

ITERATED TRILINEAR FOURIER INTEGRALS WITH ARBITRARY SYMBOLS

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My research interests fall under the broad categories of harmonic analysis and partial differential equations, with a particular focus on multilinear harmonic analysis with time-frequency analysis techniques. I have been particularly interested in understanding the relationship between L^p estimates of multilinear singular operators and the dimension of the singularity sets of their symbols as well as applications to non-linear partial differential equations and other fields.

In this paper, I prove L^p estimates for trilinear multiplier operators with singular symbols. These operators arise in the study of iterated trilinear Fourier integrals, which are trilinear variants of the bilinear Hilbert transform. Specifically, I consider trilinear operators determined by multipliers that are products of two functions $m_1(\xi_1, \xi_2)$ and $m_2(\xi_2, \xi_3)$, such that the singular set of m_1 lies in the hyperplane $\xi_1 = \xi_2$ and that of m_2 lies in the hyperplane $\xi_2 = \xi_3$. While previous work [15] requires that the multipliers satisfy $\chi_{\xi_1 < \xi_2} \cdot \chi_{\xi_2 < \xi_3}$, my results allow for the case of the arbitrary multipliers, which have common singularities.

BIOGRAPHICAL SKETCH

Joeun Jung was born in Busan, South Korea and raised in Seoul, South Korea. In 2004, she obtained a Bachelor of Science in Mathematics with the minor in Mathematics Education from Chung-Ang University in Seoul and graduated Summa Cum Laude from the university. In 2004, she received the teacher certification and passed one of the competitive tests in South Korea to be assigned as a secondary school teacher. For two years, as a mathematics teacher in a secondary school, she was in charge of training the advanced students who prepared for the competitive mathematics olympiad in South Korea. In 2007, she entered the mathematics Ph.D. program at Cornell University. At Cornell, she conducted research on Multilinear Harmonic Analysis under the supervision of her advisor, Professor Camil Muscalu. In August 2015, she will become a researcher at PARC(PDE and Functional Analysis Research Center) of Seoul National University in Seoul, South Korea.

I dedicate this work to my grandmother, Chae-Ja Im,
who taught me the meaning of unconditional love and sacrifices.

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Chapter 1

Introduction

1.1 Background

L^p estimates for multilinear singular operators and their connections to other fields such as partial differential equations, ergodic theory, and probability, have been the focus of a great deal of mathematical activity in recent years.

In 1979, Coifman and Meyer [3] investigated the bilinear operators defined by:

$$B(f_1, f_2)(x) := \int_{\mathbf{R}^2} m(\xi_1, \xi_2) \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) e^{2\pi i x(\xi_1 + \xi_2)} d\xi_1 d\xi_2, \quad (1.1)$$

where the multiplier m satisfies the classical Marcinkiewicz-Mikhlin-Hörmander condition

$$|\partial^\alpha(m(\xi))| \lesssim \frac{1}{|\xi|^{|\alpha|}}$$

for sufficiently many multi-indices α and where $\xi := (\xi_1, \xi_2)$. Here, \widehat{f} denotes the Fourier transform, which is defined by $\widehat{f}(\eta) = \int_{\mathbf{R}} e^{-2\pi i x \eta} f(x) dx$, and $A \lesssim B$ denotes the assertion that $A \leq CB$ for some large constant C . Specifically, they obtained the following results:

Theorem 1.1.1. *The operator B maps $L^{p_1} \times L^{p_2} \rightarrow L^{p_3}$, provided $1 < p_1, p_2 \leq \infty$ and $0 < p_3 < \infty$, where $1/p_1 + 1/p_2 = 1/p_3$.*

If m is identically equal to one, then the operator B is the pointwise product operator. Thus the wide range of L^p estimates for the operators B serves as a generalization of Hölder's inequality in which products are replaced by classical paraproducts, which satisfies the classical Marcinkiewicz-Mikhlin-Hörmander condition. Theorem 1.1.1 is critical to understanding the general Korteweg-de Vries equation, which describes weakly non-linear shallow water waves, and the Leibnitz rule, which refers to inequalities of the type

$$\|D^\alpha(fg)\|_p \lesssim \|D^\alpha f\|_{p_1} \|g\|_{q_1} + \|f\|_{p_2} \|D^\alpha g\|_{q_2}, \quad (1.2)$$

where $1 < p_i, q_i \leq \infty$, $1/p_i + 1/q_i = 1/p$, for $i = 1, 2$, and $1/(1 + \alpha) < p < \infty$.

If the multiplier m in (1.1) is replaced with $\chi_{\xi_1 < \xi_2}$, which is singular along the line $\{\xi_1 = \xi_2\}$, one establishes the bilinear Hilbert transform(BHT) given by the equation

$$BHT(f_1, f_2)(x) := \int_{\xi_1 < \xi_2} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) e^{2\pi i x(\xi_1 + \xi_2)} d\xi_1 d\xi_2, \quad (1.3)$$

modulo minor modification, where f_1, f_2 are test functions on \mathbb{R} . The bilinear Hilbert transform was introduced by Calderón when he understood the first Calderón commutator in connection to the Cauchy integral on Lipschitz curves. Lacey and Thiele, [9] in 1997 and [10] in 1999, provide a wide range of L^p estimates for the bilinear Hilbert transform. Specifically, they announced the following theorem:

Theorem 1.1.2. *The operator BHT maps $L^{p_1} \times L^{p_2} \rightarrow L^{p_3}$, provided $1 < p_1, p_2 \leq \infty$*

and $2/3 < p_3 < \infty$, where $1/p_1 + 1/p_2 = 1/p_3$.

Work of Muscalu, Tao, Thiele [15] in 2004 provides Hölder type L^p estimates on a trilinear variant of the bilinear Hilbert transform given by

$$T(f_1, f_2, f_3)(x) := \int_{\xi_1 < \xi_2 < \xi_3} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) e^{2\pi i x(\xi_1 + \xi_2 + \xi_3)} d\xi_1 d\xi_2 d\xi_3. \quad (1.4)$$

The operator T may be considered to be the continuous analogue of an iterated Fourier series with a product of three functions. The operator T has deep connections to AKNS systems (named after Ablowitz-Kaup-Newell-Segur), which describe various models in partial differential equations, including the Korteweg-de Vries and non-linear Schrödinger equations.

1.2 Statement of Result

We now construct trilinear operators $T_{m_1 m_2}$ with more generic symbols with the same singularity sets as the one of T in (1.4). We define such an operator by:

$$T_{m_1 m_2}(f_1, f_2, f_3)(x) := \int_{\mathbb{R}^3} m_1(\xi_1, \xi_2) m_2(\xi_2, \xi_3) \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) e^{2\pi i x(\xi_1 + \xi_2 + \xi_3)} d\xi_1 d\xi_2 d\xi_3, \quad (1.5)$$

where $m_i(\xi_i, \xi_{i+1})$, for each $i = 1, 2$, is smooth away from the line

$$\Gamma_i = \{(\xi_i, \xi_{i+1}) \in \mathbb{R}^2 : \xi_i = \xi_{i+1}\} \quad (1.6)$$

and satisfies the condition

$$|\partial^\alpha(m_i(\xi))| \lesssim \frac{1}{\text{dist}(\Gamma_i, \xi)^{|\alpha|}}$$

for every $\xi \in \mathbb{R}^2 \setminus \Gamma_i$ and sufficiently many multi-indices α .

If we consider the singular symbol $\chi_{\xi_1 < \xi_2 < \xi_3}$ of T as described in (1.4) as $\chi_{\xi_1 < \xi_2} \cdot \chi_{\xi_2 < \xi_3}$, then it is easy to see that the symbol $m_1(\xi_1, \xi_2) \cdot m_2(\xi_2, \xi_3)$ is a natural variant of $\chi_{\xi_1 < \xi_2} \cdot \chi_{\xi_2 < \xi_3}$ with the same singular sets along two hyperplanes, $\xi_1 = \xi_2$ and $\xi_2 = \xi_3$. A simple example for this symbol is $a(\xi_1 - \xi_2) \cdot b(\xi_2 - \xi_3)$ where a and b are the classical Marcinkiewicz-Mikhlin-Hörmander symbols. If we replace both $m_1(\xi_1, \xi_2)$ and $m_2(\xi_2, \xi_3)$ with classical symbols $a(\xi_1, \xi_2)$ and $b(\xi_2, \xi_3)$ satisfying the classical condition, then we obtain an example of flag paraproducts

$$T_{ab}(f_1, f_2, f_3)(x) := \int_{\mathbb{R}^3} a(\xi_1, \xi_2) b(\xi_2, \xi_3) \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) e^{2\pi i x(\xi_1 + \xi_2 + \xi_3)} d\xi_1 d\xi_2 d\xi_3,$$

which were introduced by Muscalu [11] in 2007, who proved a variety of Hölder type L^p estimates for this family. Flag paraproducts arise naturally in many settings, including non-linear partial differential equations and probability theory. In fact, we may consider the operator $T_{m_1 m_2}$ as an infinite sum of flag paraproducts. This is why we need to introduce wave packets associated to tiles, which are discussed later, as is done in the proof in [9] of L^p estimates for the bilinear Hilbert transform, in order to overcome this complexity.

Before stating the main result of this paper, we review the estimates proved by Muscalu, Tao, Thiele [15] in 2004 for the closely related operator T in (1.4).

Let us consider the 3-dimensional affine hyperspace

$$S := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbf{R}^4 \mid \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1\}.$$

We denote by \mathbf{D}' the open interior of the convex hull of the following twelve extremal points A_1, \dots, A_{12} , which belong to S :

$$\begin{aligned} A_1 &= (1, 1/2, 1, -3/2) & A_2 &= (1/2, 1, 1, -3/2) & A_3 &= (1/2, 1, -3/2, 1) \\ A_4 &= (1, 1/2, -3/2, 1) & A_5 &= (1, -1/2, 0, 1/2) & A_6 &= (1, -1/2, 1/2, 0) \\ A_7 &= (1/2, -1/2, 0, 1) & A_8 &= (1/2, -1/2, 1, 0) & A_9 &= (-1/2, 1, 0, 1/2) \\ A_{10} &= (-1/2, 1, 1/2, 0) & A_{11} &= (-1/2, 1/2, 1, 0) & A_{12} &= (-1/2, 1/2, 0, 1). \end{aligned}$$

In addition, we denote by \mathbf{D}'' the open interior of the convex hull of the twelve extremal points, $\tilde{A}_1, \dots, \tilde{A}_{12}$, in S , where the points \tilde{A}_j are given by exchanging the 1st coordinate and the 3rd coordinate of A_j for $j = 1, \dots, 12$. Then, we set $\mathbf{D} := \mathbf{D}' \cap \mathbf{D}''$.

Theorem 1.2.1. *Let $1 < p_1, p_2, p_3 \leq \infty$ and $0 < p_4 < \infty$. The operator T maps $L^{p_1} \times L^{p_2} \times L^{p_3} \rightarrow L^{p_4}$, provided $(1/p_1, 1/p_2, 1/p_3, 1 - 1/p_4) \in \mathbf{D}$.*

In particular, T maps $L^{p_1} \times L^{p_2} \times L^{p_3} \rightarrow L^{p_4}$ as long as $1 < p_1, p_2, p_3 \leq \infty$ and $1 \leq p_4 < \infty$.

One of the essential ideas in [15] is to consider the symbol $\chi_{\xi_1 < \xi_2} \cdot \chi_{\xi_2 < \xi_3}$ as $\chi_{\xi_1 < \xi_2} \cdot \chi_{\frac{\xi_1 + \xi_2}{2} < \xi_3}$ in the region $\{|\xi_3 - \xi_2| \gg |\xi_2 - \xi_1|\}$, and, similarly, to consider $\chi_{\xi_1 < \xi_2} \cdot \chi_{\xi_2 < \xi_3}$ as $\chi_{\xi_1 < \frac{\xi_2 + \xi_3}{2}} \cdot \chi_{\xi_2 < \xi_3}$ in the region $\{|\xi_3 - \xi_2| \ll |\xi_2 - \xi_1|\}$. This observation enabled the authors to decompose the multiplier operator corresponding to each region with a composition of two bilinear Hilbert transforms with some constraint on the inner bilinear Hilbert transform. This decomposi-

tion enabled the authors to establish its L^p estimates by applying the idea from the corresponding proof for the bilinear Hilbert transform. However, we can not apply this technique to $T_{m_1 m_2}$, because the symbols $m_1(\xi_1, \xi_2) \cdot m_2(\xi_2, \xi_3)$ are truly dependent on its variables, so we cannot rely on the benefit from the characteristic functions; This makes the whole proof here is more challenging.

The main purpose of this paper is to obtain a large set of L^p estimates for $T_{m_1 m_2}$. Specifically, we obtain the following result:

Theorem 1.2.2. *The operator $T_{m_1 m_2}$ maps*

$$T_{m_1 m_2} : L^{p_1} \times L^{p_2} \times L^{p_3} \rightarrow L^{p_4}, \quad (1.7)$$

provided $(1/p_1, 1/p_2, 1/p_3, 1 - 1/p_4) \in \mathbf{D}$, where $1 < p_1, p_2, p_3 \leq \infty$, $0 < p_4 < \infty$, and $1/p_1 + 1/p_2 + 1/p_3 = 1/p_4$.

The notation and techniques used here to prove Theorem 1.2.2 are inspired by the work of Muscalu, Tao and Thiele in [14] and [15].

Chapter 2

Restricted weak-type Interpolation

In this chapter, we will review the restricted weak-type interpolation theorems from [17]. These theorems allow us to reduce multilinear L^p estimates, such as those in Theorem 1.2.2, to certain restricted weak-type estimates.

To prove the L^p estimates on $T_{m_1 m_2}$, it is convenient to use duality to convert (1.7) from a trilinear operator estimate to a quadrilinear form estimate because this makes the estimate more symmetric. We shall investigate the following quadrilinear form Λ associated to $T_{m_1 m_2}$:

$$\Lambda(f_1, f_2, f_3, f_4) := \int_{\mathbf{R}} T_{m_1 m_2}(f_1, f_2, f_3)(x) f_4(x) dx.$$

The assertion that the operator $T_{m_1 m_2}$ maps from $L^{p_1} \times L^{p_2} \times L^{p_3}$ to L^{p_4} is equivalent to the assertion that Λ is bounded on $L^{p_1} \times L^{p_2} \times L^{p_3} \times L^{p'_4}$, for $1 < p_4 < \infty$, where $1/p_4 + 1/p'_4 = 1$. For $p_4 \leq 1$, this simple duality relationship breaks down, but the interpolation arguments in [17] will enable us to reduce the statement in (1.7) to certain restricted type estimates on Λ . As in [17], we find it more convenient to work with the quantities $\alpha_i = 1/p_i$, $i = 1, 2, 3$, and

$\alpha_4 = 1/p'_4$, where p_i stands for the exponent of L^{p_i} .

Definition 2.0.3. A tuple $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with $-\infty < \alpha_i < 1$, for all $1 \leq i \leq 4$, is called admissible if $\sum_{i=1}^4 \alpha_i = 1$ and there is at most one index j such that $\alpha_j < 0$. We call an index i good if $\alpha_i \geq 0$, and we call it bad if $\alpha_i < 0$. A good tuple is an admissible tuple which has no bad indices and a bad tuple is an admissible tuple with one bad index.

Definition 2.0.4. Let E be a subset $E \subset \mathbb{R}$ with finite measure and E' be a subset of E . Define that E' is a major subset of E , provided $2|E'| \geq |E|$.

Definition 2.0.5. Let E be a subset $E \subset \mathbb{R}$ with finite measure. $X(E)$ denotes the space of all functions f with $|f| \leq \chi_E$ almost everywhere.

Definition 2.0.6. Fix an admissible tuple $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. A quadrilinear form Λ is of restricted type α if for each tuple (E_1, E_2, E_3, E_4) of subsets of \mathbb{R} with finite measure, there exists a constant C and a major subset E'_j of E_j , for each (one or none) bad index j , such that

$$|\Lambda(f_1, f_2, f_3, f_4)| \leq C|E_1|^{\alpha_1}|E_2|^{\alpha_2}|E_3|^{\alpha_3}|E_4|^{\alpha_4}$$

for each tuple (f_1, f_2, f_3, f_4) of functions with $f_j \in X(E'_j)$ when j is a bad index (if exists), and $f_i \in X(E_i)$ when i is a good index.

The following restricted type result will be proved in the last chapter:

Theorem 2.0.7. For every vertex A_i, \tilde{A}_i , $i = 1, \dots, 12$, there exists an admissible tuple α arbitrarily close to the vertex such that the form Λ is of restricted type α .

