FINITENESS PROPERTIES AND PIECEWISE PROJECTIVE HOMEOMORPHISMS

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In 1929 the mathematician and physicist John von Neumann isolated an analytic property of groups from the Banach-Tarski paradox. This property is now known as *amenability*. He observed that groups which contain the free group of rank 2, or F_2, are nonamenable and asked whether all nonamenable groups contain F_2. This was disproved by Ol'shankii [27] and independently by Adyan [2][3] in 1979. The first finitely presented counterexample was constructed in 2003 by Olshanskii and Sapir [28]. In [26] Monod introduced a new family of counterexamples. However, Monod's examples are not finitely presentable (or even finitely generated).

In this thesis we will isolate a finitely presentable nonamenable subgroup of Monod's group $H(\mathbb{R})$. (This is joint work with Justin Moore [24].) The group is generated by $a(t) = t + 1$ together with:

$$b(t) = \begin{cases} t & \text{if } t \leq 0 \\ \frac{t}{1-t} & \text{if } 0 \leq t \leq \frac{1}{2} \\ 3 - \frac{1}{t} & \text{if } \frac{1}{2} \leq t \leq 1 \\ t+1 & \text{if } 1 \leq t \end{cases} \qquad c(t) = \begin{cases} \frac{2t}{1+t} & \text{if } 0 \leq t \leq 1 \\ t & \text{otherwise} \end{cases}$$

An isomorphism between the group $\langle a, b \rangle$ and Thompson's group F was discovered by Thurston in the late 1970's by considering the reals as continued fractions. We extend Thurston's work to $\langle a, b, c \rangle$, and use this as a combinatorial framework to show the following:

Theorem. The group $\langle a, b, c\rangle$ is finitely presented with 3 generators and 9 relations.

We also obtain a natural infinite presentation, a *normal form*, and an algorithm for converting a given word to a word in normal form using the relations. Further, we show the following:

Theorem. The group $\langle a, b, c\rangle$ acts on a connected cell complex X by cell permuting homeomorphisms such that the following holds.

1. X is contractible.
2. The quotient $X/\langle a, b, c\rangle$ has finitely many cells in each dimension.
3. The stabilizers of each cell are of type F_∞.

It follows that $\langle a, b, c\rangle$ is of type F_∞.

This provides the first example of a group that is of type F_∞, nonamenable and does not contain F_2.

Monod's technique for proving nonamenability of his groups relies on the nonamenability of the associated orbit equivalence relations. In forthcoming work we characterize all piecewise projective groups of homeomorphisms that are isomorphic to F and study the associated orbit equivalence relations.

This thesis also contains the following unrelated construction which gives a new illustration of a phenomenon first exhibited by Brady [6].

Theorem. Let Γ be the complete bipartite graph with 22 vertices in each set of the partition. There is a subcomplex X of the product $\Gamma \times \Gamma \times \Gamma$ and a map

$$\chi : X \to S^1$$

such that $\pi_1(X)$ is hyperbolic and the kernel

$$H = \mathrm{Ker}(\chi_* : \pi_1(X) \to \mathbb{Z})$$

is finitely presented but not of type FP_3. In particular, H is not hyperbolic.

BIOGRAPHICAL SKETCH

Yash Lodha was born in Jaipur, India on May 16 1987. He attended Maharaja Sawai Man Singh Vidhyalaya and then moved to Binghamton University in January 2006 to study for a BA in Mathematics, graduating in May 2009. He arrived at Cornell University in August 2009 and received a Ph.D. in Mathematics in January 2015.

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CHAPTER 1

INTRODUCTION

The notion of an *amenable group* was introduced by von Neumann as an abstract means for preventing the existence of paradoxical decompositions of the group: a discrete group is *amenable* if it admits a finitely additive translation invariant probability measure. At the heart of Banach and Tarski's paradoxical decomposition of the sphere is the existence of a paradoxical decomposition of the free group on two generators. Since subgroups of amenable groups are easily seen to be amenable, it is natural to ask whether every nonamenable group contains a free group on two generators. Day was the first to pose this problem in print [18], where he attributed it to John von Neumann.

In 1980, Ol'shanskii solved the von Neumann-Day problem by producing a counterexample [27]. Soon after, Adyan showed that certain Burnside groups are also counterexamples [2][3]. These examples are not finitely presented and the restriction of the von Neumann-Day problem to the class of finitely presented groups remained open until Ol'shanskii and Sapir constructed an example in 2003 [28]. Shortly after, Ivanov published another finitely presented counterexample [22], that is somewhat simpler but similar in spirit to the Ol'shanskii-Sapir example. Both examples were produced by elaborate inductive constructions and are difficult to analyze. It is also interesting to note that both of these examples are based on the constructions of nonamenable torsion groups (they are torsion-by-cyclic) and in particular are far from being torsion free.

The elaborate inductive construction of the group is however, large and unwieldy. The construction of this example is also quite involved, however it is simpler than the Olshanskii and Sapir example.

Amenability continues to play an important role in many areas of modern mathematics such as descriptive set theory, ergodic theory, geometric group theory, and representation theory. There is considerable interest in the construction of new examples and in finding new reasons why a group can fail to be amenable.

In his recent article [26], Monod produced a new family of counterexamples to the von Neumann-Day problem. They are all subgroups of the group H consisting of all piecewise projective transformations of the real projective line which fix the point at infinity. Monod demonstrated that H does not contain nonabelian free subgroups by adapting the method of Brin and Squier [12].

In this thesis, we will isolate a finitely presented nonamenable subgroup of Monod's group H. This provides the first finitely presented torsion free counterexample to the von Neumann-Day problem. (This is joint work with Justin Moore [24].) Moreover, our presentations for this group are very explicit: it has a presentation with 3 generators and 9 relations as well as a natural infinite presentation. The group is generated by $a(t) = t + 1$ together with the following two homeomorphisms of $\mathbb{R}$:

$$b(t) = \begin{cases} t & \text{if } t \leq 0 \\ \frac{t}{1-t} & \text{if } 0 \leq t \leq \frac{1}{2} \\ 3 - \frac{1}{t} & \text{if } \frac{1}{2} \leq t \leq 1 \\ t+1 & \text{if } 1 \leq t \end{cases} \qquad c(t) = \begin{cases} \frac{2t}{1+t} & \text{if } 0 \leq t \leq 1 \\ t & \text{otherwise} \end{cases}$$

The main result of this chapter is the following. (This is joint work with Justin Moore [24].)

Theorem 1.0.1. *The group G_0 generated by the functions $a(t)$, $b(t)$, and $c(t)$ is* *nonamenable and finitely presented.*

Since it is a subgroup of H, G_0 does not contain a nonabelian free subgroup. We claim no originality in our proof that G_0 is nonamenable; this is a routine modification of the methods of [26] which in turn relies on [14] [15].

It is interesting to note that, by an unpublished result of Thurston, $a(t)$ and $b(t)$ generate the subgroup $P(\mathbb{Z}) \leq H$, consisting of those homeomorphisms which are C^1 and piecewise $\mathrm{PSL}_2(\mathbb{Z})$, which he moreover showed is a copy of Richard Thompson's group F. In fact the methods of [26] easily show that $t \mapsto t + 1/2$ and $b(t)$ generate a nonamenable group, although at present it is unclear whether this group is finitely presented. While there are strong parallels between the group $\langle a, b, c \rangle$ and F, neither [26] nor [24] seems to shed any light on whether F is amenable.

In 1979 Geoghegan made the following conjectures about Thompson's group F.

1. F has type F_∞.
2. F has no nonabelian free subgroups.
3. F is nonamenable.
4. All homotopy groups of F at infinity are trivial.

Conjectures (1) and (4) were proved by Brown and Geoghegan [11]. Conjecture (2) was proved by Brin and Squier [12]. The status of (3) still remains open. There is considerable interest in Conjecture (3) especially because if F is nonamenable it would be an elegant counterexample to the von Neumann-Day problem.

It is natural to inquire about the higher finiteness conditions satisfied by $\langle a, b, c \rangle$, especially in light of Geoghegan's conjectures about Thompson's group F. For a topologist, a finite group presentation describes a finite, connected, 2-dimensional CW complex. An Eilenberg-Maclane complex for a group G is a connected CW complex X which is aspherical and satisfies that $\pi_1(X) = G$. A group is said to be of type F_∞ if it admits an Eilenberg-Maclane complex with finitely many cells in each dimension.

In Chapter 4 we will establish that this group is of type F_∞. More particularly, we will show the following.

Theorem 1.0.2. *The group $G = \langle a, b, c \rangle$ acts on a connected cell complex X by cell permuting homeomorphisms such that the following holds.*

1. *X is contractible.*
2. *The quotient X/G has finitely many cells in each dimension.*
3. *The stabilizers of each cell are of type F_∞.*

It follows that the group G is of type F_∞.

CHAPTER 2

PRELIMINARIES

2.1 Piecewise projective homeomorphisms and amenable equivalence relations

The text below which establishes the notation we will use is mostly a reproduction of the corresponding text in [24]. Our analysis of the group G_0 will closely parallel the now well-established analysis of Thompson's group F. We direct the reader to the standard reference [4] for the properties of F; additional information can be found in [13]. We shall mostly follow the notation and conventions of [4] [12].

We will take $\mathbb{N}$ to include 0; in particular all counting will start at 0. Let $2^{\mathbb{N}}$ denote the collection of all infinite binary sequences and let $2^{<\mathbb{N}}$ denote the collection of all finite binary sequences. If $i \in \mathbb{N}$ and u is a binary sequence of length at least i, we will let $u \restriction i$ denote the initial part of u of length i. If s and t are finite binary sequences, then we will write $s \subseteq t$ if s is an initial segment of t and $s \subset t$ if s is a proper initial segment of t. If neither $s \subseteq t$ nor $t \subseteq s$, then we will say that s and t are *incompatible*. The set $2^{<\mathbb{N}}$ is equipped with a lexicographic order defined by $s <_{\text{lex}} t$ if either $t \subset s$ or s and t are incompatible and $s(i) < t(i)$ where i is minimal such that $s(i) \neq t(i)$.

If s and t are two sequences (s is finite but t may be infinite), then $s^\frown t$ will be used to denote the concatenation of s and t. In some circumstances, $^\frown$ will be omitted; for instance we will often write $s\mathtt{01}$ instead of $s^\frown\mathtt{01}$. If ξ and η are infinite sequences, then we will say that ξ and η are *tail equivalent* if there are s, t, and ζ such that $\xi = s^\frown\zeta$ and $\eta = t^\frown\zeta$. Given an infinite sequence s, $\tilde{s}$ is the sequence satisfying $\tilde{s}(i) = \mathtt{0}$ if $s(i) = \mathtt{1}$ and $\tilde{s}(i) = \mathtt{1}$ if $s(i) = \mathtt{0}$. The constant sequences $\mathtt{000}...$, $\mathtt{111}...$ are denoted by $\bar{\mathtt{0}}, \bar{\mathtt{1}}$ respectively. Sequences $s, t \in 2^{<\mathbb{N}}$ are said to be *consecutive* if there is a sequence $u \in 2^{<\mathbb{N}}$ and numbers $n_1, n_2 \in \mathbb{N} \cup \{0\}$ such that $s = u\mathtt{01}^{n_1}, t = u\mathtt{10}^{n_2}$.

Let $\mathcal{T}$ denote the collection of all finite rooted ordered binary trees. More concretely, we view elements T of $\mathcal{T}$ as *prefix* sets — those sets T of finite binary sequences with the property that every infinite binary sequence has a unique initial segment in T. Observe that, for each m, there are only finitely many elements of $\mathcal{T}$ with m elements. There is also a natural ordering on $\mathcal{T}$, which we will refer to as *dominance*: if every element of S has an extension in T, then we say that S is dominated by T. Notice that if S is dominated by T, then $|S| \leq |T|$. If A is a finite set of binary sequences, then there is a unique minimal element T of $\mathcal{T}$ (with respect to the order of dominance) such that every element of A has an extension in T.

A *tree diagram* is a pair (L, R) of elements of $\mathcal{T}$ with the property that $|L| = |R|$. A tree diagram describes a map of infinite binary sequences as follows:

$$s_i{}^\frown\xi \mapsto t_i{}^\frown\xi$$

where s_i and t_i are the ith elements of L and R respectively and ξ is any binary sequence. The collection of all such functions from $2^{\mathbb{N}}$ to $2^{\mathbb{N}}$ defined in this way is *Thompson's group F*. This map is also defined on any finite binary sequence u such that u has a prefix in L. We will follow [12] and write $s.f$ for the result of applying an automorphism f to the input s. The operation of F is therefore defined as $s.(fg) = (s.f).g$. Thompson's group F is generated by the following functions.

$$\xi.a = \begin{cases} 0\eta & \text{if } \xi = 00\eta \\ 10\eta & \text{if } \xi = 01\eta \\ 11\eta & \text{if } \xi = 1\eta \end{cases} \qquad \xi.b = \begin{cases} 0\eta & \text{if } \xi = 0\eta \\ 10\eta & \text{if } \xi = 100\eta \\ 110\eta & \text{if } \xi = 101\eta \\ 111\eta & \text{if } \xi = 11\eta \end{cases}$$

Recall that the *real projective line* is the set of all lines in $\mathbb{R}^2$ which pass through the origin. Such lines can naturally be identified with elements of $\mathbb{R} \cup \{\infty\}$ via the x-coordinate of their intersection with the line $y = 1$. It will be useful to represent points on the real projective line by binary sequences derived from their continued fractions expansion. Define a map $\Phi : 2^{\mathbb{N}} \to \mathbb{R} \cup \{\infty\}$ as follows. First define $\phi : 2^{\mathbb{N}} \to [0, \infty]$ by

$$\phi(0\xi) = \frac{1}{1 + \frac{1}{\phi(\xi)}} \qquad \phi(1\xi) = 1 + \phi(\xi)$$

and set

$$\Phi(0\xi) = -\phi(\tilde{\xi}) \qquad \Phi(1\xi) = \phi(\xi).$$

This function is one-to-one except at ξ which are eventually constant (i.e. $r_k = \infty$ for some k). On sequences which are eventually constant, the map is two-to-one: $\Phi(s0\bar{1}) = \Phi(s1\bar{0})$ and $\Phi(\bar{0}) = \Phi(\bar{1}) = \infty$.

In the mid 1970s, Thurston observed that the functions a and b from the introduction become the generators a and b for Thompson's group F defined above when "conjugated" by Φ. Moreover, the elements of F correspond exactly to those homeomorphisms f of $\mathbb{R}$ which are piecewise $\mathrm{PSL}_2(\mathbb{Z})$ and which have continuous derivatives. We will generally take the viewpoint that Φ provides just another way of describing the real projective line, just as decimal expansions allow us to describe real numbers. In particular, we will regard the definitions of a and b in the introduction and the definitions given above in terms of sequences as being two ways of describing the *same functions*.

Since we will be proving that a group is finitely presented, it will be useful to deal with formal words over formal alphabets. If G is a group and A is a subset of G, an A-word is a finite sequence of elements of the set $A \times (\mathbb{Z} \setminus \{0\})$. We typically denote a pair (a, n) as a^n, but we emphasize here that it is formally distinct from the group element a^n. The *word length* of an A-word is the sum of the absolute values of the exponents which occur in it.

