal S-regular sequences).
Throughout this chapter, \( S := \mathbb{k}[x_1, \ldots, x_n] \) is a standard graded polynomial ring on \( n \) variables over an infinite field \( \mathbb{k} \). All ideals, whether contained in \( S \) or otherwise, are assumed to be homogeneous and proper, and for any minimal set of generators of a given ideal, we always assume the generators are homogeneous. For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), write \( x^\alpha \) to mean the monomial \( x^{\alpha_1} \cdots x^{\alpha_n} \) in \( S \). Recall that if \( A = \{p_1, \ldots, p_r\} \) is a collection of forms in \( S \), then we write \( \langle A \rangle \) or \( \langle p_1, \ldots, p_r \rangle \) to mean the ideal of \( S \) generated by \( A \). Also, recall that if \( I, J \) are ideals of a ring \( R \), then the ideal quotient of \( I \) and \( J \) is the ideal \( (I : J) := \{r \in R : rJ \subseteq I\} \subseteq R \).

### 10.1 Licci and minimally licci ideals

Recall that a complete intersection (CI) ideal of \( S \) is an ideal generated by an \( S \)-regular sequence of forms. If \( J = \langle f_1, \ldots, f_r \rangle \subseteq S \) is a CI ideal such that \( \deg(f_1) \leq \cdots \leq \deg(f_r) \), then we say \( \mathbf{e} := (\deg(f_1), \ldots, \deg(f_r)) \in \mathbb{P}^r \) is the type of \( J \) (as a CI ideal). Given a positively graded \( \mathbb{k} \)-algebra \( A \), and a finitely generated \( A \)-module \( M \), the socle of \( M \) is the submodule of \( M \) given by

\[
\text{Soc}(M) := (0 :_M \mathfrak{m}_A) = \{m \in M : \mathfrak{m}_A m = 0\} \cong \text{Hom}_A(\mathbb{k}, M).
\]

The socle of \( M \) is also a \( \mathbb{k} \)-vector space. In particular, \( \dim_{\mathbb{k}}(\text{Soc}(M)) < \infty \), since \( M \) is finitely generated as an \( A \)-module. We say \( M \) is a Gorenstein \( A \)-module if it is a Cohen–Macaulay \( A \)-module of Krull dimension \( d \), and

\[
\dim_{\mathbb{k}} \text{Soc}(M/\langle f_1, \ldots, f_d \rangle R) = 1
\]

for any (equivalently all) maximal \( A \)-regular sequences \( f_1, \ldots, f_d \) (in \( \mathfrak{m}_A \)). We say \( A \) is a Gorenstein ring if \( A \) is Gorenstein as an \( A \)-module. Note that complete intersection ideals are Gorenstein.
Definition 10.1.1. Let $R$ be a Cohen–Macaulay graded $k$-algebra, and let $I, I' \subset R$ be ideals of height $r$. If there exists a CI ideal $J \subset R$ of height $r$ satisfying $J \subseteq I \cap I'$, $I = J : I'$ and $I' = J : I$, then we say $I$ and $I'$ are (algebraically) 
\textit{directly linked} (by $J$), and we write $I \sim I'$, or simply $I \sim I'$ (if the ideal $J$ is not important). This binary relation $\sim$ is called \textit{direct linkage}, and the ideal $J$ is called a \textit{link}.

Direct linkage is symmetric, but not necessarily reflexive or transitive. Taking its transitive closure, we get an equivalence relation called \textit{liaison} (or \textit{linkage}). Equivalently, we have the following definition:

Definition 10.1.2. Let $R$ be a Cohen–Macaulay graded $k$-algebra, and let $I, I' \subset R$ be ideals of height $r$. If there exists a finite sequence of ideals $I_0, I_1, \ldots, I_s$ ($s \in \mathbb{P}$) of height $r$, such that $I_0 = I$, $I_s = I'$, and $I_0 \sim I_1 \sim \cdots \sim I_s$, then we call “$I_0 \sim I_1 \sim \cdots \sim I_s$” a \textit{sequence of links} from $I$ to $I'$, we say $s$ is the \textit{length} of this sequence, and we say $I$ and $I'$ are (algebraically) \textit{linked}. This binary relation (that $I$ and $I'$ are linked) is an equivalence relation called \textit{liaison} (or \textit{linkage}), and each equivalence class is called a \textit{liaison class}. An ideal that is in the liaison class\footnote{Any two CI ideals of the same height are linked, so for a fixed height, the liaison class of a CI ideal is unique; see, e.g. [108], for a proof.} of a CI ideal is called a \textit{licci} ideal.

In our definition of liaison, we require that the links are CI ideals. This can be generalized by allowing links to be Gorenstein ideals, which yields the notion of \textit{Gorenstein liaison}. The prefixes ‘CI-’ and ‘G-’ (for complete intersection and Gorenstein respectively) are usually attached to distinguish between the two definitions, e.g. CI-liaison, G-linked, etc.. Currently, Gorenstein liaison theory is an area of active research [61, 66, 67, 81, 93, 94], and there are many open problems.
on whether results in CI-liaison theory can be extended analogously in G-liaison theory; see [81]. In this paper, we only use CI-liaison theory and hence do not use the ‘CI-’ prefix. See [95] for details.

**Remark 10.1.3.** The notion of liaison is (more commonly) defined for projective schemes as follows: Let $V_1, V_2$ be equidimensional subschemes of projective $n$-space (over an algebraically closed field $k$) of codimension $r$, and let $X$ be a complete intersection scheme of codimension $r$ containing both $V_1$ and $V_2$. If the defining ideals $I_{V_1}, I_{V_2}, I_X$ (of $V_1, V_2, X$ respectively) satisfy $I_X \subseteq I_{V_1} \cap I_{V_2}$, $I_X : I_{V_1} = I_{V_2}$ and $I_X : I_{V_2} = I_{V_1}$, then we say $V_1$ and $V_2$ are (algebraically) directly linked, and we write $V_1 \sim_X V_2$, or simply $V_1 \sim V_2$. The equivalence relation generated by (the transitive closure of) $\sim$ is called liaison, and we can define link, liaison class, etc. analogously.

**Definition 10.1.4.** Let $R$ be a Cohen–Macaulay graded $k$-algebra, and let $I, I', I'' \subset R$ be ideals of height $r$. If $I$ minimally contains an $(a_1, \ldots, a_r)$-regular sequence of forms, and $I$ and $I'$ are directly linked by a CI ideal $J$ of type $(a_1, \ldots, a_r)$, then we say $J$ is a minimal link. If $I$ and $I''$ are linked, and there exists a sequence of links from $I$ to $I''$ such that each link is minimal, then we say $I$ is minimally linked to $I''$. An ideal that is minimally linked to a CI ideal is called minimally licci.

Given linked ideals $I$ and $I'$ of $S$, there are many possible sequences of links from $I$ to $I'$ of varying lengths, and much work has been done on understanding licci ideals and their corresponding sequences of links. Gaeta [56] showed that every Cohen–Macaulay ideal of height two is minimally licci, while Peskine and Szpiro [103] proved that an ideal of height two is licci if and only if it is Cohen–Macaulay. However, these results do not extend to ideals of height $\geq 3$. Not
every Cohen–Macaulay ideal of height three is licci [73], and for every \( r \geq 3 \), there are licci ideals of height \( r \) that are not minimally licci [72]. Nevertheless, Watanabe [128] showed that every Gorenstein ideal of height three is licci, while Migliore and Nagel [94] recently proved that every Gorenstein ideal of height three is minimally licci.

10.2 Sequentially bounded licci ideals

In this section, we weaken the notion of ‘minimally licci ideals’ and define a subclass of licci ideals that we call ‘sequentially bounded’. In particular, minimally licci ideals are sequentially bounded licci ideals. As the main result of this section, we prove that the EGH conjecture (Conjecture 9.3.1) holds for every sequentially bounded licci ideal that ‘admits a minimal first link’, which we now define precisely:

**Definition 10.2.1.** Let \( I \) be a licci ideal of height \( r \). Suppose there exists a sequence of links \( I_0 \sim J_1 \sim I_1 \sim \cdots \sim J_s \sim I_s \) from \( I_0 = I \) to a CI ideal \( I_s \), such that \( J_1, \ldots, J_s \) (as CI ideals) have types \( a^{(1)}, \ldots, a^{(s)} \in \mathbb{P}^r \) respectively and satisfy \( a^{(1)} \geq \cdots \geq a^{(s)} \). Then \( I \) is called a *sequentially bounded* licci ideal. Furthermore, if \( J_1 \) is a minimal link, then we say \( I \) is a sequentially bounded licci ideal that *admits a minimal first link*. More generally, if \( J_1 \) is a link that satisfies a certain property \( \mathfrak{P} \) (e.g. \( J_1 \) has a certain type), then we say \( I \) is a sequentially bounded licci ideal that *admits a first link with property \( \mathfrak{P} \).

**Theorem 10.2.2.** Let \( 2 \leq e_1 \leq \cdots \leq e_n \) be integers. If \( I \subsetneq S \) is a sequentially bounded licci ideal that admits a minimal first link and minimally contains an \((e_1, \ldots, e_n)\)-regular sequence of forms, then there exists a monomial ideal \( J \subsetneq S \) containing \( x_1^{e_1}, \ldots, x_n^{e_n} \) such that \( I \) and \( J \) have the same Hilbert function.
Before we prove Theorem 10.2.2, we need the following useful theorem on Hilbert functions under liaison.

**Theorem 10.2.3 ([40]).** Let $I, J$ be (homogeneous) ideals of $S$ such that $J \subseteq I$. If $S/J$ is an Artinian Gorenstein ring, and $s := \max\{t \in \mathbb{N} : H(S/J, t) \neq 0\}$, then $H(S/I, t) = H(S/J, t) - H(S/(J : I), s - t)$ for all $0 \leq t \leq s$.

**Proof of Theorem 10.2.2.** Let $I_0 \sim J_1 \sim I_1 \sim J_2 \sim I_2 \sim \ldots \sim J_s \sim I_s$ be a sequence of links from $I_0 = I$ to a CI ideal $I_s$, where $J_1$ is a minimal link. Let $a^{(1)}, \ldots, a^{(s)}, a^{(s+1)} \in \mathbb{P}^n$ denote the types of $J_1, \ldots, J_s, I_s$ respectively (as CI ideals), and assume $a^{(1)} \geq \ldots \geq a^{(s)}$. Note also that $a^{(s)} \geq a^{(s+1)}$, since $J_s$ and $I_s$ are CI ideals satisfying $J_s \subseteq I_s$. For each $i \in [s + 1]$, write $a^{(i)} = (a_{i,1}, \ldots, a_{i,n})$, and let $\Gamma_i$ be the collection of all monomials in $S$ that divide $x^{(a^{(i)}-1_n)}$. Also, define the collections of monomials $\tilde{\Gamma}_{s+1}, \tilde{\Gamma}_s, \ldots, \tilde{\Gamma}_1$ recursively as follows: Let $\tilde{\Gamma}_{s+1} = \Gamma_{s+1}$, and for each $i \in [s]$, define

$$\tilde{\Gamma}_{s+1-i} := \left\{ q \in \Gamma_{s+1-i} : \frac{x^{(a^{(s+1-i)}-1_n)}}{q} \notin \tilde{\Gamma}_{s+2-i} \right\}.$$

Recall from Chapter 2.7 that a multicomplex $M$ is a collection of monomials that is closed under divisibility (i.e. if $m \in M$ and $m'$ divides $m$, then $m' \in M$).

**Claim.** $\tilde{\Gamma}_i$ is a multicomplex, and $H(S/I_{i-1}, t) = \left| \{ q \in \tilde{\Gamma}_i : \deg(q) = t \} \right|$, for all $i \in [s + 1]$, $t \in \mathbb{N}$.

**Proof of Claim.** We shall prove both assertions of the claim simultaneously by induction on $s + 2 - i$ (for $i \in [s + 1]$). The base case is trivial: $\tilde{\Gamma}_{s+1}$ is clearly a multicomplex, and

$$H(S/I_s, t) = H(S/\langle x_1^{a_{n+1,1}}, \ldots, x_n^{a_{n+1,n}} \rangle, t) = \left| \{ q \in \tilde{\Gamma}_{s+1} : \deg(q) = t \} \right| \quad (10.1)$$

for all $t \in \mathbb{N}$. In particular, $\Gamma_{s+1} = \tilde{\Gamma}_{s+1}$ forms a $\mathbb{k}$-basis for $S/\langle x_1^{a_{n+1,1}}, \ldots, x_n^{a_{n+1,n}} \rangle$.  

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For the induction step, let \( m_{s+1-i} := x^{(a^{(s+1-i)}-1_n)} \). Since \( a^{(1)} \geq \cdots \geq a^{(s+1)} \) implies \( \tilde{\Gamma}_{i+1} \subseteq \Gamma_{i+1} \subseteq \Gamma_i \) for all \( i \in [s] \), it then follows from Theorem 10.2.3 that \( H(S/I_{s-i}, t) \) equals

\[
H(S/J_{s+1-i}, t) - H(S/I_{s+1-i}, |a^{(s+1-i)}| - n - t)
\]

\[
= |\{ q \in \Gamma_{s+1-i} : \deg(q) = t \}| - |\{ q' \in \tilde{\Gamma}_{s+2-i} : \deg(q') = |a^{(s+1-i)}| - n - t \}|
\]

\[
= |\{ q \in \Gamma_{s+1-i} : \deg(q) = t \}| - \left| \left\{ \frac{m_{s+1-i}}{q'} \in \Gamma_{s+1-i} : q' \in \tilde{\Gamma}_{s+2-i}, \deg\left( \frac{m_{s+1-i}}{q'} \right) = t \} \right| \]

\[
= |\{ q \in \Gamma_{s+1-i} : \deg(q) = t \}| - \left| \left\{ q \in \Gamma_{s+1-i} : \frac{m_{s+1-i}}{q} \in \tilde{\Gamma}_{s+2-i}, \deg(q) = t \} \right| \]

\[
= |\{ q \in \tilde{\Gamma}_{s+1-i} : \deg(q) = t \}|
\]

for every \( t \in \mathbb{N} \).

Next, we prove that \( \tilde{\Gamma}_{s+1-i} \) is a multicomplex. Choose an arbitrary \( q \in \Gamma_{s+1-i} \) such that \( q \notin \tilde{\Gamma}_{s+1-i} \), and suppose \( qx_j \in \Gamma_{s+1-i} \) for some \( j \in [n] \). Clearly \( \Gamma_{s+1-i} \) is a multicomplex containing \( \tilde{\Gamma}_{s+1-i} \), hence to prove that \( \tilde{\Gamma}_{s+1-i} \) is a multicomplex, it suffices to show that \( qx_j \notin \tilde{\Gamma}_{s+1-i} \). Now, \( qx_j \in \Gamma_{s+1-i} \) implies \( x_j^{(a^{(s+1-i)},j-1)} \) does not divide \( q \), which means \( x_j \) divides \( \frac{m_{s+1-i}}{q} \), so since \( \tilde{\Gamma}_{s+2-i} \) is a multicomplex by induction hypothesis, we thus get \( \frac{m_{s+1-i}}{qx_j} \in \tilde{\Gamma}_{s+2-i} \) which yields \( qx_j \notin \tilde{\Gamma}_{s+1-i} \). \( \blacksquare \)

With the above claim, we now complete the proof of Theorem 10.2.2. Let \( J \subset S \) be the ideal spanned by monomials in \( S \) that are not contained in \( \tilde{\Gamma}_1 \). Using the claim, \( \tilde{\Gamma}_1 \) is a multicomplex, hence \( \tilde{\Gamma}_1 \) forms a \( \mathbb{k} \)-basis for \( S/J \), and we get

\[
H(S/I, t) = H(S/I_0, t) = |\{ q \in \tilde{\Gamma}_1 : \deg(q) = t \}| = H(S/J, t).
\]

Finally, since \( J_1 \) is a minimal link, we have \((e_1, \ldots, e_n) = a^{(1)}\), so \( \tilde{\Gamma}_1 \subseteq \Gamma_1 \) implies \( J \) contains \( x_1^{e_1}, \ldots, x_n^{e_n} \). \( \square \)

**Corollary 10.2.4.** Let \( 2 \leq e_1 \leq \cdots \leq e_n \) be integers. If \( I \subset S \) is a minimally licci ideal that minimally contains an \((e_1, \ldots, e_n)\)-regular sequence of forms, then
there exists a monomial ideal in $S$ containing $x_1^{e_1}, \ldots, x_n^{e_n}$ with the same Hilbert function as $I$.

Proof. Given any sequence $I_0 \overset{J_1}{\sim} I_1 \overset{J_2}{\sim} I_2$ of minimal links, where $J_1$ and $J_2$ have types $a$ and $b$ respectively, the definition of liaison yields $I_0 \overset{J_1}{\sim} I_1 \overset{J_2}{\sim} I_0$, hence the minimality of $J_2$ as a link implies $b \leq a$. Consequently, a minimally licci ideal is a sequentially bounded licci ideal that admits a minimal first link, and the assertion follows from Theorem 10.2.2.

\begin{theorem}
\label{thm:minimallicci}
Let $2 \leq e_1 \leq e_2 \leq e_3$ be integers. If $I \subset \mathbb{k}[x_1, x_2, x_3]$ is a homogeneous Gorenstein ideal that minimally contains an $(e_1, e_2, e_3)$-regular sequence of forms, then there exists a monomial ideal $J$ in $\mathbb{k}[x_1, x_2, x_3]$ containing $x_1^{e_1}, x_2^{e_2}, x_3^{e_3}$, such that $I$ and $J$ have the same Hilbert function.
\end{theorem}

Theorem 10.2.5 is an immediate consequence of Corollary 10.2.4, since every Gorenstein ideal of height three is minimally licci [94]. In fact, Migliore and Nagel [94] proved a stronger result: If $I$ is a Gorenstein ideal that is not a CI ideal, then linking $I$ minimally twice gives a Gorenstein ideal with two fewer generators than $I$. Their proof uses Buchsbaum-Eisenbud’s structure theorem [24], which says that every Gorenstein ideal of height three is generated by the submaximal Pfaffians of an alternating matrix. Note that Watanabe [128] previously showed the minimal number of generators of every Gorenstein ideal of height three is odd.

\begin{remark}[The Two Variables Case]
Every ideal $I \subset S$ containing a maximal $S$-regular sequence is Artinian and hence Cohen–Macaulay. Since Gaeta’s theorem [56] says every Cohen–Macaulay ideal of height two is minimally licci, Corollary 10.2.4 thus yields a different proof (cf. [106], [29, Remark 14]) that the EGH conjecture holds for $n = 2$; see Chapter 9.2.
\end{remark}
10.3 EGH conjecture for non-maximal regular sequences

Previously in Chapter 9.3, we discussed equivalent versions of the EGH conjecture. In particular, we know that Caviglia–Maclagan [29] showed the equivalence of $\text{EGH}_{n,e,r}$ and $\text{EGH}_{e,n}$, which are the versions with not necessarily maximal $S$-regular sequences, and maximal $S$-regular sequences, respectively. Using a modification of their proof, we show that $\text{EGH}_{\min \deg}$ holds for every sequentially bounded licci ideal that admits a minimal first link whenever $n \geq r$.

**Lemma 10.3.1.** Let $I_1, I_2 \subsetneq S$ be ideals of height $r < n$, and let $f_1, \ldots, f_r, g$ be an $S$-regular sequence of forms. Define $J := \langle f_1, \ldots, f_r \rangle$ and $J' := \langle f_1, \ldots, f_r, g \rangle$.

If $I_1 \sim J I_2$, then

$$\left((I_1 : g^j) + \langle g \rangle\right) \sim \left((I_2 : g^j) + \langle g \rangle\right)$$

for every $j \in \mathbb{N}$.

**Proof.** For convenience, write $I'_i := \left((I_i : g^j) + \langle g \rangle\right)$ for each $i \in \{1, 2\}$. The case $j = 0$ is trivial, so assume $j \geq 1$. Observe that the identity $I_2 = J : I_1$ yields

$$I'_2 = \frac{1}{g^j} \left((J : I_1) \cap \langle g^j \rangle\right) + \langle g \rangle = \left\{ \frac{s'}{g^j} \in S : s' \in \langle g^j \rangle, s'I_1 \subseteq J \right\} + \langle g \rangle$$

$$= \left\{ s \in S : sg^j I_1 \subseteq J \right\} + \langle g \rangle,$$

while the identity $I_1 = J : (J : I_1)$ yields

$$J' : I'_1 = \left\{ s \in S : s \cdot \frac{1}{g^j} (I_1 \cap \langle g^j \rangle) \subseteq J \right\} + \langle g \rangle$$

$$= \left\{ s \in S : s \cdot \frac{1}{g^j} ((J : (J : I_1)) \cap \langle g^j \rangle) \subseteq J \right\} + \langle g \rangle$$

$$= \left\{ s \in S : ss' \in J \text{ for every } s' \in S' \right\} + \langle g \rangle,$$

where $S' := \{ s' \in S : s' t g^j \in J \text{ for all } t \in S \text{ such that } tI_1 \subseteq J \}$.  

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A routine check gives $J' : I_1' \subseteq I_2'$. To show the reverse inclusion $J' : I_1' \supseteq I_2'$, note that $sp \in J$ for every non-zero $s \in S$ and non-zero $p \in I_1$ satisfying $sp^j p \in J$, since otherwise $g^j$ would be a zero-divisor of $S/J$, which contradicts the assumption that $f_1, \ldots, f_r, g$ is an $S$-regular sequence. By the same argument, every $s' \in S'$ satisfies $s't$ for all $t \in S$ such that $tI_1 \subseteq J$. Consequently, each $s \in I_2'$ satisfies $sI_1 \subseteq J$ and hence satisfies $ss' \in J$ for all $s' \in S'$, which gives the reverse inclusion, so we conclude $J' : I_1' = I_2'$. A symmetric argument yields $J' : I_2' = I_1'$.

Finally, since $f_1, \ldots, f_r, g$ is an $S$-regular sequence, we get $J \cap \langle g^j \rangle = \langle g^j f_1, \ldots, g^j f_r \rangle$. It then follows from $J \subseteq I_1 \cap I_2$ that

$$J' = \frac{1}{g^j}(J \cap \langle g^j \rangle) + \langle g \rangle \subseteq I_1' \cap I_2', \quad (10.3)$$

therefore $I_1' \sim J' \sim I_2'$.

**Theorem 10.3.2.** Let $r \in [n]$, let $2 \leq e_1 \leq \cdots \leq e_r$ be integers, and let $I \subsetneq S$ be an ideal (of height $r$) that minimally contains an $(e_1, \ldots, e_r)$-regular sequence of forms. If $I \subsetneq S$ is a sequentially bounded licci ideal that admits a minimal first link (for example, $I$ could be a minimally licci ideal), then there exists a monomial ideal $J \subsetneq S$ containing $x_1^{e_1}, \ldots, x_r^{e_r}$ such that $I$ and $J$ have the same Hilbert function.

**Proof.** We follow Caviglia-Maclagan’s proof and similarly prove our proposition by induction on $n - r \in \mathbb{N}$, where the base case $n - r = 0$ is equivalent to Theorem 10.2.2. Assume $r < n$, fix the integers $2 \leq e_1 \leq \cdots \leq e_r$, and let $I = I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_s$ be a sequence of links from $I_0$ to a CI ideal $I_s$, such that $J_1, \ldots, J_s$ (as CI ideals) have types $a^{(1)}, \ldots, a^{(s)}, a^{(s+1)} \in \mathbb{P}^r$ respectively and satisfy $a^{(1)} \geq \cdots \geq a^{(s+1)}$. In particular, $J_s$ and $I_s$ are CI ideals satisfying $J_s \subseteq I_s$, so the last inequality $a^{(s)} \geq a^{(s+1)}$ is guaranteed. Also, assume $J_1$ is a minimal link and $a^{(1)} = (e_1, \ldots, e_r)$. Since $\mathbb{k}$ is infinite and $r < n$, we can choose some linear
form $g$ that is a non-zero-divisor on each of $S/J_1, \ldots, S/J_s, S/I_s$. Consequently, for each $i \in [s]$, we can define the CI ideal $J_i' := J_i + \langle g \rangle$ of type $(1, a^{(i)}) \in \mathbb{P}^{r+1}$.

Next, let $N \in \mathbb{P}$ be sufficiently large so that $(I : g^\infty) = (I : g^N)$. For each $i \in \{0, 1, \ldots, s\}$ and $j \in \{0, 1, \ldots, N\}$, define the ideal $I^{(j)}_i := (I_i : g^j) + \langle g \rangle$. By Lemma 10.3.1, we get the sequence of links

$$I_0^{(j)} \sim I_1^{(j)} \sim \ldots \sim I_s^{(j)}$$

(10.4)
for every $j \in \{0, 1, \ldots, N\}$, and we observe that each $I_s^{(j)}$ is a CI ideal.

For each $i \in [s]$, write $J_i = \langle f_1^{(i)}, \ldots, f_r^{(i)} \rangle$ so that

$$a^{(i)} = (\deg(f_1^{(i)}), \ldots, \deg(f_r^{(i)})).$$

Also, write $I_s = \langle f_1^{(s+1)}, \ldots, f_r^{(s+1)} \rangle$ so that

$$a^{(s+1)} = (\deg(f_1^{(s+1)}), \ldots, \deg(f_r^{(s+1)})).$$

The quotient ring $R := S/\langle g \rangle$ is isomorphic to a polynomial ring on $n-1$ variables, and by the natural quotient map $\pi : S \to R$, the $S$-regular sequence $f_1^{(i)}, \ldots, f_r^{(i)}$ descends to an $R$-regular sequence $\pi(f_1^{(i)}), \ldots, \pi(f_r^{(i)})$ for each $i \in [s+1]$. In particular, $J_i'' := \pi(J_i')$ is a CI ideal in $R$ for all $i \in [s]$. By applying $\pi$ to the ideals in (10.4), we get the sequence of links

$$\pi(I_0^{(j)}) \sim \pi(I_1^{(j)}) \sim \ldots \sim \pi(I_s^{(j)})$$

(10.5)
for every $j \in \{0, 1, \ldots, N\}$.

Now $J_1'', \ldots, J_s''$ (as CI ideals) have types $a^{(1)}, \ldots, a^{(s)}$ respectively, and for every $j \in \{0, 1, \ldots, N\}$, the ideal $\pi(I_s^{(j)})$ is a CI ideal of type $a^{(s+1)}$, thus $\pi(I_0^{(j)})$ is a sequentially bounded licci ideal in $R$ that admits a minimal first
Using $R \cong k[x_1, \ldots, x_{n-1}]$, the induction hypothesis then says that for every $j \in \{0, 1, \ldots, N\}$, there exists a monomial ideal in $k[x_1, \ldots, x_{n-1}]$ containing $x_1^{e_1}, \ldots, x_r^{e_r}$ with the same Hilbert function as $\pi(I_0^{(j)})$. Let $M_j$ be the lex-plus-powers ideal in $k[x_1, \ldots, x_{n-1}]$ containing $x_1^{e_1}, \ldots, x_r^{e_r}$ with this same Hilbert function (which exists by Clements-Lindström theorem [36]; see Chapter 2.8), and let $K_j \subseteq k[x_1, \ldots, x_n]$ be the set of monomials $K_j := \{mx_j^m : m \in M_j\}$.

Finally, consider the ideal $K$ generated by the monomials in $\bigcup_{j=0}^N K_j$. As shown by Caviglia and Maclagan [29, Proposition 10], the ideal $K$ contains $x_1^{e_1}, \ldots, x_r^{e_r}$ and has the same Hilbert function as $I$, and their proof holds verbatim.

**Corollary 10.3.3.** **Gorenstein ideal and EGH conjecture** Let $n \geq 3$, and let $2 \leq e_1 \leq e_2 \leq e_3$ be integers. If $I \subseteq S$ is a homogeneous Gorenstein ideal (of height three) that minimally contains an $(e_1, e_2, e_3)$-regular sequence of forms, then there exists a monomial ideal $J \subseteq S$ containing $x_1^{e_1}, x_2^{e_2}, x_3^{e_3}$, such that $I$ and $J$ have the same Hilbert function.

**Proof.** This follows from Theorem 10.3.2, since every Gorenstein ideal of height three is minimally licci [94]; cf. Theorem 10.2.5.

There are several other equivalent versions of the EGH conjecture that we discussed in Chapter 9.3. We next state an equivalent formulation of Theorem 10.3.2, which follows from (an analogue of) Proposition 9.3.2 and the following lemma:

**Lemma 10.3.4.** Let $r \in [n]$, and let $2 \leq e_1 \leq \cdots \leq e_r$ be integers. If $I \subseteq S$ is a sequentially bounded licci ideal that admits a first link $J_1$ of type $(e_1, \ldots, e_r)$, then $I$ must contain an $S$-regular sequence $f_1, \ldots, f_r$ of forms of degrees $e_1, \ldots, e_r$ respectively.
Proof. Clearly $J_1 \subseteq I$ by the definition of a link, hence $I$ contains an $S$-regular sequence $f'_1, \ldots, f'_r$ of degrees $e'_1, \ldots, e'_r$ respectively for some $n$-tuple of integers $(e'_1, \ldots, e'_r) \leq (e_1, \ldots, e_r)$. Note that $k$ is infinite by assumption. We then use Theorem 3.4.6(i), Theorem 3.4.6(iv), and Theorem 3.4.6(ii) iteratively to conclude that there exist forms $y_1, \ldots, y_r \in S$ of degrees $(e_1 - e'_1), \ldots, (e_r - e'_r)$ respectively, such that $f'_1 y_1, \ldots, f'_r y_r$ is an $S$-regular sequence of forms of degrees $e_1, \ldots, e_r$ respectively, which we check is contained in $I$.

Theorem 10.3.5. Let $r \in [n]$, and let $2 \leq e_1 \leq \cdots \leq e_r$ be integers. If $I \not\subseteq S$ is a sequentially bounded licci ideal (of height $r$) that admits a first link of type $\leq (e_1, \ldots, e_r)$, then there exists a monomial ideal $J \not\subseteq S$ containing $x_1^{e_1}, \ldots, x_r^{e_r}$ such that $I$ and $J$ have the same Hilbert function.

Notes

By considering only homogeneous ideals in $S$, we know from the definitions that the sequence of inclusions in Figure 10.1 hold.

![Figure 10.1: Inclusions involving licci ideals.](image)

As we have discussed at the end of Chapter 10.1, every Cohen–Macaulay ideal of height two is minimally licci (Gaeta [56]), while an ideal of height two is licci if and only if it is Cohen–Macaulay (Peskine–Szpiro [103]), thus for ideals of height two, all the inclusions in Figure 10.1 are in fact equalities. However, for every $r \geq 3$, Huneke–Migliore–Nagel–Ulrich [72] in 2007 constructed licci ideals of height $r$ that are not minimally licci, thus for (homogeneous) ideals of height $r \geq 3$, at least one of the inclusions in Figure 10.1 is strict.

Question 10.3.6. For homogeneous ideals of height $r \geq 3$, which of the inclusions in Figure 10.1 are strict?
In this chapter, we have only used results in CI-liaison theory. It is then natural to ask if we can apply results in G-liaison theory to obtain an analogue of Theorem 10.2.2 involving glicci ideals. Specifically, we pose the following questions:

**Question 10.3.7.** What does it mean to be “minimally glicci” or “sequentially bounded glicci”? Do these notions make sense? How do we compare consecutive G-links?

Also, we note that sequentially bounded licci ideals are defined combinatorially. Do these ideals have any geometric significance?

Finally, we end this chapter/part with a personal anecdote by the author.