By interpolation of restricted weak-type estimates in [17], we thus obtain that Theorem 2.0.7 implies that Λ is of restricted type α for any admissible tuples $\alpha \in \mathbf{D}$.

It only remains to convert these restricted type estimates into strong type estimates (1.7). To do this, one can just apply (exactly as in [17]) the multilinear Marcinkiewicz interpolation theorem in [8] for good tuples and the interpolation lemma 3.11 in [17] for bad tuples. This completes the proof of Theorem 1.2.2. Therefore, it remains to prove Theorem 2.0.7 here.

Chapter 3

Discretization

In order to prove Theorem 2.0.7, we first discretize the continuous form Λ and reduce the relevant result to estimates for a discretized variant involving sums of inner products with wave packets. We begin this chapter by recalling some standard definitions from [15].

3.1 Definitions

For any interval I , we denote by $|I|$ the Lebesgue measure of I and by cI with $c > 0$ the interval with the same center as I but c times the side-length. For any (quasi) cube and square Q , we denote by $|Q|$ its side-length and by cQ with $c > 0$ the cube with the same center as Q but with the side-length $c|Q|$.

Definition 3.1.1. Fix a positive integer n and a n -tuple $\sigma \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$. Define the shifted n -dyadic grid $D = D_\sigma^n$ to be the collection of cubes of the form

$$D_\sigma^n := \{2^j(k + (0, 1)^n + (-1)^j\sigma) \mid j \in \mathbf{Z}, k \in \mathbf{Z}^n\}.$$

Definition 3.1.2. A subset D' of a shifted n -dyadic grid D is called sparse if it satisfies the following properties, for any two cubes Q, Q' in D with $Q \neq Q'$:

1. $|Q| < |Q'|$ implies $10^9|Q| < |Q'|$.
2. $|Q| = |Q'|$ implies $10^9Q \cap 10^9Q' = \emptyset$.

We observe that any subset of a shifted n -dyadic grid with $n \leq 3$ can be split into $O(1)$ sparse subsets.

Definition 3.1.3. Let a 3-tuple $\sigma \in \{0, \frac{1}{3}, \frac{2}{3}\}^3$ be a shift. A collection $\mathbf{Q} \subset D_\sigma^3$ of cubes is said to have rank 1 if it satisfies the following properties for all $Q, Q' \in \mathbf{Q}$:

1. If $Q \neq Q'$, then $Q \cap Q' = \emptyset$.
2. If $Q \neq Q'$, then $Q_i \neq Q'_i$ for all $i = 1, 2, 3$.
3. If $3Q'_i \subset 3Q_i$ for some $i = 1, 2, 3$, then $10^7Q'_j \subset 10^7Q_j$ for all $j = 1, 2, 3$.
4. If $|Q'| < 10^9|Q|$ and $3Q'_i \subset 3Q_i$ for some $i = 1, 2, 3$, then $3Q'_i \cap 3Q_j = \emptyset$ for all $j \neq i$.

Definition 3.1.4. Fix $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \{0, \frac{1}{3}, \frac{2}{3}\}^3$ and fix $1 \leq i \leq 3$. An i -tile with shift σ_i is a rectangle $P = I_P \times \omega_P$ in the phase plane such that $I_P \in D_0^1$, $\omega_P \in D_{\sigma_i}^1$, and $|I_P| \cdot |\omega_P| = 1$. A tri-tile with shift σ is a 3-tuple $\vec{P} = (P_1, P_2, P_3)$ such that each P_i is an i -tile with shift σ_i , and each I_{P_i} is independent of i , which we denote $I_{\vec{P}}$. The frequency cube $Q_{\vec{P}}$ of a tri-tile \vec{P} is defined by $\prod_{i=1}^3 \omega_{P_i}$.

We shall sometimes refer to i -tiles with shift σ_i just as i -tiles, or even as tiles, if the parameters σ_i, i are insignificant.

Definition 3.1.5. A set $\vec{\mathbf{P}}$ of tri-tiles is called sparse, if all tri-tiles in $\vec{\mathbf{P}}$ have the same shift and the set of the frequency cube $Q_{\vec{P}}$ with $\vec{P} \in \vec{\mathbf{P}}$ is sparse.

Similarly, observe that any set of tri-tiles can be split into $O(1)$ sparse subsets.

Definition 3.1.6. Let P and P' be tiles. We define the following notations:

1. $P' < P$ if $I_{P'} \subsetneq I_P$ and $3\omega_{P'} \subseteq 3\omega_P$.
2. $P' \leq P$ if $P' < P$ or $P' = P$.
3. $P' \lesssim P$ if $I_{P'} \subseteq I_P$ and $10^7\omega_{P'} \subseteq 10^7\omega_P$.
4. $P' \lesssim' P$ if $P' \lesssim P$ and $P' \not\leq P$.

Definition 3.1.7. A collection $\vec{\mathbf{P}}$ of tri-tiles is said to have rank 1 if it satisfies the following properties for all $\vec{P}, \vec{P}' \in \vec{\mathbf{P}}$:

1. If $\vec{P} \neq \vec{P}'$, then $P_i \neq P'_i$ for all $i = 1, 2, 3$.
2. If $P'_i \leq P_i$ for some $i = 1, 2, 3$, then $P'_j \lesssim P_j$ for all $j = 1, 2, 3$.
3. If $P'_i \leq P_i$ for some $i = 1, 2, 3$ and furthermore $|I_{\vec{P}'}| < 10^9|I_{\vec{P}}|$, then we have $P'_j \lesssim' P_j$ for all $j \neq i$.

Definition 3.1.8. Let P be a tile. A wave packet Φ_P adapted to P is a function such that $\widehat{\Phi}_P$ is supported in $\frac{9}{10}\omega_P$ and

$$|\Phi_P(x)| \lesssim |I_P|^{-1/2} \widetilde{\chi}_{I_P}(x)^M \quad (3.1)$$

for all $M > 0$, with the implicit constant depending on M . Here, $\widetilde{\chi}_{I_P}$ denotes the approximate cutoff function of the interval I_P , which is defined by

$$\widetilde{\chi}_{I_P}(x) := \left(1 + \left(\frac{|x - x_{I_P}|}{|I_P|}\right)^2\right)^{-1/2},$$

where x_{I_P} is the center of the interval I_P .

3.2 Transition to Discretized Model Operator

The discretized variant of Theorem 2.0.7 is as follows:

Theorem 3.2.1. *Let $\sigma, \sigma' \in \{0, \frac{1}{3}, \frac{2}{3}\}^3$ be shifts, and let $\vec{\mathbf{P}}, \vec{\mathbf{Q}}$ be finite collections of tri-tiles with shifts σ, σ' , respectively, such that both $\vec{\mathbf{P}}$ and $\vec{\mathbf{Q}}$ have rank 1. For each $\vec{P} \in \vec{\mathbf{P}}$, and $\vec{Q} \in \vec{\mathbf{Q}}$, let $\Phi_{P_i} = \Phi_{P_i, i}$ and $\Phi_{Q_i} = \Phi_{Q_i, i}$ be wave packets on P_i and Q_i , $i = 1, 2, 3$, respectively. Define the forms $\Lambda_{\vec{\mathbf{P}}, \vec{\mathbf{Q}}}$ and $\Lambda_{\vec{\mathbf{P}}, \vec{\mathbf{Q}}}^\#$ by*

$$\Lambda_{\vec{\mathbf{P}}, \vec{\mathbf{Q}}}^\#(f_1, f_2, f_3, f_4) := \sum_{\vec{P} \in \vec{\mathbf{P}}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle B_{P_1}^\#(f_1, f_2), \Phi_{P_1} \rangle \langle f_3, \Phi_{P_2} \rangle \langle f_4, \Phi_{P_3} \rangle,$$

where

$$B_{P_1}^\#(f_1, f_2) := \sum_{\substack{\vec{Q} \in \vec{\mathbf{Q}}: \\ \omega_{Q_3} \subset \omega_{P_1}, 2^\# |\omega_{Q_3}| \sim |\omega_{P_1}|}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_1, \Phi_{Q_1} \rangle \langle f_2, \Phi_{Q_2} \rangle \Phi_{Q_3}$$

and

$$\Lambda_{\vec{\mathbf{P}}, \vec{\mathbf{Q}}}(f_1, f_2, f_3, f_4) := \sum_{\ell=1}^M \sum_{\# \geq 1000} 2^{-(\#+1)\ell} \Lambda_{\vec{\mathbf{P}}, \vec{\mathbf{Q}}}^\#(f_1, f_2, f_3, f_4)$$

for $\# \geq 1000$ and a sufficiently big number $M \in \mathbf{Z}$. Then, $\Lambda_{\vec{\mathbf{P}}, \vec{\mathbf{Q}}}^\#$ is of restricted type α , modulo the extra factor $2^{\#/2}$, for all admissible tuples $\alpha \in \mathbf{D}$, uniformly in the parameters $\sigma, \sigma', \vec{\mathbf{P}}, \vec{\mathbf{Q}}, \Phi_{P_i}, \Phi_{Q_i}$. Therefore, for any positive integer M , $\Lambda_{\vec{\mathbf{P}}, \vec{\mathbf{Q}}}$ is of restricted type α , for all admissible tuples $\alpha \in \mathbf{D}$, uniformly in the parameters $\sigma, \sigma', \vec{\mathbf{P}}, \vec{\mathbf{Q}}, \Phi_{P_i}, \Phi_{Q_i}$. Furthermore, in the case that α has a bad index j , the major subset E_j' can be chosen independently of the parameters $\sigma, \sigma', \vec{\mathbf{P}}, \vec{\mathbf{Q}}, \Phi_{P_i}, \Phi_{Q_i}$.

Note that once we obtain the first conclusion in Theorem 3.2.1, say

$$|\Lambda_{\vec{\mathbf{P}}, \vec{\mathbf{Q}}}^\#(f_1, f_2, f_3, f_4)| \lesssim 2^{\#/2} |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3} |E_4|^{\alpha_4} \quad (3.2)$$

for given tuple (E_1, E_2, E_3, E_4) of subsets of \mathbb{R} with finite measure and for each tuple (f_1, f_2, f_3, f_4) of functions with $f_j \in X(E'_j)$ when j (if exists) is a bad index, and $f_i \in X(E_i)$ when i is a good index, then we obtain the following conclusion:

$$\begin{aligned}
|\Lambda_{\vec{P}, \vec{Q}}(f_1, f_2, f_3, f_4)| &\leq \sum_{\ell=1}^M \sum_{\#\geq 1000} 2^{-(\#+1)\ell} |\Lambda_{\vec{P}, \vec{Q}}^{\#}(f_1, f_2, f_3, f_4)| \\
&\lesssim \sum_{\ell=1}^M \sum_{\#\geq 1000} 2^{-(\#+1)\ell} 2^{\#/2} |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3} |E_4|^{\alpha_4} \\
&\leq |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3} |E_4|^{\alpha_4}
\end{aligned}$$

for any positive integer M . Henceforth, we shall focus on showing the first conclusion.

The remainder for this section is devoted to showing how Theorem 2.0.7 can be deduced from Theorem 3.2.1. First, we divide the symbol $m_1 m_2$ in (1.5) as

$$\begin{aligned}
m_1(\xi_1, \xi_2) &= \sum_Q m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2), \\
m_2(\xi_2, \xi_3) &= \sum_{Q'} m_2(\xi_2, \xi_3) \phi_{Q'}(\xi_2, \xi_3)
\end{aligned}$$

by a standard partition of unity, where Q, Q' are the shifted dyadic squares such that $Q = Q_1 \times Q_2$ and $Q' = Q'_1 \times Q'_2$. Here, Q_j, Q'_j are shifted dyadic intervals $2^k(n + (1, 0) + \sigma)$, for any $k, n \in \mathbb{Z}$ and $\sigma \in \{0, 1/3, 2/3\}$, satisfying the

property

$$\text{dist}(Q, \Gamma_1) \simeq C_0|Q|,$$

$$\text{dist}(Q', \Gamma_2) \simeq C'_0|Q'|,$$

where C_0, C'_0 are fixed large constants and Γ_j , for each $j = 1, 2$, is defined by (1.6). Here, each $\phi_Q, \phi_{Q'}$ are smooth bumps adapted to Q, Q' and supported in $\frac{8}{10}Q, \frac{8}{10}Q'$, respectively. Then, we obtain

$$\begin{aligned} m_1(\xi_1, \xi_2)m_2(\xi_2, \xi_3) &= \sum_{Q, Q'} m_1(\xi_1, \xi_2)\phi_Q(\xi_1, \xi_2) \cdot m_2(\xi_2, \xi_3)\phi_{Q'}(\xi_2, \xi_3) \\ &= \sum_{|Q| \ll |Q'|} + \sum_{|Q| \sim |Q'|} + \sum_{|Q| \gg |Q'|} := m_{\text{I}} + m_{\text{III}} + m_{\text{III}}, \end{aligned}$$

which is essentially similar to looking at the symbol separately in the three regions $\chi_{|\xi_2 - \xi_1| \ll |\xi_3 - \xi_2|}$, $\chi_{|\xi_2 - \xi_1| \sim |\xi_3 - \xi_2|}$ and $\chi_{|\xi_2 - \xi_1| \gg |\xi_3 - \xi_2|}$.

The multiplier operator T_{III} defined by

$$T_{\text{III}}(f_1, f_2, f_3)(x) := \int_{\mathbb{R}^3} m_{\text{III}}(\xi_1, \xi_2, \xi_3) \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) e^{2\pi i x(\xi_1 + \xi_2 + \xi_3)} d\xi_1 d\xi_2 d\xi_3$$

is similar to the Bilinear Hilbert Transform with three functions. Specifically, it satisfies

$$|\partial^\alpha(m_{\text{III}}(\xi))| \lesssim \frac{1}{\text{dist}(\Gamma, \xi)^{|\alpha|}}$$

for every $\xi \in \mathbb{R}^3 \setminus \Gamma$ and sufficiently many multi-indices α , where Γ is given by the line $\Gamma := \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 = \xi_2 = \xi_3\}$. In fact, such operators were studied in [17] and Muscalu, Tao and Thiele established a wider range of L^p estimates for them than those in Theorem 1.2.2.

Now we claim that the multiplier operator $T_{\mathbb{I}}$ given by

$$T_{\mathbb{I}}(f_1, f_2, f_3)(x) := \int_{\mathbf{R}^3} m_{\mathbb{I}}(\xi_1, \xi_2, \xi_3) \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) e^{2\pi i x(\xi_1 + \xi_2 + \xi_3)} d\xi_1 d\xi_2 d\xi_3$$

maps $L^{p_1} \times L^{p_2} \times L^{p_3} \rightarrow L^{p_4}$, provided $(1/p_1, 1/p_2, 1/p_3, 1-1/p_4) \in \mathbf{D}'$, where $1 < p_1, p_2, p_3 \leq \infty, 0 < p_4 < \infty$, and $1/p_1 + 1/p_2 + 1/p_3 = 1/p_4$. And similarly, claim that the operator $T_{\mathbb{III}}$ with the singular symbol $m_{\mathbb{III}}$ has the same L^p estimates as long as $(1/p_1, 1/p_2, 1/p_3, 1 - 1/p_4) \in \mathbf{D}''$. If these claims for $T_{\mathbb{I}}$ and $T_{\mathbb{III}}$ hold, then we finally obtain Theorem 1.2.2. We shall only prove the claim for $T_{\mathbb{I}}$, as the claim for $T_{\mathbb{III}}$ follows by a permutation of the indices 1 and 3.

Consider

$$\begin{aligned} m_{\mathbb{I}} &= \sum_{|Q| \ll |Q'|} m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \cdot m_2(\xi_2, \xi_3) \phi_{Q'}(\xi_2, \xi_3) \\ &= \sum_Q m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \left(\sum_{Q': |Q| \ll |Q'|} m_2(\xi_2, \xi_3) \phi_{Q'}(\xi_2, \xi_3) \right) \\ &= \sum_Q m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \left(\sum_{Q': |Q| \ll |Q'|} \sum_{\mathbf{n} \in \mathbf{Z}^2} C_{\mathbf{n}}^{Q'} \phi_{Q'_1, \mathbf{n}, 1}(\xi_2) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3) \right). \end{aligned} \quad (3.3)$$

We establish the last equality by computing a double Fourier series of $m_2(\xi_2, \xi_3) \phi_{Q'}(\xi_2, \xi_3)$, where $\phi_{Q'_i, \mathbf{n}, i}$ is a bump function adapted to Q'_i and supported to $\frac{9}{10}Q'_i$ uniformly in n_i for $(n_1, n_2) := \mathbf{n} \in \mathbf{Z}^2$. Here, the Fourier coefficient is given by

$$C_{\mathbf{n}}^{Q'} = \frac{1}{|Q'|^2} \int_{\mathbf{R}^2} m_2(\xi_2, \xi_3) \phi_{Q'}(\xi_2, \xi_3) e^{-2\pi i n_1 \frac{\xi_2}{|Q'|}} e^{-2\pi i n_2 \frac{\xi_3}{|Q'|}} d\xi_2 d\xi_3. \quad (3.4)$$

We now assert that $|C_{\mathbf{n}}^{Q'}| \lesssim C(\mathbf{n})$, where the implicit constant does not depend on each Q' , and $C(\mathbf{n})$ is a rapidly decreasing sequence. This assertion is

justified later in Chapter 4.