In order to prove the nonamenability of G_0, we will need to employ Zimmer's theory of amenable equivalence relations. Let X be a Polish space and let $E \subseteq X^2$ be an equivalence relation which is Borel and which has countable equivalence classes. E is *μ-amenable* if, after discarding a μ-measure 0 set, E is the orbit equivalence relation of an action of $\mathbb{Z}$. (This is not the standard definition, but it is equivalent by [15].) We will need the following two results.

Theorem 2.1.1. *[34] If Γ is a countable amenable group acting by Borel automorphisms on a Polish space X and μ is any σ-finite Borel measure on X, then the orbit equivalence relation is μ-amenable.*

Theorem 2.1.2. *[14] (see also the discussion in [26]) If Γ is a countable dense subgroup of $\mathrm{PSL}_2(\mathbb{R})$, then the action of Γ on the real projective line induces an orbit equivalence relation which is not amenable with respect to Lebesgue measure.*

We refer the reader to [23] for further information on amenable equivalence relations.

We will conclude this section by sketching a proof that the group G_0 from the introduction is nonamenable. Let K denote the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Viewed as fractional linear transformations, K is generated by $t+1$, $2t$, and $-1/t$. Since K contains $\mathrm{PSL}_2(\mathbb{Z})$ as a proper subgroup, it is dense in $\mathrm{PSL}_2(\mathbb{R})$ and hence by Theorem 2.1.2, the orbit equivalence relation of its action on the real projective line is not amenable with respect to Lebesgue measure. By Theorem 2.1.1, it is sufficient to show that G_0 induces the same orbit equivalence relation on $\mathbb{R} \setminus \mathbb{Q}$. To see this, it can be verified that the element $bca^{-1}c^{-1}a$ coincides with $t \mapsto 2t$ on the interval $[0, 1]$. From this and the identity $2(r - n) + 2n = 2r$ it follows that the orbits of G_0 include the orbits of the action of $\langle t \mapsto t + 1, t \mapsto 2t\rangle$ on $\mathbb{R} \setminus \mathbb{Q}$. Finally aba and ba^{-3} coincide with $t \mapsto -1/t$ on the intervals $[-1, -1/2]$ and $[1/2, 1]$ respectively. The assertion about orbits now follows from the fact that for any $r \in \mathbb{R}$, there is a $n \in \mathbb{Z}$ such that $2^n r$ is in $[-1, -1/2] \cup [1/2, 1]$ and $(2^n)(-1/(2^n r)) = -1/r$.

2.2 Finiteness properties of groups.

The classical finiteness properties of groups are that of being finitely generated and finitely presented. These notions were generalized by C.T.C. Wall [33]. In this chapter we are concerned with the properties *type* F_n. These properties are quasi-isometry invariants of groups [1]. In order to discuss these properties first we need to define Eilenberg-Maclane complexes.

An Eilenberg-Maclane complex for a group G, or a $K(G, 1)$, is a connected CW-complex X such that $\pi_1(X) = G$ and $\widetilde{X}$ is contractible. It is a fact that for any group G, there is an Eilenberg-Maclane complex X which is unique up to homotopy type. A group is said to be *of type* F_n if it has an Eilenberg-Maclane complex with a finite n-skeleton. Clearly, a group is finitely generated if and only if it is of type F_1, and finitely presented if and only if it is of type F_2. (For more details see [19].)

For a group G, consider the group ring $\mathbb{Z}G$. We view $\mathbb{Z}$ as a $\mathbb{Z}G$ module where the action of G is trivial, i.e. $g \cdot 1 = 1$ for every $g \in G$. A module is called *projective* if it is the direct summand of a free module. The group G is said to be of type FP_n if there is a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$ (an exact sequence):

$$\ldots \to P_n \to P_{n-1} \to \ldots \to P_0 \to \mathbb{Z}$$

of the trivial $\mathbb{Z}G$ module $\mathbb{Z}$ such that for each $1 \le i \le n$, P_i is finitely generated as a $\mathbb{Z}G$ module.

A group is of type FP_1 if and only if it is finitely generated. If a group is of type F_n then it is of type FP_n. In general for $n > 1$ type FP_n does not imply type F_n. (There are examples due to Bestvina-Brady [5].) However, whenever a group is finitely presented and of type FP_n, it is also of type F_n. For a detailed exposition about homological finiteness properties, we refer the reader to [19].

The following is a special case of a well known result (Proposition 1.1 in [10]).

Proposition 2.2.1. *Let Γ be a group that acts on a cell complex X by cell permuting homeomorphisms such that the following holds:*

1. *X is acyclic, i.e. $\tilde{H}_n(X) = 0$ for each $n \ge 0$.*
2. *The quotient X/Γ has finitely many cells in each dimension.*
3. *The stabilizers of each cell are of type FP_∞.*

Then Γ *is of type* FP_∞.

We remark that if a group is finitely presented and of type FP_∞, then it is of type F_∞. So for a finitely presented group Γ we can replace type FP_∞ with type F_∞ in the above proposition. We shall be concerned with the property F_∞. It will be our goal to construct a CW-complex X and an action of G on X which satisfies the hypothesis of the above proposition.

2.3 CAT(0) Cube complexes

Although the complex we construct in this section is not a cube complex, cube complexes will play an important role in the proof of contractibility of our complex.

By a *regular n-cube* $\Box^n$ we mean a cube in $\mathbb{R}^n$ which is isometric to the cube $[0,1]^n$ in $\mathbb{R}^n$. Informally a cube complex is a cell complex of regular Euclidean cubes glued along their faces by isometries. More formally, a cube complex is a cell complex X that satisfies the following conditions. The metric on such cube complexes is the piecewise Euclidean metric. (see [8] for details).

Definition 2.3.1. Given a face f of a regular cube $\Box^n$, let x be the center of this face. The link $Lk(f, \Box^n)$ is the set of unit tangent vectors at x that are orthogonal to f and point in $\Box^n$. This is a subset of the unit sphere S^{n-1} which is homeomorphic to a simplex of dimension $n - \dim(f) - 1$. This admits a natural spherical metric, in which the dihedral angles are right angles.

Definition 2.3.2. Let f be a cell in X and let

$$S = \{C \mid C \text{ is a cube in } X \text{ that contains } f \text{ as a face }\}$$

The link $Lk(f,X) = \bigcup_{C \in S} Lk(f,C)$. This is a complex of spherical "all right" simplices glued along their faces by isometries. This admits a natural piecewise spherical metric.

Given a geodesic metric space X, a triple $x, y, z \in X$ forms a *geodesic* triangle $\triangle(x,y,z)$ obtained by joining x, y, z pairwise by geodesics. Let $\triangle_{\mathbb{E}^2}(x', y', z')$ be a Euclidean triangle such that the Euclidean distance between each pair (x', y'), (x', z'), (y', z') is equal to the distance between the corresponding pair $(x, y), (x, z), (y, z)$. The triangle $\triangle_{\mathbb{E}^2}(x', y', z')$ is called a *comparison triangle* for $\triangle(x, y, z)$. If p is a point on the geodesic joining x, y, there is a point p' in the geodesic joining x', y' such that $d_X(x,p) = d_{\mathbb{E}^2}(x', p')$. This is called a *comparison point.*

X is said to be CAT(0) if for any geodesic triangle $\triangle(x, y, z)$ and a pair $p, q \in \triangle(x, y, z)$, the corresponding comparison points $p', q' \in \triangle_{\mathbb{E}^n}(x', y', z')$ have the property that $d_X(p, q) \leq d_{\mathbb{E}^2}(p', q')$. It is a well known fact that CAT(0) spaces are contractible.

Gromov gave the following characterizations of CAT(0) and CAT(-1) cube complexes by combinatorial conditions on the links of vertices in [20]. A nice survey of these results can be found in [16] and [17] (see Proposition I.6.8).

Definition 2.3.3. A simplicial complex is said to satisfy the *no* $\square$*-condition* if there are no 4-cycles in the 1-skeleton for which none of the pairs of opposite vertices of the cycle are connected by an edge. A simplicial complex Z is called a "flag" complex if any set $v_1, ..., v_n$ of vertices of Z that are pairwise connected by an edge span a simplex. This is also known as the "no empty triangles" condition.

Definition 2.3.4. A cube complex X is said to be nonpositively curved if the link of each vertex is a flag complex.

Theorem 2.3.5. *(Gromov) A cube complex X is* $\mathrm{CAT}(0)$ *if and only if it is nonpositively curved and simply connected. Furthermore, X admits a* $\mathrm{CAT}(-1)$ *metric if and only if it is* $\mathrm{CAT}(0)$ *and the link of each vertex satisfies the no $\square$-condition.*

Theorem 2.3.6. *(Gromov, Eberlin, Bridson) A* $\mathrm{CAT}(0)$ *metric space with a cocompact group of isometries is hyperbolic if and only if it does not contain isometrically embedded flat planes.*

We remark that any nonpositively curved cube complex C has the property that the universal cover $\widetilde{C}$ is contractible, and so in particular the group $\pi_i(C)$ is trivial for each $i > 1$.

2.4 Vertex transitive graphs

The following is a natural generalization of Cayley graphs. Let G be a group and H be a subgroup of G. Let S be a finite set of elements of G such that:

1. $S \subseteq G \setminus H$.
2. For each $s \in S$ we have $s^{-1} \in S$.
3. $S \cup H$ generates G.

Then we can form the so called *coset graph* $\mathrm{Cos}(G, H, S)$ as follows. The vertices of $\mathrm{Cos}(G, H, S)$ are the right cosets of H in G, and two cosets Hg_1, Hg_2 are connected by an edge if $g_1 g_2^{-1} \in H(S)H = \cup_{s \in S} HsH$. Here $HsH = \{h_1 s h_2 \mid h_1, h_2 \in H\}$ is a double coset.

The group G acts on the graph $\mathrm{Cos}(G, H, S)$ on the right, and this action is vertex transitive. In particular, the group of graph automorphisms of $\mathrm{Cos}(G, H, S)$

is vertex transitive. By the following theorem (see Theorem 3.8 in [25]), in fact every vertex transitive graph can be described in this way.

Theorem 2.4.1. *Let* Ω *be some vertex transitive graph. Then* Ω *is isomorphic to some coset graph* $\mathrm{Cos}(G, H, S)$.

The 1-skeleton of our complex will emerge as a vertex transitive graph in this way.

CHAPTER 3

FINITE PRESENTATIONS AND NORMAL FORMS

The results in Sections 3.1 and 3.2 have been obtained in joint work with Justin Moore [24].

3.1 The presentations

In this section we will describe both a finite and infinite presentation of the group G_0 defined in the introduction. We start with the following two primitive functions defined on binary sequences:

$$\xi.x = \begin{cases} \mathtt{0}\eta & \text{if } \xi = \mathtt{00}\eta \\ \mathtt{10}\eta & \text{if } \xi = \mathtt{01}\eta \\ \mathtt{11}\eta & \text{if } \xi = \mathtt{1}\eta \end{cases} \qquad \xi.y = \begin{cases} \mathtt{0}(\eta.y) & \text{if } \xi = \mathtt{00}\eta \\ \mathtt{10}(\eta.y^{-1}) & \text{if } \xi = \mathtt{01}\eta \\ \mathtt{11}(\eta.y) & \text{if } \xi = \mathtt{1}\eta \end{cases}$$

(The function x is nothing but the function a described in the previous section.) From these functions, we define families of functions x_s ($s \in 2^{<\mathbb{N}}$) and y_s ($s \in 2^{<\mathbb{N}}$) which act just as x and y, but localized to those binary sequences which extend s.

$$\xi.x_s = \begin{cases} s^\frown(\eta.x) & \text{if } \xi = s^\frown\eta \\ \xi & \text{otherwise} \end{cases} \qquad \xi.y_s = \begin{cases} s^\frown(\eta.y) & \text{if } \xi = s^\frown\eta \\ \xi & \text{otherwise} \end{cases}$$

If s is the empty-string, it will be omitted as a subscript. The relationship between these functions and the functions a, b, and c of the introduction is expressed by the following proposition.

Proposition 3.1.1. *For all ξ in $2^{\mathbb{N}}$, $\phi(\xi.y) = 2\phi(\xi)$ and*

$$\Phi(\xi).a = \Phi(\xi.x) \qquad \Phi(\xi).b = \Phi(\xi.x_{\mathtt{1}}) \qquad \Phi(\xi).c = \Phi(\xi.y_{\mathtt{10}}).$$

Remark 3.1.2. The effects of doubling on continued fractions was first worked out by Hurwitz [21]. Raney introduced transducers for making calculations such as these in [30].

Proof. We will only prove the identities $\phi(\xi.y) = 2\phi(\xi)$ and $\Phi(\xi).c = \Phi(\xi.y_{\mathtt{10}})$; the remaining verifications are similar and left to the reader. We will first verify the identity $\phi(\xi.y) = 2\phi(\xi)$. Observe that, since ϕ and y are continuous, it suffices to verify this equality for sequences which are eventually constant. The proof is now by induction on the minimum digit beyond which the sequence is constant. For the base case we have:

$$\phi(\bar{\mathtt{0}}.y) = \phi(\bar{\mathtt{0}}) = 0 = 2 \cdot 0 \qquad \phi(\bar{\mathtt{1}}.y) = \phi(\bar{\mathtt{1}}) = \infty = 2 \cdot \infty.$$

In the inductive step, we have three cases:

$$\phi(\mathtt{00}\xi.y) = \phi(\mathtt{0}(\xi.y)) = \frac{1}{1+\frac{1}{\phi(\xi.y)}} = \frac{1}{1+\frac{1}{2\phi(\xi)}} = \frac{2}{2+\frac{1}{\phi(\xi)}} = 2\phi(\mathtt{00}\xi)$$

$$\phi(\mathtt{01}\xi.y) = \phi(\mathtt{10}(\xi.y^{-1})) = 1 + \frac{1}{1+\frac{1}{\phi(\xi.y^{-1})}} = 1 + \frac{1}{1+\frac{2}{\phi(\xi)}} = \frac{2}{1+\frac{1}{1+\phi(\xi)}} = 2\phi(\mathtt{01}\xi)$$

$$\phi(\mathtt{1}\xi.y) = \phi(\mathtt{11}(\xi.y)) = 2 + \phi(\xi.y) = 2 + 2\phi(\xi) = 2(1+\phi(\xi)) = 2\phi(\mathtt{1}\xi).$$

Next we turn to the verification of $\Phi(\xi).c = \Phi(\xi.y_{\mathtt{10}})$. Observe that if ξ does not extend $\mathtt{10}$, then $\Phi(\xi)$ is outside the interval $(0,1)$ and we have $\Phi(\xi).c = \Phi(\xi) = \Phi(\xi.y_{\mathtt{10}})$. The remaining case follows from the identity we have already established, noting that $\frac{2t}{t+1} = \frac{2}{1+1/t}$:

$$\Phi(\mathtt{10}\xi).c = \left(\frac{1}{1+\frac{1}{\phi(\xi)}}\right).c = \frac{2}{2+\frac{1}{\phi(\xi)}} = \frac{1}{1+\frac{1}{2\phi(\xi)}} = \frac{1}{1+\frac{1}{\phi(\xi.y)}} = \Phi(\mathtt{10}\xi.y_{\mathtt{10}}).$$

□

From this point forward, we will identify a, b, and c with x, $x_{\mathtt{1}}$, and $y_{\mathtt{10}}$, respectively and suppress all mention of Φ.