Before the author tried his hands on the EGH conjecture, he had never heard of liaison theory. He had, however, knowledge of colon ideals (ideal quotients), and when he thought about the EGH conjecture, he tried working backwards from Theorem 3.4.6(v) by taking quotients with CI ideals. This seemingly naïve idea gave rise to Theorem 10.2.2, although his original formulation of Theorem 10.2.2 definitely did not contain any liaison theoretic terminology. What he had was a sequence of quotients by CI ideals starting with a CI ideal, and he suspected that such sequences ought to be well-studied, if not well-known.

After relooking at the papers relevant to the EGH conjecture, the author noticed that Theorem 3 in a paper by Davis–Geramita–Orecchia [40] was labeled with the words “Hilbert functions under liaison”. He understood what Theorem 3 said, but he did not understand what “liaison” meant in this theorem. Doing an extensive literature search, the author learned about liaison theory, a whole new branch of algebraic geometry that he previously did not know existed. (To be fair, Eisenbud does discuss linkage theory in his book [45, Chap. 21.10].) Finding the link to linkage theory was definitely a pleasant surprise for the author, and he soon realized that what he had proven was closely related to minimally licci ideals. It was an even more pleasant surprise when the author learned about Migliore–Nagel’s recent result [94] that every Gorenstein ideal of height three is minimally licci, which led to Theorem 10.2.5.

The author is very grateful to Irena Peeva for introducing the EGH conjecture in her lectures in such an intriguing manner that he became curious and tried working on it.
Part IV

Generalized Macaulay representations and the flag $f$-vectors of generalized colored complexes
CHAPTER 11
COLORED ANALOGUES OF COMPRESSION AND SHIFTING

In this chapter, we look at colored analogues of compression and shifting (cf. Chapter 2.5). Chapter 11.1 deals with color compression, while Chapter 11.2 deals with color shifting and colored algebraic shifting.

Color compression was first introduced by Björner, Frankl and Stanley [19] in 1987 in the context of colored multicomplexes, although under the same original name of ‘compression’. Around the same time, the idea of color shifting was also floating around within the combinatorics community [100]. However, it was only in 2006 that the notion of color shifting first appeared in publication, during which Babson and Novik [8] developed a remarkable theory of colored algebraic shifting for \(a\)-colored complexes. In fact, colored algebraic shifting is a colored analogue of symmetric algebraic shifting proposed by Kalai [75, 76], and one of the most important properties of colored algebraic shifting is that it preserves the Cohen–Macaulayness of an \(a\)-balanced Cohen–Macaulay complex, provided a suitable linear order of its vertices is chosen; see Theorem 11.2.9.

Throughout this chapter, let \(X\) be a set of variables, and let \(\pi = (X_1, \ldots, X_n)\) be an ordered partition of \(X\) such that each \(X_i = \{x_{i,1}, x_{i,2}, \ldots\}\) is linearly ordered by \(x_{i,1} < x_{i,2} < \cdots\). For every \(i \in [n]\), let \(\phi_i : X_i \to \mathbb{P}\) be an arbitrary map, and define \(e_i = (e_{i,1}, e_{i,2}, \ldots) := (\phi_i(x_{i,1}), \phi_i(x_{i,2}), \ldots)\). Recall that \(\mathcal{M}_\pi(e_1, \ldots, e_n)\) denotes the collection of all monomials \(m\) in variables \(X\) such that \(m_{x_{i}} \in \mathcal{M}_{X_{i}}(e_{i})\) for every \(i \in [n]\), and when \(e_i = (\infty, \infty, \ldots)\) for every \(i \in [n]\), we simply write \(\mathcal{M}_\pi(e_1, \ldots, e_n)\) as \(\mathcal{M}_\pi\).
11.1 Color compression

The main goal of this section is to prove that color compression preserves the fine $f$-vectors of colored multicomplexes in $\mathcal{M}_\pi(e_1, \ldots, e_n)$ (Theorem 11.1.3). The ideas involved are not new, and our proof follows from a slight modification of the proof of [19, Theorem 1], which considered colored multicomplexes in $\mathcal{M}_\pi$. Note that Theorem 11.1.3 is analogous to how the Clements–Lindström theorem [36] (introduced in Chapter 2.8) extends both the Kruskal–Katona theorem (see Chapter 2.6) and Macaulay’s theorem (see Chapter 3.6).

First of all, recall that an order ideal of a poset $(P, \leq)$ is a subset $I \subseteq P$ such that if $x \in P$ and $y \leq x$, then $y \in I$. If $\leq$ is a well-ordering on $P$, then the order ideal $I$ is called an initial segment of $P$ (with respect to $\leq$). Observe that for every $i \in [n]$, $d \in \mathbb{N}$, we can order the monomials in $\mathcal{M}_d X_i(e_i)$ by treating $\mathcal{M}_d X_i(e_i)$ as a subposet of $\mathcal{M}_d^d X_i(e_i)$ with the induced lex order. This induced lex order is clearly a well-ordering on $\mathcal{M}_d^d X_i(e_i)$, and we write $I_d^{d,e_i}(N)$ to denote the lex initial segment of $\mathcal{M}_d^d X_i(e_i)$ of size $N$.

Let $(M, \pi)$ be a colored multicomplex of type $a \in \mathbb{P}^n$ in $\mathcal{M}_\pi(e_1, \ldots, e_n)$, and fix some $t \in [n]$. Every monomial $m \in M$ can be factorized as $m = m_{X_t} m_{X-X_t}$, hence we can write $M$ as the disjoint union

$$M = \bigsqcup_{\tilde{m} \in M_{X_t}} \left( \bigsqcup_{d \in \mathbb{N}} \left\{ m \in M : m_{X-X_t} = \tilde{m}, \deg(m/\tilde{m}) = d \right\} \right). \quad (11.1)$$

For each $d \in \mathbb{N}$, $\tilde{m} \in M_{X_t}$, let $f_d(\tilde{m})$ be the number of $m \in M$ such that $m_{X-X_t} = \tilde{m}$ and $\deg(m/\tilde{m}) = d$. Define the operation $C_t$ on $M$ by

$$C_t(M) := \bigsqcup_{\tilde{m} \in M_{X_t}} \left( \bigsqcup_{d \in \mathbb{N}} \left\{ \tilde{m} m^* : m^* \in I_d^{d,e_i}(f_d(\tilde{m})) \right\} \right). \quad (11.2)$$
Note that when $M_{X_t} = M$, this specializes to the usual notion of compression for multicomplexes. The Clements–Lindström theorem [36] is equivalent to the statement that if $e_{t,1} \geq e_{t,2} \geq \cdots$, and $M_{X_t} = M$, then $\mathcal{C}_t(M)$ is a multicomplex; cf. Chapter 2.8. Using this version of Clements–Lindström theorem, we prove the following important lemma.

**Lemma 11.1.1.** If $e_{t,1} \geq e_{t,2} \geq \cdots$, then $\mathcal{C}_t(M)$ is a multicomplex.

**Proof.** Let $p$ be an arbitrary monomial in $\mathcal{M}_\pi(e_1, \ldots, e_n)$, and suppose there is some $x \in X$ such that $p' := px$ is in $\mathcal{C}_t(M)$. To show $\mathcal{C}_t(M)$ is a multicomplex, we have to show that $p \in \mathcal{C}_t(M)$.

Suppose $x \in X - X_t$. Let $q := p_{X-X_t}$, $q' := p'_{X-X_t}$, $d := \deg(p/q)$, and note that $q' = qx$, $\deg(p'/q') = d$. Every $m \in M$ satisfying $m_{X-X_t} = q'$ must also satisfy $m/x \in M$ and $(m/x)_{X-X_t} = q$, hence $f_d(q') \leq f_d(q)$. This means $\mathcal{T}_{X_t}^{\text{den}}(f_d(q')) \subseteq \mathcal{T}_{X_t}^{\text{den}}(f_d(q))$, therefore $p \in \mathcal{C}_t(M)$.

Suppose instead $x \in X_t$. Again let $q := p_{X-X_t}$, and observe that $p'_{X-X_t} = q$. For any subcollection $M' \subseteq \mathcal{M}_\pi$, define $M'/q := \{m/q : m \in M', m_{X-X_t} = q\}$. We check that $M/q$ is a subcomplex of $M$ satisfying $(M/q)_{X_t} = M/q$. Since $\mathcal{C}_t(M/q) = \mathcal{C}_t(M)/q$, it follows from $p' \in \mathcal{C}_t(M)$ and $p'_{X-X_t} = q$ that $p'/q \in \mathcal{C}_t(M/q)$. Now $e_{t,1} \geq e_{t,2} \geq \cdots$, so the Clements–Lindström theorem says $\mathcal{C}_t(M/q)$ is a multicomplex, thus $p/q \in \mathcal{C}_t(M/q)$, which implies $p \in \mathcal{C}_t(M)$. □

Clearly, $(\mathcal{C}_t(M), \pi)$ is an $a$-colored multicomplex in $\mathcal{M}_\pi(e_1, \ldots, e_n)$ with the same fine $f$-vector as $(M, \pi)$. If $\mathcal{C}_t(M) = M$ for all $t \in [n]$, then we say $(M, \pi)$ is *color-compressed*. Equivalently, we have the following definition.

**Definition 11.1.2.** Let $a \in \mathbb{P}^n$, and let $(M, \pi)$ be an $a$-colored multicomplex
in $\mathcal{M}_\pi(e_1, \ldots, e_n)$. We say $(M, \pi)$ is color-compressed if for each $t \in [n]$ and $m \in M_{X-X_t}$, the set $\{m' \in M_{X_t} : mm' \in M, \deg(m') = k\}$ is a lex initial segment of $\mathcal{M}^k_{X_t}(e_t)$ for all $k \geq 0$.

**Theorem 11.1.3.** Let $a \in \mathbb{P}^n$, and let $f = \{f_b\}_{b \leq a}$ be an array of integers. If $e_{i,1} \geq e_{i,2} \geq \cdots$ for every $i \in [n]$, then the following are equivalent:

(i) $f$ is the fine $f$-vector of an $a$-colored multicomplex in $\mathcal{M}_\pi(e_1, \ldots, e_n)$.

(ii) $f$ is the fine $f$-vector of a color-compressed $a$-colored multicomplex in $\mathcal{M}_\pi(e_1, \ldots, e_n)$.

**Proof.** Let $(M, \pi)$ be an $a$-colored multicomplex in $\mathcal{M}_\pi(e_1, \ldots, e_n)$ that is not color-compressed. Starting with $M_0 = M$, iteratively construct a sequence $M_0, M_1, M_2, \ldots$ of multicomplexes such that $M_i = C_t(M_{i-1})$ for each $i \geq 1$, where $t_i \in [n]$ is chosen so that $M_i \neq M_{i-1}$. Given $t \in [n]$ and any $a$-colored multicomplex $(M', \pi)$ in $\mathcal{M}_\pi(e_1, \ldots, e_n)$, we get

$$\sum_{p \in M'} \sum_{i \in [n]} |\{q \in M_{X_i}(e_i) : q \leq_{\text{lex}} p_{X_i}\}| \geq \sum_{p \in C_t(M')} \sum_{i \in [n]} |\{q \in M_{X_i}(e_i) : q \leq_{\text{lex}} p_{X_i}\}|,$$

with equality holding if and only if $C_t(M') = M'$. Thus, the sequence $M_0, M_1, M_2, \ldots$ must terminate, say with last term $M_k$, and $(M_k, \pi)$ is a color-compressed $a$-colored multicomplex in $\mathcal{M}_\pi(e_1, \ldots, e_n)$ with the same fine $f$-vector as $(M, \pi)$, which proves (i) $\Rightarrow$ (ii). The converse (ii) $\Rightarrow$ (i) is trivial. \(\square\)

**Definition 11.1.4.** Let $(\Delta, \rho)$ be an $a$-colored complex for some $a \in \mathbb{P}^n$, where $\rho = (V_1, \ldots, V_n)$ is an ordered partition of the vertex set $V$ of $\Delta$, and assume $V_i$ is linearly ordered for each $i \in [n]$. We say $(\Delta, \rho)$ is color-compressed if for all $F \in \Delta$ and $t \in [n]$, the set $\{F' \cap V_i : F' \in \Delta, \lvert F' \cap V_i \rvert = \lvert F \cap V_i \rvert, F' \cap (V \setminus V_i) = F \cap (V \setminus V_i)\}$ is a colex initial segment of $(\langle V_i \rangle_{F \cap V_i})$. 

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By setting $e_i = (1, 1, \ldots)$ for each $i \in [n]$, Theorem 11.1.3 yields the following:

**Corollary 11.1.5.** Let $a \in \mathbb{P}^n$, and let $f = \{f_b\}_{b \leq a}$ be an array of integers. The following are equivalent:

(i) $f$ is the fine $f$-vector of an $a$-colored complex.

(ii) $f$ is the fine $f$-vector of a color-compressed $a$-colored complex.

In the proof of Theorem 11.1.3, we started with an $a$-colored multicomplex $(M, \pi)$ in $\mathcal{M}_\pi(e_1, \ldots, e_n)$ that is not color-compressed, and we constructed a finite sequence of multicomplexes in $\mathcal{M}_\pi(e_1, \ldots, e_n)$:

$$M, C_{t_1}(M), C_{t_2}(C_{t_1}(M)), \ldots, C_{t_k}(C_{t_{k-1}}(\cdots C_{t_1}(M))).$$ (11.4)

All the terms in this sequence are distinct, and the last term corresponds to a color-compressed colored multicomplex. Clearly if $n = 1$ (i.e. in the “uncolored” case), the last term is uniquely determined by $M$, known as the compression of $M$. However, if $n > 1$, then in general, different choices of $t_1, \ldots, t_k$ determine different color-compressed colored multicomplexes, as the following example shows.

**Example 11.1.6.** Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ be sets of variables, each linearly ordered by $x_1 < x_2 < x_3$ and $y_1 < y_2 < y_3$ respectively, and set $\pi = (X, Y)$. Consider the colored multicomplex $(M, \pi)$ of type $(1, 1)$ in $\mathcal{M}_\pi(1_3, 1_3)$, given by $M = \{x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_3y_1, x_3y_3\}$. Note that $(C_1(M), \pi)$ and $(C_2(M), \pi)$ are both color-compressed, yet $C_1(M) \neq C_2(M)$.

Let $(M, \pi)$ and $(M', \pi)$ be colored multicomplexes in $\mathcal{M}_\pi(e_1, \ldots, e_n)$, and suppose $(M', \pi)$ is color-compressed. If $M' = C_{t_k}(C_{t_{k-1}}(\cdots C_{t_1}(M)))$ for some finite sequence $t_1, \ldots, t_k$ of integers in $[n]$, then $(M', \pi)$ is called a color compression of $(M, \pi)$. It is easy to see that the integers $t_1, \ldots, t_k$ in (11.4) are distinct, hence
every colored multicomplex in $\mathcal{M}_\pi(e_1, \ldots, e_n)$ has at most $n!$ color compressions. We leave the reader to verify that if each $X_i$ is infinite, then in fact almost every colored multicomplex (in the probabilistic sense) in $\mathcal{M}_\pi(e_1, \ldots, e_n)$ has $n!$ color compressions.

For $n \neq 1$, the non-uniqueness of color compressions of colored multicomplexes and the non-uniqueness of color-compressed multicomplexes with a given fine $f$-vector [19] suggest the usual definition of the $i$-th Macaulay representation of a positive integer $N$ (which is uniquely determined given $N$ and $i$) is inadequate for the task of numerically characterizing the fine $f$-vectors of color-compressed a-colored multicomplexes in $\mathcal{M}_\pi(e_1, \ldots, e_n)$. Later in Chapter 13, we will introduce generalized Macaulay representations to address these non-uniqueness issues.

### 11.2 Color shifting and colored algebraic shifting

The notion of shifting (resp. color shifting), which is usually defined for simplicial complexes (resp. colored complexes), can naturally be extended to multicomplexes (resp. colored multicomplexes). Let $Y = \{y_1, y_2, \ldots\}$ be a set of variables linearly ordered by $y_1 < y_2 < \cdots$, let $\phi : Y \to \mathbb{P}$ be an arbitrary map, and define $e = (e_1, e_2, \ldots) := (\phi(y_1), \phi(y_2), \ldots)$. A multicomplex $M' \in \mathcal{M}_Y(e)$ is called shifted (in $\mathcal{M}_Y(e)$) if every $m' \in M'$ satisfies the following property: If $y_j$ divides $m'$ and $y_i^{e_i}$ does not divide $m'$ for some integers $1 \leq i < j$, then $m'y_i/y_j \in M'$. In the case when $M'$ is finite and $e = (1, 1, \ldots)$, we can identify $M'$ with a simplicial complex, and this notion of ‘shifted’ coincides with the usual notion of a shifted complex; see Chapter 2.5. Given $a \in \mathbb{P}^n$, an a-colored multicomplex $(M, \pi)$ in $\mathcal{M}_\pi(e_1, \ldots, e_n)$ is called color-shifted if $M_{X_i}$ is shifted in $\mathcal{M}_{X_i}(e_i)$ for every $i \in [n]$.  

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A color-shifted colored complex of arbitrary type can be analogously defined as follows.

**Definition 11.2.1.** Let \((\Delta, \rho)\) be an \(a\)-colored complex for some \(a \in \mathbb{P}^n\), and let 
\[
\rho = (V_1, \ldots, V_n). 
\]
For each \(i \in [n]\), assume 
\[
V_i = \{v_{i,1}, \ldots, v_{i,\lambda_i}\} 
\]
has \(\lambda_i\) elements linearly ordered by 
\[
v_{i,1} < \cdots < v_{i,\lambda_i}. 
\]
Then we say \((\Delta, \rho)\) is color-shifted if every 
\(F \in \Delta\) and \(i \in [n]\) satisfy the following property: If 
\(v_{i,j} \in F\) and \(v_{i,j'} \not\in F\) for some integers 
\(1 \leq j' < j \leq \lambda_i\), then 
\((F \setminus \{v_{i,j}\}) \cup \{v_{i,j'}\} \in \Delta.\)

**Remark 11.2.2.** Let \(a \in \mathbb{P}^n\), and let \((M, \pi)\) be an \(a\)-colored multicomplex in 
\(\mathcal{M}_\pi(e_1, \ldots, e_n)\). It clearly follows from definition that if \((M, \pi)\) is color-compressed, 
then \((M, \pi)\) is color-shifted. In general, the converse is not true. However, if \(a = 1_n\), then 
\((M, \pi)\) being color-shifted is equivalent to \((M, \pi)\) being color-compressed, 
i.e. the notions of ‘color-shifted’ and ‘color-compressed’ are equivalent, and we 
leave this easy exercise to the reader.

The rest of this section deals with the notion of colored algebraic shifting introduced by Babson and Novik [8]. Assume for the rest of this section that \(X\) is 
a finite set. Recall we have defined a linear order on \(X_i\) for each \(i \in [n]\), so that 
\(X\) as a poset is a disjoint union of \(n\) chains. Let \(\prec\) be any linear extension of this 
partial order on \(X\), and let \(\prec_{rt} \) be the degree-revlex order on 
\(\mathcal{M}_\pi\) induced by \(\prec\). The main idea of colored algebraic shifting is that given \(\prec\), \(a \in \mathbb{P}^n\), and any 
\(a\)-colored complex \((\Gamma, \pi \cap V)\) (where \(V \subseteq X\) denotes the vertex set of \(\Gamma\)), we can 
construct another simplicial complex \(\tilde{\Delta}_\prec(\Gamma)\) with vertex set \(V' \subseteq X\), such that 
\((\tilde{\Delta}_\prec(\Gamma), \pi \cap V')\) is a color-shifted \(a\)-colored complex having the same fine \(f\)-vector 
as \((\Gamma, \pi \cap V)\). This map \((\Gamma, \pi \cap V) \mapsto (\tilde{\Delta}_\prec(\Gamma), \pi \cap V')\) is called colored algebraic shifting, and the \(a\)-colored complex 
\((\tilde{\Delta}_\prec(\Gamma), \pi \cap V')\) is called the colored algebraic shifting of \((\Gamma, \pi \cap V)\) (with respect to \(\prec\)). If the context is clear, we simply say
\(\tilde{\Delta}_\prec(\Gamma)\) is the colored algebraic shifting of \(\Gamma\).

To define \(\tilde{\Delta}_\prec(\Gamma)\), we first have to fix some notation. Let \(|X_i| = \lambda_i \in \mathbb{P}\) for each \(i \in [n]\), and write \(\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{P}^n\). In particular, \(\lambda_i < \infty\) for all \(i \in [n]\), since we assumed that \(X\) is finite. Let \(k\) be a field of characteristic zero, and let \(k[X]\) denote the polynomial ring over \(k\) with the set of variables \(X\), such that \(k[X]\) is \(\mathbb{N}^n\)-graded by multidegree \(\text{Deg}_\pi\). Note that \(\prec_{\text{lex}}\) is a monomial order (see Chapter 2.4). Given \(f \in k[X]\) and a monomial order \(<\) on \(k[X]\), define the initial term of \(f\), written in \(<\>(f)\), to be the greatest term of \(f\) with respect to \(<\). If \(I \subseteq k[X]\) is a homogeneous ideal, then the initial ideal of \(I\) is the monomial ideal \(\{\text{in}_<(f) : f \in I\}\).

Let \(G := GL_{\lambda_1}(k) \times \cdots \times GL_{\lambda_n}(k)\). Every \(g = (g_1, \ldots, g_n) \in G\) defines an \(\mathbb{N}^n\)-graded automorphism of \(k[X]\) via \(g(x_{i,j}) = \sum_{t=1}^{\lambda_i} g_t^{(i,j)} x_{t,j}\) for all \(t \in [n]\), \(j \in [\lambda_t]\), where \(g_t^{(i,j)}\) denotes the \((i, j)\)-th entry of the matrix \(g_t \in GL_{\lambda_t}(k)\).

**Theorem 11.2.3** ([8, Theorem 5.3]). Let \(I \subseteq k[X]\) be a homogeneous ideal, and let \(<\) be any monomial order on \(k[X]\). Then there exists a Zariski open set \(U \subseteq G\) and an ideal \(J\) such that \(\text{in}_<(uI) = J\) for all \(u \in U\).

The monomial ideal \(J\) defined in Theorem 11.2.3 is called the \(G\)-generic initial ideal of \(I\) (with respect to the monomial order \(<\)), and we denote it by \(\text{gin}_{<}^G(I)\).

In the case \(n = 1\), \(G\)-generic initial ideals are called generic initial ideals; see [45, Section 15.9] for a good introduction to generic initial ideals.

**Definition 11.2.4.** Given \(m \in M_\pi\), let \(\text{Max}(m)\) be the \(n\)-tuple \((c_1, \ldots, c_n)\) in \([\lambda_1] \times \cdots \times [\lambda_n]\), where \(c_i = \max\{t \in [\lambda_i] : x_{i,t} \text{ divides } m_{X_i}\}\) for each \(i \in [n]\). Define the set of monomials

\[A_\pi := \left\{ m \in M_\pi : \text{Deg}_\pi(m) + \text{Max}(m) \leq \lambda + 1_n \right\}.\]
The color-squarefree map \( \tilde{\Phi} : \mathcal{A}_\pi \rightarrow \mathcal{M}_\pi(\mathbf{1}_{\lambda_1}, \ldots, \mathbf{1}_{\lambda_n}) \) is the map given by

\[
\prod_{i \in [n]} \left( x_{i,t_{i,1}} \cdots x_{i,t_{i,\mu_i}} \right) \mapsto \prod_{i \in [n]} \left( x_{i,t_{i,1}} x_{i,(t_{i,2}+1)} \cdots x_{i,(t_{i,\mu_i}+\mu_i-1)} \right),
\]

where \( t_{i,1} \leq \cdots \leq t_{i,\mu_i} \) for all \( i \in [n] \). It is easy to see \( \tilde{\Phi} \) is a bijective map that preserves the \( \mathbb{N}^n \)-grading.

**Definition 11.2.5.** Let \( a \in \mathbb{P}^n \), let \( \prec \) be a linear extension of the partial order on \( X \), and let \( \prec_{r\text{lex}} \) be the degree-revlex order on \( \mathcal{M}_\pi \) induced by \( \prec \). If \( \Gamma \) is a simplicial complex on a vertex set \( V \subseteq X \) such that \( (\Gamma, \pi \cap V) \) is an \( a \)-colored complex, and \( I_\Gamma \subseteq k[X] \) is the Stanley–Reisner ideal of \( \Gamma \), then define

\[
\tilde{\Delta}_\prec(\Gamma) := \left\{ \tilde{\Phi}(m) : m \in \left( \mathcal{M}_\pi - \text{gin}^G_{\prec_{r\text{lex}}}(I_\Gamma) \right) \cap \mathcal{A}_\pi \right\}.
\]

**Theorem 11.2.6** ([8, Theorem 5.6]). Let \( \prec \) be any linear extension of the partial order on \( X \). If \( \Gamma \) is a simplicial complex on a vertex set \( V \subseteq X \) such that \( (\Gamma, \pi \cap V) \) is an \( a \)-colored complex, then \( \tilde{\Delta}_\prec(\Gamma) \) is a simplicial complex.\(^1\) Furthermore, if \( V' \subseteq X \) is the vertex set of \( \tilde{\Delta}_\prec(\Gamma) \), then \( (\tilde{\Delta}_\prec(\Gamma), \pi \cap V') \) is a color-shifted \( a \)-colored complex that has the same fine \( f \)-vector as \( (\Gamma, \pi \cap V) \).

**Remark 11.2.7.** Note that the proof of Theorem 11.2.6 as given in [8, Theorem 5.6] requires the assumptions that \( k \) is a field of characteristic zero, and that \( X \) is a finite set. Also, the significance of the degree-revlex order \( \prec_{r\text{lex}} \) can be explained by the fact that the degree-revlex order satisfies two key properties: If \( x_0 \) is the smallest element in \( X \) with respect to \( \prec \) (which exists since \( X \) is finite), then \( \text{in}_{\prec_{r\text{lex}}} (I + \langle x_0 \rangle) = \text{in}_{\prec_{r\text{lex}}} (I) + \langle x_0 \rangle \) and \( \text{in}_{\prec_{r\text{lex}}} (I : x_0) = (\text{in}_{\prec_{r\text{lex}}} (I) : x_0) \) for every homogeneous ideal \( I \subseteq k[X] \), where the ideal \( (I : x_0) \) is defined by \( \left\{ m \in k[X] : mx_0 \in I \right\} \); see [45, Section 15.7].

\(^1\)More precisely, \( \tilde{\Delta}_\prec(\Gamma) \subseteq \mathcal{M}_\pi(\mathbf{1}_{\lambda_1}, \ldots, \mathbf{1}_{\lambda_n}) \) is a multicomplex consisting of only squarefree monomials, hence \( \tilde{\Delta}_\prec(\Gamma) \) can be identified with a simplicial complex whose vertex set is contained in \( X \), and we simply say \( \tilde{\Delta}_\prec(\Gamma) \) is a simplicial complex without any ambiguity.
**Theorem 11.2.8** ([97]). If $\Gamma$ is a simplicial complex on a vertex set $V \subseteq X$ such that $(\Gamma, \pi \cap V)$ is a color-shifted $a$-colored complex, then $\tilde{\Delta}_\prec(\Gamma) = \Gamma$ for any linear extension $\prec$ of the partial order on $X$.

**Theorem 11.2.9** ([8, Theorem 6.5]). If $\Gamma$ is a simplicial complex on a vertex set $V \subseteq X$ such that $(\Gamma, \pi \cap V)$ is an $a$-balanced Cohen–Macaulay complex, then there exists a particular linear extension $\prec$ of the partial order on $X$ such that $\tilde{\Delta}_\prec(\Gamma)$ (computed over a field of characteristic zero) is Cohen–Macaulay.

**Notes**

The notion of “fine $f$-vector” requires an ordered partition $\pi$ of some set (of vertices or variables) for it to even make sense. Given any $a \in \mathbb{P}^n$, we recall that the definition of an $a$-colored complex (resp. multicomplex) comes with a given ordered partition $\pi'$ of some set of vertices (resp. variables). Thus, when defining the fine $f$-vector of an $a$-colored complex (resp. multicomplex), it is implicitly assumed that $\pi = \pi'$.

It is still possible to define the fine $f$-vector of an arbitrary simplicial complex or multicomplex that is not necessarily colorable of type $a$. For this notion to make sense, we will have to specify a particular ordered partition $\pi$. Similarly, it is also possible to define the operation $C_t$ on an arbitrary multicomplex (not necessarily colorable of type $a$). As with the case of fine $f$-vectors, the definition of $C_t$ requires an ordered partition of a set of variables to make sense, so when we apply $C_t$ to arbitrary multicomplexes in $\mathcal{M}_X$, there is an implicitly assumed ordered partition of $X$. In addition, if the corresponding ordered partition of $X$ is $(X_1, \ldots, X_n)$, then $C_t$ also requires that we specify some maps $\phi_1, \ldots, \phi_n$, where $\phi_i : X_i \to \mathbb{P}$ for each $i \in [n]$, otherwise the lex initial segments in the definition of $C_t$ would be ambiguous. Given some ordered partition $(X_1, \ldots, X_n)$ and the maps $\phi_1, \ldots, \phi_n$, we check that the proof of Lemma 11.1.1 still holds for arbitrary multicomplexes $M$ (not necessarily colorable of type $a$).

Consequently, $C_t(M)$ is a multicomplex, and it makes sense to talk about a color compression of an arbitrary multicomplex, as well as a color-compressed multicomplex (in this general sense). The proof of Theorem 11.1.3 still holds true when we omit every instance of “$a$-colored”, hence an array of integers is the fine $f$-vector (in the general sense) of a multicomplex (resp. simplicial complex) if and only if it is the fine $f$-vector (again in the general sense) of a color-compressed multicomplex (resp. simplicial complex). By the same reasoning, it also makes sense to define “color-shifting” and “color-shifted” for arbitrary simplicial complexes or
multicomplexes by specifying a particular ordered partition \((X_1, \ldots, X_n)\) and the corresponding maps \(\phi_1, \ldots, \phi_n\).
In this chapter, we introduce the notion of $a$-Macaulay decomposability for simplicial complexes, which depends on the parameter $a \in \mathbb{N}^n$. We begin in Chapter 12.1 by reviewing the closely related notions of vertex-decomposability and shellability for simplicial complexes. In Chapter 12.2, we define $a$-Macaulay decomposability and study its basic properties. In particular, $a$-Macaulay decomposable simplicial complexes are always vertex-decomposable, and hence shellable. In Chapter 12.3, we prove the main result of this chapter, that pure color-shifted $a$-balanced complexes are $a$-Macaulay decomposable (Theorem 12.3.1), and we will look at some of the consequences of this result. The most important consequence will only come in Chapter 13, where we see that Theorem 12.3.1 is the main reason why a generalized notion of Macaulay representations is even possible.

12.1 Vertex-decomposability and shellability

The notion of vertex-decomposability was originally defined for pure complexes by Provan and Billera [104] and subsequently generalized to non-pure complexes by Björner and Wachs [21, Section 11]. Throughout, we will always use the generalized definition by Björner–Wachs for vertex-decomposability, which is given as follows.

**Definition 12.1.1.** A simplicial complex $\Delta$ with vertex set $V$ is (recursively) defined to be *vertex-decomposable* if either (i) $\Delta$ is a simplex or $\Delta = \{\emptyset\}$; or

(ii) there exists a vertex $x \in V$, called a *shedding vertex* of $\Delta$, such that

(a) $dl_\Delta(x)$ and $lk_\Delta(x)$ are vertex-decomposable, and

(b) no facet of $lk_\Delta(x)$ is a facet of $dl_\Delta(x)$. 