Then, we can majorize (3.3) by

$$\sum_{\mathbf{n} \in \mathbb{Z}^2} C(\mathbf{n}) \sum_{Q, Q'; |Q| \ll |Q'|} m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \phi_{Q'_1, \mathbf{n}, 1}(\xi_2) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3). \quad (3.5)$$

Because of the big decaying factor $C(\mathbf{n})$, once we have L^p estimates of the corresponding trilinear operator with the symbol

$$\sum_{Q, Q'; |Q| \ll |Q'|} m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \phi_{Q'_1, \mathbf{n}, 1}(\xi_2) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3), \quad (3.6)$$

then we can control the last summation with respect to $\mathbf{n} \in \mathbb{Z}^2$.

Now by applying the Taylor series to $\phi_{Q'_1, \mathbf{n}, 1}(\xi_2)$, we obtain that

$$\phi_{Q'_1, \mathbf{n}, 1}(\xi_2) = \sum_{\ell=0}^M \phi_{Q'_1, \mathbf{n}, 1}^{(\ell)} \left(\frac{\xi_1 + \xi_2}{2} \right) \left(\frac{\xi_2 - \xi_1}{2} \right)^\ell \frac{1}{\ell!} + R_M(\xi_1, \xi_2)$$

for a sufficiently big number $M \in \mathbb{Z}$. Here $R_M(\xi_1, \xi_2)$ is the remainder term given by

$$R_M(\xi_1, \xi_2) = \frac{1}{M!} \phi_{Q'_1, \mathbf{n}, 1}^{(M)}(\xi_\theta) \left(\frac{\xi_2 - \xi_1}{2} \right)^M, \quad (3.7)$$

where $\xi_\theta = (1 - \frac{\theta}{2})\xi_2 + \frac{\theta}{2}\xi_1$ for some θ between 0 and 1.

We now split (3.6) into three terms with $\ell = 0$, with $1 \leq \ell < M$, and with the remainder term $R_M(\xi_1, \xi_2)$ as following:

$$\begin{aligned}
& \sum_{Q, Q'; |Q| \ll |Q'|} m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \phi_{Q'_1, \mathbf{n}, 1} \left(\frac{\xi_1 + \xi_2}{2} \right) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3) \\
& + \sum_{\ell=1}^{M-1} \sum_{Q, Q'; |Q| \ll |Q'|} m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \phi_{Q'_1, \mathbf{n}, 1}^{(\ell)} \left(\frac{\xi_1 + \xi_2}{2} \right) \left(\frac{\xi_2 - \xi_1}{2} \right)^\ell \frac{1}{\ell!} \phi_{Q'_2, \mathbf{n}, 2}(\xi_3) \\
& + \sum_{Q, Q'; |Q| \ll |Q'|} m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) R_M(\xi_1, \xi_2) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3) \\
& := m_{\mathbb{I}, \{\ell=0\}} + \sum_{\ell=1}^{M-1} m_{\mathbb{I}, \ell} + m_{\mathbb{I}, R_M}. \tag{3.8}
\end{aligned}$$

We can easily check that $m_{\mathbb{I}, \{\ell=0\}}$ is similar to the symbol of the operator T in (1.4) after splitting $m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2)$ as a double Fourier series in ξ_1, ξ_2 (modulo extra outer summation and a rapidly decaying factor from the Fourier coefficients, like $C(\mathbf{n})$ in (3.5)). Thus, we now consider the other two cases in (3.8) more carefully.

For $1 \leq \ell < M$, we rewrite $m_{\mathbb{I}, \ell}$ as

$$\sum_{\substack{\# \geq 1000; \\ Q, Q'; \\ k_2 - k_1 = \#}} m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \phi_{Q'_1, \mathbf{n}, 1}^{(\ell)} \left(\frac{\xi_1 + \xi_2}{2} \right) \left(\frac{\xi_2 - \xi_1}{2} \right)^\ell \frac{1}{\ell!} \phi_{Q'_2, \mathbf{n}, 2}(\xi_3) \tag{3.9}$$

by letting $|Q| = 2^{k_1}$ and $|Q'| = 2^{k_2}$, for $k_1, k_2 \in \mathbf{Z}$, and by assuming that $k_2 = k_1 + \#$, with $\# \geq 1000$ as $k_1 \ll k_2$. Then (3.9) is equal to

$$\sum_{\substack{\# \geq 1000; \\ Q, Q'; \\ k_2 - k_1 = \#}} 2^{(k_1-1)\ell} \left[m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \left(\frac{\xi_2 - \xi_1}{2^{k_1}} \right)^\ell \frac{1}{\ell!} \right] \phi_{Q'_1, \mathbf{n}, 1}^{(\ell)} \left(\frac{\xi_1 + \xi_2}{2} \right) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3)$$

$$= \sum_{\# \geq 1000} \sum_{\substack{Q, Q'; \\ 2^{\#|Q|} \sim |Q'|}} \frac{2^{(k_1-1)\ell}}{2^{k_2\ell}} \left[m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \left(\frac{\xi_2 - \xi_1}{2^{k_1}} \right)^\ell \frac{1}{\ell!} \right] \cdot 2^{k_2\ell} \phi_{Q'_1, \mathbf{n}, 1}^{(\ell)} \left(\frac{\xi_1 + \xi_2}{2} \right) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3)$$

$$= \sum_{\# \geq 1000} 2^{-(\#+1)\ell} \sum_{\substack{Q, Q'; \\ 2^{\#|Q|} \sim |Q'|}} \left[\sum_{\mathbf{s} \in \mathbb{Z}^2} C_{\mathbf{s}}^{Q, \ell} \phi_{Q_1, \mathbf{s}, 1}(\xi_1) \phi_{Q_2, \mathbf{s}, 2}(\xi_2) \right] \cdot 2^{k_2\ell} \phi_{Q'_1, \mathbf{n}, 1}^{(\ell)} \left(\frac{\xi_1 + \xi_2}{2} \right) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3)$$

by computing a double Fourier series of $m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \left(\frac{\xi_2 - \xi_1}{2^{k_1}} \right)^\ell \frac{1}{\ell!}$, where $\phi_{Q_i, \mathbf{s}, i}$ is a bump function adapted to Q_i and supported to $\frac{9}{10}Q_i$ uniformly in s_i for $(s_1, s_2) := \mathbf{s} \in \mathbb{Z}^2$. Here, the Fourier coefficient $C_{\mathbf{s}}^{Q, \ell}$ is given by

$$C_{\mathbf{s}}^{Q, \ell} = \frac{1}{2^{2k_1}} \int_{\mathbb{R}^2} m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \left(\frac{\xi_2 - \xi_1}{2^{k_1}} \right)^\ell \frac{1}{\ell!} e^{-2\pi i s_1 \frac{\xi_1}{2^{k_1}}} e^{-2\pi i s_2 \frac{\xi_2}{2^{k_1}}} d\xi_1 d\xi_2. \quad (3.10)$$

We now assert that $|C_{\mathbf{s}}^{Q, \ell}| \lesssim C(\mathbf{s})$, where the implicit constant does not depend on Q and ℓ , and $C(\mathbf{s})$ is a rapidly decreasing sequence. This assertion is justified later in Chapter 4. Then, we can majorize $m_{\mathbb{I}, \ell}$ by

$$\sum_{\mathbf{s} \in \mathbb{Z}^2} C(\mathbf{s}) \sum_{\# \geq 1000} 2^{-(\#+1)\ell} \sum_{\substack{Q, Q'; \\ 2^{\#|Q|} \sim |Q'|}} \phi_{Q_1, \mathbf{s}, 1}(\xi_1) \phi_{Q_2, \mathbf{s}, 2}(\xi_2) \cdot 2^{k_2\ell} \phi_{Q'_1, \mathbf{n}, 1}^{(\ell)} \left(\frac{\xi_1 + \xi_2}{2} \right) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3) \quad (3.11)$$

Here again, because of the big decaying factor $C(\mathbf{s})$, once we have L^p estimates

of the corresponding trilinear operator with the symbol

$$\sum_{\#\geq 1000} 2^{-(\#+1)\ell} \sum_{\substack{Q, Q'; \\ 2^\#|Q|\sim|Q'|}} \phi_{Q_1, s, 1}(\xi_1) \phi_{Q_2, s, 2}(\xi_2) 2^{k_2\ell} \phi_{Q'_1, n, 1}^{(\ell)}\left(\frac{\xi_1 + \xi_2}{2}\right) \phi_{Q'_2, n, 2}(\xi_3), \quad (3.12)$$

then we can control the last summation with respect to $\mathbf{s} \in \mathbf{Z}^2$.

We now claim that $2^{k_2\ell} \phi_{Q'_1, n, 1}^{(\ell)}(\cdot)$ is also a bump function adapted to a dyadic interval Q'_1 . Observe that if a dyadic interval Q'_1 is given by $2^{k_2}[m, m+1]$, for some $m \in \mathbf{Z}$, then we can denote the bump function $\phi_{Q'_1, n, 1}(\xi)$ adapted to Q'_1 in (3.12) by

$$\phi_{Q'_1, n, 1}(\xi) := \phi\left(\frac{\xi}{2^{k_2}} - m\right) \cdot e^{2\pi i n_1 \frac{\xi}{2^{k_2}}}, \quad (3.13)$$

where ϕ is a bump function adapted to $[0, 1]$, and then (3.13) is equal to

$$\phi\left(\frac{\xi}{2^{k_2}} - m\right) \cdot e^{2\pi i n_1 (\frac{\xi}{2^{k_2}} - m)} = \phi_{n_1}\left(\frac{\xi}{2^{k_2}} - m\right),$$

where $\phi_{n_1}(\xi)$ is given by $\phi(\xi)e^{2\pi i n_1 \xi}$, which is a bump function adapted to $[0, 1]$.

Then, we have

$$2^{k_2\ell} \phi_{Q'_1, n, 1}^{(\ell)}(\xi) = \phi_{n_1}^{(\ell)}\left(\frac{\xi}{2^{k_2}} - m\right), \quad (3.14)$$

which is a bump function adapted to $Q'_1 = 2^{k_2}[m, m+1]$, for all $1 \leq \ell < M$, so that we denote $2^{k_2\ell} \phi_{Q'_1, n, 1}^{(\ell)}(\xi)$ by $\tilde{\phi}_{Q'_1, n, 1}(\xi)$, since each number ℓ is unimportant here.

Thus, (3.12) can be written as

$$\sum_{\#\geq 1000} 2^{-(\#+1)\ell} \sum_{\substack{Q, Q'; \\ 2^\#|Q|\sim|Q'|}} \phi_{Q_1, s, 1}(\xi_1) \phi_{Q_2, s, 2}(\xi_2) \tilde{\phi}_{Q'_1, n, 1}\left(\frac{\xi_1 + \xi_2}{2}\right) \phi_{Q'_2, n, 2}(\xi_3)$$

$$= \sum_{\#\geq 1000} 2^{-(\#+1)\ell} \sum_{\substack{Q, Q'; \\ 2^{\#}|Q|\sim|Q'|}} \phi_{Q_1, \mathbf{s}, 1}(\xi_1) \phi_{Q_2, \mathbf{s}, 2}(\xi_2) \tilde{\phi}_{Q'_1, \mathbf{n}, 1}(\xi_1 + \xi_2) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3), \quad (3.15)$$

where $Q'_1 = 2^{k_2+1}[m, m+1]$. Henceforth, we shall redefine Q'_1 to be Q''_1 and redefine $\tilde{\phi}_{Q'_1, \mathbf{n}, 1}$ accordingly for simplicity.

In the last expression (3.15), we see that $\xi_1 \in \frac{9}{10}Q_1$ and $\xi_2 \in \frac{9}{10}Q_2$, which follows that $\xi_1 + \xi_2 \in \frac{9}{10}Q_1 + \frac{9}{10}Q_2$. As a consequence, one can find a shifted dyadic interval Q_3 with the properties that $\frac{9}{10}Q_1 + \frac{9}{10}Q_2 \subseteq \frac{7}{10}Q_3$ and $|Q_1| = |Q_2| \simeq |Q_3|$. In particular, there exists bump functions $\phi_{Q_3, \mathbf{n}, 3}$ adapted to Q_3 uniformly in $\mathbf{n} \in \mathbf{Z}^2$ and supported in $\frac{9}{10}Q_3$, such that $\phi_{Q_3, \mathbf{n}, 3} \equiv 1$ on $\frac{9}{10}Q_1 + \frac{9}{10}Q_2$.

Similarly, we can find a shifted dyadic interval Q'_3 satisfying $\frac{9}{10}Q'_1 + \frac{9}{10}Q'_2 \subseteq \frac{7}{10}Q'_3$ and $|Q'_1| \simeq |Q'_2| \simeq |Q'_3|$, and bump functions $\phi_{Q'_3, \mathbf{s}, 3}$ adapted to Q'_3 uniformly in $\mathbf{s} \in \mathbf{Z}^2$ and supported in $\frac{9}{10}Q'_3$, such that $\phi_{Q'_3, \mathbf{s}, 3} \equiv 1$ on $\frac{9}{10}Q'_1 + \frac{9}{10}Q'_2$.

Thus (3.15) can be written as

$$\sum_{\#\geq 1000} 2^{-(\#+1)\ell} \sum_{\substack{Q, Q'; \\ 2^{\#}|Q|\sim|Q'|}} \phi_{Q_1, \mathbf{s}, 1}(\xi_1) \phi_{Q_2, \mathbf{s}, 2}(\xi_2) \phi_{Q_3, \mathbf{s}, 3}(\xi_1 + \xi_2) \cdot \\ \tilde{\phi}_{Q'_1, \mathbf{n}, 1}(\xi_1 + \xi_2) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3) \phi_{Q'_3, \mathbf{n}, 3}(\xi_1 + \xi_2 + \xi_3), \quad (3.16)$$

where shifted dyadic quasi-cubes Q, Q' in \mathbb{R}^3 are defined by $Q := Q_1 \times Q_2 \times Q_3$, $Q' := Q'_1 \times Q'_2 \times Q'_3$. Since any set of shifted dyadic quasi-cubes in \mathbb{R}^3 can be split into $O(1)$ sparse subsets, we can assume that the sum (3.16) runs over sparse collections of Q and Q' modulo finitely many such corresponding expressions. Then we can see that, for each shifted dyadic quasi-cube Q in such a sparse collection, there exists a unique shifted dyadic cube \tilde{Q} in \mathbb{R}^3 such that