We now return to our discussion of the generators. It is straightforward to verify that the following relations are satisfied by these elements, where s and t are finite binary sequences:

1. $x_s^2 = x_{s0} x_s x_{s1}$;
2. if $t.x_s$ is defined, then $x_t x_s = x_s x_{t.x_s}$;
3. if $t.x_s$ is defined, then $y_t x_s = x_s y_{t.x_s}$;
4. if s and t are incompatible, then $y_s y_t = y_t y_s$;
5. $y_s = x_s y_{s0} y_{s10}^{-1} y_{s11}$.

We will refer to these relations collectively as R. The first two groups of relations are known to give a presentation for F: the function x_{1^n} corresponds to the nth generator in the standard infinite presentation of F. We will use F to denote the group generated by $\{x_s : s \in 2^{<\mathbb{N}}\}$. We denote the elements of the set

$$\{y_s^t \mid s \in 2^{<\mathbb{N}}, s \neq 0^k \text{ or } 1^k, t \in \{1, -1\}\}$$

as *percolating elements*.

Notice that any y_s is conjugate by an element of F to exactly one of y, y_0, y_1, or y_{10}. Define $X = \{x_s : s \in 2^{<\mathbb{N}}\}$, $Y = \{y_s : s \in 2^{<\mathbb{N}}\}$, and Y_0 to be the set of all y_s such that s is not a constant binary sequence. Observe that Y_0 consists of those elements of Y which are conjugate to y_{10} by an element of F. The group G_0 defined in the introduction is therefore generated by the (redundant) generating set $S_0 = X \cup Y_0$.

Let R_0 be those relations in R which only refer to generators in S_0 and let G be the group generated by $S = X \cup Y$. Most of this chapter will focus on proving the following theorem. (This is joint work with Justin Moore and appears in [24].)

Theorem 3.1.3. *The relations R give a presentation for G and the relations R_0 give a presentation for G_0. Moreover, G and G_0 admit finite presentations.*

In the remainder of this section, we will prove that G and G_0 are finitely presented assuming that R and R_0 give presentations for these groups. The finite generating sets for these groups are $\{x, x_1, y_0, y_1, y_{10}\}$ and $\{a, b, c\} = \{x, x_1, y_{10}\}$, respectively. Before proceeding, it will be necessary to define the other generators as words in terms of these generators; these definitions will be compatible with equalities which hold in G. We begin by declaring

$$y = xy_0y_{10}^{-1}y_{11} \qquad x_0 = x^2x_1^{-1}x^{-1} \qquad x_{10} = x_1^2x^{-1}x_1^{-1}xx_1^{-1}.$$

Observe that $0.x^{-n} = 0^{n+1}$ and $1.x^n = 1^{n+1}$ and set

$$x_{0^{n+1}} = x^n x_0 x^{-n} \qquad x_{1^{n+1}} = x^{-n}x_1x^n$$

$$y_{0^{n+1}} = x^n y_0 x^{-n} \qquad y_{1^{n+1}} = x^{-n}y_1x^n.$$

If $s \in 2^{<\mathbb{N}}$ is nonconstant, fix a word f_s in $\{x, x_1\}$ such that $10.f_s = s$ and define

$$x_s = f_s^{-1}x_{10}f_s \qquad y_s = f_s^{-1}y_{10}f_s.$$

Next we note the following two standard properties of F.

Proposition 3.1.4. *If g is any element of F and s is a finite binary sequence such that $s.g$ is defined, then $x_s g = g x_{s.g}$. In particular if g, x_s and $x_{s.g}$ are expressed as words in $\{x, x_1\}$, then the above equality is derivable from the relations in (1) and (2) above.*

Proposition 3.1.5. *If $u <_{\text{lex}} v$ are incompatible binary sequences, then there is a g in F and $s <_{\text{lex}} t$ each of length at most 3 such that $s.g = u$ and $t.g = v$.*

From these facts it follows that every relation in (4) is conjugate via an element of F to a relation in (4) indexed by sequences of length at most 3. The relations in (5) are conjugate via elements of F to a relation $y_s = x_s y_{s0} y_{s10}^{-1} y_{s11}$ where $s \in \{0, 10, 1\}$. The relations in (3) can be expressed as $y_s g = g y_s$ where $s \in \{0, 10, 1\}$ and $g \in F$ such that $s.g = s$. These can be derived from relations $y_s x_t = x_t y_s$ where $s.x_t = s$ and s, t are binary sequences of length at most 3. In particular G and G_0 are finitely presented. In the case of G_0, one can check that the following list of 9 relations actually suffice:

$$x_1 x^{-2} x_1 x = x^{-1} x_1 x x_1 x^{-1} \qquad x_1 x^{-3} x_1 x^2 = x^{-2} x_1 x^2 x_1 x^{-1}$$

$$y_{10} x_0 = x_0 y_{10} \qquad y_{10} x_{01} = x_{01} y_{10}$$

$$y_{10} x_{11} = x_{11} y_{10} \qquad y_{10} x_{111} = x_{111} y_{10}$$

$$y_{01} y_{10} = y_{10} y_{01} \qquad y_{001} y_{10} = y_{10} y_{001}$$

$$y_{10} = x_{10} y_{100} y_{1010}^{-1} y_{1011}.$$

(Notice that all of the above relations except the last assert that a pair of elements of the group commute. In each case this is because they are supported on disjoint sets, where the support g is the set of x such that $x.g \neq x$.) When expressed in terms of the original generators, these become:

$$ba^{-2}ba = a^{-1}baba^{-1} \qquad ba^{-1}a^{-2}ba^2 = a^{-2}ba^2ba^{-1}$$

$$ca^2b^{-1}a^{-1} = a^2b^{-1}a^{-1}c \qquad cb^2a^{-1}bab = b^2a^{-1}babc$$

$$ca^{-1}ba = a^{-1}bac \qquad ca^{-2}ba^2 = a^{-2}ba^2c$$

$$caca^{-1} = aca^{-1}c \qquad ca^2ca^{-2} = a^2ca^{-2}c$$

$$c = b^2a^{-1}b^{-1}acb^{-2}ab^{-1}c^{-1}ba^{-1}bab^{-1}ab^{-1}cba^{-1}ba^{-1}.$$

3.2 Sufficiency of the relations

In this section, we will prove that the relations in R and R_0 are sufficient to give presentations for G and G_0. We will use without proof that the relations in R which only refer to the generators in X give a presentation for F (see [4] [13]). The strategy of the proof is as follows. First, we will argue that any S-word can be put into a *standard form* by applying the relations. Standard forms are not unique, but are organized in a way which better facilitates further symbolic manipulations. We will then define the notion of a *sufficiently expanded* standard form, argue that every standard form can be sufficiently expanded by applying the relations in R, and that any sufficiently expanded standard form which represents an element of F is an X-word.

We will begin by defining some terminology. In what follows, we will say that an S-word Ω_1 is *derived from* an S-word Ω_0 if it is the result of applying substitutions of the following forms:

$$y_t^i x_s^{\pm 1} \Rightarrow x_s^{\pm 1} y^i_{t.x_s^{\pm 1}} \qquad y_s \Rightarrow x_s y_{s0} y_{s10}^{-1} y_{s11}$$

$$y_u y_v \Leftrightarrow y_v y_u \qquad x^{i+j} \Leftrightarrow x^i x^j \qquad y^{i+j} \Leftrightarrow y^i y^j$$

$$\text{delete an occurrence of } y^i y^{-i}$$

where $s, t, u, v \in 2^{<\mathbb{N}}$ are such that $t.x_s$ is defined and u and v are incompatible, and i, j are nonzero integers of the same sign. We will write this symbolically as $\Omega_0 \Rightarrow \Omega_1$. Notice that each of these substitutions corresponds either to a relation in R or to a group-theoretic identity. Also observe that only S_0-words can be derived from S_0-words.

Definition 3.2.1. An S-word Ω is in *standard form* if it is the concatenation of a X-word followed by a Y-word and whenever $\Omega(i) = y_s^m$, $\Omega(j) = y_t^n$, and $s \subseteq t$, then $j \leq i$. We will write *standard form* to mean an S-word in standard form. The *depth* of a standard form Ω is the least l such that there is binary sequence s of length l such that y_s occurs in Ω (if Ω is an X-word, then we say that Ω has infinite depth).

Notice in particular that a given y_s can occur at most once in a standard form (although possibly with an exponent other than ± 1). Observe that any group element which is expressible by a word in standard form allows us to describe the group element via a labeled tree diagram in the sense of the previous section.

Lemma 3.2.2. *For every $s \in 2^{<\mathbb{N}}$ and every $l \in \mathbb{N}$, there is a standard form Ω which can be derived from $y_s^{\pm 1}$ such that:*

1. *if x_u occurs in Ω, then u extends s;*
2. *if y_u occurs in Ω, then u extends s, has length at least l, and the exponent of y_u is ± 1;*
3. *if y_u and y_v occur in Ω and $u \neq v$, then u and v are incompatible.*

Proof. The proof is by induction on $l - |s|$. If $l - |s| = 0$, there is nothing to do since y_s already satisfies the conclusion of the lemma. If $l - |s| > 0$, then $y_s \Rightarrow x_s y_{s0} y_{10}^{-1} y_{11}$ and we can apply the induction hypothesis to obtain Ω_{s0}, Ω_{s10}, Ω_{s11} which satisfy the conclusion of the lemma for y_{s0}, y_{s10}^{-1}, and y_{s11} respectively for the same value of l. By conclusion 1 of the lemma, we can apply substitutions of the form $y_v x_u \Rightarrow x_u y_v$ for incompatible u and v move the occurrence of x_u in

$$x_s \Omega_{s0} \Omega_{s10} \Omega_{s11}$$

to the left, placing the word a standard form which satisfies the conclusions of the lemma. The case of y_s^{-1} is handled similarly using the substitution $y_s^{-1} \Rightarrow x_s^{-1} y_{s00}^{-1} y_{s01} y_{s1}^{-1}$. □

Lemma 3.2.3. *If Ξ is an X-word, then there is an l_0 such that if Ω is a standard form of depth $l \geq l_0$, then $\Omega\Xi \Rightarrow \Omega'$ for some standard form Ω' of depth at least $l - k$ where k is the word length of Ξ.*

Proof. If $\Xi = x_s^{\pm 1}$, then observe that if t is a finite binary sequence of length at least $l + 2$, then $t.x_s^{\pm 1}$ is defined and its length differs from the length of Ω by at most 1. Thus by repeated applications of the substitution $y_t^i x_s^{\pm 1} \Rightarrow x_s^{\pm 1} y^i_{t.x_s^{\pm 1}}$, the final occurrence of x_s in $\Omega x_s^{\pm 1}$ can be moved to the left of all occurrences of a y_t. This results in a standard form in which the depth is changed by at most 1. The general case now follows by induction. □

Lemma 3.2.4. *If Ω is any S-word and $l \in \mathbb{N}$, then $\Omega \Rightarrow \Omega'$ for some standard form Ω' of depth at least l.*

Proof. The proof is by double induction: first on the word length n of Ω and then on l. The case $n = 1$ is handled by Lemma 3.2.2. By making a substitution of the form $a^{\pm(k+1)} \Rightarrow a^{\pm k} a^{\pm 1}$ if necessary, we may assume that $\Omega = \Omega_0 \Omega_1$ where Ω_i is an S-word of positive length. By our induction hypothesis $\Omega_1 \Rightarrow \Xi\Upsilon$, where Ξ and Υ are X- and Y-words respectively and Υ has depth l. Let k be the word length of Ξ and let $m \geq l$ be such that if y_u occurs in Υ, u has length less than m. By our induction hypothesis, there is a standard form Ω_0' of depth at least $m + k$ such that $\Omega_0 \Rightarrow \Omega_0'$. By Lemma 3.2.3 we have that $\Omega_0'\Xi \Rightarrow \Omega_0''$ for some standard form Ω_0'' of depth at least m, we have:

$$\Omega \Rightarrow \Omega_0 \Omega_1 \Rightarrow \Omega_0 \Xi \Upsilon \Rightarrow \Omega_0' \Xi \Upsilon \Rightarrow \Omega_0'' \Upsilon$$

Finally, notice that since the depth of Ω_0'' is at least m, $\Omega' = \Omega_0''\Upsilon$ is a standard form of depth at least l, as desired. □

If Ω is standard form and y_s occurs in Ω, we say that s is *exposed in* Ω if there is a finite binary sequence u extending s such that if t is a binary sequence compatible with u and y_t occurs in Ω, then t is an initial part of s.

Definition 3.2.5. A standard form Ω is *sufficiently expanded* if whenever y_s occurs in Ω and s is not exposed in Ω, then:

- y_{s0} occurs in Ω if y_s occurs positively in Ω;
- y_{s1} occurs in Ω if y_s occurs negatively in Ω.

The motivation for this definition is as follows. Suppose that Ω is a standard form which is not sufficiently expanded and that this is witnessed by $\Omega(i) = y_s^n$ for $n > 0$. If we substitute

$$x_s y_{s0} y_{s10}^{-1} y_{s11} y_s^{n-1}$$

for y_s^n in Ω, then whenever y_t occurs before y_s in Ω, $t.x_s$ is defined. A similar conclusion holds — with x_s^{-1} replacing x_s — if $n < 0$ and the substitution

$$x_s^{-1} y_{s00}^{-1} y_{s01} y_{s1}^{-1} y_s^{n+1}$$

is applied. This plays an important role in the proof of the next lemma.

Lemma 3.2.6. *If Ω is a standard form, then there is a sufficiently expanded standard form which can be derived from Ω.*

Proof. We will prove the lemma by defining a *well-founded* partial ordering $\lhd$ on the set of standard forms and a notion of expansion on standard forms which are not sufficiently expanded in such a way that produces a smaller standard form in this ordering. Here a partial order is *well-founded* if it has no infinite strictly decreasing sequences. We will define the ordering first.

If Ω is an S-word, let $T(\Omega)$ denote the minimal prefix set which has the property that if y_t occurs in Ω, then t has an extension in T. If Ω_0 and Ω_1 are standard forms, define $\Omega_0 \triangleleft \Omega_1$ if $|T(\Omega_0)| < |T(\Omega_1)|$ or $|T(\Omega_0)| = |T(\Omega_1)|$ and $|k_0| < |k_1|$ where k_i is the exponent in Ω_i of the $\leq_{\mathrm{lex}}$-maximal s such that y_s occurs in at least one of Ω_0 or Ω_1 and for which $k_0 \neq k_1$ (if y_s occurs in only one of the Ω's, then the other exponent is 0). Notice that for a fixed m there are only finitely many prefix sets of cardinality m. In particular the collection F of all finite binary sequences which have an extension in a prefix set of cardinality m is finite. Since the lexicographic ordering on $\mathbb{N}^F$ is a well-order, $\triangleleft$ is well-founded.