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Given any shedding vertex $x$ of a vertex-decomposable simplicial complex $\Delta$, condition (ii)(b) (in Definition 12.1.1) implies that
\[
\dim(\text{lk}_\Delta(x)) < \dim(\text{dl}_\Delta(x)) = \dim \Delta.
\] (12.1)
When $\Delta$ is pure, the notion of vertex-decomposability specializes to the original definition given by Provan and Billera [104]. In fact, for pure simplicial complexes, we have the following result.

**Proposition 12.1.2 ([104]).** If $\Delta$ is a pure $d$-dimensional vertex-decomposable complex, and $x$ is a shedding vertex of $\Delta$, then $\text{dl}_\Delta(x)$ is pure and $d$-dimensional, and $\text{lk}_\Delta(x)$ is pure and $(d - 1)$-dimensional.

**Remark 12.1.3.** By definition, the existence of a shedding vertex $v$ in a simplicial complex $\Delta$ necessarily implies that both $\text{dl}_\Delta(v)$ and $\text{lk}_\Delta(v)$ are vertex-decomposable, where no facet of $\text{lk}_\Delta(v)$ is a facet of $\text{dl}_\Delta(v)$. This also means that $\Delta$ is vertex-decomposable. Note that not every vertex $x$ of $\Delta$ satisfying condition (ii)(b) of the definition is necessarily a shedding vertex. However, we caution the reader that some authors (see, e.g., [18] [130]) instead define a shedding vertex to be a vertex $x$ satisfying condition (ii)(b), so according to their definition, $\text{dl}_\Delta(x)$ and $\text{lk}_\Delta(x)$ may not be vertex-decomposable, even if $x$ is a shedding vertex.

Similar to vertex-decomposability, the notion of shellability was originally defined for pure complexes. For our purposes, we will use the following generalized notion of shellability introduced by Björner and Wachs [20], whereby shellable simplicial complexes are not necessarily pure.

**Definition 12.1.4.** A simplicial complex is shellable if its facets can be arranged in a linear order $F_1, \ldots, F_t$ so that the subcomplex $\langle \{F_{k+1}\} \rangle \cap \langle \{F_1, \ldots, F_k\} \rangle$ is pure of dimension $\dim F_{k+1} - 1$ for all $k \in [t - 1]$. Such a linear order of its facets is called a shelling.
Theorem 12.1.5 ([21, Theorem 11.3] [104]). Every vertex-decomposable simplicial complex is shellable. In particular, let $\Delta$ be a vertex-decomposable simplicial complex, and suppose $x$ is a shedding vertex of $\Delta$. If $F_1, \ldots, F_t$ is a shelling of $\text{dl}_\Delta(x)$, and $G_1, \ldots, G_s$ is a shelling of $\text{lk}_\Delta(x)$, then $F_1, \ldots, F_t, G_1 \cup \{x\}, \ldots, G_s \cup \{x\}$ is a shelling of $\Delta$.

12.2 Basic properties of Macaulay decomposability

The notion of $a$-Macaulay decomposability, first introduced by the author in [34], is a refinement of vertex-decomposability. In this section, we will define and prove basic properties of $a$-Macaulay decomposability. We begin by introducing the notion of $a$-ribs.

Definition 12.2.1. Let $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, and let $\Delta$ be a simplicial complex with vertex set $V$. A subcomplex $\Delta'$ of $\Delta$ is called an $a$-rib of $\Delta$ if there exists an ordered partition $(V_1, \ldots, V_n)$ of $V$ such that $\Delta'$ is generated by all faces $F \in \Delta$ of the form $F = F_1 \cup \cdots \cup F_n$, where $F_i \in \binom{V_i}{a_i}$ for each $i \in [n]$, i.e.

$$\Delta' = \left\{ F_1 \cup \cdots \cup F_n \in \Delta : F_i \in \binom{V_i}{a_i} \text{ for each } i \in [n] \right\}.$$

For example, if $p, q \in \mathbb{P}$, then the complete bipartite graph $K_{p,q}$, treated as a 1-dimensional simplicial complex, is a $(1, 1)$-rib of a simplex with $p + q$ vertices. If $\Delta$ is a simplex, then an $a$-rib of $\Delta$ is the subcomplex $\left( \binom{V_1}{a_1} \right) * \cdots * \left( \binom{V_n}{a_n} \right)$ for some ordered partition $(V_1, \ldots, V_n)$. By definition, an $a$-rib of any simplicial complex $\Delta$ is always pure of dimension $|a| - 1$, and if $\Delta$ is pure, then for a fixed $a \in \mathbb{P}^n$, the union of all $a$-ribs of $\Delta$ (over all ordered partitions of $V$) forms the $(|a| - 1)$-skeleton of $\Delta$. For the case $n = 1$, the $a$-rib of every $\Delta$ is unique.
Lemma 12.2.2. Any \( a \)-rib of a simplex is vertex-decomposable.

Proof. Let \( \Delta \) be a simplex with vertex set \( V \), and let \( \Gamma \) be the \( a \)-rib of \( \Delta \) corresponding to the ordered partition \((V_1, \ldots, V_n)\) of \( V \). Given simplicial complexes \( \Sigma_1, \Sigma_2 \) with disjoint vertex sets, [104, Proposition 2.4] says \( \Sigma_1 + \Sigma_2 \) is vertex-decomposable if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are both vertex-decomposable. Since \( \Gamma = \langle (\binom{V_1}{a_1}) \rangle \cdots \langle (\binom{V_n}{a_n}) \rangle \), it then suffices to show that the \( k \)-skeleton (any \( k \in \mathbb{N} \)) of a simplex is vertex-decomposable, which is straightforward.

The notion of \( a \)-Macaulay decomposability involves the \( a \)-ribs of simplices in a fundamental way. These \( a \)-ribs of simplices form the base case of the recursive definition for \( a \)-Macaulay decomposability, analogous to how simplices form the base case of the recursive definition for vertex-decomposability. Since \( a \)-ribs of simplices are refinements of simplices, we should then think of \( a \)-Macaulay decomposability as a refinement of vertex-decomposability. Before we state the precise definition, recall that for \( x, y \in \mathbb{Z}^n \), we write \( x < y \) if \( x \leq y \) and \( x \neq y \).

Definition 12.2.3. Let \( a \in \mathbb{N}^n \). A simplicial complex \( \Delta \) with vertex set \( V \) is called \( a \)-Macaulay decomposable if either (i) \( \Delta \) is an \( a \)-rib of a simplex; or

(ii) there exists a vertex \( x \in V \) and some \( a' \in \mathbb{N}^n \), \( a' < a \), such that

(a) \( \text{dl}_\Delta(x) \) is \( a \)-Macaulay decomposable,

(b) \( \text{lk}_\Delta(x) \) is \( a' \)-Macaulay decomposable, and

(c) no facet of \( \text{lk}_\Delta(x) \) is a facet of \( \text{dl}_\Delta(x) \).

Such a vertex \( x \) in (ii), if it exists, is called a Macaulay shedding vertex of \( \Delta \).

Remark 12.2.4. In Definition 12.2.3, we do not require that \( \Delta \) is pure. For example, the non-pure simplicial complex \( \langle \{\{1,2,3\},\{1,2,4\},\{1,5\},\{2,5\}\} \rangle \) is \( (2,1) \)-Macaulay decomposable with 5 as a Macaulay shedding vertex.
Remark 12.2.5. If the simplicial complex $\Delta$ in Definition 12.2.3 is pure, then condition (ii)(c) of the definition can be omitted. Furthermore, if condition (ii) holds, then $a' \prec a$ (recall that $a' \prec a$ if $a' < a$ and $|a| = |a'| + 1$), and both $\text{dl}_\Delta(x)$ and $\text{lk}_\Delta(x)$ are pure; cf. Proposition 12.1.2. Conversely, if there exists some vertex $x'$ of a simplicial complex $\Delta'$ such that $\text{dl}_{\Delta'}(x')$ is pure and $a$-Macaulay decomposable; and $\text{lk}_{\Delta'}(x')$ is pure and $a'$-Macaulay decomposable for some $a' \in \mathbb{N}^n$ satisfying $a' \prec a$, then $\Delta'$ is pure and $a$-Macaulay decomposable.

The following proposition shows that a Macaulay shedding vertex is a shedding vertex of a vertex-decomposable simplicial complex.

Proposition 12.2.6. Let $a \in \mathbb{N}^n$, and let $\Delta$ be a simplicial complex. If $\Delta$ is $a$-Macaulay decomposable, then $\Delta$ is vertex-decomposable, and a Macaulay shedding vertex of $\Delta$ is a shedding vertex of $\Delta$.

Proof. We prove by double induction on $|a|$ and $|V|$, where $V$ denotes the vertex set of $\Delta$. The cases $|a| = 0$ and $|V| \leq 2$ are trivial, and in view of Lemma 12.2.2, we assume $\Delta$ is not an $a$-rib of any simplex. Let $x$ be a Macaulay shedding vertex of $\Delta$. Note that $\text{dl}_\Delta(x)$ is an $a$-Macaulay decomposable simplicial complex with its vertex set contained in $V \setminus \{x\}$, while $\text{lk}_\Delta(x)$ is $a'$-Macaulay decomposable for some $a' < a$, hence $\text{dl}_\Delta(x)$ and $\text{lk}_\Delta(x)$ are both vertex-decomposable by induction hypothesis. Also, no facet of $\text{lk}_\Delta(x)$ is a facet of $\text{dl}_\Delta(x)$ by definition, hence $\Delta$ is vertex-decomposable, and we easily conclude that $x$ is also a shedding vertex. \qed

In view of Theorem 12.1.5, we immediately have the following implications:

\[ a\text{-Macaulay decomposable} \Rightarrow \text{vertex-decomposable} \Rightarrow \text{shellable}. \quad (12.2) \]
These implications hold for all \( a \in \mathbb{N}^n \). In particular, the second implication is strict, even for pure complexes [104, Remark 3.4.5]. The following corollary gives a partial converse to the first implication when \( n = 1 \). Note however that for \( n > 1 \), the first implication is strict; see Proposition 12.2.11.

**Corollary 12.2.7.** Let \( \Delta \) be a \((d - 1)\)-dimensional simplicial complex. Then \( \Delta \) is vertex-decomposable if and only if \( \Delta \) is \((d)\)-Macaulay decomposable.

**Proof.** Let \( \Delta \) be vertex-decomposable with vertex set \( V \). In view of Proposition 12.2.6, we only need to show that \( \Delta \) is \((d)\)-Macaulay decomposable, which easily follows by induction on \( |V| \). \( \square \)

**Proposition 12.2.8.** Let \( n, m \in \mathbb{P} \), let \( a \in \mathbb{N}^n \), \( b \in \mathbb{N}^m \), and let \( \Delta, \Delta' \) be simplicial complexes with disjoint vertex sets \( V, V' \) respectively. If \( \Delta \) is \( a\)-Macaulay decomposable and \( \Delta' \) is \( b\)-Macaulay decomposable, then \( \Delta \ast \Delta' \) is \((a, b)\)-Macaulay decomposable.

**Proof.** We prove by double induction on \(|a|+|b|\) and \(|V|+|V'|\), where the cases \(|a|+|b| \leq 2\) or \(|V|+|V'| \leq 2\) are trivial. Suppose \( \Delta \) is not an \( a\)-rib of a simplex, and let \( x \) be a Macaulay shedding vertex of \( \Delta \). Since \( dl_{\Delta}(x) \) is \( a\)-Macaulay decomposable with its vertex set contained in \( V \setminus \{x\} \), while \( lk_{\Delta}(x) \) is \( a'\)-Macaulay decomposable for some \( a' < a \), it follows from the induction hypothesis that \( dl_{\Delta}(x) \ast \Delta' \) is \((a, b)\)-Macaulay decomposable and \( lk_{\Delta}(x) \ast \Delta' \) is \((a', b)\)-Macaulay decomposable. It is easy to show that \( dl_{\Delta}(x) \ast \Delta' = dl_{\Delta \ast \Delta'}(x) \) and \( lk_{\Delta}(x) \ast \Delta' = lk_{\Delta \ast \Delta'}(x) \). Also, if a facet \( F \) of \( lk_{\Delta \ast \Delta'}(x) \) is a facet of \( dl_{\Delta \ast \Delta'}(x) \), then \( F \cap V \) is a facet of both \( dl_{\Delta}(x) \) and \( lk_{\Delta}(x) \), which is a contradiction. Consequently, \( \Delta \ast \Delta' \) is \((a, b)\)-Macaulay decomposable by definition in this case. The case when \( \Delta' \) is not a \( b\)-rib of a simplex is similar.
Finally, if $\Delta$ is an $a$-rib of a simplex $\Sigma$ with vertex set $X$, and $\Delta'$ is a $b$-rib of a simplex $\Sigma'$ with vertex set $X'$, then $\Delta \ast \Delta' = \langle (X_1^{a_1}) \ast \cdots \ast (X_n^{a_n}) \rangle \ast \langle (X_1'^{b_1}) \ast \cdots \ast (X_m'^{b_m}) \rangle$, where $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_m)$, and $(X_1, \ldots, X_n)$, $(X_1', \ldots, X_m')$ are the corresponding ordered partitions of $X$, $X'$ respectively. Thus, $\Delta \ast \Delta'$ is an $(a, b)$-rib of a simplex with vertex set $X \cup X'$ and hence $(a, b)$-Macaulay decomposable.

Let $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and $b = (b_1, \ldots, b_m) \in \mathbb{N}^m$, where $m \leq n$ are positive integers. If there are integers $0 = i_0 < i_1 < \cdots < i_{m-1} < i_m = n$ such that $b_t = a_{i_{t-1}+1} + \cdots + a_{i_t}$ for all $t \in [m]$, then we say $a$ is an ordered refinement of $b$. In particular, $a$ is an ordered refinement of $b$ implies $|a| = |b|$. If there exists a permutation $\phi$ on $[n]$ such that $(a_{\phi(1)}, \ldots, a_{\phi(n)})$ is an ordered refinement of $b$, then we say $a$ is a permuted refinement of $b$, and we denote this by $a \leq_{pr} b$.

**Definition 12.2.9.** Let $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, and let $\Sigma$ be the simplex with vertex set $V = \{(i, j) \in \mathbb{P}^2 : j \in [n], i \in [a_j + 1]\}$. For each $t \in [n]$, define $V_t := \{(i, t) \in \mathbb{P}^2 : i \in [a_t + 1]\} \subseteq V$.

Then the unique $a$-rib of $\Sigma$ corresponding to the ordered partition $(V_1', \ldots, V_n')$ of $V$ is called the $a$-ribcage.

For example, the $(1, 1)$-ribcage is isomorphic to the complete bipartite graph $K_{2,2}$ (treated as a 1-dimensional simplicial complex), while the $(2)$-ribcage is isomorphic to the empty triangle $\langle \{\{1, 2\}, \{2, 3\}, \{3, 1\}\} \rangle$.

**Lemma 12.2.10.** Let $b \in \mathbb{P}^m$ and let $\Delta$ be the $b$-ribcage. If $a \in \mathbb{P}^n$ is an ordered refinement of $b$ such that $\Delta$ is $a$-Macaulay decomposable, then $a = b$.

**Proof.** Write $a = (a_1, \ldots, a_n)$ and suppose $\Delta$ is $a$-Macaulay decomposable corresponding to the ordered partition $(V_1', \ldots, V_n')$ of its vertex set $V$. Note that
\[ |V| = |b| + m = |a| + m, \text{ so if } n > m, \text{ then } |V'_j| = a_j \text{ for some } j \in [n], \text{ which implies } V'_j \text{ is a non-empty set of conepoints of } \Delta. \] (Recall that } v \in V \text{ is a conepoint of } \Delta \text{ if } v \in F \text{ for all facets } F \text{ of } \Delta. \) However, by the definition of the } b \text{-ribcage, } \Delta \text{ has no conepoints, thus } n = m, \text{ which then forces } a = b. \]

**Proposition 12.2.11.** Let } m, n \in \mathbb{P}, \text{ let } a \in \mathbb{N}^n, \text{ and let } \Delta \text{ be a simplicial complex. If } a \preceq pr b, \text{ then } \Delta \text{ is } a \text{-Macaulay decomposable implies } \Delta \text{ is } b \text{-Macaulay decomposable. Furthermore, if } a \in \mathbb{P}^n, \text{ } b \in \mathbb{P}^m \text{ and } n \neq m, \text{ then this implication is strict.} \]

**Proof.** Observe that if we can show every } a \text{-rib of a simplex is } b \text{-Macaulay decomposable for all } b \succeq pr a, \text{ then by the definition of Macaulay decomposability, we can prove the first assertion using double induction on } |a| \text{ and the number of vertices of } \Delta. \text{ Write } a = (a_1, \ldots, a_n), \text{ let } \Delta' \text{ be a simplex with vertex set } V, \text{ and } \text{let } \Gamma \text{ be an } a \text{-rib of } \Delta' \text{ corresponding to the ordered partition } (V_1, \ldots, V_n) \text{ of } V. \] Choose some } b = (b_1, \ldots, b_m) \succeq pr a. \text{ Without loss of generality, assume } a \text{ is an ordered refinement of } b, \text{ and let } 0 = i_0 < i_1 < \cdots < i_{m-1} < i_m = n \text{ such that } b_t = a_{i_{t-1}+1} + \cdots + a_{i_t} \text{ for all } t \in [m]. \text{ For each } t \in [m], \text{ define}

\[
\Gamma_t := \left( \left( \frac{V_{i_{t-1}+1}}{a_{i_{t-1}+1}} \right) \right) \ast \cdots \ast \left( \left( \frac{V_{i_t}}{a_{i_t}} \right) \right), \tag{12.3}
\]

and note that } \Gamma_t \text{ is an } (a_{i_{t-1}+1}, \ldots, a_{i_t}) \text{-rib of the simplex with the vertex set } V_{i_{t-1}+1} \cup \cdots \cup V_{i_t}, \text{ so } \Gamma_t \text{ is vertex-decomposable by Lemma 12.2.2 and hence } (b_t)\text{-Macaulay decomposable by Corollary 12.2.7. Now since } \Gamma = \Gamma_1 \ast \cdots \ast \Gamma_m, \text{ it then follows from Proposition 12.2.8 that } \Gamma \text{ is } b \text{-Macaulay decomposable, which proves the first assertion. Finally, the second assertion follows immediately from Lemma 12.2.10.} \]

**Proposition 12.2.12.** Let } a \in \mathbb{N}^n. \text{ If } \Delta \text{ is an } a \text{-Macaulay decomposable simplicial complex, then } \dim \Delta = |a| - 1.
Proof. The assertion follows by double induction on $|a|$ and $|V|$, where $V$ denotes the vertex set of $\Delta$. \hfill \Box

### 12.3 Pure color-shifted implies Macaulay decomposable

**Theorem 12.3.1.** Let $a \in \mathbb{P}^n$ and let $(\Delta, \pi)$ be a pure $a$-balanced complex. If $(\Delta, \pi)$ is color-shifted, then $\Delta$ is $a$-Macaulay decomposable.

**Proof.** Write $a = (a_1, \ldots, a_n)$, $|a| = k$, and let $\pi = (V_1, \ldots, V_n)$ be the ordered partition of the vertex set $V$ of $\Delta$. For each $i \in [n]$, assume $V_i = \{v_{i,1}, \ldots, v_{i,\lambda_i}\}$ has $\lambda_i$ elements linearly ordered by $v_{i,1} < \cdots < v_{i,\lambda_i}$. Without loss of generality, assume $\Delta$ is not an $a$-rib of a simplex, which implies $\lambda_t > a_t$ for some $t \in [n]$. Define $a' := a - \delta_{t,n}$, and let $W, W'$ be the vertex sets of $d\Delta(v_t, \lambda_t)$, $l\Delta(v_t, \lambda_t)$ respectively. Since $(\Delta, \pi)$ is color-shifted, it follows that $(d\Delta(v_t, \lambda_t), \pi \cap W)$ is a color-shifted $a$-colored complex, while $(l\Delta(v_t, \lambda_t), \pi \cap W')$ is a color-shifted $a'$-colored complex. We now prove this theorem by double induction on $k$ and $|V|$.

Suppose there is a facet $F$ of $d\Delta(v_t, \lambda_t)$ such that $\dim F < k - 1$. Since $\Delta$ is pure of dimension $k - 1$, we have $F \cup \{v_t, \lambda_t\} \in \Delta$, so $(\Delta, \pi)$ is color-shifted implies $F \cup \{v_t, j\} \in \Delta$ for every $j \in [\lambda_t - 1]$ satisfying $v_{t,j} \notin F$. Such a $j$ exists since $\lambda_t > a_t$, and in particular, $F \cup \{v_{t,j}\} \in d\Delta(v_t, \lambda_t)$, which contradicts the assumption that $F$ is a facet of $d\Delta(v_t, \lambda_t)$. This means $(d\Delta(v_t, \lambda_t), \pi \cap W)$ is a pure color-shifted $a$-balanced complex, hence $d\Delta(v_t, \lambda_t)$ is $a$-Macaulay decomposable by induction hypothesis. A similar argument shows $l\Delta(v_t, \lambda_t)$ is pure of dimension $k - 2$ and hence $a'$-Macaulay decomposable by induction hypothesis and Proposition 12.2.11, therefore $\Delta$ is $a$-Macaulay decomposable. \hfill \Box
Remark 12.3.2. Both the hypotheses in Theorem 12.3.1, i.e. that \((\Delta, \pi)\) is pure and \(a\)-balanced (instead of just \(a\)-colored), are necessary. Murai [97] gave an example of a non-pure \(1_5\)-balanced color-shifted complex that is not shellable, so since all \(1_5\)-Macaulay decomposable simplicial complexes are shellable (cf. Proposition 12.2.6), this same example fails to be \(1_5\)-Macaulay decomposable. Also, the pure simplicial complex \(\langle \{\{1, 2\}, \{3, 4\}\} \rangle\), together with the ordered partition \((\{1\}, \{2\}, \{3\}, \{4\})\) of its vertex set [4], is clearly color-shifted and \(1_4\)-colored, although it is not \(1_4\)-balanced. However, it fails to be \(1_4\)-Macaulay decomposable. In general, Proposition 12.2.12 says an \(a\)-colored complex that is not \(a\)-balanced can never be \(a\)-Macaulay decomposable.

Theorem 12.3.1 generalizes Murai’s result [97, Proposition 4.2], which states that a pure \(a\)-balanced color-shifted complex is shellable. From Theorem 12.3.1 and its proof, we get the following useful corollaries. Note in particular that the proof of the next corollary is contained in the proof of Theorem 12.3.1.

Corollary 12.3.3. Let \(a = (a_1, \ldots, a_n) \in \mathbb{P}^n\), let \((\Delta, \pi)\) be a pure color-shifted \(a\)-balanced complex, and let \(\pi = (V_1, \ldots, V_n)\) be the corresponding ordered partition of its vertex set. If \(\Delta\) is not an \(a\)-rib of a simplex, then the maximal element of \(V_i\) is a Macaulay shedding vertex of \(\Delta\) for every \(i \in [n]\) satisfying \(|V_i| > a_i\).

Corollary 12.3.4. Let \(a \in \mathbb{P}^n\), and let \((\Delta, \pi)\) be a color-shifted \(a\)-balanced complex. Then the following are equivalent:

\(\text{(i) } \Delta \text{ is pure.}\)

\(\text{(ii) } \Delta \text{ is pure } a\text{-Macaulay decomposable.}\)

\(\text{(iii) } \Delta \text{ is pure vertex-decomposable.}\)

\(\text{(iv) } \Delta \text{ is pure shellable.}\)

\(\text{(v) } \Delta \text{ is Cohen–Macaulay (over any field).}\)
Proof. Theorem 12.3.1 yields (i) ⇒ (ii), Proposition 12.2.6 yields (ii) ⇒ (iii), while the implications (iii) ⇒ (iv) ⇒ (v) ⇒ (i) are well-known (see [119]). □

Remark 12.3.5. Corollary 12.3.4 generalizes a result by Babson and Novik [8, Theorem 6.2], which states that given $(\Delta, \pi)$ a color-shifted $a$-balanced complex, $\Delta$ is Cohen–Macaulay (over any field) if and only if $\Delta$ is pure.

Corollary 12.3.6. Let $a \in \mathbb{P}^n$, and let $f = \{f_b\}_{b \leq a}$ be an array of integers. Then the following are equivalent:

(i) $f$ is the fine $f$-vector of an $a$-balanced Cohen–Macaulay complex.
(ii) $f$ is the fine $f$-vector of an $a$-balanced pure vertex-decomposable complex.
(iii) $f$ is the fine $f$-vector of an $a$-balanced pure color-shifted complex.
(iv) $f$ is the fine $f$-vector of an $a$-balanced color-shifted Cohen–Macaulay complex.

Proof. Let $(\Gamma, \pi)$ be an $a$-balanced Cohen–Macaulay complex, and assume each $V_i$ in the ordered partition $\pi = (V_1, \ldots, V_n)$ is linearly ordered, so $V := \bigcup_{i \in [n]} V_i$ as a poset is a disjoint union of $n$ chains. By Theorem 11.2.9, there is a linear extension $\prec$ of this partial order on $V$ such that the colored algebraic shifting $(\tilde{\Delta}_{\prec}(\Gamma), \pi')$ is a color-shifted $a$-balanced Cohen–Macaulay complex with the same fine $f$-vector as $(\Gamma, \pi)$, hence (i) ⇒ (iv). The implications (iv) ⇒ (iii) ⇒ (ii) follow from Corollary 12.3.4, Theorem 12.3.1 and Proposition 12.2.6, while (ii) ⇒ (i) is trivial. □

Biermann and Van Tuyl [13] proved that the $f$-vector of a $1_n$-colored complex is the $h$-vector of a $1_n$-balanced pure vertex-decomposable complex. Their proof gives an explicit construction and uses a generalization of the “whiskering” construction of graphs introduced by Villarreal [126], and Cook and Nagel [37]. In particular, note that a simplicial complex with $n$ vertices is trivially $1_n$-colorable, hence the
$f$-vector of a simplicial complex is the $h$-vector of a completely balanced pure vertex-decomposable complex.

Although Biermann and Van Tuyl did not mention flag $f$-vectors in their paper, it follows from their explicit construction that the flag $f$-vector of a $1_n$-colored complex is the flag $h$-vector of a $1_n$-balanced pure vertex-decomposable complex. By the Björner–Frankl–Stanley theorem [19] (cf. Theorem 14.2.10), an array of integers is the flag $f$-vector of a $1_n$-colored complex if and only if it is the flag $h$-vector of a $1_n$-balanced Cohen–Macaulay complex. Thus, the flag $f$-vector version of Biermann–Van Tuyl’s result is equivalent to the statement that the flag $f$-vector of a $1_n$-balanced Cohen–Macaulay complex is the flag $f$-vector of a $1_n$-balanced pure vertex-decomposable complex. Corollary 12.3.6 thus generalizes this statement to the $a$-balanced case: The fine $f$-vector of an $a$-balanced Cohen–Macaulay complex is the fine $f$-vector of an $a$-balanced pure vertex-decomposable complex.

Notes

All pure shellable simplicial complexes are Cohen–Macaulay ([119, Thm. III.2.5]). For an excellent treatment on shellability, see [20] and [21].

The experienced reader would notice that Macaulay decomposability is a slight extension of the notion of vertex-decomposability. Despite the fact that this extension is seemingly simple, it turns out that Macaulay decomposability is the “correct” notion in the context of (generalized) Macaulay representations, which we will discuss in detail in the next chapter. To illustrate what we mean, consider the following example: Let $\Delta$ be the (unique) pure 2-dimensional compressed simplicial complex with 6 facets and linearly ordered vertices $x_1 < \cdots < x_5$. Using Macaulay decomposability, we can decompose $\Delta$ into (iterations of) the deletions and links of $\Delta$ at Macaulay shedding vertices. Note that the number $f_k(\Delta)$ of $k$-dimensional faces of $\Delta$ equals $f_k(dl_{\Delta}(x)) + f_{k-1}(lk_{\Delta}(x))$. The (usual) 3rd Macaulay representation of $f_2(\Delta) = 6$ is $6 = \binom{4}{3} + \binom{2}{2} + \binom{1}{1}$, and notice that the three summands $\binom{4}{3}, \binom{2}{2}, \binom{1}{1}$ equal the number of 2-dimensional faces in $dl_{\Delta}(x_5)$, the
number of edges in $dl_{lk\Delta}(x_3)$, and the number of vertices in $lk_{lk\Delta}(x_3)$ respectively. The vertices $x_3, x_5$ are precisely the Macaulay shedding vertices of $\Delta$ that yield this “decomposition” of $\Delta$ into three subcomplexes corresponding to the summands.

In the next chapter, we will use Theorem 12.3.1 as our starting point: Given a pure color-shifted $a$-balanced complex $\Delta$ for some $a \in \mathbb{P}^n$, Theorem 12.3.1 tells us that $\Delta$ is $a$-Macaulay decomposable, so we can decompose $\Delta$ into (iterations) of its deletions and links at Macaulay shedding vertices. We keep track of these (iterations of the) deletions and links and encode the relevant combinatorial data as a suitably labeled binary tree. The left and right branches of these trees correspond to deletions and links respectively. These labeled binary trees are essentially what we will call generalized Macaulay representations.
CHAPTER 13

GENERALIZED MACAULAY REPRESENTATIONS

In this chapter, we introduce the notion of generalized \( \mathbf{a} \)-Macaulay representations of positive integers. The central idea that makes generalized \( \mathbf{a} \)-Macaulay representations possible is the notion of ‘\( \mathbf{a} \)-Macaulay decomposability’, which was introduced in Chapter 12. Later in Chapter 14, we will use these generalized \( \mathbf{a} \)-Macaulay representations to obtain numerical characterizations of the flag \( f \)-vectors of completely balanced Cohen–Macaulay complexes, and more generally, of the fine \( f \)-vectors of \( \mathbf{a} \)-colored complexes for arbitrary \( \mathbf{a} \in \mathbb{P}^n \).

The usual Macaulay representations, first introduced in Chapter 2.6 in the context of the Kruskal–Katona theorem, are sums of binomial coefficients, where the top components of its binomial coefficients are strictly decreasing integers, and the corresponding bottom components are consecutive decreasing integers. Consequently, every Macaulay representation is completely determined by the top components and the first bottom component of its binomial coefficients.

The key insight to \( f \)-vector enumeration that we offer in this chapter is that we should think of such Macaulay representations not as sums of binomial coefficients, but as (rooted) binary trees whose root is labeled by the first bottom component, and whose remaining leaves are labeled by the top components. For example, the (usual) 3rd Macaulay representation of the integer 6 is given by \( \binom{4}{3} + \binom{2}{2} + \binom{1}{1} \). We can then encode all the data given in this Macaulay representation by the labeled binary tree illustrated in Figure 13.1.

Our eventual goal is to understand how the color structure of \( \mathbf{a} \)-colored complexes influences the possible the fine \( f \)-vectors they can have. Already at the end
of Chapter 11.1, we have the first hint of why the usual Macaulay representations are inadequate in numerically characterizing those fine $f$-vectors. Hence, we want to generalize these Macaulay representations in a suitable manner. By thinking of Macaulay representations as labeled binary trees, there is a natural way to generalize them: Instead of labeling the leaves of the tree by integers, we label them by $n$-tuples of integers. We can also label the interior vertices of the tree by integers in $[n]$. Such ‘general’ labeled binary trees are essentially what generalized $a$-Macaulay representations are, and they correspond to sums of products of binomial coefficients. Just as how the top components of the binomial coefficients in Macaulay representations are strictly decreasing, there are also restrictions on the possible labels in generalized $a$-Macaulay representations.