$Q \subseteq \frac{7}{10}\tilde{Q}$ and $|Q| \sim |\tilde{Q}|$. Thus we can now assume that the sum (3.16) runs over sparse collections \mathbf{Q}, \mathbf{Q}' of shifted dyadic cubes Q, Q' , respectively. Then, we can see that the multipliers of the type (3.16) are well localized, which allows us to simplify the corresponding trilinear operator $T_{m_{\#, \ell}}$ with the symbol $m_{\#, \ell}$, where $m_{\#, \ell}$ denotes the inner sum in (3.16). More specifically, in order to establish L^p estimates for the trilinear operator $T_{m_{\#, \ell}}$, for each $\# \geq 1000$, we consider the quadrilinear form $\Lambda_{\#, \ell}$ associated to $T_{m_{\#, \ell}}$ defined by

$$\begin{aligned}
& \int_{\mathbf{R}} T_{m_{\#, \ell}}(f_1, f_2, f_3)(x) f_4(x) dx \\
&= \int_{\mathbf{R}} \left(\int_{\mathbf{R}^3} m_{\#, \ell}(\xi_1, \xi_2, \xi_3) \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) e^{2\pi i x(\xi_1 + \xi_2 + \xi_3)} d\xi_1 d\xi_2 d\xi_3 \right) f_4(x) dx \\
&= \int_{\mathbf{R}^3} m_{\#, \ell}(\xi_1, \xi_2, \xi_3) \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) \widehat{f}_4(-\xi_1 - \xi_2 - \xi_3) d\xi_1 d\xi_2 d\xi_3 \\
&= \sum_{\substack{Q \in \mathbf{Q}, Q' \in \mathbf{Q}' \\ 2^\# |Q| \sim |Q'|}} \int_{\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0} \phi_{Q_1, s, 1}(\xi_1) \phi_{Q_2, s, 2}(\xi_2) \phi_{Q_3, s, 3}(\xi_1 + \xi_2) \widetilde{\phi}_{Q'_1, n, 1}(\xi_1 + \xi_2) \cdot \\
&\quad \phi_{Q'_2, n, 2}(\xi_3) \phi_{Q'_3, n, 3}(\xi_1 + \xi_2 + \xi_3) \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) \widehat{f}_4(\xi_4) d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
&= \sum_{\substack{Q \in \mathbf{Q}, Q' \in \mathbf{Q}' \\ 2^\# |Q| \sim |Q'|}} \int_{\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0} \widehat{f_1 * \check{\phi}_{Q_1, s, 1}(\xi_1)} \widehat{f_2 * \check{\phi}_{Q_2, s, 2}(\xi_2)} \widehat{\check{\phi}_{Q_3, s, 3} * \widetilde{\check{\phi}}_{Q'_1, n, 1}(\xi_1 + \xi_2)} \\
&\quad \widehat{f_3 * \widetilde{\check{\phi}}_{Q'_2, n, 2}(\xi_3)} \widehat{f_4 * \widetilde{\check{\phi}}_{Q'_3, n, 3}(\xi_4)} d\xi_1 d\xi_2 d\xi_3 d\xi_4
\end{aligned}$$

$$\begin{aligned}
&= \sum_{Q' \in \mathbf{Q}'} \int_{\mathbf{R}} \left(\left[\sum_{\substack{Q \in \mathbf{Q}; \\ 2^{\#}|Q| \sim |Q'|}} ((f_1 * \check{\phi}_{Q_1, s, 1})(x) (f_2 * \check{\phi}_{Q_2, s, 2})(x)) * \check{\phi}_{Q_3, s, 3} \right] * \check{\phi}_{Q'_1, n, 1} \right) (x) \\
&\quad (f_3 * \check{\phi}_{Q'_2, n, 2})(x) (f_4 * \check{\phi}_{Q'_3, n, 3})(x) dx \tag{3.17}
\end{aligned}$$

by Plancherel's Theorem. Here, $\check{\phi}_{Q'_3, n, 3}(\xi) := \phi_{Q'_3, n, 3}(-\xi)$ and we redefine Q'_3 to be $-Q'_3$, which is the selected interval in \mathbf{R} about the origin. Furthermore, (3.17) is equal to

$$\begin{aligned}
&\sum_{Q' \in \mathbf{Q}'} |Q'|^{\frac{3}{2}} \int_{\mathbf{R}} \left\langle \left[\sum_{\substack{Q \in \mathbf{Q}; \\ 2^{\#}|Q| \sim |Q'|}} |Q|^{\frac{3}{2}} \int_{\mathbf{R}} \langle f_1, \Phi_{x'}^{Q_1, s, 1} \rangle \langle f_2, \Phi_{x'}^{Q_2, s, 2} \rangle \overline{\Phi_{x'}^{Q_3, s, 3}} dx' \right], \Phi_x^{Q'_1, n, 1} \right\rangle \\
&\quad \langle f_3, \Phi_x^{Q'_2, n, 2} \rangle \langle f_4, \Phi_x^{Q'_3, n, 3} \rangle dx, \tag{3.18}
\end{aligned}$$

where $\Phi_x^{Q_j, s, j}(y)$ is defined by $|Q|^{-1/2} \overline{\check{\phi}_{Q_j, s, j}(x-y)}$ and we define $\Phi_{x'}^{Q'_j, n, j}(y)$ accordingly. Finally, by defining $\Phi_{P_j, t, n, j} := \Phi_{x_{\vec{P}} + |I_{\vec{P}}|t}^{Q'_j, n, j}$ and $\Phi_{Q_j, t', s, j} := \Phi_{x_{\vec{Q}} + |I_{\vec{Q}}|t'}^{Q_j, s, j}$ where $x_{\vec{P}}$ is the center of $I_{\vec{P}}$, we have that (3.18) is equal to

$$\int_0^1 \int_0^1 \sum_{\vec{P}: Q_{\vec{P}} \in \mathbf{Q}'} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle B_{P_1, s, t'}(f_1, f_2), \Phi_{P_1, t, n, 1} \rangle \langle f_3, \Phi_{P_2, t, n, 2} \rangle \langle f_4, \Phi_{P_3, t, n, 3} \rangle dt dt',$$

where

$$B_{P_1, s, t'}(f_1, f_2) = \sum_{\substack{\vec{Q}: Q_{\vec{Q}} \in \mathbf{Q}, \\ -\omega_{Q_3} \subset \omega_{P_1}, 2^{\#}|\omega_{Q_3}| \sim |\omega_{P_1}|}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_1, \Phi_{Q_1, t', s, 1} \rangle \langle f_2, \Phi_{Q_2, t', s, 2} \rangle \overline{\Phi_{Q_3, t', s, 3}}$$

and where \vec{P} and \vec{Q} range over all tri-tiles with frequency cubes $Q_{\vec{P}} := Q'$ and $Q_{\vec{Q}} := Q$, respectively. Observe that the collection of tri-tiles \vec{P} and that of tri-tiles \vec{Q} have rank 1 by construction. The condition $-\omega_{Q_3} \subset \omega_{P_1}$ is automatic

for nonzero summands. By redefining Q_3 to be $-Q_3$ and redefining $\Phi_{Q_3,t',s,3}$ accordingly, we can replace $\bar{\Phi}_{Q_3,t',s,3}$ by $\Phi_{Q_3,t',s,3}$ and the constraint $-\omega_{Q_3} \subset \omega_{P_1}$ by $\omega_{Q_3} \subset \omega_{P_1}$ in the expression of $B_{P_1,s,t'}$. Then Theorem 2.0.7 follows by integrating the conclusion of Theorem 3.2.1 over t, t' because of the uniformity assumption of Theorem 3.2.1 (modulo L^p estimates for the operator corresponding to $m_{\mathbb{I},R_M}$ in (3.8), which we show in Chapter 5). By the usual limiting argument, we can get rid of the finiteness condition on the collections \vec{P} and \vec{Q} of \vec{P} and \vec{Q} , respectively. Finally, we have an operator, as those in Theorem 3.2.1, in terms of wave packets which are perfectly localized in frequency, but not in time space.

Chapter 4

Fourier Coefficient

In this chapter, we will prove the boundedness conditions of the Fourier coefficients $C_{\mathbf{n}}^{Q'}$ and $C_{\mathbf{s}}^{Q,\ell}$ in (3.4) and (3.10), respectively, which were necessary properties in the previous chapter. First, recall that

$$C_{\mathbf{n}}^{Q'} = \frac{1}{|Q'|^2} \int_{\mathbf{R}^2} m_2(\xi_2, \xi_3) \phi_{Q'}(\xi_2, \xi_3) e^{-2\pi i n_1 \frac{\xi_2}{|Q'|}} e^{-2\pi i n_2 \frac{\xi_3}{|Q'|}} d\xi_2 d\xi_3,$$

and

$$C_{\mathbf{s}}^{Q,\ell} = \frac{1}{|Q|^2} \int_{\mathbf{R}^2} m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \left(\frac{\xi_2 - \xi_1}{|Q|} \right)^\ell \frac{1}{\ell!} e^{-2\pi i s_1 \frac{\xi_1}{|Q|}} e^{-2\pi i s_2 \frac{\xi_2}{|Q|}} d\xi_1 d\xi_2.$$

Lemma 4.0.2. $|C_{\mathbf{n}}^{Q'}| \lesssim C(\mathbf{n})$, which is not depending on Q' , where $\mathbf{n} := (n_1, n_2) \in \mathbf{Z}^2$. Similarly, $|C_{\mathbf{s}}^{Q,\ell}| \lesssim C(\mathbf{s})$, which is not depending on Q and ℓ , where $\mathbf{s} := (s_1, s_2) \in \mathbf{Z}^2$.

Proof. Let $|Q| = 2^{k_1}$ and $|Q'| = 2^{k_2}$ for $k_1, k_2 \in \mathbf{Z}$. We denote $m_2(\xi_2, \xi_3) \phi_{Q'}(\xi_2, \xi_3)$ by $m_{2,Q'}(\xi_2, \xi_3)$, which is the smooth restriction for the symbol m_2 to the cube Q' . Since $m_{2,Q'}$ is smooth, we can integrate by parts as much as we want. Before

doing that, we have

$$C_{\mathbf{n}}^{Q'} = \int_{\mathbb{R}^2} m_{2,Q'}(2^{k_2}\xi_2, 2^{k_2}\xi_3) e^{-2\pi i(n_1\xi_2+n_2\xi_3)} d\xi_2 d\xi_3$$

and because of the support Q' of $m_{2,Q'}$, we can see that $|2^{k_2}\xi_2 - 2^{k_2}\xi_3| \sim 2^{k_2}$ for any $(\xi_2, \xi_3) \in Q'$. Then after taking integration by parts sufficiently many times, we have

$$|C_{\mathbf{n}}^{Q'}| \lesssim \frac{1}{(1+|n_1|)^M} \frac{1}{(1+|n_2|)^M} \leq \frac{1}{(1+|n_1|+|n_2|)^M}$$

for some large constant $M \in \mathbb{Z}$, because we have that

$$|\partial^\alpha(m_{2,Q'}(\xi, \eta))| \leq \frac{1}{\text{dist}(\Gamma_2, (\xi, \eta))^{|\alpha|}} \sim \frac{1}{2^{k_2|\alpha|}},$$

where $\Gamma_2 = \{(\xi_2, \xi_3) \in \mathbb{R}^2 : \xi_2 = \xi_3\}$ so that $|\partial^\alpha(m_{2,Q'}(2^{k_2}\xi_2, 2^{k_2}\xi_3))| \sim 1$. We note that $\frac{1}{(1+|n_1|+|n_2|)^M}$ is a rapidly decreasing sequence depending only on \mathbf{n} , and not on Q' .

Similarly, we can see that

$$C_{\mathbf{s}}^{Q,\ell} = \int_{\mathbb{R}^2} m_1(2^{k_1}\xi_1, 2^{k_1}\xi_2) \phi_Q(2^{k_1}\xi_1, 2^{k_1}\xi_2) (\xi_2 - \xi_1)^\ell \frac{1}{\ell!} e^{-2\pi i(s_1\xi_1+s_2\xi_2)} d\xi_1 d\xi_2.$$

Because of the support Q of ϕ_Q , we have $|2^{k_1}\xi_1 - 2^{k_1}\xi_2| \sim 2^{k_1}$. Thus, we can easily see that

$$C_{\mathbf{s}}^{Q,\ell} \lesssim \frac{1}{(1+|s_1|+|s_2|)^{M'}}$$

for some large constant $M' \in \mathbb{Z}$. We also note that $\frac{1}{(1+|s_1|+|s_2|)^{M'}}$ is a rapidly decreasing sequence depending only on \mathbf{s} , and not on Q, ℓ . \square

Chapter 5

Remainder term

In this chapter, we will obtain L^p estimates for the operator corresponding to $m_{\mathbb{I}, R_M}$ in (3.8). Let us recall from (3.8) that $m_{\mathbb{I}, R_M}$ be given by

$$\sum_{Q, Q'; |Q| \ll |Q'|} m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) R_M(\xi_1, \xi_2) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3). \quad (5.1)$$

Here, the remainder term in the Taylor series in (3.7) is defined by

$$R_M(\xi_1, \xi_2) = \frac{1}{M!} \phi_{Q'_1, \mathbf{n}, 1}^{(M)}(\xi_\theta) \left(\frac{\xi_2 - \xi_1}{2} \right)^M$$

for a big number $M \in \mathbf{Z}$ and where $\xi_\theta = (1 - \frac{\theta}{2})\xi_2 + \frac{\theta}{2}\xi_1$ for some θ between 0 and 1.

Let $|Q| = 2^{k_1}$ and $|Q'| = 2^{k_2}$ for $k_1, k_2 \in \mathbf{Z}$. Then, we have that (5.1) is equal to

$$\sum_{\# \geq 1000} \sum_{\substack{Q, Q' \\ k_2 - k_1 = \#}} m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \phi_{Q'_1, \mathbf{n}, 1}^{(M)}(\xi_\theta) \left(\frac{\xi_2 - \xi_1}{2} \right)^M \frac{1}{M!} \phi_{Q'_2, \mathbf{n}, 2}(\xi_3)$$

$$\begin{aligned}
&= \sum_{\#\geq 1000} \sum_{\substack{Q,Q' \\ k_2-k_1=\#}} 2^{(k_1-1)M} 2^{-k_2M} \left(m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \left(\frac{\xi_2 - \xi_1}{2^{k_1}} \right)^M \frac{1}{M!} \right) \\
&\quad 2^{k_2M} \phi_{Q'_1, \mathbf{n}, 1}^{(M)}(\xi_\theta) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3). \tag{5.2}
\end{aligned}$$

We compute a double Fourier series of $m_1(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \left(\frac{\xi_2 - \xi_1}{2^{k_1}} \right)^M \frac{1}{M!}$ to see that (5.2) is equal to

$$\sum_{\#\geq 1000} \sum_{\substack{Q,Q' \\ k_2-k_1=\#}} 2^{-(\#+1)M} \left(\sum_{\mathbf{s} \in \mathbb{Z}^2} C_{\mathbf{s}}^Q \phi_{Q_1, \mathbf{s}, 1}(\xi_1) \phi_{Q_2, \mathbf{s}, 2}(\xi_2) \right) 2^{k_2M} \phi_{Q'_1, \mathbf{n}, 1}^{(M)}(\xi_\theta) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3), \tag{5.3}$$

where $C_{\mathbf{s}}^Q = C_{\mathbf{s}}^{Q, \ell}$ in (3.10) when $\ell = M$. Thus, we can obtain that $|C_{\mathbf{s}}^{Q, M}| \lesssim C(\mathbf{s})$ by the same proof in the chapter 4. Furthermore, we observe that if a dyadic interval Q'_1 is given by $2^{k_2}[m, m+1]$, for some $m \in \mathbb{Z}$, then $2^{k_2M} \phi_{Q'_1, \mathbf{n}, 1}^{(M)}(\xi_\theta)$ can be denoted by $\tilde{\phi}_{n_1}^{(M)}\left(\frac{\xi_\theta}{2^{k_2}} - m\right)$, where $\tilde{\phi}_{n_1}(\eta) := \phi(\eta) e^{2\pi i n_1 \eta}$ for a bump function ϕ adapted to $[0, 1]$. Therefore, (5.3) can be majorized by

$$\sum_{\mathbf{s} \in \mathbb{Z}^2} C(\mathbf{s}) \sum_{\#\geq 1000} 2^{-(\#+1)M} \sum_{\substack{Q,Q' \\ k_2-k_1=\#}} \phi_{Q_1, \mathbf{s}, 1}(\xi_1) \phi_{Q_2, \mathbf{s}, 2}(\xi_2) \tilde{\phi}_{n_1}^{(M)}\left(\frac{\xi_\theta}{2^{k_2}} - m\right) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3). \tag{5.4}$$

For simplicity, we denote

$$\tilde{m}_\# := \sum_{\substack{Q,Q' \\ k_2-k_1=\#}} \phi_{Q_1, \mathbf{s}, 1}(\xi_1) \phi_{Q_2, \mathbf{s}, 2}(\xi_2) \tilde{\phi}_{n_1}^{(M)}\left(\frac{\xi_\theta}{2^{k_2}} - m\right) \phi_{Q'_2, \mathbf{n}, 2}(\xi_3).$$

Then, it is easy to remark that the operator corresponding to $m_{\mathbb{I}, R_M}$ can be majorized by $\sum_{\mathbf{s} \in \mathbb{Z}^2} C(\mathbf{s}) \sum_{\#\geq 1000} 2^{-(\#+1)M} T_\#$, where $T_\#$ has a symbol $\tilde{m}_\#$. Now,

we claim that the symbol $\tilde{m}_\#$ satisfies

$$|\partial^\alpha(\tilde{m}_\#(\xi))| \lesssim 2^{\#\alpha} \frac{1}{\text{dist}(\Gamma, \xi)^{|\alpha|}} \quad (5.5)$$

for many multi-indices α , where $\Gamma = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 = \xi_2 = \xi_3\}$. If this claim holds, then the Coifman-Meyer Theorem implies that $T_\#$ is bounded with a bound that is of type $O(2^{100\#})$, say. This completes the proof for the desired L^p estimates for the operator corresponding to $m_{\mathbb{I}, R_M}$ because of the big decaying factor $2^{-(\#+1)M}$ in (5.4). These arguments show that after justifying the claim (5.5), Theorem 3.2.1 is the only one that remains to be proved.