Now suppose that Ω is a standard form which is not sufficiently expanded as witnessed by $\Omega(i) = y_s^n$. For simplicity, suppose that $n > 0$ and apply the substitution

$$y_s^n \Rightarrow x_s y_{s0} y_{s10}^{-1} y_{s11} y_s^{n-1}$$

(if $n = 1$, the y_s^{n-1} term is omitted) followed by substitutions of the form $y_t^m x_s \Rightarrow x_s y_{t.x_s}^m$ to move x_t to the left, forming a new word Ω' which is the concatenation of a X-word followed by a Y-word. At this point, the only thing preventing Ω' from being a standard form is the newly introduced occurrences of y_{s0}, y_{s10}, and y_{s11}. Observe that y_{s1} can not occur in Ω'; for this to happen, $s1$ would have to equal $t.x_s$ for some t such that $t.x_s$ is defined, and such a t does not exist. Furthermore, if y_t occurs in Ω' and t properly extends one of the sequences $s0$, $s10$, or $s11$, then y_t must occur before any occurrence of y_{s0}, y_{s10}, or y_{s11} in Ω'. Similarly, if t is a proper initial part of $s0$, $s10$, and $s11$ and y_t occurs in Ω', then t is actually an initial part of s and thus the occurrence is after the point of the substitution. We may therefore apply substitution of the form $y_u y_v \Leftrightarrow y_v y_u$ for incompatible u and v in order to move any two distinct occurrences of y_{s0}, y_{s10}, or y_{s11} to the same position in Ω', resulting in a word Ω'' which is now in standard form.

It now suffices to show that $\Omega'' \lhd \Omega$. Since s was not exposed in Ω, each of $s\mathtt{00}$, $s\mathtt{01}$, and $s\mathtt{1}$ has an extension in $T(\Omega)$; recall that, by assumption, $y_{s\mathtt{0}}$ does not occur in Ω. It follows that $t.x_s$ is defined for every element t of $T(\Omega)$ and that $T(\Omega') = \{t.x_s : t \in T(\Omega)\}$. Hence $T(\Omega)$ and $T(\Omega')$ have the same cardinality. Notice that if y_s occurs in Ω'', it occurs in Ω' and hence $T(\Omega')$ dominates $T(\Omega'')$. It follows that $T(\Omega'')$ has cardinality at most that of $T(\Omega)$. If $T(\Omega'')$ has the same cardinality as $T(\Omega)$, then s is the $\leq_{\text{lex}}$-maximal sequence such that the exponent of y_s in Ω and Ω'' differs and in this case, it decreases by one in absolute value. Thus we have shown $\Omega'' \lhd \Omega$. □

Let B denote the set $\{\mathtt{0}, \mathtt{1}, y, y^{-1}\}$ and let $B^{<\mathbb{N}}$ denote the collection of all finite strings of elements of B. If Λ is in $B^{<\mathbb{N}}$ and $\Lambda(i)$ is either y or y^{-1}, we will say that $\Lambda(i)$ is an *occurrence of* $y^{\pm}$. We will use B-words to analyze the evaluation of standard forms at binary sequences. The following symbolic manipulations correspond to the recursive definition of the function $y : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$.

Definition 3.2.7. Suppose that Λ is in $B^{<\mathbb{N}}$. An application of one of the substitutions

$$y\mathtt{00} \Rightarrow \mathtt{0}y \qquad y\mathtt{01} \Rightarrow \mathtt{10}y^{-1} \qquad y\mathtt{1} \Rightarrow \mathtt{11}y$$

$$y^{-1}\mathtt{0} \Rightarrow \mathtt{00}y^{-1} \qquad y^{-1}\mathtt{10} \Rightarrow \mathtt{01}y \qquad y^{-1}\mathtt{11} \Rightarrow \mathtt{1}y^{-1}$$

at an occurrence of $y^{\pm}$ is said to *advance* that symbol. If several advances of occurrences of $y^{\pm}$ are applied to Λ, resulting in Λ', then we say that Λ *can be advanced to* Λ', denoted $\Lambda \Rightarrow \Lambda'$.

Definition 3.2.8. Suppose that Λ is in $B^{<\mathbb{N}}$. An occurrence of $y^{\pm}$ is a *potential cancellation* in Λ if repeatedly advancing it results in an occurrence of the substring yy^{-1} or $y^{-1}y$ in the modified word.

Lemma 3.2.9. *Suppose that Λ is in $B^{<\mathbb{N}}$ and contains no potential cancellations. Then advancing any occurrence of a $y^{\pm}$ results in a word with no potential cancellations.*

Proof. Suppose that Λ is given and that the ith occurrence of $y^{\pm}$ is advanced to create Λ'. The only possibility for introducing a potential cancellation is if $i > 1$ and a potential cancellation occurs at the $i-1$st occurrence of a $y^{\pm}$ in Λ'. Return to Λ and advance the $i-1$st occurrence of $y^{\pm}$ as much as possible to produce Λ''. Suppose for a moment that after advancing, this occurrence is a y; notice that the next symbol is either $\mathtt{0}$ or y. We now have the following cases:

$$yy\mathtt{00} \Rightarrow y\mathtt{0}y$$

$$yy\mathtt{01} \Rightarrow y\mathtt{10}y^{-1} \Rightarrow \mathtt{11}y\mathtt{0}y^{-1}$$

$$yy\mathtt{1} \Rightarrow y\mathtt{11}y \Rightarrow \mathtt{1111}yy$$

$$y\mathtt{0}y\mathtt{00} \Rightarrow y\mathtt{00}y \Rightarrow 0yy$$

$$y\mathtt{0}y\mathtt{01} \Rightarrow y\mathtt{010}y^{-1} \Rightarrow \mathtt{10}y^{-1}0y^{-1} \Rightarrow \mathtt{1000}y^{-1}y^{-1}$$

$$y\mathtt{0}y\mathtt{1} \Rightarrow y\mathtt{011}y \Rightarrow \mathtt{10}y^{-1}\mathtt{1}y$$

$$y\mathtt{0}y^{-1}\mathtt{0} \Rightarrow y\mathtt{000}y^{-1} \Rightarrow \mathtt{0}y\mathtt{0}y^{-1}$$

$$y\mathtt{0}y^{-1}\mathtt{10} \Rightarrow y\mathtt{001}y \Rightarrow \mathtt{0}y\mathtt{1}y \Rightarrow \mathtt{011}yy$$

$$y\mathtt{0}y^{-1}\mathtt{11} \Rightarrow y\mathtt{01}y^{-1} \Rightarrow \mathtt{10}y^{-1}y^{-1}$$

The above lines list the possible contexts for two occurrences of $y^{\pm}$ in the $i-1$st and ith in Λ'' where the first occurrence is positive. In the above cases, the ith occurrence of $y^{\pm}$ is advanced in Λ'' and then the $i-1$st occurrence is advanced as much as possible, demonstrating that a cancellation does not occur. The case in which the $i-1$st occurrence of $y^{\pm}$ in Λ'' is y^{-1} is handled by symmetry — the rules for advancement and potential cancellation are invariant under the following involution:

$$y \Leftrightarrow y^{-1} \qquad \mathtt{0} \Leftrightarrow \mathtt{1}.$$

□

Lemma 3.2.10. *Suppose that Λ is in $B^{<\mathbb{N}}$ and contains no potential cancellations. There is a finite binary sequence u such that $\Lambda^\frown u$ can be advanced to $s^\frown y^n$ for some binary sequence s, where n is the number of occurrences of $y^{\pm}$ in Λ.*

Proof. The proof is by induction on the number of occurrences of $y^{\pm}$ in Λ. If there is only one occurrence, advance the occurrence as many times as possible, resulting in $s^\frown y$, $s^\frown y^{-1}$, $s^\frown y^\frown \mathtt{0}$, or $s^\frown y^{-1\frown}\mathtt{1}$ for some finite binary sequence s. In the first case we are finished; in the remaining cases, the choices $u = \mathtt{10}$, $u = \mathtt{0}$, and $u = \mathtt{0}$ work. Now suppose that Λ contains $n+1$ occurrences of $y^{\pm}$. Induction and Lemma 3.2.9 reduce the general case to the two special cases $y\mathtt{0}y^n$ and $y^{-1}\mathtt{1}y^n$. In these cases, use $u = \mathtt{0}^{2^n}$, observing that $y^n\mathtt{0}^{2^n}$ can be advanced to $\mathtt{0}y^n$. □

Lemma 3.2.11. *If Ω is a sufficiently expanded standard form then either Ω is an X-word or else Ω does not have the same evaluation as an X-word.*

Proof. Notice that it is sufficient to prove the lemma when Ω is a sufficiently expanded standard form which is a Y-word of positive length. Suppose that such an Ω is given and let $g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the evaluation of Ω in G. It will be sufficient to construct finite binary sequences u and v such that the value of g at $u^\frown\xi$ is $v^\frown y^n(\xi)$ for some $n > 0$. This is because if $\xi = \mathtt{0}^{2^n}\mathtt{1}\mathtt{0}^{2^n}\mathtt{1}\ldots$, then the value of g at $u^\frown\xi$ is $v^\frown\mathtt{0}\mathtt{1}^{2^n}\mathtt{0}\mathtt{1}^{2^n}\ldots$, which is not tail equivalent to $u^\frown\xi$.

The finite binary sequence u will be constructed by a recursive procedure. Let $u \restriction i_0$ be the finite binary sequence such that the last entry of Ω is a power of $y_{u \restriction i_0}$. Suppose that $u \restriction i$ has been defined and that $y_{u\restriction i}$ occurs in Ω. If $u \restriction i$ is exposed in Ω, then let $u \restriction l$ be any finite binary sequence extending $u \restriction i$ which witnesses this. Otherwise, define $u(i) = \mathtt{0}$ if $y_{u\restriction i}$ occurs positively in Ω and $u(i) = \mathtt{1}$ if $y_{u\restriction i}$ occurs negatively in Ω. Define Λ to be the result of simultaneously inserting y^n after $s \restriction i$ whenever $y^n_{s\restriction i}$ occurs in Ω. Notice that by the choice of our sequence u, Λ does not contain potential cancellations: any occurrence of y except for the final occurrence of $y^{\pm}$, is followed by $\mathtt{0}y^{\pm}$ and any occurrence of y^{-1} except for the final occurrence of $y^{\pm}$ is followed by $\mathtt{1}y^{\pm}$. It follows from Lemma 3.2.10 that there is a sequence s such that $\Lambda^\frown s$ can be advanced to vy^n for some binary sequence v, where n is the number of occurrences of $y^{\pm}$ in Λ (this number coincides with the number of steps of the recursive procedure above, which is at least 1). Set u to be the concatenation of $u \restriction l$ followed by s.

Recall now that we have assumed that Ω is a Y-word; $g = y^{n_k}_{t_k} \ldots y^{n_1}_{t_1}$.

$$\xi.g = \xi.(y^{n_1}_{t_1} \cdots y^{n_k}_{t_k}) = (\cdots(\xi.y^{n_1}_{t_1}).\cdots).y^{n_k}_{t_k}$$

Let ξ_i be the result of applying the $y^{n_1}_{t_1} \cdots y^{n_{i-1}}_{t_{i-1}}$ to ξ. Observe that if t_{i+1} is an initial part of ξ, then it is still an initial part of ξ_i. This follows from the fact that if $j \le i$, then t_j either extends t_{i+1} or else is incompatible with t_{i+1}. In particular, if t_{i+1} is not an initial part of ξ, then $\xi_{i+1} = \xi_i$. If t_{i+1} is an initial part of ξ_i, then $\xi_{i+1} = t_{i+1}{}^\frown y^{n_i}(\eta_i)$, where $\xi_i = t_i{}^\frown \eta_i$. It follows from $\Lambda^\frown s \Rightarrow vy^n$ that $\xi.g = v^\frown(\eta.y^n)$, where η is such that $\xi = u^\frown \eta$. □

To see that this finishes the proof of the main theorem, suppose that Ω is an S-word which evaluates to the identity function in G. By Lemma 3.2.4, $\Omega \Rightarrow \Omega'$ for some word Ω' in standard form. By Lemma 3.2.6, $\Omega' \Rightarrow \Omega''$ for some word Ω''

which is in standard form and which is sufficiently expanded. In particular, Ω'' is equivalent to Ω by the relations in R; if Ω was an S_0-word, then Ω'' is an S_0 word and the derivation $\Omega \Rightarrow \Omega' \Rightarrow \Omega''$ utilizes only relations in R_0. By Lemma 3.2.11, Ω'' is an X-word. Since R includes a presentation for F, Ω'' can be reduced to the identity using the relations in R_0.

3.3 Normal Forms

Recall that for a given group presentation, a normal form is a canonical, unique choice of a word representing each group element. In this section we shall describe a normal form for $\langle S \mid R \rangle$. This normal form will be useful in the sections to follow, but is also of independent interest since it can be used as a tool to study the group. We will also define a notion of percolating support of a normal form and prove some of its basic properties. This will be crucial in the proof that the higher homotopy groups of X are trivial. First, we define the notions of calculation and a potential cancellation.

Associated with a standard form $f\lambda$ and a sequence $\sigma \in 2^\omega$ is a *calculation*, which is a string in letters $y, y^{-1}, \mathtt{0}, \mathtt{1}$. The evaluation of $f\lambda$ on σ comprises of a prefix replacement that is determined by the transformation $f \in F$ followed by an infinite sequence of applications of the tranformations described by the percolating elements. (Recall that the elements of the set $\{y_s^t \mid s \in 2^{<\mathbb{N}}, s \text{ is nonconstant }, t \in \{1, -1\}\}$ are called percolating elements.) The latter is encoded as an infinite string with letters $y, y^{-1}, 0, 1$. The calculation is equipped with the following substitutions:

$$y\mathtt{00} \to \mathtt{0}y \qquad y\mathtt{01} \to \mathtt{10}y^{-1} \qquad y\mathtt{1} \to \mathtt{11}y$$

$$y^{-1}\mathtt{0} \to \mathtt{00}y^{-1} \qquad y^{-1}\mathtt{10} \to \mathtt{01}y \qquad y^{-1}\mathtt{11} \to \mathtt{1}y^{-1}$$

For example, for the word $y^{-1}_{100}y_{10}$ and the binary sequence 1001111... the calculation string is $10y0y^{-1}1111$.... The output of the evaluation of the word on the binary string is the limit of the strings obtained from performing these substitutions. A calculation has a *potential cancellation* if upon performing a finite set of substitutions we encounter a substring of the form yy^{-1} or $y^{-1}y$. When a calculation has no potential cancellations, we say that it has *exponent* n if n is the number of occurences of the symbols $y^{\pm}$. Note that there is no potential cancellation in the example above, and the exponent is 2.

We extend this notion to standard forms in the following definition.

Definition 3.3.1. (Potential cancellation) A calculation is said to have a *potential cancellation* if an application of a finite sequence of permissible substitutions produces an occurence of a subword yy^{-1} or $y^{-1}y$. A standard form $fy^{t_1}_{s_1}...y^{t_n}_{s_n}$ is said to have a *potential cancellation* if there is an infinite binary sequence τ such that the calculation of $y^{t_1}_{s_1}...y^{t_n}_{s_n}$ on τ contains a potential cancellation.