Finding suitable restrictions on these labels is the main difficulty that we will address in the rest of this chapter. Our overall approach can be roughly described as follows: We take a pure color-compressed $a$-balanced complex and construct a labeled binary tree that encodes all relevant combinatorial data, including its fine $f$-vector. We study the properties of such constructed labeled binary trees. We then ‘reverse’ the process by defining a generalized $a$-Macaulay representation to be a labeled binary tree satisfying a list of conditions. It is a priori not obvious that such generalized $a$-Macaulay representations exist and are well-defined based solely on the list of conditions, hence a lot of work will be done showing that they are well-defined (without relying on the existence of certain complexes), that they
exist, and that in fact they correspond bijectively to the isomorphism classes of pure color-compressed $a$-balanced complexes. Effectively, our list of conditions captures all necessary and sufficient combinatorial data required to numerically characterize the fine $f$-vectors of pure color-compressed $a$-balanced complexes. Later in the next chapter, we will explain how such generalized $a$-Macaulay representations relate to the fine $f$-vectors of arbitrary (not necessarily pure) $a$-colored complexes.

We first begin in Chapter 13.1 by describing additional terminology concerning rooted trees. This builds upon the graph theory terminology that we have already covered in Chapter 4, and will be pertinent when we define generalized Macaulay representations.

Chapters 13.2–13.5 form the technical heart of this chapter. In Chapter 13.2, we construct a labeled binary tree $(T, \phi)$ from every pure color-shifted $a$-balanced complex $(\Delta, \pi)$. We call this construction the shedding tree of $(\Delta, \pi)$. Next, in Chapter 13.3, we introduce the notion of Macaulay trees. Given some $n$-tuple $a \in \mathbb{N}^n \setminus \{0_n\}$ and any positive integer $N$, we will define a purely numerical notion of an $a$-Macaulay tree of $N$. If $a \in \mathbb{P}^n$, then every pure color-shifted $a$-balanced complex with $N$ facets corresponds to an $a$-Macaulay tree of $N$, while conversely, every $a$-Macaulay tree of $N$ corresponds to a pure $a$-balanced complex, although not necessarily color-shifted.

In Chapter 13.4, we define what it means for an $a$-Macaulay tree to be compressed-like and compatible, and we characterize pure color-compressed $a$-balanced complexes in terms of certain compressed-like compatible $a$-Macaulay trees. Finally, in Chapter 13.5, we define the notion of generalized $a$-Macaulay representations of non-negative integers, over all $n$-tuples $a \in \mathbb{N}^n \setminus \{0_n\}$.
Throughout Chapters 13.2–13.5 unless otherwise stated, \( n \) is a positive integer, and \( \mathbf{a} = (a_1, \ldots, a_n) \) is an \( n \)-tuple in \( \mathbb{N}^n \). Most of the time, we will require that \( \mathbf{a} \neq 0_n \), and sometimes, we may require that \( \mathbf{a} \in \mathbb{P}^n \). When \( \mathbf{a} \neq 0_n \), we will always let \((\Delta, \pi)\) be an \( \mathbf{a} \)-balanced complex (or equivalently an \( \mathbf{a} \)-balanced complex, in the case when \( \mathbf{a} \in \mathbb{P}^n \)), where \( \pi = (V_1, \ldots, V_n) \) is the corresponding ordered partition of the vertex set \( V \) of \( \Delta \). In particular, recall that \( \mathbf{a} \) refers to the \( m \)-tuple \((a_{i_1}, \ldots, a_{i_m})\) of positive integers, where \( 1 \leq i_1 < \cdots < i_m \leq n \) are the indices of all the non-zero entries of \( \mathbf{a} \). For each \( i \in [n] \), write \( \lambda_i = |V_i| \), and assume \( V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,\lambda_i}\} \) is linearly ordered by \( v_{i,1} < v_{i,2} < \cdots < v_{i,\lambda_i} \). Set \( \lambda = (\lambda_1, \ldots, \lambda_n) \).

For every \( \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}^n \) and every subcomplex \( \Delta' \) of \( \Delta \) with vertex set \( V' \), let \( \mathcal{F}_b(\Delta') \) be the set of all faces \( F \in \Delta' \) such that \( |F \cap V_i| = b_i \) for all \( i \in [n] \), and define \( f_b(\Delta') = |\mathcal{F}_b(\Delta')| \). By default, set \( f_b(\Delta) = 0 \) if \( \mathbf{b} \not\in \mathbb{N}^n \). Observe that \( \{f_b(\Delta')\}_{\mathbf{b} \leq \mathbf{a}} \) is by definition the fine \( f \)-vector of \((\Delta', \pi \cap V')\). Recall that \( \pi \cap V' = (V_1 \cap V', \ldots, V_n \cap V') \).

For arbitrary \( n \)-tuples of integers \( \mathbf{x} = (x_1, \ldots, x_n) \) and \( \mathbf{y} = (y_1, \ldots, y_n) \), define \( \binom{\mathbf{x}}{\mathbf{y}} := \binom{x_1}{y_1} \binom{x_2}{y_2} \cdots \binom{x_n}{y_n} \). Recall that \( \mathbf{x} < \mathbf{y} \) if \( \mathbf{x} \leq \mathbf{y} \) and \( \mathbf{x} \neq \mathbf{y} \). By convention, \( \binom{x_i}{y_i} \) equals 0 if \( y_i < 0 \). Note that \( \mathbf{x} < \mathbf{y} \) if and only if \( \mathbf{x} = \mathbf{y} - \mathbf{\delta}_{i,n} \) for some \( i \in [n] \). If \( \varphi \) is a function whose image is contained in \( \mathbb{Z}^n \), then \( \varphi \) has \( n \) component functions, and for each \( t \in [n] \), let \( \varphi^{[t]} \) be the \( t \)-th component function of \( \varphi \).

### 13.1 Rooted trees

Let \( G = (V, E) \) be a (simple) graph. A labeling of a subset \( U \subseteq V \) is a map \( \phi : U \to S \), where \( S \) is an arbitrary set whose elements are called \( \phi \)-labels, or simply labels if the context is clear. A vertex labeling of \( G \) is a labeling of \( V \).
Given an edge \( e = \{u, v\} \) of \( G \), the graph \( G \setminus e := (V, E \setminus \{e\}) \) is said to be obtained from \( G \) by deleting the edge \( e \), and the graph constructed from \( G \) by identifying the vertices \( u \) and \( v \) and then deleting edge \( e \), which we denote by \( G/e \), is said to be obtained from \( G \) by contracting \( e \). If \( \phi \) is a vertex labeling of \( G \) such that \( \phi(u) = \phi(v) \), then \( \phi \) induces a vertex labeling of \( G/e \) in the obvious way, and we denote this vertex labeling of \( G/e \) by \( \phi/e \).

Every vertex of degree 1 is called a leaf, and every vertex of degree 3 is called a trivalent vertex. If every non-leaf of \( G \) is trivalent, then we call \( G \) trivalent. When \( G \) is a tree, we say that \( G \) is rooted if it has a distinguished vertex \( r_0 \), which we call the root. Given this root \( r_0 \) and any vertices \( x, y \in V \) (possibly \( x = y \)), if \( x \) is contained in the path from \( r_0 \) to \( y \), then we say \( x \) is an ancestor of \( y \), and \( y \) is a descendant of \( x \). If in addition \( \{x, y\} \in E \), then we say \( x \) is a parent of \( y \), and \( y \) is a child of \( x \). Note that in a rooted tree, the root has no parents, and every other vertex has a unique parent. A planted tree is a rooted tree whose root is a leaf, i.e. the root has a unique child. A binary tree is a rooted tree such that every vertex has \( \leq 2 \) children. For convenience, we assume that all binary trees are trivalent, so in particular they are planted. Also, we shall assume that all binary trees are plane trees, i.e. trees that are embedded on a plane.

Suppose \( G = (V, E) \) is a binary tree with root \( r_0 \). Let \( Y \subseteq V \) be the set of all trivalent vertices, and let \( L \) be the set of all leaves in \( V \) distinct from \( r_0 \), whose elements are called terminal vertices. For convenience, define the \((\text{root}, 1, 3)\)-triple of \( G \) as the triple \((r_0, L, Y)\), and for any vertex \( x \) of \( G \), let \( D_G(x) \) denote the set of all descendants of \( x \) in \( G \). By definition, every \( y \in Y \) has exactly 2 children, which we call the left child and right child. The distinction between the left and right children of every \( y \in Y \) is well-defined by the assumption that all binary
trees are embedded on a plane, and we always attach subscripts ‘left’ and ‘right’
to y (i.e. \( y_{\text{left}} \) and \( y_{\text{right}} \)) to denote the left and right children of \( y \) respectively. The
\textit{depth-first order} on \( V \) is the linear order \( \leq \) on \( V \) such that \( r_0 \) is the unique minimal
element, and \( y < y_{\text{left}} < y_{\text{right}} \) for every \( y \in Y \). An enumeration \( x_1, \ldots, x_k \) of a
subset \( X \subseteq V \) of size \( k \) is said to be \textit{arranged in depth-first order} if \( x_1 < \cdots < x_k \)
(with respect to the depth-first order). For every \( v \in Y \cup L \), it is easy to show
there is a unique sequence of vertices \( v_1, \ldots, v_m \) in \( V \) \((m \in \mathbb{P})\) such that \( v_1 = v \),
\( v_m \in L \), and \( v_i \) is the left child of \( v_{i-1} \) whenever \( i \in [m] \) and \( i > 1 \). We call
this sequence \( v_1, \ldots, v_m \) the \textit{left relative sequence} of \( v \) in \( G \), and we say \( v_m \) is the
\textit{left-most relative} of \( v \) in \( G \). Define \textit{right relative sequence} and \textit{right-most relative}
 analogously.

Suppose \( T = (V, E) \) and \( T' = (V', E') \) are binary trees with (root,1,3)-triples
\((r_0, L, Y)\) and \((r'_0, L', Y')\) respectively. Let \( r_1 \) be the unique child of \( r_0 \) in \( T \), and
let \( r'_1 \) be the unique child of \( r'_0 \) in \( T' \). Then, we say \( T, T' \) are \textit{isomorphic as binary
trees} if there exists a bijection \( \beta : V \to V' \) such that \( \beta(r_0) = r'_0 \), \( \beta(r_1) = r'_1 \), and
every \( y \in Y \) satisfies \( \beta(y) \in Y' \), \( \beta(y_{\text{left}}) = \beta(y)_{\text{left}} \) and \( \beta(y_{\text{right}}) = \beta(y)_{\text{right}} \).

\section{Construction of shedding trees}

Let \( a \in \mathbb{P}^n \), and let \( (\Delta, \pi) \) be a pure color-shifted \( a \)-balanced complex. If \( \Delta \) is an
\( a \)-rib of a simplex (cf. Chapter 12.2), then the fine \( f \)-vector of \( (\Delta, \pi) \) is easy to
compute: Since \( \Delta = \langle \binom{V_i}{a_i} \rangle \ast \cdots \ast \langle \binom{V_n}{a_n} \rangle \), we get \( f_b(\Delta) = \binom{\lambda}{b} \) for every \( b \in \mathbb{N}^n \)
satisfying \( b \leq a \). If \( \Delta \) is not an \( a \)-rib of a simplex, then by Corollary 12.3.3, there
is some \( i \in [n] \) such that \( \lambda_i > a_i \), and \( v_{i, \lambda_i} \) is a Macaulay shedding vertex of \( \Delta \).
Observe that $\Delta = \text{dl}_\Delta(v) \cup \{ F \cup \{v\} : F \in \text{lk}_\Delta(v) \}$ for every vertex $v$ of $\Delta$, hence

$$\mathcal{F}_b(\Delta) = \mathcal{F}_b(\text{dl}_\Delta(v)) \cup \left\{ F \cup \{v_i, \lambda_i\} : F \in \mathcal{F}_{b-\delta_{i,n}}(\text{lk}_\Delta(v_i, \lambda_i)) \right\} \quad (13.1)$$

for all $b \in \mathbb{N}^n$ satisfying $b \leq a$, which yields

$$f_b(\Delta) = f_b(\text{dl}_\Delta(v)) + f_{b-\delta_{i,n}}(\text{lk}_\Delta(v)) \quad (13.2)$$

for all $b \in \mathbb{N}^n$ satisfying $b \leq a$. Furthermore, if we let $W, W'$ be the vertex sets of $\text{dl}_\Delta(v), \text{lk}_\Delta(v)$ respectively, then by Corollary 12.3.3, $(\text{dl}_\Delta(v), \pi \cap W)$ is a pure color-shifted $a$-balanced complex, while $(\text{lk}_\Delta(v), \overline{\pi \cap W})$ is a pure color-shifted $(a-\delta_{i,n})$-balanced complex, so these two balanced subcomplexes are $a$-Macaulay decomposable and $(a-\delta_{i,n})$-Macaulay decomposable respectively by Theorem 12.3.1 and Proposition 12.2.11.

The main idea of this section is to start with the pure color-shifted $a$-balanced complex $(\Delta, \pi)$, and for every pure color-shifted $a'$-balanced subcomplex $(\Delta', \pi')$ of $(\Delta, \pi)$ that is not an $a'$-rib of a simplex (where $\pi' = (V'_1, \ldots, V'_m)$, $a' = (a'_1, \ldots, a'_m) \in \mathbb{P}^m$ and $m \leq n$), choose some $t \in [m]$ such that $|V'_t| > a'_t$, replace $(\Delta', \pi')$ with two pure color-shifted balanced subcomplexes $(\text{dl}_{\Delta'}(v'_{t,\text{max}}), \pi' \cap W)$ and $(\text{lk}_{\Delta'}(v'_{t,\text{max}}), \overline{\pi' \cap W})$ (where $v'_{t,\text{max}}$ is the maximal element of $V'_t$, and $W, W'$ are the vertex sets of $\text{dl}_{\Delta'}(v'_{t,\text{max}})$, $\text{lk}_{\Delta'}(v'_{t,\text{max}})$ respectively), and repeat the process until every remaining pure color-shifted balanced subcomplex, say of type $a''$, is an $a''$-rib of a simplex. By repeatedly using (13.2) on these subcomplexes of $\Delta$, we can then compute $f_b(\Delta)$ for each $b \in \mathbb{N}^n$ satisfying $b \leq a$. We now make this idea rigorous.

**Definition 13.2.1.** Let $t \in [n]$, let $S = (S_1, \ldots, S_n)$ be an $n$-tuple of sets, and let $x = (x_1, \ldots, x_n) \in \mathbb{N}^n$. A simplicial complex $\Sigma$ is called $t$-factorizable with respect to $(S, x)$ if $\Sigma = \left\langle \left( S_i \right) \right\rangle \ast \Sigma'$ for some (non-empty) simplicial complex $\Sigma'$. If $x \in \mathbb{P}^n$, 173
then let \( \kappa(S, x) \) be the \( n \)-tuple whose \( i \)-th entry is \( |S_i| \) if \( S_i \) is non-empty, and \( x_i \) otherwise.

Let \( T_0 \) be the planted tree with 2 vertices, where \( r_0 \) is the root, and \( r_1 \) is the unique child of \( r_0 \). Define the vertex labeling \( \phi_0 \) of \( T_0 \) by \( \phi_0(r_0) = \phi_0(r_1) = (a, \Delta, \pi, \lambda) \), i.e. the \( \phi_0 \)-labels are 4-tuples, where the first entry is in \( \mathbb{N}^n \), the second entry is a simplicial complex, the third entry is an \( n \)-tuple of subsets of \( V \), and the last entry is in \( \mathbb{P}^n \). Starting with the pair \( (T_0, \phi_0) \), we shall construct a sequence 'seq' of pairs \( (T_0, \phi_0), (T_1, \phi_1), (T_2, \phi_2), \ldots \) via the following algorithm:

\[
\text{seq} \leftarrow (T_0, \phi_0).
\]
\[
t \leftarrow n.
\]
\[
\textbf{while } t > 0 \textbf{ do}
\]
\[
(T', \phi') \leftarrow \text{last pair in seq}.
\]
\[
\mathcal{L} \leftarrow \text{set of terminal vertices of } T'.
\]
\[
\textbf{if } \exists v \in \mathcal{L} \text{ with } \phi'(v) = (a', \Delta', \pi', \lambda') \text{ such that } \Delta' \text{ is not } t\text{-factorizable with respect to } (\pi', a') \text{ then}
\]
\[
\text{Choose any such } v \text{ with } \phi'(v) = (a', \Delta', \pi', \lambda').
\]
\[
\text{Write } a' = (a'_1, \ldots, a'_n), \pi' = (V'_1, \ldots, V'_n).
\]
\[
\text{Let } v'_{t, \max} \text{ be the maximal element of } V'_t.
\]
\[
\text{Let } W, W' \text{ denote the vertex sets of } dl_{\Delta'}(v'_{t, \max}), lk_{\Delta'}(v'_{t, \max}) \text{ respectively.}
\]
\[
\text{Construct a binary tree } T \text{ from } T' \text{ by adding two children } v_{\text{left}} \text{ and } v_{\text{right}} \text{ to } v.
\]
\[
\text{Treat } T' \text{ as a subgraph of } T. \text{ Let } V' \text{ denote the vertex set of } T'.
\]
\[
\text{Define the vertex labeling } \phi \text{ of } T \text{ by}
\]
\[
\phi(u) = \begin{cases}
\phi'(u), & \text{if } u \in V' \setminus \{v\}; \\
t, & \text{if } u = v;
\end{cases}
\]
\[
\begin{cases}
(a', dl_{\Delta'}(v'_{t, \max}), \pi' \cap W, \kappa(\pi' \cap W, \lambda' - \delta_{t,n})), & \text{if } u = v_{\text{left}}; \\
(a' - \delta_{t,n}, lk_{\Delta'}(v'_{t, \max}), \pi' \cap W', \kappa(\pi' \cap W', \lambda' - \delta_{t,n})), & \text{if } u = v_{\text{right}}.
\end{cases}
\]
\[
\text{seq} \leftarrow (\text{seq}, (T, \phi)) \text{ (i.e. seq with the term } (T, \phi) \text{ appended).}
\]
\[
\text{else}
\]
\[
t \leftarrow t - 1.
\]
\[
\text{end if}
\]
\[
\text{end while}
\]

Clearly \( |V'_t| = a'_t \) or \( a'_t = 0 \) implies \( \Delta' \) is \( t \)-factorizable with respect to \( (\pi', a') \), so if \( \Delta' \) is not \( t \)-factorizable with respect to \( (\pi', a') \), then \( |V'_t| > a'_t > 0 \), and
Corollary 12.3.3 implies $v_{t, \text{max}} \, \prime$ is a Macaulay shedding vertex of $\Delta'$. Since $\Delta$ is a-Macaulay decomposable, this algorithm must eventually terminate, independent of the choices of $v$ made, i.e. the sequence ‘seq’ has finitely many terms. Let $(T_0, \phi_0), (T_1, \phi_1), \ldots, (T_k, \phi_k)$ be such a sequence constructed from $(\Delta, \pi)$. It is easy to see that the last pair $(T_k, \phi_k)$ (possibly $(T_0, \phi_0)$) is uniquely determined by $(\Delta, \pi)$, independent of the choices of $v$ made. Note however that the other pairs in the sequence do depend on the choices of $v$ at each iteration in the algorithm. We call any such sequence a shedding sequence of $(\Delta, \pi)$, and we say the last pair $(T_k, \phi_k)$ is the shedding tree of $(\Delta, \pi)$.

By construction, each $T_i$ is a binary tree with $T_{i-1}$ as a subgraph, and each $\phi_i$ is a vertex labeling of $T_i$, thus each pair $(T_i, \phi_i)$ is a labeled binary tree. The trivalent vertices of $T_i$ are labeled with integers in $[n]$, such that $\phi_i(y) \geq \phi_i(y')$ for every pair $y, y'$ of trivalent vertices in $T_i$ satisfying $y' \in D_{T_i}(y)$, i.e. $y'$ is a descendant of $y$. Also, every terminal vertex of $T_i$ is labeled with a 4-tuple $(a'_1, \ldots, a'_n, \Delta', \pi', \lambda')$, where $a' = (a'_1, \ldots, a'_n) \in \mathbb{N}^n$, $\Delta'$ is a (non-empty) simplicial complex, $\pi' = (V'_1, \ldots, V'_n)$ is an $n$-tuple of subsets of $V$, and $\lambda' \in \mathbb{P}^n$.

We can check that $(\Delta', \pi')$ is a pure $\overline{a'}$-balanced color-shifted complex, and $\Delta'$ is $a'$-Macaulay decomposable. Furthermore, if $(T_i, \phi_i) = (T_k, \phi_k)$ is the shedding tree of $(\Delta, \pi)$, then $\Delta'$ is $t$-factorizable with respect to $(\pi', a')$ for every $t \in [n]$, hence $\Delta'$ is the $\overline{a'}$-rib of a simplex corresponding to the ordered partition $\overline{\pi'}$ in this case.

**Example 13.2.2.** Let $\tau = (P, Q)$ such that $P = \{p_1, p_2, p_3\}$ and $Q = \{q_1, q_2, q_3, q_4\}$ satisfy $p_1 < p_2 < p_3$ and $q_1 < q_2 < q_3 < q_4$. Consider the pure color-shifted $(1, 1)$-balanced complex $(\Sigma, \tau)$, where

\[
\Sigma = \left\{ \left\{ \{p_1, q_1\}, \{p_2, q_1\}, \{p_3, q_1\} \right\}, \left\{ \{p_1, q_2\}, \{p_2, q_2\} \right\}, \left\{ \{p_1, q_3\}, \{p_2, q_3\}, \{p_1, q_4\} \right\} \right\}.
\]
Then its shedding tree is given in Figure 13.2.

![Shedding tree diagram](image-url)

Figure 13.2: The shedding tree of the $I_2$-balanced complex $(\Sigma, \tau)$ given in Example 13.2.2.

Given any term $(T_i, \phi_i)$ of a shedding sequence of $(\Delta, \pi)$, the restriction of $\phi_i$ to the terminal vertices of $T_i$ has four component functions, which we denote by $\phi_i^{(L,a)}$, $\phi_i^{(L,\Delta)}$, $\phi_i^{(L,\pi)}$, $\phi_i^{(L,\lambda)}$ respectively. Suppose $(T, \phi)$ is the shedding tree of $(\Delta, \pi)$, and let $(r_0, L, Y)$ be the $(\text{root}, 1, 3)$-triple of $T$. Note that $\phi_i^{(L,\pi)}(u) \in \mathbb{N}^n$ and $\phi_i^{(L,\lambda)}(u) \in \mathbb{P}^n$ for all $u \in L$. By the construction of $(T, \phi)$, it is easy to show that $0_n < \phi_i^{(L,a)}(u) \leq \phi_i^{(L,\lambda)}(u)$ for all $u \in L$. Also, by the repeated use of (13.2), we have the following.

**Proposition 13.2.3.** For each $b \in \mathbb{N}^n$ satisfying $b \leq a$,

$$f_b(\Delta) = \sum_{u \in L} \left( \frac{\phi_i^{(L,\lambda)}(u)}{\phi_i^{(L,a)}(u) + b - a} \right). \quad (13.3)$$

For example, the balanced complex $(\Sigma, \tau)$ in Example 13.2.2 satisfies $f_{(1,1)}(\Sigma) = 8$, $f_{(0,1)}(\Sigma) = 3$, $f_{(1,0)}(\Sigma) = 4$, and we check that indeed $8 = \binom{3,2}{(1,1)} + \binom{1,2}{(1,0)} + \binom{1,3}{(1,0)}$, $3 = \binom{3,2}{(1,0)} + \binom{1,2}{(1,-1)} + \binom{1,3}{(1,-1)}$, and $4 = \binom{3,2}{(0,1)} + \binom{1,2}{(0,0)} + \binom{1,3}{(0,0)}$.

**Remark 13.2.4.** The entire Chapter 13.2 is still true if we replace every instance of “color-shifted” with “color-compressed”. In particular, if $(\Delta, \pi)$ is color-compressed and $(T_i, \phi_i)$ is a term in any shedding sequence of $(\Delta, \pi)$, then the colored multicomplex $(\phi_i^{(L,\Delta)}(u), \phi_i^{(L,\pi)}(u))$ is also color-compressed for every terminal vertex $u$ of $T_i$. 176
13.3 Macaulay trees

Let $a \in \mathbb{N}^n \setminus \{0_n\}$. Given any pure color-shifted $\overline{a}$-balanced complex $(\Delta, \pi)$, it is clear from Proposition 13.2.3 that the shedding tree $(T, \phi)$ of $(\Delta, \pi)$ encodes superfluous information for computing the fine $f$-vector of $\Delta$. Motivated by the desire to retain only the necessary numerical information needed to compute the fine $f$-vector of $\Delta$, we shall define the notion of an $a$-Macaulay tree of any positive integer $N$, which does not depend on the existence of any $\overline{a}$-balanced complexes.

First, we introduce a series of related definitions.

**Definition 13.3.1.** Let $a \in \mathbb{N}^n$. An $a$-splitting tree is a triple $(T, \mu, \nu)$ such that

(i) $T = (V, E)$ is a binary tree with (root, 1, 3)-triple $(r_0, L, Y)$;
(ii) $\mu : Y \to [n]$ is a labeling of the trivalent vertices of $T$; and
(iii) $\nu : V \to \mathbb{Z}^n$ is a vertex labeling of $T$ recursively defined as follows.

(a) Define $\nu(r_0) = \nu(r_1) = a$, where $r_1$ is the unique child of the root $r_0$.
(b) For every $y \in Y$, if $\nu(y)$ was already defined, while $\nu(y_{\text{left}}), \nu(y_{\text{right}})$ are both not yet defined, then define $\nu(y_{\text{left}}) = \nu(y), \nu(y_{\text{right}}) = \nu(y) - \delta_{\mu(y), n}$.

Note that $\nu : V \to \mathbb{Z}^n$ is completely determined by $T, \mu, a$, and we say $(T, \mu, \nu)$ is the $a$-splitting tree induced by $(T, \mu)$.

**Definition 13.3.2.** Let $T$ be a binary tree with (root, 1, 3)-triple $(r_0, L, Y)$, and let $\varphi$ be any vertex labeling of $T$ such that $\varphi(Y) \subseteq [n]$ and $\varphi(L) \subseteq \mathbb{P}^n$. The *left-weight labeling* of $(T, \varphi)$ is the map $\omega : Y \cup L \to \mathbb{P}^n$ defined by

$$\omega(v) := \left(\varphi(v_m) + \sum_{i=1}^{m-1} \delta_{\varphi(v_i), n}\right),$$

where $v_1, \ldots, v_m$ is the left relative sequence of $v$ in $T$, and we say $\omega(v)$ is the *left-weight* of $v$ in $(T, \varphi)$. In particular, if $v \in L$, then $\omega(v) = \varphi(v)$. We say $\omega$ is *proper* if $\omega(y_{\text{left}}) \geq \omega(y_{\text{right}})$ for every $y \in Y$. 

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Definition 13.3.3. Let $a \in \mathbb{N}^n \setminus \{0_n\}$, and let $N \in \mathbb{P}$. An $a$-Macaulay tree of $N$ is any pair $(T, \varphi)$ such that

(i) $T$ is a binary tree with $(\text{root}, 1, 3)$-triple $(r_0, L, Y)$; and

(ii) $\varphi$ is a vertex labeling of $T$ satisfying all of the following conditions.

(a) $\varphi(r_0) = a$, $\varphi(Y) \subseteq [n]$ and $\varphi(L) \subseteq \mathbb{P}^n$.

(b) If $y, y' \in Y$ satisfies $y' \in D_T(y)$, i.e. $y'$ is a descendant of $y$, then $\varphi(y) \geq \varphi(y')$.

(c) If $y \in Y$ satisfies $\varphi(y) < n$, then $\varphi^{[t]}(u) = \varphi^{[t]}(u')$ for all $u, u' \in L \cap D_T(y)$ and all $t \in [n] \setminus \lfloor \varphi(y) \rfloor$.

(d) The left-weight labeling of $(T, \varphi)$ is proper.

(e) The $a$-splitting tree $(T, \varphi|_Y, \nu)$ induced by $(T, \varphi|_Y)$ satisfies $\varphi(u) \geq \nu(u) > 0_n$ for all $u \in L$.

(f) If $y \in Y$ satisfies $\nu^{[\varphi(y)]}(y) = 1$, then $\omega^{[\varphi(y)]}(x) = \omega^{[\varphi(y)]}(y) - 1$ for all $x \in D_T(y_{\text{right}})$.

(g) $N$ can be written as the sum

$$N = \sum_{u \in L} \left( \varphi(u) \nu(u) \right). \quad (13.4)$$

An $a$-Macaulay tree is an $a$-Macaulay tree of $N$ for some positive integer $N$, and a Macaulay tree is an $a$-Macaulay tree for some $a \in \mathbb{N}^n \setminus \{0_n\}$. Suppose $(T, \varphi)$ is a Macaulay tree, and let $(r_0, L, Y)$ be the $(\text{root}, 1, 3)$-triple of $T$. The weight of $(T, \varphi)$ is the left-weight of the unique child of $r_0$.

Definition 13.3.4. Given any $t \in \{0, 1, \ldots, n\}$, we say $x \in Y \cup L$ is a $t$-leading vertex of $(T, \varphi)$ if it satisfies the following two conditions:

(i) If $x \in Y$, then $\varphi(x) \leq t$; and

(ii) If $\tilde{x}$ is the parent of $x$ and $\tilde{x} \neq r_0$, then $\varphi(\tilde{x}) > t$.  

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Let $L_{(T,\varphi)}(t)$ be the set of all $t$-leading vertices of $(T,\varphi)$.

Note that a $t$-leading vertex could possibly be a $t'$-leading vertex for distinct $t,t' \in \{0,1,\ldots,n\}$, while not every vertex in $Y \cup L$ is necessarily a $t$-leading vertex for some $t \in \{0,1,\ldots,n\}$. It is easy to show that for every $t \in \{0,1,\ldots,n\}$ and every path $v_0,v_1,\ldots,v_\ell$ from $v_0 = r_0$ to some terminal vertex $v_\ell \in L$, there exists a unique $i_\ell \in [\ell]$ such that $v_{i_\ell}$ is a $t$-leading vertex. In particular, the unique child of $r_0$ is always an $n$-leading vertex, and it is the only $n$-leading vertex in $Y \cup L$, while every terminal vertex is a 0-leading vertex, i.e. $L_{(T,\varphi)}(0) = L$.

**Definition 13.3.5.** Let $(T,\varphi)$ be a Macaulay tree, and let $(r_0,L,Y)$ be the (root,1,3)-triple of $T$. A trivalent vertex $y \in Y$ is called a cloning vertex of $(T,\varphi)$ if it satisfies the following two conditions:

(i) The two subgraphs of $T$ induced by $D_T(y_{\text{left}})$ and $D_T(y_{\text{right}})$ are isomorphic as binary trees, and they have the same corresponding $\varphi$-labels, i.e. if $\beta : D_T(y_{\text{left}}) \to D_T(y_{\text{right}})$ is the corresponding isomorphism, then $\varphi(\beta(x)) = \varphi(x)$ for all $x \in D_T(y_{\text{left}})$.

(ii) Both $y_{\text{left}}$ and $y_{\text{right}}$ are $(\varphi(y) - 1)$-leading vertices of $(T,\varphi)$.

If $(T,\varphi)$ has no cloning vertices, then we say $(T,\varphi)$ is condensed. In particular, $(T,\varphi)$ is condensed implies $\varphi(y_{\text{left}}) > \varphi(y_{\text{right}})$ for every $y \in Y$ such that $y_{\text{left}}$ and $y_{\text{right}}$ are both leaves.

