ESSAYS ON FUNDING LIQUIDITY AND CREDIT RISK DECOMPOSITION

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

Andrey Krishenik

May 2015

ESSAYS ON FUNDING LIQUIDITY AND CREDIT RISK DECOMPOSITION

Andrey Krishenik, Ph.D.

Cornell University 2015

The thesis consists of three essays on Funding Liquidity and Credit Risk Decomposition.

The recent crisis has shown the importance of the illiquidity component of credit risk. We model, in a multi-period setting, the funding liquidity of a borrower that finances its operations through short-term debt. The short-term debt is provided by a continuum of agents with heterogeneous beliefs about the prospects of the borrower. In each period, the creditors observe the borrower's fundamentals and decide on the amount they invest in its short-term debt. We formalize this problem as a coordination game and show that it features multiple Nash equilibria. This leads to an ambiguity about the value of the barrier for the liquid net worth which would trigger the default of the borrower. Removing weakly dominated strategies allows us to eliminate the multiplicity of equilibria and determine the default barrier. The unique equilibrium is shown to be of a threshold type, a property assumed in the previous literature.

We then extend the model and show the existence of the equilibria in which the company holds cash reserves in order to increase its debt capacity. We prove the existence of the optimal amount of cash on the company's balance sheet and offer a theoretical explanation for the cash holding puzzle.

The third part of the thesis connects the sovereign bond spreads and the Debt-to-GDP ratio for the Eurozone countries. The problem of sovereign spreads modeling is well studied in the literature, however their dynamics is

rarely considered as a function of macroeconomic parameters. We determine an explicit dependence of sovereign bond spread on Debt-to-GDP ratio, and, based on the bonds yield curve, compute the market perception of Debt-to-GDP ratio.

BIOGRAPHICAL SKETCH

Andrey grew up in Chernogolovka, Moscow Region, the city that had a great academic environment that significantly influenced his views on the science. In 2004 he entered Moscow Institute of Physics and Technology, being exempt from the entrance examinations. He received B.Sc and M.Sc degrees from MIPT in 2008 and 2010 respectively. In 2010 Andrey began his graduate studies at Cornell University, where he was awarded M.Sc and Ph.D degrees.

Following completion of the dissertation Andrey joins Guggenheim Partners Investment Company as a Quantitative Researcher.

To my parents

Olga V. Krishenik, Ph.D

and

Petr M. Krishenik, D.Sc

ACKNOWLEDGEMENTS

I was lucky to join the department of ORIE at Cornell University, where I had a chance to meet many great and interesting people.

I would like to express my deepest gratitude to my advisor, professor Andreea Minca, for her excellent guidance, caring, patience, and providing me with the motivation for doing research. I would never have been able to finish my dissertation without her support.

I want to thank my co-author and Committee Member, Robert Jarrow, who motivated me to learn many concepts of finance and economics. I have to admit, that his views on the scientific aspects of finance influenced me a lot.

I am also very grateful to Johannes Wissel, who was not only my co-author but a friend who encouraged me to develop new ideas. Our collaboration has always been exceptional.

I also would like to thank my Committee Member, Leonard Gross, who taught me the most advanced course in Mathematics I have ever taken. He introduced to me several Mathematical concepts, that I would not have learned otherwise.

My sincere thanks also goes to my first year advisor, Gennady Samorodnitsky, a great scientist and a person who helped me to adapt at Cornell in my first year.

I am also thankful to ORIE professors James Renegar, Dawn Woodard, Mark Lewis and David Shmoys who taught me the first courses at Cornell, Kathryn King, who was solving literally all administrative issues, and, of course, I will never forget our discussions with Dennis Panagitsas.

My last thanks go to my parents. I have always known that I would receive an unconditional support from them, and this knowledge has always been pushing me forward.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

In the thesis we describe several approaches to model credit risk. Credit risk is the risk of loss of principal or loss of a financial reward stemming from a borrower's failure to repay a loan or otherwise meet a contractual obligation. A natural approach to credit risk modeling and evaluation is a decomposition the credit risk into several components. The most important components are the illiquidity component, that represents the risk to be unable to pay debts as they fall due in the usual course of business or day-by-day operations and the insolvency component, that represents the risk of having liabilities in excess of a reasonable market value of assets held.

The financial crisis that started in 2007 resulted in failures of many financial companies in the US, including one of the biggest investment banks, Lehman Brothers, which held over $600 billion in assets, filed for Chapter 11 bankruptcy protection on September 15, 2008. This bankruptcy remains the largest bankruptcy filing in the US history. The bankruptcy was preceded by a $60 billion investors' run, and resulted in the inability of Lehman Brothers to meet the short-term obligations. Therefore, the bankruptcy was thought of being driven by the illiquidity component.

During economic crises, determining the market value of assets held is a challenging task. Therefore, in general the state of the firms is undetermined, since the market value of their asset may be unknown and the bankruptcy claims are triggered by illiquidity.

We develop a model that allows us to investigate an illiquidity component of the credit risk stemming from the creditor's beliefs about the borrower's peformance. We introduce the concept of *liquid net worth*, that is liquid assets minus all liabilities and prove the existence of the illiquidity barrier for the liquid net worth of the company.

We then extend the model and prove the existence of the optimal level of cash for the company, and provide the explanation to a 'cash holding puzzle': a firm may hold both debt and a positive cash position, since cash holdings can increase the debt capacity.

The thesis also covers another credit risk problem: the evaluation of sovereign bond risks and the dependence of this risks on Macroeconomic parameters. A natural measure of a sovereign bond risk is a risk spread. In the thesis we determine an explicit dependence of sovereign bond spread on Debt-to-GDP ratio.

The thesis is organized as follows. Chapter 1 provides the motivation for the research presented in the dissertation. In Chapter 2 we introduce the model that allows us to consider funding liquidity risk from Game Theoretical perspective. In Chapter 3 we extend the model and investigate the optimal strategies for the borrower. Chapter 4 stands separately from the previous two chapters and provides the theoretical model that shows the explicit dependence of sovereign risk spread on Debt-to-GDP ratio for certain countries. The proofs for the results of the Chapters 2 and 3 are in the Appendices A and B respectively.

CHAPTER 2

LIQUIDITY RISK MODELING

2.1 Introduction

By 2012, sovereign borrowers have accumulated around 50 trillion US dollars in debt, and the downturns of their economies resulted in more than 10 sovereign defaults within the last 15 years.

Funding liquidity risk is by no means specific to sovereign borrowers, it is also a considerable risk to financial institutions that fund a large fraction of their assets through short term debt (see e.g., (11, 25, 19, 2)). The defaults of Lehman Brothers and Bear Stearns are the recent examples of the institutions with adequate capital, which failed to meet their obligations when their short-term creditors withdrew funding.

While inability to refinance for a country and for a corporate borrower differ in many ways, their creditors face a common decision problem: Given the observation of the borrower's fundamentals and their beliefs about the future dynamics of these fundamentals, creditors must decide how much of their capital they are willing to lend.

To answer this question, we set up a model in which strategic creditors with heterogeneous beliefs invest into short term debt of a borrower. For models explaining why a large fraction of the debt is short term, we refer the reader to e.g., (10, 13). In this thesis, this fact is a model assumption.

Market participants are assumed to be risk neutral and myopic. All creditors observe the same borrower's fundamentals, but they differ in their beliefs about the future evolution of the fundamentals. We interpret fundamentals as an abstraction for public surplus or deficit (for a sovereign borrower) or P&L

(for a company). For their investment decisions, creditors take into account not only their own beliefs on the future evolution of the fundamentals, but also on the higher order beliefs, that is, beliefs about other creditors' beliefs, etc. In our model, we assume that higher order beliefs are given via the cross distribution of first order beliefs, and that this cross distribution is known to each creditor.

We represent the agents' allocation problem as a multi-period game with a continuum of agents in which each individual creditor has negligible weight relative to the total weight of all creditors. We assume that agents do not observe the strategy of other individual agents. Since coalitions of a finite number of agents have a negligible outcome on the default time in our setting, creditors do not form coalitions. At the beginning of a period, agents observe the evolution of the borrower's fundamentals up to present time and decide on their current allocation. At the end of the period, they learn the total short-term debt raised by the borrower. Since the default time depends not only on the evolution of the fundamentals, but also on the aggregate creditor decisions, this default time is a priori not observable for any agent at the time she makes her decision.

As usual in coordination games, the game we consider features multiple Nash equilibria. A key quantity in the analysis of the Nash equilibria is the *liquid net worth* of the borrower. We show existence by explicit construction of a certain Nash equilibrium. This equilibrium is of a threshold type, in which agents invest all their available capital if their belief is above the threshold, and do not invest at all if their belief is below the threshold. The threshold depends on the borrower's current liquid net worth and the cross distribution of beliefs. As soon as the liquid net worth falls below a certain negative value, called the *illiquidity barrier*, the short term debt jumps to zero, which corresponds to a run by the short term creditors, and the borrower defaults.

We, then, characterize all possible Nash equilibria and assume that the state of the market is characterized by one of the them. While the set of possible equilibria is known, the *played* equilibrium is not observable for any agent. Moreover, we make no assumptions about the probabilities assigned to these equilibria: We say that there is *ambiguity* about the played equilibrium. We show that the ambiguity is eliminated, if we assume that creditors do not play *weakly dominated* strategies.

Our concept of weak dominance can be explained as follows. Since the capital of each agent is small compared to the total available capital, the actions of an individual agent alone cannot affect the strategies played by the others. Consequently, no single agent can have an impact on the equilibrium played by the other agents. It is therefore natural to assume that when the market is in one of the equilibria, a reasonable agent will not play strategy A if strategy B gives her at least the same payoff as strategy A in *all equilibria of the other agents*, and a *strictly better payoff* in at least one such equilibrium. We say that strategy A is weakly dominated by strategy B in this case, and we call those equilibria in which agents do not use weakly dominated strategies *reasonable*.

Therefore, our assumptions that the market is in equilibrium and that agents do not play weakly dominated strategies imply that this unique reasonable equilibrium must be the played equilibrium, thereby eliminating equilibrium ambiguity. This result, consequently, removes the ambiguity about the total short term debt raised by the borrower.

This model complements two strands of literature: the literature on game theoretic models on rollover risk and the literature on structural credit risk models.

Ever since the seminal paper by (17), one of the main problems within the literature on game theoretic models on rollover risk has been the existence of multiple equilibria for the roll-over decisions of the creditors. One approach to this problem in the literature is the global games framework of (39) and (12), as for example adapted in (46, 24, 40). In this setting, agents observe fundamentals with a noise such that each agent has the same posterior distribution of the ranking, among all agents, of her observation. Uniqueness of equilibrium in these models relies on the crucial assumption that strategies are of a threshold type. Recently, (26) introduced a staggered debt structure which leads to a unique equilibrium. The main assumption which allows uniqueness is the fact that, at any time, the number of maturing creditors who do not roll over is negligible, and thus insufficient to cause the borrower to default due to illiquidity. The amount of debt rolled over is assumed to be observed in continuous time. Combined, these assumptions imply that the time of default due to illiquidity is measurable with respect to the information flow generated by the fundamentals. In contrast to these models, our analysis does not require noisy observations, a threshold structure, or the assumption that a creditor cannot lose her investment due to immediate illiquidity of the borrower. In our setup, we only assume that agents do not use weakly dominated strategies; this is sufficient to obtain a unique equilibrium.

The first generation of structural credit risk models, pioneered by (35) and (8), assume that default occurs when the net worth reaches an exogenously given barrier. A major merit of the recent work in this direction, e.g., (40, 26, 34, 32), is the focus on the interplay between insolvency and illiquidity risk (in the sense of funding illiquidity risk). Here we do not model insolvency risk, but we push further the analysis of the strategic interaction among creditors leading to funding illiquidity. We also obtain an endogenous illiquidity barrier. However, unlike in previous work, the relevant quantity in our model is the *liquid net worth, and not the net worth*. (34, 32) avoid the equilibrium ambiguity by assuming an exogenous model for the bank run probability. In our analysis, this model assumption is not needed, and we obtain the bank-run probability endogenously from the agents' beliefs about the borrower's fundamentals. Moreover, we do not assume a "representative agent" as in (34, 32), but we allow heterogeneous beliefs of creditors about the prospects of the borrower

2.2 The model

We consider a large borrower that funds itself through short term debt provided by a continuum of creditors. We consider a discrete time setup, with a finite horizon T for the economy. We model creditors as elements of a nonatomic agent space $(A = (0, 1), \mathcal{B}(0, 1), \gamma)$, where γ represents the capital measure of the agents. For every Borel set $B \subset A$, $\gamma(B)$ represents the total capital of the agents in B.

There is a *source of exogenous risk* in the economy, modeled as a probability space endowed with a public information filtration and a family of probability measures $(\Omega, \mathcal{F}, (\mathcal{F})_{t=1,\dots,T}, (\mathbb{P}^a)_{a\in A})$. Specifically, we let the stochastic process $(\theta_t), t = 1, \dots, T$ with $\theta_1 = 0$ model the borrower's fundamentals, that we can think of the P&L process for a corporate borrower or as the difference between government revenues and expenditures for a sovereign borrower in the period $(t-1, t)$. We let $\mathcal{F}_t := \sigma(Y_t)$, with

$$Y_t := (\theta_1, \dots, \theta_t),\ t = 1, \dots, T. \tag{2.1}$$

We assume that the "true" measure $\mathbb{P}$ and the distribution of the borrower's profit and loss θ_t under $\mathbb{P}$ are not known to any agent. Instead, each creditor a

has a *belief* on the distribution of the stochastic process θ given by a probability measure $\mathbb{P}^a$ equivalent to $\mathbb{P}$. In other words, creditor a believes that the real-world measure is $\mathbb{P}^a$. We assume that there exists a map

$$b : A \to \mathbb{R}$$

such that for any creditor $a \in A$, the random variables θ_t are iid with normal distribution $\mathcal{N}(b(a), 1)$ under the probability measure $\mathbb{P}^a$. Slightly abusing the terminology, we will simultaneously use the term *belief* for $\mathbb{P}^a$ and for the mean $b(a)$. We let C the image measure of γ induced by the map b, representing the creditors' capital distribution across beliefs,

$$C(b) := \gamma(a \in (0,1) \,\big|\, b(a) \geq b), \; b \in \mathbb{R}. \tag{2.2}$$

While no creditor knows the belief of any other creditor, the function $C(\cdot)$ modeling the distribution of beliefs across creditors is known to all creditors.

At each time, each creditor $a \in A$ allocates a fraction of her wealth to the borrower's debt. We assume that the creditors cannot short the borrower's debt, nor can they allocate more than 100% of their wealth to the borrower. We thus consider the following *family of allocation strategies for the creditors* (which are themselves functions of the agents and the trajectory of the stochastic process Y_t)

$$\mathcal{X} := \{\pi = (\pi_t)_{t=1,\ldots,T} \mid \pi_t : A \times \mathbb{R}^t \to [0,1], \; \pi_T(a, \cdot) = 0\}. \tag{2.3}$$

The creditors select collectively an element of $\mathcal{X}$ as an outcome of a non-cooperative game. In a game theoretic terminology, the creditors are the *players*, the family of allocation strategies represents the *action profiles*, where the *actions* of creditor a are given by the set of functions $\{\pi_t(a, \cdot) \,|\, \pi \in \mathcal{X}\}$. The *payoff function* given in Section 2.2.3 will complete the definition of the game.

A priori there is *ambiguity* about which element of $\mathcal{X}$ will be selected. To determine this outcome we will not rely on classical control theory, since the payoff of any creditor depends not only on her selected strategy, but on the element of $\mathcal{X}$ selected collectively. Instead, in Section 2.3, we will determine the outcome of this game using the standard solution concept of Nash equilibrium and by introducing a refinement of the concept of *weakly dominated strategies* in our setting.

It is a standing assumption that all creditors are rational and know that all other creditors are rational, where rational creditors can only choose elements of $\mathcal{X}$ which are Nash equilibria and in which no creditor uses weakly dominated strategies. By showing that there exists a unique such element of $\mathcal{X}$, we will remove ambiguity.

We assume that the risk free rate is zero, the short term debt offered by the borrower is r and that all market participants are neutral to the exogenous risk and myopic. We assume that the borrower takes all the debt that is offered to it. It would be an important extension of the model to let the borrower control the amount of debt its takes by using the interest rate as a control. Our focus is to solve the game of the creditors, and we leave the controlled game for future research.

In the sequel, capital letters denote random variables, while small letters denote deterministic functions.

2.2.1 Borrower's dynamics under a given allocation strategy

Fix an allocation strategy $\pi \in \mathcal{X}$. At time 1, the borrower starts with initial liquidity $x_1 \geq 0$. At each time period $t = 1, 2, \ldots T$, each creditor a allocates a fraction $\pi_t(a, Y_t)$ of her wealth to the borrower's debt. Thus, the borrower raises the following amount short term debt bearing interest rate r which is due at time $t + 1$,

$$D_t^\pi := \int_A \pi_t(a, Y_t)\gamma(da) = d_t^\pi(Y_t), \tag{2.4}$$

where $d_t^\pi : \mathbb{R}^t \to \mathbb{R}^+$ denotes the debt function at time t

$$d_t^\pi(y_t) := \int_A \pi_t(a, y_t)\gamma(da), \ y_t \in \mathbb{R}^t. \tag{2.5}$$

The borrower then pays back the short term debt due at time t plus the interest rate r on this debt, i.e., $D_{t-1}^\pi(1 + r)$, to the short term creditors.

We can now define the default time of the borrower. Denote by M_t^π the total liquidity available to the borrower at time t. We can observe that the total liquidity evolves according to the following equation

$$M_1^\pi = x_1 + D_1^\pi \tag{2.6}$$

$$M_t^\pi = M_{t-1}^\pi + \theta_t + D_t^\pi - D_{t-1}^\pi(1 + r), \ t \geq 2. \tag{2.7}$$

We define the time the borrower becomes illiquid as the first time the total liquidity available to the borrower becomes negative

$$T^\pi := \inf\{t = 2, \ldots, T \mid M_t^\pi < 0\},$$

with the usual convention that $\inf \emptyset = \infty$. We assume that the borrower operates until it becomes illiquid or until time T, whichever comes first. In particular, we assume that $D_T = 0$.

It will be convenient in the sequel to express the dynamics of the borrower in terms of the *liquid net worth* process

$$X_t^\pi = M_t^\pi - D_t^\pi,$$

which evolves according to the equations

$$\begin{cases} X_1^\pi = x_1, \\ X_t^\pi = X_{t-1}^\pi + \theta_t - r d_{t-1}^\pi(Y_{t-1}), \quad t = 2, \ldots, T^\pi \wedge T. \end{cases} \tag{2.8}$$

The default time then fulfills

$$T^\pi = \inf \{ t = 2, \ldots, T \mid X_t^\pi + d_t^\pi(Y_t) < 0 \}. \tag{2.9}$$

It is easy to see that the above equations implicitly define (recursively) deterministic functions $x_1^\pi \equiv x_1$ and $x_t^\pi : \Gamma_{t-1}(\pi) \times \mathbb{R} \to \mathbb{R}$, where $\Gamma_1(\pi) = \{0\}$ and

$$\Gamma_t(\pi) = \left\{ y_t \in \mathbb{R}^t \,\middle|\, x_k^\pi(y_k) + d_k^\pi(y_k) \geq 0 \;\forall\, k \leq t \right\}, \tag{2.10}$$

satisfying $X_t^\pi = x_t^\pi(Y_t)$ and $\{T^\pi > t\} = \{Y_t \in \Gamma_t(\pi)\}$, for $t = 1, \ldots T$. We will call the set $\Gamma_t(\pi)$ the *survival set (up to time t)* of the process Y_t under the allocation strategy π. We also define the *default sets*

$$\Delta_t(\pi) := \{ y_t = (\theta_1, \ldots, \theta_t) \in \mathbb{R}^t \,\big|\, (\theta_1, \ldots, \theta_{t-1}) \in \Gamma_{t-1}(\pi), (\theta_1, \ldots, \theta_t) \notin \Gamma_t(\pi) \} \tag{2.11}$$

which represents the set of trajectories of Y_t in which the default occurs at time t.

2.2.2 Creditors' expected payoff under a given allocation strategy

In the previous section we have expressed the dynamics of the borrower under a given creditor allocation strategy. We are now ready to specify the creditors' payoff under a given allocation strategy $\pi \in \mathcal{X}$.

Suppose that $t \leq T^\pi$. We assume for simplicity that every creditor receives zero recovery value in case of default. Zero cash-flows to the creditors in case of default is a common assumption in the literature that models the illiquidity of a borrower, see e.g, (27, 38, 7).

We assume that at time t, every short term creditor consumes the interest payment. The interest payment equals

$$C_t^{\pi,a} =_{\{T^\pi>t\}} \pi_{t-1}(a, Y_{t-1})r. \tag{2.12}$$

If default happens at time t, the creditor loses the fraction $\pi_{t-1}(a, Y_{t-1})$ and any additional exposure obtained at time t. Therefore, the creditor's wealth at time $t+1$ (per dollar at time t) including the interest payment at $t+1$ is

$$\begin{aligned}
V_{t+1}^{\pi,a} &=_{\{T^\pi=t\}} \Big(1 - (\pi_{t-1}(a, Y_{t-1}) \vee \pi_t(a, Y_t))\Big) \\
&\quad +_{\{T^\pi=t+1\}} \Big(1 - (\pi_t(a, Y_t) \vee \pi_{t+1}(a, Y_{t+1}))\Big) \\
&\quad +_{\{T^\pi>t+1\}} \Big(1 + \pi_t(a, Y_t)r\Big) \\
&= 1 -_{\{T^\pi=t\}} (\pi_{t-1}(a, Y_{t-1}) \vee \pi_t(a, Y_t)) \\
&\qquad -_{\{T^\pi=t+1\}} (\pi_t(a, Y_t) \vee \pi_{t+1}(a, Y_{t+1})) +_{\{T^\pi>t+1\}} \pi_t(a, Y_t)r.
\end{aligned} \tag{2.13}$$

The expected payoff from allocation strategy π for agent a at time t is given by

$$G_t^{\pi,a} := \mathbb{E}^a[C_T^{\pi,a} + V_{T+1}^{\pi,a} \,\big|\, Y_t] = g_t^{\pi,a}(Y_t), \tag{2.14}$$

for a deterministic function $g_t^{\pi,a} : \Gamma_{t-1}(\pi) \times \mathbb{R} \to \mathbb{R}^+$.

All quantities defined so far *under any given allocation strategy* π are $\sigma(Y_t)$-adapted, in particular the debt process D_t^π. We stress that this does not contradict the assumption that agents do not observe the debt at time t when making their *actual* investment decisions at time t. Although D_t^π is observable to agent *a given the allocation strategy* π, the *actual strategy played* by the agents is not assumed to be observable to any agent at time t. Indeed, we do not assume any probabilistic structure or law on the set of allocation strategies $\mathcal{X}$. The actual strategy played by the creditors will result from the *solution of the game*.

2.2.3 Payoff function for the creditors' game

To complete the definition of the creditors' mathematical game, it remains to specify the *payoff function* of any agent at any time $t = 1, \dots, T$. As standard in game theory, see e.g, (41), the payoff function takes as an argument an allocation strategy, i.e., is defined on the set $\mathcal{X}$.

Definition 1. *Let $a \in A$. We define the payoff function of agent a as the following map on $\mathcal{X}$*

$$\pi \mapsto (g_t^{\pi,a})_{t=1,\dots T}.$$

Note that in our setting with a continuum of creditors, two strategies may lead to the same debt functions and survival sets, even if they are different for a negligible set of creditors. We therefore identify two allocation strategies if, for almost all players, they are equal strictly before default, and they yield the same payoffs at default. We give the following formal definition.

Definition 2 (Identical strategies)**.** *Two allocation strategies π and π' are called identical if*

1. *For all $t = 1, \dots, T$, we have $\Gamma_t(\pi) = \Gamma_t(\pi')$.*
2. *For any $1 \leq t \leq T$, any $y_t \in \Gamma_t(\pi)$ and almost all creditors a, we have*

$$\pi_t(a, y_t) = \pi'_t(a, y_t).$$

3. *For any $1 \leq t \leq T$, $y_t \in \Delta_t(\pi)$ and almost all creditors a, we have $g_t^{\pi,a}(y_t) = g_t^{\pi',a}(y_t)$.*

The above definition introduces an equivalence relation on the space of allocation strategies $\mathcal{X}$. From now on we will work under an abuse of terminology, by identifying an allocation strategy with an element of an equivalence class and the space $\mathcal{X}$ with the quotient space.

2.3 The solution for the creditors' game

In Section 2.3.1 we define the Nash equilibria for our model. We provide an explicit construction of a family of threshold type equilibria in Section 2.3.2. The key properties of our model's Nash equilibria are established in Section 2.3.3. In Section 2.3.4, we introduce the concept of weak dominance, and we show that there exists a unique equilibrium if creditors do not use weakly dominated strategies.