Definition 3.3.2. (Neighboring pairs) Let $fy^{t_1}_{s_1}...y^{t_n}_{s_n}$ be a standard form. A pair $y^{t_i}_{s_i}, y^{t_j}_{s_j}$ is said to be a *neighboring pair* if:

1. $s_i \neq s_j$, and s_i is an initial segment of s_j, i.e. $s_i \subset s_j$.
2. for any finite binary sequence u such that $s_i \subset u \subset s_j$ we have that $u \neq s_k$ for all $1 \leq k \leq n$.

Such a neighboring pair is said to contain a *potential cancellation* if the associated calculation of the subword $y^{t_j}_{s_j}y^{t_i}_{s_i}$ on some (or any) binary sequence v that extends s_j contains a potential cancellation.

Lemma 3.3.3. *Any standard form can be converted to a standard form with no potential cancellations using the relations.*

Proof. We prove this by induction on the number of percolating elements in the standard form. The base case is trivial. Let this be true for $n-1$. Let $fy_{s_1}^{t_1}...y_{s_n}^{t_n}$ be a standard form such that $t_i \in \{1, -1\}$. We remark that the condition for t_i's is simply a matter of notation, i.e. for example we write y_s^2 as $y_s y_s$.

By the inductive hypothesis it follows that we can convert $fy_{s_1}^{t_1}...y_{s_{n-1}}^{t_{n-1}}$ into a standard form with no potential cancellations using the relations. Moreover, from Lemmas 4.4 and 4.9 in [24] it follows that this can further be converted into a standard form $gy_{p_1}^{q_1}...y_{p_m}^{q_m}$ with no potential cancellations and depth $l > |s_n|$. It follows that $gy_{p_1}^{q_1}...y_{p_m}^{q_m}y_{s_n}^{t_n}$ is a standard form.

Let τ be an infinite binary sequence for which the associated calculation is a potential cancellation. It follows that the potential cancellation must occur between the first two occurences of $y^{\pm,}$ in the associated calculation. In particular, the potential cancellation is produced by a neighboring pair $y_{s_n}^{t_n}, y_{p_i}^{q_i}$ for some $1 \leq i \leq m$. It follows that we can convert $gy_{p_1}^{q_1}...y_{p_m}^{q_m}y_{s_n}^{t_n}$ into a standard form $h(y_{r_1}^{k_1}...y_{r_l}^{k_l})(y_{u_1}^{v_1}...y_{u_o}^{v_o})y_{s_n}^{t_n}$ such that

1. $|r_i| > |u_j|$ for each $1 \leq i \leq l, 1 \leq j \leq o$.
2. Each pair $y_{s_n}^{t_n}, y_{u_j}^{v_j}$ is a neighboring pair that has a potential cancellation.
3. No neighboring pair of the form $y_{s_n}^{t_n}, y_{r_i}^{k_i}$ contains a potential cancellation.

These conditions allow us to perform a sequence of expansion and rearranging substitutions on $y_{s_n}^{t_n}$, followed by cancellations to obtain a standard form with no potential cancellations. □

Definition 3.3.4. A standard form is said to contain a *potential contraction* if either of the following holds.

1. It contains an occurence of a subword of the form $y_{s0}y_{s10}^{-1}y_{s11}$ but no occurences of $y_{s1}^{\pm}$.
2. It contains an occurence of a subword of the form $y_{s00}^{-1}y_{s01}y_{s1}^{-1}$ but no occurences of $y_{s0}^{\pm}$.

If a standard form $fy_{s_1}^{t_1}...y_{s_n}^{t_n}$ contains a potential contraction of the form $y_{s0}y_{s10}^{-1}y_{s11}$ (but no occurences of $y_{s1}^{\pm}$), we can replace the subword with the word $x_s^{-1}y_s$ and then apply a rearranging substitution to move the x_s^{-1} to the left of the percolating elements. Note that since there are no occurences of $y_{s0}^{\pm}$ it follows that the resulting word is a standard form. We call this a *contraction substitution.* The contraction substitution for the other case is defined in a similar way.

Lemma 3.3.5. *Any standard form can be converted into a standard form with no potential contractions and no potential cancellations.*

Proof. By Lemma 3.3.3 we can convert our standard form into a standard form with no potential cancellations. On the resulting standard form, we perform contraction substitutions, one by one, until no contraction substitutions can be performed. This process terminates, since upon performing a contraction substitution we obtain a standard form which is smaller in the well founded ordering on standard forms that was defined in the proof of Lemma 4.6 in [24] (see the second paragraph.) Moreover, it is clear that we do not introduce any potential cancellations in this process. (This is a consequence of Lemma 4.9 in [24].) □

Lemma 3.3.6. *Let $y_u^v y_s^t$ be a standard form such that $v, t \in \{1, -1\}$. Now let $fy_{s_1}^{t_1}...y_{s_n}^{t_n}$ be a standard form such that:*

1. $fy_{s_1}^{t_1}...y_{s_n}^{t_n} = (y_u^v y_s^t)^{-1}$.

2. $fy_{s_1}^{t_1}...y_{s_n}^{t_n}$ *is obtained by performing only expansion substitutions on* y_s^{-t}, *and its offsprings, in the word* $y_s^{-t}y_u^{-v}$.

If $y_u^v y_s^t$ *has no potential cancellation, then* $fy_{s_1}^{t_1}...y_{s_n}^{t_n}$ *has no potential cancellations.*

Proof. By Lemma 4.9 in [24] it suffices to show this for the following cases.

1. $t = 1, u = s0, v = 1$.
2. $t = 1, u = s0, v = -1$.
3. $t = -1, u = s1, v = 1$.
4. $t = -1, u = s1, v = -1$.

Further, by considering the involution $y \leftrightarrow y^{-1}$ $0 \leftrightarrow 1$ described in [24] (see the paragraph before Lemma 4.10 in [24]), it suffices to show this for the first two cases.

Case (1): $(y_{s0}y_s)^{-1} = y_s^{-1}y_{s0}^{-1} = x_s^{-1}y_{s00}^{-1}y_{s01}y_{s1}^{-1}y_{s0}^{-1}$ This word has no potential cancellations.

Case (2):$(y_{s0}^{-1}y_s)^{-1} = y_s^{-1}y_{s0} = x_s^{-1}y_{s00}^{-1}y_{s01}y_{s1}^{-1}y_{s0}$. This word has no potential cancellations. □

Lemma 3.3.7. *Let* $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ *be a standard form with no potential cancellations. The process for converting the word* $(y_{s_1}^{t_1}...y_{s_n}^{t_n})^{-1} = y_{s_n}^{-t_n}...y_{s_1}^{-t_1}$ *into a standard form does not involve any cancellation substitution and produces a standard form with no potential cancellations.*

Proof. We proceed by induction on the length of the word. Assume as before that $t_i \in \{1, -1\}$, and that the inductive hypothesis is satisfied by the subword $(y_{s_1}^{t_1}...y_{s_{n-1}}^{t_{n-1}})^{-1}$. Let the resulting standard form for this word be $fy_{u_1}^{v_1}...y_{u_m}^{v_m}$. It follows that

$$(y_{s_1}^{t_1}...y_{s_n}^{t_n})^{-1} = y_{s_n}^{-t_n}(fy_{u_1}^{v_1}...y_{u_m}^{v_m})$$

Let $y_{s_n}^{-t_n} = gy_{p_1}^{q_1}...y_{p_l}^{q_l}$ be a standard form such that f acts on p_i for each $1 \leq i \leq l$ and $p_i \cdot f = p_i'$ has the property that for each $1 \leq j \leq m$ either $u_j \subset p_i'$ or p_i', u_j are incompatible.

It follows that

$$(y_{s_1}^{t_1}...y_{s_n}^{t_n})^{-1} = gf(y_{p_1'}^{q_1}...y_{p_l'}^{q_l})(y_{u_1}^{v_1}...y_{u_m}^{v_m})$$

and that $gf(y_{p_1'}^{q_1}...y_{p_l'}^{q_l})(y_{u_1}^{v_1}...y_{u_m}^{v_m})$ is a standard form.

Since any potential cancellation is witnessed by a pair of neighboring elements, it suffices to show that no neighboring pair $y_{u_i}^{v_i}, y_{p_j'}^{q_j}$ contains a potential cancellation. This follows from the previous lemma. □

Recall from [13] the normal form for Thompson's group F. Note that this normal form uses the infinite generating set $X' = \{x_0, x_1, x_{11}, x_{111}, ...\} \subseteq X$.

Theorem 3.3.8. *For an element of G, let $fy_{s_1}^{t_1}...y_{s_n}^{t_n}$ be a word representing the group element such that*

1. *It is a standard form with no potential contractions or potential cancellations.*
2. *If $i < j$ then $s_i <_{lex} s_j$.*
3. *$f \in F$ is an X'-word in a normal form for F.*

Such a word exists and is unique. Moreover, for any finite binary sequence s such that $s_i \subset s$ for some $1 \leq i \leq n$ it follows that there is an element of $[s\bar{0}, s\bar{1}]$ on which the action of $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ does not preserve tail equivalence.

Proof. From Lemma 3.3.3 and 3.3.5, it follows that for any given group element we can find a word satisfying the hypothesis of the theorem. It remains to show that this is unique and that it satisfies the property concerning tail equivalence.

By way of contradiction assume that $fy_{s_1}^{t_1}...y_{s_n}^{t_n}, gy_{p_1}^{q_1}...y_{p_m}^{q_m}$ are two standard forms that satisfy conditions (1) and (2) and represent the same element of G_0. We can assume without loss of generality that $s_n <_{\text{lex}} p_m$, or else we can perform a cancellation on the right until we achieve this condition or else contradict our assumption that the words are distinct or represent the same group element. We also assume that $q_m > 0$. (The proof for $q_m < 0$ is similar.) By our assumption it follows that the word $(y_{s_1}^{t_1}...y_{s_n}^{t_n}y_{p_m}^{-1})(y_{p_m}^{-(q_m-1)}y_{p_{m-1}}^{-q_{m-1}}...y_{p_1}^{-q_1}) \in F$.

First observe that since $s_n <_{\text{lex}} p_m$, the word $(y_{s_1}^{t_1}...y_{s_n}^{t_n}y_{p_m}^{-1})$ is in standard form. Moreover, it is true that there is an infinite binary sequence ψ such that the calculation of $y_{s_1}^{t_1}...y_{s_n}^{t_n}y_{p_m}^{-1}$ on the binary sequence $p_m\psi$ does not contain any potential cancellations. If this were not the case, then this would contradict the assumption that the word $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ does not have any potential contractions.

Now let $g_1y_{u_1}^{v_1}...y_{u_k}^{v_k}$ be a standard form for the word $y_{p_m}^{-(q_m-1)}y_{p_{m-1}}^{-q_{m-1}}...y_{p_1}^{-q_1}$. Let l_1, l_2 be sufficiently large numbers such that $l_2 > |u_j|$ for every $1 \leq j \leq m$ and g_1 acts on every finite binary sequence of length at least l_1 to produce a finite binary sequence of length at least l_2.

Now by Lemma 4.4 in [24] we can convert the word $(y_{s_1}^{t_1}...y_{s_n}^{t_n}y_{p_m}^{-1})$ into a standard form $g_2y_{w_1}^{z_1}...y_{w_j}^{z_j}$ with depth greater than l_1 and no potential cancellations. By our observation above it follows that this word is not empty and there is a sequence $w_j\nu$ such that the associated calculation of $y_{w_1}^{z_1}...y_{w_j}^{z_j}$ on this sequence has no potential cancellation.

Now by our assumptions about depth it follows that g_1 acts properly on $w_1, ..., w_j$, and that

$$(g_2 y_{w_1}^{z_1}...y_{w_j}^{z_j})(g_1 y_{u_1}^{v_1}...y_{u_k}^{v_k})$$
$$= g_2 g_1 (y_{w_1\cdot g_1}^{z_1}...y_{w_j\cdot g_1}^{z_j})(y_{u_1}^{v_1}...y_{u_k}^{v_k})$$

is a standard form.

By Lemma 3.3.7 it follows that the calculation of the standard form

$$(y_{w_1\cdot g_1}^{z_1}...y_{w_j\cdot g_1}^{z_j})(y_{u_1}^{v_1}...y_{u_k}^{v_k})$$

on the sequence $(w_j \cdot g_1)\nu$ has at least an occurence of y, y^{-1} and no potential cancellations. By Lemmas 4.10 and 4.11 from [24] this contradicts the assumption that this word is an element of F. So the original standard forms must have been the same to begin with.

The statement about tail equivalence follows from the no potential cancellation assumption together with Lemmas $4.10, 4.11$ from [24]. □

The following is immediate.

Corollary 3.3.9. *Let $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ be a Y-word in normal form. Then if $y_{u_1}^{v_1}...y_{u_m}^{v_m}$ is a Y-word in normal form such that $y_{u_1}^{v_1}...y_{u_m}^{v_m}$ is an element of the right coset $F(fy_{s_1}^{t_1}...y_{s_n}^{t_n})$ then it follows that $y_{u_1}^{v_1}...y_{u_m}^{v_m} = y_{s_1}^{t_1}...y_{s_n}^{t_n}$.*

This provides a normal form representative for right cosets of F in G. This normal form will be useful in the next chapter.

CHAPTER 4

HIGHER FINITENESS PROPERTIES

4.1 Right cosets of F in G and Special forms

In this section we will define *special forms*, which are standard forms that will play a crucial role in both the definition our complex as well the group action.

Definition 4.1.1. Let Ω be the set of right cosets of F in G. The group G acts naturally on Ω by right multiplication. A Y-standard form $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ is called a *special form* if the following holds:

1. Each pair s_i, s_{i+1} is a consecutive pair of finite binary sequences. (i.e. there is a sequence $u \in 2^{<\mathbb{N}}$ and numbers $n_1, n_2 \in \mathbb{N} \cup \{0\}$ such that $s_i = u\mathtt{0}\mathtt{1}^{n_1}, s_{i+1} = u\mathtt{1}\mathtt{0}^{n_2}$.)
2. For each $1 \leq j \leq n$, $t_j \in \{1, -1\}$ and $t_{j+1} = (-1)t_j$.

Some basic examples of special forms are y_s, y_s^{-1}, $y_{s0}y_{s10}^{-1}$, and $y_{s0}y_{s10}^{-1}y_{s11}$. A special form $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ is of *type 1* if $t_1 = -1$ and *type 2* if $t_1 = 1$. The *parity* of the special form is the parity of n. In particular, the special form is said to have *odd* or *even* length if n is odd or even.

If $\lambda = y_{s_1}^{t_1}...y_{s_n}^{t_n}$ is a special form, then $\lambda^{-1} = y_{s_n}^{-t_n}...y_{s_1}^{-t_1}$ is not a special form. However since $y_{s_i}^{-t_i}, y_{s_j}^{-t_j}$ commute in G, we have that the group element described by λ^{-1} can be represented by the special form $y_{s_1}^{-t_1}...y_{s_n}^{-t_n}$. Note that if λ is of type 1 then this special form representation of λ^{-1} is of type 2 and vice versa.