**Definition 13.3.6.** Let $(T,\varphi)$ be an $a$-Macaulay tree of some positive integer $N$, where $a \in \mathbb{N}^n \setminus \{0_n\}$. The condensation of $(T,\varphi)$ is a condensed $a$-Macaulay tree $(\tilde{T},\tilde{\varphi})$ of $N$ that is constructed from $(T,\varphi)$ via the following algorithm:

$(\tilde{T},\tilde{\varphi}) \leftarrow (T,\varphi)$.

while $\exists$ a cloning vertex of $(\tilde{T},\tilde{\varphi})$ do
Choose any cloning vertex $y$ of $(T, \varphi)$.

$\tilde{\varphi}(y) \leftarrow \varphi(y_{\text{left}})$.

$\varphi \leftarrow \varphi|_{\tilde{V}\setminus \mathcal{D}(y_{\text{right}})}$ (where $\tilde{V}$ is the vertex set of $\tilde{T}$).

$\tilde{T} \leftarrow$ subgraph of $T$ induced by $\tilde{V}\setminus \mathcal{D}(y_{\text{right}})$.

$e \leftarrow \{y, y_{\text{left}}\}$.

$(\tilde{T}, \tilde{\varphi}) \leftarrow (T/e, \varphi/e)$.

end while

In defining the condensation $(\tilde{T}, \tilde{\varphi})$ of $(T, \varphi)$, the number of vertices of $\tilde{T}$ becomes strictly smaller in each iteration of the while loop, hence the while loop is not an infinite loop. It is also easy to see that the pair $(\tilde{T}, \tilde{\varphi})$ obtained from the above algorithm is uniquely determined by $(T, \varphi)$, independent of the choice of the cloning vertex in each iteration of the while loop, hence $(\tilde{T}, \tilde{\varphi})$ is well-defined, and in particular, the condensation of a condensed Macaulay tree is itself. The fact that $(\tilde{T}, \tilde{\varphi})$ is a condensed $a$-Macaulay tree of $N$ is straightforward and left to the reader as an exercise.

For the rest of this section, let $a \in \mathbb{P}^n$, let $(\Delta, \pi)$ be a pure color-shifted $a$-balanced complex, and let $(T_0, \phi_0), (T_1, \phi_1), \ldots, (T_k, \phi_k)$ be a shedding sequence of $(\Delta, \pi)$, where $(r_0, L_i, Y_i)$ denotes the (root, 1, 3)-triple of each $T_i$. Clearly $r_0$ is the common root of $T_0, T_1, \ldots, T_k$, and observe that

$$Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_k; \quad Y_0 \cup L_0 \subseteq Y_1 \cup L_1 \subseteq \cdots \subseteq Y_k \cup L_k. \tag{13.5}$$

For every $y \in Y_k$, let $\alpha(y)$ be the largest integer $< k$ such that $y \in L_{\alpha(y)}$, while for every $u \in L_k$, set $\alpha(u) = k$. Also, for each $i \in \{0, 1, \ldots, k\}$, define the vertex labeling $\varphi_i$ of $T_i$ as follows.

$$\varphi_i(u) = \begin{cases} a, & \text{if } u = r_0; \\ \phi_i(u), & \text{if } u \in Y_i; \\ \phi_i^{[a, \lambda]}(u), & \text{if } u \in L_i. \end{cases}$$

By construction, every $u \in Y_i$ is labeled with an integer in $[n]$, while every $u \in L_i$.
is labeled with an \( n \)-tuple in \( \mathbb{P}^n \), hence the left-weight labeling of \( (T_i, \varphi_i) \) is well-defined, and we denote this left-weight labeling by \( \omega_i \).

**Lemma 13.3.7.** If \( v \in Y_k \), then \( \omega_{\alpha(v)}(v) = \omega_{\alpha(v)+1}(v) \).

**Proof.** Let \( \phi_{\alpha(v)}(v) = (a', \Delta', \pi', \lambda') \), where \( a' = (a'_1, \ldots, a'_n) \), and \( \pi' = (V'_1, \ldots, V'_n) \) is an ordered partition of the vertex set \( V' \) of \( \Delta' \). Let \( t = \phi_{\alpha(v)+1}(v) \), let \( v'_{t,\text{max}} \) be the maximal element of \( V'_t \), and denote the vertex set of \( d\Delta'(v'_{t,\text{max}}) \) by \( W \). Also, write \( \pi' \cap W = (V'\pi_1, \ldots, V'\pi_n) \). Since \( \Delta' \) is color-shifted, every vertex \( v'' < v'_{t,\text{max}} \) is in \( W \), hence \( |V'_t| = |V'_t| - 1 \). Note that \( |V'_t| > a'_t > 0 \) yields \( |V''_t| > 0 \), while every vertex in \( V' \setminus V_t \) is in \( W \), thus \( \kappa(\pi' \cap W, \lambda' - \delta_{t,n}) = \lambda' - \delta_{t,n} \). Now,

\[
\varphi_{\alpha(v)+1}(v_{\text{left}}) = \phi_{\alpha(v)+1}^{[\mathcal{L}, \lambda]}(v_{\text{left}}) = \kappa(\pi' \cap W, \lambda' - \delta_{t,n})
\]  

(13.6)

by construction, and \( v, v_{\text{left}} \) is the left relative sequence of \( v \) in \( T_{\alpha(v)+1} \), therefore

\[
\omega_{\alpha(v)+1}(v) = \varphi_{\alpha(v)+1}(v_{\text{left}}) + \delta_{t,n} = \lambda' = \phi_{\alpha(v)}^{[\mathcal{L}, \lambda]}(v) = \varphi_{\alpha(v)}(v) = \omega_{\alpha(v)}(v).
\]  

(13.7)

**Proposition 13.3.8.** For every \( i \in \{0, 1, \ldots, k\} \), we have the following:

(i) \( \omega_i(v) = \omega_j(v) \) for all \( v \in Y_i \cup L_i \) and all integers \( j \) satisfying \( i \leq j \leq k \).

(ii) \( \omega_i \) is proper.

(iii) If the children of some \( y \in Y_i \) are both in \( L_k \), then \( \varphi_i(y_{\text{left}}) > \varphi_i(y_{\text{right}}) \).

(iv) If \( \varphi_i(x) < n \) for some \( x \in Y_i \), then \( \omega_i^{[t]}(x') = \omega_i^{[t]}(x) \) for all \( x' \in D_{T_i}(x) \) and all \( t \in [n] \setminus \varphi_i(x) \).

**Proof.** First of all, (i) follows from Lemma 13.3.7 by the algorithmic construction of a shedding sequence. Next, we prove (ii) and (iii) concurrently by induction on \( i \), with the base case \( i = 0 \) being vacuously true since \( Y_0 = \emptyset \).
Choose an arbitrary \( y \in Y_i \) and let \( q := \alpha(y) < i \). If \( q < i - 1 \), then both \( y_{\text{left}} \) and \( y_{\text{right}} \) are vertices in \( T_{q+1} \), and \( \omega_{q+1} \) is proper by induction hypothesis, hence \( \omega_{q+1}(y_{\text{left}}) \geq \omega_{q+1}(y_{\text{right}}) \). Using statement (i), we thus get \( \omega_i(y_{\text{left}}) \geq \omega_i(y_{\text{right}}) \).

Furthermore, if \( y_{\text{left}} \) and \( y_{\text{right}} \) are both in \( L_k \), then the induction hypothesis also yields \( \varphi_{q+1}(y_{\text{left}}) > \varphi_{q+1}(y_{\text{right}}) \), which implies \( \varphi_i(y_{\text{left}}) > \varphi_i(y_{\text{right}}) \).

Suppose \( q = i - 1 \). Let \( \phi_q(y) = (a', \Delta', \pi', \lambda') \), and write \( a' = (a'_1, \ldots, a'_n) \), \( \pi' = (V'_1, \ldots, V'_n) \). Let \( t = \phi_i(y) \), let \( v'_{t,\text{max}} \) be the maximal element of \( V'_t \), and let \( W, W' \), be the vertex sets of \( \text{dl}_{\Delta'}(v'_{t,\text{max}}) \), \( \text{lk}_{\Delta'}(v'_{t,\text{max}}) \) respectively. By the maximality of \( q = \alpha(y) \), we have

\[
\omega_i(y_{\text{left}}) = \varphi_i(y_{\text{left}}) = \kappa(\pi' \cap W, \lambda' - \delta_{t,n}); \quad (13.8)
\]
\[
\omega_i(y_{\text{right}}) = \varphi_i(y_{\text{right}}) = \kappa(\pi' \cap W', \lambda' - \delta_{t,n}). \quad (13.9)
\]

Clearly \( \text{lk}_{\Delta'}(v'_{t,\text{max}}) \subseteq \text{dl}_{\Delta'}(v'_{t,\text{max}}) \) implies \( W' \subseteq W \). Note also that \( \text{dl}_{\Delta'}(v'_{t,\text{max}}) \) is \( \mathbf{a}' \)-balanced, while \( \text{lk}_{\Delta'}(v'_{t,\text{max}}) \) is \( (\mathbf{a}' - \delta_{t,n}) \)-balanced, so \( |V'_t| > a'_i > 0 \) implies the \( \emptyset \) entries of \( \pi' \cap W \) and \( \pi' \cap W' \) are identical, thus \( \kappa(\pi' \cap W, \lambda' - \delta_{t,n}) \geq \kappa(\pi' \cap W', \lambda' - \delta_{t,n}) \), i.e. \( \omega_i(y_{\text{left}}) \geq \omega_i(y_{\text{right}}) \).

Suppose further that \( y_{\text{left}} \) and \( y_{\text{right}} \) are both in \( L_k \), and recall we already have

\[
\varphi_i(y_{\text{left}}) = \omega_i(y_{\text{left}}) \geq \omega_i(y_{\text{right}}) = \varphi_i(y_{\text{right}}). \quad (13.10)
\]

If \( \varphi_i(y_{\text{left}}) = \varphi_i(y_{\text{right}}) \), then \( \pi' \cap W = \pi' \cap W' \). Since \( y_{\text{left}}, y_{\text{right}} \in L_k \) implies \( \text{dl}_{\Delta'}(v'_{t,\text{max}}) \) is an \( \mathbf{a}' \)-rib of a simplex and \( \text{lk}_{\Delta'}(v'_{t,\text{max}}) \) is an \( (\mathbf{a}' - \delta_{t,n}) \)-rib of a simplex, it follows that \( \Delta' \) is an \( \mathbf{a}' \)-rib of a simplex, which contradicts the fact that \( y \) is not a terminal vertex in \( T_k \). Hence, we must have \( \varphi_i(y_{\text{left}}) > \varphi_i(y_{\text{right}}) \) in this case, therefore completing the induction step for statements (ii) and (iii).

Finally, we prove statement (iv). Fix some \( i > 0 \), assume \( \varphi_i(x) < n \) for some \( x \in Y_i \), and write \( m := \varphi_i(x) \). Let \( \phi_{\alpha(x)}(x) = (\mathbf{\tilde{a}}, \tilde{\Delta}, \tilde{\pi}, \tilde{\lambda}) \), write \( \mathbf{\tilde{a}} = (\tilde{a}_1, \ldots, \tilde{a}_n) \),
\(\pi = (\bar{V}_1, \ldots, \bar{V}_n), \bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n)\), and let \(\bar{V} := \bigcup_{i \in [n]} \bar{V}_i\) be the vertex set of \(\bar{D}\).

From the construction of the shedding sequence, \(\bar{D}\) is \(t\)-factorizable with respect to \((\pi, \bar{\lambda})\) for all \(t \in [n] \setminus [m]\), hence

\[
\bar{D} = \left\langle \frac{\bar{V}_n}{a_n} \right\rangle * \left\langle \frac{\bar{V}_{n-1}}{a_{n-1}} \right\rangle * \cdots * \left\langle \frac{\bar{V}_{m+1}}{a_{m+1}} \right\rangle * \Sigma \tag{13.11}
\]

for some (non-empty) simplicial complex \(\Sigma\) with vertex set \(\bar{V}_1 \cup \cdots \cup \bar{V}_m\), which yields

\[
dl_{\bar{D}}(v) = \left\langle \frac{\bar{V}_n}{a_n} \right\rangle * \left\langle \frac{\bar{V}_{n-1}}{a_{n-1}} \right\rangle * \cdots * \left\langle \frac{\bar{V}_{m+1}}{a_{m+1}} \right\rangle * dl_{\Sigma}(v), \tag{13.12}
\]

\[
lk_{\bar{D}}(v) = \left\langle \frac{\bar{V}_n}{a_n} \right\rangle * \left\langle \frac{\bar{V}_{n-1}}{a_{n-1}} \right\rangle * \cdots * \left\langle \frac{\bar{V}_{m+1}}{a_{m+1}} \right\rangle * lk_{\Sigma}(v), \tag{13.13}
\]

for all \(v \in \bar{V}_1 \cup \cdots \cup \bar{V}_m\). For every \(y \in \mathcal{D}_{T_i}(x) \cap Y\), since \(\varphi_i(y) \leq \varphi_i(x) = m\) implies the Macaulay shedding vertex used to construct the children of \(y\) in \(T_{\alpha(y)+1}\) is chosen from \(\bar{V}_1 \cup \cdots \cup \bar{V}_m\), it then follows inductively that for every \(x' \in \mathcal{D}_{T_i}(x)\), the simplicial complex \(\phi_{\alpha(x')}^{[\ell, \Delta]}(x')\) satisfies

\[
\phi_{\alpha(x')}^{[\ell, \Delta]}(x') = \left\langle \frac{\bar{V}_n}{a_n} \right\rangle * \left\langle \frac{\bar{V}_{n-1}}{a_{n-1}} \right\rangle * \cdots * \left\langle \frac{\bar{V}_{m+1}}{a_{m+1}} \right\rangle * \Sigma' \tag{13.14}
\]

for some (non-empty) simplicial complex \(\Sigma'\) (dependent on the choice of \(x'\)) with its vertex set contained in \(\bar{V}_1 \cup \cdots \cup \bar{V}_m\). Consequently, statement (i) yields

\[
\omega_{\alpha(x')}^{[t]}(x') = \omega_{\alpha(x')}^{[t]}(x') = \phi_{\alpha(x')}^{[\ell, \Delta]}(x') = \bar{\lambda}_t = \omega_{\alpha}^{[t]}(x) \tag{13.15}
\]

for all \(x' \in \mathcal{D}_{T_i}(x)\) and all \(t \in [n] \setminus [m]\).

\[\square\]

**Theorem 13.3.9.** \((T_i, \varphi_i)\) is an \(a\)-Macaulay tree for each \(i \in \{0, 1, \ldots, k\}\). Furthermore, if \(\Delta\) has \(N\) facets, then \((T_k, \varphi_k)\) is a condensed \(a\)-Macaulay tree of \(N\).

**Proof.** Choose any \(i \in \{0, 1, \ldots, k\}\), and let \((T_i, \varphi|_{Y_i}, \nu_i)\) be the \(a\)-splitting tree induced by \((T_i, \varphi|_{Y_i})\). Clearly \(\varphi_i(r_0) = a, \varphi_i(Y_i) \subseteq [n]\) and \(\varphi_i(L_i) \subseteq \mathbb{P}^n\) by the
definition of \( \varphi_i \), while the construction of a shedding sequence yields \( \varphi_i(y) \geq \varphi_i(y') \) for every \( y, y' \in Y_i \) satisfying \( y' \in D_{T_i}(y) \). It is also easy to show that

\[
\varphi_i(u) = \phi_i^{[\mathcal{L}, \mathcal{X}]}(u) \geq \phi_i^{[\mathcal{L}, a]}(u) = \nu_i(u) > 0_n
\]  

(13.16)

for all \( u \in L_i \). Proposition 13.3.8 says \( \omega_i \) is proper, and if \( \varphi_i(x) < n \) for some \( x \in Y_i \), then \( \omega_i^{[t]}(x') = \omega_i^{[t]}(x) \) for all \( x' \in D_{T_i}(x) \) and all \( t \in [n] \setminus [\varphi_i(x)] \). This implies \( \varphi_i^{[t]}(u) = \varphi_i^{[t]}(u') \) for all \( u, u' \in L_i \cap D_{T_i}(x) \) and all \( t \in [n] \setminus [\varphi_i(x)] \).

Furthermore, if \( y \in Y_i \) satisfies \( \nu_i^{[\varphi_i(y)]}(y) = 1 \), then \( \nu_i^{[\varphi_i(y)]}(y_{\text{right}}) = 0 \), so by the definitions of \( \phi_i^{[\mathcal{L}, \mathcal{X}]}(y) + 1 \) and \( \kappa \), we get that \( \omega_i^{[\varphi_i(y)]}(y_{\text{right}}) = \varphi_i^{[\varphi_i(y)]}(y_{\text{right}}) \) is the \( \varphi_i(y) \)-th entry of \( \phi_i^{[\mathcal{L}, \mathcal{X}]}(y_{\text{right}}) \), which equals \( \varphi_i^{[\varphi_i(y)]}(y) - 1 = \omega_i^{[\varphi_i(y)]}(y) - 1 \). Hence, \( \omega_i^{[\varphi_i(y)]}(x) = \omega_i^{[\varphi_i(y)]}(y) - 1 \) for all \( x \in D_{T_i}(y_{\text{right}}) \), and we conclude that \( (T_i, \varphi_i) \) is an \( a \)-Macaulay tree.

Next, suppose \( (T_k, \varphi_k) \) is not condensed, let \( z \) be a cloning vertex of \( (T_k, \varphi_k) \), and let \( \varphi_k(z) = q \). Also, let \( \phi_{\alpha(z)}(z) = (a', \Delta', \pi', \mathcal{X}') \), and write \( a' = (a'_1, \ldots, a'_n) \), \( \pi' = (V'_1, \ldots, V'_n) \), \( \mathcal{X}' = (\lambda'_1, \ldots, \lambda'_{q}) \). If \( z_{\text{left}} \) and \( z_{\text{right}} \) are both in \( L \), then Proposition 13.3.8(iii) says \( \varphi_k(z_{\text{left}}) > \varphi_k(z_{\text{right}}) \), which contradicts the assumption that \( z \) is a cloning vertex. This forces \( z_{\text{left}}, z_{\text{right}} \in Y \) and \( \varphi_k(z_{\text{left}}) = \varphi_k(z_{\text{right}}) = q \).

By assumption, the two subgraphs of \( T_k \) induced by \( D_{T_k}(z_{\text{left}}) \) and \( D_{T_k}(z_{\text{right}}) \) are isomorphic as binary trees, and they have the same corresponding \( \varphi_k \)-labels, hence Proposition 13.3.8(iv) implies \( \omega_k^{[q]}(x) = \omega_k^{[q]}(z_{\text{left}}) = \lambda'_{q} - 1 \) for all \( x \in (D_{T_k}(z)) \setminus \{z\} \). Now, let \( v'_{q, \lambda_q} \) be the maximal element in \( V'_q \). Then

\[
dl_{\Delta'}(v'_{q, \lambda_q}) = \langle \langle V'_n \rangle \rangle_{a'_n} * \cdots * \langle \langle V'_{q+1} \rangle \rangle_{a'_{q+1}} * \langle \langle V'_q \setminus \{v'_{q, \lambda_q}\} \rangle \rangle_{a'_q} * \Sigma, \tag{13.17}
\]

\[
\lk_{\Delta'}(v'_{q, \lambda_q}) = \langle \langle V'_n \rangle \rangle_{a'_n} * \cdots * \langle \langle V'_{q+1} \rangle \rangle_{a'_{q+1}} * \langle \langle V'_q \setminus \{v'_{q, \lambda_q}\} \rangle \rangle_{a'_q - 1} * \Sigma, \tag{13.18}
\]

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ψ define the simplicial complex set is a subset of
\[ \{(\omega_0, \omega_1, \ldots, \omega_n) : v \in X, \text{ both } t \text{ and its right child are ancestors of } u \}. \]

Recall that \( \omega^J \) denotes the \( J \)-th component function of \( \omega \) for any \( J \in [n] \). For each \( u \in L \), define the set \( \psi(v, u) = \{(\omega_0, \omega_1, \ldots, \omega_n) : v \in X, \text{ both } t \text{ and its right child are ancestors of } u \} \), whose vertex set is a subset of \( P \times [n] \). The complexes \( \psi(v, u) \), \( \{(\psi(v, u), (u)) \} \), and \( \psi(v, (u)) \) have disjoint vertex sets.

**Definition 13.3.11.** Let \( a \in \mathbb{N} \setminus \{0\} \), let \( (T, \phi) \) be an a-Macaulay tree, and \( \phi_i \) denote the (root, 1)-tree of \( T \) induced by \( (v_0, L, y) \). Also, let \( (T, \phi) \) be the a-Macaulay tree induced by \( (v_0, L, y) \), and let \( \omega \) be the left-leaf labeling of \( (T, \phi) \).

For \( m \in \mathbb{N} \), define \( [m] = \{(1, 1, 2), \ldots, [m] \} \subseteq P \times [n] \). Also, define the simplicial complex \( \psi(v, u) = \{(\omega_0, \omega_1, \ldots, \omega_n) : v \in X, \text{ both } t \text{ and its right child are ancestors of } u \} \). For each \( u \in L \), define the set \( \psi(v, u) = \{(\omega_0, \omega_1, \ldots, \omega_n) : v \in X, \text{ both } t \text{ and its right child are ancestors of } u \} \).

**Remark 13.3.10.** In view of Theorem 13.3.9, we say \( (T, \phi) \) is the Macaulay tree induced by \( (v_0, L, y) \).

\( \Delta_N = \left( \langle \delta_0, \delta_1, \ldots, \delta_n \rangle \right) \ast \left( \langle \delta_0, \delta_1, \ldots, \delta_n \rangle \right) \ast \cdots \ast \langle \delta_0, \delta_1, \ldots, \delta_n \rangle \) yields \( N \) vertices for some common simplicial complex \( \Delta \) with vertex set \( \bigcup_{i=1}^N \{ \}$$. This implies that \( \Delta \) is a factorizable with respect to \( (v, a) \). However, the construction of the shedding sequence would then force \( \phi(e) \neq q \), which is a contradiction, therefore \( \Delta \) must be condensed. Finally, if \( \Delta \) has \( N \) facets, then Proposition 13.2.3 yields...
sets, so $Ψ_{(T, ϕ)}(u)$ is well-defined. Let $Δ_{(T, ϕ)}$ be the union of all simplicial complexes $Ψ_{(T, ϕ)}(u)$ over all possible terminal vertices $u ∈ L$. Let $(s_1, . . . , s_n) ∈ ℙ^n$ be the weight of $(T, ϕ)$ (i.e. the left-weight of the unique child of the root of $T$), and define the ordered partition $π_{(T, ϕ)} := ([s_1]_1, . . . , [s_n]_n)$.

**Proposition 13.3.12.** Given $a ∈ ℙ^n$, let $(T, ϕ)$ be an $a$-Macaulay tree of some positive integer $N$. Then $(Δ_{(T, ϕ)}, π_{(T, ϕ)})$ is a pure $a$-balanced complex with $N$ facets.

**Proof.** Follow the notation as above. Choose an arbitrary $u ∈ L$, and let $V(u)$ be the vertex set of $Ψ_{(T, ϕ)}(u)$. For each $t ∈ [n]$, define

$$b_t(u) := \{(i, j) ∈ ψ_{(T, ϕ)}(u) : j = t\}, \quad (13.22)$$

$$B_t(u) := \{(i, j) ∈ Ψ_{(T, ϕ)}(u) : j = t\}. \quad (13.23)$$

Observe that $(|b_1(u)|, . . . , |b_n(u)|) = a − ν(u)$ by the construction of an $a$-splitting tree, while $⟨ν(u)⟩$ is the $ν(u)$-rib of a simplex corresponding to the ordered partition $([ϕ^{[1]}(u)]_1, [ϕ^{[2]}(u)]_2, . . . , [ϕ^{[n]}(u)]_n)$. Since $B_t(u) ⊆ [s_i]_t$, for each $i ∈ [n]$ implies $π_{(T, ϕ)} ∩ V(u) = (B_1(u), . . . , B_n(u))$, it follows that $(Ψ_{(T, ϕ)}(u), π_{(T, ϕ)} ∩ V(u))$ is an $a$-balanced complex. Also, the $ν(u)$-rib $⟨ν(u)⟩$ and the simplex $\{ψ_{(T, ϕ)}(u)\}$ are both pure, so $Ψ_{(T, ϕ)}(u)$ is pure. Furthermore, $ψ_{(T, ϕ)}(u) = ψ_{(T, ϕ)}(u')$ if and only if $u = u'$, thus

$$Δ_{(T, ϕ)} = \bigcup_{u ∈ L} Ψ_{(T, ϕ)}(u), \quad (13.24)$$

which has $∑_{u ∈ L} (ϕ_{ν(u)}) = N$ facets, by the definition of $(T, ϕ)$. Finally, let $r_1$ denote the unique child of $r_0$, let $v_1, . . . , v_m$ be the left relative sequence of $r_1$, and let $u_i ∈ L$ be the right-most relative of $v_i$ for each $i ∈ [m]$. Since $ω(r_1) = (s_1, . . . , s_n)$, the construction of $ω$ yields

$$\bigcup_{i ∈ [n]} [s_i]_i ⊆ \bigcup_{i ∈ [m]} V(u_i) ⊆ \bigcup_{i ∈ [n]} [s_i]_i, \quad (13.25)$$

therefore $(Δ_{(T, ϕ)}, π_{(T, ϕ)})$ is a pure $a$-balanced complex with $N$ facets. \hfill \Box
Figure 13.3: A $(1, 1)$-Macaulay tree $(T, \varphi)$ where $(\Delta_{(T, \varphi)}, \pi_{(T, \varphi)})$ is not color-shifted.

![Figure 13.3](image)

Remark 13.3.13. The pure $a$-balanced complex $(\Delta_{(T, \varphi)}, \pi_{(T, \varphi)})$ is not necessarily color-shifted. For example, the $(1, 1)$-Macaulay tree $(T, \varphi)$ in Figure 13.3 yields

$$
\Delta_{(T, \varphi)} = \left\{ \{(1, 1), (1, 2)\}, \{(2, 1), (1, 2)\}, \{(3, 1), (1, 2)\}, \{(1, 1), (3, 2)\}, \{(2, 1), (3, 2)\} \right\},
$$

yet $\{(2, 1), (2, 2)\} \not\in \Delta_{(T, \varphi)}$, which implies $(\Delta_{(T, \varphi)}, \pi_{(T, \varphi)})$ is not color-shifted. In this example, $(T, \varphi)$ is condensed, so $(T, \varphi)$ being condensed does not necessarily imply $(\Delta_{(T, \varphi)}, \pi_{(T, \varphi)})$ is color-shifted.

Remark 13.3.14. If $(\Delta, \pi)$ is a pure color-shifted $a$-balanced complex such that $(\Delta, \pi) = (\Delta_{(T, \varphi)}, \pi_{(T, \varphi)})$ for some $a$-Macaulay tree $(T, \varphi)$, then knowing $(\Delta, \pi)$ does not uniquely determine $(T, \varphi)$. For example, in Figure 13.4, $(T, \varphi)$ and $(T', \varphi')$ are two distinct $(2, 2)$-Macaulay trees such that $(\Delta_{(T, \varphi)}, \pi_{(T, \varphi)}) = (\Delta_{(T', \varphi')}, \pi_{(T', \varphi')})$.

However, the condensations of $(T, \varphi)$ and $(T', \varphi')$ are identical, and by the definition of a shedding sequence, it is straightforward to show (e.g. by induction on the number of vertices of $\Delta$) that the condensation of $(T, \varphi)$ is the $a$-Macaulay tree induced by the shedding tree of $(\Delta_{(T, \varphi)}, \pi_{(T, \varphi)})$.  

Figure 13.4: Distinct $(2, 2)$-Macaulay trees $(T, \varphi)$ and $(T', \varphi')$ such that $(\Delta_{(T, \varphi)}, \pi_{(T, \varphi)}) = (\Delta_{(T', \varphi')}, \pi_{(T', \varphi')})$.

![Figure 13.4](image)
13.4 Compressed-like compatible Macaulay trees

Let \( a \in \mathbb{P}^n \). Given any pure color-shifted \( a \)-balanced complex \((\Delta, \pi)\), we can construct the \( a \)-Macaulay tree induced by the shedding tree of \((\Delta, \pi)\), and we showed this Macaulay tree must be condensed. Conversely, given any \( a \)-Macaulay tree \((T, \varphi)\) that is not necessarily condensed, we can construct the pure \( a \)-balanced complex \((\Delta_{(T,\varphi)}, \pi_{(T,\varphi)})\). (See Chapter 13.3 for details.) As shown in Remark 13.3.13, \((\Delta_{(T,\varphi)}, \pi_{(T,\varphi)})\) is not necessarily color-shifted, independent of whether \((T, \varphi)\) is condensed. In this section, we will introduce the notions of ‘compressed-like’ and ‘compatible’ for \( a \)-Macaulay trees, and we will establish the bijection

\[
\left\{ \text{isomorphism classes of} \right. \\
\text{pure color-compressed} \\
\text{\( a \)-balanced complexes} \\
\text{with \( N \) facets} \left\} \longleftrightarrow \left\{ \begin{array}{c} \\
\text{condensed} \\
\text{compressed-like compatible} \\
\text{\( a \)-Macaulay trees of \( N \)} \end{array} \right. \right. 
\]

with the maps \((\Delta, \pi) \mapsto a\)-Macaulay tree induced by the shedding tree of \((\Delta, \pi)\), and \((\Delta_{(T,\varphi)}, \pi_{(T,\varphi)}) \mapsto (T, \varphi)\).

**Definition 13.4.1.** Let \( a \in \mathbb{N}^n \setminus \{0_n\} \) and let \((T, \varphi)\) be an \( a \)-Macaulay tree. Denote the left-weight labeling of \((T, \varphi)\) by \( \omega \), and let \((r_0, L, Y)\) be the \((\text{root}, 1, 3)\)-triple of \( T \). We say \((T, \varphi)\) is compressed-like if the following conditions hold for every \( t \in [n] \) and every \( y \in Y \) satisfying \( \varphi(y) = t \):

(i) If \( y \) is a descendant of \( x_{\text{left}} \) for some \( x \in Y \) satisfying \( \varphi(x) = t \), and \( x_0, x_1, \ldots, x_\ell \) is the path (of length \( \ell \geq 2 \)) from \( x_0 = x \) to \( x_\ell = y_{\text{right}} \), then \( \omega^{[\ell]}(x_i) = \omega^{[\ell]}(x) - i \) for all \( i \in [\ell] \).

(ii) Let \( y_1, \ldots, y_m \) be the right relative sequence of \( y_{\text{left}} \) in \( T \). If \( k \in [m] \) is the smallest integer such that \( y_k \in L \) or \( \varphi(y_k) \neq t \), then \( \omega^{[t']}(y_k) \geq \omega^{[t']}(y_{\text{right}}) \) for all \( t' \in [n] \setminus \{t\} \).
Lemma 13.4.2. Given $k \in \mathbb{P}$, let $\Delta$ be a pure compressed $(k-1)$-dimensional simplicial complex with vertex set $V$. Then the Macaulay tree induced by the shedding tree of $(\Delta, V)$ is compressed-like.

Proof. Let $(T, \varphi)$ be the $k$-Macaulay tree induced by the shedding tree of $(\Delta, V)$, let $(r_0, L, Y)$ be the (root, 1, 3)-triple of $T$, and let $r_1$ be the unique child of $r_0$. Without loss of generality, assume $V = [n]$ and $n > k$. By definition, $dl_\Delta(n)$ and $lk_\Delta(n)$ are both pure and compressed. In particular, $dl_\Delta(n) = \langle \binom{n-1}{k} \rangle$, hence the left child of $r_1$ is a terminal vertex. By induction on $n$, the left child of every $y \in Y$ is a terminal vertex, thus $(T, \varphi)$ is trivially compressed-like. \hfill \Box

Proposition 13.4.3. For any positive integers $k$ and $N$, there exists a unique condensed compressed-like $k$-Macaulay tree $(T, \varphi)$ of $N$. Furthermore, the left child of every trivalent vertex of $T$ is a terminal vertex.