In order to prove the claim (5.5), fix $\xi_o \in \mathbb{R}^3$. Then, there exists a unique shifted dyadic quasi-cube $Q_1 \times Q_2 \times Q'_2$ containing ξ_o in \mathbb{R}^3 so that we obtain

$$|\partial^\alpha(\tilde{m}_\#(\xi_o))| \leq \left| \partial^\alpha \left[\phi_{Q_1, s, 1}(\xi_1) \phi_{Q_2, s, 2}(\xi_2) \tilde{\phi}_{n_1}^{(M)} \left(\frac{\xi_\theta}{2^{k_2}} - n \right) \phi_{Q'_2, n, 2}(\xi_3) \right] \right|_{\xi=\xi_o}$$

because of localization of ϕ_Q functions. Then, it is easy to check that

$$|\partial^\alpha(\tilde{m}_\#(\xi_o))| \lesssim 2^{\#\alpha} \cdot 2^{-k_2|\alpha|} \lesssim 2^{\#\alpha} \frac{1}{\text{dist}(\Gamma, \xi_o)^{|\alpha|}}$$

as $\text{dist}(\Gamma, \xi_o) \lesssim 2^{k_2}$, where $\xi_o \in Q_1 \times Q_2 \times Q'_2$. This completes the proof of the claim (5.5).

Chapter 6

Tile Norms and Known Estimates

In order to establish the estimates (3.2) for the forms $\Lambda_{\vec{P}, \vec{Q}}^\#$, the standard approach is to organize our collections of tri-tiles \vec{P}, \vec{Q} into trees as in [6] and consider the standard tile norms. We first review standard definitions and comments for trees from [15]. We will henceforth assume that \vec{P} and \vec{Q} are sparse of rank 1.

6.1 Definitions of Trees and Tile Norms

Definition 6.1.1. For any $i = 1, 2, 3$ and a tri-tile $\vec{P}_T \in \vec{P}$, define a i -tree with top \vec{P}_T to be a collection $T \subseteq \vec{P}$ of tri-tiles such that

$$P_i \leq P_{T,i} \text{ for all } \vec{P} \in T,$$

where $P_{T,i}$ is the i component of \vec{P}_T . We write I_T and $\omega_{T,i}$ for $I_{\vec{P}_T}$ and $\omega_{P_T,i}$ respectively. We say that T is a tree if it is a i -tree for some $i = 1, 2, 3$.

Observe that a tree T does not necessarily have to contain its top \vec{P}_T .

Definition 6.1.2. Fix $1 \leq i \leq 3$. Two trees T, T' are said to be strongly i -disjoint if

1. $P_i \neq P'_i$ for all $\vec{P} \in T, \vec{P}' \in T'$.
2. For any $\vec{P} \in T, \vec{P}' \in T'$ with $2\omega_{P_i} \cap 2\omega_{P'_i} \neq \emptyset$, one obtains $I_{\vec{P}} \cap I_T = \emptyset$ and $I_{\vec{P}} \cap I_{T'} = \emptyset$.

Note that if T and T' are strongly i -disjoint, then we can see that two tiles $I_P \times 2\omega_{P_i}$ and $I_{P'} \times 2\omega_{P'_i}$ are disjoint for all $\vec{P} \in T, \vec{P}' \in T'$. Given that $\vec{\mathbf{P}}$ is sparse, we can see that if T is an i -tree, then we have either $\omega_{P_j} = \omega_{P'_j}$ or $2\omega_{P_j} \cap 2\omega_{P'_j} = \emptyset$ for all $\vec{P}, \vec{P}' \in T$ and $j \neq i$.

We now recall the standard tile norms from [14] and [15]. In the remainder of this paper we shall estimate expressions of the form

$$\left| \sum_{\vec{P} \in \vec{\mathbf{P}}} \frac{1}{|I_{\vec{P}}|^{1/2}} a_{P_1}^{(1)} a_{P_2}^{(2)} a_{P_3}^{(3)} \right|, \quad (6.1)$$

where $\vec{\mathbf{P}}$ is a collection of tri-tiles and where $a_{P_i}^{(i)}$ are complex numbers for P_i , where $\vec{P} = (P_1, P_2, P_3) \in \vec{\mathbf{P}}$.

In some cases such as the associated expression for the Bilinear Hilbert transform, we just have

$$a_{P_i}^{(i)} = \langle f_i, \Phi_{P_i} \rangle, \quad (6.2)$$

for all $i = 1, 2, 3$, however, here we will have more complicate sequences $a_{P_3}^{(3)}$ when dealing with $\Lambda_{\vec{\mathbf{P}}, \vec{\mathbf{Q}}}^\#$.

In [14], the following standard tile norms were introduced:

Definition 6.1.3. Suppose that $\vec{\mathbf{P}}$ is a finite collection of tri-tiles. For $i = 1, 2, 3$, suppose that $(a_{P_i}^{(i)})_{\vec{P} \in \vec{\mathbf{P}}}$ is a sequence of complex numbers. We define the size of the

sequence by

$$\text{size}_i((a_{P_i}^{(i)})_{\vec{P} \in \vec{\mathbf{P}}}) := \sup_{T \subset \vec{\mathbf{P}}} \left(\frac{1}{|I_T|} \sum_{\vec{P} \in T} |a_{P_i}^{(i)}|^2 \right)^{1/2},$$

where the supremum ranges over all j -trees T in $\vec{\mathbf{P}}$ for some $j \neq i$.

We also define the energy of the sequence by

$$\text{energy}_i((a_{P_i}^{(i)})_{\vec{P} \in \vec{\mathbf{P}}}) := \sup_{n \in \mathbf{Z}} \sup_{\mathbf{T}} 2^n \left(\sum_{T \in \mathbf{T}} |I_T| \right)^{1/2},$$

where the inner supremum ranges over all collections \mathbf{T} of strongly i -disjoint trees in $\vec{\mathbf{P}}$ such that

$$\left(\sum_{\vec{P} \in T} |a_{P_i}^{(i)}|^2 \right)^{1/2} \geq 2^n |I_T|^{1/2}$$

for all $T \in \mathbf{T}$, and

$$\left(\sum_{\vec{P} \in T'} |a_{P_i}^{(i)}|^2 \right)^{1/2} \leq 2^{n+1} |I_{T'}|^{1/2}$$

for all sub-trees $T' \subset T \in \mathbf{T}$.

Observe that the complex number $a_{P_i}^{(i)}$ in (6.2) is only associated with the i -tile P_i rather than the full tri-tile \vec{P} .

6.2 Known Estimates

Since the size can be thought of as a phase-space variant of the BMO norm, we obtain the following relevant variant of the John-Nirenberg inequality for the size:

Lemma 6.2.1. *Suppose that $\vec{\mathbf{P}}$ is a finite collection of tri-tiles and that, for $i = 1, 2, 3$,*

$(a_{P_i}^{(i)})_{\vec{P} \in \vec{\mathbf{P}}}$ is a sequence of complex numbers. Then

$$\text{size}_i((a_{P_i}^{(i)})_{\vec{P} \in \vec{\mathbf{P}}}) \sim \sup_{T \subset \vec{\mathbf{P}}} \frac{1}{|I_T|} \left\| \left(\sum_{\vec{P} \in T} |a_{P_i}^{(i)}|^2 \frac{\chi_{I_{\vec{P}}}}{|I_{\vec{P}}|} \right)^{1/2} \right\|_{L^{1,\infty}(I_T)},$$

where the supremum ranges over all j -trees $T \subset \vec{\mathbf{P}}$ for some $j \neq i$.

Proof. See Lemma 4.2 in [14]. □

By duality argument we can see that the following lemma:

Lemma 6.2.2. *Suppose that $\vec{\mathbf{P}}$ is a finite collection of tri-tiles and that $(a_{P_i}^{(i)})_{\vec{P} \in \vec{\mathbf{P}}}$, for $i = 1, 2, 3$, is a sequence of complex numbers. Then, there exists a collection \mathbf{T} of strongly i -disjoint trees, and complex coefficients c_{P_i} , for all $\vec{P} \in \bigcup_{T \in \mathbf{T}} T$, such that*

$$\text{energy}_i((a_{P_i}^{(i)})_{\vec{P} \in \vec{\mathbf{P}}}) \sim \left| \sum_{T \in \mathbf{T}} \sum_{\vec{P} \in T} a_{P_i}^{(i)} \bar{c}_{P_i} \right|,$$

and such that

$$\sum_{\vec{P} \in T'} |c_{P_i}|^2 \lesssim \frac{|I_{T'}|}{\sum_{T \in \mathbf{T}} |I_T|}$$

for all $T \in \mathbf{T}$ and all sub-trees $T' \subseteq T$ of T .

Proof. The proof follows by setting that n, \mathbf{T} are the extremizers in the definition of the energy and that $c_{P_i} := 2^n (\sum_{T \in \mathbf{T}} |I_T|)^{-1/2} a_{P_i}^{(i)}$ for all $\vec{P} \in \bigcup_{T \in \mathbf{T}} T$. □

To establish the estimates on (6.1), we recall the following standard combinatorial tool from [14]:

Proposition 6.2.3. *Suppose that $\vec{\mathbf{P}}$ is a finite collection of tri-tiles and that $(a_{P_i}^{(i)})_{\vec{P} \in \vec{\mathbf{P}}}$, for $i = 1, 2, 3$, is a sequence of complex numbers. Then*

$$\left| \sum_{\vec{P} \in \vec{\mathbf{P}}} \frac{1}{|I_{\vec{P}}|^{1/2}} a_{P_1}^{(1)} a_{P_2}^{(2)} a_{P_3}^{(3)} \right| \lesssim \prod_{i=1}^3 \text{size}_i((a_{P_i}^{(i)})_{\vec{P} \in \vec{\mathbf{P}}})^{\theta_i} \text{energy}_i((a_{P_i}^{(i)})_{\vec{P} \in \vec{\mathbf{P}}})^{1-\theta_i}$$

for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$, with the implicit constant depending on the θ_i .

Proof. See the Appendix [15]. □

In order to use Proposition 6.2.3, we will need some estimates on size and energy. We have the following lemmas from [15] for $a_{P_i}^{(i)}$, $i = 1, 2, 3$, defined in (6.2):

Lemma 6.2.4. *Suppose that $\vec{\mathbf{P}}$ is a finite collection of tri-tiles. Suppose that E_i , for $i = 1, 2, 3$, is a subset of \mathbb{R} with finite measure and f_i is a function in $X(E_i)$. Then, we have*

$$\text{size}_i(\langle \langle f_i, \Phi_{P_i} \rangle \rangle_{\vec{\mathbf{P}} \in \vec{\mathbf{P}}}) \lesssim \sup_{\vec{\mathbf{P}} \in \vec{\mathbf{P}}} \int_{E_i} \frac{\tilde{\chi}_{I_{\vec{\mathbf{P}}}}^M}{|I_{\vec{\mathbf{P}}}|}$$

for all M , where the implicit constant depends on M .

Proof. See Lemma 6.8 in [15]. □

Lemma 6.2.5. *Suppose that $\vec{\mathbf{P}}$ is a finite collection of tri-tiles. Suppose that E_i , for $i = 1, 2, 3$, is a subset of \mathbb{R} with finite measure and f_i is a function in $X(E_i)$. Then, we have*

$$\text{energy}_i(\langle \langle f_i, \Phi_{P_i} \rangle \rangle_{\vec{\mathbf{P}} \in \vec{\mathbf{P}}}) \leq \|f_i\|_2.$$

Proof. See Lemma 6.7 in [15]. □

Chapter 7

Estimates of Size and Energy

In this chapter, we will provide the relevant size and energy estimates for the form $\Lambda_{\vec{P}, \vec{Q}}^\#$ in Theorem 3.2.1. In order to obtain more complicated size and energy estimates than those of $a_{P_i}^{(i)} = \langle f_i, \Phi_{P_i} \rangle$ described in Lemma 6.2.4 and Lemma 6.2.5, it shall be more convenient to rewrite $\Lambda_{\vec{P}, \vec{Q}}^\#$ as

$$\Lambda_{\vec{P}, \vec{Q}}^\#(f_1, f_2, f_3, f_4) = \sum_{\vec{Q} \in \vec{\mathcal{Q}}} \frac{1}{|I_{\vec{Q}}|^{1/2}} a_{Q_1}^{(1)} a_{Q_2}^{(2)} a_{Q_3}^{(3), \#},$$

where

$$\begin{aligned} a_{Q_1}^{(1)} &:= \langle f_1, \Phi_{Q_1} \rangle \\ a_{Q_2}^{(2)} &:= \langle f_2, \Phi_{Q_2} \rangle \\ a_{Q_3}^{(3), \#} &:= \sum_{\substack{\vec{P} \in \vec{\mathcal{P}}: \omega_{Q_3} \subseteq \omega_{P_1} \\ 2^\# |\omega_{Q_3}| \sim |\omega_{P_1}|}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_3, \Phi_{P_2} \rangle \langle f_4, \Phi_{P_3} \rangle \langle \Phi_{P_1}, \Phi_{Q_3} \rangle \end{aligned} \quad (7.1)$$

by reversing the order of summation. As a consequence, the inner summation has a stronger spatial localized than the outer summation because the \vec{P} tiles have a narrower spatial interval than the \vec{Q} tiles by the constraint $\omega_{Q_3} \subseteq \omega_{P_1}$.

Compared to the form $\Lambda_{\vec{P}, \vec{Q}}$ in Theorem 4.8 of [15], we have the form $\Lambda_{\vec{P}, \vec{Q}}^\#$ with extra constraint $2^\# |\omega_{Q_3}| \sim |\omega_{P_1}|$. Because of this constraint, we are no longer able to apply one of the essential techniques to estimate the tile norms of the type of $a_{Q_3}^{(3)}$ (described in Lemma 8.2 of [15] and Lemma 6.1 of [14]). This technique exploits the symmetry of \vec{P} and \vec{Q} to decouple them. Hence we have

$$\begin{aligned} a_{Q_3}^{(3)} &:= \sum_{\vec{P} \in \vec{P} : \omega_{Q_3} \subseteq \omega_{P_1}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_3, \Phi_{P_2} \rangle \langle f_4, \Phi_{P_3} \rangle \langle \Phi_{P_1}, \Phi_{Q_3} \rangle \\ &= \sum_{\vec{P} \in \vec{P}'} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_3, \Phi_{P_2} \rangle \langle f_4, \Phi_{P_3} \rangle \langle \Phi_{P_1}, \Phi_{Q_3} \rangle, \end{aligned}$$

where \vec{P}' is the sub-collection of \vec{P} , which only depends on a fixed i -tree $T \subseteq \vec{Q}$, for some $i = 1, 2$, and not on each $\vec{Q} \in \vec{Q}$. Thus, here, we will need a new trick to deal with the tile norms of the type of $a_{Q_3}^{(3), \#}$ by taking advantage of the extra constraint $2^\# |\omega_{Q_3}| \sim |\omega_{P_1}|$.

We first set out some general definitions, which shall be useful in this chapter. Define that a collection $\{\omega\}$ of intervals is *lacunary around the frequency* ξ if we have $\text{dist}(\xi, \omega) \sim |\omega|$ for all ω in the collection. We also define a *modulated Calderón-Zygmund operator* to be any operator T_K of the form

$$T_K f(x) = \int K(x, y) f(y) dy,$$

where $x, y \in \mathbb{R}$, and the (possibly vector-valued) kernel K satisfies the estimates

$$|K(x, y)| \lesssim 1/|x - y|$$

and

$$|\nabla_{x, y}(e^{2\pi i(x\xi + y\eta)} K(x, y))| \lesssim 1/|x - y|^2$$

for all $x \neq y$ and for some $\xi, \eta \in \mathbf{R}$. Observe that a modulated Calderón-Zygmund operator is the composition of an ordinary Calderón-Zygmund operator with modulation operators such as $f \mapsto e^{2\pi i \xi} \cdot f$. Thus we see that T_K is bounded on L^p , for all $1 < p < \infty$, and is also weak-type $(1, 1)$ by standard Calderón-Zygmund theory.