2.3.1 Definition of equilibria

We assume that short term creditors are rational, and the outcome of the game introduced in the previous section is a Nash equilibrium.

From our payoff definition (2.14), we can see that creditors must take into account the strategies of all other players not only at the current moment, but also at future allocation times. At any time t, creditors choose an allocation strategy which is an equilibrium for the subgame of decisions starting from time t. Such a strategy is a *Nash equilibrium*. In such an equilibrium, no agent can increase her payoff by changing her strategy from time t on, if the other players keep their strategies. This is formalized as follows in our model.

Definition 3 (Nash equilibrium)**.** *Let $\bar{A} \subseteq A$ and $t \in \{1, ..., T-1\}$. By convention, any allocation strategy is a Nash equilibrium on $\bar{A}$ at time T. An allocation strategy $\pi^* \in \mathcal{X}$ is called a* Nash equilibrium on *$\bar{A}$ at time t if we have that*

- *it is a Nash equilibrium on $\bar{A}$ at time $t+1$, and*
- *for each $a \in \bar{A}$ and any allocation strategy π which satisfies $\pi_s(a', \cdot) = \pi^*_s(a', \cdot)$ for all $s = 1, \ldots, t-1$ and all $a' \in A$, and $\pi_s(a', \cdot) = \pi^*_s(a', \cdot)$ for all $s = t, t+1, \ldots, T$*

and all $a' \in A \setminus \{a\}$, we have

$$g_t^{\pi^*,a}(y_t) \geq g_t^{\pi,a}(y_t) \quad \forall y_t \in \Gamma_{t-1}(\pi^*) \times \mathbb{R}. \tag{2.15}$$

An allocation strategy $\pi^* \in \mathcal{X}$ is called a Nash equilibrium on $\bar{A}$ *if it is a* Nash equilibrium on $\bar{A}$ *at time* 1.

In the following we shall say "Nash equilibrium" or "equilibrium" when we mean "Nash equilibrium on A" .

2.3.2 Threshold type Nash equilibria

In the sequel, we shall often encounter threshold type allocation strategies of the form

$$\pi_t(a, y_t) = \begin{cases} 1 & \text{if } \ b(a) \geq b_t(y_t) \\ 0 & \text{if } \ b(a) < b_t(y_t) \end{cases}$$

for suitable deterministic functions $b_t(\cdot)$. Such an allocation strategy depends on a creditor only through her belief, and the creditor will invest into the short term debt if and only if her belief is equal or above a *critical threshold* given by $b_t(y_t)$. It then follows from (2.2) and (2.5) that

$$d_t^{\pi}(y_t) = C(b_t(y_t)).$$

In the following we will identify a family of Nash equilibria of such threshold type. Define the function

$$\beta(q) = rC(q) - q \tag{2.16}$$

and note that $\beta(\cdot)$ is strictly decreasing. We can now state our first main result.

Theorem 1 (Existence of Nash equilibria)**.** *Consider a sequence of threshold functions $b_t(\cdot)$ for $t = 1, ..., T$ which satisfy the following backward recursion.*

1. *For $t = T$ we have*

$$b_t(\cdot) \equiv +\infty. \tag{2.17}$$

2. *Given $b_{t+1}(\cdot)$ for $t < T$, with the function $C(b_{t+1}(\cdot))$ representing the short term debt as a function of the liquid net worth, and the function $m_{t+1}(\cdot)$ representing the total liquidity available to the borrower*

$$m_{t+1}(x) = C(b_{t+1}(x)) + x, \tag{2.18}$$

we assume that $b_t(\cdot)$ satisfies

$$q(t, x) = \beta^{-1}\Big(x - m_{t+1}^{-1}(0) + \Phi^{-1}\Big(\frac{r}{1 + r}\Big)\Big) \tag{2.19}$$

and

$$b_t(x) = \begin{cases} q(t, x) & \text{if } \; x + C(b_t(x)) \geq 0, \\ +\infty & \text{if } \; x + C(b_t(x)) < 0. \end{cases} \tag{2.20}$$

Define now a threshold type allocation strategy $\bar{\pi}$ by

$$\bar{\pi}_t(a, y_t) = \left.\begin{cases} 1 & \text{if } \; b(a) \geq b_t(x_t^{\bar{\pi}}(y_t)) \\ 0 & \text{if } \; b(a) < b_t(x_t^{\bar{\pi}}(y_t)) \end{cases}\right\}, \quad y_t \in \Gamma_{t-1}^{\pi} \times \mathbb{R}, \tag{2.21}$$

for $t = 1, ..., T$. (Here the function $\bar{\pi}_t$ in (2.21) is well-defined through recursion since the function $x_t^{\bar{\pi}}(\cdot)$ is defined in terms of $\bar{\pi}_1, ..., \bar{\pi}_{t-1}$, cf. Section 2.2.1.) Then $\bar{\pi}$ is a Nash equilibrium.

It is not hard to see that equation (2.20) allows multiple solutions for the threshold function $b_t(x)$. Hence there exist multiple Nash equilibria $\bar{\pi}$; we will discuss this phenomenon further below. Note that the allocation strategy $\bar{\pi}_t(a, \cdot)$ in (2.21) depends on y_t only through the observed liquid net worth $x_t^{\bar{\pi}}(y_t)$. By (2.21), at any time t, the observed short term debt of the borrower is of the form

$$D_t^{\bar{\pi}} = C(b_t(X_t^{\bar{\pi}})), \tag{2.22}$$

i.e., depends on the liquid net worth at time t only. The value $m_{t+1}^{-1}(0)$ in (2.19) is well-defined by the following result.

Lemma 1. *For any sequence of functions satisfying (2.17) – (2.20) and each $t \leq T$, the function $b_t(\cdot)$ is decreasing and $m_t(\cdot)$ is strictly increasing. For each $x \in \mathbb{R}$, we have that $b_{t-1}(x) \leq b_t(x)$ and $m_t^{-1}(0) \leq m_{t+1}^{-1}(0)$ for all t.*

We can interpret the thresholds $b_t(\cdot)$ and $q(t, \cdot)$ in (2.21) as follows. Suppose first that creditors do not take into account the possibility of default at the time when they make the allocation, and worry only about the future probability of default. Remark that in this case, creditors will invest all their capital in the borrower if, according to their measure, the probability of default in the next period is less or equal than $\frac{r}{1+r}$, and they will invest zero capital otherwise. The threshold $q(t, \cdot)$ is equal to the mean of θ_t under the probability measure which assigns exactly $\frac{r}{1+r}$ to the probability of default in the next period; this is the critical threshold in case the borrower survives at time t. By next taking into account the possibility of default at the time of the allocation, creditors finally reach the equilibrium given by the threshold $b_t(\cdot)$.

By (2.22) and (2.18), we have that $X_t^{\bar{\pi}} + D_t^{\bar{\pi}} = m_t(X_t^{\bar{\pi}})$. So it follows from (2.9) that the default time for the strategy $\bar{\pi}$ is given by

$$T^{\bar{\pi}} = \inf \left\{ t = 1, \ldots, T \mid X_t^{\bar{\pi}} < m_t^{-1}(0) \right\}.$$

This means that $m_t^{-1}(0)$ is the *illiquidity barrier* for the liquid net worth process $X_t^{\bar{\pi}}$ when creditors play the strategy $\bar{\pi}$, and by (2.8), the liquid net worth process has the Markovian dynamics

$$X_t^{\bar{\pi}} = X_{t-1}^{\bar{\pi}} + \theta_t - rC(b_{t-1}(X_{t-1}^{\bar{\pi}})).$$

The value of the barrier is computed via the recursion (2.17) – (2.20), and is a function of the creditors' belief and capital distribution $C(\cdot)$, the interest rate r, and the distance to horizon $T - t$.

We have already noted that there exist multiple series of threshold functions $b_t(\cdot)$ satisfying (2.17) – (2.20), and hence multiple Nash equilibria by Theorem 1. We now present two extreme examples $\hat{\pi}$ and $\breve{\pi}$ of the threshold type equilibria introduced in Theorem 1. The first example $\hat{\pi}$ will play a critical role in the sequel. In Section 2.3.4, we shall introduce the notion of *weakly dominated* strategies, and call an equilibrium *reasonable* if it is not weakly dominated for any creditor. In Theorem 3, we will show that equilibria other than $\hat{\pi}$ cannot occur if creditors do not use weakly dominated strategies, or in other words, that $\hat{\pi}$ is the unique reasonable Nash equilibrium.

Definition 4. *We define a series of threshold functions $\hat{b}_t(\cdot)$ recursively by (2.17) – (2.19) and*

$$\hat{b}_t(x) = \begin{cases} q(t, x) & \text{if } \ x + C(q(t, x)) \geq 0, \\ +\infty & \text{if } \ x + C(q(t, x)) < 0. \end{cases} \tag{2.23}$$

We easily check that the threshold function defined by the backward recursion (2.23) fulfills (2.20). Letting

$$\hat{\pi}_t(a, x) :={}_{b(a)\geq \hat{b}_t(x)}, \tag{2.24}$$

$$\hat{d}_t(x) :=C(\hat{b}_t(x))), \tag{2.25}$$

we have that the strategy $\hat{\pi}$ defined forward recursively by the functions $(a, y_t) \mapsto \hat{\pi}_t(a, x_t^{\hat{\pi}}(y_t))$ for $t = 1, ..., T$, is a Nash equilibrium by Theorem 1, and we have $d_t^{\hat{\pi}}(y_t) = \hat{d}_t(x_t^{\hat{\pi}}(y_t))$.

The next example introduces another Nash equilibrium in which all creditors withdraw their funds as soon as the liquid net worth becomes negative, causing the borrower to default. This is an equilibrium in which default happens in every state of the liquid net worth in which it is technically possible.

We define a series of threshold functions $\breve{b}_t(\cdot)$ recursively by (2.17) – (2.19) and

$$\breve{b}_t(x) = \begin{cases} q(t,x) & \text{if } \; x \geq 0, \\ +\infty & \text{if } \; x < 0. \end{cases} \tag{2.26}$$

We easily check that the threshold function defined by the backward recursion (2.26) fulfills (2.20). The resulting strategy $\breve{\pi}$ defined forward recursively by

$$\breve{\pi}_t(a, y_t) = \begin{cases} 1 & \text{if } \; b(a) \geq \breve{b}_t(x_t^{\breve{\pi}}(y_t)) \\ 0 & \text{if } \; b(a) < \breve{b}_t(x_t^{\breve{\pi}}(y_t)) \end{cases}$$

is a Nash equilibrium by Theorem 1.

One can easily construct infinitely many equilibria for which the illiquidity barrier lies between the barriers of $\hat{\pi}$ and of $\breve{\pi}$. Indeed, in any equilibrium as in Theorem 1, the critical threshold $b_t(x)$ jumps from $q(t,x)$ to ∞ once the liquid net worth x crosses the illiquidity barrier from above, and this barrier can be any value between 0 and the (unique) zero of the function $x \mapsto x + C(q(t,x))$.

2.3.3 Properties of Nash equilibria

We have seen in the last section that there exists multiple equilibria. However, the family of equilibria identified in Theorem 1 does not contain all Nash equilibria. In this section, we prove a key result on the set of *all* Nash equilibria. We show that the equilibrium $\hat{\pi}$ in Definition 4 yields, among all equilibria, the maximal amount of short term debt for a given liquid net worth.

We start with a result which says that in any Nash equilibrium, no creditor increases her investment into the short term debt at the time of default.

Lemma 2. *Let π a Nash equilibrium at time t, for a time $t \in \{1, ..., T\}$. For any $y_t \in \Delta_t(\pi)$, we have $\pi_t(a, y_t) \leq \pi_{t-1}(a, y_{t-1})$ for any creditor $a \in A$.*

As a consequence of Lemma 2 we obtain that in all equilibria in which the borrower survives at time t, creditors either do not invest at all, or invest all their capital into the borrower.

Proposition 1. *Let π be a Nash equilibrium at time t. For almost all creditors $a \in A$ and all $y_t \in \Gamma_t(\pi)$ we have $\pi_t(a, y_t) \in \{0, 1\}$. More precisely, on the set $\{Y_t \in \Gamma_t(\pi)\}$ we have*

$$\pi_t(a, Y_t) = \begin{cases} 1 & \text{if } \mathbb{P}^a[T^\pi > t+1 | Y_t] > \frac{1}{1+r} \\ 0 & \text{if } \mathbb{P}^a[T^\pi > t+1 | Y_t] < \frac{1}{1+r}. \end{cases}$$

Our second main result says that in any equilibrium, the short term debt is at most equal to the short term debt in equilibrium $\hat{\pi}$ under the same liquid net worth. Recall the function $\hat{d}_t(\cdot)$ in (2.25).

Theorem 2. *Let π be a Nash equilibrium at time t, for some $t \in \{1, ..., T\}$. We have $d_t^\pi(y_t) \leq \hat{d}_t(x_t^\pi(y_t))$ for all $y_t \in \Gamma_t(\pi)$.*

We immediately obtain

Corollary 1. *Let π be an equilibrium at time t, for a $t \in \{1, ..., T\}$. On the set $\{T^\pi > t-1\}$, we have that $X_t^\pi < \hat{m}_t^{-1}(0)$ implies $T^\pi = t$.*

2.3.4 Uniqueness of a reasonable Nash equilibrium

We have seen in Section 2.3.2 that there exist multiple equilibria. We assume that creditors take into account *ambiguity* about the actual played equilibrium: a creditor that observes the played equilibrium only through the past trajectory of the borrower's debt and models the allocation strategy of all creditors using an equilibrium strategy π takes into consideration the fact that she can be mistaken and the actual equilibrium allocation strategy is π^*, with π and π^* indistinguishable from her observation of the past debt trajectory. To hedge against this uncertainty, the creditor will eliminate dominated strategies.

In games with a finite number of players it is common to use iterated elimination of dominated strategies to eliminate multiple equilibria. However, in games with a large number of "small" creditors, this usual iterative procedure will not work. Indeed, eliminating dominated strategies for one player will have, by itself, no effect on the set of possible outcomes of the game. Therefore, it will not prompt the other players to eliminate any of their strategies. In games with a continuum of creditors in a non-observed equilibrium, any player eliminates weakly dominated strategies *as if the set of Nash equilibria is exogenously given by the equilibria reached by the other players*. Consider an creditor who is strictly worse off by playing strategy A, as opposed to strategy B, in at least one Nash equilibrium of the other players, and is indifferent between the two strategies in all other Nash equilibria of the other players. We can assume that in this case any rational creditor who cannot observe the equilibrium played by the other players, will not play strategy A, which we will call *weakly dominated* by strategy B.

Before giving the precise definition of a weakly dominated strategy in the context of games with a continuum of players, it is helpful to introduce the following notation for an allocation strategy obtained from another allocation strategy by modifying the strategy of a single player. Let $a \in A$, π be a strategy on A, and π^* be a strategy on $A \backslash \{a\}$. We denote by $\langle \pi^* | \pi(a) \rangle$ the strategy on A defined by

$$\langle \pi^* | \pi(a) \rangle (a', \cdot) = \begin{cases} \pi(a, \cdot) & \text{if } a' = a \\ \pi^*(a', \cdot) & \text{if } a' \in A \backslash \{a\}. \end{cases}$$

To distinguish the liquid net worth trajectories resulting from allocation strategies, the following notation will be useful. Let $t \in \{1, ..., T\}$. We introduce the set of allocation strategies consistent with an observation of a trajectory of the borrower's fundamentals and a trajectory of the debt up to but excluding time t. For $y_t \in \mathbb{R}^t$ and $z_{t-1} \in \mathbb{R}^{t-1}$ we set

$$O_t(y_t, z_{t-1}) = \{\pi \in \mathcal{X} \mid y_{t-1} \in \Gamma_{t-1}(\pi), d_s^\pi(y_s) = z_s, s = 1, \ldots, t-1\}. \tag{2.27}$$

Remark 1. *In particular, note that for a strategy $\pi \in \mathcal{X}$, for a time $t \in \{1, ..., T\}$, a trajectory $y_t \in \Gamma_t(\pi)$ and a strategy $\pi^* \in O_t(y_t, d_1^\pi(y_1), \ldots, d_{t-1}^\pi(y_t))$ we have that the liquid net worth trajectory under π^* coincides with the liquid net worth trajectory under π,*

$$x_s^\pi(y_s) = x_s^{\pi^*}(y_s), \; s = 1, \ldots, t.$$

Definition 5 (Weak dominance)**.** *Let $\pi \in \mathcal{X}$, and fix some $a \in A$. We say by convention that π is not weakly dominated at time T, since for all $\tilde{\pi} \in \mathcal{X}$, $\tilde{\pi}_T(a, \cdot) = 0$. We define the concept of weak dominance backward recursively for $t = T-1, \ldots, 1$ as follows.*

Let $\tilde{\pi} \in \mathcal{X}$ with $\tilde{\pi}_s = \pi_s$ for all $s \leq t-1$. We say that π is weakly dominated by $\tilde{\pi}$ for creditor *a at time t if for every $y_t \in \Gamma_{t-1}(\pi) \times \mathbb{R}$ and for every $\pi^* \in O_t\big(y_t, (d_1^\pi(y_1), \ldots, d_{t-1}^\pi(y_{t-1}))\big)$ which is an equilibrium on $A \backslash \{a\}$ at time t and is not weakly dominated for any creditor in $A \backslash \{a\}$ at any time $t = t+1, \ldots, T$ we have*

$$g_t^{\langle \pi^* | \pi(a) \rangle, a}(y_t) \leq g_t^{\langle \pi^* | \tilde{\pi}(a) \rangle, a}(y_t)$$

and if this inequality is strict for at least one such y_t and π^*. We say that π is weakly dominated for creditor *a* at time *t if it is weakly dominated by some strategy $\tilde{\pi}$ for creditor a at time t.*

We say that π is weakly dominated for creditor *a if it is weakly dominated for creditor a at some time $t = 1, \ldots, T-1$.*

The base step of the recursion is set at the time horizon T, when by definition all strategies are set to 0, so by convention no strategy is weakly dominated. Then, at any time $t < T$, creditors decide on a multi-period allocation strategy for times $t, \ldots, T$. In order to eliminate multi-period weakly dominated strategies played at times $t, \ldots, T$, any creditor compares her strategy at time t against multi-period Nash equilibria in which the other players are guaranteed not use weakly dominated strategies starting with time $t+1$.

We shall define reasonable Nash equilibria as those equilibria in which weakly dominated strategies are not played for any $t \in \{1, \ldots, T\}$.

Definition 6 (Reasonable Nash equilibrium)**.** *A Nash equilibrium π^* on A is called a* reasonable Nash equilibrium *if π^* is not weakly dominated for any creditor $a \in A$.*

In this section we show, that $\hat{\pi}$ is the unique Nash equilibrium (up to an identity in the sense of Definition 2) played when creditors do not use weakly dominated strategies. This implies that, when we assume that almost all creditors $a \in A$ do not use weakly dominated strategies, they will make their decisions according to the strategy $\hat{\pi}(a, \cdot)$.

Theorem 3 (Uniqueness)**.** *The equilibrium $\hat{\pi}$ is the unique reasonable Nash equilibrium (up to identity in the sense of Definition 2).*

In other words, the strategy $\hat{\pi}$ thus represents the *unique solution of the creditors' game*, under our solution concept of reasonable Nash equilibrium.

2.4 Conclusions

We have modeled the investment decisions of a large borrower's creditors as a dynamic non-cooperative game. We have shown that this game has a unique solution and we have characterized this solution as a threshold type strategy in terms of the liquid net worth.

CHAPTER 3

OPTIMAL CASH HOLDINGS

3.1 Introduction

The cash holdings of US firms stand today at 5 trillion dollars. Financial institutions, required to hold minimal reserves, have sharply increased these reserves after the recent crisis. Yet, large cash holdings are not a phenomenon specific to crises, nor to financial institutions (non-financial institutions hold today roughly 1.5 trillion dollars): over the last decades average ratios of cash-to-assets have steadily increased.

To explain large cash holdings, the literature has mainly focused on the *precautionary* motive. As the reasons for the precautionary hoarding may be structurally different for banks and corporations, they are usually treated separately.

One line of literature, for instance (17), explains hoarding as a method for reducing bank run risks: The modern version of bank run risk, in the form of rollover risk, is the rationale behind cash hoarding and restricted term lending in (3). In (23), banks hoard liquidity in an equilibrium in which they weigh the risk of future liquidations of an illiquid asset against holding cash.

Like banks that may see their funding not rolled over, firms may also see their funding restricted, either directly by the debt markets or as the banks restricting lending. The precautionary motive for firms, documented in (6), comes from the firms' need to insure against future liquidation of their projects, in periods of restricted lending. Cash holdings allow firms to pursue investment opportunities when other sources of funding dry up. (28) propose a theoretical model in a dynamic setting, where firms' cash policy, investment and financing are driven by the (exogenous) risk in the capital supply. (4) verifies empirically that the propensity to save cash occurs systematically only for financially constrained firms.

Besides the precautionary motive, (21) document a tax motive; (5) document a competitive motive, i.e., firms often hold cash to rapidly take new investment opportunities. More recently, the competitive motive is empirically tested in (20). Contrary to the precautionary motive, which is related to financial constraints, the latter motives also apply to firms that do not necessarily need to raise funds to finance their operations.

Within the literature that relates cash holdings to firm characteristics, (44) find that firms with lower, more volatile cash flows and higher growth opportunities also have higher cash holdings, which they interpret as a stronger precautionary motive for such firms. Focusing on the value of cash holdings rather than the level, (18) find that the value of cash exhibits wide variation across firms with strong dependence on the level and quality of governance.

We introduce a *static* model in which both cash holdings and the debt level arise endogenously and to give conditions under which the company optimally chooses to hold a positive level of cash in order to select a creditors' equilibrium with higher debt provision. As in (31), the heterogeneity of beliefs is a critical component of our model. The company is assumed to be optimistic about the evolution of the risky asset, yet the creditors differ in their beliefs about this evolution.

This can be interpreted as the company and its creditors having asymmetric information: in absence of full information on the company's investment strategies, some creditors assign lower value to the asset. This phenomenon would disappear if the company released its information. In our model the company chooses to keep the information private, e.g., for competitive reasons. We can also interpret this heterogeneity as disagreement: even under full information, some creditors are pessimistic. In either case, for a given position in cash and the risky asset, the valuation of the assets differs from creditor to creditor depending on her beliefs. The optimistic borrower faces the following tradeoff: the risky asset is the best opportunity for investing all available funds. However, positive cash holdings increase the value assigned to the borrower's assets by pessimistic creditors, thereby, increasing its debt capacity. The cash holding level acts as a creditors' equilibrium selection mechanism.

When the distribution of capital across the creditors' beliefs is diffuse, the company invests all available funds in the risky asset. More explicitly, the company is optimistic about the asset and, therefore, buys as many units of the risky asset as possible. Since the distribution of capital is diffuse, the drop in debt capacity can be made arbitrarily small by choosing a sufficiently small increase in the asset position. Therefore, if the company has a non-zero cash position, it is always optimal to buy an additional positive quantity of the asset, without violating the capital constraints. The result on zero cash position depends on our assumption of uniform capital distribution across beliefs, but can be extended to the case when the capital distribution function decreases superlinearly with the belief, as opposed to linearly for the uniform case. In all these cases, the increase of debt capacity with respect to the risky investment is small in relation to the profits expected by the optimistic borrower by the increase in the risky investment. In other words, there are not sufficiently many creditors who would change their strategies in order for the benefit to exceed the cost of holding cash.

In contrast, when the distribution of capital across the creditors' beliefs contains an atom, representing a "large" creditor, under certain conditions on the beliefs of the large creditor, there exists an optimal and strictly positive level of cash holdings for the company. If the company chooses a higher investment in the risky asset, i.e, keeps less funds in cash, then the pessimistic large creditor will withdraw and the smaller creditors will coordinate into a different equilibrium. There will be a decrease in debt capacity, even larger than the capital of the large creditor. In this new equilibrium, the feasible level of investment in the risky asset declines, which implies a decrease in shareholder value. If, on the other hand, the company chooses a lower value of investment in the asset, i.e. keeps more funds in cash, then this is not optimal since the company misses valuable investment opportunities.

Our findings suggest that firms that exist in markets with a large number of small investors are less likely to hold cash than firms that borrow from a small number of large investors. This is a new empirically testable implication that awaits future testing.

Unlike the previous literature, in our model cash holdings emerge endogenously even in absence of liquidity shocks. Like the precautionary motive, our theory applies to firms that need to raise external funds for their operations. It applies particularly to banks and firms with high growth opportunities. Our model would not apply to firms that do not have funding constraints, where the tax or competitive motives are more appropriate.

Our work relates to several strands of literature. We have discussed above our contribution to the literature studying why firms hold cash. Within the literature on the optimal level of cash holdings and based on a precautionary motive, (1) propose a model in which cash is used to reduce debt or is saved in order to hedge against an adverse reduction in debt capacity, which built on the model introduced in (22) the context of risk management under (exogenous) risk associated with external funding. In their setting, creditors are not strategic and the debt capacity is exogenous.