Proof. First of all, Lemma 13.4.2 and Theorem 13.3.9 imply the existence of a condensed compressed-like $k$-Macaulay tree $(T, \varphi)$ of $N$. Let $(r_0, L, Y)$ be the (root, 1, 3)-triple of $T$, and note that $\varphi(y) = 1$ for all $y \in Y$. Let $(T, \varphi|_Y, \nu)$ be the $k$-splitting tree induced by $(T, \varphi|_Y)$, let $\omega$ be the left-weight labeling of $(T, \varphi)$, and let $u_1, \ldots, u_m$ be all the terminal vertices (i.e. $|L| = m$), arranged in depth-first order. Clearly $|Y| = m - 1$, and we let $y_1, \ldots, y_{m-1}$ be all the trivalent vertices arranged in depth-first order.

Next, we show that the left child of every trivalent vertex in $T$ is a terminal vertex. Suppose not, and say $z := y_{\text{left}} \not\in L$ for some $y \in Y$. Let $z = z_1, \ldots, z_\ell$ be the right relative sequence of $z$ in $T$ for some $\ell \geq 2$, and let $z'$ be the left child of $z_{\ell-1}$. Note that $z_\ell$ is the right child of $z_{\ell-1}$, so $(T, \varphi)$ is compressed-like implies $\omega(z_\ell) = \omega(z_{\ell-1}) - 1$, yet the construction of $\omega$ yields $\omega(z') = \omega(z_{\ell-1}) - 1$, which
contradicts the assumption that \((T, \varphi)\) is condensed. Therefore, the left child of every \(y \in Y\) is in \(L\) as claimed.

Consequently, \(y_i\) is the parent of \(u_i\) for every \(i \in [m - 1]\), while \(y_{m-1}\) is the parent of \(u_m\). If we define \(y_m := u_m\), then the right relative sequence of \(y_1\) is precisely \(y_1, \ldots, y_m\). Thus, \(T\) is uniquely determined by \(m\), and \(\nu(u_i) = k + 1 - i\) for every \(i \in [m]\). Since \(\omega\) is proper implies \(\omega(u_i) \geq \omega(y_{i+1})\) for all \(i \in [m - 1]\), we get \(\varphi(u_i) = \omega(u_i) \geq \omega(y_{i+1}) - 1 = \omega(u_{i+1}) = \varphi(u_{i+1})\) for all \(i \in [m - 2]\). Also, \((T, \varphi)\) is condensed, so \(\varphi(u_{m-1}) = \omega(u_{m-1}) > \omega(u_m) = \varphi(u_m)\). This means

\[
N = \sum_{i=1}^{m} \left( \frac{\varphi(u_i)}{\nu(u_i)} \right) = \sum_{i=1}^{m} \left( \frac{\varphi(u_i)}{k + 1 - i} \right), \tag{13.26}
\]

where \(\varphi(u_1) > \cdots > \varphi(u_m) \geq \nu(u_m) \geq 1\), therefore the uniqueness of \((T, \varphi)\) follows from the uniqueness of the \(k\)-th Macaulay representation of \(N\). \(\Box\)

**Theorem 13.4.4.** If \((\Delta, \pi)\) is a pure color-compressed \(a\)-balanced complex for some \(a \in \mathbb{P}^n\), then the Macaulay tree induced by the shedding tree of \((\Delta, \pi)\) is compressed-like.

**Proof.** For convenience, identify each vertex \(v_{i,j}\) of \(\Delta\) with the pair \((j, i)\), so that each \(V_i\) in \(\pi = (V_1, \ldots, V_n)\) is identified with \([\lambda_i]_i\). Choose any shedding sequence \((T_0, \phi_0), (T_1, \phi_1), \ldots, (T_k, \phi_k)\) of \((\Delta, \pi)\), let \((r_0, L_i, Y_i)\) be the \((\text{root}, 1, 3)\)-triple of each \(T_i\), let \((T, \phi)\) be the Macaulay tree induced by \((T, \phi) := (T_k, \phi_k)\), and denote the left-weight labeling of \((T, \phi)\) by \(\omega\). For every \(y \in Y_k\), let \(\alpha(y)\) be the largest integer \(< k\) such that \(y \in L_{\alpha(y)}\), while for every \(u \in L_k\), set \(\alpha(u) = k\).

Clearly \((T, \varphi)\) is compressed-like when \(k = 0\), so assume \(k \geq 1\). Fix \(t \in [n]\), choose some \(x \in Y_k\) such that \(\varphi(x) = t\), and let \(x_1, \ldots, x_\ell\) be the right relative sequence of \(x_{\text{left}}\). If \(x_{\text{left}} \not\in L_k\), then choose any \(\ell' \in [\ell - 1]\) such that \(\varphi(x_{\ell'}) = t\).
while if $x_{\text{left}} \in L_k$, then set $\ell' = 0$. In either case, we claim that $\omega^{[\ell]}(x_i) = \omega^{[\ell]}(x) - i$ for all $i \in [\ell' + 1]$.

The claim is trivially true if $x_{\text{left}} \in L_k$, so assume $x_{\text{left}} \not\in L_k$. The construction of a shedding sequence yields $\varphi(x_j) = t$ for every $j \in [\ell']$. Let $\phi_{\alpha(x)}(x) = (a', \Delta', \pi', \lambda')$, where $a' = (a'_1, \ldots, a'_n)$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_n)$. Define $\Delta_i := \phi_{\alpha(x_i)}[\ell, \Delta](x_i)$ for each $i \in [\ell]$, and note that $\Delta_1 = \delta \lambda_1((\lambda'_1, t))$. Since $(\Delta', \pi')$ is color-compressed, every $a'_i$-set in $[\lambda'_i - 1]_t$ is contained in (some facet of) $\Delta_1$, thus by induction on $i \in [\ell' + 1]$, every $(a'_i + 1 - i)$-set in $[\lambda'_i - 1]_{t-i}$ is contained in (some facet of) $\Delta_i$. This implies $\omega^{[\ell]}(x_i) = t$-th entry of $\phi_{\alpha(x_i)}[\ell, \lambda](x_i) = \lambda'_i - i = \omega^{[\ell]}(x) - i$ for all $i \in [\ell' + 1]$, so our claim is true in both cases. Condition (i) in Definition 13.4.1 holds by repeatedly using this claim and the definition of $\omega$.

Condition (ii) in Definition 13.4.1 is vacuously true if $n = 1$, so assume $n > 1$. Fix some $t' \in [n] \setminus \{t\}$, choose any $y \in Y$ such that $\varphi(y) = t$, let $y_1, \ldots, y_m$ be the right relative sequence of $y_{\text{left}}$ in $T$, and let $q \in [m]$ be the smallest integer such that $y_q \in L$ or $\varphi(y_q) \neq t$. For this condition to hold, we need to show that $\omega^{[\ell]}(y_q) \geq \omega^{[\ell]}(y_{\text{right}})$. If $q = 1$, then $\omega$ is proper implies $\omega^{[\ell]}(y_q) = \omega^{[\ell]}(y_{\text{left}}) \geq \omega^{[\ell]}(y_{\text{right}})$, and we are done, so assume $q \geq 2$. For the rest of this proof, let $\phi_{\alpha(y_{\text{right}})}(a''', \Delta'', \pi''', \lambda''')$ and let $\phi_{\alpha(y)}(a, \Delta, \pi, \lambda)$, where $a'' = (a''_1, \ldots, a''_n)$, $a = (a_1, \ldots, a_n)$, $\pi'' = (V''_1, \ldots, V''_n)$, $\pi = (V_1, \ldots, V_n)$, $\lambda'' = (\lambda''_1, \ldots, \lambda''_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$.

If $V''_\nu = \emptyset$, then $a''_\nu = 0$. Since $\varphi(y) \neq t'$ implies $a''_\nu = 0$, we have $\tilde{V}_\nu = \emptyset$, so for all descendants $u$ of $y$ in $T$, the $t'$-th entry in $\phi_{\alpha(u)}^{[\ell, \pi]}(u)$ must be the empty set. By the construction of a shedding sequence, we get $\omega^{[\ell]}(u) = t'$-th entry of $\phi_{\alpha(u)}^{[\ell, \lambda]}(u) = \tilde{\lambda}_\nu$, for all descendants $u$ of $y$. In particular, $\omega^{[\ell]}(y_q) = \omega^{[\ell]}(y_{\text{right}})$, and we are done.
If instead $V''_i \neq \emptyset$, then $(\lambda''_i, t')$ is the maximal element of $V''_i$, and we choose a facet $F$ of $\Delta''$ that contains this vertex $(\lambda''_i, t')$. Note that $\Delta'' = \text{lk}_{\Delta}((\omega'[t](y), t))$ is $(\tilde{a} - \delta_{t,n})$-balanced, which implies $F \cup \{(\omega'[t](y), t)\}$ is a facet of $\tilde{\Delta}$, thus the definition of $q$ and the construction of a shedding sequence together yield $|F \cap V_i| \geq q - 2$. Let $F'$ be the set of the $q - 2$ largest elements in $F \cap V_i$, and let $u_i = (\omega'[t_i](y_i), t)$ for each $i \in [q - 1]$. Since $\tilde{\Delta}$ is color-compressed, $(F \setminus F') \cup \{u_1, \ldots, u_{q-1}\}$ must be a facet of $\tilde{\Delta}$, thus $F \setminus F'$ is a facet of $\phi_{\alpha(y_q)}^{[\mathcal{L}, \Delta]}(y_q)$. In particular, $(\lambda''_t, t')$ is contained in $F \setminus F'$, so $\phi_{\alpha(y_q)}^{[\mathcal{L}, \Delta]}(y_q)$ is color-compressed implies the $t'$-entry of $\phi_{\alpha(y_q)}^{[\mathcal{L}, \Delta]}(y_q)$ is a set containing $[\lambda''_t]_{t'}$. Consequently, $\omega'[t'](y_q) = t'$-th entry of $\phi_{\alpha(y_q)}^{[\mathcal{L}, \Delta]}(y_q) \geq \lambda''_t = \omega'[t'](y_{right}). \Box$

For the rest of this section, let $a \in \mathbb{N}^n \setminus \{0_n\}$, and let $(T, \varphi)$ be an $a$-Macaulay tree. Let $(r_0, L, Y)$ be the (root, 1, 3)-triple of $T$, denote the left-weight labeling of $(T, \varphi)$ by $\omega$, and let $(T, \varphi|_Y, \nu)$ be the $a$-splitting tree induced by $(T, \varphi|_Y)$. For any $x \in Y \cup L$, recall that $\mathcal{D}_T(x)$ is the set of all descendants of $x$ in $T$.

**Definition 13.4.5.** Suppose $t \in [n]$ and $x \in Y \cup L$. Let $x_1, \ldots, x_\ell$ be the right relative sequence of $x$ in $T$, and let $k$ be the smallest integer in $[\ell]$ such that $x_k \in L$ or $\varphi(x_k) \neq t$. We then define the following sets:

\[
\psi_{(T, \varphi)}(x) := \left\{ \left( \omega^{[\varphi(v)]}(v), \varphi(v) \right) : v \in Y, \text{ both } v \text{ and its right child are ancestors of } x \right\};
\]

\[
\psi_{(T, \varphi)}^{(t)}(x) := \left\{ s \in \mathbb{P} : (s, t) \in \psi_{(T, \varphi)}(x) \right\};
\]

\[
\hat{\psi}_{(T, \varphi)}^{(t)}(x) := \left\{ \omega'[t](x_i) : i \in [k] \right\};
\]

\[
\xi_{(T, \varphi)}^{(t)}(x) := \text{largest } (a_\ell - |\psi_{(T, \varphi)}^{(t)}(x)| - k)-\text{subset of } [\omega'[t](x_k) - 1] \text{ w.r.t. colex order};
\]

\[
\xi_{(T, \varphi)}(x) := \psi_{(T, \varphi)}^{(t)}(x) \cup \hat{\psi}_{(T, \varphi)}^{(t)}(x) \cup \xi_{(T, \varphi)}^{(t)}(x).
\]

Observe that the sets $\psi_{(T, \varphi)}^{(t)}(x), \hat{\psi}_{(T, \varphi)}^{(t)}(x), \xi_{(T, \varphi)}^{(t)}(x)$ are pairwise disjoint, so $\xi_{(T, \varphi)}^{(t)}(x)$ is an $a_t$-set, and we call this $a_t$-set $\xi_{(T, \varphi)}^{(t)}(x)$ the $t$-signature of $x$ in $(T, \varphi)$. In particular, if $x$ is a $t$-leading vertex, then $\psi_{(T, \varphi)}^{(t)}(x) = \emptyset$. 

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**Definition 13.4.6.** Let \( t \in \{0, 1, \ldots, n - 1\} \), and let \( u_1, \ldots, u_m \) (for some \( m \in \mathbb{P} \)) be all the \( t \)-leading vertices in \( L_{(T, \varphi)}(t) \) arranged in depth-first order. For every \( u \in L_{(T, \varphi)}(t) \) such that the \( (t + 1) \)-th entry of \( \mathbf{a} - \nu(u) \) is positive, let \( j \) be the unique integer in [\( m \)] satisfying \( u = u_j \), and define \( \zeta_t^{j+1}(u) := u_{j-1} \). In particular, since \( \nu(u_1) = \mathbf{a} \), we necessarily have \( j > 1 \), hence \( \zeta_t^{j+1}(u) \) is well-defined.

Let \( v \) be the unique common ancestor of \( u_{j-1} \) and \( u_j \) in \( Y \cup L \) such that neither child of \( v \) is a common ancestor of \( u_{j-1} \) and \( u_j \). The path from \( r_0 \) to the right-most relative of \( v_{\text{left}} \) and the path from \( r_0 \) to the left-most relative of \( v_{\text{right}} \) share the common vertex \( v \), and these two paths must each contain some \( t \)-leading vertex. The definition of \( t \)-leading vertices implies \( \varphi(v) > t \), so by the definition of the depth-first order, \( u_{j-1} \) is contained in the right relative sequence of \( v_{\text{left}} \) in \( T \), and \( u_j \) is contained in the left relative sequence of \( v_{\text{right}} \) in \( T \). Furthermore, since the \( (t+1) \)-th entry of \( \mathbf{a} - \nu(u_j) \) is positive, the definition of a Macaulay tree forces \( \varphi(v) = t+1 \). Consequently, if \( (T, \varphi) \) is compressed-like, then Condition (ii) in Definition 13.4.1 yields \( \xi_{(T, \varphi)}(t)(u_j) \leq_{ct} \xi_{(T, \varphi)}(t)(u_{j-1}) \), or equivalently, \( \zeta_t^{(t)}(u) \leq_{ct} \zeta_t^{(t)}(\zeta_t^{j+1}(u)) \).

**Definition 13.4.7.** Let \( i, j \in \mathbb{N} \) satisfy \( j + 1 < i \leq n \). For every \( u \in L_{(T, \varphi)}(j + 1) \) such that the \( i \)-th entry of \( \mathbf{a} - \nu(u) \) is positive, assume \( \zeta_j^{i+1}(u) \) has already been defined, and suppose \( \xi_{(T, \varphi)}^{(j+1)}(u) \leq_{ct} \xi_{(T, \varphi)}^{(j+1)}(\zeta_j^{i+1}(u)) \). Then for any \( x \in L_{(T, \varphi)}(j) \), let \( \tilde{x} \) be the unique ancestor of \( x \) that is a \( (j + 1) \)-leading vertex, set \( \tilde{y} := \zeta_j^{(j+1)}(\tilde{x}) \), and define \( \zeta_j^{i}(x) \) as the smallest vertex in

\[
U := \{ z \in L_{(T, \varphi)}(j) \cap \mathcal{D}_T(\tilde{y}) : \xi_{(T, \varphi)}^{(j+1)}(z) \geq_{ct} \xi_{(T, \varphi)}^{(j+1)}(x) \}
\]

with respect to the depth-first order. Note that if \( y \) is the unique \( j \)-leading vertex in the right relative sequence of \( \tilde{y} \), then the definition of a \( (j + 1) \)-signature yields

\[
\xi_{(T, \varphi)}^{(j+1)}(x) \leq_{ct} \xi_{(T, \varphi)}^{(j+1)}(\tilde{x}) \leq_{ct} \xi_{(T, \varphi)}^{(j+1)}(\tilde{y}) = \xi_{(T, \varphi)}^{(j+1)}(y), \quad (13.27)
\]

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thus $y \in U$ (i.e. $U$ is non-empty), so $\zeta^j_i(x)$ is well-defined.

**Remark 13.4.8.** Let $i, j \in \mathbb{N}$ satisfy $j < i \leq n$, and let $x \in L_{(T,\varphi)}(j)$ such that the $i$-th entry of $a - \nu(x)$ is positive. If $\zeta^j_i(x)$ is well-defined, then $\zeta^j_i(x)$ can be determined from $x$ via the following algorithm.

$(x_0, x_1, \ldots, x_\ell) \leftarrow$ path from $r_0$ to $x$ (which has length $\ell$).
$q \leftarrow$ largest integer in $[\ell - 1]$ such that $\varphi(x_q) = i$ and $x_{q+1}$ is the right child of $x_q$.
$z \leftarrow$ left child of $x_q$.
$(y_1, \ldots, y_{\ell'}) \leftarrow$ right relative sequence of $z$ (which has $\ell'$ terms).
$k \leftarrow$ smallest integer in $[\ell']$ such that $y_k \in L$ or $\varphi(y_k) \neq i$.

**Remark 13.4.8.** Let $i, j \in \mathbb{N}$ satisfy $j < i \leq n$, and let $x \in L_{(T,\varphi)}(j)$ such that

- $(x_0, x_1, \ldots, x_\ell) \leftarrow$ path from $r_0$ to $x$ (which has length $\ell$).
- $q \leftarrow$ largest integer in $[\ell - 1]$ such that $\varphi(x_q) = i$ and $x_{q+1}$ is the right child of $x_q$.
- $z \leftarrow$ left child of $x_q$.
- $(y_1, \ldots, y_{\ell'}) \leftarrow$ right relative sequence of $z$ (which has $\ell'$ terms).
- $k \leftarrow$ smallest integer in $[\ell']$ such that $y_k \in L$ or $\varphi(y_k) \neq i$.

**Remark 13.4.8.** Let $i, j \in \mathbb{N}$ satisfy $j < i \leq n$, and let $x \in L_{(T,\varphi)}(j)$ such that

$(x_0, x_1, \ldots, x_\ell) \leftarrow$ path from $r_0$ to $x$ (which has length $\ell$).
$q \leftarrow$ largest integer in $[\ell - 1]$ such that $\varphi(x_q) = i$ and $x_{q+1}$ is the right child of $x_q$.
$z \leftarrow$ left child of $x_q$.
$(y_1, \ldots, y_{\ell'}) \leftarrow$ right relative sequence of $z$ (which has $\ell'$ terms).

**Remark 13.4.8.** Let $i, j \in \mathbb{N}$ satisfy $j < i \leq n$, and let $x \in L_{(T,\varphi)}(j)$ such that $y_k \in L$ or $\varphi(y_k) \neq i$.

Notice that $\zeta^j_i(x)$ is well-defined only if the set $U$ in each iteration of the while loop is non-empty.

**Proposition 13.4.9.** Let $(T,\varphi)$ be compressed-like, let $s, t \in \mathbb{N}$ satisfy $t < s \leq n$, and let $x \in L_{(T,\varphi)}(t)$ such that the $s$-th entry of $a - \nu(x)$ is positive. If $\zeta^s_i(x)$ is well-defined, then we have the following:

- $(i)$ $\psi^{(s)}_{T,\varphi}(x) = \{q_1, \ldots, q_m\}$ for some $m \in [a_s]_s$ such that $a_s - m + 1 < q_1 < \cdots < q_m$.
- $(ii)$ $\xi^{(s)}_{(T,\varphi)}(u) = \{q_i - i : i \in [a_s - m + 1] \} \cup \{q_2, \ldots, q_m\}$ for every $u \in D_T(\zeta^s_i(x))$.
- $(iii)$ If $i$ is an integer satisfying $s < i \leq n$, then $\xi^{(i)}_{(T,\varphi)}(u) = \zeta^i_{(T,\varphi)}(x)$ for every $u \in D_T(\zeta^s_i(x))$. 

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Proof. First of all, since the $s$-th entry of $a - \nu(x)$ is positive, $\psi_{(T,\varphi)}^{(s)}(x)$ must be non-empty, so we can write $\psi_{(T,\varphi)}^{(s)}(x)$ as the set $\{q_1, \ldots, q_m\}$ for some $m \in [a_s]$, such that $q_1 < \cdots < q_m$. For each $i \in [m]$, let $y_i \in Y$ be the unique ancestor of $x$ such that $(\omega^{\varphi(y_i)}(y_i), \varphi(y_i)) = (q_i, s)$. Also, let $z$ be the left child of $y_1$. The subset consisting of the $m$ largest elements in $\zeta_{(T,\varphi)}^{(s)}(x) \in \mathbb{P}^{a_s}$ is precisely $\psi_{(T,\varphi)}^{(s)}(x)$, hence $q_1 \geq a_s - m + 1$. If $q_1 = a_s - m + 1$, then the definition of the left-weight labeling $\omega$ yields $\omega^{[s]}(z) = a_s - m$, while the construction of the $a$-splitting tree $(T, \varphi|_Y, \nu)$ yields $\nu^{[s]}(z) = a_s - m + 1$. This means the left-most relative of $z$, which we denote by $z_{\text{left-most}}$, satisfies $\varphi(z_{\text{left-most}}) \geq \nu(z_{\text{left-most}})$, hence contradicting the definition of a Macaulay tree. Consequently, $q_1 > a_s - m + 1$, i.e. statement (i) is true.

Let $z_1, \ldots, z_{\ell}$ be the right relative sequence of $z$, and let $k$ be the smallest integer in $[\ell]$ such that $z_k \in L$ or $\varphi(z_k) \neq s$. Since $(T, \varphi)$ is compressed-like, Condition (i) in Definition 13.4.1 yields

$$\psi_{(T,\varphi)}^{(s)}(\zeta^{(s)}_t(x)) = \{q_2, \ldots, q_m\} \cup \{\omega^{[s]}(z_i) : i \in [k-1]\} = \{q_2, \ldots, q_m\} \cup \{q_1 - i : i \in [k-1]\}.$$  \hfill (13.28)

Next, let $z'$ be the left-most relative of $z_k$. The vertex $z'$ and all vertices in $\mathcal{D}_T(\zeta^{(s)}_t(x)) \cap L$ are contained $\mathcal{D}_T(z_k) \cap L$, so it follows from the definition of a Macaulay tree that $\varphi^{[s]}(u) = \varphi^{[s]}(z') = \omega^{[s]}(z_k) = q_1 - k$ for every $u \in \mathcal{D}_T(\zeta^{(s)}_t(x)) \cap L$. Since $t < s$ and $\zeta^{(s)}_t(x)$ is by construction a $t$-leading vertex of $(T, \varphi)$, we get

$$\xi^{(s)}_{(T,\varphi)}(u) = \{q_1 - i : i \in [a_s - m + 1]\} \cup \{q_2, \ldots, q_m\}$$  \hfill (13.29)

for every $u \in \mathcal{D}_T(\zeta^{(s)}_t(x))$, thus proving statement (ii).

Finally, if $s < i \leq n$ ($i \in \mathbb{P}$), then since both $\zeta^{(s)}_t(x)$ and $x$ are in $\mathcal{D}_T(y_1)$, it follows from the definition of a Macaulay tree and $\varphi(y_1) = s < i$ that $\psi_{(T,\varphi)}^{(i)}(u) = \psi_{(T,\varphi)}^{(i)}(x)$ and $\omega^{[i]}(u) = \omega^{[i]}(x)$ for all $u \in \mathcal{D}_T(\zeta^{(s)}_t(x))$, which proves (iii). \hfill $\Box$
Definition 13.4.10. Define \((T, \varphi)\) to be always \(n\)-compatible. Suppose \(n > 1\), \(t \in [n - 1]\), and \((T, \varphi)\) is compressed-like. Then we say \((T, \varphi)\) is \(t\)-compatible if

(i) \((T, \varphi)\) is \(s\)-compatible for all \(s \in [n]\); and

(ii) \(\zeta_{(T, \varphi)}(i) \leq_{ct} \zeta_{(T, \varphi)}(\zeta_i(x))\) for every \(x \in L_{(T, \varphi)}(t)\), \(i \in [n]\) such that the \(i\)-th entry of \(a - \nu(x)\) is positive.

In particular, condition (i) ensures \(\zeta_{(T, \varphi)}(x)\) is well-defined. If \((T, \varphi)\) is \(1\)-compatible, which is identical to \((T, \varphi)\) being \(j\)-compatible for all \(j \in [n]\), then we say \((T, \varphi)\) is compatible. Equivalently, \((T, \varphi)\) is compatible if \(\zeta_{(T, \varphi)}(i) \leq_{ct} \zeta_{(T, \varphi)}(\zeta_i(x))\) for every \(i, t \in [n]\) satisfying \(t < i\), and every \(x \in L_{(T, \varphi)}(t)\) such that the \(i\)-th entry of \(a - \nu(x)\) is positive.

Lemma 13.4.11. Let \(0 \leq t < s \leq n\) be integers, and let \(x \in L_{(T, \varphi)}(t)\) such that \(a - \nu(x)\) has a positive \(s\)-th entry. Suppose \((T, \varphi)\) is compressed-like and \((t + 1)\)-compatible. Then \(\zeta_{(T, \varphi)}(x)\) is well-defined, and for each integer \(k\) such that \(t < k < s\),

(i) \(\psi_{(T, \varphi)}(u) \cup \zeta_{(T, \varphi)}(x)\) for all \(u \in D_T(\zeta_{(T, \varphi)}(x))\); and

(ii) \(\zeta_{(T, \varphi)}(u) \geq_{ct} \zeta_{(T, \varphi)}(x)\) for all \(u \in D_T(\zeta_{(T, \varphi)}(x))\).

Proof. First of all, \(\zeta_{(T, \varphi)}(x)\) is well-defined by the definition of \((t + 1)\)-compatible. Let \(x_0, x_1, \ldots, x_\ell\) be the path from \(r_0\) to \(x\) in \(T\), and let \(q\) be the largest integer in \([\ell - 1]\) such that \(\varphi(x_q) = s\) and \(x_{q+1}\) is the right child of \(x_q\). For each integer \(i\) satisfying \(t \leq i \leq n\), let \(q_i\) be the unique integer in \([\ell]\) such that \(x_{q_i}\) is an \(i\)-leading vertex. Also, denote the left child of \(x_q\) by \(z\), let \(z_1, \ldots, z_m\) be the path in \(T\) from \(z\) to \(\zeta_{(T, \varphi)}(x)\), and for each integer \(i\) satisfying \(t \leq i < s\), let \(j_i\) be the unique integer in \([m]\) such that \(z_{j_i}\) is an \(i\)-leading vertex. For example, \(q_{\ell} = \ell\) and \(j_{m} = m\), i.e. \(x_{q_{\ell}} = x\) and \(z_{j_{m}} = \zeta_{(T, \varphi)}(x)\).

Suppose \(k \in \mathbb{P}\) satisfies \(t < k < s\). Then \((T, \varphi)\) is \((k + 1)\)-compatible, hence \(\zeta_{(T, \varphi)}(x_{q_k})\) is well-defined, and Remark 13.4.8 yields \(\zeta_{(T, \varphi)}(x_{q_k}) = z_{j_k}\). Furthermore, since
$(T, \varphi)$ is $k$-compatible, we have

$$\zeta^{(k)}_{(T, \varphi)}(z_{j_k}) = \zeta^{(k)}_{(T, \varphi)}(\zeta^{(k)}_k(x_{q_k})) \geq c_T \zeta^{(k)}_{(T, \varphi)}(x_{q_k}) \geq c_T \zeta^{(k)}_{(T, \varphi)}(x). \quad (13.30)$$

Now, define vertex $z'$ algorithmically as follows: Starting with $z' = z_{j_k}$, as long as $z' \not\in L_{(T, \varphi)}(k-1)$, replace $z'$ with $z'_{\text{right}}$ if $\omega^{[k]}(z') \in \zeta^{(k)}_{(T, \varphi)}(x)$, and replace $z'$ with $z'_{\text{left}}$ otherwise. Repeat process until $z' \in L_{(T, \varphi)}(k-1)$; the resulting $z'$ is the vertex we want. Using Remark 13.4.8 and condition (i) of Definition 13.4.1, we then get $z' = z_{j_{k-1}}$, thus $\psi^{(k)}_{(T, \varphi)}(z_{j_{k-1}}) = \psi^{(k)}_{(T, \varphi)}(z') \subseteq \xi^{(k)}_{(T, \varphi)}(x)$. Since $z_{j_{k-1}}$ is a $(k-1)$-leading vertex, we have $\psi^{(k)}_{(T, \varphi)}(u) \subseteq \xi^{(k)}_{(T, \varphi)}(x)$ for all $u \in D_T(z_{j_{k-1}})$, so statement (i) follows from the fact that $\zeta^{(k)}_\ell(x) \in D_T(z_{j_{k-1}})$.

Next, we prove statement (ii). If $|\psi^{(k)}_{(T, \varphi)}(\zeta^{(k)}_\ell(x))| = a_k$, then it follows from statement (i) that $\xi^{(k)}_{(T, \varphi)}(u) = \psi^{(k)}_{(T, \varphi)}(u) = \xi^{(k)}_{(T, \varphi)}(x)$ for all $u \in D_T(\zeta^{(k)}_\ell(x))$ and we are done. Assume $|\psi^{(k)}_{(T, \varphi)}(\zeta^{(k)}_\ell(x))| < a_k$, and define

$$p := \max \left( \xi^{(k)}_{(T, \varphi)}(x) \setminus \psi^{(k)}_{(T, \varphi)}(\zeta^{(k)}_\ell(x)) \right).$$

The construction of $\zeta^{(k)}_\ell(x)$ yields $\xi^{(k)}_{(T, \varphi)}(z_{j_{k-1}}) \geq c_T \xi^{(k)}_{(T, \varphi)}(x)$, so in particular, $\omega^{[k]}(z_{j_{k-1}}) \geq p$. Since $z_{j_{k-1}}$ is a $(k-1)$-leading vertex, the definition of a Macaulay tree tells us that the left-most relative of $z_{j_{k-1}}$, which we denote by $z''$, satisfies $\omega^{[k]}(z'') = \omega^{[k]}(z_{j_{k-1}}) \geq p$. Thus, $\omega^{[k]}(u) \geq p$ for all $u \in D_T(z_{j_{k-1}})$, therefore the definition of a $k$-signature, together with statement (i), yields $\zeta^{(k)}_{(T, \varphi)}(u) \geq c_T \xi^{(k)}_{(T, \varphi)}(x)$ for all $u \in D_T(\zeta^{(k)}_\ell(x)) \subseteq D_T(z_{j_{k-1}})$.

**Theorem 13.4.12.** Let $a \in \mathbb{P}^n$. If $(\Delta, \pi)$ is a pure color-compressed $a$-balanced complex, then the $a$-Macaulay tree induced by the shedding tree of $(\Delta, \pi)$ is compatible.