Definition 4.1.2. Let $\Gamma \subset \Omega$ be the cosets that contain elements that can be represented by special forms. If τ_1, τ_2 are special forms such that $F\tau_1 = F\tau_2$ then we say that $\tau_1 \sim \tau_2$, and that τ_1, τ_2 are *equivalent special forms.*

We describe some basic manipulations of special forms.

Definition 4.1.3. Given a special form $\lambda = y_{s_1}^{t_1}...y_{s_n}^{t_n}$, an *expansion substitution* at $y_{s_i}^{t_i}$ entails replacing $y_{s_i}^{t_i}$ in λ by $y_{s_i 0}y_{s_i 10}^{-1}y_{s_i 11}$ if $t_i = 1$ or replacing $y_{s_i}^{t_i}$ by $y_{s_i 00}^{-1}y_{s_i 01}y_{s_i 1}^{-1}$ if $t_i = -1$. Replacing $y_{s0}y_{s10}^{-1}y_{s11}$ by y_s or replacing $y_{s00}^{-1}y_{s01}y_{s1}^{-1}$ by y_s^{-1} is called a *contraction substitution.*

The following is a basic property of special forms.

Lemma 4.1.4. *The following conditions hold for special forms.*

1. *Performing contraction and expansion substitutions on a special form produces special forms that belong to the same coset.*
2. *Given any pair of special forms that belong to the same coset, one can be obtained from the other by performing a sequence of expansion and contraction substitutions.*

Proof. Let $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ be a special form. If a substitution $y_s \to x_s y_{s0}y_{s10}^{-1}y_{s11}$ (or $y_s^{-1} = x_s^{-1}y_{s00}^{-1}y_{s01}y_{s1}^{-1}$) is performed, then x_s (or x_s^{-1}) commutes with every percolating element of $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ that occurs to its left in the word. This means that x_s (or x_s^{-1}) can be deleted from the word to obtain a special form that lies in the same coset.

Let $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ and $y_{p_1}^{q_1}...y_{p_m}^{q_m}$ be special forms that lie in the same coset. By applying expansion rules we can assume that whenever s_i, p_j are compatible, s_i extends

p_j. By our assumption $(y_{s_1}^{t_1}...y_{s_n}^{t_n})(y_{p_1}^{q_1}...y_{p_m}^{q_m})^{-1} = (y_{s_1}^{t_1}...y_{s_n}^{t_n})(y_{p_1}^{-q_1}...y_{p_m}^{-q_m})$ can be reduced to an X-word by performing substitutions. Using the procedure in Chapter 3, we perform expansion substitutions on the letters $y_{p_i}^{q_i}$ and their offsprings one by one, until all the offsprings are cancelled. This means that the special form $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ can be obtained from $y_{p_1}^{q_1}...y_{p_m}^{q_m}$ by performing expansion substitutions. □

Corollary 4.1.5. *The parity and type of special forms that lie in the same coset is the same.*

The following lemma produces a unique canonical choice of special form for each equivalence class of Γ. This is an immediate corollary of Theorem 3.3.8.

Lemma 4.1.6. *If λ is a special form then there is a unique special form τ in $F\lambda$ such that any special form λ' in $F\lambda$ can be derived from τ by expansion substitutions. Moreover, given a special form $\psi_1 \in F\lambda$ there is a special form $\psi_2 \in F\lambda$ that can be obtained from both ψ_1, λ by expansion substitutions.*

Proof. Given λ, we perform a sequence of contraction substitutions, one by one, whenever possible until no contraction substitution can be performed. This produces a special form $\tau \in F\lambda$ that is unique by Theorem 3.3.8. For the second statement of the Lemma, observe that we can perform a finite sequence of expansion substitutions on τ to obtain the required special form ψ_2. □

We now discuss the action of F on special forms.

Definition 4.1.7. Let $\nu = y_{s_1}^{t_1}...y_{s_n}^{t_n}$ be a special form. We say that an element $f \in F$ *acts on* ν if it acts on each sequence in the set $\{s_1, ..., s_n\}$. In particular, we obtain that

$$(F\nu) \cdot f = F(y_{s_1}^{t_1}...y_{s_n}^{t_n} f) = F(y_{f(s_1)}^{t_1}...y_{f(s_n)}^{t_n})$$

We remark that if in the above f does not act on ν then we can use the relations $y_s \to x_s y_{s0} y_{s10}^{-1} y_{s11}$ and $y_s^{-1} = x_s^{-1} y_{s00}^{-1} y_{s01} y_{s1}^{-1}$ to obtain a word $g\nu' = \nu$ such that:

1. $g \in F$ and ν' is a special form.
2. f acts on ν' and $\nu' f = h\nu''$ for a special form ν''.

It follows that

$$\nu f = g\nu' f = gh\nu''$$

and so in particular $F\nu \cdot f = F\nu''$.

Lemma 4.1.8. *Consider the action of F on Ω. The following conditions hold.*

1. *Γ is invariant under the action of F on Ω. Also, the type and parity of the special forms representing the equivalence class is preserved under this action.*
2. *The action of F on elements Γ of the same type and parity is transitive. The action on Γ has precisely four orbits, one for each combination of type and parity.*

Proof. If τ is a special form and $f \in F$ then by performing expansion substitutions on τ we obtain special form $\tau' = y_{s_1}^{t_1}...y_{s_n}^{t_n}$ such that $\tau' \in F\tau$, f acts on $s_1, ..., s_n$ and $F\tau \cdot f = F(y_{f(s_1)}^{t_1}...y_{f(s_n)}^{t_n})$. Now $y_{f(s_1)}^{t_1}...y_{f(s_n)}^{t_n}$ is a special form since the conditions on the t_i's hold and $f(s_1), ..., f(s_n)$ are consecutive leaves of a finite binary tree if $s_1, ..., s_n$ have this property. Observe that the type and parity are preserved under the action of F.

Now we prove the second claim. Let $y_{s_1}^{t_1}...y_{s_m}^{t_m}, y_{p_1}^{q_1}...y_{p_n}^{q_n}$ be two special forms that are of the same type and parity. This means that in particular $t_i = q_i$ for $1 \leq i \leq n$.

By performing expansion substitutions we can assume that $n = m$. There is an $f \in F$ such that $f(s_i) = p_i$ for $1 \le i \le n$. So we get $F(y_{s_1}^{t_1}...y_{s_n}^{t_n}) \cdot f = F(y_{p_1}^{q_1}...y_{p_n}^{q_n})$. Our claim follows from this and the observation that the type and parity of a special form are invariant under the action of F. □

The following is an immediate consequence.

Corollary 4.1.9. *Let $f \in F$ and $\lambda_1 \in G$. Then $F(\lambda f) \in \Gamma$ if and only if $F\lambda \in \Gamma$.*

Now we study the stabilizers of the action of G, F on Ω. First we make a simple observation. For each $\lambda \in G$,

$$\mathrm{Stab}_G(F\lambda) = \{\lambda^{-1} f \lambda \mid f \in F\} \cong F$$

Now we describe the F-stabilizers of special forms. Recall the Higman-Thompson groups $F_{3,r}$ [10]. It is well known that these are of type F_∞ [10].

Lemma 4.1.10. *If λ is a special form then* $\mathrm{Stab}_F(F\lambda) \cong F_3 \times F \times F$.

Proof. Let $\tau \in F\lambda$ be the special form from Lemma 4.1.6, i.e. the unique special form satisfying the property that if $\lambda' \in F\lambda$ is a special form then λ' can be obtained from τ by performing expansion substitutions. Let $f \in \mathrm{Stab}_F(F\lambda)$. Now $F\lambda \cdot f = F\nu$ where $\lambda f = g\nu$ for $g \in F$ for a special form ν. By our assumption $\nu \sim \lambda$, and hence ν can be obtained from τ by performing expansion substitutions.

The set of special forms obtained by performing expansion substitutions on $\tau = y_{s_1}^{t_1}...y_{s_n}^{t_n}$ is in a natural bijective correspondence with finite 3-branching forests with n roots.

Let F_τ be the set of elements of F that fix the binary sequences $s_1, ..., s_n$. We note that F_τ forms a group under composition which is isomorphic to $F \times F$. (Recall that sequences of the form 0^k or 1^k cannot occur as subscripts of the percolating elements.)

Let F'_τ be the set of elements $f \in F$ such that $F\tau \cdot f = F\tau$ and f fixes each binary sequence that does not extend a sequence in the set $\{s_1, ..., s_n\}$. There are special forms $\nu_1 = y_{p_1}^{q_1}...y_{p_m}^{q_m}, \nu_2 = y_{h_1}^{j_1}...y_{h_m}^{j_m}$ which can be obtained from τ by performing expansion substitutions such that $f(p_i) = h_i$. A tree diagram for an element $f \in F'_\tau$ can be obtained by considering a tree pair (P, Q) where $p_1, ..., p_m$ occur as leaves of P, $h_1, ..., h_m$ occur as leaves of Q, p_i is mapped to h_i and every leaf of P that is not in the set $\{p_1, ..., p_m\}$ is mapped to a leaf with the same address in Q. Let P', Q' be the forest of subtrees of P, Q with roots $s_1, ..., s_n$ that are obtained by deleting every vertex that has an address that does not extend $s_1, ..., s_n$. The tree pair P', Q' describes an element of $F_{2,n}$ which naturally corresponds to an element of $F_{3,n}$ in the following way: The expansion substitution $y_s \to y_{s0}y_{s10}^{-1}y_{s11}$ (or $y_s^{-1} = y_{s00}^{-1}y_{s01}y_{s1}^{-1}$) corresponds naturally to a "3-branching" that is uniquely determined by the exponent of y_s. So the trees P', Q' correspond naturally to a tree pair U, V in $F_{3,n}$. This correspondence is well defined under composition and inverses and hence produces a group homomorphism between F'_τ and $F_{3,n}$ which is easily seen to be a group isomorphism. Recall that for each $n \in \mathbb{N}$, $F_{3,n} \cong F_{3,1} = F_3$ and $F_{2,n} \cong F_{2,1} = F_2 = F$. The stabilizer $\mathrm{Stab}_F(F\tau)$ has a natural product decomposition as

$$F_\tau \times F'_\tau \cong F_3 \times F \times F$$

□

Definition 4.1.11. Two special forms $\tau_1 = y_{s_1}^{t_1}...y_{s_n}^{t_n}, \tau_2 = y_{p_1}^{q_1}...y_{p_n}^{q_n}$ are said to be *independent* if each pair s_i, p_j is *incompatible.*

The definition of incompatibility extends naturally to cosets of special forms.

We make this precise in the next lemma.

Lemma 4.1.12. *If ν_1, ν_2 are independent special forms and λ_1, λ_2 are special forms such that $F\lambda_1 = F\nu_1, F\lambda_2 = F\nu_2$, then λ_1, λ_2 are independent special forms.*

Proof. This follows immediately from Lemma 4.1.4 and the observation that $y_s^{\pm}, y_t^{\pm}$ are incompatible if and only if the pairs

$$\{s, t0\}, \{s, t00\}, \{s, t10\}, \{s, t01\}, \{s, t1\}, \{s, t11\}$$

are all incompatible. □

The following lemma is a straightforward generalization of Lemma 4.1.10.

Lemma 4.1.13. *Let $\tau_1, ..., \tau_n$ be special forms that are pairwise independent. Then $\bigcap_{1 \leq i \leq n} Stab_F(F\tau_i)$ is a finite product of copies of F and F_3 and in particular is of type F_∞.*

Now we discuss some more basic properties of products of special forms. These properties will be crucial in the construction of the cell complex X in the next section.

Definition 4.1.14. Let $\tau_1, ..., \tau_n$ be a list of special forms that are pairwise independent. Such a list is said to be *sorted* if for each $1 \leq i < n$

$$\tau_i = y_{s_1}^{t_1} ... y_{s_l}^{t_l} \qquad \tau_{i+1} = y_{p_1}^{q_1} ... y_{p_m}^{q_m}$$

has the property that $s_l <_{\text{lex}} p_1$. (In particular, $s_k <_{\text{lex}} p_j$ for each $1 \leq k \leq l, 1 \leq j \leq m$.)

We say that a sorted list of special forms is *consecutive* if for each $1 \leq i < n$, $\tau_i = y_{s_1}^{t_1} ... y_{s_l}^{t_l}, \tau_{i+1} = y_{p_1}^{q_1} ... y_{p_m}^{q_m}$ satisfies that $s_l <_{\text{lex}} p_1$ and s_l, p_1 are consecutive.

We say that a sorted list of special forms is of *alternating sign* if for each $1 \leq i < n$, $\tau_i = y_{s_1}^{t_1}...y_{s_l}^{t_l}, \tau_{i+1} = y_{p_1}^{q_1}...y_{p_m}^{q_m}$ satisfy that $t_l = -q_1$.

The following lemmas follow immediately from the definitions.

Lemma 4.1.15. *If $\tau_1, ..., \tau_n$ is an sorted list of pairwise independent special forms then $\tau = \tau_1...\tau_n$ is a special form if and only if our list is consecutive and of alternating sign.*

Lemma 4.1.16. *Let $\lambda_1, ..., \lambda_n$ be special forms such that $\lambda_1...\lambda_n$ is a special form. Let τ be a special form and $1 \leq i < n$ be such that λ_i, τ are consecutive and of alternating type. Then the Y-word $\tau\lambda_1...\lambda_n$ can be converted into a standard form $f\nu$ with the property that there is an infinite binary string u such that the calculation of $f\nu$ on u contains no potential cancellations and two occurrences of y or y^{-1}.*

Proof. Let $\lambda_i = y_{s_1}^{t_1}...y_{s_l}^{t_l}$, $\lambda_{i+1} = y_{s_{l+1}}^{t_{l+1}}....y_{s_n}^{t_n}$ and τ. By performing expansion substitutions we obtain $\tau = fy_{u_1}^{v_1}...y_{u_m}^{v_m}$ where $f \in F$ and $u_1 = s_{l+1}v$ where v is a finite binary string consisting only of 0's. By our assumptions it follows that $t_l = -t_{l+1} = -v_1$ and that the calculation of the associated standard form $f\nu$ on the sequence $s_{l+1}\bar{0}$ has no potential cancellation and two occurences of y^{v_1}. $\square$

Lemma 4.1.17. *Let $\nu_1, ..., \nu_n$ and $\lambda_1, ..., \lambda_n$ be sorted lists of pairwise independent special forms. Then there is an $f \in F$ such that $F\nu_i \cdot f = F\lambda_i$ if and only if for each $1 \leq i \leq n$ the following holds:*

1. *The forms ν_i has the same type and parity as λ_i.*
2. *For $i < n$, the pair ν_i, ν_{i+1} is consecutive if and only if λ_i, λ_{i+1} is consecutive.*

Proof. We proceed by induction on n. The case $n = 1$ follows from Lemma 4.1.8. The action of F preserves the type and parity. First we observe that the conditions are necessary. Let ψ_1, ψ_2 be special forms and $f \in F$. Let $\psi_i' \sim \psi_i$ be special forms such that f acts on ψ_i' and $\psi_i' f = f\psi_i''$ for a special form ψ_i''. It follows that ψ_1', ψ_2' are independent and sorted and that $F\psi_i \cdot f = F\psi_i''$. By Lemma 4.1.8 we know that ψ_i'' has the same type and parity as ψ_i.