As in (1), the existence of an optimal level of cash holdings implies that cash is not a substitute for negative debt, an issue raised in (43). In our model, cash holdings bear the similarity with moral hazard based capital requirements, since in equilibrium cash holdings mitigate the risk perceived by pessimistic investors. Despite this similarity, since cash is not negative debt, cash and capital are also not fully fungible. Seen as a function of the capital, the optimal level of cash, in conjunction with an optimal level of debt, implies the existence of optimal capital ratios, with and without cash subtracted from capital (which are different in general).

Importantly, the existence of optimal capital ratios is related to the existence of an optimal capital structure. Our results are obtained by modeling the creditors with different beliefs about the risks inherent not just in the company's investment, but in the whole capital structure of the borrower.

There is a large literature on optimal investment under market frictions. This vast literature began with seminal papers by (14, 15, 48), and continued with recent monographs (30, 47). Most models in this literature are from the perspective of a small investor who derives utility from consumption and final wealth. These models are reduced form models in the sense that the evolution of prices and the different frictions are exogenous. We complement this literature by providing an equilibrium model in which an important friction -funding constraints- is endogenous.

3.2 The model

We consider an optimal investment problem for a company that funds itself with some initial capital and with debt provided by a continuum of creditors.

The company can invest in a risky asset and in cash. Without loss of generality we can assume that the risk-free rate is equal to zero, since the relevant quantity in our model is the difference between the rate offered by the company and the risk-free rate.

In the sequel, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that models the evolution of the risky asset. A measurable space for the creditors is fixed and given by $(A = (0,1), \mathcal{B})$, where $\mathcal{B}$ denotes the Borel σ-algebra. The company and all creditors are referred to as creditors and are all assumed risk-neutral.

We consider a one period setup, $t = 0, 1$. At time 0 the company starts operations with an initial capital $c > 0$. It raises an amount s of debt, due at time 1 and bearing interest rate $r > 0$. The company invests in $y > c$ units of the risky asset, whose price at time 0 is normalized to 1. The remaining funds are held in cash, assumed nonnegative at time 0, i.e.,

$$m := c + s - y \geq 0. \tag{3.1}$$

At time 1 the price of the risky asset changes to a price, assumed positive, $P \geq 0$ and consequently the value of the risky asset holdings of the company becomes Py. The capital at time 1 is given by

$$m - s(1 + r) + yP = c - rs + y(P - 1). \tag{3.2}$$

The company is solvent if the capital is positive and insolvent otherwise. If the company is insolvent at time 1, it is completely liquidated and the proceeds distributed to the creditors proportionally to their outstanding debt. It follows that for each dollar invested in the company's debt at time 0 the creditors receive at time 1 an amount

$$\frac{m + yP}{s} \wedge (1 + r) = \frac{c + s + y(P - 1)}{s} \wedge (1 + r). \tag{3.3}$$

Note that the assumption that $P \geq 0$ implies that the asset-to-debt ratio at time 1 is greater than 0. Having established the basic setup, we now establish the company's debt capacity and investment in the risky asset that emerges from strategic interaction with the creditors at time 0.

3.2.1 The creditors

The creditors differ in their beliefs about the future prospects of the borrower, in their endowed capital, and in their capital allocation strategy. We now describe these differences.

Creditors' beliefs. We assume that each creditor $a \in A$ has a *belief* given by a probability measure $\mathbb{P}^a$ under which the random variable P has a log-normal distribution $b(a)exp(N - \frac{1}{2})$, with $b(a) \in \mathbb{R}^+$ and N following a standard normal distribution. This means that creditors measure events under equivalent probability measures: they agree about the volatility of the asset, but they disagree about the expectation of P. Creditor a believes that this expectation is given by $b(a)$. We assume that the function b is a surjection on $\mathbb{R}^+$.

Intuitively, creditor a believes that the real-world measure is $\mathbb{P}^a$. Slightly abusing the terminology, we will simultaneously use the term *belief* for $\mathbb{P}^a$ and for the mean $b(a)$.

Borrower's belief. We assume that the borrower has a *belief* given by a probability measure $\mathbb{P}^{\tilde{b}}$ under which P has a log-normal distribution $\tilde{b}\, exp(N - \frac{1}{2})$ (with N a standard normal random variable).

Furthermore, we assume that the borrower is optimistic about the risky asset, so $\tilde{b} > 1$. Therefore the borrower desires as much debt as offered by the creditors.

Creditors' capital. The creditors are endowed with a capital measure C. We assume that the image measure of C induced by the map b on $(\mathbb{R}_+, \mathbb{B}(\mathbb{R}_+))$ is known to all creditors. This image measure represents the distribution of capital across beliefs and we denote by

$$S(b) := C(\{a \in A \mid b(a) \geq b\}),\ b \in \mathbb{R}_+$$

the distribution function.

We assume that the company's initial funds satisfy $c > rC(A)$, which immediately implies

$$c > rs. \tag{3.4}$$

This means that the company's initial capital must be sufficient to pay the interest on its debt. This simplifying assumption is realistic, since in practice the short-term interest rate r is very low.

Creditors' allocation strategies. Each creditor chooses between investing her capital in riskless cash, or in the borrower's risky debt which pays interest $r > 0$.

Definition 3.2.1. *We will call a function* $\pi : A \times \mathbb{R}^+ \to [0, 1]$ *an* allocation strategy*.*

$\pi(a, y)$ represents for creditor the fraction of the wealth of creditor a, that she invests into the company's debt, given the company's investment in the risky asset is y.

Under this allocation strategy, the debt that the company raises at time 0 depends on the company's investment and is given by

$$s(\pi, y) = \int_A \pi(a, y) da.$$

We assume that the company cannot raise more funds than those offered by the creditors. Moreover, we assume that the investment of the company in the risky asset is bounded away from c (otherwise the company does need to raise debt), in mathematical terms this means that there exists an ϵ, such that the investment of the company is greater than $c + \epsilon$, thus we restrict the domain of π to $A \times (c + \epsilon, c + C)$.

3.2.2 The problem setting

In this section we introduce the definition of a Nash Equilibrium.

We first define the payoff for a given allocation strategy. In the sequel we consider only allocation strategies under which the debt is bounded from below by an $\epsilon > 0$.

Creditors' payoffs. For a given allocation strategy π, the expected return of creditor $a \in A$ per dollar invested into the company is given, using (3.3), by the following

$$\mathbb{E}^{b(a)}\left[\frac{c + s(\pi, y) + y(P-1)}{s(\pi, y)} \wedge (1+r)\right].$$

It will be convenient in the sequel to introduce the function $g : \mathbb{R}_+^3 \to \mathbb{R}_+$ that gives the return per dollar to a creditor with belief x, when the debt is z and the company's investment is y.

$$\begin{aligned} g(x, y, z) :&= \mathbb{E}^x\left[\frac{c + z + y(P-1)}{z} \wedge (1+r)\right] \\ &= 1 + r - \frac{1}{z}\mathbb{E}^x\left[(c + y(P-1) - rz)^-\right] \\ &= 1 + r - \frac{1}{z}Put(yx, y + rz - c) \end{aligned} \tag{3.5}$$

where $Put(x, k)$ denotes the Black-Scholes price of a European Put Option, with initial value of the underlying x and strike $k = y + rz - c$ (under zero interest rate, volatility 1 and time to maturity 1). Note that the strike is strictly positive: $y + rz - c > 0$ since we have $y > c$.

The put option analogy to a creditor's debt is a well-known idea (ever since the initial paper of (9) and the subsequent developments by (36)). An important difference exists in our model because each creditor's value for the defaultable bond differs based on their private belief. More importantly, the recovery rate is endogenous and depends on the equilibrium strategies of the company and all the creditors. This is similar to the literature that endogenizes a company's debt-equity ratio decision, see (49) and the references therein.

We will use the standard notation (for our choice of parameters)

$$d_{1,2}(x, k) := ln\frac{x}{k} \pm \frac{1}{2},$$

Φ the cumulative standard normal density function and ϕ the standard normal distribution function. *Call* denotes the price of an European Call Option.

Note that for all $y, z \in \mathbb{R}_+$ we have $x \to g(x, y, z)$ is continuous and strictly increasing.

Using the return per dollar given in (3.5), we can then write the total payoff of creditor a under strategy π as a function of y:

$$G^{a,\pi}(y) := \pi(a, y)g\left(b(a), y, s(\pi)\right) + (1 - \pi(a, y)). \tag{3.6}$$

Nash equilibrium. Before defining a Nash equilibrium, we introduce some notation. We let $(\pi_{-a}, \pi') : (0, 1) \times (c, c + C] \to [0, 1]$ be defined by

$$(\pi_{-a}, \pi')(a', y) = \begin{cases} \pi'(y) & \text{if } a' = a \\ \pi(a', y) & \text{if } a' \in (0, 1)\backslash\{a\}. \end{cases}$$

In other words, the map (π_{-a}, π') coincides with the map π on $(0, 1)\backslash\{a\} \times (c, c + C]$.

Remark 3.2.2. *Since only one creditor changes her strategy, the total debt stays the same*

$$s(\pi, y) = s\big((\pi_{-a}, \pi'), y\big).$$

We now define the Nash equilibrium. In this definition we are concerned only with creditors's allocation, for a given choice of the company. Importantly, we do not impose a constraint on the creditor's allocation that makes the company's investment feasible.

Definition 3.2.3. *An allocation strategy π is called Nash Equilibrium (NE) if for every $a \in (0, 1)$, any $y \in (c, c + C]$, and any $\pi' : A \to [0, 1]$:*

$$G^{a,\pi}(y) \geq G^{a,((\pi_{-a},\pi')}(y). \tag{3.7}$$

We let $\mathcal{C}$ the set of NE.

In other words, in equilibrium, given the strategy of the company and of the other creditors, no creditor has an incentive to deviate from her strategy.

We now define the following sets:

$$\mathcal{Y}_0 = \Big\{y \in (c, c + C],\ \forall \pi \in \mathcal{C}\ y \leq c + s(\pi, y)\Big\} \tag{3.8}$$

$$\mathcal{Y}_1 = \Big\{y \in (c, c + C],\ \exists \pi \in \mathcal{C}\ y \leq c + s(\pi, y)\Big\} \tag{3.9}$$

Using Equation (3.2) we obtain the company's expected capital at time 1,

$$\begin{aligned} H^{\tilde{b},\pi}(y) &:= \mathbb{E}^{\tilde{b}}\Big[\Big(c + y(P - 1) - rs(\pi)\Big)^+\Big] \\ &= Call(y\tilde{b}, y + rz - c). \end{aligned} \tag{3.10}$$

We assume that under optimistic scenario the company chooses an investment that is feasible under any Nash Equilibrium and results into the maximum expected capital.

Assumption 3.2.4. *The company under optimistic scenario chooses*

$$y^* \in \underset{y \in \mathcal{Y}_1}{argmax} \, (\underset{\pi \in C}{min} \, H^{\tilde{b},\pi}(y)). \tag{3.11}$$

We assume that under pessimistic scenario the company chooses an investment that can be feasible under some Nash Equilibrium, and results in maximum expected capital

Assumption 3.2.5. *The company under pessimistic scenario chooses*

$$y^* \in \underset{y \in \mathcal{Y}_0}{argmax} \, (\underset{\pi \in C, y \leq c + s(\pi, y)}{max} H^{\tilde{b},\pi}(y)). \tag{3.12}$$

3.3 Nash Equilibria under a diffuse capital distribution.

The goal of this section is to investigate the properties of the Nash Equilibrium defined in the previous section. We find the conditions for the existence of a Nash Equilibrium. We show that the Nash Equilibrium is unique, thus the company's policy under optimistic scenario is the same as under pessimistic scenario. We find this unique optimal strategy for the company.

Throughout this section we assume that the capital distribution across beliefs is diffuse, i.e., we assume that the distribution function $S(\cdot)$ is continuous.

3.3.1 Nash Equilibria Properties

We begin with showing that any NE has the following monotone property: for any company's investment y there exists a creditor that invests a positive fraction of her wealth in the company , then any other creditor with higher beliefs invests all of her wealth in the company as well. Formally, we have the following proposition.

Proposition 3.3.1. *Suppose π is a Nash Equilibrium and there exists an creditor a' such that $\pi(a', y) > 0$. Then $\pi(a, y) = 1$ for every creditor a such that $b(a) > b(a')$.*

The proof of the Proposition Threshold type 1 is the Appendix B.

In light of the Proposition 3.3.1, it is natural to consider the type of strategies under which every creditor with a belief above a threshold invests all her money in the company, and there are no creditors with the belief below the threshold that invest in the company.

Definition 3.3.2. *The allocation strategy π is said to be of a* Threshold Type *if it has the following property: there exists a function $b^* : (c, c + C] \to [0, 1]$ such that*

$$\pi(a, y) = \begin{cases} 1, & \text{if } b(a, y) > b^*(y) \\ 0, & \text{if } b(a, y) < b^*(y). \end{cases} \tag{3.13}$$

We will denote this strategy π_{b^*} and call the function $b^*(\cdot)$ the critical belief function.

For a threshold type allocation strategy π_{b^*}, the debt provided by the creditors is given using the distribution function:

$$s(\pi_{b^*}, y) = S(b^*(y)). \tag{3.14}$$

Furthermore we have the following payoff for the creditors

$$G^{a,\pi_{b^*}}(y) = \begin{cases} g(b(a), y, S(b^*)) & \text{if } b(a) > b^*(y) \\ 1 & \text{otherwise.} \end{cases}$$

From Definition 3.3.2 and Proposition 3.3.1, it immediately follows that any Nash Equilibrium π we can be written as

$$\pi = \pi_{b^*},$$

where

$$b^*(y) = \inf\{b(a) \mid \pi(a, y) > 0\}.$$

We have the following corollary.

Corollary 3.3.3. *If an allocation strategy π is a Nash Equilibrium, then it is of the Threshold Type.*

Let's fix the company's investment y and name $b^*(y)$ the critical belief. In light of this corollary, we can express any Nash Equilibrium in terms of the critical belief. Intuitively, the critical belief is the belief under which the expected return per dollar from investing in the company's debt is the same as the return under the risk free investment, i.e., it is equal to 1. Note however that the return per dollar invested does not have a trivial monotonicity in the critical belief, since the debt itself depends on the critical belief. In general, such a critical belief is not unique.

The reminder of the section is dedicated to proving the existence of a Nash Equilibrium and the uniqueness of the company's strategy under certain conditions. In particular, existence holds when the distribution of capital has full support, see the Remark 3.3.5.

Proposition 3.3.4. *Suppose there exists $\epsilon > 0$ in the image of S, such that*

$$\frac{\sup_{y\in[c,c+C]} Put(yS^{-1}(\epsilon), y + r\epsilon - c)}{\epsilon} < r. \tag{3.15}$$

Then for any $y \in [c, c + C]$ there exists a critical belief $\beta < S^{-1}(\epsilon)$ such that

$$g\left(\beta, y, S\left(\beta\right)\right) = 1. \tag{3.16}$$

The proof of Proposition 3.3.4 is in the Appendix B.

Remark 3.3.5. *Assumption* (3.15) *holds when S has full support. Indeed, assuming that S is differentiable, we have for all $y \in [c, c + C]$*

$$\lim_{\epsilon\to 0}\frac{\sup_{y\in[c,c+C]} Put(yS^{-1}(\epsilon), y+r\epsilon-c)}{\epsilon} \leq \lim_{\epsilon\to 0}\frac{Put(cS^{-1}(\epsilon), C+r\epsilon)}{\epsilon}$$
$$= \lim_{\epsilon\to 0} -c\Phi(-d_1)S^{-1}(\epsilon)' + r\Phi(-d_2)$$
$$= 0$$

where in the first inequality we use the fact that the put option price is decreasing in its first argument and increasing in its second argument, in the second line we use l'Hôpital's rule and in the last line we use $\lim_{\epsilon\to 0}\Phi(-d_1(yS^{-1}(\epsilon), y+r\epsilon-c)) = \lim_{\epsilon\to 0}\Phi(-d_1(yS^{-1}(\epsilon), y+r\epsilon-c)) = 0$.

We now consider the case when the distribution of beliefs has bounded support, for a sufficiently large bound. In this case, if the distribution of beliefs is uniform, then there exists a unique critical belief for which the expected return on the dollar is one and the debt is bounded away from zero.

Proposition 3.3.6. *Fix $\epsilon > 0$. There exists an $M = M(\epsilon) > 0$ such that the hypothesis of Proposition 3.3.4 is satisfied for the following uniform distribution of capital across beliefs, $S : [0, M] \to \mathbb{R}$ with $S(b) = \frac{C}{M}(M-b)$. Then for any $y \in [c, c+C]$ there exists a unique $\beta < M(1-\frac{\epsilon}{C})$ such that*

$$g(\beta, y, S(\beta)) = 1. \tag{3.17}$$

The proof of Proposition 3.3.6 is in the Appendix B.

In the sequel we fix $\epsilon > 0$ and let $M(\epsilon)$ be as defined in Proposition 3.3.6 implying that the uniform distribution is on $[0, M(\epsilon)]$. We will restrict ourselves to investment strategies of the company with $y \geq c + \epsilon$.

Proposition 3.3.7. *A strategy $\pi_b(y)$ is a Nash Equilibrium if and only if for any $y \in (c+\epsilon, c+C)$*

$$g\left(b(y), y, S\left(b(y)\right)\right) = 1 \tag{3.18}$$

and $S(b) > \epsilon$.

The proof of Proposition 3.3.7 is in the Appendix B.

By Proposition 3.3.6 and Proposition 3.3.7, we can conclude there exists a unique equilibrium for the creditors.

We have the following proposition that states that a debt capacity function is well defined. By abuse of notation, we will denote this function s.

Proposition 3.3.8 (Existence of a debt capacity function)**.** *For every $y \geq c+\epsilon$ there exists a unique $\beta(y)$ such that $\pi_{\beta(y)} \in \mathcal{C}$. The debt capacity as a function of y is given by*

$$s(y) := s(\pi_{\beta(y)}) = S\left(\beta(y)\right).$$

From Proposition 3.3.4 there exists a unique $\beta = \beta(y)$ such that

$$g\left(\beta(y), y, S\left(\beta(y)\right)\right) = 1. \tag{3.19}$$

and $S\left(\beta(y)\right) \geq \epsilon$.

From Proposition 3.3.7, we have that $\pi_{\beta(y)}$ is the unique Nash Equilibrium.

3.3.2 The company's optimal decision

It now remains to study the optimal investment of the company. Recall that in the previous section we only require that the creditors reach an equilibrium. In this section we need to first check if the the investment of the company is feasible, in other words, if the debt offered by the creditors in equilibrium is sufficient for the company to make its investment.

We begin by noting that the function β that uniquely determines the creditor's equilibrium as a function of the company's investment is a continuous function. This in turn implies the following proposition.

Proposition 3.3.9. *The debt capacity of the company $s(y) = S(\beta(y))$ is a continuous function of y.*

The following proposition is critical to determining the optimal choice of the company. Any choice of the company lies in the set $\{c < y \leq c + s(y)\}$. In this set we have that more investment in the risky asset decreases the debt capacity.

Proposition 3.3.10. *The map $y \to \beta(y)$ is strictly increasing (the map $y \to s(y)$ is strictly decreasing) over the set of feasible strategies of the company $\{c < y \leq c + s(y)\}$.*

The proof of Proposition 3.3.10 is in Appendix B.

The company has the following optimization problem.

$$\underset{c<y\leq c+s(y)}{Maximize}\, H^{\tilde{b},\pi_{\beta(y)}}(y)$$

i.e.,

$$\underset{c<y\leq c+s(y)}{Maximize}\, Call(y\tilde{b}, y + rs(y) - c) \tag{3.20}$$

The main result of the paper, namely that there exists a unique solution to the company's optimization problem is based on the following proposition.

Proposition 3.3.11. *The function $y \to Call(y\tilde{b}, y + rs(y) - c)$ is increasing over the feasible set $\{c \leq y \leq c + s(y)\}$.*

The proof of Proposition 3.3.11 is in the Appendix B.

Theorem 1. *There exists a unique solution y^* to the Problem 3.20 given by*

$$y^* := \sup\{c < y \leq c + s(y)\} \tag{3.21}$$

We have that $(\pi_{\beta(y^*)}, y^*)$ *is the unique company's strategy. Moreover, we have that the unique solution saturates the funding constraint:*

$$y^* = c + s(y^*). \tag{3.22}$$

The proof of Theorem 1 is in the Appendix B.

The first part of the theorem states that the optimal choice of the company is the maximum feasible investment.

Proposition 3.3.10 ensures that the the debt capacity of company decreases when y increases, and that the debt capacity function $s(y)$ is a continuous function of y. These imply that at the optimal choice the constraint is saturated, i.e., the company invests all the available funds in the risky asset. Intuitively, the company is optimistic about the asset and, therefore, buys as many units of the risky asset as possible. Since the distribution of capital is diffuse, the drop in debt capacity can be made arbitrarily small by choosing a sufficiently small increase in the asset position. Therefore, if the company has a non-zero cash position, it is always optimal to buy an additional positive quantity of the asset, without violating the capital constraints.

Of course, that it is optimal for the company to increase the risky asset position in the first place stems from the first part of Theorem 1, and depends on our assumptions on the belief distribution. In particular, we rely on Proposition 3.3.11 which states that, under a uniform capital distribution across beliefs, the increase of debt capacity with respect to the risky investment is small in relation to the profits expected by the optimistic borrower by the increase in the risky investment.

Theorem 1 guarantees that the company invests all available funds in the risky asset. The same results would hold if the company would not commit to any investment strategy. Note that in this case, the rational creditors would assume that all the debt is invested into the risky asset. This is consistent with the unique optimal strategy of the company, which is to hold zero cash.

In the next section we will study the same problem in presence of a large creditor.

3.4 Nash Equilibria given a large creditor

In this section we assume that besides "small" creditors, the company can borrow money from a "large" creditor α.

The rate for the large creditor is $R \geq r$. Creditor α has a belief about the risky asset given by $b(\alpha)$. The capital available to the large creditor is given by $C(\alpha)$, while the total capital stays as in the previous setting equal to C.

We let $M_0 = M(1 - \frac{C(\alpha)}{C})$. We assume that $b(\alpha) \in (0, M_0)$. The distribution function for the capital is given by $S : [0, M] \to \mathbb{R}_+$, where

$$S(b) := \frac{C - C(\alpha)}{M_0}(M_0 - b) + C(\alpha)_{b \leq b(\alpha)}. \tag{3.23}$$

For simplicity, we only consider the case of a single atom. Allowing for any finite number of atoms would not change our main results concerning the existence of equilibria with positive cash holdings.

3.4.1 Nash Equilibria Properties

The behavior of the "large" creditor affects the overall debt of the company and, therefore, significantly affects the decisions of the "small" creditors. We investigate the behavior of a "large" creditor and find that she either invests all of her available capital in the company or invests nothing. More formally we have the following proposition.

Proposition 3.4.1. *If π is a Nash Equilibrium, then $\pi(\alpha, y) \in \{0, 1\}$. Moreover, for any strategy of the "small" creditors $\pi_{-\alpha}$, the "large" creditor maximizes her expected gain either if she invests nothing or if she invests all her available capital $C_{b(\alpha)}$ in the company.*

The proof of Proposition 3.4.1 is in the Appendix B.

Similar to the case considered in the Section 3.3, the following monotone property holds when there is a "large" creditor. If there exists a "small" creditor that invests a positive fraction of her wealth in the company, then any other "small" creditor with higher beliefs invests all her wealth in the company as well. Formally, the modification of Proposition 3.3.1 holds under a non-diffuse capital distribution.

Proposition 3.4.2. *If π is a Nash Equilibrium and there exists a "small" creditor a', such that $\pi(a', y) > 0$, then $\pi(a, y) = 1$ for every "small" creditor a such that $b(a) > b(a')$.*

The proof of the statement mimics the proof of Proposition 3.3.1.

3.4.2 Nash Equilibria in non discriminative case

We now consider the case when the company pays the same rate r to the "small" and "large" creditors.

Proposition 3.4.3. *If a strategy π is a Nash Equilibrium, then there exists a function $b^* : [c + \epsilon, c + C] \to \mathbb{R}^+$, such that, given the company's investment is y, any creditor $a \in A$ with the belief $b(a) > b^*(y)$ invests in the company, while every creditor a' with the belief $b(a') < b^*(y)$ does not invest.*

The proof of Proposition 3.4.3 is in the Appendix B.

It then immediately follows:

Corollary 3.4.4. *If a strategy π, is a Nash Equilibrium, then it is of a Threshold Type.*

Any Nash Equilibrium can be denoted as π_{b^*}: the strategy under which, given the company's investment y, all creditors with beliefs $b > b^*(y)$ invest all available money in the company, and all creditors with beliefs $b < b^*(y)$ do not invest in the company.

Similarly to Section 3.3 the following proposition holds.