In order to estimate the size of $a_{Q_3}^{(3), \#}$, we need the following lemma:

Lemma 7.0.6. *Let $\vec{\mathbf{P}}$ and $\vec{\mathbf{Q}}$ be finite collections of tri-tiles and $T \subset \vec{\mathbf{Q}}$ be a i -tree for some $i = 1, 2$. Define the collections of intervals, $\{\omega_{Q_3}\}_T$ and $\{\omega_{P_i}\}_{\vec{\mathbf{P}}, T}$, for $i = 1, 2, 3$, by*

$$\{\omega_{Q_3}\}_{\vec{\mathbf{Q}} \in T} := \{\omega_{Q_3} : \vec{Q} \in T\}$$

and $\{\omega_{P_i}\}_{\vec{\mathbf{P}}, T} := \{\omega_{P_i} : \vec{P} \in \vec{\mathbf{P}}, \omega_{Q_3} \subset \omega_{P_1}, 2^\# |\omega_{Q_3}| \sim |\omega_{P_1}| \text{ where } \vec{Q} \in T\}$.

Then, the collections $\{\omega_{Q_3}\}_{\vec{\mathbf{Q}} \in T}$, $\{\omega_{P_2}\}_{\vec{\mathbf{P}}, T}$, and $\{\omega_{P_3}\}_{\vec{\mathbf{P}}, T}$ are lacunary, however the collection $\{\omega_{P_1}\}_{\vec{\mathbf{P}}, T}$ is non-lacunary.

Proof. Without loss of generality, we shall assume that T is 1-tree in this proof.

First, to show that the collection $\{\omega_{P_1}\}_{\vec{\mathbf{P}}, T}$ is non-lacunary, pick any $\xi_o \in \omega_{T,1}$. It is easy to see that $\xi_o \in \omega_{T,1} \subset 3\omega_{Q_1}$ for all $\vec{Q} \in T$, henceforth there exists a fixed number C_o such that $\xi_o \in C_o \omega_{Q_3}$ for all $\vec{Q} \in T$. Therefore, we conclude that $\xi_o \in C_o \omega_{P_1}$ for all $\omega_{P_1} \in \{\omega_{P_1}\}_{\vec{\mathbf{P}}, T}$, and hence the collection $\{\omega_{P_1}\}_{\vec{\mathbf{P}}, T}$ is non-lacunary.

Then, it follows that the collection $\{\omega_{P_i}\}_{\vec{\mathbf{P}}, T}$, for each $i = 2, 3$, is lacunary with the same ξ_o , because we know that, for each $\vec{P} \in \vec{\mathbf{P}}$, ω_{P_2} and ω_{P_3} are a few steps away from ω_{P_1} .

Now, we prove that $\{\omega_{Q_3}\}_{\vec{\mathbf{Q}} \in T}$ is lacunary: For any $\vec{Q} \in T$, we have $Q_1 \leq Q_{T,1}$. Then, because we have rank 1, we obtain $Q_2 \lesssim Q_{T,2}$ and $Q_3 \lesssim Q_{T,3}$. In

particular, if we furthermore assume $Q_1 < Q_{T,1}$, then we have $|10^9 \omega_{T,1}| < |\omega_{Q_1}|$ by sparseness. Thus, we obtain $Q_3 \lesssim Q_{T,3}$ and $Q_3 \not\lesssim Q_{T,3}$. Thence, we have

$$10^7 \omega_{T,3} \subset 10^7 \omega_{Q_3} \text{ but } 3\omega_{T,3} \not\subset 3\omega_{Q_3}.$$

Again, pick any $\xi_o \in \omega_{T,1}$. Then, we conclude that $\text{dist}(\xi_o, \omega_{Q_3}) \approx C_o |\omega_{Q_3}|$. \square

Now we can obtain the size estimates for $a_{Q_3}^{(3),\#}$.

Lemma 7.0.7. *For $j = 3, 4$, let E_j be sets of finite measure in \mathbf{R} and f_j be functions in $X(E_j)$. Then, we have*

$$\text{size}_3((a_{Q_3}^{(3),\#})_{\vec{Q} \in \vec{\mathcal{Q}}}) \lesssim \sup_{\vec{Q} \in \vec{\mathcal{Q}}} \left(\int_{E_3} \frac{\tilde{\chi}_{I_{\vec{Q}}}^M}{|I_{\vec{Q}}|} \right)^{1-\theta} \left(\int_{E_4} \frac{\tilde{\chi}_{I_{\vec{Q}}}^M}{|I_{\vec{Q}}|} \right)^\theta \quad (7.2)$$

for any $0 < \theta < 1$ and $M > 0$, with the implicit constant depending on θ, M .

Proof. We first recall that $a_{Q_3}^{(3),\#}$ has been defined by

$$a_{Q_3}^{(3),\#} = \sum_{\substack{\vec{P} \in \vec{\mathcal{P}}: \omega_{Q_3} \subseteq \omega_{P_1} \\ 2^\# |\omega_{Q_3}| \sim |\omega_{P_1}|}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_3, \Phi_{P_2} \rangle \langle f_4, \Phi_{P_3} \rangle \langle \Phi_{P_1}, \Phi_{Q_3} \rangle.$$

By Lemma 6.2.1, it suffices to show that, for any 1 or 2-tree $T \subset \vec{Q}$, we have

$$\frac{1}{|I_T|} \left\| \left(\sum_{\vec{Q} \in T} |a_{Q_3}^{(3),\#}|^2 \frac{\chi_{I_{\vec{Q}}}}{|I_{\vec{Q}}|} \right)^{1/2} \right\|_{L^{1,\infty}(I_T)} \lesssim \sup_{\vec{Q} \in \vec{\mathcal{Q}}} \frac{1}{|I_{\vec{Q}}|} \left(\int_{E_3} \tilde{\chi}_{I_{\vec{Q}}}^M \right)^{1-\theta} \left(\int_{E_4} \tilde{\chi}_{I_{\vec{Q}}}^M \right)^\theta.$$

We may assume that T contains its top P_T , since in the other case we could decompose T into disjoint trees containing its top. Henceforth, it suffices to

estimate

$$\left\| \left(\sum_{\vec{Q} \in T} |a_{Q_3}^{(3),\#}|^2 \frac{\chi_{I_{\vec{Q}}}}{|I_{\vec{Q}}|} \right)^{1/2} \right\|_{L^1, \infty(I_T)} \lesssim \left(\int_{E_3} \tilde{\chi}_{I_T}^M \right)^{1-\theta} \left(\int_{E_4} \tilde{\chi}_{I_T}^M \right)^\theta \quad (7.3)$$

for any 1 or 2-tree $T \subset \vec{Q}$. Now we fix T . To prove (7.3), we first consider the case when both f_3 and f_4 are supported on $5I_T$. We may then assume that $E_3, E_4 \subset 5I_T$. By Plancherel's Theorem we see that

$$\langle \Phi_{P_1}, \Phi_{Q_3} \rangle = \langle \widehat{\Phi}_{P_1}, \widehat{\Phi}_{Q_3} \rangle. \quad (7.4)$$

Now we pick a Schwartz function $\psi_{\omega_{Q_3}}$ so that $\text{supp } \widehat{\psi}_{\omega_{Q_3}} \subseteq \omega_{Q_3}$ and $\widehat{\psi}_{\omega_{Q_3}} \equiv 1$ on $\frac{9}{10}\omega_{Q_3}$. Then, (7.4) can be rewritten as

$$\begin{aligned} \langle \widehat{\Phi}_{P_1}, \widehat{\Phi}_{Q_3} \cdot \widehat{\psi}_{\omega_{Q_3}} \rangle &= \langle \widehat{\Phi}_{P_1} * \widehat{\psi}_{\omega_{Q_3}}, \widehat{\Phi}_{Q_3} \rangle \\ &= 2^{-\frac{\#}{2}} \langle 2^{\frac{\#}{2}} \widehat{\Phi}_{P_1} * \widehat{\psi}_{\omega_{Q_3}}, \widehat{\Phi}_{Q_3} \rangle \\ &:= 2^{-\frac{\#}{2}} \langle \tilde{\Phi}_{\bar{P}_1}, \Phi_{Q_3} \rangle, \end{aligned}$$

where $\tilde{\Phi}_{\bar{P}_1} := 2^{\frac{\#}{2}} \widehat{\Phi}_{P_1} * \widehat{\psi}_{\omega_{Q_3}}$. We note that $\tilde{\Phi}_{\bar{P}_1}$ is an L^2 -normalized bump adapted to $I_{\bar{P}, 2^\#}$, where $I_{\bar{P}, 2^\#}$ denotes the unique dyadic interval of length $2^\# |I_{\bar{P}}|$ containing $I_{\bar{P}}$. Thus, we can see that

$$a_{Q_3}^{(3),\#} = 2^{-\frac{\#}{2}} \sum_{\substack{\bar{P} \in \vec{\mathbf{P}}: \omega_{Q_3} \subseteq \omega_{P_1} \\ 2^\# |\omega_{Q_3}| \sim |\omega_{P_1}|}} \frac{1}{|I_{\bar{P}}|^{1/2}} \langle f_3, \Phi_{P_2} \rangle \langle f_4, \Phi_{P_3} \rangle \langle \tilde{\Phi}_{\bar{P}_1}, \Phi_{Q_3} \rangle$$

Now we claim that

$$a_{Q_3}^{(3),\#} = 2^{-\frac{\#}{2}} \sum_{\vec{P}' \in \vec{\mathbf{P}}'} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_3, \Phi_{P_2} \rangle \langle f_4, \Phi_{P_3} \rangle \langle \tilde{\Phi}_{\tilde{P}_1}, \Phi_{Q_3} \rangle,$$

where the collection $\vec{\mathbf{P}}'$ of tri-tiles is defined by

$$\vec{\mathbf{P}}'(T) = \{ \vec{P}' = \tilde{P}_1 \times P_2 \times P_3 : \vec{P} = P_1 \times P_2 \times P_3 \in \vec{\mathbf{P}} \text{ and } \tilde{P}_1 = I_{\vec{P}, 2\#} \times \omega_{Q_3} \}$$

for some $\vec{Q} \in T$ with $\omega_{Q_3} \subseteq \omega_{P_1}$ and $2^\# |\omega_{Q_3}| \sim |\omega_{P_1}|$.

We claim that the sub-collection $\vec{\mathbf{P}}' \subset \vec{\mathbf{P}}$ depends on $\vec{\mathbf{P}}$ and $T \subset \vec{\mathbf{Q}}$, and not on each $\vec{Q} \in T$. We will prove this claim in Lemma 7.0.8. Using the claim, we can see that

$$a_{Q_3}^{(3),\#} = 2^{-\frac{\#}{2}} \langle B_{\vec{\mathbf{P}}'}(f_3, f_4), \Phi_{Q_3} \rangle,$$

where

$$B_{\vec{\mathbf{P}}'} := \sum_{\vec{P}' \in \vec{\mathbf{P}}'} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_3, \Phi_{P_2} \rangle \langle f_4, \Phi_{P_3} \rangle \tilde{\Phi}_{\tilde{P}_1}.$$

Thus, in order to prove (7.3), it suffices to show that

$$\left\| \left(\sum_{\vec{Q} \in T} |2^{-\frac{\#}{2}} \langle B_{\vec{\mathbf{P}}'}(f_3, f_4), \Phi_{Q_3} \rangle|^2 \frac{\tilde{\chi}_{I_{\vec{Q}}}^{100}}{|I_{\vec{Q}}|} \right)^{1/2} \right\|_{L^{1,\infty}} \lesssim |E_3|^{1-\theta} |E_4|^\theta.$$

The vector-valued operator

$$F \mapsto \left(\langle F, \Phi_{Q_3} \rangle \frac{\tilde{\chi}_{I_{\vec{Q}}}^{50}}{|I_{\vec{Q}}|^{1/2}} \right)_{\vec{Q} \in T}$$

is a modulated Calderón-Zygmund operator since we know that the collection

$\{\omega_{Q_3}\}_{\vec{Q} \in T}$ of intervals is lacunary, so it suffices to show that

$$2^{-\frac{\#}{2}} \|B_{\vec{P}'}(f_3, f_4)\|_1 \lesssim \|f_3\|_{1/(1-\theta)} \|f_4\|_{1/\theta}.$$

We observe that

$$\begin{aligned} 2^{-\frac{\#}{2}} \|B_{\vec{P}'}(f_3, f_4)\|_1 &= 2^{-\frac{\#}{2}} \sum_{\vec{P} \in \vec{P}'} \frac{1}{|I_{\vec{P}}|^{1/2}} |\langle f_3, \Phi_{P_2} \rangle| |\langle f_4, \Phi_{P_3} \rangle| \int_{\mathbf{R}} |\tilde{\Phi}_{\vec{P}_1}| dx \\ &= \sum_{\vec{P} \in \vec{P}'} |\langle f_3, \Phi_{P_2} \rangle| |\langle f_4, \Phi_{P_3} \rangle| \end{aligned}$$

as $\frac{\tilde{\Phi}_{\vec{P}_1}(x)}{2^{\frac{\#}{2}} |I_{\vec{P}}|^{1/2}}$ is L^1 -normalized. Then, it is equal to

$$\begin{aligned} &\sum_{\vec{P} \in \vec{P}'} \frac{|\langle f_3, \Phi_{P_2} \rangle|}{|I_{\vec{P}}|^{1/2}} \cdot \frac{|\langle f_4, \Phi_{P_3} \rangle|}{|I_{\vec{P}}|^{1/2}} \cdot |I_{\vec{P}}| \\ &\lesssim \int_{\mathbf{R}} \sum_{\vec{P} \in \vec{P}'} \frac{|\langle f_3, \Phi_{P_2} \rangle|}{|I_{\vec{P}}|^{1/2}} \cdot \frac{|\langle f_4, \Phi_{P_3} \rangle|}{|I_{\vec{P}}|^{1/2}} \cdot \tilde{\chi}_{I_{\vec{P}}}^{100M}(x) dx \\ &\lesssim \int_{\mathbf{R}} \left(\sum_{\vec{P} \in \vec{P}'} \frac{|\langle f_3, \Phi_{P_2} \rangle|^2}{|I_{\vec{P}}|} \cdot \tilde{\chi}_{I_{\vec{P}}}^{100M} \right)^{1/2} \left(\sum_{\vec{P} \in \vec{P}'} \frac{|\langle f_4, \Phi_{P_3} \rangle|^2}{|I_{\vec{P}}|} \cdot \tilde{\chi}_{I_{\vec{P}}}^{100M} \right)^{1/2} dx \\ &\lesssim \int_{\mathbf{R}} |S_2(f_3)(x)| |S_3(f_4)(x)| dx, \end{aligned} \tag{7.5}$$

where $S_j f := \left(\langle f, \Phi_{P_j} \rangle \frac{\tilde{\chi}_{I_{\vec{P}}}^{50M}}{|I_{\vec{P}}|^{1/2}} \right)_{\vec{P} \in \vec{P}'}$. Then, since we know that the collections of intervals, $\{\omega_{P_2}\}_{\vec{P}, T}$ and $\{\omega_{P_3}\}_{\vec{P}, T}$, are lacunary, we can majorize (7.5) by

$$(7.5) \lesssim \|S_2(f_3)(x)\|_{1/(1-\theta)} \|S_3(f_4)(x)\|_{1/\theta} \lesssim \|f_3\|_{1/(1-\theta)} \|f_4\|_{1/\theta}$$

for any $0 \leq \theta < 1$.