Now ψ_1', ψ_2' are consecutive if and only if ψ_1, ψ_2 are consecutive. Let

$$\psi_1' = y_{s_1}^{t_1}...y_{s_l}^{t_l}, \psi_2' = y_{u_1}^{v_1}...y_{u_m}^{v_m}$$

Then $s_1, ..., s_l, u_1, ..., u_m$ are consecutive leaves in a finite rooted binary tree if and only if $f(s_1), ..., f(s_l), f(u_1), ..., f(u_m)$ have this property. It follows that ψ_1'', ψ_2'' are consecutive if and only if ψ_1, ψ_2 are consecutive.

To see that the conditions are sufficient, observe that by applying expansion substitutions we can find special forms $\lambda_i' \sim \lambda_i, \nu_i' \sim \nu_i$ such that λ_i' and ν_i' have the same number of percolating elements. The conditions allow us to construct an element of F which maps the indices of the percolating elements of λ_i' (which are finite binary sequences) to the indices of the percolating elements ν_i' in the $<_{\text{lex}}$ order simultaneously for each i. □

4.2 The complex X

In this section we will build a CW complex X and an action of G on X by cell permuting homeomorphisms. First we will describe the 1-skeleton of X. Then we will introduce a class of finite subgraphs of $X^{(1)}$ called *clusters*.

Definition 4.2.1. We define $X^{(0)} = \Omega$. For $\lambda_1, \lambda_2 \in G$ the 0-cells $F\lambda_1, F\lambda_2$ of

$X^{(0)}$ are connected by a 1-cell in $X^{(1)}$ if and only if $F(\lambda_1\lambda_2^{-1}) \in \Gamma$. By Lemma 4.1.9 this condition is well defined for any choice of coset representatives of $F\lambda_1, F\lambda_2$. The right action of G on $X^{(0)}$ extends to a right action of G on $X^{(1)}$.

Recall the discussion about vertex transitive graphs from subsection 2.4. Let $\Psi = \{y_{10}, y_{10}^{-1}, y_{100}y_{101}^{-1}, y_{100}^{-1}y_{101}\}$. For $\lambda_1, \lambda_2 \in G$ the condition $F\lambda_1\lambda_2^{-1} \in \Gamma$ is equivalent to the condition that $\lambda_1\lambda_2^{-1} \in F\Psi F$. This means that $X^{(1)} = \mathrm{Cos}(G, F, \Psi)$ by Theorem 2.4.1 $X^{(1)}$ is vertex transitive.

It will often be convenient to choose a coset representative $f\lambda$ of a 0-cell which is a standard form, with $f \in F$ and λ a Y-standard form.

Now we will show that the action of G on $X^{(1)}$ is cocompact and the stabilizers of cells are of type F_∞. The action of G on the 0-cells is easily seen to be transitive. We now study the orbits of the action on the 1-cells.

Lemma 4.2.2. *The action of G on the set of 1-cells of $X^{(1)}$ has precisely two orbits.*

Proof. Given a 1-cell $\{F\nu_1, F\nu_2\}$,

$$\{F\nu_1, F\nu_2\} \cdot \nu_1^{-1} = \{F, F\nu_2\nu_1^{-1}\}$$

Further, if $\{F, F\lambda\}$ is a 1-cell and λ is a type 2 special form, then $\{F, F\lambda\} \cdot \lambda^{-1} = \{F\lambda^{-1}, F\}$ is an 1-cell incident to F and λ^{-1} is a type 1 special form. This means that any 1-cell contains in its G-orbit a 1-cell $\{F, F\tau\}$ where τ is of type 1.

So it suffices to show that given $\{F, F\lambda_1\}$, $\{F, F\lambda_2\}$, where λ_1, λ_2 are special forms of type 1 there is an $f \in F$ such that $\{F \cdot f, F\lambda_1 \cdot f\} = \{F, F\lambda_2\}$ if and only if λ_1, λ_2 have the same parity. This is true by Lemma 4.1.8. □

Lemma 4.2.3. *For each $\tau \in X^{(0)}, e \in X^{(1)}$ the following is true:*

1. $\text{Stab}_G(v)$ *is isomorphic to* F.
2. $\text{Stab}_G(e)$ *is isomorphic to* $F_3 \times F \times F$.

Proof. The fact (1) is true by construction. For (2), first observe that $\text{Stab}_G(\{F\lambda_1, F\lambda_2\})$ is conjugate in G to $\text{Stab}_G(\{F\lambda_1\lambda_2^{-1}, F\})$.

So it suffices to understand the stabilizer of a 1-cell $\{F\lambda, F\}$. where λ is a special form. If $F\lambda \cdot \nu = F$ then $\lambda\nu = f$ for some $f \in F$ and $\nu = \lambda^{-1}f$. We can perform expansion substitutions on percolating elements of λ^{-1} to obtain a special form ψ such that:

1. $\lambda^{-1} = f_1\psi$ for $f_1 \in F$.
2. f acts on ψ.
3. $\psi f = f\psi'$ for a special form ψ'. Note that ψ' must have the same parity and type as both ψ and λ^{-1}.

So $F \cdot \nu = F\psi' \neq F\lambda$, since λ, ψ' have different types. This means that if $\nu \in G$ has the property that $\{F\lambda, F\} \cdot \nu = \{F\lambda, F\}$ then in fact $F\lambda \cdot \nu = F\lambda, F\emptyset \cdot \nu = F$. So it follows that

$$\text{Stab}_G(\{F\lambda, F\}) = \text{Stab}_G(F\lambda) \cap F = \text{Stab}_F(F\lambda) \cong F_3 \times F \times F$$

by Lemma 4.1.10. □

Now we define a class of finite subgraphs of $X^{(1)}$ called *clusters*. Recall the definition of a sorted list of special forms from the previous section. (Special forms in a sorted list are in particular pairwise independent.) Also recall that special forms τ_1, τ_2 are said to be equivalent, or $\tau_1 \sim \tau_2$ if $F\tau_1 = F\tau_2$.

Definition 4.2.4. Let $\lambda_1, ..., \lambda_n$ be a sorted list of pairwise independent special forms and $\tau \in G$. The subgraph of $X^{(1)}$ consisting of 0-cells in the set

$$I = \{F(\prod_{i \in A} \lambda_i)\tau \mid A \subseteq \{1, ..., n\}\}$$

and 1-cells

$$J = \{e \mid e = \{u, v\} \text{ is a 1-cell of } X^{(1)} \text{ such that } u, v \in I\}$$

is called an *n-cluster*. For this *parametrization* of the n-cluster the group element τ is said to be the *basepoint*, $\lambda_1, ..., \lambda_n$ are said to be *parameters*, and the 0-cell $F\tau$ the *base-vertex*. Informally, this parametrization of the n-cluster is said to be *based at* τ and *parametrized by* $\lambda_1, ..., \lambda_n$.

Observe that each 1-cell in $X^{(1)}$ is a 1-cluster. In particular any 1-cell e in $X^{(1)}$ can be described as $\{F\lambda\tau, F\tau\}$ by a pair λ, τ where λ is a special form and $\tau \in G$. We remark that if $f \in F$ the pairs λ, τ and $\lambda, f\tau$ do not necessarily describe the same 1-cell since $F\lambda(f\tau), F\lambda\tau$ can be distinct 0-cells even though $F(f\tau) = F\tau$.

Definition 4.2.5. Consider the cluster in the previous definition. If $\tau = f\tau'$ for $f \in F$ and $\lambda_i f = f_i \lambda'_i$ for $f_i \in F$ and special forms λ'_i, then this cluster has the following *equivalent parametrization*: It is based at τ' and parametrized by $\lambda'_1, ..., \lambda'_n$. These parametrizations are related by the fact that $\tau \sim \tau'$.

Another equivalent parametrization is of the following form: Let $\tau' = (\prod_{i \in A} \lambda_i)\tau$ for some $A \subseteq \{1, ..., n\}$. Then the cluster based at τ' is parametrized by $\nu_1, ..., \nu_n$ where $\nu_i = \lambda_i$ if $i \notin A$ and $\nu_i = \lambda_i^{-1}$ if $i \in A$ is the same as above.

Moreover, a third equivalent parametrization can be obtained by combining the first with the second, i.e when the basepoint τ' satisfies that $\tau' \sim (\prod_{i\in A}\lambda_i)\tau$ for some $A \subseteq \{1, ..., n\}$.

Definition 4.2.6. Let Δ be a cluster based at τ with parameters $\lambda_1, ..., \lambda_n$. We say that the parametrization is *balanced* if $\lambda_1, ..., \lambda_n$ are of alternating sign. If this is a balanced parametrization and λ_1 is of type 1 (or type 2), then τ is said to be a type 1 (or a type 2) basepoint for Δ. We say that this parametrization is *consecutive* if $\lambda_1, ..., \lambda_n$ are consecutive.

In the next lemma we show that these notions are invariant under equivalent parametrizations of clusters provided the new basepoint is $\sim$-equivalent to the old basepoint.

Lemma 4.2.7. *Let Δ be a cluster at τ with parameters $\lambda_1, ..., \lambda_n$. Let $\tau = f\tau'$ and ν_i be special forms such that $\lambda_i f = g_i f \nu_i$ for $g_i \in F$. Then the following holds:*

1. *$\nu_1, ..., \nu_n$ is an sorted list of pairwise independent special forms.*
2. *The equivalent parametrization based at $f\tau'$ and parametrized by $\nu_1, ..., \nu_n$ is either consecutive or of alternating sign if and only if the original parametrization is respectively consecutive or of alternating sign.*

Proof. For any $A \subseteq \{1, ..., n\}$ observe that

$$(\prod_{i\in A}\lambda_i)\tau = (\prod_{i\in A}\lambda_i)f\tau'$$
$$= (\prod_{i\in A} g_i)f(\prod_{i\in A}\nu_i)\tau'$$

such that g_i is an element of F that is a product of words x_s, x_s^{-1} obtained by replacing y_s by $x_s y_{s0} y_{s10}^{-1} y_{s11}$ or y_s^{-1} by $x_s^{-1} y_{s00}^{-1} y_{s01} y_{s1}^{-1}$ where y_s or y_s^{-1} is a percolating element of λ_i or offsprings of such percolating elements. Clearly each g_i commutes with λ_j for each $j \neq i$. In particular,

$$F((\prod_{i\in A}\lambda_i)\tau) = F((\prod_{i\in A}\nu_i)\tau')$$

Recall that the action of F on sorted lists of pairwise independent special forms preserves the properties of consecutive and alternating sign. So $\nu_1, ..., \nu_n$ are consecutive or of alternating sign if and only if $\lambda_1, ..., \lambda_n$ have the respective property. □

We may occasionally drop the reference to n above and call such a subgraph a *cluster*. Clusters will be denoted by the symbol Δ, often with subscripts. The set of clusters are ordered by inclusion.

Lemma 4.2.8. *For any cluster there are precisely two balanced parametrizations. A cluster base-vertex $F\tau$ with a balanced parametrization by basepoint τ and parameters $\lambda_1, ..., \lambda_n$ also admits the base-vertex $F(\prod_{i\in\{1,...,n\}}\lambda_i)\tau$ and a balanced parameterization with basepoint $(\prod_{i\in\{1,...,n\}}\lambda_i)\tau$ and parameters $\lambda_1^{-1}, ..., \lambda_n^{-1}$. Moreover, this cluster does not admit any other base-vertex for which there is a balanced parametrization.*

Proof. Recall that a special form τ is of type 1 if and only if τ^{-1} is of type 2. Let τ be a basepoint of a cluster parametrized by $\lambda_1, ..., \lambda_n$. There is a unique set of numbers $l_1, ..., l_n \in \{1, -1\}$ such that $\prod_{1\leq i\leq n}\lambda_i^{l_i}$ and $\prod_{1\leq i\leq n}\lambda_i^{-l_i}$ are balanced. Our assertion follows immediately. □

4.2.1 Subclusters

The set of clusters is ordered by inclusion as subgraphs of $X^{(1)}$. In this subsection we study subclusters of clusters.

Definition 4.2.9. A cluster Δ_1 is said to be a *subcluster* of a cluster Δ_2 if there is an inclusion $\Delta_1 \subseteq \Delta_2$ as graphs in $X^{(1)}$.

First we show that if we have clusters Δ_1, Δ_2 such that $\Delta_1 \subseteq \Delta_2$ as graphs, then there is a natural inclusion of these clusters in the sense of parameters. We then proceed to define some types of subclusters and show that any subcluster of a cluster must be of one of these types.

Lemma 4.2.10. *Let Δ_1, Δ_2 be clusters such that $\Delta_1 \subseteq \Delta_2$. By reparametrizing if necessary, we can assume that they have a common basepoint τ and parameters $\lambda_1, ..., \lambda_n$ and $\nu_1, ..., \nu_m$ respectively (not necessarily balanced). Then there are $C_1, ..., C_n \subseteq \{1, ..., m\}$ that have the following properties:*

1. *The sets $C_1, ..., C_n$ are pairwise disjoint.*
2. *Each $C_i = [a_i, b_i] \cap \{1, ..., m\}$ such that $b_i < a_j$ for $i < j$.*
3. *$(\prod_{j \in C_i} \nu_j)$ are special forms.*
4. *$\lambda_i \sim (\prod_{j \in C_i} \nu_j)$.*

Proof. Let $e_1, ..., e_n$ be the 1-cells incident to $F\tau$ in Δ_1 such that $e_i = \{F(\lambda_i \tau), F\tau\}$. Since these 1-cells belong in Δ_2 as well, for each edge e_i there is a set $C_i \subseteq \{1, ..., m\}$ such that $\prod_{j \in C_i} \nu_i \sim \lambda_i$ and $e_i = \{F((\prod_{j \in C_i} \nu_i)\tau), F\tau\}$. By Lemma 4.1.12 it follows that $C_i \cap C_j = \emptyset$. Now $\prod_{j \in C_i} \nu_i \sim \lambda_i$ is a special form and so the elements of each set $\{\nu_j \mid j \in C_i\}$ listed in increasing order of index must be consecutive and of alternating sign. In particular the conditions (2) and (3) must hold. The order $b_i < a_j$ for $i < j$ in condition (2) is satisfied since both $\lambda_1, ..., \lambda_n$ and $\nu_1, ..., \nu_m$ are sorted lists. $\square$

Now we introduce the *types* of subclusters.

Definition 4.2.11. Let Δ be a cluster at τ parametrized by $\lambda_1, ..., \lambda_n$. A subcluster Δ' of Δ is a *face* of Δ if there are sets $A, B \subseteq \{1, 2, ..., n\}$ such that:

1. $A \cap B = \emptyset$.
2. Δ' has a basepoint $\tau' = (\prod_{i \in B} \lambda_i)\tau$.
3. The parameters of Δ' are elements of the set $\{\lambda_i \mid i \in A\}$.