Proposition 3.4.5. *If a strategy π_b is a Nash Equilibrium, for any $y \geq c + \epsilon$*

$$g\,(b, y, S\,(b)) = 1 \tag{3.24}$$

and $S\,(b) > \epsilon$.

The proof of the proposition mimics the proof of the first part of Proposition 3.3.7 in Section 3.3.

We distinguish the following cases.

Proposition 3.4.6.

1. *For any fixed $y \geq c + \epsilon$, there exists either one or two roots $b^* = b^*(y)$ such that $g(b^*, y, S\,(b^*)) = 1$.*
2. *If Equation 3.24 has two solutions, b^* and b^{**} ($b^* < b^{**}$), then $b^* \in (0, b(\alpha)]$, $b^{**} \in [b(\alpha)^+, M_0(1 - \frac{\epsilon}{C - C(\alpha)})]$*

Proposition 3.4.7. *If π_β is a Nash Equlibrium then for any y only following cases may arise*

1. *$\beta = b^*$ the unique root of Equation 3.24.*
2. *$\beta = b^* \in (0, b(\alpha))$.*
3. *$\beta = b^{**} \in [b(\alpha)^+, M_0(1 - \frac{\epsilon}{C - C(\alpha)})]$, and $g\,(b(\alpha), y, s(\pi_{b^{**}}, y) + C(\alpha)) \leq 1$.*

3.4.3 The company's optimal decision in non discriminative case

Similarly to Section 3.3, this section studies the company's optimal investment policy given the response of the creditors as a function of the investment policy. The key difference with a "large" creditor is a discontinuity introduced when the large creditor invests or not (we have seen in the previous section that she invests all or nothing of her available capital). Moreover, this discontinuity in the debt may be larger than the capital of the large creditor due to the presence of multiple equilibria.

We will show that under certain conditions, the optimal policy of the company leads to non-zero cash holdings. Indeed, the company will maximize the investment in the asset subject to the debt capacity constraint. Due to the jump down in the debt capacity, it will follow that the risky asset investment will not saturate the debt capacity constraint.

We have to take into account the behavior of the large creditor separately. We begin by determining the existence of an investment policy of the company for which the large investor is the critical creditor and an investment policy for which the large creditor does not invest but all small creditors with higher beliefs than her invest.

Proposition 3.4.8. *There exists at most one $y_1 > c$ and respectively at most one $y_0 > c$ such that:*

$$g\left(b(\alpha), y_1, S\left(b(\alpha)\right)\right) = 1. \tag{3.25}$$

$$g\left(b(\alpha), y_0, S\left(b(\alpha)^+\right)\right) = 1. \tag{3.26}$$

Moreover, if both Equations (3.25) and (3.26) have solutions y_1 and y_0, then $y_1 \geq y_0$.

The proof of Proposition 3.4.8 is in Appendix B.

Note also that $g\,(b(\alpha), y, S\,(b(\alpha))) > 1$, when $y_1 \to c$, and $g\,(b(\alpha), y, S\,(b(\alpha)))$ is continuous in the second argument. Therefore if Equation (3.25) does not have a solution, then $g\,(b(\alpha), y, S\,(b(\alpha))) > 1$, for any y. In this case, we set $y_1 = c + C$.

Similarly, if (3.26) does not have a solution, we set $y_0 = c + C$.

We define two maps corresponding to the debt provided in the two cases (π_{b^*} and $\pi_{b^{**}}$) of Proposition 3.4.6.

1. $y \to \beta_1(y)$, $y \to s_1(y)$: $[c + \epsilon, y_1] \to (0, b(\alpha)]$. We choose $\beta_1(y)$ such that:

$$g(\beta_1(y), y, S\,(\beta_1(y))) = 1. \tag{3.27}$$

   The solution to Equation (3.27) on $(0, b(\alpha)]$ exists since for any $y \in [c + \epsilon, y_1]$

$$g(b(\alpha), y, S\,(b(\alpha))) \geq 1,$$

   and

$$g(0, y, S\,(0)) < 1.$$

   From Proposition 3.4.6 the solution is unique. We let

$$s_1(y) = S\,(\beta_1(y)).$$

2. $y \to \beta_0(y)$: $(y_0, c + C] \to (b(\alpha), M_0(1 - \frac{\epsilon}{C - C(\alpha)})]$. We choose $\beta_0(y)$ such that:

$$g(\beta_0(y), y, S\,(\beta_0(y))) = 1. \tag{3.28}$$

   The solution to Equation (3.28) on $(b(\alpha), M_0(1 - \frac{\epsilon}{C - C(\alpha)}))$ exists as for any $y \in (y_0, c + C]$.

$$g(b(\alpha), y, S\,(b(\alpha))) \leq 1,$$

   and

$$g(x, y, S\,(x)) > 1,$$

   for $x = M_0(1 - \frac{\epsilon}{C - C(\alpha)})$. From Proposition 3.4.6 the solution is unique. We let

$$s_0(y) = S\,(\beta_0(y)).$$

The debt capacity functions associated with the possible NEs are thus well defined. As expected, these debt capacity functions have the following monotonicity property.

Proposition 3.4.9. *The maps $y \to \beta_{0,1}(y)$ are strictly increasing (the maps $y \to s_{0,1}(y)$ are strictly decreasing) over the respective sets of feasible strategies of the company $\{c < y \leq c + s_{0,1}(y)\}$.*

The proof of Proposition 3.4.9 mimics the proof of Proposition 3.3.10.

Note that the debt capacity functions have different domains, and the monotonicity property above is stated separately for each function.

However, their domains intersect in the interval $(y_0, y_1]$, so over this interval we are able to compare the two debt capacity functions. Indeed, since $\beta_1(y) \leq b(\alpha) < \beta_0(y)$ for any $y \in (y_0, y_1]$, we have

$$s_1(y) > s_0(y),\ y \in (y_0, y_1]. \tag{3.29}$$

Proposition 3.4.7, 3 establishes a criterion where the NE, associated with the debt capacity function s_0, is indeed a NE by considering the payoff from a full investment by the "large" creditor. To be able to verify this criterion we first establish the following proposition.

Proposition 3.4.10. *The equation*

$$g\Big(b(\alpha), y, s_0(y) + C(\alpha)\Big) = 1 \tag{3.30}$$

has at most one solution y_0' on $(y_0, y_1]$.

The proof of Proposition 3.4.10 is in the Appendix B.

If Equation (3.30) does not have a solution on $(y_0, y_1]$, we set $y_0' = y_0$. We are now able to establish for a $y \in (y_0, y_1]$ which of the two possible debt capacity functions corresponds to a NE: y_0' represents the maximum investment amount of the company under which the smaller debt capacity function s_0 (with the critical belief function β_0) does not correspond to a NE. Formally, we have the following Corollary which follows immediately from Propositions 3.4.6, 3.4.7 and 3.4.10.

Corollary 3.4.11. *If π_β is a Nash Equilibrium*

1. *For every $y \in (y_0', y_1]$, $\beta = \beta_0(y)$.*
2. *For every $y \in (y_0, y_0')$, $\beta \neq \beta_0(y)$.*

We assume that the company has the information about the preferences of the "large" creditor. We then denote

$$\tilde{s}_0(y) = \begin{cases} s_1(y), \text{ when } y \in (c_0, y_0'], \\ s_0(y), \text{ when } y \in (y_0', c + C_0], \end{cases}$$

and

$$\tilde{s}_1(y) = \begin{cases} s_1(y), \text{ when } y \in (c_0, y_1], \\ s_0(y), \text{ when } y \in (y_1, c + C_0]. \end{cases}$$

The function $\tilde{s}_0(y)$ represents the debt provided by the creditors, given that the "large" creditor decides not to invest in the company when $y > y_0'$, while the function $\tilde{s}_1(y)$ represents the debt provided by the creditors, given that the "large" creditor decides to invest in the company when $y \in (y_0', y_1]$.

Under pessimistic scenario the optimization problem (3.11) can be formulated as follows:

$$y^* \in \underset{\pi \in C, y \leq c + \tilde{s}_0(y)}{argmax} (G^{\tilde{b},\pi}(y)), \tag{3.31}$$

otherwise

$$y^{**} \in \underset{\pi \in C, y \leq c + \tilde{s}_1(y)}{argmax} (G^{\tilde{b},\pi}(y)). \tag{3.32}$$

As $\tilde{s}_0(y)$ is monotonically decreasing function and $\lim_{y\to c}\tilde{s}_0(y) > c$ and $\lim_{y\to+\infty}\tilde{s}_0(y) = 0$, the problem (3.31) always has a unique solution (π^*, y^*) that we will call a Nash Equilibrium under pessimistic scenario. If $y^* \in [y_0', y_1]$, then $\tilde{s}_1(y_0') > \tilde{s}_0(y_0') \geq y_0'$ and for any $\delta > 0$, $\tilde{s}_1(y_1+\delta) = \tilde{s}_0(y_1+\delta) \leq y_1$, therefore the problem (3.32) has a unique solution (π^{**}, y^{**}), different from (π^*, y^*), that we will call a Nash Equilibrium under optimistic scenario and $y^{**} \in [y_0', y_1]$.

3.4.4 The optimal cash holdings in non discriminative case

It now remains to synthesize the previous results and establish for each possible investment strategy of the company the debt provided in equilibrium by the creditors.

If Equation (3.25) does not have a solution, in other words $y_1 = c + C$, then the results of the optimization problems (3.31) and (3.32) coincide, and the "large" creditor invests under both scenarios. The company does not hold cash.

Otherwise, we consider the equations:

$$y = c + s_1(y), \text{ on } y \in (c, y_1]. \tag{3.33}$$

$$y = c + s_0(y), \text{ on } y \in (y_0', c + C). \tag{3.34}$$

Since the maps $y \to s_{0,1}(y)$ are strictly decreasing, both equations have not more than one solution. We have the following case distinctions.

1.
$$y_0' \geq c + s_1(y_0').$$

Then there exists a unique $y^* \in (c, y_0']$, such that $y^* = c + s_1(y^*)$, and there is a unique strategy for the company under both optimistic and pessimistic scenarios y^*. **The company does not hold cash.**

2.

$$c + s_0(y_0') < y_0' < c + s_1(y_0'),$$

and

$$y_1 < c + s_1(y_1).$$

In this case the solution to the company's problem under pessimistic scenario is y_0' and under optimistic scenario is y_1. **The company holds positive cash in both cases.** This is immediate since the inequalities above are strict so the risky asset investments do not saturate the debt capacity.

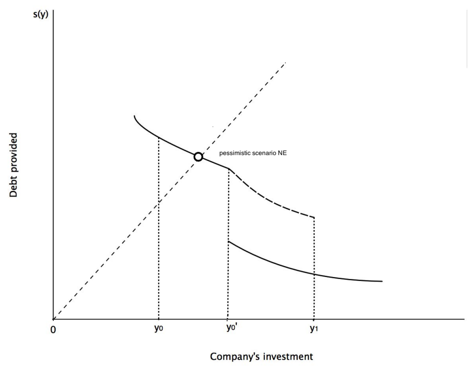

Figure 3.1: Case (i) illustration.

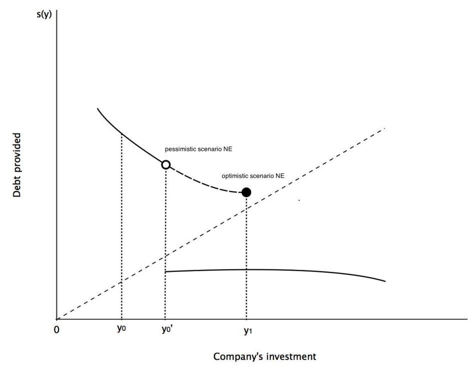

Figure 3.2: Case (ii) illustration.

3.

$$c + s_0(y_0') < y_0' < c + s_1(y_0'),$$

and

$$y_1 \geq c + s_1(y_1).$$

Then there exists a unique $y^* \in (y_0', y_1]$, such that $y^* = c + s_1(y^*)$. the solution to the company's problem under pessimistic scenario is y_0' and under optimistic scenario is y^*. **The company holds positive cash under pessimistic scenario.**

4.

$$y_0' \leq c + s_0(y_0'),$$

and

$$y_1 \geq c + s_1(y_1).$$

Then there exists a unique $y^* \in [y_0', y_1]$, such that $y^* = c + s_0(y^*)$, and there exist a unique a unique $y^{**} \in (y^*, y_1]$, such that $y^{**} = c + s_1(y^{**})$. the solution to the company's problem under pessimistic scenario is y^* and under optimistic scenario is y^{**}. **The company holds zero cash under both scenarios.**

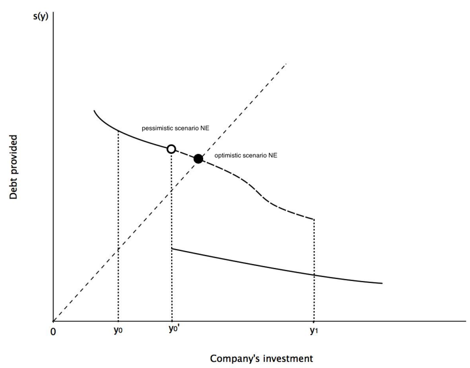

Figure 3.3: Case (iii) illustration.

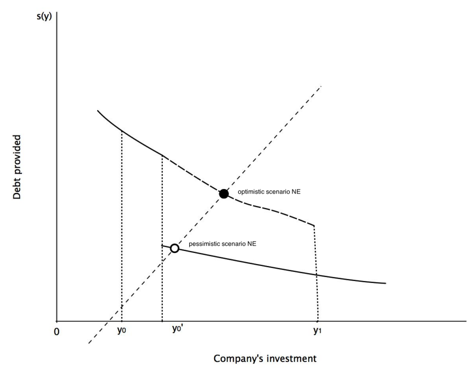

Figure 3.4: Case (iv) illustration.

5.

$$y_0' \leq c + s_0(y_0'),$$

and

$$y_1 + s_0(y_1) \leq y_1 < c + s_1(y_1).$$

Then there exists a unique $y^* \in (y_0', y_1]$, such that $y^* = c + s_0(y^*)$ the solution to the company's problem under pessimistic scenario is y^* and under optimistic scenario is y_1. **The company holds positive cash under optimistic scenario**.

6.

$$y_1 < c + s_0(y_1).$$

Then there exists a unique $y^* \in (y_1, c + C)$, such that $y^* = c + s_0(y^*)$, and y^* is a unique strategy for the company under both optimistic and pessimistic scenarios, when **the company does not hold cash.**

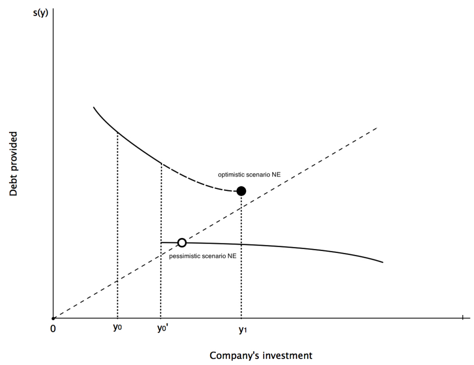

Figure 3.5: Case (v) illustration.

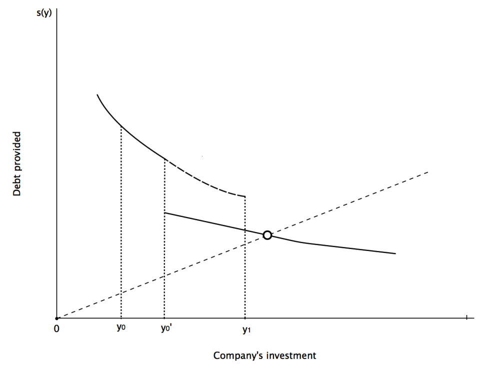

Figure 3.6: Case (vi) illustration.

3.4.5 Discriminative case

In this subsection we consider the case when the company can offer the rate $R \geq r$ to a "large" creditor. In Subsection 3.4.4 we illustrated the cases when the company has positive cash holdings in order to convince the "large" creditor to invest in the company. Now, it can choose an optimal combination of the investment y and the rate R. A larger investment leads to a higher payoff of the company, since it is optimistic about the asset. However, this higher investment in the asset is accompanied by a decrease in debt capacity, so larger investments may not be feasible. Higher interest rates allow the company to maintain debt capacity that it would otherwise lose, but come at a cost. The company must solve this tradeoff by choosing the optimal pair of asset investment and interest rate offered to the large creditor. A critical question is whether, this optimal pair will still lead to positive cash holdings, or, on the contrary, there exists an interest rate that maximizes the company's payoff while maintaining its debt capacity.

We show that positive cash holdings are optimal even if the company optimizes both y and R. In other words, cash holdings are a powerful tool for the company to increase its debt capacity and may be preferred to increasing the interest rate beyond a certain point. The company will use a combination of cash holdings and interest rate increase to convince its large creditor.

For simplicity, we assume $C(\alpha) = C$, i.e. small creditors do not have any capital. In this case we assume that $r \geq 0$ is the minimum market rate.

We define $l : [c+\epsilon, c+C(\alpha)] \times [r, +\infty) \to \mathbb{R}$ an expected payoff of the "large" creditor, if she invests in the company, given the investment of the company is y and the rate is R.

$$\begin{aligned} l(y,R) &= \mathbb{E}^{x}\left[\frac{c + C(\alpha) + y(P-1)}{c + C(\alpha)} \wedge (1+r)\right] C(\alpha) \\ &= (1+R)C(\alpha) - Put(yb(\alpha), y + RC(\alpha) - c). \end{aligned} \tag{3.35}$$

We define the set of admissible strategies of the company $\mathcal{A} \in \mathbb{R}^2$.

$$\mathcal{A} = \left\{(y,R) \middle| l(y,R) \geq C(\alpha), y \leq c + C(\alpha)\right\} \tag{3.36}$$

The expected payoff of the company, given the investment is y and the rate is R

$$h(y,R) = \mathbb{E}^{\tilde{b}}\Big[\Big(c + y(P-1) - RC(\alpha)\Big)^{+}\Big] = Call(y\tilde{b}, y + RC(\alpha) - c).$$

The company, thus, solves the following optimization problem.

$$(y^*, R^*) \in \underset{\mathcal{A}}{argmax}\,(h(y,R)). \tag{3.37}$$

Since h is a bounded function on a compact set $\mathcal{A}$ and the function $h(y, R)$ is monotonically decreasing in R, while the function $l(y, R)$ is monotonically increasing in R, the following result holds.

Proposition 3.4.12. *The solution (y^*, R^*) to the problem (3.37) exists and either*

$$l(y^*, R^*) = C(\alpha), \text{ or}$$

$$l(c + C(\alpha), r) \geq C(\alpha),$$

and $(c + C(\alpha), r)$ is a solution to the problem (3.37).

The latter Equation corresponds to the case, when the minimum rate r is sufficient to convince the "large" creditor to invest in the company.

We, now, assume $l(c + C(\alpha), r) < 1$. We define the set

$$\mathcal{A}_1 = \left\{(y, R)\middle| l(y, R) = C(\alpha), y \leq c + C(\alpha)\right\},$$

The problem (3.37) is, then, equivalent to

$$(y^*, R^*) \in \underset{\mathcal{A}_1}{argmax}\,(h(y, R)). \tag{3.38}$$

To ensure that there exist positive cash holdings, it is sufficient to show, that the expected payoff of the company, given zero cash holdings is strictly less, than the payoff at some point $(y^0, R^0) \in \mathcal{A}_1$ (not necessarily optimal), when the cash holdings are not zero, i.e. the optimization problem (3.38) has an interior point solution. We consider the following examples.

Example 3.4.13. *We consider the following example $c = 5$, $\tilde{b} = 1.03$, $C(\alpha) = 100$, $b(\alpha) = 0.99$, $r = 0.01$, $\sigma = 0.25$.*

Then if $y = c + C(\alpha)$, $R = 25.24\%$. In this case the payoff of the company is $h(y, R) = 5.23$.

However, if the company offers $R^0 = 6.77\%$ it can invest $y^0 = 55$ in the risky asset, and get the payoff of $h(y^0, R^0) = 5.58$. This result is robust with respect to the choice of volatillity, and holds for realistic parameters, as the following example shows.

Example 3.4.14. *We consider the following example* $c = 5$, $\tilde{b} = 1.03$, $C(\alpha) = 500$, $b(\alpha) = 0.995$, $r = 0.01$, $\sigma = 0.1$.

Then if $y = c + C(\alpha)$, $R = 14.34\%$. In this case the payoff of the company is $h(y, R) = 5.03$.

However, if the company offers $R^0 4.45\%$ it can invest $y^0 = 255$ in the risky asset, keeping 250 in cash, and get the payoff of $h(y^0, R^0) = 6.54$.

The Examples 3.4.13 and 3.4.14 show that the company has the incentive to have significant positive cash holdings even if it has an option to optimize the rate.

3.5 Conclusions

We have introduced a novel theoretical model that explains the cash holdings puzzle. We set up a static one period model, in which cash becomes a means to select an equilibrium of creditors with different valuations of the company's assets. What drives our results is not the need to protect against adverse future liquidity shocks as in previous studies, but the role of cash in increasing the debt capacity: the net debt increases with cash holdings.

In the equilibria with positive cash holdings, any decrease of cash holdings (or equivalently, any increase in the investment in the asset) may lead to a negative jump in the debt capacity.

There is an important difference between markets with diffuse investors and markets with a large investor. In the former, no single investor can have a sufficiently high impact on the overall debt provision to prompt the borrower to bear the cost of holding cash. In the latter, the strategy of the large creditor acts itself as a selection mechanism between different equilibria of the creditors. The company may choose to hold cash when the large creditor is sufficiently pessimistic, and this will have the benefit of keeping the large creditor and also a positive fraction of the small creditors. Our results hold when the company can vary the interest rate. We show that cash holdings are a powerful tool for the company to increase its debt capacity and may be preferred to increasing the interest rate beyond a certain point.

CHAPTER 4

SOVEREIGN RISK SPREAD MODELING

4.1 Introduction

Sovereign credit risk modeling is a common and well-studied topic, and many new modes have appeared in the recent years. As the sovereign credit risk is influenced by various factors, such as Debt-to-GDP ratio, political stability, unemployment rate, the amount of the outstanding debt, availability of the certain instruments of financial control and many others, searching for an explicit dependence on these parameters is a challenging task, and, is typically omitted in the models.

The classical example is provided in (42), the first parsimonious model, where the yield and, thus, the price, depend on the time to maturity only. The model has a lot of flexibility but cannot match all yield curves, however the simplicity made it very popular among practitioners and inspired various extensions.

If we look at the factors that affect the sovereign bond prices closer, we will see that some of the factors are country-specific, while the others are global. The recent works (33), (37) and (45) show the significant impact of the global factors on the bond prices. However during the latest European Credit Crisis the variation of the European countries' long-term yields raised from less than 2% in October, 2009 to over 20% in November, 2011, that shows the importance of studying the influence of the country-specific factors.

The influence of the certain internal factors is studied in (29), where the authors analyze the consequences of the default, such as change in the country's export, GDP growth and unemployment rate, and, then, allow the government to default if it is profitable for the country. The model directly links the internal parameters with the default risk, however the recent European Crisis showed that the political risks that were hard to quantify could noticeably influence the country's choice whether to default or not, making the decision much more complicated than the bareer-type default problem. We, thus, do not study the mechanism of sovereign default, due to complexity of the event.

In order to investigate the dependence of the yield curve on macroeconomic parameters we decompose the sovereign credit risk into two independent components, one is responsible for the risk-free rate and the other for the risk spread. We use the results of , (42) that expands the setting from (50), and connect the inflation and the unemployment rate with the risk-free rate, and study the risk spread component only.

The work (16) shows that the risk spreads for the developing countries' bonds, that are issued in foreign currency have a significant correlation with Debt-to-GDP ratio. The authors claim that for the developed countries like the USA, the UK or Canada this correlation is not statistically significant, however this correlation for the Eurozone countries is similar to the developing countries, not to developed non-Eurozone ones. We, thus, consider the risk spread as a function of Debt-to-GDP ratio.

We model both the dynamics of the sovereign spread and the Debt-to-GDP ratio. The Debt-to-GDP ratio data is updated once a year, and, therefore it is reasonable to model it as a piecewise constant function. The risky spread dynamics cannot be observed directly from the market data, which makes calibration and pricing a challenging task.

4.2 The model

4.2.1 Introduction

We decompose the instantenious sovereign bonds rate to the sum of the risk-free rate and the risk spread. The price of zero-coupon bond traded at time t with the maturity T will be

$$B(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T (-r_s - \lambda_s)ds\right)\right].$$

The risk spread depends on many parameters, however we assume that the parameter $R_s = \frac{D_s}{X_s}$, which represents the ratio of the Debt-to-GDP, is the most important one.

We consider the risk spread as $\lambda_s = f(R_s, s)$. Since the dependence of the default events and, thus, the risk spread on the Debt-to-GDP ratio has a non trivial nature, we do not model it explicitly.