Now we consider the relatively easy case $f_3 \equiv 0$ on $5I_T$. A proof of this

case is essentially similar to the proof of Lemma 9.1 in [15] modulo the extra constraint $2^\# |\omega_{Q_3}| \sim |\omega_{P_1}|$ from the definition of $a_{Q_3}^{(3),\#}$. This extra constraint does not affect the proof in this case because we know that the collections of intervals, $\{\omega_{P_2}\}_{\vec{P},T}$ and $\{\omega_{P_3}\}_{\vec{P},T}$, are lacunary. For fixed $\vec{Q} \in T$, because of the decay of Φ_{P_1} and Φ_{Q_3} , we have

$$\begin{aligned} |a_{Q_3}^{(3),\#}| &\lesssim |I_{\vec{P}}|^{-1/2} \sum_{\substack{\vec{P} \in \vec{\mathbf{P}}: \omega_{Q_3} \subseteq \omega_{P_1} \\ 2^\# |\omega_{Q_3}| \sim |\omega_{P_1}|}} |\langle f_3, \Phi_{P_2} \rangle| |\langle f_4, \Phi_{P_3} \rangle| \int \frac{\tilde{\chi}_{I_{\vec{P}}}^{100M}}{|I_{\vec{P}}|^{1/2}} \frac{\tilde{\chi}_{I_{\vec{Q}}}^{100M}}{|I_{\vec{Q}}|^{1/2}} dx \\ &= |I_{\vec{Q}}|^{-1/2} \int \sum_{\substack{\vec{P} \in \vec{\mathbf{P}}: \omega_{Q_3} \subseteq \omega_{P_1} \\ 2^\# |\omega_{Q_3}| \sim |\omega_{P_1}|}} \left(|\langle f_3, \Phi_{P_2} \rangle| |\langle f_4, \Phi_{P_3} \rangle| \frac{\tilde{\chi}_{I_{\vec{P}}}^{100M}}{|I_{\vec{P}}|} \right) \tilde{\chi}_{I_{\vec{Q}}}^{100M} dx. \end{aligned}$$

Furthermore, by applying Cauchy-Schwarz inequality, we thus have

$$|a_{Q_3}^{(3),\#}| \lesssim |I_{\vec{Q}}|^{-1/2} \int |S_2 f_3| |S_3 f_4| \tilde{\chi}_{I_{\vec{Q}}}^{100M} dx,$$

where the square function S_j , for $j = 2, 3$, is the vector-valued quantity

$$S_j f := \left(\langle f, \Phi_{P_j} \rangle \frac{\tilde{\chi}_{I_{\vec{P}}}^{50M}}{|I_{\vec{P}}|^{1/2}} \right)_{\vec{P} \in \vec{\mathbf{P}}: \omega_{Q_3} \subseteq \omega_{P_1}, 2^\# |\omega_{Q_3}| \sim |\omega_{P_1}|}.$$

We claim the weighted square-function estimate

$$\|S_j f\|_{L^p(\tilde{\chi}_{I_{\vec{Q}}}^{100M} dx)} \lesssim \|f\|_{L^p(\tilde{\chi}_{I_{\vec{Q}}}^M dx)},$$

for all $1 < p < \infty$ and $j = 2, 3$, by Hölder, which follows because the collection $\{\omega_{P_j}\}_{\vec{P},T}$ of intervals, for each $j = 2, 3$, is lacunary around some frequency ξ_o .

So S_j is a modulated Calderón-Zygmund operator whose kernel is

$$K_j(x, y) = \bar{\Phi}_{P_j}(y) \frac{\tilde{\chi}_{I_{\bar{P}}}^{50M}(x)}{|I_{\bar{P}}|^{1/2}},$$

which decays like $O(|I_{\bar{Q}}|^{-1}(|x - y|/|I_{\bar{Q}}|)^{-50M})$ for all $|x - y| \gg |I_{\bar{Q}}|$.

Therefore, we see that

$$\begin{aligned} |a_{Q_3}^{(3),\#}| &\lesssim |I_{\bar{Q}}|^{-1/2} \|f_3\|_{L^{\frac{1}{1-\theta}}(\tilde{\chi}_{I_{\bar{Q}}}^M dx)} \|f_4\|_{L^{\frac{1}{\theta}}(\tilde{\chi}_{I_{\bar{Q}}}^M dx)} \\ &\lesssim |I_{\bar{Q}}|^{-1/2} \left(\int_{E_3} \tilde{\chi}_{I_{\bar{Q}}}^M \right)^{1-\theta} \left(\int_{E_4} \tilde{\chi}_{I_{\bar{Q}}}^M \right)^\theta \\ &\lesssim |I_{\bar{Q}}|^{-1/2} \left(\frac{|I_{\bar{Q}}|}{|I_T|} \right)^{M(1-\theta)} \left(\int_{E_3} \tilde{\chi}_{I_T}^M \right)^{1-\theta} \left(\int_{E_4} \tilde{\chi}_{I_T}^M \right)^\theta \end{aligned}$$

for all $\vec{Q} \in T$; the claim (7.3) then follows by square-summing in \vec{Q} . This proves the estimate (7.3) when $f_3 \equiv 0$ on $5I_T$. A similar argument gives the estimate (7.3) when $f_4 \equiv 0$ on $5I_T$. \square

We will now prove the claim in the proof of Lemma 7.0.7.

Lemma 7.0.8. *Let $T \subset \vec{Q}$ be sparse i -tree for some $i = 1, 2$. And let*

$$\vec{P}'(T) = \{\vec{P}' = \tilde{P}_1 \times P_2 \times P_3 : \vec{P} = P_1 \times P_2 \times P_3 \in \vec{P} \text{ and } \tilde{P}_1 = I_{\tilde{P}, 2^\#} \times \omega_{Q_3}\}$$

$$\text{for some } \vec{Q} \in T \text{ with } \omega_{Q_3} \subseteq \omega_{P_1} \text{ and } 2^\# |\omega_{Q_3}| \sim |\omega_{P_1}|\},$$

where $I_{\tilde{P}, 2^\#}$ is the unique dyadic interval of length $2^\# |I_{\tilde{P}}|$ containing $I_{\tilde{P}}$. Then by assuming $\vec{Q} \in T$ and $\vec{P} \in \vec{P}$, we have the following statement:

$$\omega_{Q_3} \subseteq \omega_{P_1} \text{ and } 2^\# |\omega_{Q_3}| \sim |\omega_{P_1}| \quad \text{if and only if} \quad \vec{P}' \in \vec{P}' \text{ and } \langle \tilde{\Phi}_{\tilde{P}_1}, \Phi_{Q_3} \rangle \neq 0.$$

We remark that the collection $\vec{\mathbf{P}}'$ depends only on $\vec{\mathbf{P}}$ and $T \subset \vec{\mathbf{Q}}$, and not on each $\vec{Q} \in T$.

Proof. Let $\vec{Q} \in T$ and $\vec{P} \in \vec{\mathbf{P}}$. And let $\vec{P}' \in \vec{\mathbf{P}}'$ such that $\langle \tilde{\Phi}_{\vec{P}'}, \Phi_{Q_3} \rangle \neq 0$. Then by definition of $\vec{\mathbf{P}}'$, we know that there exists a $\vec{Q}^* \in T$ such that $\omega_{Q_3^*} \subseteq \omega_{P_1}$, $2^\# |\omega_{Q_3^*}| \sim |\omega_{P_1}|$ and $\tilde{P}_1 = I_{\vec{P}, 2^\#} \times \omega_{Q_3^*}$. Since $\langle \tilde{\Phi}_{\vec{P}'}, \Phi_{Q_3} \rangle \neq 0$, we have that $\omega_{Q_3^*} \cap \omega_{Q_3} \neq \emptyset$. Thus we obtain that $\omega_{Q_3^*} = \omega_{Q_3}$ because both \vec{Q} and \vec{Q}^* are in T which is sparse i -tree for some $i = 1, 2$. Therefore, we have that $\omega_{Q_3} \subseteq \omega_{P_1}$ and $2^\# |\omega_{Q_3}| \sim |\omega_{P_1}|$. This proves the “if” part. Since the opposite implication is obvious, we have completed the proof. \square

We will give the proof of the energy estimates.

Lemma 7.0.9. *For $j = 3, 4$, let E_j be sets of finite measure in \mathbf{R} and f_j be functions in $X(E_j)$. Then, we have*

$$\text{energy}_3((a_{Q_3}^{(3), \#})_{\vec{Q} \in \vec{\mathbf{Q}}}) \lesssim 2^{\#/2} \left(|E_4|^{1/2} \sup_{\vec{P} \in \vec{\mathbf{P}}} \frac{\int_{E_3} \tilde{\chi}_{I_{\vec{P}}}^M}{|I_{\vec{P}}|} \right)^{1-\theta} \left(|E_3|^{1/2} \sup_{\vec{P} \in \vec{\mathbf{P}}} \frac{\int_{E_4} \tilde{\chi}_{I_{\vec{P}}}^M}{|I_{\vec{P}}|} \right)^\theta$$

for any $0 < \theta < 1$ and $M > 0$, with the implicit constant depending on θ, M . In particular, we have $\text{energy}_3((a_{Q_3}^{(3), \#})_{\vec{Q} \in \vec{\mathbf{Q}}}) \lesssim 2^{\#/2} |E_4|^{(1-\theta)/2} |E_3|^{\theta/2}$ for any $0 < \theta < 1$, with the implicit constant depending on θ .

Proof. By Lemma 6.2.2, it suffices to show that

$$\left| \sum_{T \in \mathbf{T}} \sum_{\vec{Q} \in T} a_{Q_3}^{(3), \#} \bar{c}_{Q_3} \right| \lesssim 2^{\#/2} \left(|E_4|^{1/2} \sup_{\vec{P} \in \vec{\mathbf{P}}} \frac{\int_{E_3} \tilde{\chi}_{I_{\vec{P}}}^M}{|I_{\vec{P}}|} \right)^{1-\theta} \left(|E_3|^{1/2} \sup_{\vec{P} \in \vec{\mathbf{P}}} \frac{\int_{E_4} \tilde{\chi}_{I_{\vec{P}}}^M}{|I_{\vec{P}}|} \right)^\theta \quad (7.6)$$

for all collections \mathbf{T} of strongly 3-disjoint trees and all coefficients c_{Q_3} such that

$$\sum_{\tilde{Q} \in \tilde{\mathbf{T}}} |c_{Q_3}|^2 \lesssim \frac{|I_{\tilde{\mathbf{T}}}|}{\sum_{T \in \mathbf{T}} |I_T|}$$

for all $T \in \mathbf{T}$ and all sub-trees $\tilde{T} \subseteq T$. Now fix \mathbf{T} and c_{Q_3} . Then,

$$\begin{aligned} \left| \sum_{T \in \mathbf{T}} \sum_{\tilde{Q} \in \tilde{\mathbf{T}}} a_{Q_3}^{(3), \#} \bar{c}_{Q_3} \right| &= \left| \sum_T \sum_{\tilde{Q}} \sum_{\substack{\tilde{P} \in \tilde{\mathbf{P}}; \omega_{Q_3} \subset \omega_{P_1} \\ 2^{\#} |\omega_{Q_3}| \sim |\omega_{P_1}|}} \frac{1}{|I_{\tilde{P}}|^{1/2}} \langle f_3, \Phi_{P_2} \rangle \langle f_4, \Phi_{P_3} \rangle \langle \Phi_{P_1}, \Phi_{Q_3} \rangle \bar{c}_{Q_3} \right| \\ &= \left| \sum_{\tilde{P} \in \tilde{\mathbf{P}}} \frac{1}{|I_{\tilde{P}}|^{1/2}} b_{P_1}^{(1)} \langle f_3, \Phi_{P_2} \rangle \langle f_4, \Phi_{P_3} \rangle \right|, \end{aligned} \quad (7.7)$$

where

$$b_{P_1}^{(1)} := \sum_T \sum_{\substack{\tilde{Q} \in \tilde{\mathbf{T}}; \omega_{Q_3} \subset \omega_{P_1} \\ 2^{\#} |\omega_{Q_3}| \sim |\omega_{P_1}|}} \langle \Phi_{P_1}, c_{Q_3} \Phi_{Q_3} \rangle.$$

Then, by Proposition 6.2.3, (7.7) can be majorized by

$$\begin{aligned} &\text{size}_1((b_{P_1}^{(1)})_{\tilde{P} \in \tilde{\mathbf{P}}})^{\theta_1} \text{energy}_1((b_{P_1}^{(1)})_{\tilde{P} \in \tilde{\mathbf{P}}})^{1-\theta_1} \cdot \text{size}_2(\langle f_3, \Phi_{P_2} \rangle_{\tilde{P} \in \tilde{\mathbf{P}}})^{\theta_2} \\ &\quad \text{energy}_2(\langle f_3, \Phi_{P_2} \rangle_{\tilde{P} \in \tilde{\mathbf{P}}})^{1-\theta_2} \text{size}_3(\langle f_4, \Phi_{P_3} \rangle_{\tilde{P} \in \tilde{\mathbf{P}}})^{\theta_3} \text{energy}_3(\langle f_4, \Phi_{P_3} \rangle_{\tilde{P} \in \tilde{\mathbf{P}}})^{1-\theta_3} \end{aligned} \quad (7.8)$$

for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$. Furthermore, by Lemma 6.2.4 and Lemma 6.2.5, we obtain that (7.8) can be majorized by

$$\begin{aligned} &\text{size}_1((b_{P_1}^{(1)})_{\tilde{P} \in \tilde{\mathbf{P}}})^{\theta_1} \text{energy}_1((b_{P_1}^{(1)})_{\tilde{P} \in \tilde{\mathbf{P}}})^{1-\theta_1} \left(\sup_{\tilde{P} \in \tilde{\mathbf{P}}} \frac{\int_{E_3} \tilde{\chi}_{I_{\tilde{P}}}^M}{|I_{\tilde{P}}|} \right)^{\theta_2} \left(\sup_{\tilde{P} \in \tilde{\mathbf{P}}} \frac{\int_{E_4} \tilde{\chi}_{I_{\tilde{P}}}^M}{|I_{\tilde{P}}|} \right)^{\theta_3} \\ &\quad \|f_3\|_2^{1-\theta_2} \|f_4\|_2^{1-\theta_3}. \end{aligned} \quad (7.9)$$

By setting $\theta_1 = 0$ and $\theta_3 := \theta$, we have that (7.9) is equal to

$$\begin{aligned} & \text{energy}_1((b_{P_1}^{(1)})_{\vec{P} \in \vec{\mathbf{P}}}) \left(\sup_{\vec{P} \in \vec{\mathbf{P}}} \frac{\int_{E_3} \tilde{\chi}_{I_{\vec{P}}}^M}{|I_{\vec{P}}|} \right)^{1-\theta} \left(\sup_{\vec{P} \in \vec{\mathbf{P}}} \frac{\int_{E_4} \tilde{\chi}_{I_{\vec{P}}}^M}{|I_{\vec{P}}|} \right)^\theta \|f_3\|_2^\theta \|f_4\|_2^{1-\theta} \\ & \lesssim \text{energy}_1((b_{P_1}^{(1)})_{\vec{P} \in \vec{\mathbf{P}}}) \left(|E_4|^{1/2} \sup_{\vec{P} \in \vec{\mathbf{P}}} \frac{\int_{E_3} \tilde{\chi}_{I_{\vec{P}}}^M}{|I_{\vec{P}}|} \right)^{1-\theta} \left(|E_3|^{1/2} \sup_{\vec{P} \in \vec{\mathbf{P}}} \frac{\int_{E_4} \tilde{\chi}_{I_{\vec{P}}}^M}{|I_{\vec{P}}|} \right)^\theta. \end{aligned}$$

So, to establish (7.6), we are left to show that

$$\text{energy}_1((b_{P_1}^{(1)})_{\vec{P} \in \vec{\mathbf{P}}}) \lesssim 2^{\# / 2}.$$

By Lemma 6.2.2 again, we obtain that there exist a collection \mathbf{T}' of strongly 1-disjoint trees and a complex coefficient d_{P_1} , for all $\vec{P} \in \bigcup_{T' \in \mathbf{T}'} T'$, such that

$$\text{energy}_1((b_{P_1}^{(1)})_{\vec{P} \in \vec{\mathbf{P}}}) \sim \left| \sum_{T' \in \mathbf{T}'} \sum_{\vec{P} \in T'} b_{P_1} \bar{d}_{P_1} \right|$$

and such that

$$\sum_{\vec{P} \in \tilde{T}''} |d_{P_1}|^2 \lesssim \frac{|I_{T''}|}{\sum_{T' \in \mathbf{T}'} |I_{T'}|}$$

for all $T' \in \mathbf{T}'$ and all sub-trees $T'' \subseteq T'$ of T' .

Then, we reduce to showing that

$$\left| \sum_{T' \in \mathbf{T}'} \sum_{\vec{P} \in T'} b_{P_1} \bar{d}_{P_1} \right| = \left| \sum_{T' \in \mathbf{T}'} \sum_{\vec{P} \in T'} \sum_{T \in \mathbf{T}} \sum_{\substack{\vec{Q} \in T; \omega_{Q_3} \subset \omega_{P_1} \\ 2^{\#} |\omega_{Q_3}| \sim |\omega_{P_1}|}} c_{Q_3} d_{P_1} \langle \Phi_{P_1} \Phi_{Q_3} \rangle \right| \lesssim 2^{\# / 2}. \quad (7.10)$$

From the decay of the Φ_{Q_3} we have

$$|\langle \Phi_{P_1} \Phi_{Q_3} \rangle| \lesssim \frac{|I_{\vec{P}}|^{1/2}}{|I_{\vec{Q}}|^{1/2}} \left(1 + \frac{\text{dist}(I_{\vec{Q}}, I_{\vec{P}})}{|I_{\vec{Q}}|} \right)^{-100}$$

$$\lesssim 2^{-\# / 2} 2^{-100k},$$

if we assume $\text{dist}(I_{\vec{Q}}, I_{\vec{P}}) \sim 2^k |I_{\vec{Q}}|$ for some $k \in \mathbf{Z}$.