**Proof.** For convenience, identify each vertex $v_{i,j}$ of $\Delta$ with the pair $(j, i)$, so that
each \( V_i \) in \( \pi = (V_1, \ldots, V_n) \) is identified with \([\lambda_i]_j\). Let \((T_0, \phi_0), (T_1, \phi_1), \ldots, (T_k, \phi_k)\) be a shedding sequence of \((\Delta, \pi)\). For each \( i \in [k] \), let \((r_0, L_i, Y_i)\) denote the \((\text{root}, 1, 3)\)-triple of \( T_i \); let \((T_i, \varphi_i)\) be the Macaulay tree induced by \((T_i, \phi_i)\), and let \( \omega_i \) be the left-weight labeling of \((T_i, \varphi_i)\). Let \((T_k, \varphi_k|_{Y_k}, \psi)\) be the \( a \)-splitting tree induced by \((T_k, \varphi_k|_{Y_k})\). Also, for every \( y \in Y_k \), let \( \alpha(y) \) be the largest integer \(< k\) such that \( y \in L_{\alpha(y)} \), while for every \( u \in L_k \), set \( \alpha(u) = k \). Note that for \((T_k, \varphi_k)\) to be compatible, there is an implicit assumption of \((T_k, \varphi_k)\) being compressed-like, which we can assume by Theorem 13.4.4.

Suppose \((T_k, \varphi_k)\) is not compatible, and let \( t \in [n - 1] \) be maximal such that \((T_k, \varphi_k)\) is not \( t \)-compatible. Choose \( x \in L_{(T_k, \varphi_k)}(t), s \in [n] \setminus [t] \) such that \( a - \nu(x) \) has a positive \( s \)-th entry, and \( \xi^{(t)}_{(T_k, \varphi_k)}(x) \not\in \xi^{(t)}_{(T_k, \varphi_k)}(\xi^s(x)) \). In particular, \( \zeta^s_t(x) \) is well-defined since \((T_k, \varphi_k)\) is \((t + 1)\)-compatible. Next, define

\[
F' := \bigcup_{j \in [n] \setminus [t]} \{(i, j) : i \in \xi^{(j)}_{(T_k, \varphi_k)}(x)\}.
\]

By Proposition 13.3.8(iv), \( \omega_k^{[j]}(u) = \omega_k^{[j]}(x) \) for all \( j \in [n] \setminus [t] \) and \( u \in D_{T_k}(x) \), hence \( F \cup F' \in \Delta \) for all \( u \in D_{T_k}(x) \) and \( F \in \phi^{[L, \Delta]}_{\alpha(u)}(u) \) satisfying \( F \subseteq \bigcup_{i \in [t]} V_i \). Note that \( F_t := \{(i, t) : i \in \xi^{(t)}_{(T_k, \varphi_k)}(x)\} \subseteq V_t \) is a face of \( \phi^{[L, \Delta]}_{\alpha(x)}(x) \), so \( F' \cup F_t \in \Delta \).

Recall the definition of the operation \( C_j \) (for any \( j \in [n] \)) on colored multicomplexes defined in Chapter 11.1, and define \( C_j \) on colored complexes analogously. Proposition 13.4.9(ii) yields \( \xi^{(s)}_{(T_k, \varphi_k)}(\xi^s_t(x)) \not\in \xi^{(s)}_{(T_k, \varphi_k)}(\xi^s_t(x)) \), so since \((\Delta, \pi)\) is color-compressed implies \( C_s(\Delta) = \Delta \), it then follows from \( F' \cup F_t \in \Delta \) that \( \widetilde{F} \cup F_t \in \Delta \), where

\[
\widetilde{F} := \left( F' \setminus \{(i, s) : i \in \xi^{(s)}_{(T_k, \varphi_k)}(x)\} \right) \cup \{(i, s) : i \in \xi^{(s)}_{(T_k, \varphi_k)}(\xi^s_t(x))\}.
\]

Now, Lemma 13.4.11(i) says \( \psi^{(i)}_{(T_k, \varphi_k)}(\xi^s_t(x)) \subseteq \xi^{(i)}_{(T_k, \varphi_k)}(\xi^s_t(x)) \) for all integers \( t < i < s \) (if any), while Proposition 13.4.9(iii) says \( \psi^{(i)}_{(T_k, \varphi_k)}(\xi^s_t(x)) = \psi^{(i)}_{(T_k, \varphi_k)}(x) \) for
all integers \( s < i \leq n \) (if any). Consequently, \( \psi_{(T_k,\varphi_k)}(\xi^s_t(x)) \subseteq \tilde{F} \), hence it follows from \( \tilde{F} \cup F_i \in \Delta \) that
\[
(\tilde{F} \setminus \psi_{(T_k,\varphi_k)}(\xi^s_t(x))) \cup F_i \in \phi_{\alpha(\xi^s_t(x))}^\varepsilon(\xi^s_t(x)).
\] (13.31)

Looking at vertices in \( V \), this means \( \xi^{(t)}_{(T_k,\varphi_k)}(\xi^s_t(x)) \gtrless \xi^{(t)}_{(T_k,\varphi_k)}(x) \), which contradicts our original assumption that \( \xi^{(t)}_{(T_k,\varphi_k)}(\xi^s_t(x)) \gtrless \xi^{(t)}_{(T_k,\varphi_k)}(x) \).

Recall that the weight of \((T,\varphi)\) is the left-weight of the unique child of \( r_0 \).

**Lemma 13.4.13.** Let \((T,\varphi)\) have weight \((s_1,\ldots,s_n) \in \mathbb{P}^n\). For each \( t \in [n] \), let \( G_t \in \binom{[n]}{n_t} \) and define \( F_t := \{(i,t) : i \in G_t\} \). If there exists some \( u \in L \) such that \( \psi_{(T,\varphi)}^{(t)}(u) \subseteq G_t \) for all \( t \in [n] \), then \( F_1 \cup \cdots \cup F_n \) is contained in \( \Psi_{(T,\varphi)}(u) \) if and only if \( \xi^{(t)}_{(T,\varphi)}(u) \gtrless \xi^{(t)}_{(T,\varphi)} \) for all \( t \in [n] \).

**Proof.** Suppose there exists some \( u \in L \) as described. Definition 13.3.11 yields \( \Psi_{(T,\varphi)}(u) = \langle \varphi^{(u)} \rangle \ast \langle \{\psi_{(T,\varphi)}(u)\} \rangle \), so \( F_1 \cup \cdots \cup F_n \in \Psi_{(T,\varphi)}(u) \) if and only if
\[
F_t \setminus \{(i,t) : i \in \psi_{(T,\varphi)}^{(t)}(u)\} \in \langle \left( \left\lfloor \frac{\varphi^{(t)}(u)}{\nu^{(t)}(u)} \right\rfloor_t \right) \rangle
\] (13.32)
for all \( t \in [n] \). Note that \( |\psi_{(T,\varphi)}^{(t)}(u)| = a_t - \varphi^{(t)}(u) \), thus it suffices to show that max \( \{G_t \setminus \psi_{(T,\varphi)}^{(t)}(u)\} \leq \varphi^{(t)}(u) = \omega^{(t)}(u) \) for all \( t \in [n] \) satisfying \( |\psi_{(T,\varphi)}^{(t)}(u)| < a_t \) (since the case \( |\psi_{(T,\varphi)}^{(t)}(u)| = a_t \) is trivial), which is equivalent to \( G_t \subseteq \xi^{(t)}_{(T,\varphi)}(u) \). □

**Theorem 13.4.14.** Let \((T,\varphi)\) be a compressed-like, compatible \( a \)-Macaulay tree. Then \((\Delta_{(T,\varphi)}, \pi_{(T,\varphi)})\) is a pure color-compressed \( a \)-balanced complex.

**Proof.** Denote the \((\text{root},1,3)\)-triple of \( T \) by \((r_0,L,Y)\), let \((T,\varphi)_{\downarrow Y},\nu\) be the \( a \)-splitting tree induced by \((T,\varphi)_{\downarrow Y}\), and let \( \omega \) be the left-weight labeling of \((T,\varphi)\). We know from Proposition 13.3.12 that \((\Delta_{(T,\varphi)}, \pi_{(T,\varphi)})\) is a pure \( a \)-balanced complex, so we are left to show that \((\Delta_{(T,\varphi)}, \pi_{(T,\varphi)})\) is color-compressed.
Let $F$ be an arbitrary facet of $\Delta_{(T,\varphi)}$. By construction, there exists a unique terminal vertex in $L$, which we denote by $u_F$, such that $F \in \Psi_{(T,\varphi)}(u_F)$. Recall that $[s]_i := \{(1,i),(2,i),\ldots,(s,i)\}$ for any $s \in \mathbb{P}$, $i \in [n]$, and note that $\pi_{(T,\varphi)} = ([s_1],\ldots,[s_n])$, where $s_1,\ldots,s_n \in \mathbb{P}$ is the weight of $(T,\varphi)$. For each $i \in [n]$, define $F_i := F \cap [s]_i \in (\{s\}^i)$, and define $\hat{F}_i := \{s \in \mathbb{P} : (s,i) \in F_i\} \subseteq [s_i]$. If $\hat{F}_i \neq [a_i]$, then let $\hat{F}_i < \hat{F}$ denote the immediate predecessor of $\hat{F}_i$ in $(\mathbb{P}^n)$ with respect to the colex order, while if $\hat{F}_i = [a_i]$, then let $\hat{F}_i := [a_i]$. In either case, define $F'_i := \{(s,i) : s \in \hat{F}_i\}$ and $F'(i) := (F \setminus F_i) \cup F'_i$. Let

$$S_{(T,\varphi)} := \{F \in \Delta_{(T,\varphi)} : F \text{ is a facet of } \Delta_{(T,\varphi)}, \text{ and } \exists i \in [n] \text{ such that } F'(i) \not\in \Delta_{(T,\varphi)}\}.$$  

By the definition of color compression, $(\Delta_{(T,\varphi)}, \pi_{(T,\varphi)})$ is color-compressed if and only if $S_{(T,\varphi)} = \emptyset$.

Suppose $S_{(T,\varphi)} \neq \emptyset$. Choose any $F \in S_{(T,\varphi)}$ such that $u_F$ is minimal with respect to the depth-first order, and let $t \in [n]$ satisfy $F'(t) \not\in \Delta_{(T,\varphi)}$. If $\psi_{(T,\varphi)}^{(t)}(u_F) = \emptyset$, then $\hat{F}_t \subseteq [\varphi^{[t]}(u_F)]$ by definition, which implies $\hat{F}_t \subseteq [\varphi^{[t]}(u_F)]$, thus $F'(t) \in \Psi_{(T,\varphi)}(u_F)$, which is a contradiction. Consequently, we can assume $\psi_{(T,\varphi)}^{(t)}(u_F) \neq \emptyset$, so write $\psi_{(T,\varphi)}^{(t)}(u_F) = \{q_1,\ldots,q_m\}$ (where $1 \leq m \leq a_t$) such that $q_1 < \cdots < q_m$. For each $i \in [m]$, let $y_i \in Y$ be the unique ancestor of $u_F$ such that $(\omega^{[\varphi^{(y_i)}]}(y_i),\varphi(y_i)) = (q_i,t)$. If $m < a_t$ and $\hat{F}_t \psi_{(T,\varphi)}^{(t)}(u_F) \neq [a_t - m]$, then $\psi_{(T,\varphi)}^{(t)}(u_F) \subseteq \hat{F}_t$ by the definition of the colex order, which means $F'(t) \in \Psi_{(T,\varphi)}(u_F)$, and again we have a contradiction, hence we can further assume that $\hat{F}_t = \{q_1,\ldots,q_m\} \cup [a_t - m]$.

Note that $\psi_{(T,\varphi)}^{(t)}(u_F) \neq \emptyset$ implies the $t$-th entry of $\mathbf{a} - \nu(u_F)$ is positive, so $\zeta_{(t)}(u_F)$ is well-defined, and Proposition 13.4.9(i) tells us $q_1 > a_t - m + 1$. By the definition of colex order, we then get

$$\hat{F}'_t = \{q_1 - i : i \in [a_t - m + 1]\} \cup \{q_2,\ldots,q_m\}, \quad (13.33)$$
thus Proposition 13.4.9(ii) gives \( \xi^{(t)}_{(T, \varphi)}(\zeta^t_0(u_F)) = \hat{\mathcal{F}}_t' \). Also, Proposition 13.4.9(iii) says \( \xi^{(t)}_{(T, \varphi)}(\zeta^t_0(u_F)) = \xi^{(i)}_{(T, \varphi)}(u_F) \) for all integers \( t < i \leq n \) (if any), while Lemma 13.4.11 says \( s^{(i)}_{(T, \varphi)}(\zeta^t_0(u_F)) \subseteq s^{(i)}_{(T, \varphi)}(u_F) \) and \( s^{(i)}_{(T, \varphi)}(\zeta^t_0(u_F)) \geq_{ct} s^{(i)}_{(T, \varphi)}(u_F) \) for all \( i \in [t - 1] \). Now, define

\[
G := F'_t \cup \left( \bigcup_{i \in [n] \setminus \{t\}} \{ (s, i) : s \in s^{(i)}_{(T, \varphi)}(\zeta^t_0(u_F)) \} \right),
\]

and observe that \( \hat{F}'_{(t)} \leq_{ct} \hat{G}_i \) for all \( i \in [n] \). However, Lemma 13.4.13 tells us that \( G \in \Psi_{(T, \varphi)}(\zeta^t_0(u_F)) \), and \( \zeta^t_0(u_F) \) is strictly smaller than \( u_F \) in the depth-first order, so the minimality of \( u_F \) implies \( F'_{(t)} \in \Delta_{(T, \varphi)} \) (although not necessarily in \( \Psi_{(T, \varphi)}(\zeta^t_0(u_F)) \)), which is a contradiction. \( \square \)

**Theorem 13.4.15.** Given \( \mathbf{a} \in \mathbb{P}^n \), let \( (T, \varphi) \) be an \( \mathbf{a} \)-Macaulay tree of some positive integer \( N \). Then \( (T, \varphi) \) is the \( \mathbf{a} \)-Macaulay tree induced by the shedding tree of some pure color-compressed \( \mathbf{a} \)-balanced complex with \( N \) facets if and only if \( (T, \varphi) \) is condensed, compressed-like, and compatible.

**Proof.** If there exists a pure color-compressed \( \mathbf{a} \)-balanced complex \( (\Delta, \pi) \) with \( N \) facets, such that \( (T, \varphi) \) is the \( \mathbf{a} \)-Macaulay tree induced by the shedding tree of \( (\Delta, \pi) \), then Theorem 13.3.9 says \( (T, \varphi) \) is condensed, Theorem 13.4.4 says \( (T, \varphi) \) is compressed-like, and Theorem 13.4.12 tells us \( (T, \varphi) \) is compatible. Conversely, if instead \( (T, \varphi) \) is condensed, compressed-like, and compatible, then Proposition 13.3.12 gives the pure \( \mathbf{a} \)-balanced complex \( (\Delta_{(T, \varphi)}, \pi_{(T, \varphi)}) \) with \( N \) facets, Theorem 13.4.14 tells us \( (\Delta_{(T, \varphi)}, \pi_{(T, \varphi)}) \) is color-compressed, while by Remark 13.3.14, the \( \mathbf{a} \)-Macaulay tree induced by the shedding tree of \( (\Delta_{(T, \varphi)}, \pi_{(T, \varphi)}) \) is precisely \( (T, \varphi) \). \( \square \)
13.5 Generalized $\mathbf{a}$-Macaulay representations

Definition 13.5.1. Let $\mathbf{a} \in \mathbb{N}^n \setminus \{0_n\}$ and let $N \in \mathbb{P}$. A generalized $\mathbf{a}$-Macaulay representation of $N$ is a condensed compressed-like compatible $\mathbf{a}$-Macaulay tree of $N$. Explicitly, it is a pair $(T, \varphi)$ such that

(i) $T$ is a binary tree with (root, 1, 3)-triple $(r_0, L, Y)$.

(ii) $\varphi$ is a vertex labeling of $T$ satisfying all of the following conditions.
   (a) $\varphi(r_0) = \mathbf{a}$, $\varphi(Y) \subseteq [n]$ and $\varphi(L) \subseteq \mathbb{P}^n$.
   (b) If $y, y' \in Y$ satisfies $y' \in D_T(y)$, then $\varphi(y) \geq \varphi(y')$.
   (c) If $y \in Y$ satisfies $\varphi(y) < n$, then $\varphi^{[t]}(u) = \varphi^{[t]}(u')$ for all $u, u' \in L \cap D_T(y)$ and all $t \in [n] \setminus [\varphi(y)]$.
   (d) The left-weight labeling $\omega$ of $(T, \varphi)$ satisfies $\omega(y_{\text{left}}) \geq \omega(y_{\text{right}})$ for all $y \in Y$.
   (e) The $\mathbf{a}$-splitting tree $(T, \varphi|_Y, \nu)$ induced by $(T, \varphi|_Y)$ satisfies $\varphi(u) \geq \nu(u) > 0_n$ for all $u \in L$.
   (f) If $y \in Y$ satisfies $\nu[\varphi(y)](y) = 1$, then $\omega[\varphi(y)](x) = \omega[\varphi(y)](y) - 1$ for all $x \in D_T(y_{\text{right}})$.
   (g) $N$ can be written as the sum

$$N = \sum_{u \in L} \binom{\varphi(u)}{\nu(u)}.$$  (13.34)

(iii) There does not exist any $y \in Y$ satisfying the following two conditions:
   (a) The two subgraphs of $T$ induced by $D_T(y_{\text{left}})$ and $D_T(y_{\text{right}})$ are isomorphic as binary trees, and they have the same corresponding $\varphi$-labels, i.e. if $\beta : D_T(y_{\text{left}}) \to D_T(y_{\text{right}})$ is the corresponding isomorphism, then $\varphi(\beta(x)) = \varphi(x)$ for all $x \in D_T(y_{\text{left}})$.
   (b) Both $y_{\text{left}}$ and $y_{\text{right}}$ are $(\varphi(y) - 1)$-leading vertices of $(T, \varphi)$. 

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(iv) For all $t \in [n]$ and $y \in Y$ such that $\varphi(y) = t$, the following conditions hold:

(a) If $y$ is a descendant of $x_{\text{left}}$ for some $x \in Y$ satisfying $\varphi(x) = t$, and $x_0, x_1, \ldots, x_\ell$ is the path (of length $\ell \geq 2$) from $x_0 = x$ to $x_\ell = y_{\text{right}}$, then $\omega^{[t]}(x_i) = \omega^{[t]}(x) - i$ for all $i \in [\ell]$.

(b) Let $y_1, \ldots, y_m$ be the right relative sequence of $y_{\text{left}}$ in $T$. If $k \in [m]$ is the smallest integer such that $y_k \in L$ or $\varphi(y_k) \neq t$, then $\omega^{[t']} (y_k) \geq \omega^{[t']} (y_{\text{right}})$ for all $t' \in [n] \setminus \{t\}$.

(v) $\xi^{(t)}_{T,\varphi}(x) \leq_{\alpha} \xi^{(t)}_{T,\varphi}(\xi_i(x))$ for every $i, t \in [n]$ satisfying $t < i$ and every $t$-leading vertex $x$ of $(T, \varphi)$ such that the $i$-th entry of $\mathbf{a} - \nu(x)$ is positive.

**Remark 13.5.2.** Trying to understand the definition of generalized $a$-Macaulay representations without reading Chapters 13.2–13.4 is like trying to learn algebraic geometry for the first time by starting with the definition of schemes. There is a lot of new terminology to internalize. Once we have deciphered the meaning of all the terminology used in this given list of conditions, we would find that every condition involves only comparing the labels and left-weights of certain pairs of vertices in $T$. Thus, generalized Macaulay representations are purely numerical notions and do not depend on the existence of any colored complexes.

**Remark 13.5.3.** Conditions (ii), (iii) and (iv) can be easily checked by hand (at least when $T$ is not too large), while condition (v) is somewhat tedious to check. For large $n$, we suggest checking these conditions algorithmically.

For $N = 0$, the **generalized $a$-Macaulay representation of 0** is defined to be the pair $\alpha_0 := (T_0, \varphi_0)$, where $T_0$ is the null graph (i.e., the unique graph with zero vertices), and $\varphi_0 : \emptyset \to \emptyset$ is the (unique) function whose domain and codomain are empty sets. A **generalized $a$-Macaulay representation** is a generalized $a$-Macaulay representation of $N$ for some $N \in \mathbb{N}$, and a **generalized Macaulay representation**
is a generalized $a$-Macaulay representation for some $a \in \mathbb{N}^n \setminus \{0_n\}$. The $n$-tuple $a$ is called the type of this generalized Macaulay representation. A generalized Macaulay representation $\alpha$ is called non-trivial if $\alpha \neq \alpha_\emptyset$, and it is called trivial if $\alpha = \alpha_\emptyset$.

In general, a generalized $a$-Macaulay representation of $N$ is not uniquely determined by $a$ and $N$ (cf. Chapter 11.1). However, for $n = 1$, every compressed-like $k$-Macaulay tree is compatible (for any $k \in \mathbb{P}$), so Proposition 13.4.3 says the generalized $k$-Macaulay representation of $N$ is unique given $k, N \in \mathbb{P}$. For arbitrary $a \in \mathbb{P}^n$, Theorem 13.4.15 gives the bijection

\[
\begin{align*}
\{ \text{isomorphism classes of} & \text{ pure color-compressed} \\
& \text{a-balanced complexes} \\
& \text{with } N \text{ facets} \} 
\quad \longleftrightarrow \quad 
\{ \text{generalized } a\text{-Macaulay} \\
& \text{representations of } N \}.
\end{align*}
\]

**Notes**

As already mentioned in Chapter 1, it was pointed out by Björner–Frankl–Stanley [19] that part of the difficulty in characterizing the fine $f$-vectors of $a$-colored complexes (for $a \in \mathbb{P}^n$) lies in the non-uniqueness of color-compressed $a$-colored complexes with a given fine $f$-vector when $n > 1$. For arbitrary $a \in \mathbb{P}^n$, there could be many non-isomorphic color-compressed $a$-colored complexes with the same fine $f$-vector. As Frohmader [54] has showed, it is not possible to numerically characterize the fine $f$-vectors of color-compressed $1_n$-colored complexes through stronger restrictions on the color-selected subcomplexes. We now explain the significance of Frohmader’s result.

Observe that compressed simplicial complexes form a strictly smaller subclass of the simplicial complexes, yet the collection of their $f$-vectors equals the collection of $f$-vectors of arbitrary simplicial complexes. Similarly, color-compressed $a$-colored complexes form a strictly smaller subclass of the $a$-colored complexes, yet the collection of their fine $f$-vectors equals the collection of fine $f$-vectors of arbitrary $a$-colored complexes. We could ask if there is an even strictly smaller subclass of $a$-colored complexes with the same collection of fine $f$-vectors. Frohmader proved that if we insist on imposing additional restrictions on the color structure of (the subcomplexes of) $a$-colored complexes beyond color compression, then no
such strictly smaller subsubclass exists. Short of enumerating all possible color-compressed a-colored complexes and then choosing \( \geq 1 \) complex for each possible fine \( f \)-vector, the smallest subclass that we have to work with is the subclass of color-compressed a-colored complexes; see [54] for details. Consequently, the non-uniqueness issue raised by Björner–Frankl–Stanley is inherent in any numerical characterization of the fine \( f \)-vectors of a-colored complexes, and this issue cannot be avoided.

Generalized a-Macaulay representations are non-unique. They are inherently non-unique because they correspond bijectively to isomorphism classes of pure color-compressed a-balanced complexes. Different a-Macaulay representations of a positive integer \( N \) correspond to different sums of products of binomial coefficients that add up to \( N \). In the next chapter, we shall see that such generalized a-Macaulay representations are very well-suited for numerically characterizing a-colored complexes.
In this final chapter, we present complete numerical characterizations for the
flag $f$-vectors of completely balanced Cohen–Macaulay complexes (Chapter 14.1),
and more generally, for the fine $f$-vectors of $a$-colored complexes for arbitrary $a \in \mathbb{P}^n$ (Chapter 14.2). Both of these numerical characterizations are stated in
terms of generalized Macaulay representations, so a complete understanding of
these characterizations would require knowledge of previous chapters (especially
Chapter 13). In fact, most of the hard work needed to characterize the flag $f$-
vectors of completely balanced Cohen–Macaulay complexes has already been done
in Chapter 13.

In contrast, characterizing the fine $f$-vectors of arbitrary colored complexes
requires more work. The main new idea introduced in Chapter 14.2 is that we can
define a partial order $\preceq$ on generalized Macaulay representations of different types.
(Recall that the type of a generalized $a$-Macaulay representation of any positive
integer is the $n$-tuple $a$.) It is easy to see that any numerical characterization of
the fine $f$-vectors of arbitrary colored complexes must include the Kruskal–Katona
theorem as a special case. Using $\preceq$, we can succinctly state the Kruskal–Katona
theorem as follows: $(f_0, \ldots, f_{d-1}) \in \mathbb{P}^d$ is the $f$-vector of a $(d - 1)$-dimensional
simplicial complex if and only if $\alpha_d \preceq \cdots \preceq \alpha_1$, where each $\alpha_k$ is the unique
generalized $k$-Macaulay representation of $f_{k-1}$; see Theorem 14.2.8.

In other words, the Kruskal–Katona theorem is equivalent to the existence of
a chain of (generalized) Macaulay representations ordered by $\preceq$. For the general
case of the fine $f$-vectors of $a$-colored complexes, our numerical characterization is
equivalent to the existence of a subposet of generalized Macaulay representations (ordered by \( \preceq \)) with a specific poset structure: It must be anti-isomorphic to the closed interval \([0_n, a] \) contained in the poset of all \( n \)-tuples in \( \mathbb{N}^n \) ordered by \( \preceq \); see Theorem 14.2.7.

14.1 Flag \( f \)-vectors of completely balanced Cohen–Macaulay complexes

The Kruskal–Katona theorem (cf. Chapter 2.6) is stated in terms of the (usual) Macaulay representations, together with suitably defined differentials on them. Analogously, we can also define differentials for generalized Macaulay representations. Since every generalized \( a \)-Macaulay representation is an \( a \)-Macaulay tree, we shall define differentials more generally on Macaulay trees as follows.

**Definition 14.1.1.** Let \( a \in \mathbb{P}^n \), let \( \alpha = (T, \varphi) \) be an \( a \)-Macaulay tree of a positive integer \( N \), let \((r_0, L, Y)\) be the (root, 1, 3)-triple of \( T \), and let \((T, \varphi|_Y, \nu)\) be the \( a \)-splitting tree induced by \((T, \varphi|_Y)\). Then for any \( x \in \mathbb{Z}^n \), define

\[
\partial_{[x]}(\alpha) := \sum_{u \in L} \left( \frac{\varphi(u)}{\nu(u) + x} \right).
\]

**Theorem 14.1.2.** Let \( a \in \mathbb{P}^n \), and let \( f = \{f_b\}_{b \leq a} \) be an array of integers. Then the following are equivalent:

(i) \( f \) is the fine \( f \)-vector of a pure color-compressed \( a \)-balanced complex.

(ii) \( f_a > 0 \), and there is a generalized \( a \)-Macaulay representation \( \alpha \) of \( f_a \) such that \( f_b = \partial_{[b-a]}(\alpha) \) for all \( b \leq a \).

**Proof.** This immediately follows from Theorem 13.4.15 and Proposition 13.2.3. \( \square \)
Theorem 14.1.3. Let $d \in \mathbb{P}$, let $f = \{f_S\}_{S \subseteq [d]}$ be an array of integers, and define $\phi : 2^{[d]} \to \{0,1\}^d$ by $S \mapsto (s_1, \ldots, s_d)$, where $s_i = 1$ if and only if $i \in S$. Then the following are equivalent:

(i) $f$ is the flag $f$-vector of a $(d - 1)$-dimensional completely balanced Cohen–Macaulay complex.

(ii) $f_{[d]} > 0$, and there is a generalized $1_d$-Macaulay representation $\alpha$ of $f_{[d]}$ such that $f_S = \partial_{\phi(S)-1_d}(\alpha)$ for all $S \subseteq [d]$.

Proof. As mentioned in Remark 11.2.2, the notion of ‘color-compressed’ is equivalent to the notion of ‘color-shifted’ in the specific case of type $1_d$, hence the assertion follows from Corollary 12.3.6 and Theorem 14.1.2. \hfill \Box

Remark 14.1.4. In view of Theorem 14.2.10, there is also a different but equivalent numerical characterization of the flag $h$-vectors of completely balanced Cohen–Macaulay complexes; see Corollary 14.2.11.

Example 14.1.5. By the Frankl–Füredi–Kalai theorem [15], the vector $(7, 11, 5)$ is the $f$-vector of a completely balanced Cohen–Macaulay complex, since the corresponding $h$-vector $(1, 4, 0, 0)$ is the $f$-vector (including the $f_{-1}$ term) of a $1_3$-colored complex. One possible refinement of $(7, 11, 5)$ is the array $F = \{F_S\}_{S \subseteq [3]}$, given by

$F_{\{1,2,3\}} = 5$, $F_{\{1,2\}} = 5$, $F_{\{1,3\}} = 3$, $F_{\{2,3\}} = 3$, $F_{\{1\}} = 3$, $F_{\{2\}} = 2$, $F_{\{3\}} = 2$, $F_{\emptyset} = 1$,

and we would like to know if $F$ is the flag $f$-vector of a completely balanced Cohen–Macaulay complex. Figure 14.1 gives a list of all 24 generalized $1_3$-Macaulay representations of 5. Using Theorem 14.1.3, we then conclude that up to permutations of the 3 colors such that $f_{\{1,2\}} \geq f_{\{1,3\}} \geq f_{\{2,3\}}$, the possible flag $f$-vectors $\{f_S\}_{S \subseteq [3]}$ of a 2-dimensional completely balanced Cohen–Macaulay complex satisfying $f_{[3]} = 5$ (and $f_{\emptyset} = 1$) are given by:
Therefore $F$ is not the flag $f$-vector of a completely balanced Cohen–Macaulay complex.

14.2 Fine $f$-vectors of generalized colored complexes

In this section, we will obtain a numerical characterization of the fine $f$-vectors of colored complexes of arbitrary type. Note in particular that if $\Delta$ is a $(d - 1)$-dimensional simplicial complex with vertex set $V$, then the pair $(\Delta, V)$ is trivially a $(d)$-colored complex, where $V$ here is treated as the trivial partition of itself (into 1 subset). It is also clear that the fine $f$-vector of $(\Delta, V)$ is identically the $f$-vector of $\Delta$ (up to a shift in the indexing). Consequently, our numerical characterization includes the Kruskal–Katona theorem as a special case.

The Kruskal–Katona theorem says a vector $(f_0, \ldots, f_{d-1}) \in \mathbb{P}^d$ is the $f$-vector of a $(d - 1)$-dimensional simplicial complex if and only if $\partial_{i+1}(f_i) \leq f_{i-1}$ for all $i \in [d-1]$. Our main new idea in this section is that we can define a partial order $\preceq$ on generalized Macaulay representations of different types. (Recall that the type of a generalized $a$-Macaulay representation of any positive integer is the $n$-tuple $a$.) In the case of “uncolored” simplicial complexes, the Kruskal–Katona theorem can be succinctly stated as follows: $(f_0, \ldots, f_{d-1}) \in \mathbb{P}^d$ is the $f$-vector of a $(d - 1)$-dimensional simplicial complex if and only if $\alpha_d \preceq \cdots \preceq \alpha_1$, where each $\alpha_k$ is the unique generalized $k$-Macaulay representation of $f_{k-1}$; see Theorem 14.2.8. In
Figure 14.1: All 24 generalized \(1_3\)-Macaulay representations of 5.
other words, the Kruskal–Katona theorem is equivalent to the existence of a chain of (generalized) Macaulay representations ordered by $\leq$. In general, our numerical characterization of the fine $f$-vectors of $a$-colored complexes is equivalent to the existence of a subposet of generalized Macaulay representations (ordered by $\preceq$), such that it is anti-isomorphic to the subposet of all $n$-tuples in $\mathbb{N}^n$ ordered by $\leq$ bounded above by $a$ (with respect to $\leq$).