We make the following assumptions

1. The economy faces the shocks according to compound Poisson process with the rate $\kappa_1(s, R)$.
2. Each shock has a magnitude that is random and follows exponential distribution with the parameter κ_2.
3. The economy can absorb any small shock, i.e. any shock with the magnitude less than $C(R)$.
4. The country declares default on the bonds of a certain type and enters the restructuring procedure, if the shock is large, i.e. larger than $C(R)$.
5. The higher the Debt-to-GDP ratio is, the more vulnerable the country to default as a result of a shock. We assume $C_s(R) = -C_s' R + C_s''(t)$, where $C_s^1 > 0$ and $s + t$ is a maturity of the bond.

6. Usually, the shocks and their magnitude represent the global problems and, therefore, their intensity do not depend on R, i.e. $\kappa_1(s,R) = \kappa_1(t)$ and $\kappa_2(t,R) = \kappa_2$.

7. For simplicity, we assume that the investor receives nothing, if the restructuring procedure has been launched. This assumption does not affect the results and can be relaxed.

If we denote the default time as τ, we can find:

$$\mathbb{P}(\tau < T) = \sum_{n=0}^{+\infty} \frac{e^{-\int_0^T \kappa_1(s)} \int_0^T \left(\kappa_1(s)(1 - e^{-\kappa_2 C_s(R_s)})\right)^n ds}{n!} = \exp\left(-\int_0^T \kappa_1(s) e^{-\kappa_2 C_s(R_s)} ds\right). \tag{4.1}$$

On the other hand, risk-neutral pricing requires:

$$B(t,T) = \mathbb{E}^{\mathbb{Q}}\left[exp(\int_t^T (-r_s - \lambda_s)ds)\right] = \mathbb{E}^{\mathbb{Q}} \exp\left[\int_t^T \Big(-\kappa_1(s)e^{-\kappa_2 C_s(R_s)} - r_s\Big)ds\right], \tag{4.2}$$

where $\int_t^T \exp(-\bar{r}_s)ds$ is the price of a zero coupon bond, issued at time t, that matures at time T, and $\bar{r}_s = r_s + \lambda_s$ is the total rate.

Equation (4.2) is valid for any t and T, therefore:

$$\lambda_s = \kappa_1(s) \exp(\kappa_2 C_s(R_s)). \tag{4.3}$$

Equation (4.3) yields the following dependence of the implied and actual risk spread from the Debt-to-GDP ratio:

$$\lambda_s = C_1(s) \exp(C_2 R_s). \tag{4.4}$$

4.2.2 Macroecomic parameters modeling

Since the values of R_s are updated once a year, we assume that it can be described as a piecewise constant function of time.

$$R_s = R_i, \text{ for } s \in [i, i+1). \tag{4.5}$$

The risk spread dynamics is represented through the Equation (4.4), however the dynamics of λ_s affects the Debt-to-GDP ratio. Since λ_s is a continuous function, while R_s makes jumps at the integer points, the Equation (4.4) yields that $C_1(s)$ must also make jumps in the integer points.

We assume that at any integer point i: $C_1(i) = C_1(0)$.

We then assume that R_0 is known and

$$R_{i+1} = \frac{1}{C_2} \ln \frac{\lambda_i}{C_1(0)} \tag{4.6}$$

Equation (4.6) shows the explicit dependence of the Debt-to-GDP ratio on the risk spread.

4.2.3 Parameters' dynamics

We consider the Vasicek model for $C_1(s)$. We again assume that for any integer i:

$$C_1(i) = C_1(0), \tag{4.7}$$

and for $s \in [i, i+1)$:

$$dC_1(s) = \theta(\mu - C_1(s))ds + \sigma dW_s. \tag{4.8}$$

Equations (4.4) and (4.8) yield:

$$d\lambda_s = \theta(\mu \exp(C_2 R_s) - \lambda_s)ds + \sigma \exp(C_2 R_s) dW_s \tag{4.9}$$

The long term dynamics of the Debt-to-GDP ratio is a process that can be found recursively.

$$R_{i+1} = C(0)e^{-\theta} + \mu(1 - e^{-\theta}) + \sqrt{\frac{e^{2\theta} - 1}{2\theta}} W_1 \tag{4.10}$$

For $s \in [i, i+1)$:

$$R_s = R_i \tag{4.11}$$

4.3 Pricing technique

We have shown the recursive dynamics of the Debt-to-GDP ratio, however the analytical solution of a zero-coupon price is a challenging task, and in this subsection we are considering the quantities R_i as model parameters.

Then, if $t \in [0, 1)$ and $T \in [i, i+1)$, a zero-coupon bond $B(t, T)$ can be priced.

We denote:

$$A(T_1, T_2) = \frac{1 - e^{-\theta(T_2 - T_1)}}{\theta} \tag{4.12}$$

$$D(T_1, T_2) = \left(\mu \exp\left(C_2 R_{T_1}\right) - \frac{\left(\sigma \exp\left(C_2 R_{T_1}\right)\right)^2}{2\theta^2}\right)\Big(A(T_1, T_2) - (T_1 - T_2)\Big) - \frac{\left(\sigma \exp\left(C_2 R_{T_1}\right)\right)^2 A(T_1, T_2)^2}{4\theta} \tag{4.13}$$

Then the following integrals can be evaluated as follows:

1. If $T' \in (t, 1]$:

$$I(t, T') = \mathbb{E}^{\mathbb{Q}}\left[\exp \int_t^{T'} -\lambda_s ds\right] = \exp\left[-A(t, T')C_1(t)\exp(C_2 R_0) + D(t, T')\right], \tag{4.14}$$

2. For any integer j:

$$I(j, j{+}1) = \mathbb{E}^{\mathbb{Q}}\left[\exp \int_{j}^{j+1} -\lambda_s ds\right] = \exp\left[-A(j, j+1)C_1(0)\exp(C_2 R_j) + D(j, j+1)\right], \tag{4.15}$$

3. If $T' \in [i, i+1)$:

$$I(i, T') = \mathbb{E}^{\mathbb{Q}}\left[\exp \int_{i}^{T'} -\lambda_s ds\right] = \exp\left[-A(i, T')C_1(0)\exp(C_2 R_i) + D(i, T')\right], \tag{4.16}$$

If we assume no recovery and no risk-free interest zero-coupon bond price can be found as follows:

If $t < 1 \leq i \leq T < i + 1$:

$$B^{RF}(t, T) = I(t, 1)I(i, T)\prod_{j=1}^{j=i-1} I(j, j+1) =$$

$$I(t, 1)I(i, T)\exp\left(\sum_{j=1}^{j=i-1}\Big(-A(j, j+1)C_1(0)\exp(C_2 R_j) + D(j, j+1)\Big)\right) \tag{4.17}$$

The price of a Zero Coupon Bond with the recovery rate δ_r and non zero risk free rate is

$$B^{ZCB}(t, T) = (1 - \delta_r)B^{RF}(t, T) + \delta_r \tag{4.18}$$

If the bond pays annual coupons C_a, then the total bond price is

$$B(t, T) = \begin{cases} B^{ZCB}(t, T) + C_a \sum_{j=0}^{j=i-1} B^{ZCB}(t, T - j) + C_a(T - t - i)B^{ZCB}(t, T - i), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{when } T - i > t, \\ B^{ZCB}(t, T) + C_a \sum_{j=0}^{j=i-2} B^{ZCB}(t, T - j) + C_a(T - t - i + 1)B^{ZCB}(t, T - i + 1), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{when } T - i < t. \end{cases} \tag{4.19}$$

If the bond pays semestial coupons C_h, then the total bond price is

$$B(t,T) = \begin{cases} B^{ZCB}(t,T) + \frac{C_h}{2}\sum_{j=0}^{j=2i+1} B^{ZCB}(t, T-\frac{j}{2}) - C_h(i+t+1-T), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{when } T-i-0.5 > t, \\ B^{ZCB}(t,T) + \frac{C_h}{2}\sum_{j=0}^{j=2i} B^{ZCB}(t, T-\frac{j}{2}) - C_h(t-i+\frac{1}{2}-T), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{when } T-i-0.5 \leq t < T-i, \\ B^{ZCB}(t,T) + \frac{C_h}{2}\sum_{j=0}^{j=2i-1} B^{ZCB}(t, T-\frac{j}{2}) - C_h(t+i-T), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{when } T-i \leq t. \end{cases} \tag{4.20}$$

4.4 Calibration

4.4.1 Overview and methodology

We model the market perceptions of the Debt-to-GDP ratio. We use a sequential scheme to estimate the model parameters.

1. We fix the quantities R_i at the level that was observed at the time of trade and the speed of mean-reversion θ at 1.
2. We consider the coupon bonds that mature within 4 years and solve the optimization problem for μ, σ and C_2.
3. We search for an implied Debt-to-GDP ratio, that gives the best match to the market data.
4. We assume Debt-to-GDP ratio is a piecewise constant function of time, that can be characterized with 6 numbers: current Debt-to-GDP ratio, the ratio in 1 year, the ratio between years 2 and 5, the ratio between years 6 and 10, the ratio between years 11 and 15, and the ratio after 15 years.

4.4.2 Results

We analayze Italian and Spanish bonds' prices data from 2006 to 2014 and estimate the market perception of Debt-to-GDP ratio over time.

The figures 4.1 - 4.6 shows the model predictions and the actual yields to maturiy for Italian and Spanish bonds in 2013-2014, and the figures 4.7 - 4.10 show the market perception of Debt-to-GDP Ratio for Italy and Spain.

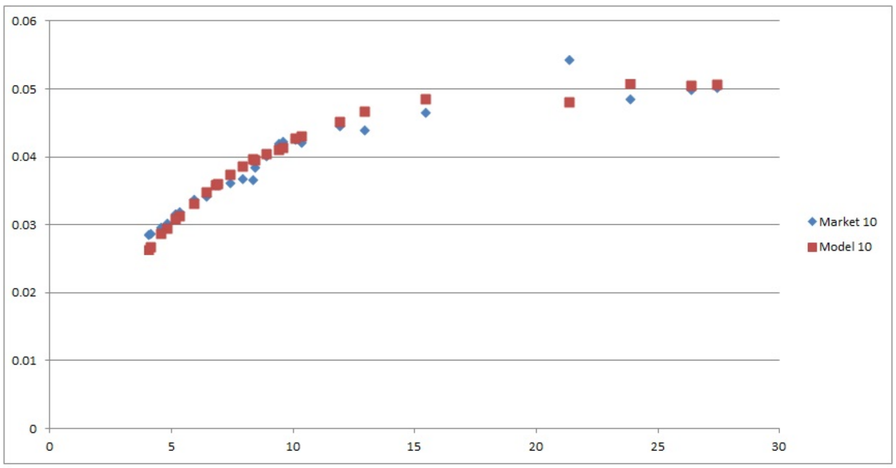

Figure 4.1: Yield To Maturity, Italy, April 2013

MSE= 0.323 (on 100 scale)

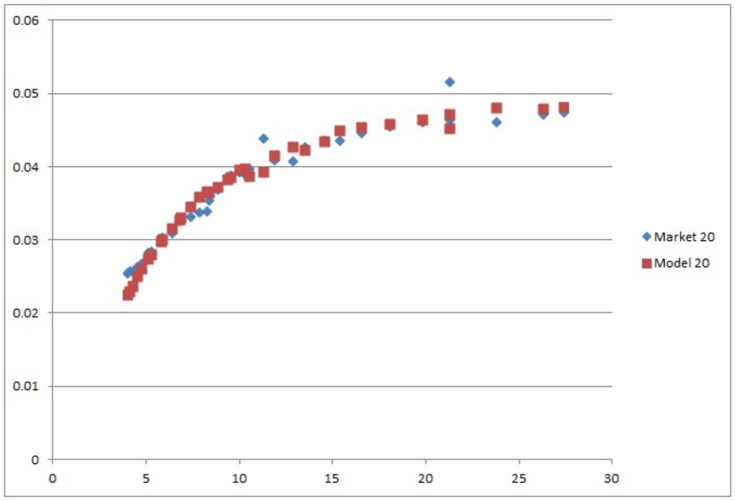

Figure 4.2: Yield To Maturity, Italy, May 2013

MSE= 0.322 (on 100 scale)

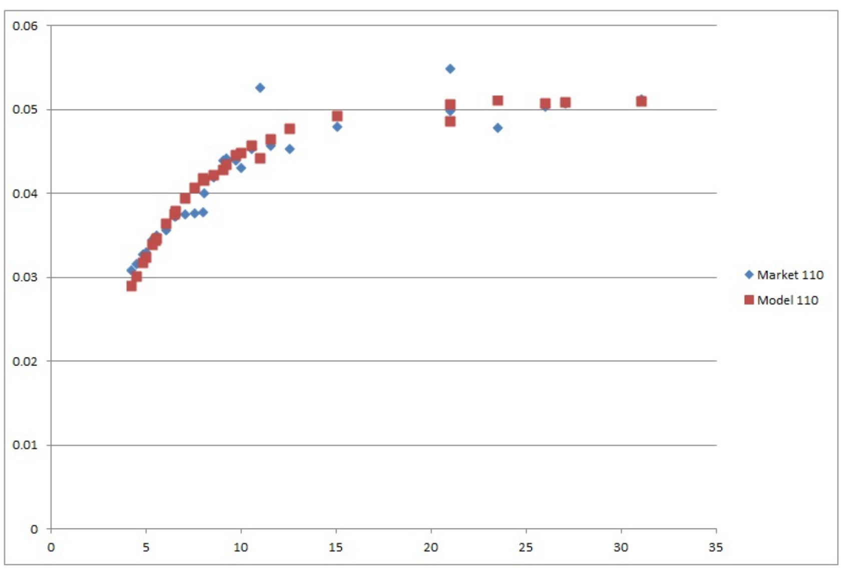

Figure 4.3: Yield To Maturity, Italy, September 2013

MSE= 0.324 (on 100 scale)

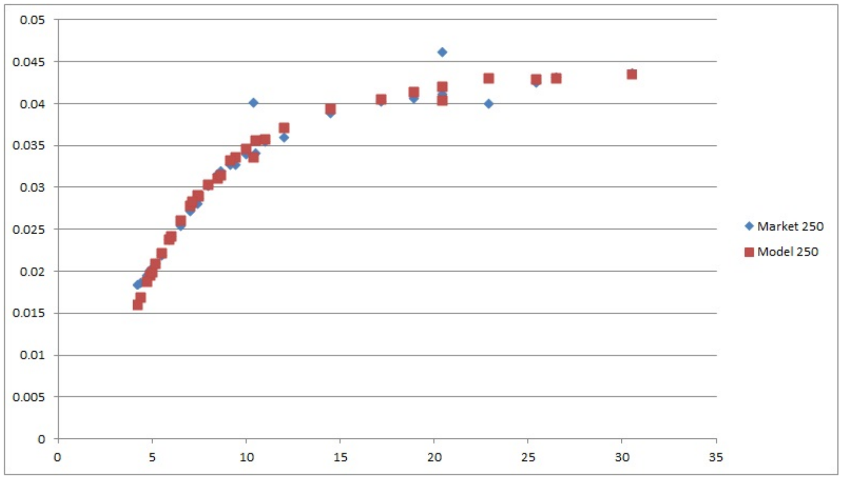

Figure 4.4: Yield To Maturity, Italy, April 2014

MSE= 0.333 (on 100 scale)

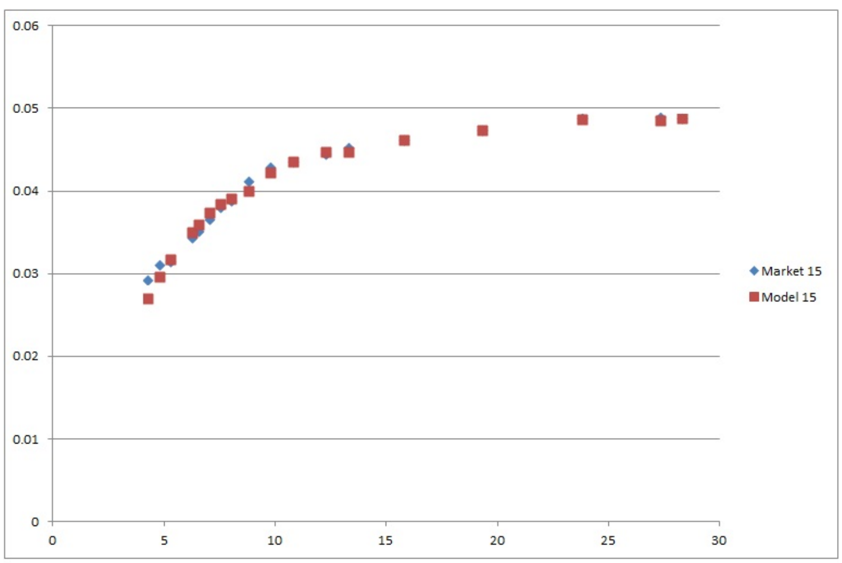

Figure 4.5: Yield To Maturity, Spain, May 2013

MSE= 0.327 (on 100 scale)

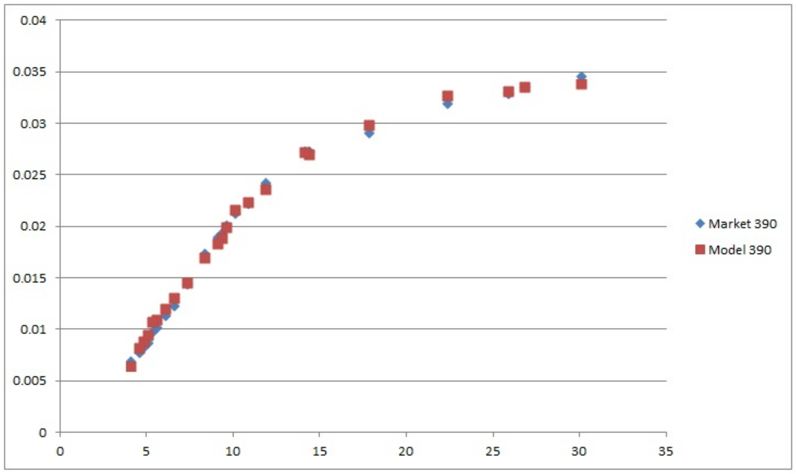

Figure 4.6: Yield To Maturity, Spain, October 2014

MSE= 0.348 (on 100 scale)

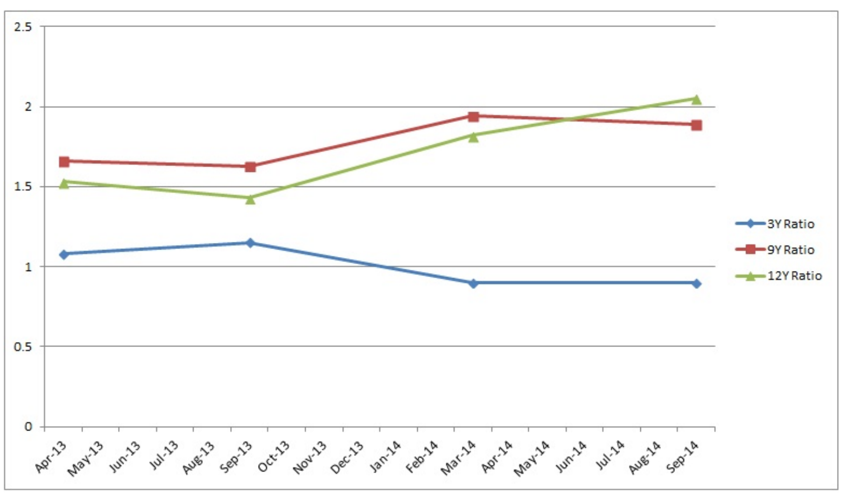

Figure 4.7: Italy, Market Perception of Debt-to-GDP Ratio, 2013-2014

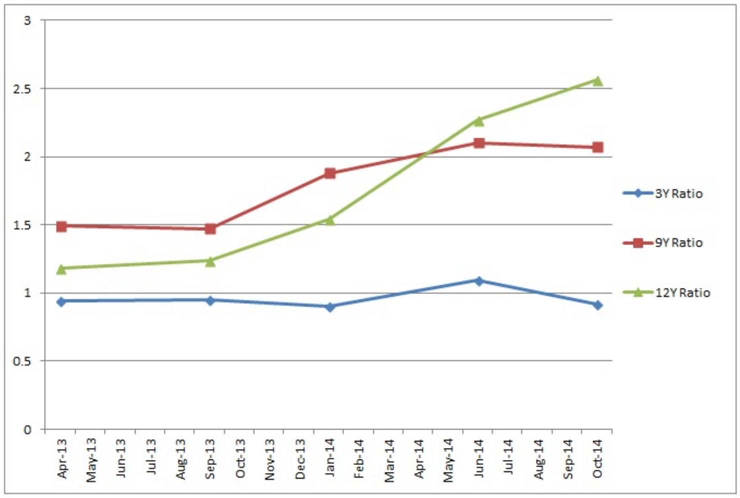

Figure 4.8: Spain, Market Perception of Debt-to-GDP Ratio, 2013-2014

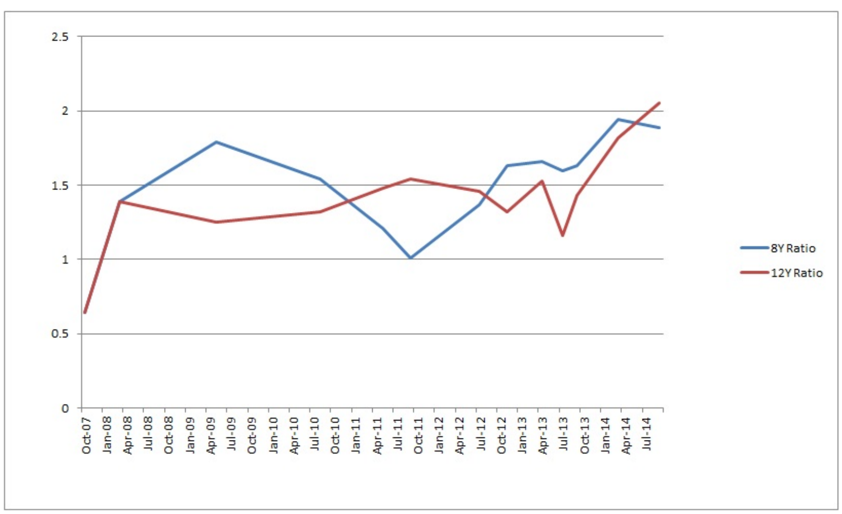

Figure 4.9: Italy, Market Perception of Debt-to-GDP Ratio, 2007-2014

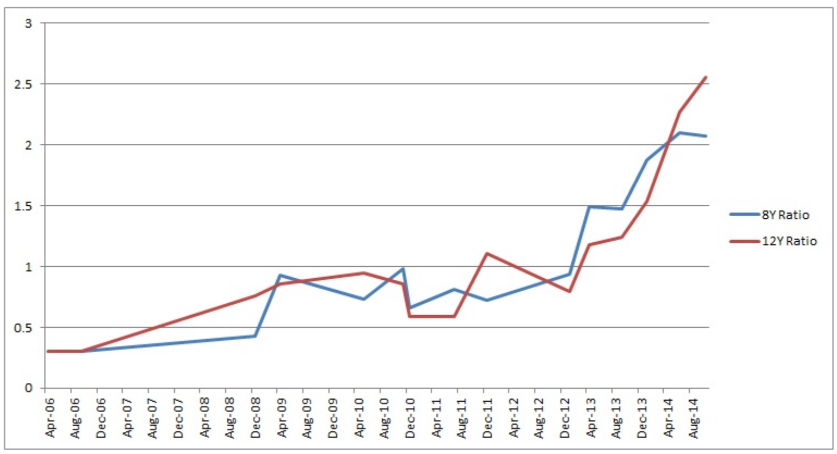

Figure 4.10: Spain, Market Perception of Debt-to-GDP Ratio, 2006-2014

APPENDIX A

Before we come to the proof of Theorem 1 we first provide the

Proof of Lemma 1. First note that the function $b_t(\cdot)$ is decreasing for each t since $\beta(\cdot)$ and hence $\beta^{-1}(\cdot)$ is decreasing. Hence $C(b_t(\cdot))$ is increasing, and hence $m_t(\cdot)$ is strictly increasing by (2.18) for each t.

By definition, $b_T(\cdot) = \infty$. Set $m_{T+1}^{-1}(0) := \infty$. We show by backward induction on $t \leq T$ that $b_{t-1}(x) \leq b_t(x)$ and $m_t^{-1}(0) \leq m_{t+1}^{-1}(0)$. The initial step $t = T$ is clear. Now let $t < T$ and assume $b_t(x) \leq b_{t+1}(x)$ and $m_{t+1}^{-1}(0) \leq m_{t+2}^{-1}(0)$. Then

$$m_t(x) = C\big(b_t(x)\big) + x \geq C\big(b_{t+1}(x)\big) + x = m_{t+1}(x)$$

for all x. So $0 = m_t(m_t^{-1}(0)) \geq m_{t+1}(m_t^{-1}(0))$ and thus $m_{t+1}^{-1}(0) \geq m_t^{-1}(0)$ since $m_{t+1}^{-1}(\cdot)$ is increasing. This implies $b_t(x) \geq b_{t-1}(x)$ since $\beta(\cdot)$ is decreasing, finishing the induction step. ∎

We next show that under an allocation strategy $\bar{\pi}$ of the structure as in Theorem 1, no creditor increases her investment into the short term debt at the time of default.