Now, choose any $T' \in \mathbf{T}'$ and pick any $\vec{P} \in T'$. Then, we construct sets $\mathcal{A}_{\vec{P}, \mathbf{T}}^k$ defined by $\mathcal{A}_{\vec{P}, \mathbf{T}}^k := \{\vec{Q} \in \bigcup_{T \in \mathbf{T}} T : \omega_{Q_3} \subset \omega_{P_1}, 2^\# |\omega_{Q_3}| \sim |\omega_{P_1}| \text{ and } \text{dist}(I_{\vec{Q}}, I_{\vec{P}}) \sim 2^k |I_{\vec{Q}}|\}$ for $k \geq 1$. Then, we can see that there is a bounded number $C_{\#, k}$ for \vec{Q} satisfying all assumptions in the set for each $\vec{P} \in T'$. More specifically, we establish that $C_{\#, k} \lesssim 2^k \cdot 2^\#$.

Hence, using the Cauchy-Schwarz inequality, we can estimate the corresponding piece of (7.10) by

$$\begin{aligned} & \left| \sum_{T' \in \mathbf{T}'} \sum_{\vec{P} \in T'} \sum_{\vec{Q} \in \mathcal{A}_{\vec{P}, \mathbf{T}}^k} c_{Q_3} d_{P_1} \langle \Phi_{P_1} \Phi_{Q_3} \rangle \right| \\ & \lesssim C_{\#, k} 2^{-\# / 2} 2^{-100k} \left(\sum_{T' \in \mathbf{T}'} \sum_{\vec{P} \in T'} |d_{P_1}|^2 \right)^{1/2} \left(\sum_{T \in \mathbf{T}} \sum_{\vec{Q} \in T} |c_{Q_3}|^2 \right)^{1/2} \\ & \lesssim 2^{\# / 2} 2^{-99k}. \end{aligned} \tag{7.11}$$

Summing over $k \geq 1$, we have

$$\left| \sum_{T' \in \mathbf{T}'} \sum_{\vec{P} \in T'} \sum_{T \in \mathbf{T}} \sum_{\substack{\vec{Q} \in T; \omega_{Q_3} \subset \omega_{P_1} \\ 2^\# |\omega_{Q_3}| \sim |\omega_{P_1}| \\ \text{dist}(I_{\vec{P}}, I_{\vec{Q}}) \gtrsim |I_{\vec{Q}}|}} c_{Q_3} d_{P_1} \langle \Phi_{P_1} \Phi_{Q_3} \rangle \right| \lesssim 2^{\# / 2}.$$

If $\text{dist}(I_{\vec{P}}, I_{\vec{Q}}) \lesssim |I_{\vec{Q}}|$, then we have $I_{\vec{P}} \subseteq 3I_{\vec{Q}}$ and thus

$$|\langle \Phi_{P_1} \Phi_{Q_3} \rangle| \lesssim 2^{-\# / 2}.$$

Hence we can estimate the corresponding piece of (7.10) by

$$\left| \sum_{T' \in \mathbf{T}'} \sum_{\vec{P} \in T'} \sum_{T \in \mathbf{T}} \sum_{\substack{\vec{Q} \in T; \omega_{Q_3} \subset \omega_{P_1} \\ \text{dist}(I_{\vec{P}}, I_{\vec{Q}}) \leq |I_{\vec{Q}}|}} c_{Q_3} d_{P_1} \langle \Phi_{P_1} \Phi_{Q_3} \rangle \right| \lesssim 3 \cdot 2^\# 2^{-\#/2} \lesssim 2^{\#/2}.$$

This completes the proof of the energy estimates. □

Chapter 8

Proof of Theorem 2.0.7

We now use the estimates from the previous two chapters to prove the main theorem. We have few cases to consider depending on the bad index of each vertex A_i, \tilde{A}_i for $1 < i < 12$. Here, we only consider the case of vertex A_i because the other case can be done analogously.

(Case 1: vertices with a bad index 1): the bad index 2 case can be dealt with similarly.

Fix an admissible tuple $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ near A_i for some $9 \leq i \leq 12$. Fix also the finite collections $\vec{\mathbf{P}}, \vec{\mathbf{Q}}$ of tri-tiles and an arbitrary tuple (E_1, E_2, E_3, E_4) of subsets of \mathbb{R} with finite measure. We need to find a major subset E'_1 of E_1 such that

$$|\Lambda_{\vec{\mathbf{P}}, \vec{\mathbf{Q}}}^\#(f_1, f_2, f_3, f_4)| \lesssim 2^{\#/2} |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3} |E_4|^{\alpha_4}$$

for all tuple (f_1, f_2, f_3, f_4) of functions with $f_1 \in X(E'_1)$ and $f_i \in X(E_i)$, $i = 2, 3, 4$.

Define the exceptional set Ω_{C_1} by

$$\Omega_{C_1} := \bigcup_{j=1}^4 \{M_{\chi_{E_j}} > C_1 |E_j| / |E_1|\}$$

for a large constant C_1 , where M is the dyadic Hardy-Littlewood maximal function. By the classical Hardy-Littlewood inequality, we have $|\Omega_{C_1}| < |E_1|/2$ if C_1 is sufficiently large. Thus, if we set $E'_1 := E_1 \setminus \Omega_{C_1}$ for a sufficiently large constant C_1 , then E'_1 is a major subset of E_1 . Now, we shall show that, for fixed $f_1 \in X(E'_1)$ and $f_i \in X(E_i)$, for $i = 2, 3, 4$, we establish the estimate

$$\left| \sum_{\vec{Q} \in \vec{\mathbf{Q}}} \frac{1}{|I_{\vec{Q}}|^{1/2}} a_{Q_1}^{(1)} a_{Q_2}^{(2)} a_{Q_3}^{(3),\#} \right| \lesssim 2^{\#/2} |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3} |E_4|^{\alpha_4}, \quad (8.1)$$

where the $a_{Q_1}^{(1)}$, $a_{Q_2}^{(2)}$ and $a_{Q_3}^{(3),\#}$ are defined by (7.1).

We first consider a decomposition of $\vec{\mathbf{Q}}$ as $\bigcup_{k \geq 0} \vec{\mathbf{Q}}_k$ where, for each $k \geq 0$, the finite collection $\vec{\mathbf{Q}}_k$ of tri-tiles is given by

$$\vec{\mathbf{Q}}_k := \left\{ \vec{Q} \in \vec{\mathbf{Q}} : 1 + \frac{\text{dist}(I_{\vec{Q}}, \mathbf{R} \setminus \Omega_{C_1})}{|I_{\vec{Q}}|} \sim 2^k \right\}.$$

We will establish such corresponding estimate of $\vec{\mathbf{Q}}_k$ as (8.1) with an additional factor 2^{-k} on the right-hand side, which enables us to complete our proof after summing over k .

Fix $k \geq 0$ and then, for any $\vec{Q} \in \vec{\mathbf{Q}}_k$, we have

$$\frac{\int_{E'_1} \tilde{\chi}_{I_{\vec{Q}}}^{M_1}}{|I_{\vec{Q}}|} \lesssim 2^{-(M_1+C_1)k},$$

and, for $j = 2, 3, 4$,

$$\frac{\int_{E_j} \tilde{\chi}_{I_{\vec{Q}}}^{M_1}}{|I_{\vec{Q}}|} \lesssim 2^k \frac{|E_j|}{|E_1|}$$

for a sufficiently big number M_1 .

By Lemma 6.2.4 and Lemma 7.0.7, we then have that

$$\begin{aligned} \text{size}_1((a_{Q_1}^{(1)})_{\vec{Q} \in \vec{\mathbf{Q}}_k}) &\lesssim 2^{-(M_1+C_1)k} \\ \text{size}_2((a_{Q_2}^{(2)})_{\vec{Q} \in \vec{\mathbf{Q}}_k}) &\lesssim 2^k \frac{|E_2|}{|E_1|} \\ \text{size}_3((a_{Q_3}^{(3),\#})_{\vec{Q} \in \vec{\mathbf{Q}}_k}) &\lesssim 2^k \frac{|E_3|^{1-\theta} |E_4|^\theta}{|E_1|} \end{aligned}$$

for some $0 < \theta < 1$. Similarly, by Lemma 6.2.5 and Lemma 7.0.9, we obtain that

$$\begin{aligned} \text{energy}_1((a_{Q_1}^{(1)})_{\vec{Q} \in \vec{\mathbf{Q}}_k}) &\lesssim |E_1|^{1/2} \\ \text{energy}_2((a_{Q_2}^{(2)})_{\vec{Q} \in \vec{\mathbf{Q}}_k}) &\lesssim |E_2|^{1/2} \\ \text{energy}_3((a_{Q_3}^{(3),\#})_{\vec{Q} \in \vec{\mathbf{Q}}_k}) &\lesssim 2^{\# / 2} |E_3|^{(1-\theta)/2} |E_4|^{\theta/2}. \end{aligned}$$

By Proposition 6.2.3, for a sufficiently large M_1 , we establish that

$$\begin{aligned} & \left| \sum_{\vec{Q} \in \vec{\mathbf{Q}}_k} \frac{1}{|I_{\vec{Q}}|^{1/2}} a_{Q_1}^{(1)} a_{Q_2}^{(2)} a_{Q_3}^{(3),\#} \right| \\ & \lesssim 2^{\# / 2} 2^{-k} \frac{|E_1|^{(1+\theta_1)/2} |E_2|^{(1+\theta_2)/2} (|E_3|^{(1-\theta)/2} |E_4|^{\theta/2})^{(1+\theta_3)/2}}{|E_1|} \end{aligned}$$

for $0 < \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$. Now, by choosing $\theta_1 := 2\alpha_1 + 1$, $\theta_2 := 2\alpha_2 - 1$, $\theta_3 := 2(\alpha_3 + \alpha_4) - 1$, and $\theta := \frac{\alpha_4}{(\alpha_3 + \alpha_4)}$, we obtain the corresponding estimates for $\vec{\mathbf{Q}}_k$ in (8.1).

(Case 2: vertices with a bad index 4): the bad index 3 case can be dealt with similarly.

Fix an admissible tuple $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ near A_i for some $i = 1, 2$. Fix also the finite collections $\vec{\mathbf{P}}, \vec{\mathbf{Q}}$ of tri-tiles and an arbitrary tuple (E_1, E_2, E_3, E_4) of subsets of \mathbf{R} with finite measure. As before, define the exceptional set Ω_{C_2} by

$$\Omega_{C_2} := \bigcup_{j=1}^4 \{M_{\chi_{E_j}} > C_2 |E_j| / |E_4|\}$$

for a constant C_2 , and then set $E'_4 := E_4 \setminus \Omega_{C_2}$. Hence E'_4 is a major subset of E_4 , if C_2 is sufficiently large. Now, we will show that, for fixed $f_4 \in X(E'_4)$ and $f_i \in X(E_i)$, for $i = 1, 2, 3$, we establish the estimate

$$\left| \sum_{\vec{Q} \in \vec{\mathbf{Q}}} \frac{1}{|I_{\vec{Q}}|^{1/2}} a_{Q_1}^{(1)} a_{Q_2}^{(2)} a_{Q_3}^{(3), \#} \right| \lesssim 2^{\#/2} |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3} |E_4|^{\alpha_4}, \quad (8.2)$$

where the $a_{Q_1}^{(1)}$, $a_{Q_2}^{(2)}$ and $a_{Q_3}^{(3), \#}$ are given by (7.1).

We now obtain a decomposition of $\vec{\mathbf{Q}}$ as $\bigcup_{k \geq 0} \vec{\mathbf{Q}}_k$, where, for each $k \geq 0$, the finite collection $\vec{\mathbf{Q}}_k$ of tri-tiles is given by

$$\vec{\mathbf{Q}}_k := \left\{ \vec{Q} \in \vec{\mathbf{Q}} : 1 + \frac{\text{dist}(I_{\vec{Q}}, \mathbf{R} \setminus \Omega_{C_2})}{|I_{\vec{Q}}|} \sim 2^k \right\},$$

and similarly obtain a decomposition of $\vec{\mathbf{P}}$ as $\bigcup_{k' \geq 0} \vec{\mathbf{P}}_{k'}$, where, for each $k' \geq 0$, the finite collection $\vec{\mathbf{P}}_{k'}$ of tri-tiles is given by

$$\vec{\mathbf{P}}_{k'} := \left\{ \vec{P} \in \vec{\mathbf{P}} : 1 + \frac{\text{dist}(I_{\vec{P}}, \mathbf{R} \setminus \Omega_{C_2})}{|I_{\vec{P}}|} \sim 2^{k'} \right\}.$$

We will establish such corresponding estimate of $\vec{\mathbf{Q}}_k$ and $\vec{\mathbf{P}}_{k'}$ as (8.2) with an additional factor $2^{-k-k'}$ on the right-hand side, which enables us to complete

our proof after summing over k and k' .

For fixed $k \geq 0$ and for any $\vec{Q} \in \vec{\mathbf{Q}}_k$, we have that

$$\frac{\int_{E'_4} \tilde{\chi}_{I_{\vec{Q}}}^{M_2}}{|I_{\vec{Q}}|} \lesssim 2^{-(M_2+C_2)k}$$

$$\frac{\int_{E_j} \tilde{\chi}_{I_{\vec{Q}}}^{M_2}}{|I_{\vec{Q}}|} \lesssim 2^k \frac{|E_j|}{|E_4|}$$

for $j = 1, 2, 3$, and for a sufficiently big number M_2 .

By Lemma 6.2.4 and Lemma 7.0.7 we have that

$$\text{size}_1((a_{Q_1}^{(1)})_{\vec{Q} \in \vec{\mathbf{Q}}_k}) \lesssim 2^k \frac{|E_1|}{|E_4|}$$

$$\text{size}_2((a_{Q_2}^{(2)})_{\vec{Q} \in \vec{\mathbf{Q}}_k}) \lesssim 2^k \frac{|E_2|}{|E_4|}$$

$$\text{size}_3((a_{Q_3}^{(3),\#})_{\vec{Q} \in \vec{\mathbf{Q}}_k}) \lesssim 2^{-(M_2+C_2)k}$$

for some $0 < \theta < 1$. Similarly, by Lemma 6.2.5 and Lemma 7.0.9, we have

$$\text{energy}_1((a_{Q_1}^{(1)})_{\vec{Q} \in \vec{\mathbf{Q}}_k}) \lesssim |E_1|^{1/2}$$

$$\text{energy}_2((a_{Q_2}^{(2)})_{\vec{Q} \in \vec{\mathbf{Q}}_k}) \lesssim |E_2|^{1/2}.$$

Now fix $k' \geq 0$. For any $\vec{P} \in \vec{\mathbf{P}}_{k'}$, we obtain that

$$\frac{\int_{E_3} \tilde{\chi}_{I_{\vec{P}}}^{M_2}}{|I_{\vec{P}}|} \lesssim 2^{k'} \frac{|E_3|}{|E_4|}$$

$$\frac{\int_{E'_4} \tilde{\chi}_{I_{\vec{P}}}^{M_2}}{|I_{\vec{P}}|} \lesssim 2^{-(M_2+C_2)k'}.$$

By Lemma 7.0.9, we thus have

$$\text{energy}_3((a_{Q_3}^{(3),\#})_{\vec{Q} \in \vec{\mathbf{Q}}_k}, \vec{P} \in \vec{\mathbf{P}}_{k'}) \lesssim 2^{\#/2} 2^{-(M_2+C_2)k'} |E_3|^{(2-\theta)/2} |E_4|^{(\theta-1)/2}$$

for some $0 < \theta < 1$ to be chosen later.

By Proposition 6.2.3, for a sufficiently large M_2 , we establish that

$$\begin{aligned} & \left| \sum_{\vec{Q} \in \vec{\mathbf{Q}}_k, \vec{P} \in \vec{\mathbf{P}}_{k'}} \frac{1}{|I_{\vec{Q}}|^{1/2}} a_{Q_1}^{(1)} a_{Q_2}^{(2)} a_{Q_3}^{(3),\#} \right| \\ & \lesssim 2^{\#/2} 2^{-k-k'} \frac{|E_1|^{(1+\theta_1)/2} |E_2|^{(1+\theta_2)/2}}{|E_4|^{1-\theta_3}} (|E_3|^{(2-\theta)/2} |E_4|^{(\theta-1)/2})^{1-\theta_3} \end{aligned}$$

for $0 < \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$. By setting $\theta_1 := 2\alpha_1 - 1$, $\theta_2 := 2\alpha_2 - 1$, $\theta_3 := 2(\alpha_3 + \alpha_4) - 1$, and $\theta := \frac{(3\alpha_3+2\alpha_4)}{\alpha_3+\alpha_4}$, we obtain the corresponding estimates for $\vec{\mathbf{Q}}_k$ and $\vec{\mathbf{P}}_{k'}$ in (8.2).

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