Throughout, let $a, a' \in \mathbb{N}^n \setminus \{0_n\}$ satisfy $a' \leq a$, and let $\alpha = (T, \varphi)$ be an $a$-Macaulay tree. Denote the (root, 1, 3)-triple of $T$ by $(r_0, L, Y)$, and let $(T, \varphi|_Y, \nu)$ be the $a$-splitting tree induced by $(T, \varphi|_Y)$.

**Definition 14.2.1.** Let $a, a' \in \mathbb{N}^n \setminus \{0_n\}$ satisfy $a' \leq a$, and let $\alpha = (T, \varphi)$ be an $a$-Macaulay tree. Denote the (root, 1, 3)-triple of $T$ by $(r_0, L, Y)$, and let $(T, \varphi|_Y, \nu)$ be the $a$-splitting tree induced by $(T, \varphi|_Y)$. Then, the $a'$-twin of $(T, \varphi)$ is the pair $(T', \varphi')$ constructed from $(T, \varphi)$ via the following algorithm:

1. $S \leftarrow \{x \in (Y \cup L \cup \{r_0\}) : \nu(x) > a - a'\}.$
2. $S_0 \leftarrow \{y \in Y : \nu(y) > a - a', \nu(y_{\text{right}}) \not> a - a'\}.$
3. $T' \leftarrow$ subgraph of $T$ induced by $S$.
4. $\varphi' \leftarrow \varphi|_S$.
5. $\varphi'(r_0) \leftarrow a'$.
6. **while** $S_0 = \emptyset$ **do**
   - $z \leftarrow$ largest vertex in $S_0$ with respect to the depth-first order.
   - $z_{\text{child}} \leftarrow$ unique child of $z$ in $T'$.
   - $S_0 \leftarrow S_0 \setminus \{z\}$.
   - $L' \leftarrow$ set of leaves of $T'$.
   - **if** $z_{\text{child}} \in L'$ **then**
     - $\varphi'(z) \leftarrow \varphi'(z_{\text{child}}) + \delta_{\varphi'}(z, n)$.
     - $\varphi'(z_{\text{child}}) = \varphi'(z)$.
   - **else**
     - $t' \leftarrow \varphi'(z)$.
     - $\varphi'(z) \leftarrow \varphi'(z_{\text{child}})$.
     - **for** $x \in D_T(z_{\text{child}}) \cap L'$ **do**
       - $(\varphi')^{[t']}(x) \leftarrow (\varphi')^{[t']}(x) + 1$.
     - **end for**
   - **end if**
   - $(T', \varphi') \leftarrow (T'/\{z, z_{\text{child}}\}, \varphi'/\{z, z_{\text{child}}\})$.
   - **end while**
Remark 14.2.2. It is easy to verify that the pair \((T', \varphi')\) in Definition 14.2.1 is an \(a'\)-Macaulay tree of \(\partial_{[a' - a]}(\alpha)\).

For the rest of this section, let \(a, a' \in \mathbb{N} \setminus \{0\}\) satisfy \(a' \leq a\), let \(\alpha = (T, \varphi)\) be an \(a\)-Macaulay tree, let \(\alpha' = (T', \varphi')\) be an \(a'\)-Macaulay tree, and let \(\alpha'' = (T'', \varphi'')\) be the condensation of the \(a'\)-twin of \(\alpha\). Denote the \((\text{root}, 1, 3)\)-triples of \(T'\) and \(T''\) by \((r_0', L', Y')\) and \((r_0'', L'', Y'')\) respectively. Next, construct a graph \(\tilde{T}\) as follows: Take the disjoint union of \(T'\) and \(T''\), identify the roots \(r_0', r_0''\) as a single vertex \(\hat{r}\), then attach a leaf \(\tilde{r}\) to \(\hat{r}\). Note that \(\tilde{T}\) is a binary tree with root \(\tilde{r}\), and treat \(T'\) and \(T''\) as subgraphs of \(\tilde{T}\). Now, let \(\tilde{a} := (a', 2) \in \mathbb{N}^{n+1} \setminus \{0_{n+1}\}\), and define the vertex labeling \(\tilde{\varphi}\) of \(\tilde{T}\) as follows:

\[
\tilde{\varphi}(x) = \begin{cases} 
\tilde{a}, & \text{if } x = \tilde{r}; \\
n + 1, & \text{if } x = \hat{r}; \\
\varphi'(x), & \text{if } x \in Y'; \\
(\varphi'(x), 2) \in \mathbb{P}^{n+1}, & \text{if } x \in L'; \\
\varphi''(x), & \text{if } x \in Y''; \\
(\varphi''(x), 1) \in \mathbb{P}^{n+1}, & \text{if } x \in L''.
\end{cases}
\]

We check that \((\tilde{T}, \tilde{\varphi})\) is an \(\tilde{a}\)-Macaulay tree, which we denote by \(\alpha \land \alpha'\). Furthermore, \(\alpha \land \alpha'\) is compressed-like if both \(\alpha\) and \(\alpha'\) are compressed-like, and \(\alpha \land \alpha'\) is condensed if \(\alpha'\) is condensed.

![Diagram](image.png)

(a) \((1, 1)\)-Macaulay tree \(\alpha'\)  
(b) \((2, 2)\)-Macaulay tree \(\alpha\)  
(c) \((1, 1)\)-twin of \(\alpha\)  
(d) \((1, 1, 2)\)-Macaulay tree \(\alpha \land \alpha'\)

Figure 14.2: Macaulay trees \(\alpha\), \(\alpha'\), \(\alpha''\), and \(\alpha \land \alpha'\).

Example 14.2.3. Figure 14.2 shows an example of a \((1, 1)\)-Macaulay tree \(\alpha'\), a
(2, 2)-Macaulay tree \( \alpha \), the (1, 1)-twin \( \alpha'' \) of \( \alpha \) (which is its condensation), and the corresponding (1, 1, 2)-Macaulay tree \( \alpha \wedge \alpha' \).

**Definition 14.2.4.** Let \( \alpha, \alpha' \) be non-trivial generalized Macaulay representations. If \( \alpha \wedge \alpha' \) is compatible, or equivalently, if \( \alpha \wedge \alpha' \) is a generalized Macaulay representation, then we write \( \alpha \preceq \alpha' \). Also, write \( \alpha \emptyset \preceq \alpha'' \) for all generalized Macaulay representations \( \alpha'' \).

**Proposition 14.2.5.** \( \preceq \) is a partial order on generalized Macaulay representations.

**Proof.** Given \( n \)-tuples \( \rho = ([s_1], \ldots, [s_n]) \) and \( \rho' = ([s'_1], \ldots, [s'_n]) \) for some positive integers \( s_1, \ldots, s_n, s'_1, \ldots, s'_n \), write \( \rho \leq \rho' \) to mean \( (s_1, \ldots, s_n) \leq (s'_1, \ldots, s'_n) \).

Let \( a, b, c \in \mathbb{N} \setminus \{0_n\} \) satisfy \( c \leq b \leq a \), and let \( \alpha, \beta, \gamma \) be arbitrary generalized Macaulay representations of types \( a, b, c \) respectively. Without loss of generality, assume \( \alpha, \beta, \gamma \) are all non-trivial. Clearly \( \alpha \preceq \alpha \), since \( \alpha \) is compatible implies \( \alpha \wedge \alpha \) is compatible.

Next, suppose \( \alpha \preceq \beta \) and \( \beta \preceq \alpha \). This means \( \alpha \wedge \beta \) and \( \beta \wedge \alpha \) are well-defined, which implies \( a = b \), hence the condensation of the \( a \)-twin of \( \beta \) is \( \beta \), and the condensation of the \( b \)-twin of \( \alpha \) is \( \alpha \). Since \( \alpha \wedge \beta \) is compressed-like and compatible, Theorem 13.4.14 yields \( (\Delta_{\alpha \wedge \beta}, \pi_{\alpha \wedge \beta}) \) is color-compressed, hence by the construction of \( \alpha \wedge \beta \), we get \( \Delta_{\alpha} \subseteq \Delta_{\beta} \) and \( \pi_{\alpha} \leq \pi_{\beta} \). A symmetric argument also gives \( \Delta_{\beta} \subseteq \Delta_{\alpha} \) and \( \pi_{\beta} \leq \pi_{\alpha} \), thus \( (\Delta_{\alpha}, \pi_{\alpha}) = (\Delta_{\beta}, \pi_{\beta}) \). Now \( \alpha \) and \( \beta \) are condensed, so Theorem 13.4.15 yields \( \alpha = \beta \).

Finally, suppose instead that \( \alpha \preceq \beta \) and \( \beta \preceq \gamma \). Let \( \alpha' \) be the \( b \)-twin of \( \alpha \), let \( \tilde{\alpha} \) be the \( c \)-twin of \( \alpha \), and let \( \tilde{\beta} \) be the \( c \)-twin of \( \beta \). Since \( \alpha \wedge \beta \) and \( \beta \wedge \gamma \) are compressed-like and compatible, Theorem 13.4.14 says \( (\Delta_{\alpha \wedge \beta}, \pi_{\alpha \wedge \beta}) \) and \( (\Delta_{\beta \wedge \gamma}, \pi_{\beta \wedge \gamma}) \) are both color-compressed. By the construction of \( \alpha \wedge \beta \), we get \( \Delta_{\alpha'} \subseteq \Delta_{\beta}, \Delta_{\tilde{\alpha}} \subseteq \Delta_{\tilde{\beta}}, \)
\[ \pi_{\alpha'} \leq \pi_{\beta}, \text{ and } \pi_{\alpha} \leq \pi_{\beta}. \]

Similarly, the construction of \( \beta \land \gamma \) yields \( \Delta_{\beta} \subseteq \Delta_{\gamma} \) and \( \pi_{\beta} \leq \pi_{\gamma}, \text{ thus } \Delta_{\alpha} \subseteq \Delta_{\gamma} \) and \( \pi_{\alpha} \leq \pi_{\gamma}. \)

Consequently, the construction of \( \alpha \land \gamma \) yields \( (\Delta_{\alpha \land \gamma}, \pi_{\alpha \land \gamma}) \) is a pure color-compressed \((c, 2)\)-balanced complex, so since \( \alpha \land \gamma \) is condensed, Remark 13.3.14 tells us the Macaulay tree induced by the shedding tree of \( (\Delta_{\alpha \land \gamma}, \pi_{\alpha \land \gamma}) \) is precisely \( \alpha \land \gamma \), and Theorem 13.4.15 then yields \( \alpha \land \gamma \) is compatible, i.e. \( \alpha \preceq \gamma \).

\[ \square \]

**Proposition 14.2.6.** Let \( k, N, N' \in \mathbb{P} \), let \( \alpha \) be the generalized \((k + 1)\)-Macaulay representation of \( N \), and let \( \alpha' \) be the generalized \( k \)-Macaulay representation of \( N' \). Then \( \alpha \preceq \alpha' \) if and only if \( \partial_{[-1]}(\alpha) \leq N' \).

**Proof.** Let the condensation of the \( k \)-twin of \( \alpha \) be \( \alpha'' \), which by construction is a generalized \( k \)-Macaulay representation of \( \partial_{[-1]}(\alpha) \). By definition, \( \alpha \preceq \alpha' \) if and only if \( \alpha \land \alpha' \) is compatible, or equivalently, \( \Delta_{\alpha''} \subseteq \Delta_{\alpha'} \). Now, \( \alpha', \alpha'' \) are both generalized \( k \)-Macaulay representations, so by Theorem 13.4.15, \( \Delta_{\alpha'}, \Delta_{\alpha''} \) are both pure compressed \((k - 1)\)-dimensional simplicial complexes and hence each completely determined by its number of facets. This implies \( \Delta_{\alpha''} \subseteq \Delta_{\alpha'} \) if and only if \( f_{k-1}(\Delta_{\alpha''}) \leq f_{k-1}(\Delta_{\alpha'}) \), which by Proposition 13.3.12 is equivalent to \( \partial_{[-1]}(\alpha) \leq N' \).

Finally, we characterize the fine \( f \)-vectors of colored complexes.

**Theorem 14.2.7.** Let \( a \in \mathbb{P}^n \), and let \( f = \{ f_b \}_{b \leq a} \) be an array of integers. Then the following four conditions are equivalent:

(i) \( f \) is the fine \( f \)-vector of an \( a \)-colored complex.

(ii) \( f \) is the fine \( f \)-vector of a color-compressed \( a \)-colored complex.

(iii) \( f \) is the fine \( f \)-vector of a color-shifted \( a \)-colored complex.
(iv) $f_{0_n} = 1$, $f_{a'} = 0$ for all $a' \not\geq 0_n$, $f_{\delta_{i,n}} > 0$ for all $i \in [n]$, and there is an array $\{\alpha_b\}_{0_n < b \leq a}$ such that each $\alpha_b$ is a generalized $b$-Macaulay representation of $f_b$, and $\alpha_b \preceq \alpha_{b'}$, for all $b, b'$ satisfying $0_n < b' < b \leq a$.

**Proof.** The equivalences (i) $\iff$ (ii) $\iff$ (iii) follow immediately from Corollary 11.1.5 and Remark 11.2.2, thus we are left to show that (i) $\iff$ (iv). Let $(\Delta, \pi)$ be an arbitrary colored-compressed $a$-colored complex, and assume without loss of generality that $\pi = ([\lambda_1], \ldots, [\lambda_n])$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{P}$. Clearly $f_{0_n}(\Delta) = 1$ and $f_{a'}(\Delta) = 0$ for all $a' \not\geq 0_n$. By definition, $f_{\delta_{i,n}}(\Delta) = \lambda_i > 0$ for each $i \in [n]$.

For any $b \in \mathbb{N}^n$ satisfying $0_n < b \leq a$, define $\Delta_b := \langle F_b(\Delta) \rangle$, let $U_b$ denote the vertex set of $\Delta_b$, and let $\pi_b := \pi \cap U_b$. By default, set $U_b = \emptyset$ if $F_b(\Delta) = \emptyset$.

Notice that if $F_b(\Delta) \neq \emptyset$, then $(\Delta_b, \pi_b)$ is a pure color-compressed $b$-balanced complex, thus by Theorem 13.4.15, the $b$-Macaulay tree induced by the shedding tree of $(\Delta_b, \pi_b)$ is a generalized $b$-Macaulay representation, which we denote by $\alpha_b = (T_b, \varphi_b)$. We can then extend each such $\alpha_b$ to a generalized $b$-Macaulay representation $\alpha_b = (T_b, \varphi_b)$ as follows.

Given $b = (b_1, \ldots, b_n)$, let $m$ be the number of non-zero entries in $b$, and let $I := \{i_1, \ldots, i_m\} \subseteq [n]$ be the set of indices with $i_1 < \cdots < i_m$, such that $b_{i_1}, \ldots, b_{i_m}$ are all the non-zero entries of $b$. Let $(r_0, L, Y)$ be the (root, 1, 3)-triple of $T_b$, and define the vertex labeling $\varphi_b$ of $T_b$ so that

- $\varphi_b(r_0) = b$;
- $\varphi_b(y) = i_{\langle \varphi_b(y) \rangle} \in I$ for all $y \in Y$;
- $(\varphi_b)^{[t]}(u) = (\bar{\varphi}_b)^{[t]}(u)$ for all $t \in [m]$, $u \in L$;
- $(\varphi_b)^{[j]}(u) = \lambda_j$ for all $j \in [n] \setminus I$, $u \in L$.

We can check that $\alpha_b$ is a $b$-Macaulay tree of $f_b(\Delta)$, and the fact that $\alpha_b$ is a generalized Macaulay representation easily implies $\alpha_b$ is also a generalized Macaulay representation. As for the case $F_b(\Delta) = \emptyset$, define $\alpha_b := \alpha_\emptyset$. 

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Let \( b, b' \in \mathbb{N}^n \) satisfy \( 0_n < b' \prec b \leq a \). We want to show that \( \alpha_b \preceq \alpha_{b'} \).

Clearly, if \( \mathcal{F}_b(\Delta) = \emptyset \), then \( \alpha_b = \alpha_\emptyset \preceq \alpha_{b'} \). Since \( \mathcal{F}_{b'}(\Delta) = \emptyset \) implies \( \mathcal{F}_b(\Delta) = \emptyset \), we can assume \( \mathcal{F}_b(\Delta) \) and \( \mathcal{F}_{b'}(\Delta) \) are non-empty, i.e. \( \alpha_b \) and \( \alpha_{b'} \) are both non-trivial. Let \( \tilde{\alpha}_{b',b} \) denote the \( b' \)-twin of \( \alpha_b \). Theorem 13.4.15 tells us \( \Delta_{b'} = \Delta_{\tilde{\alpha}_{b',b}} \)
and \( \langle \mathcal{F}_{b'}(\Delta_b) \rangle = \Delta_{\tilde{\alpha}_{b',b}} \), while it follows from definition that \( \langle \mathcal{F}_b(\Delta_b) \rangle \subseteq \Delta_{b'} \).

Thus, by the construction of \( \alpha_b \land \alpha_{b'} \), we get that \( (\Delta_{\alpha_b \land \alpha_{b'}}, \pi_{\alpha_b \land \alpha_{b'}}) \) is a pure color-compressed \((b', 2)\)-balanced complex, therefore Theorem 13.4.15 tells us \( \alpha_{b' \land \alpha_{b'}} \) is compatible, i.e. \( \alpha_b \preceq \alpha_{b'} \).

Conversely, suppose instead statement (iv) of the theorem is true. By assumption, \( \alpha_b \preceq \alpha_{b'} \) for every \( b, b' \in \mathbb{N}^n \) satisfying \( 0_n < b' \prec b \leq a \), so since \( \preceq \) is a partial order (Proposition 14.2.5), it follows that \( \alpha_b \preceq \alpha_c \) for every \( b, c \in \mathbb{N}^n \) satisfying \( 0_n < c \leq b \leq a \). Define \( \pi_\Gamma := ([f_{b,1}], \ldots, [f_{b,n}]) \), and let \( \Gamma \) be the union of all simplicial complexes \( \Delta_{\alpha_b} \) over all \( n \)-tuples \( b \) satisfying \( 0_n < b \leq a \), \( f_b > 0 \). For each such \( b \), the \( b \)-balanced complex \( (\Delta_{\alpha_b}, \pi_b) \) is color-compressed, hence \( (\Gamma, \pi_\Gamma) \) is a color-compressed \( a \)-colored complex.

Now, choose any \( b, c \in \mathbb{N}^n \) such that \( 0_n < c \leq b \leq a \). Note that \( \alpha_\emptyset \preceq \alpha_b \), while \( \alpha_b = \alpha_\emptyset \) if and only if \( f_b = 0 \), so \( \alpha_b = 0 \) if and only if \( f_b(\Gamma) = 0 \). Consequently, if \( f_b > 0 \), then \( f_c > 0 \) and \( \mathcal{F}_c(\Delta_{\alpha_b}) \subseteq \Delta_{\alpha_c} \), which imply \( \mathcal{F}_c(\Gamma) = \mathcal{F}_c(\Delta_{\alpha_c}) \). Therefore by Proposition 13.3.12, \( \alpha_b \) is a generalized \( b \)-Macaulay representation of \( f_b(\Gamma) \) for every \( b \in \mathbb{N}^n \) satisfying \( 0_n < b \leq a \), i.e. \( f \) is the fine \( f \)-vector of \( (\Gamma, \pi_\Gamma) \). \( \square \)

In the special case \( n = 1 \), i.e. \( a = (a_1) \), an \( a \)-colored complex is precisely a \((k - 1)\)-dimensional simplicial complex for some \( k \in [a_1] \), so in view of Theorem 14.2.7 (cf. Proposition 14.2.6), we can restate the Kruskal-Katona theorem (see Chapter 2.6) as follows:
Theorem 14.2.8. Let $d \in \mathbb{P}$, and let $f = (f_0, \ldots, f_{d-1}) \in \mathbb{Z}^d$. Then the following are equivalent:

(i) $f$ is the $f$-vector of a $(d-1)$-dimensional simplicial complex.
(ii) $f$ is the $f$-vector of a compressed $(d-1)$-dimensional simplicial complex.
(iii) $f$ is the $f$-vector of a shifted $(d-1)$-dimensional simplicial complex.
(iv) $f \in \mathbb{P}^d$ and $\alpha_d \preceq \cdots \preceq \alpha_1$, where each $\alpha_k$ is the (unique) generalized $k$-Macaulay representation of $f_{d-1}$.

As for the case $a = 1_n$, Theorem 14.2.7 yields the following corollary.

Corollary 14.2.9. Let $n \in \mathbb{P}$, let $f = \{f_S\}_{S \subseteq [n]}$ be an array of integers, and define $\phi : 2^n \to \{0,1\}^n$ by $S \mapsto (s_1, \ldots, s_n)$, where $s_i = 1$ if and only if $i \in S$. Then the following are equivalent:

(i) $f$ is the flag $f$-vector of a $1_n$-colored complex.
(ii) $f_\emptyset = 1$, $f_{\{i\}} > 0$ for all $i \in [n]$, and there is an array $\{\alpha_S\}_{S \subseteq [n], S \neq \emptyset}$ such that each $\alpha_S$ is a generalized $\phi(S)$-Macaulay representation of $f_S$, and $\alpha_S \preceq \alpha_{S'}$ for all non-empty $S, S' \subseteq [n]$ satisfying $|S| = |S'| + 1$.

In 1987, building upon an earlier paper by Stanley [116], Björner–Frankl–Stanley [19] proved the following combinatorial characterization:

Theorem 14.2.10. Let $a \in \mathbb{P}^n$, and let $f = \{f_b\}_{b \preceq a}$ be an array of integers. The following are equivalent:

(i) $f$ is the fine $h$-vector of an $a$-balanced CM complex.
(ii) $f$ is the fine $h$-vector of an $a$-balanced pure shellable complex.
(iii) $f$ is the fine $f$-vector of an $a$-colored multicomplex.
(iv) $f$ is the fine $f$-vector of a color-compressed $a$-colored multicomplex.
Consequently, the flag $h$-vector of a $(d - 1)$-dimensional completely balanced Cohen–Macaulay complex is the flag $f$-vector of a $1_d$-colored complex, thus by Corollary 14.2.9, we get the following different but equivalent numerical characterization of completely balanced Cohen–Macaulay complexes.

**Corollary 14.2.11.** Let $d \in \mathbb{P}$, let $h = \{h_S\}_{S \subseteq [d]}$ be an array of integers, and define $\phi : 2^d \to \{0, 1\}^d$ by $S \mapsto (s_1, \ldots, s_d)$, where $s_i = 1$ if and only if $i \in S$. Then the following are equivalent:

(i) $h$ is the flag $h$-vector of a $(d - 1)$-dimensional completely balanced Cohen–Macaulay complex.

(ii) $h_\emptyset = 1$, $h_{\{i\}} > 0$ for all $i \in [d]$, and there is an array $\{\alpha_S\}_{S \subseteq [d], S \neq \emptyset}$ such that each $\alpha_S$ is a generalized $\phi(S)$-Macaulay representation of $h_S$, and $\alpha_S \preceq \alpha_{S'}$ for all non-empty $S, S' \subseteq [d]$ satisfying $|S| = |S'| + 1$.

**Notes**

Arguably the most important (and most difficult) open problem in $f$-vector theory is the $g$-conjecture. The $g$-conjecture originally referred to a numerical characterization of the $f$-vectors of simplicial polytopes proposed by McMullen [87] in 1971. This original version was proven to be true by Stanley [117] (necessity) and Billera–Lee [16] (sufficiency) in 1980, and it is now known as the $g$-theorem. Today, the $g$-conjecture instead refers to a more general conjecture, that the $f$-vectors of triangulated spheres (or more generally, homology spheres) are also characterized by the same conditions proposed by McMullen. For an excellent overview of the status and latest developments concerning the (still open) $g$-conjecture, including possible further generalizations to homology manifolds, see [124].

While the $f$-vectors of simplicial polytopes can be considered to be well-understood, the closely related problem of finding a numerical characterization for the $f$-vectors of general (not necessarily simplicial) polytopes is still open. As stated by Björner and Billera [15], it is believed that “the best route to an eventual characterization of $f$-vectors of general polytopes lies in an understanding of their flag $f$-vectors.”
APPENDIX A

CLAIM 4.2(ii) IN FRANKL–FÜREDI–KALAI’S PAPER IS NOT TRUE

The original proof of the Frankl–Füredi–Kalai theorem in [53] is incorrect as stated. In particular, Claim 4.2(ii) in [53] is not true. As stated in the paper, Claim 4.2 (which consists of two parts (i) and (ii)) is crucial for the proof of their main theorem. Indeed, we check that Claim 4.2(ii) is used several times in the proof of Claim 4.3 and the main theorem (Theorem 4.1). The purpose of this appendix is to have on record a justification of why Claim 4.2(ii) is not true. We explain why the claim is not true in two ways: We explain where and why the argument for this claim breaks down; and we give an explicit counterexample to Claim 4.2(ii).

Nevertheless, the actual non-numerical version of the Frankl–Füredi–Kalai theorem is true: London [83], Engel [48, Chap. 8], and Mermin–Murai [89] gave three different proofs of the non-numerical version, while in Chapter 6 of this dissertation, we gave another (fourth) different proof (and in fact, a generalization) of the Frankl–Füredi–Kalai theorem.

As a side remark, we note that the numerical version of the Frankl–Füredi–Kalai theorem as stated in [53] is also incorrect. The uniqueness claim in Lemma 1.1 of the paper is not true as stated, which makes the $\partial_k^{(r)}(\cdot)$ operator introduced in [53] not well-defined. This was pointed out by Billera–Björner [15]. There is an easy fix to make the $\partial_k^{(r)}(\cdot)$ operator well-defined, and this fix was already known earlier by J. Eckhoff (around the 1990s). The numerical version of the Frankl–Füredi–Kalai theorem stated in [15] includes this fix.

Throughout, we shall use, without further explanation, the terminology found
in [53] (which is mostly very different from ours). In particular, they use \( \mathbb{N} \) to denote positive integers. To avoid confusion, we instead use \( \mathbb{Z}^+ \) for positive integers, but we keep their notation that \( \mathbb{N}_i \) denotes the set \( \{n \in \mathbb{Z}^+ : n \equiv i \pmod{r}\} \).

Claim 4.2(ii) asserts that

\[
\Delta \hat{F}(\ell) \subseteq \hat{F}(\ell + 1). \tag{A.1}
\]

We instead claim the following: We can only conclude that every \( F \in \Delta \hat{F}(\ell) \) is contained in \( \hat{F}(\ell + t) \) for some \( t \in [k - \ell] \), which is a weaker assertion.

Since \( S \in F(\ell) \) if and only if \( \hat{S} \in \hat{F}(\ell) \), any element of \( \Delta \hat{F}(\ell) \) is of the form \( \hat{S}\backslash a \) for some \( S \in F(\ell) \) and some \( a \in \hat{S} = S\backslash K \). Thus, to show \( \Delta \hat{F}(\ell) \subseteq \hat{F}(\ell + 1) \), it suffices to show that for any \( S \in F(\ell) \) and any \( a \in \hat{S} \), we have

\[
\hat{S}\backslash a \in \hat{F}(\ell + 1) = \{\hat{W} : W \in F(\ell + 1)\}. \tag{A.2}
\]

This is equivalent to showing the existence of some \( T \in F(\ell + 1) \) such that \( T\backslash K(T) = \hat{S}\backslash a = (S\backslash\{a\})\backslash K \). (Note that \( a \not\in K \) implies \( K \subseteq S\backslash\{a\} \).) Define \( R = S\backslash\{a\} \), and note that if such a \( T \) exists, then we necessarily have that \( R \subseteq T \).

Since \( |R| = k - 1 \) and \( |T| = k \), such a \( T \) must be of the form \( T = R \cup \{b\} \) for some \( b \in \mathbb{Z}^+ \backslash R \), thus it suffices to find some \( b \in \mathbb{Z}^+ \backslash R \) such that \( T = R \cup \{b\} \in F(\ell + 1) \) and \( T\backslash K(T) = R\backslash K \). Note that

\[
T\backslash K(T) = R\backslash K \iff R \cup \{b\}\backslash K(T) = R\backslash K \iff K(T) = K \cup \{b\}, \tag{A.3}
\]

and in particular, since \( K = K(S) \subseteq S \) and \( a \not\in K \) together imply \( K \subseteq S\backslash a = R \), we have \( b \not\in K \) (note that \( b \not\in R \) by definition). This yields

\[
|K \cup \{b\}| = |K| + 1 = \ell + 1. \tag{A.4}
\]

Consequently, if we can find some \( b \in \mathbb{Z}^+ \backslash R \) such that \( T = R \cup \{b\} \) satisfies \( K(T) = K \cup \{b\} \), then we necessarily have \( T \in F(\ell + 1) \) and \( T\backslash K(T) = R\backslash K \) and we would be done.
Now, consider $R_c = [r] \setminus \overline{R}$, and let $x$ be the $(j+1)$-th element in $R_c$. Note that $x \in \mathbb{Z}^+ \setminus R$ by definition. The asserted proof as stated in [53] is equivalent to the claim that $K(R \cup \{x\}) = K \cup \{x\}$. However, this is not true in general.

As an explicit counterexample, let $j = 3$, $k = 6$, $r = 10$. Consider

$$S = \{4, 6, 8, 9, 10, 15\}.$$

Let $\mathcal{F} \subseteq \mathcal{M}(6, 10, 3)$ be a shifted family generated by $S$. Note that

$$\hat{S} = \{8, 9, 10, 15\} \in \hat{\mathcal{F}}(2).$$

Take $a = 10$, and note that $\hat{S} \setminus \{a\} = \{8, 9, 15\} \in \Delta \hat{\mathcal{F}}(2)$.

Consider the following question: Is $\{8, 9, 15\} \in \hat{\mathcal{F}}(3)$? Suppose there is some $t \geq 3$ and some $T \in \mathcal{F}(t)$ such that $\hat{T} = \hat{S} \setminus \{a\}$. Since $T \in \mathcal{F}$ and $\mathcal{F}$ is generated by $S$, we must have $s(T) = 4$ (recall that $s(T)$ means the smallest element in $T$), so the assumption that $T \in \mathcal{F}(t)$ for some $t \geq 3$ forces $T \cap \mathbb{N}_7 \neq \emptyset$. Say $w \in T \cap \mathbb{N}_7$. Note that if $w \geq 17$, then $T \not\leq_p S$. Thus we are forced to have $w = 7$, which then yields $T \in \mathcal{F}(5)$ and $t = 5$. Therefore, $\{8, 9, 15\} \not\in \hat{\mathcal{F}}(3)$, which contradicts (A.1), i.e. Claim 4.2(ii) is not true.

We also tried to fix the rest of the proof of Theorem 4.1 in [53]. Unfortunately, there is no obvious fix, since another intermediate claim in the proof is also false (whose inaccuracy is an implication of applying Claim 4.2(ii)). Specifically, the statement "Then $\Delta \hat{I}(\ell - 1) = \hat{I}(\ell)$ holds for all $\ell$ except $\ell = \ell_0$" on line 6 of page 175 of the paper is not true. We can only conclude that

$$\hat{I}(\ell) \subseteq \Delta \hat{I}(\ell - 1) \subseteq \bigcup_{t=0}^{k-\ell} \mathcal{I}(\ell + t) \quad \text{(A.5)}$$

for $\ell \in [k]$, $\ell \neq \ell_0$. 

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