Lemma 3. *If $y_t \in \Delta_t(\bar{\pi})$, $t > 1$, then for any creditor a, we have $\bar{\pi}_{t-1}(a, y_{t-1}) \geq \bar{\pi}_t(a, y_t)$.*

Proof of Lemma 3. Since $m_t(\cdot)$ is strictly increasing, $y_t \in \Delta_t(\bar{\pi})$ and (2.18) imply

$$x_t^{\bar{\pi}}(y_t) < m_t^{-1}(0),$$

$$x_{t-1}^{\bar{\pi}}(y_{t-1}) \geq m_{t-1}^{-1}(0).$$

Combined with Lemma 1 these equations imply

$$x_t^{\bar{\pi}}(y_t) < m_t^{-1}(0) \leq m_{t-1}^{-1}(0) \leq x_{t-1}^{\bar{\pi}}(y_{t-1}).$$

Therefore again from Lemma 1, we obtain

$$b_{t-1}(x_{t-1}^{\bar{\pi}}(y_{t-1})) \leq b_t(x_{t-1}^{\bar{\pi}}(y_{t-1})) \leq b_t(x_t^{\bar{\pi}}(y_t)),$$

which proves the statement. ∎

Proof of Theorem 1. We prove the claim by backward induction. The induction base at T holds by convention. Assume now that π^* is an equilibrium at time $t+1$. Fix $a \in A$, and take any allocation strategy π which satisfies $\pi_s(a', \cdot) = \bar{\pi}_s(a', \cdot)$ for all creditors a' and all $s = 0, \ldots, t-1$, and $\pi_s(a', \cdot) = \bar{\pi}_s(a', \cdot)$ for all creditors $a' \neq a$ and all $s = t, t+1, \ldots, T$. The strategy π is obtained from $\bar{\pi}$ by modifying the strategy of player a alone, from time t on.

We have to show that

$$g_t^{\bar{\pi},a}(Y_t) \geq g_t^{\pi,a}(Y_t)$$

for each $Y_t \in \Gamma_{t-1}(\bar{\pi}) \times \mathbb{R}$. First note that since the set $\{a\}$ has measure zero, changing the strategy of a single player will have no impact on the default time, that is,

$$T^{\pi} = T^{\bar{\pi}}. \tag{A.1}$$

By (A.1), and since consumption depends only on allocations up to $t-1$ by (2.12), we obtain $C_t^{\bar{\pi},a} = C_t^{\pi,a}$, so it suffices to show

$$\mathbb{E}^a[V_{t+1}^{\bar{\pi},a} \,|\, Y_t] \geq \mathbb{E}^a[V_{t+1}^{\pi,a} \,|\, Y_t]. \tag{A.2}$$

We now distinguish two cases.

Case i) Suppose that $Y_t \in \Delta_t(\pi) = \Delta_t(\bar{\pi})$. By Lemma 3 we have $\bar{\pi}_{t-1}(a, Y_{t-1}) \geq \bar{\pi}_t(a, Y_t)$, and the allocations π and $\bar{\pi}$ are the same until time $t-1$. It follows that

$$\begin{aligned}\mathbb{E}^a[V_{t+1}^{\bar{\pi},a} \,|\, Y_t] &= 1 - (\bar{\pi}_{t-1}(a, Y_{t-1}) \vee \bar{\pi}_t(a, Y_t)) = 1 - \bar{\pi}_{t-1}(a, Y_{t-1}) \\ &\geq 1 - (\bar{\pi}_{t-1}(a, Y_{t-1}) \vee \pi_t(a, Y_t)) = 1 - (\pi_{t-1}(a, Y_{t-1}) \vee \pi_t(a, Y_t)) = \mathbb{E}^a[V_{t+1}^{\pi,a} \,|\, Y_t].\end{aligned}$$

So we have (A.2).

Case ii) Suppose that $Y_t \in \Gamma_t(\pi) = \Gamma_t(\bar{\pi})$. On $\{T^{\bar{\pi}} = t+1\}$, we have $0 > X^{\bar{\pi}}_{t+1} + C(b_{t+1}(X^{\bar{\pi}}_{t+1})) = X^{\bar{\pi}}_{t+1} + C(b_{t+1}(X^{\bar{\pi}}_{t+1}))$. Hence $b_{t+1}(X^{\bar{\pi}}_{t+1}) = \infty$ and thus $\bar{\pi}_{t+1}(a, Y_{t+1}) = 0$ on $\{T^{\bar{\pi}} = t+1\}$ by (2.21). So

$$\begin{aligned}
\mathbb{E}^a\big[V^{\bar{\pi},a}_{t+1} \,\big|\, Y_t\big] &= 1 + \mathbb{E}^a\big[\, -_{\{T^{\bar{\pi}}=t+1\}}\, (\bar{\pi}_t(a, Y_t) \vee \bar{\pi}_{t+1}(a, Y_{t+1})) +_{\{T^{\bar{\pi}}>t+1\}}\, \bar{\pi}_t(a, Y_t) r \,\big|\, Y_t\big] \\
&= 1 + \mathbb{E}^a\big[\, -_{\{T^{\bar{\pi}}=t+1\}}\, \bar{\pi}_t(a, Y_t) +_{\{T^{\bar{\pi}}>t+1\}}\, \bar{\pi}_t(a, Y_t) r \,\big|\, Y_t\big] \\
&= 1 + \bar{\pi}_t(a, Y_t)\Big(\mathbb{P}^a\big[T^{\bar{\pi}} > t+1 \,\big|\, Y_t\big](1+r) - 1\Big). \qquad \text{(A.3)}
\end{aligned}$$

Since $T^{\pi} = T^{\bar{\pi}}$, we have that $\mathbb{P}^a[T^{\pi} > t+1 \,|\, Y_t]$ and $\mathbb{P}^a[T^{\bar{\pi}} > t+1 \,|\, Y_t]$ coincide, so as in (A.3) we obtain

$$\begin{aligned}
\mathbb{E}^a\big[V^{\pi,a}_{t+1} \,\big|\, Y_t\big] &= 1 + \mathbb{E}^a\big[\, -_{\{T^{\pi}=t+1\}}\, (\pi_t(a, Y_t) \vee \pi_{t+1}(a, Y_{t+1})) +_{\{T^{\pi}>t+1\}}\, \pi_t(a, Y_t) r \,\big|\, Y_t\big] \\
&\le 1 + \mathbb{E}^a\big[\, -_{\{T^{\pi}=t+1\}}\, \pi_t(a, Y_t) +_{\{T^{\pi}>t+1\}}\, \pi_t(a, Y_t) r \,\big|\, Y_t\big] \\
&= 1 + \pi_t(a, Y_t)\Big(\mathbb{P}^a\big[T^{\pi} > t+1 \,\big|\, Y_t\big](1+r) - 1\Big) \\
&= 1 + \pi_t(a, Y_t)\Big(\mathbb{P}^a\big[T^{\bar{\pi}} > t+1 \,\big|\, Y_t\big](1+r) - 1\Big). \qquad \text{(A.4)}
\end{aligned}$$

We also have the equivalence

$$\begin{aligned}
T^{\bar{\pi}} > t+1 &\Leftrightarrow X^{\bar{\pi}}_{t+1} + D^{\bar{\pi}}_{t+1} \ge 0 \\
&\Leftrightarrow X^{\bar{\pi}}_{t+1} + C(b_{t+1}(X^{\bar{\pi}}_{t+1})) \ge 0 \\
&\Leftrightarrow X^{\bar{\pi}}_{t+1} \ge m^{-1}_{t+1}(0) \\
&\Leftrightarrow X^{\bar{\pi}}_t + \theta_{t+1} - rC(b_t(X^{\bar{\pi}}_t)) \ge m^{-1}_{t+1}(0) \\
&\Leftrightarrow -\theta_{t+1} + b(a) \le X^{\bar{\pi}}_t - rC(b_t(X^{\bar{\pi}}_t)) - m^{-1}_{t+1}(0) + b(a)
\end{aligned}$$

which yields

$$\mathbb{P}^a[T^{\bar{\pi}} > t+1 \,|\, Y_t] = \Phi\Big(X^{\bar{\pi}}_t - rC(b_t(X^{\bar{\pi}}_t)) - m^{-1}_{t+1}(0) + b(a)\Big). \qquad \text{(A.5)}$$

Since $T^{\bar{\pi}} > t$ we have $X^{\bar{\pi}}_t + C(b_t(X^{\bar{\pi}}_t)) = X^{\bar{\pi}}_t + C(b_t(X^{\bar{\pi}}_t)) \ge 0$ and thus $b_t(X^{\bar{\pi}}_t) = q(t, X^{\bar{\pi}}_t)$.

So it follows from (A.5) that we have the equivalence

$$
\begin{aligned}
& && \mathbb{P}^a[T^{\bar{\pi}} > t+1 | Y_t] \geq \tfrac{1}{1+r} \\
\Leftrightarrow & && \Phi^{-1}(\tfrac{r}{1+r}) \geq -X_t^{\bar{\pi}} + rC(q(t, X_t^{\bar{\pi}})) + m_{t+1}^{-1}(0) - b(a) \\
\Leftrightarrow & && b(a) \geq -X_t^{\bar{\pi}} + rC(q(t, X_t^{\bar{\pi}})) + m_{t+1}^{-1}(0) - \Phi^{-1}(\tfrac{r}{1+r}) = q(t, X_t^{\bar{\pi}}) = b_t(X_t^{\bar{\pi}}) \\
\Leftrightarrow & && \bar{\pi}_t(a, Y_t) = 1, \qquad\qquad (A.6)
\end{aligned}
$$

where we used (2.16) and (2.19) in the first of the last two equalities, and (2.21) in the last equivalence. Combining (A.3) and (A.4) with (A.6) and $\pi_t(a, Y_t) \in [0,1]$ yields (A.2). ∎

Proof of Lemma 2. We prove the statement by contradiction. Let $y_t \in \Delta_t(\pi)$. Assume that there exists an creditor $a \in A$ such that $\pi_t(a, y_t) > \pi_{t-1}(a, y_{-1})$ and consider the strategy

$$
\pi'_t(a, y_t) = \begin{cases} \pi_s(a, y_s) & \text{if } s \neq t, \\ \pi_{s-1}(a, y_{s-1}) & \text{if } s = t. \end{cases}
$$

Then we have

$$
g_t^{\pi,a}(y_t) = 1 - \pi_t(a, y_t) < 1 - \pi_{t-1}(a, y_{t-1}) = g_t^{\langle \pi | \pi'(a) \rangle, a}(y_t)
$$

in contradiction to the definition of a Nash equilibrium. ∎

Proof of Proposition 1. On the set $\{T^\pi > t\}$ we have

$$
\mathbb{E}^a\big[V_{t+1}^{\pi,a} \,\big|\, Y_t\big] = 1 + \mathbb{E}^a\big[-_{\{T^\pi = t+1\}}(\pi_t(a, Y_t) \vee \pi_{t+1}(a, Y_{t+1})) +_{\{T^\pi > t+1\}} \pi_t(a, Y_t) r \,\big|\, Y_t\big].
$$

Since π is an equilibrium at time t, Lemma 2 yields

$$
_{\{T^\pi = t+1\}}(\pi_t(a, Y_t) \vee \pi_{t+1}(a, Y_{t+1})) =_{\{T^\pi = t+1\}} \pi_t(a, Y_t)
$$

and therefore we obtain

$$\begin{aligned}
\mathbb{E}^a\big[V^{\pi,a}_{t+1}\,\big|\,Y_t\big] &= 1 + \pi_t(a, Y_t)\mathbb{E}^a\big[\,-_{\{T^\pi=t+1\}} +_{\{T^\pi>t+1\}} r\,\big|\,Y_t\big] \\
&= 1 + \pi_t(a, Y_t)\big(\mathbb{P}^a[T^\pi > t+1\,\big|\,Y_t](1+r) - 1\big). \qquad \text{(A.7)}
\end{aligned}$$

If $\mathbb{P}^a[T^\pi > t+1|Y_t] > \frac{1}{1+r}$, then $\pi_t(a, Y_t) = 1$ maximizes the right hand side of (A.7), hence we have $\pi_t(a, Y_t) = 1$.

If $\mathbb{P}^a[T^\pi > t+1|Y_t] < \frac{1}{1+r}$, then $\pi_t(a, Y_t) = 0$ maximizes the right hand side of (A.7), hence we have $\pi_t(a, Y_t) = 0$. ∎

For the remained of this appendix, let $b_t(\cdot)$, $m_t(\cdot)$ and $q(t,\cdot)$ denote the functions defined by the recursion (2.17) – (2.19) and (2.23).

Proof of Theorem 2. We prove the assertion by backward induction on $t = T, ..., 1$. For $t = T$ the assertion is trivial. So let $t < T$ and assume the assertion is true for $t+1$. On $\{Y_t \in \Gamma_t(\pi)\}$, we have the sequence of implications (using that $X^\pi_{t+1} = x^\pi_{t+1}(Y_{t+1})$)

$$\begin{aligned}
Y_{t+1} \in \Gamma_{t+1}(\pi) &\Leftrightarrow X^\pi_{t+1} + d^\pi_{t+1}(Y_{t+1}) \geq 0 \\
&\Rightarrow X^\pi_{t+1} + \hat{d}_{t+1}(X^\pi_{t+1}) \geq 0 \\
&\Leftrightarrow X^\pi_{t+1} \geq m^{-1}_{t+1}(0) \\
&\Leftrightarrow X^\pi_t + \theta_{t+1} - rd^\pi_t(Y_t) \geq m^{-1}_{t+1}(0) \\
&\Leftrightarrow -\theta_{t+1} + b(a) \leq X^\pi_t - rd^\pi_t(Y_t) - m^{-1}_{t+1}(0) + b(a), \qquad \text{(A.8)}
\end{aligned}$$

where the "⇒" in the second step follows from $Y_{t+1} \in \Gamma_{t+1}(\pi)$ and the induction hypothesis. Now (A.8) yields

$$\mathbb{P}^a[T^\pi > t+1|Y_t] \leq \Phi\Big(X^\pi_t - rd^\pi_t(Y_t) - m^{-1}_{t+1}(0) + b(a)\Big).$$

Combining this with the equivalence

$$\begin{aligned} &\Phi\Big(X_t^\pi - rd_t^\pi(Y_t) - m_{t+1}^{-1}(0) + b(a)\Big) < \frac{1}{1+r} \\ \Leftrightarrow\ \ &b(a) < rd_t^\pi(Y_t) + m_{t+1}^{-1}(0) - \Phi^{-1}(\tfrac{r}{1+r}) - X_t^\pi =: h_t(Y_t), \end{aligned}$$

we obtain the implications

$$b(a) < h_t(Y_t) \ \Rightarrow\ \mathbb{P}^a[T^\pi > t+1|Y_t] < \tfrac{1}{1+r} \ \Rightarrow\ \pi_t(a, Y_t) = 0,$$

of which the last follows from Proposition 1. Therefore

$$d_t^\pi(Y_t) \le C\big(h_t(Y_t)\big) = C\big(rd_t^\pi(Y_t) + m_{t+1}^{-1}(0) - \Phi^{-1}(\tfrac{r}{1+r}) - X_t^\pi\big).$$

Setting $z = X_t^\pi - m_{t+1}^{-1}(0) + \Phi^{-1}(\frac{r}{1+r})$ and $s = d_t^\pi(Y_t)$, Lemma 4 below then implies

$$d_t^\pi(Y_t) \le C(\beta^{-1}(X_t^\pi - m_{t+1}^{-1}(0) + \Phi^{-1}(\tfrac{r}{1+r}))). \tag{A.9}$$

Since $T^\pi > t$, we obtain that $0 \le X_t^\pi + d_t^\pi(Y_t) \le X_t^\pi + C(\beta^{-1}(X_t^\pi - m_{t+1}^{-1}(0) + \Phi^{-1}(\frac{r}{1+r})))$ $= X_t^\pi + C(q(t, X_t^\pi))$. This implies $\hat{b}_t(X_t^\pi) = \beta^{-1}(X_t^\pi - m_{t+1}^{-1}(0) + \Phi^{-1}(\frac{r}{1+r}))$ and hence (A.9) becomes $d_t^\pi(Y_t) \le \hat{d}_t(X_t^\pi)$, finishing the induction step. ∎

Lemma 4. *Fix $z \in \mathbb{R}$. The function*

$$f(s) = s - C(rs - z)$$

is strictly increasing in $s \in \mathbb{R}$, and its zero is given by $s = C(\beta^{-1}(z))$.

Proof. Clearly $f(\cdot)$ is strictly increasing, and we have

$$\begin{aligned} f(s) = 0 &\Leftrightarrow rC(rs - z) - rs = 0 \\ &\Leftrightarrow \beta(rs - z) = z \\ &\Leftrightarrow s = \tfrac{1}{r}(\beta^{-1}(z) + z) = C(\beta^{-1}(z)), \end{aligned}$$

where the last equation follows from (2.16). ∎

In the following, we shall use the notation

$$\hat{\pi}(a, y_t) := \hat{\pi}_t(a, x_t^{\hat{\pi}}(y_t))$$

with the function $\hat{\pi}_t(a, x)$ introduced in (2.24). For the proof of Theorem 3, we shall use the following two immediate propositions.

Proposition 2. *Suppose that $y_t \in \Gamma_t(\hat{\pi})$. Then for all $a \in A$, we have*

$$g_t^{\langle \hat{\pi} | \hat{\pi}(a) \rangle, a}(y_t) > g_t^{\langle \hat{\pi} | \tilde{\pi}(a) \rangle, a}(y_t)$$

for any $\tilde{\pi}_t(a, y_t) \neq \hat{\pi}_t(a, y_t)$.

The following Proposition states that given that from time $t + 1$ strategy $\hat{\pi}$ is played, and conditional on survival at time t, the strategy $\hat{\pi}$ is the best response at time t for any agent.

Proposition 3. *Let $\pi \in \mathcal{X}$ and $t \in \{1, ..., T\}$ such that for all $s \geq t+1$ and $y_s \in \Gamma_{s-1}(\pi) \times \mathbb{R}$ we have $\pi_s(\cdot, y_s) = \hat{\pi}_s(\cdot, x_s^{\pi}(y_s))$. Suppose that $y_t \in \Gamma_t(\pi)$. Then for all $a \in A$, we have*

$$g_t^{\langle \pi | \hat{\pi}(a) \rangle, a}(y_t) > g_t^{\langle \pi | \tilde{\pi}(a) \rangle, a}(y_t)$$

for any $\tilde{\pi}_t(a, y_t) \neq \hat{\pi}_t(a, x_t^{\pi}(y_s))$. Moreover, if π is an equilibrium at time t then $\pi_t(\cdot, y_t) = \hat{\pi}_t(\cdot, x_t^{\pi}(y_s))$.

The last statement of Proposition 3 follows by Theorem 2.

Proof of Theorem 3. For $t = 1, \ldots, T$, let Π_t be the set of strategies which are equilibria at time t and not weakly dominated for any a at times $s = t, \ldots, T$. We shall first prove the following statement (H_t) by backward induction on $t = T, \ldots, 1$.

(H_t) We have that $\hat{\pi} \in \Pi_t$. Conversely, any $\pi \in \Pi_t$ which is an equilibrium at time $t-1$ fulfills

$$\Gamma_t(\pi) = \{y_{t-1} \in \Gamma_{t-1}(\pi), x_t^{\pi}(y_t) \geq m_t^{-1}(0)\}, \tag{A.10}$$

$$\pi_t(a, y_t) : \begin{cases} = \hat{\pi}_t(a, x_t^{\pi}(y_t)) & \text{if } \; y_{t-1} \in \Gamma_{t-1}(\pi) \text{ and } \; x_t^{\pi}(y_t) \geq m_t^{-1}(0), \\ \leq \pi_{t-1}(a, y_{t-1}) & \text{if } \; y_{t-1} \in \Gamma_{t-1}(\pi) \text{ and } \; x_t^{\pi}(y_t) < m_t^{-1}(0) \end{cases} \tag{A.11}$$

if $t \geq 2$, and $\pi_1 \equiv \hat{\pi}_1$. Note that (A.11) immediately implies for any $y_{t-1} \in \Gamma_{t-1}(\pi)$ that

$$g_{t-1}^{\langle \pi | \hat{\pi}(a) \rangle, a}(y_{t-1}) = g_{t-1}^{\langle \hat{\pi} | \hat{\pi}(a) \rangle, a}(y_{t-1}).$$

Clearly (H_T) is true since any equilibrium π fulfills $\pi_T \equiv 0$ by definition. For the induction step, let $1 \leq t < T$ and assume that (H_{t+1}) is true. We will now prove (H_t) in the following steps.

Step 1) Let us show that $\hat{\pi} \in \Pi_T$. By the induction hypothesis, $\hat{\pi}$ is not weakly dominated at times $t+1, ..., T$. We shall now show that $\hat{\pi}$ is not weakly dominated at time t. Suppose, for a contradiction, that there exists a strategy $\tilde{\pi}$ with $\tilde{\pi}_s = \hat{\pi}_s$ for all $s \leq t-1$, such that $\hat{\pi}$ is weakly dominated by $\tilde{\pi}$ for some creditor a at time t. Then we have the following two statements.

a) For all trajectories $y_t \in \Gamma_{t-1}(\hat{\pi}) \times \mathbb{R}$ we have $\hat{\pi} \in O_t\Big(y_t, (d_1^{\hat{\pi}}(y_1), \ldots, d_{t-1}^{\hat{\pi}}(y_{t-1}))\Big)$ with $\hat{\pi} \in \Pi_{t+1}^*$ and we have

$$g_t^{\langle \hat{\pi} | \hat{\pi}(a) \rangle, a}(y_t) \leq g_t^{\langle \hat{\pi} | \tilde{\pi}(a) \rangle, a}(y_t). \tag{A.12}$$

b) There exists a trajectory $y_t \in \Gamma_{t-1}(\hat{\pi}) \times \mathbb{R}$ and a strategy $\pi^* \in O_t\Big(y_t, (d_1^{\hat{\pi}}(y_1), \ldots, d_{t-1}^{\hat{\pi}}(y_{t-1}))\Big)$ with $\pi^* \in \Pi_{t+1}^*$ such that

$$g_t^{\langle \pi^* | \hat{\pi}(a) \rangle, a}(y_t) < g_t^{\langle \pi^* | \tilde{\pi}(a) \rangle, a}(y_t). \tag{A.13}$$

For y_t and π^* as in b), if $y_t \in \Gamma_t(\pi^*)$, we have by Theorem 2 that $y_t \in \Gamma_t(\hat{\pi})$. Now the induction hypothesis implies that $g_t^{\langle \pi^* | \hat{\pi}(a) \rangle, a}(y_t) = g_t^{\langle \hat{\pi} | \hat{\pi}(a) \rangle, a}(y_t)$ and $g_t^{\langle \pi^* | \tilde{\pi}(a) \rangle, a}(y_t) = g_t^{\langle \hat{\pi} | \tilde{\pi}(a) \rangle, a}(y_t)$, and so (A.13) yields the inequality $g_t^{\langle \hat{\pi} | \hat{\pi}(a) \rangle, a}(y_t) < g_t^{\langle \hat{\pi} | \tilde{\pi}(a) \rangle, a}(y_t)$, in contradiction to the statement of Proposition 2. Therefore we must have $y_t \in \Delta_t(\pi^*)$ and (A.13) becomes

$$1 - (\hat{\pi}_{t-1}(a, y_{t-1}) \vee \hat{\pi}_t(a, y_t)) < 1 - (\tilde{\pi}_{t-1}(a, y_{t-1}) \vee \tilde{\pi}_t(a, y_t)).$$

Since $\tilde{\pi}_{t-1} = \hat{\pi}_{t-1}$, it follows that $\tilde{\pi}_t(a, y_t) < \hat{\pi}_t(a, y_t)$, so in particular $\hat{\pi}_t(a, y_t) > 0$ and hence $y_t \in \Gamma_t(\hat{\pi})$ by definition of $\hat{\pi}$. Since $\hat{\pi}$ is an equilibrium, using Proposition 2, this is a contradiction to (A.12).

Step 2) Now suppose $t > 1$, and fix any $\pi \in \Pi_t$. Since $\Gamma_t(\pi) \subseteq \{y_{t-1} \in \Gamma_{t-1}(\pi), x_t^\pi(y_t) \geq m_t^{-1}(0)\}$ by Theorem 2, to show (A.10) it is sufficient to show that

$$\{y_{t-1} \in \Gamma_{t-1}(\pi), x_t^\pi(y_t) \geq m_t^{-1}(0)\} \subseteq \Gamma_t(\pi). \tag{A.14}$$

Step 2a) We first prove that for all $y_t \in \Gamma_{t-1}(\pi) \times \mathbb{R}$, we have

$$\pi_t(a, y_t) \geq \min\left(\pi_{t-1}(a, y_{t-1}), \hat{\pi}_t(a, x_t^\pi(y_t))\right). \tag{A.15}$$

To this end, take any $a \in A$ and assume, for a contradiction, that there exists a trajectory $\tilde{y}_t \in \Gamma_{t-1}(\pi) \times \mathbb{R}$ such that

$$\pi_t(a, \tilde{y}_t) < \min\left(\pi_{t-1}(a, \tilde{y}_{t-1}), \hat{\pi}_t(a, x_t^\pi(\tilde{y}_t))\right). \tag{A.16}$$

Since $\hat{\pi}_t$ is $\{0, 1\}$-valued, and π_{t-1} is $\{0, 1\}$-valued for almost all creditors by Proposition 1, for almost any creditor a which satisfies (A.16) we have that $\min(\pi_{t-1}(a, \tilde{y}_{t-1}), \hat{\pi}_t(a, x_t^\pi(\tilde{y}_t)) = 1$, which yields

$$\pi_{t-1}(a, \tilde{y}_{t-1}) = \hat{\pi}_t(a, x_t^\pi(\tilde{y}_t)) = 1, \tag{A.17}$$

$$\pi_t(a, \tilde{y}_t) < 1. \tag{A.18}$$

Define the allocation strategy $\tilde{\pi}$ by

$$\tilde{\pi}_s(a, y_s) = \begin{cases} 1 & \text{if } \ y = \tilde{y} \text{ and } \ s = t, \\ \pi_s(a, y_s) & \text{otherwise.} \end{cases} \tag{A.19}$$

Then π is weakly dominated by $\tilde{\pi}$ for creditor a at time t. To prove this, let $y_t \in \Gamma_{t-1}(\pi) \times \mathbb{R}$ and consider a strategy

$$\pi^* \in \mathcal{O}_t\Big(y_t, (d_1^{\pi}(y_1), \ldots, d_{t-1}^{\pi}(y_{t-1}))\Big) \tag{A.20}$$

which is an equilibrium on $A \backslash \{a\}$ and is not weakly dominated for any creditor in $A \backslash \{a\}$ at times $t = t + 1, ..., T$. Since $\{a\}$ has measure zero, we can extend π^* to an equilibrium on A, again denoted by π^*, which is not weakly dominated for any creditor in A at times $t = t + 1, ..., T$. We first have to show that

$$g_t^{\langle \pi^* | \pi(a) \rangle, a}(y_t) \leq g_t^{\langle \pi^* | \tilde{\pi}(a) \rangle, a}(y_t). \tag{A.21}$$

If $y \neq \tilde{y}$, then $\tilde{\pi} = \pi$ and we have equality in (A.21). So we now assume $y = \tilde{y}$. First, observe that $T^{\langle \pi^* | \pi(a) \rangle} = T^{\langle \pi^* | \tilde{\pi}(a) \rangle} = T^{\pi^*}$. Since $\tilde{\pi}_{t-1}(a, \tilde{y}_{t-1}) = \pi_{t-1}(a, \tilde{y}_{t-1})$, we have $C_t^{\langle \pi^* | \pi(a) \rangle, a} = C_t^{\langle \pi^* | \tilde{\pi}(a) \rangle, a}$. We now distinguish two cases.

i) Suppose that $\tilde{y}_t \in \Delta_t(\pi^*)$.

   Since $\tilde{\pi}_{t-1}(a, \tilde{y}_{t-1}) = \pi_{t-1}(a, \tilde{y}_{t-1}) = 1$ by (A.17), (A.19), it follows from (2.12) – (2.14) that $g_t^{\langle \pi^* | \tilde{\pi}(a) \rangle, a}(\tilde{y}_t) = g_t^{\langle \pi^* | \pi(a) \rangle, a}(\tilde{y}_t) = 0$.

ii) Suppose that $\tilde{y}_t \in \Gamma_t(\pi^*)$.

   By the induction hypothesis, we have $\pi^*_{t+1}(a, y_{t+1}) = \hat{\pi}_{t+1}(a, x^{\pi^*}_{t+1}(y_{t+1}))$ whenever $y_t \in \Gamma_t(\pi^*)$ and $x^{\pi^*}_{t+1}(y_{t+1}) \geq m^{-1}_{t+1}(0)$, and $\pi^*_{t+1}(a, y_{t+1}) \leq \pi^*_t(a, y_t)$ whenever $y_t \in \Gamma_t(\pi^*)$ and $x^{\pi^*}_{t+1}(y_{t+1}) < m^{-1}_{t+1}(0)$.

By Proposition 3, we have that $\hat{\pi}_t(a, x_t^{\pi^*}(\tilde{y}_t))$ is a strict maximizer over $\pi(a)$ of $g_t^{\langle \pi^* | \pi(a) \rangle, a}(\tilde{y}_t)$. By (A.17), (A.18), (A.19), and the consistency condition (A.20) (see Remark 1), we have

$$\tilde{\pi}_t(a, \tilde{y}_t) = 1 = \hat{\pi}_t(a, x_t^{\pi}(\tilde{y}_t)) = \hat{\pi}_t(a, x_t^{\pi^*}(\tilde{y}_t))$$
$$\pi_t(a, \tilde{y}_t) < 1 = \hat{\pi}_t(a, x_t^{\pi}(\tilde{y}_t)) = \hat{\pi}_t(a, x_t^{\pi^*}(\tilde{y}_t)).$$

Therefore

$$g_t^{\langle \pi^* | \pi(a) \rangle, a}(\tilde{y}_t) < g_t^{\langle \pi^* | \tilde{\pi}(a) \rangle, a}(\tilde{y}_t).$$

So we have shown (A.21).

Consider now a strategy ρ defined recursively by $\rho_s(a, \cdot) = \pi_s(a, \cdot)$ for $s < t$ and $\rho_s(a, y_s) = \hat{\pi}_s(a, x_s^{\rho}(y_s))$ for $s \geq t$ and $y_s \in \Gamma_s(\rho)$. We have by Step 1 that $\rho \in \Pi_t$. We have $\hat{\pi}_t(a, x_t^{\rho}(\tilde{y}_t)) = \hat{\pi}_t(a, x_t^{\pi}(\tilde{y}_t)) = 1$. This implies that $\tilde{y}_t \in \Gamma(\rho)$. So for $\pi^* = \rho$ we are in case ii) above, and the inequality in (A.21) is strict for $y_t = \tilde{y}_t$. This shows that $\tilde{\pi}$ weakly dominates π for creditor a at time t, in contradiction to our assumption. Hence we have (A.15).

Step 2b) Let $Y_{t-1} \in \Gamma_{t-1}(\pi)$. It follows immediately from step 2a) that $D_t^{\pi} \geq \min\left(\hat{d}_t(X_t^{\pi}), D_{t-1}^{\pi}\right)$. Set

$$\mu_{t-1}(y_{t-1}) := \max\left(m_t^{-1}(0), -d_{t-1}^{\pi}(y_{t-1})\right).$$

We then have

$$
\begin{aligned}
T^\pi > t \Leftrightarrow\ & X_t^\pi + D_t^\pi \geq 0 \\
\Leftrightarrow\ & X_t + d_t^\pi(Y_{t-1}, X_t^\pi) \geq 0 \\
\Leftarrow\ & X_t^\pi + C(b_t(X_t^\pi)) \geq 0 \quad \text{and} \quad X_t^\pi + d_{t-1}^\pi(Y_{t-1}) \geq 0 \\
\Leftrightarrow\ & X_t^\pi \geq m_t^{-1}(0) \quad \text{and} \quad X_t^\pi \geq -d_{t-1}^\pi(Y_{t-1}) \\
\Leftrightarrow\ & X_t^\pi \geq \mu_{t-1}(Y_{t-1}) && \text{(A.22)} \\
\Leftrightarrow\ & X_{t-1}^\pi + \theta_t - r d_{t-1}^\pi(Y_{t-1}) \geq \mu_{t-1}(Y_{t-1}) \\
\Leftrightarrow\ & -\theta_t + b(a) \leq X_{t-1}^\pi - r d_{t-1}^\pi(Y_{t-1}) - \mu_{t-1}(Y_{t-1}) + b(a), && \text{(A.23)}
\end{aligned}
$$

where the "$\Leftarrow$" in the third line follows from $D_t^\pi \geq \min\left(\hat{d}_t(X_t^\pi), D_{t-1}^\pi\right)$. Now (A.23), using that $-\theta_t + b(a) \sim \mathcal{N}(0,1)$ under $\mathbb{P}^a$, yields

$$\mathbb{P}^a[T^\pi > t | Y_{t-1}] \geq \Phi\Big(X_{t-1}^\pi - r d_{t-1}^\pi(Y_{t-1}) - \mu_{t-1}(Y_{t-1}) + b(a)\Big).$$

Combining this with the equivalence

$$
\begin{aligned}
& \Phi\Big(X_{t-1}^\pi - r d_{t-1}^\pi(Y_{t-1}) - \mu_{t-1}(Y_{t-1}) + b(a)\Big) > \frac{1}{1+r} \\
\Leftrightarrow \quad & b(a) > r d_{t-1}^\pi(Y_{t-1}) + \mu_{t-1}(Y_{t-1}) - \Phi^{-1}\left(\tfrac{r}{1+r}\right) - X_{t-1}^\pi =: h_{t-1}(Y_{t-1}), && \text{(A.24)}
\end{aligned}
$$

we obtain the implications

$$b(a) > h_{t-1}(Y_{t-1}) \;\Rightarrow\; \mathbb{P}^a[T^\pi > t | Y_{t-1}] > \tfrac{1}{1+r} \;\Rightarrow\; \pi_{t-1}(a, Y_{t-1}) = 1, \qquad \text{(A.25)}$$

of which the last follows from Proposition 1. Therefore

$$d_{t-1}^\pi(Y_{t-1}) \geq C\big(h_{t-1}(Y_{t-1})\big) = C\big(r d_{t-1}^\pi(Y_{t-1}) + \mu_{t-1}(Y_{t-1}) - \Phi^{-1}\left(\tfrac{r}{1+r}\right) - X_{t-1}^\pi\big).$$

Setting $z = X_{t-1}^\pi - \mu_{t-1}(Y_{t-1}) + \Phi^{-1}\left(\frac{r}{1+r}\right)$ and $s = d_{t-1}^\pi(Y_{t-1})$, Lemma 4 then implies

$$d_{t-1}^\pi(Y_{t-1}) \geq C\big(\beta^{-1}\big(X_{t-1}^\pi - \mu_{t-1}(Y_{t-1}) + \Phi^{-1}\left(\tfrac{r}{1+r}\right)\big)\big). \qquad \text{(A.26)}$$

Since $T^\pi > t-1$, we have $X^\pi_{t-1} + d^\pi_{t-1}(Y_{t-1}) \geq 0$. Suppose, for a contraction, that $-d^\pi_{t-1}(Y_{t-1}) > m_t^{-1}(0)$. It would follow that $X^\pi_{t-1} \geq -d^\pi_{t-1}(Y_{t-1}) = \max(m_t^{-1}(0), -d^\pi_{t-1}(Y_{t-1})) = \mu_{t-1}(Y_{t-1})$, and so we would obtain $\beta^{-1}(X^\pi_{t-1} - \mu_{t-1}(Y_{t-1}) + \Phi^{-1}(\frac{r}{1+r})) \leq \beta^{-1}(\Phi^{-1}(\frac{r}{1+r}))$ since β^{-1} is decreasing. With (A.26) we get

$$d^\pi_{t-1}(Y_{t-1}) \geq C(\beta^{-1}(\Phi^{-1}(\tfrac{r}{1+r}))) \geq C(\hat{b}_t(m_t^{-1}(0))) = -m_t^{-1}(0),$$

where the last inequality follows from $\beta^{-1}(\Phi^{-1}(\frac{r}{1+r})) \leq \beta^{-1}(m_t^{-1}(0) - m_{t+1}^{-1}(0) + \Phi^{-1}(\frac{r}{1+r})) = q(t, m_t^{-1}(0)) \leq \hat{b}_t(m_t^{-1}(0))$. This contradicts our assumption. Hence we have

$$-d^\pi_{t-1}(Y_{t-1}) \leq m_t^{-1}(0)$$

and thus $\mu_{t-1}(Y_{t-1}) = m_t^{-1}(0)$. Now (A.22) reads as (A.14), which implies (A.10).

Step 3) Let us now complete the induction step by showing (A.11).

First suppose that $t > 1$. If $Y_t \in \Gamma_{t-1}(\pi) \times \mathbb{R}$ and $X^\pi_t < m_t^{-1}(0)$, the inequality in (A.11) follows from Lemma 2 and Theorem 2. If $Y_t \in \Gamma_{t-1}(\pi) \times \mathbb{R}$ and $X^\pi_t \geq m_t^{-1}(0)$, then $Y_t \in \Gamma_t(\pi)$ by (A.10), so as in (A.7) we have

$$\begin{aligned}\mathbb{E}^a[V^{\pi,a}_{t+1} \,|\, Y_t] &= 1 + \pi_t(a, Y_t)\Big(\mathbb{P}^a[T^\pi > t+1 \,|\, Y_t](1+r) - 1\Big) \\ &= 1 + \pi_t(a, Y_t)\Big(\mathbb{P}^a[X^\pi_{t+1} < m_{t+1}^{-1}(0) \,|\, Y_t](1+r) - 1\Big), \end{aligned} \tag{A.27}$$

where the second equation follows from the induction hypothesis. Since π is an equilibrium at time t, maximizing the above expectation with respect to $\pi_t(a, Y_t)$ yields $\pi_t(a, Y_t) = \hat{\pi}_t(a, X^\pi_t)$ for almost all a, and we obtain (A.11).

Next suppose that $t = 1$. Since $Y_1 = 0 \in \Gamma_1(\pi)$, we have (A.27) as in the case $t > 1$, and again we obtain $\pi_1(a, y_1) = \hat{\pi}_1(a, y_1)$ for almost all a. This concludes the induction step, and so we have proved (H_t) for all $t = 1, \ldots, T$.

To complete the proof of the theorem, first note that $\hat{\pi}$ is a reasonable Nash equilibrium by (H_1). Conversely, let π be any reasonable Nash equilibrium. Forward induction and (A.10) immediately yield $\Gamma_t(\pi) = \Gamma_t(\hat{\pi})$. So we have condition 1 of Definition 2. Equation (A.11) yields condition 2 of Definition 2, and condition 3 of Definition 2 follows from (A.11) and Lemma 2. Hence π and $\hat{\pi}$ are identical. ∎

APPENDIX B

Proof of Proposition 3.3.1: Let's first consider the allocation strategy $(\pi_{-a'}, \pi')$, where $\pi'(a', y) = 0$ (creditor a' does not invest). Since (π, y) is a Nash Equilibrium, we have that

$$G^{b(a'),\pi}(y) \geq G^{b(a'),(\pi_{-a'},\pi')}(y) = 1,$$

where the equality $G^{b(a'),(\pi_{-a'},\pi')}(y) = 1$ holds since the payoff of an creditor who does not invest is equal to 1.

Therefore,

$$\pi(a', y)g\left(b(a'), y, s(\pi, y)\right) + (1 - \pi(a', y)) \geq 1.$$

This yields, since $\pi(a') > 0$

$$g\left(b(a'), y, s(\pi, y)\right) \geq 1. \tag{B.1}$$

Let's assume, by way of contradiction, that there exists a s.t. $b(a) > b(a')$ and $\pi(a, y) < 1$. Consider then the strategy (π_{-a}, π''), where $\pi''(a, y) = 1$. Under this strategy, the payoff of creditor a is given by

$$\begin{aligned} G^{a,(\pi_{-a},\pi'')}(y) &= g\left(b(a), y, s(\pi, y)\right) \\ &> g\left(b(a), y, s(\pi, y)\right) \\ &\geq 1, \end{aligned}$$

where the strict inequality in the second line stems from the monotonicity of the function g in its first argument and the last inequality holds from Equation (B.1).

Note now that

$$\begin{aligned} G^{a,\pi}(y) = \pi(a, y)G^{a,(\pi_{-a},1)}(y) + \left(1 - \pi(a, y)\right) \\ < G^{a,(\pi_{-a},\pi'')}(y), \end{aligned}$$

since $G^{a,(\pi_{-a},\pi'')}(y) > 1$ and $\pi(a, y) \in [0, 1)$.

This contradicts the definition of a Creditors' Nash Equilibrium. ∎

Proof of Proposition 3.3.4: Define the function $h : Supp(S) \to \mathbb{R}$

$$h(x) := g\left(x, y, S(x)\right). \tag{B.2}$$

The function h has the following properties.

1. It is a continuous function, as g is continuous in the first and the third argument and $S(x)$ is continuous in x (under the non-atomic capital distribution) .

2. $h(S^{-1}(\epsilon)) > 1$. This follows directly from Assumption (3.15) and using (3.5), since

$$h(x) = 1 + r - \frac{Put(yx, y + rS(x) - c)}{S(x)}. \tag{B.3}$$

3. $h(x) < 1$ for sufficiently small x. From Equation (3.5)

$$h(x) = 1 + r - r\Phi(-d_2) + \frac{1}{S(x)}\left(yx\Phi(-d_1) - (y - c)\Phi(-d_2)\right)$$

On the other hand

$$\lim_{x\to 0} \Phi(-d_1) = \lim_{x\to 0} \Phi(-d_2) = 1, \tag{B.4}$$

therefore,

$$\lim_{x\to 0} \frac{1}{S(x)}\left(yx\Phi(-d_1) - (y - c)\Phi(-d_2)\right) = -\frac{1}{C}(y - c) < 0,$$

which implies

$$\lim_{x\to 0} h(x) = 1 - \frac{1}{C}(y - c) < 1.$$

The statement now follows. ∎

Proof of Proposition 3.3.6:

For the uniform distribution with support $[0, M]$, we have $S^{-1}(\epsilon) = M(1 - \frac{\epsilon}{C})$. We have that

$$\lim_{M\to\infty} \frac{\sup_{y\in[c,c+C]} Put(yM(1-\frac{\epsilon}{C}), y + r\epsilon - c)}{\epsilon} \leq \lim_{M\to\infty} \frac{Put(cM(1-\frac{\epsilon}{C})C + r\epsilon)}{\epsilon}$$
$$= 0$$

There exists therefore $M(\epsilon)$ such that the hypothesis of Proposition is satisfied 3.3.4 for the uniform distribution on $[0, M(\epsilon)]$. By Proposition 3.3.4 we have that for any $y \in [c, c + C]$ there exists a critical belief $\beta < M(1 - \frac{\epsilon}{C})$ such that $g(\beta, y, S(\beta)) = 1$. We now show uniqueness of such a critical belief.

We let

$$h_1(x) := (1 - h(x))S(x) = Put(yx, K(x)) - rS(x) : [0, M] \to \mathbb{R}, \tag{B.5}$$

with $K(x) = y + rS(x) - c$. The function h_1 represents the expected aggregate loss of the creditors.

We immediately have

$$\frac{\partial}{\partial k} Put(yx, K(x)) = \Phi(-d_2)$$

and

$$\frac{\partial}{\partial x} Put(yx, K(x)) = -y\Phi(-d_1).$$

We obtain using $K'(x) = -\frac{rC}{M}$

$$\begin{aligned} \frac{d}{dx} h_1 &= \left[\frac{rC}{M} - \frac{rC}{M}\Phi(-d_2) - y\Phi(-d_1)\right] \\ &= \left[\frac{rC}{M}\Phi(d_2) - y\Phi(-d_1)\right]. \end{aligned} \tag{B.6}$$

and thus

$$\frac{d^2}{dx^2}h_1 = \left[\phi(d_2(x,z))\frac{rC}{M} + y\phi(d_1(x,z))\right]\left[\frac{1}{x} + \frac{1}{K(x)}\frac{rC}{M}\right] > 0.$$

Since the function h_1 is strictly convex and we have that $h_1(0) > 0$ and $h_1\left(M(1-\frac{\epsilon}{C})\right) < 0$, it follows that the root β is unique on $\left[0, M(1-\frac{\epsilon}{C})\right]$.

Note, however, that since $\lim_{x\to M, x<M} \frac{Put(yx, y+rS(x)-c)}{S(x)} = \infty$ there is a second root on the interval $\left(M(1-\frac{\epsilon}{C}), M\right)$ with the property that the debt provided is strictly smaller than ϵ.

∎

Proof of Proposition 3.3.7: 1. (Necessity). If π_b is a Nash Equilibrium, then Equation (3.18) holds. The statement will be proven by contradiction.

It is sufficient to consider two cases.

(a)

$$g\left(b, y, S\left(b\right)\right) > 1$$

Since g is a continuous function in its first argument, there exists $b' < b$, such that $g\left(b', y, S\left(b\right)\right) > 1$. Since the belief function b is a surjection, there exists an creditor a' such that $b(a') = b'$. Consider the allocation strategy $(\pi_{b_{-a'}}, \pi')$, where $\pi'(a', y) = 1$ in which a' invests fully. Then

$$G^{a',(\pi_{b_{-a'}},\pi')}(y) = g\left(b', y, S\left(b\right)\right)) > 1 = G^{a',\pi_b}(y),$$

where the last inequality holds since, by definition, in the equilibrium π_b we have that $\pi(a') = 0$. We thus obtain a contradiction to the definition of a CNE.

(b)

$$g(b, y, S(b)) < 1$$

Similarly as above we obtain that there exists $b' > b$, such that $g(b', y, S(b)) < 1$ and there exists creditor a' such that $b(a') = b'$. Consider the allocation strategy $(\pi_{b_{-a'}}, \pi'')$, where $\pi''(a', y) = 0$ in which a' does not invest. Then

$$G^{a',(\pi_{b_{-a'}},\pi'')}(y) = 1 > g(b', y, S(b)) = G^{a',\pi_b}(y),$$

which again contradicts to the definition of CNE.

2. (Sufficiency). If Equation (3.18) holds, then (π_b, y) is a CNE. In order to verify the statement we will consider two cases.

   (a) If $b(a) \geq b$, by the monotonicity of g in its first argument we have $g(b(a), y, s(\pi_b, y)) \geq 1$. Then for any $\pi' < 1$, using that $s(\pi_b, y) = s((\pi_{b_{-a}}, \pi'), y) = S(b)$, we obtain

$$\begin{aligned} G^{a,(\pi_b,y)} &= g(b(a), y, S(b)) = g(b(a), y, S(b)) \\ &\geq \pi' g(b(a), y, S(b)) + (1 - \pi') \\ &= G^{a,((\pi_{b_{-a}},\pi'),y)} \end{aligned} \tag{B.7}$$

   (b) If $b(a) < b$, then similarly as above we obtain for any $\pi' > 0$

$$G^{a,(\pi_b,y)} = 1 < \pi'(a) g(b(a), y, S(b)) + (1 - \pi'(a)) = G^{a,((\pi_{b_{-a}},\pi'),y)}, \tag{B.8}$$

   since $g(b(a), y, S(b)) < 1$.

The statement immediately follows from Equations (B.7) and (B.8).

■

To prove Proposition 3.3.10 we will need the following Lemma.

Lemma B.0.1. *The function h_1, defined in (B.5), has the following property.*

$$\frac{dh_1(x, y, S(x))}{dy} > 0.$$

Proof.

$$Put(yx, K(y)) = (y + rS(x) - c)\Phi(-d_2) - yx\Phi(-d_1) > 0.$$

$$-y\Big(x\Phi(-d_1) - \Phi(-d_2)\Big) > (c - rS(x))\Phi(-d_2) > 0,$$

where the second inequality follows from (3.4), which implies

$$\frac{dh_1(x, y, S(x))}{dy} = -\frac{1}{S(x)}(x\Phi(-d_1) - \Phi(-d_2)) > 0.$$

∎

Proof of Proposition 3.3.10: From Lemma B.0.1, if $y_1 < y_2$ and

$$g(x_1, y_1, S(x_1)) = g(x_2, y_2, S(x_2)) = 1,$$

then

$$h_1(x_1, y_1, S(x_1)) = h_1(x_2, y_2, S(x_2)) = 0.$$

On the other hand:

$$h_1(x_2, y_2, S(x_2)) = h_1(x_1, y_1, S(x_1)) < h_1(x_1, y_2, S(x_1)),$$

which yields

$$h_1(x_1, y_2, S(x_1)) > 0.$$

From Proposition 3.3.4 the equation $h_1(x, y, S(x)) = 0$ has a unique root for any fixed y and $h_1(x, y) < 0$ for sufficiently large x, therefore $x_2 > x_1$.

∎

Proof of Proposition 3.3.11: Let $y_1 < y_2$ with $c \leq y_{1,2} \leq s(y_{1,2})$. By Proposition 3.3.10 we have that $s(y_1) > s(y_2)$, since both strategies are feasible. Then

$$Call(y_1\tilde{b}, y_1 + rs(y_1) - c) < Call(y_1\tilde{b}, y_1 + rs(y_2) - c) \tag{B.9}$$

For a fixed $z \geq 0$ consider the function

$$y \to Call(y\tilde{b}, y + rz - c).$$

We obtain, since $\tilde{b} > 1$ (recall that the company is optimistic about the asset)

$$\frac{d}{dy} Call(y\tilde{b}, y + rz - c) = \tilde{b}\Phi(d_1) - \Phi(d_2) > 0. \tag{B.10}$$

Since $y_2 > y_1$, it follows from the above (with $z = s(y_2)$) that

$Call(y_1\tilde{b}, y_1 + rs(y_2) - c) < Call(y_2\tilde{b}, y_2 + rs(y_2) - c)$

Combined with (B.9), this yields

$$Call(y_1\tilde{b}, y_1 + rs(y_1) - c) < Call(y_2\tilde{b}, y_2 + rs(y_2) - c).$$

■

Proof of Theorem 1: We first consider the problem where the set of feasible strategies for the company is $\{c \leq y \leq c + s(y)\}$.

$$\underset{c \leq y \leq c+s(y)}{Maximize}\, Call(y\tilde{b}, y + rs(y) - c) \tag{B.11}$$

We will show that this problem has a unique solution that does not saturate the first constraint. Therefore its solution will be the unique solution to Problem 3.20 and is given by

1. Existence is guaranteed since the feasible set is compact (the function s is continuous and bounded) and the objective function is continuous.

2. Proposition 3.3.11 identifies the unique maximizer (B.11) as $\sup\{c \leq y \leq c + s(y)\}$.

3. It remains to be shown that this maximizer in is not $y = c$.

   Consider the function

$$y \to \inf_{b \geq 0} g(b, y, s(y)) : [c, \infty) \to \mathbb{R}^+. \tag{B.12}$$

   The value of this function is given by $\inf_{b \geq 0} g(b, y = c, s(c)) > 1$. Indeed since the company invests its own capital in the risky asset and keeps the rest of the funds in cash, any creditor can expect a strict gain on investment (since the price of the asset is strictly positive).

   Due to the continuity, we obtain that there exists $y_2 > c$ such that for any $c \leq y < y_2$ we have

$$\inf_{b \geq 0} g(b, y, s(c + \epsilon)) > 1,$$

   and therefore $s(y) = C$ for $c \leq y < y_2$.

   On the set $c \leq y < y_2$ we have, from (1) (with $z = C$) that the company's payoff is strictly increasing. Therefore the maximizer in (B.11) cannot be not be $y = c$.

It now remains to prove that the funding constraint is saturated.

We first show that y^* is a solution to (3.22). Suppose by way of contradiction that

$$y^* < c + s(y^*).$$

We consider the function $h : [c, \infty) \to \mathbb{R}^+$

$$h(y) := c + s(y) - y. \tag{B.13}$$

From Proposition 3.3.9 we have $h(y)$ is a continuous function and $h(y^*) > 0$, therefore there exists $\epsilon > 0$, such that $h(y^* + \epsilon) > 0$, then $y^* + \epsilon < c + s(y^* + \epsilon)$, in contradiction with (3.21).

Since any solution to equation (3.22) is feasible, uniqueness follows directly from Proposition 3.3.10.

∎

Proof of Proposition 3.4.1: Let (π, y) be a Creditors' Nash Equilibrium.

For notational simplicity we let $s = s((\pi_{-\alpha}, 0))$ the total debt provided by the small creditors in the equilibrium (π, y). Since (π, y) is a Creditors' Nash Equilibrium, the large creditor cannot increase her expected wealth by altering her strategy. We therefore have

$$G^{\alpha,(\pi,y)} \geq G^{\alpha,((\pi_{-\alpha},0),y)} = C(\alpha),$$

which implies, and that $s(\pi) = s + \pi(\alpha)C(\alpha)$,

$$f(\pi(\alpha)C(\alpha)) \geq C(\alpha),$$

where the function f gives the expected payoff of creditor α, given that she invests an amount $x \leq C(\alpha)$ in the company

$$f(x) := Rx - \frac{x(1+R)}{s(1+r) + x(1+R)} Put(yb(\alpha), K(x)) + C(\alpha), \tag{B.14}$$

for a strike $K(x) := y + rs + Rx - c$.

Since (π, y) is a CNE, we have $\pi(\alpha)C(\alpha)$ is a maximizer of the function f. Suppose, by way of contradiction, that $\pi(\alpha) \in (0, 1)$, i.e., it is an interior maximizer.

This implies

$$\frac{df(x)}{dx}\Big|_{x=\pi(\alpha)C(\alpha)} = 0. \tag{B.15}$$

We will show for a contradiction that $f'(x)|_{x=\pi(\alpha)C(\alpha)} > 0$.

We denote by $l(x) = (1+R) - \frac{1+R}{s(1+r)+x(1+R)} Put(yb(\alpha), K(x))$, the return per dollar, estimated by a "large" creditor, given her investment is x and the investment of "small" creditors is s.

$$f(x) = xl(x) + C(\alpha) - x \tag{B.16}$$

Using the standard Black and Scholes formulas we obtain

$$\begin{aligned}\frac{dl(x)}{dx} &= \frac{(1+R)^2}{\big(s(1+r)+x(1+R)\big)^2} Put(yb(\alpha), K(x)) - \frac{R(1+R)}{s(1+r)+x(1+R)} \frac{\partial Put(yb(\alpha), K(x))}{\partial K} \\ &= \frac{(1+R)}{\big(s(1+r)+x(1+R)\big)} \left(1 - l(x) + R\Phi(d_2)\right)\end{aligned}$$

where in the last equality we used the symmetry of the standard normal distribution function $(1 - \Phi(-d_2) = \Phi(d_2))$.

Therefore, we can expand

$$\begin{aligned}f'(x) = l(x) + x\frac{dl(x)}{dx} - 1 &= \Big(h(x) - 1\Big) + \frac{x(1+R)}{s(1+r)+x(1+R)}\Big(R\Phi(d_2) - \Big(h(x)-1\Big)\Big) \\ &= (l(x)-1)\left(1 - \frac{x(1+R)}{x(1+R)+s(1+r)}\right) + R\frac{x(1+R)}{x(1+R)+s(1+r)}\Phi(d_2)\end{aligned}$$

Since $l(x) - 1 \geq 0$ we have

$$f'(x)\Big|_{x=\pi(\alpha)C(\alpha)} > 0, \tag{B.17}$$

in contradiction to (B.15).

■

Proof of Proposition 3.4.3: The statement follows through for all "small" creditors a, s.t. $b(a, y) > b(a', y)$ using the same proof as for Proposition 3.3.1. We consider the case $b(\alpha, y) > b(a')$, and show that $\pi(\alpha, y) = 1$. In light of Proposition 3.4.1, it suffices to show that $\pi(\alpha, y) > 0$.

Assume by way of contradiction that the large creditor does not invest, i.e., $\pi(\alpha, y) = 0$. In this case we have $G^{\alpha,\pi}(y) = 1$.

Since g is strictly increasing in the belief we have

$$g\big(b(\alpha), y, s(\pi, y)\big) > g\big(b(a'), y, s(\pi, y)\big) \geq 1,$$

where the last inequality holds from the assumption that $\pi(a') > 0$ in equilibrium.

By continuity of g in the debt, there exists $\epsilon > 0$ such that

$$g\big(b(\alpha), y, s(\pi, y) + \epsilon\big) > 1.$$

Therefore, if we denote π^ϵ the function, such that $\pi^\epsilon(\alpha, y) = \epsilon$

$$G^{\alpha,(\pi_{-\alpha},\pi^\epsilon)}(y) = \epsilon g\big(b(\alpha), y, s(\pi, y) + \epsilon\big) + (1 - \epsilon) > 1 = G^{\alpha,\pi}(y).$$

This contradicts the definition of Nash Equilibrium and concludes the proof. ■

Proof of Propositions 3.4.6 and 3.4.7: The proof is similar to the proof of Proposition 3.3.4. We begin by stating the following Lemma:

Lemma B.0.2. *If* $g\left(b(\alpha), y, S\left(b(\alpha)^+\right)\right) > 1$, *then* $g\left(b(\alpha), y, S\left(b(\alpha)\right)\right) > 1$.

Proof. From the assumption $g\left(b(\alpha), y, S\left(b(\alpha)^+\right)\right) > 1$, there exists an $\epsilon \in (0, C(\alpha))$ such that $g\left(b(\alpha), y, S\left(b(\alpha)^+\right) + \epsilon\right) > 1$. Therefore, if we consider the function $f : [0, C(\alpha)] \to \mathbb{R}_+$, with $f(x) := xg(b(\alpha), y, S\left(b(\alpha)^+\right) + x) + (C(\alpha) - x)$ we obtain

$$f(\epsilon) > f(0) = 1.$$

From Proposition 3.4.1, f attains its maximum on $[0, C(b(\alpha)]$ either at 0 or at $C(\alpha)$. From the inequality above, the maximum is attained at $C(\alpha)$ and we have

$f(C(\alpha)) > 1$. This implies that $g(b(\alpha), y, S(b(\alpha))) > 1$. ∎

We consider the function $h(\cdot)$ defined in (B.2).

1. We begin by showing that the equation $h(x) = 1$ has a solution.

   In the proof of Proposition 3.3.4 we have shown that:

   (a) $h(x) > 1$ for $x = M_0(1 - \frac{\epsilon}{C - C(\alpha)})$.

   (b) $h(x) < 1$ for sufficiently small x.

   We consider three cases.

   (a) $h(b(\alpha)) = 1$, then $g(b(\alpha), y, S(b(\alpha))) = 1$.

   (b) $h(b(\alpha)) > 1$. Since $h(\cdot)$ is a continuous function on $(0, b(\alpha)]$, there exists $b^* \in (0, b(\alpha))$ such that $h(b^*) = 1$.

   (c) $h(b(\alpha)) < 1$. From Lemma B.0.2, $h(b(\alpha)^+) = g(b(\alpha), y, S(b(\alpha)^+)) < 1$. Since $h(\cdot)$ is a continuous function on $\left(b(\alpha), M_0(1 - \frac{\epsilon}{C - C(\alpha)})\right)$, there exists $b^* \in \left(b(\alpha), M_0(1 - \frac{\epsilon}{C - C(\alpha)})\right)$ such that $h(b^*) = 1$.

   Proposition 3.3.6 states that under uniform capital distribution there exists a unique solution to $h(x) = 1$ in which the debt is larger than ϵ. In the presence of an atom, we consider 2 cases:

2. We now show that there exists not more than one solution to the equation $h(x) = 1$ on $(0, b(\alpha)]$ and not more than one solution on $\left(b(\alpha), M_0(1 - \frac{\epsilon}{C - C(\alpha)})\right)$.

   (a) Suppose there exist two solutions to the equation $h(x) = 1$: b^* and b^{**} on $(0, b(\alpha)]$. Then there exist two solutions to this equation if the capital has the uniform distribution function $S(b) = \frac{C}{M}(M - b)$, in contradiction to Proposition 3.3.6.

(b) Suppose there exist two solutions to the equation $h(x) = 1$: b^* and b^{**} on $\left(b(\alpha), M_0(1 - \frac{\epsilon}{C-C(\alpha)})\right)$. Then there exist two solutions to this equation if the capital has the uniform distribution function $S(b) = \frac{C-C(\alpha)}{M_0}(M_0 - b)$, in contradiction to Proposition 3.3.6.

This concludes the first statement of the proposition.

Let now b^* the smallest solution to $h(x) = 1$ and b^{**} if there is a second solution. We have the following cases.

3. $b^* \in (0, b(\alpha)]$. Then $g(b, y, S(b^*)) \geq 1$ for all $b \geq b^*$, and $g(b, y, S(b^*)) < 1$ for all $b < b^*$. Therefore, no creditor ("small" or "large") can increase her expected gain by unilaterally changing her strategy. Therefore (π_{b^*}, y) is an NE.

If there exists a second solution $b^{**} \in \left(b(\alpha)^+, M_0(1 - \frac{\epsilon}{C-C(\alpha)})\right)$ to $h(x) = 1$, then:

(a) If $g(b(\alpha), y, s(\pi_{b^{**}}, y) + C(\alpha)) > 1$, then the "large" creditor can increase her expected payoff by investing all her money. Therefore b^{**} is not a CNE.

(b) Suppose now, by way of contradiction, that $g(b(\alpha), y, s(\pi_{b^{**}}, y)+C(\alpha)) \leq 1$ and b^{**} is not a CNE. Since no "small" creditor can increase her payoff by unilaterally changing her strategy, it follows that the "large" creditor can increase her payoff with a positive investment. From Corollary 3.4.1 her maximum expected payoff is achieved if she invests all available money, therefore $g(b(\alpha), y, s(\pi_{b^{**}}, y)+C(\alpha)) > 1$. Contradiction. Therefore b^{**} must be a NE.

4. $b^* > b(\alpha)$. In this case we have

$$g\left(b(\alpha), y, S\left(b(\alpha)^+\right)\right) < 1. \tag{B.18}$$

Similarly to the previous part, no "small" creditor can increase her expected gain, by unilaterally changing her strategy. Since the "large" creditor can change the debt structure of the company, we consider her separately. We prove the statement by contradiction. From Corollary 3.4.1, the "large" creditor achieves the maximum expected payoff when $\pi(\alpha) \in \{0, 1\}$. Therefore, if she can increase the expected payoff with changing the strategy, the following inequality holds:

$$g\left(b(\alpha), y, S\left(b^*\right) + C(\alpha)\right) > 1. \tag{B.19}$$

We consider three cases.

(a) $C(\alpha) < S\left(b(\alpha)^+\right) - S\left(b^*\right)$. Then there exists $\beta \in (b(\alpha), b^*)$, such that $C(\alpha) = S(\beta) - S\left(b^*\right)$, therefore, as g is increasing in the first argument:

$$g\left(\beta, y, S(\beta)\right) = g\left(\beta, y, S\left(b^*\right) + C(\alpha)\right) > g\left(b(\alpha), y, S\left(b^*\right) + C(\alpha)\right) > 1,$$

therefore there exists $b^{**} \in (b(\alpha), \beta) \subset (b(\alpha), b^*)$, such that $g\left(b^{**}, y, S\left(b^{**}\right)\right) = 1$. Since (3.24) has a unique root, we obtain a contradiction.

(b) $C(\alpha) > S\left(b(\alpha)^+\right) - S\left(b^*\right)$. In this case (B.19) writes

$$g\left(b(\alpha), y, S\left(b(\alpha)^+\right) + \left(C(\alpha) - S\left(b(\alpha)^+\right) + S\left(b^*\right)\right)\right) > 1.$$

Since the maximum payoff of the large creditor is not interior it follows from the above and from (B.18) that

$$g\left(b(\alpha), y, S\left(b(\alpha)\right)\right) > 1.$$

Therefore, there exists $b^{**} \in (0, b(\alpha))$, such that $g\left(b^{**}, y, s(\pi_{b^{**}}, y)\right) = 1$. Since (3.24) has a unique root, we obtain a contradiction.

(c) $C(\alpha) = S(b(\alpha)^{+}) - S(b^{*})$, then (B.19) writes $g(b(\alpha), y, S(b(\alpha)^{+})) > 1$ in contradiction to (B.18).

■

Proof of Proposition 3.4.8: The first part of the proposition follows from Lemma B.0.1 which states that the function $(1-g)S$ is increasing in its second argument, and thus g is decreasing in its second argument. From Lemma B.0.1

$$\frac{\partial g\left(b(\alpha), y, S(b(\alpha)^{+})\right)}{\partial y} < 0. \tag{B.20}$$

Therefore, for any $\epsilon > 0$

$$g\left(b(\alpha), y_0 - \epsilon, S(b(\alpha)^{+})\right) > 1.$$

Lemma B.0.2 implies then

$$g\left(b(\alpha), y_0 - \epsilon, S(b(\alpha))\right) > 1. \tag{B.21}$$

Since (B.21) is valid for any $\epsilon > 0$,

$$g\left(b(\alpha), y_0, S(b(\alpha))\right) \geq 1. \tag{B.22}$$

The last statement of the proposition now follows from (B.20) and (B.22). ■

Proof of Proposition 3.4.10: We again consider the function g defined in (3.5).

$$g(x, y, z) = 1 + r - \frac{1}{z} Put(yx, y + rz - c).$$

$$\frac{dg\Big(b(\alpha), y, S(\beta_0(y)) + C(\alpha)\Big)}{dy} = \frac{\partial g\Big(b(\alpha), y, S(\beta_0(y)) + C(\alpha)\Big)}{\partial y} + \frac{\partial g\Big(b(\alpha), y, S(\beta_0(y)) + C(\alpha)\Big)}{\partial z}\frac{\partial S(\beta_0(y))}{\partial y}.$$

$$\frac{\partial g(x, y, z)}{\partial z}\Big|_{g(x,y,z)=1} = \frac{r}{z}(1 - \Phi(-d_2)) > 0.$$

Therefore:

$$\frac{\partial g\Big(b(\alpha), y, S(\beta_0(y)) + C(\alpha)\Big)}{\partial z}\frac{\partial S(\beta_0(y))}{\partial y}\Big|_{g\left(b(\alpha),y,S(\beta_0(y))+C(\alpha)\right)=1} < 0.$$

Therefore:

$$\frac{dg\Big(b(\alpha), y, S(\beta_0(y)) + C(\alpha)\Big)}{dy}\Big|_{g\left(b(\alpha),y,S(\beta_0(y))+C(\alpha)\right)=1} < 0$$

Therefore Equation (3.30) has not more than one solution y_0' on $(y_0, y_1]$. ■

BIBLIOGRAPHY

[1] Viral Acharya, Heitor Almeida, and Murillo Campello. Is cash negative deb? a hedging perspective on corporate financial policies. *NBER working paper series*, 2005.

[2] VIRAL Acharya, DOUGLAS Gale, and TANJU Yorulmazer. Rollover risk and market freezes. *Journal of Finance*, 66(4):1177–1209, 2011.

[3] Viral V Acharya and David Skeie. A model of liquidity hoarding and term premia in inter-bank markets. *Journal of Monetary Economics*, 58(5):436–447, 2011.

[4] Heitor Almeida, Murillo Campello, and Michael S Weisbach. The cash flow sensitivity of cash. *Journal of Finance*, 59(4):1777–1804, 2004.

[5] Jonathan B Baskin. Corporate liquidity in games of monopoly power. *The Review of Economics and Statistics*, 69(2):312–19, 1987.

[6] Thomas W Bates, Kathleen M Kahle, and René M Stulz. Why do us firms hold so much more cash than they used to? *Journal of Finance*, 64(5):1985–2021, 2009.

[7] Bruno Biais, Thomas Mariotti, Jean-Charles Rochet, and Stéphane Villeneuve. Large risks, limited liability, and dynamic moral hazard. *Econometrica*, 78(1):73–118, 2010.

[8] Fischer Black and John C. Cox. Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance*, 31(2):351–367, 1976.

[9] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *The Journal of Political Economy*, pages 637–654, 1973.

[10] Markus K. Brunnermeier and Martin Oehmke. The maturity rat race. *Forthcoming in Journal of Finance*, 2012.

[11] Markus K. Brunnermeier and Lasse Heje Pedersen. Market liquidity and funding liquidity. *Review of Financial Studies*, 22(6):2201–2238, 2009.

[12] H. Carlsson and E. Van Damme. Global games and equilibrium selection. *Econometrica*, 61(5):989–1018, 1993.

[13] Ing-Haw Cheng and Konstantin Milbradt. The hazards of debt: Rollover freezes, incentives, and bailouts. *Review of Financial Studies*, 25(4):1070–1110, 2012.

[14] George M Constantinides. Capital market equilibrium with transaction costs. *The Journal of Political Economy*, pages 842–862, 1986.

[15] Mark HA Davis and Andrew R Norman. Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15(4):676–713, 1990.

[16] Salvatore Dell'Erba, Ricardo Hausmann, and Ugo Panizza. Debt levels, debt composition, and sovereign spreads in emerging and advanced economies. *Center for International Development at Harvard University*, 2011.

[17] Douglas W Diamond and Philip H Dybvig. Bank runs, deposit insurance, and liquidity. *Journal of Political Economy*, 91(3):401–419, June 1983.

[18] Amy Dittmar and Jan Mahrt-Smith. Corporate governance and the value of cash holdings. *Journal of Financial Economics*, 83(3):599–634, 2007.

[19] D. Duffie. *How big banks fail and what to do about it*. Princeton University Press, Princeton, NJ, 2010.

[20] Laurent Fresard. Financial strength and product market behavior: The real effects of corporate cash holdings. *Journal of Finance*, 65(3):1097–1122, 2010.

[21] C Fritz Foley, Jay C Hartzell, Sheridan Titman, and Garry Twite. Why do firms hold so much cash? a tax-based explanation. *Journal of Financial Economics*, 86(3):579–607, 2007.

[22] Kenneth A Froot, David S Scharfstein, and Jeremy C Stein. Risk management: coordinating corporate investment and financing policies. *Journal of Finance*, 48(5):1629–1658, 1993.

[23] Douglas Gale and Tanju Yorulmazer. Liquidity hoarding. *Theoretical Economics*, 8(2):291–324, 2013.

[24] I. Goldstein and A. Pauzner. Demand–deposit contracts and the probability of bank runs. *Journal of Finance*, 60(3):1293–1327, 2005.

[25] Gary Gorton and Andrew Metrick. Securitized banking and the run on repo. *Journal of Financial Economics*, 104(3):425 – 451, 2012.

[26] Zhiguo He and Wei Xiong. Rollover risk and credit risk. *Journal of Finance*, 67(2):391–430, 04 2012.

[27] Julien Huggonier and Erwan Morellec. Bank capital, liquid reserves, and insolvency risk. Technical report, Tech. report, Swiss Finance Institute and EPFL, 2014.

[28] Julien Hugonnier, Semyon Malamud, and Erwan Morellec. Capital supply uncertainty, cash holdings, and investment. *Swiss Finance Institute Research Paper No. 11*, 44, 2011.

[29] Alexandre Jeannet. The dynamics of sovereign credit risk. *Journal of Financial and Quantitative Analysis*.

[30] Yuri Kabanov and Mher Safarian. *Markets with transaction costs: Mathematical Theory*. Springer, 2009.

[31] Andrey Krishenik, Andreea Minca, and Johannes Wissel. When do creditors with heterogeneous beliefs agree to run? *Available at SSRN 2181106*, 2013.

[32] G. Liang, E. Lütkebohmert, and W. Wei. A continuous time structural model for insolvency, recovery, and rollover risks. *Working paper*, 2012.

[33] F.A. Longstaff, J. Pan, L.H.Pedersen, and K. J. Singleton. How sovereign is sovereign credit risk. *American Economic Journal: Macroeconomics*, pages 75–103, 2011.

[34] Eva Lütkebohmert, G. Liang, and Y. Xiao. A multi-period bank run model for liquidity risk. *Forthcoming in Review of Finance*, 2012.

[35] Robert C. Merton. On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, 29(2):449–470, 1974.

[36] Robert C Merton. On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, 29(2):449–470, 1974.

[37] Alain Monfort and Jean-Paul Renne. Credit and liquidity risks in euro-area and sovereign yield curves. *working paper*, 2011.

[38] Santiago Moreno-Bromberg and Jean-Charles Rochet. Market frictions and corporate finance: An overview paper. *Swiss Finance Institute Research Paper*, 2014.

[39] S. Morris and H. Shin. Global games: theory and applications. *Cowles Foundation discussion paper*, 2001.

[40] Stephen Morris and Hyun Song Shin. Illiquidity component of credit risk. *Working paper*, 2009.

[41] Roger B Myerson. *Game theory*. Harvard university press, 2013.

[42] Siegel A.F. Nelson C.R. Parsimonious modeling of yield curves. *Journal of Business*, 60:473–489, 1987.

[43] Tim Opler, Lee Pinkowitz, René Stulz, and Rohan Williamson. The determinants and implications of corporate cash holdings. *Journal of Financial Economics*, 52(1):3–46, 1999.

[44] Lee Pinkowitz, René M Stulz, and Rohan Williamson. Multinationals and the high cash holdings puzzle. Technical report, National Bureau of Economic Research, 2012.

[45] R. Rebonato, I. Saroka, and V. Putyatin. A principal-component-based affine term structure model. *working paper*, 2014.

[46] Jean-Charles Rochet and Xavier Vives. Coordination failures and the lender of last resort: Was Bagehot right after all? *Journal of the European Economic Association*, 2(6):1116–1147, 2004.

[47] Leonard CG Rogers. *Optimal investment*. Springer, 2013.

[48] Steven E Shreve, H Mete Soner, and Gan-Lin Xu. Optimal investment and consumption with transaction costs. Technical report, Carnegie Mellon University, 1990.

[49] Suresh Sundaresan. A Review of Mertons Model of the Firms Capital Structure with Its Wide Applications. *Annu. Rev. Financ. Econ.*, 5(1):21–41, 2013.

[50] Sandrine Ungari and Julien Turc. Macro-finance model, 2012.