If $\{1, ..., n\} \setminus A$ is a singleton then Δ' is a *maximal* face.

Definition 4.2.12. Let Δ be a cluster at τ parametrized by $\lambda_1, ..., \lambda_n$. A subcluster Δ' of Δ is a *diagonal subcluster* if:

1. Δ' has a basepoint at τ.
2. There are numbers $1 = a_0 < a_1 < a_2 ... < a_{m-1} < a_m = n + 1$ such that Δ' is parametrized by $\nu_1, ..., \nu_m$ where $\nu_i = \prod_{a_{i-1} \leq i < a_i} \lambda_i$.

We remark that each ν_i must be a special form and so the terms in the product $\prod_{a_{i-1} \leq i < a_i} \tau_i$ must be consecutive special forms of alternating type.

To motivate the above definition, imagine a unit Euclidean square that is subdivided to include a diagonals at a specified pair of opposite points. If we take a product of three such squares, the product contains as a subcomplex the product of the three diagonals. So this cube is sitting "diagonally" inside the product which is a subdivided 6-cube.

Let Δ be a cluster at τ that has a balanced parametrization by $\lambda_1, ..., \lambda_n$. Let $A, B \subseteq \{1, ..., n\}$ be such that the vertices $F \prod_{i \in A} \lambda_i \tau$ and $F \prod_{i \in B} \lambda_i \tau$ are connected by an edge. This means that $(\prod_{i \in A} \lambda_i)(\prod_{j \in B} \lambda_j)^{-1}$ can be reduced to a special form using the relations. Since $\lambda_1, ..., \lambda_n$ is a balanced parametrization, it must be the case that either $A \setminus B$ or $B \setminus A$ is empty. Furthermore, in the former case, from the definition of balanced and alternating type it must be true that $A \setminus B = [i, j] \cap \{1, ..., n\}$ (and in the latter case that $B \setminus A = [i, j] \cap \{1, ..., n\}$) for some $1 \leq i < j \leq n$.

Now we will show that the set of types for subclusters that we have defined is exhaustive.

Lemma 4.2.13. *Let Δ be a cluster at τ parametrized by $\lambda_1, ..., \lambda_n$. A subcluster Δ' of Δ is either a diagonal subscluster of Δ, a face of Δ or a diagonal subcluster of a face of Δ.*

Proof. We assume without loss of generality that the subcluster is also based at τ. By Lemma 4.2.10 the subcluster must have a parametrization as follows: There are sets $C_1, ..., C_k \subseteq \{1, ..., n\}$ such that

1. The sets $C_1, ..., C_k$ are pairwise disjoint.
2. $C_i = [a_i, b_i] \cap \{1, ..., m\}$ such that $b_i < a_j$ for $i < j$.
3. $\nu_i = (\prod_{j \in C_i} \lambda_j)$ are special forms.

Then our subcluster is parametrized by $\nu_1, ..., \nu_k$. The subcluster is a diagonal subcluster of Δ if and only if $C_1 \cup C_2 \cup ... \cup C_k = \{1, ..., n\}$. Otherwise, we have the following two possibilities. It is a face of Δ if and only if $C_1, ..., C_k$ are single element sets, and otherwise it is a diagonal subcluster of a face of Δ. $\square$

Now we shall show that the intersection of two clusters is a cluster. First we need to prove a technical lemma.

Lemma 4.2.14. *Let $\lambda_1, ..., \lambda_n$ and $\nu_1, ..., \nu_m$ be sorted lists of pairwise independent special forms such that*

$$F(\prod_{1 \leq i \leq n} \lambda_i) = F(\prod_{1 \leq j \leq m} \nu_j)$$

Let

$$I_1 = \{J \subseteq \{1, ..., n\} \mid \prod_{i \in J} \lambda_i \text{ is a special form.}\}$$

and

$$I_2 = \{J \subseteq \{1, ..., m\} \mid \prod_{i \in J} \nu_i \text{ is a special form.}\}$$

Let $\{K_1, ..., K_s\}$ and $\{L_1, ..., L_t\}$ be the maximal elements of I_1, I_2 respectively (with respect to inclusion). Then $s = t$ and after re-ordering the indices if necessary we have that

$$F(\prod_{i \in K_j} \lambda_i) = F(\prod_{i \in L_j} \nu_i)$$

Proof. It suffices to show that for any $1 \leq d \leq s$ there is a $1 \leq r \leq t$ such that

$$F(\prod_{i \in K_d} \lambda_i) = F(\prod_{i \in L_r} \nu_i)$$

Since

$$F(\prod_{1 \leq i \leq n} \lambda_i) = F(\prod_{1 \leq j \leq m} \nu_j)$$

it follows that we can perform expansion and contraction substitutions on the special form $\prod_{i \in K_\varrho} \lambda_i$ to obtain a subword of $F(\prod_{1 \leq j \leq m} \nu_j)$ that is a special form. It follows that there is an element $A \in I_2$ such that

$$\prod_{i \in K_\varrho} \lambda_i \sim \prod_{i \in A} \nu_i$$

Let $A = \{j_1, j_1 + 1, ..., j_1 + w_1\}$ and $K_\varrho = \{j_2, j_2 + 1, ..., j_2 + w_2\}$. We claim that $A \in \{L_1, ..., L_t\}$. If this is not the case, then either the special forms $\nu_{j-1}, \prod_{i \in A} \nu_i$

are consectutive and of alternating type, or $\prod_{i \in A} \nu_i, \nu_{j+w+1}$ are consecutive and of alternating type. Let us assume the latter (the former case is similar).

There are two possibilities.

1. We can perform expansion and contraction moves on $\prod_{j_1 \le i \le j_1+w_1+1} \nu_i$ to obtain a subword of $\prod_{j_2 \le i \le j_2+w_2+1} \lambda_i$ that properly contains $\prod_{j_2 \le i \le j_2+w_2} \lambda_i$.
2. We can perform expansion and contraction moves on $\prod_{j_2 \le i \le j_2+w_2+1} \lambda_i$ to obtain a subword of $\prod_{j_1 \le i \le j_1+w_1+1} \nu_i$ that properly contains $\prod_{j_1 \le i \le j_1+w_1} \nu_i$.

In either case we obtain that in fact $\prod_{j_2 \le i \le j_2+w_2+1} \lambda_i$ is a special form. This contradicts the assumption that K_ϱ is a maximal element of I_2.

This means that our assumption that $A \notin \{L_1, ..., L_t\}$ must be false. □

Lemma 4.2.15. *Let Δ_1, Δ_2 be clusters with basepoints τ_1, τ_2 and parameters $\lambda_1, ..., \lambda_n$ and $\nu_1, ..., \nu_m$ respectively. If $\Delta_3 = \Delta_1 \cap \Delta_2$ is nonempty, then Δ_3 is a cluster.*

Proof. By reparametrization of the clusters if necessary we can assume that $\tau_1 = \tau_2 = \tau$. Let $e_1, ..., e_k$ be the set of 1-cells incident to $F\tau$ that are contained in both Δ_1 and Δ_2. Let $C_1, ..., C_k \subseteq \{1, ..., n\}$ and $D_1, ..., D_k \subseteq \{1, ..., m\}$ be such that $\sigma_i = \prod_{j \in C_i} \lambda_j$ and $\psi_i = \prod_{j \in D_i} \nu_j$, and $e_i = \{F\sigma_i\tau, F\tau\} = \{F\psi_i\tau, F\tau\}$. By our assumption we have $\psi_i \sim \sigma_i$. Our proof consists of four steps.

Step (1): We claim that σ_i, σ_j are independent if and only if ψ_i, ψ_j are independent.

First note that $\psi_i \sim \sigma_i$ are equivalent to a special form. Since they are also expressed as products of independent special forms they must be special forms. Since $\psi_i \sim \sigma_i$ for each $1 \le i \le k$ our claim follows from Lemma 4.1.12.

Step (2): Let C_i, C_j be such that $C_i \cap C_j \neq \emptyset$. Then by step (1) it also holds that $D_i \cap D_j \neq \emptyset$. Let

$$\sigma_{i,j} = \prod_{k \in C_i \cap C_j} \lambda_k$$

and

$$\psi_{i,j} = \prod_{k \in D_i \cap D_j} \nu_k$$

In this step we will show that $\sigma_{i,j} \sim \psi_{i,j}$.

Define

$$\sigma_{i \setminus j} = \prod_{k \in C_i \setminus C_j} \lambda_k, \sigma_{j \setminus i} = \prod_{k \in C_j \setminus C_i} \lambda_k$$

Similarly define

$$\psi_{i \setminus j} = \prod_{k \in D_i \setminus D_j} \nu_k, \psi_{j \setminus i} = \prod_{k \in D_j \setminus D_i} \nu_k$$

We have $\sigma_i \sigma_j = \sigma_{i \setminus j} \sigma_{i,j}^2 \sigma_{j \setminus i}$ and $\psi_i \psi_j = \psi_{i \setminus j} \psi_{i,j}^2 \psi_{j \setminus i}$.

Recall that $\psi_j = \psi_{i,j} \psi_{j \setminus i} \sim \sigma_j = \sigma_{i,j} \sigma_{j \setminus i}$. By the second statement in Lemma 4.1.6 it follows that we can apply expansion substitutions to $\psi_{i,j} \psi_{j \setminus i}$ to obtain special forms $\rho_{i,j}, \rho_{j \setminus i}$ such that $\rho_{i,j} \sim \psi_{i,j}, \rho_{j \setminus i} \sim \psi_{j \setminus i}$ and $\rho_{i,j} \rho_{j \setminus i}$ can be obtained from the special form $\sigma_{i,j} \sigma_{j \setminus i}$ by expansion substitutions.

Let $\rho_{i,j} = y_{s_1}^{t_1} ... y_{s_l}^{t_l}$. In particular we can apply expansion substitutions to $\sigma_{i,j}$ to obtain a special form that contains a subword of the form $y_{s_1}^{t_1} ... y_{s_m}^{t_m}$ for some $1 \leq m \leq l$.

Since $\sigma_{i \setminus j} \sigma_{i,j} \sim \psi_{i \setminus j} \psi_{i,j}$ we can apply expansion substitutions $\sigma_{i,j}$ to obtain a standard form that contains a subword $y_{s_o}^{t_o} ... y_{s_l}^{t_l}$. for some $1 \leq o \leq l$. These observations together imply that we can apply expansion substitutions to $\sigma_{i,j}$ to obtain a word that contains as a subword $y_{s_1}^{t_1} ... y_{s_n}^{t_n}$. This means that a subword of $\sigma_{i,j}$ is equivalent to $\psi_{i,j}$.

But doing the same analysis above with σ and ψ interchanged we obtain that $\psi_{i,j}$ contains a subword that is equivalent to $\sigma_{i,j}$. So in fact $\psi_{i,j} \sim \sigma_{i,j}$. This also implies that $\psi_{i\setminus j} \sim \sigma_{i\setminus j}$ and $\psi_{j\setminus i} \sim \sigma_{j\setminus i}$.

In particular, it follows that these special forms satisfy $\psi_{i\setminus j}, \psi_{i,j}, \psi_{j\setminus i} \in \{\psi_1, ..., \psi_k\}$ and $\sigma_{i\setminus j}, \sigma_{i,j}, \sigma_{j\setminus i} \in \{\sigma_1, ..., \sigma_k\}$.

Step (3): The elements $\{C_1, ..., C_k\}$ (respectively $\{D_1, ..., D_k\}$) are partially ordered by inclusion. We claim that the set of minimal elements $P_1 \subseteq \{C_1, ..., C_k\}, P_2 \subseteq \{D_1, ..., D_k\}$ in this ordering satisfy the following.

Let $P_1 = \{C_{i_1}, ..., C_{i_s}\}$ and $P_2 = \{D_{j_1}, ..., D_{j_t}\}$. Then the following holds:

1. For each $1 \leq i \leq k$, C_i (and respectively D_i) can be expressed as a union of sets in P_1 (and respectively P_2).
2. The sets $C_{i_1}, ..., C_{i_s}$ (respectively $D_{j_1}, ..., D_{j_t}$) are pairwise disjoint.
3. $s = t$.

By step (2) it follows that if $C_i \cap C_j \neq \emptyset$ and $D_i \cap D_j \neq \emptyset$ then there are $1 \leq s_1, s_2, s_3 \leq k$ such that:

1. $\prod_{i \in C_{s_l}} \lambda_i \sim \prod_{i \in D_{s_l}} \nu_i$ and C_{s_l}, C_{s_m} (respectively D_{s_l}, D_{s_m}) are disjoint for $l \neq m, 1 \leq l, m \leq 3$.
2. $C_i = C_{s_1} \cup C_{s_2}$ and $D_i = D_{s_1} \cup D_{s_2}$.
3. $C_j = C_{s_2} \cup C_{s_3}$ and $D_j = D_{s_2} \cup D_{s_3}$.
4. $\sigma_{s_l} = \prod_{i \in C_{s_l}} \lambda_i \sim \psi_{s_l} = \prod_{i \in D_{s_l}} \nu_i$.

This proves our assertion. Moreover, our analysis produces a natural bijective correspondence between the sets $\{C_{a_1}, ..., C_{a_s}\}$ and $\{D_{b_1}, ..., D_{b_t}\}$. So in particular we have that $s = t$.

For the rest of the proof we let $C'_l = C_{i_l}, D'_l = D_{j_l}$, and $\sigma'_l = \sigma_{i_l}, \psi'_l = \psi_{j_l}$. Since we know that $s = t$, so we shall use s for the remainder of the argument.

Step (4): Now we claim that the subgraph $\Delta_1 \cap \Delta_2$ is precisely the cluster Δ_3 with basepoint τ and parameters $\sigma'_i = \prod_{j \in C'_i} \lambda_j \sim \psi'_i = \prod_{j \in D'_i} \nu_j$.

Let $F\zeta$ be a 0-cell in $\Delta_1 \cap \Delta_2$, and let $I_1 \subseteq \{1, ..., n\}, I_2 \subseteq \{1, ..., m\}$ such that

$$\zeta \sim (\prod_{i \in I_1} \lambda_i)\tau \sim (\prod_{j \in I_2} \nu_j)\tau$$

It suffices to show that there are $A, B \subseteq \{1, ..., s\}$ such that $I_1 = \cup_{i \in A} C'_i$ and $I_2 = \cup_{j \in B} D'_j$. It will then follow that

$$\zeta \sim (\prod_{i \in (\cup_{i \in A} C'_i)} \lambda_i)\tau \sim (\prod_{j \in (\cup_{j \in B} D'_j} \nu_j))\tau$$

and so $F\zeta \in \Delta_3$.

Let $\{J_1, ..., J_l\}, \{K_1, ..., K_{l'}\}$ be the maximal elements of the sets

$$\{I \subseteq I_1 \mid \prod_{i \in I} \lambda_i \text{ is a special form}\}$$

$$\{I \subseteq I_2 \mid \prod_{i \in I} \nu_i \text{ is a special form}\}$$

respectively.

Note that $\cup_{1 \leq i \leq l} J_i = I_1, \cup_{1 \leq j \leq l'} K_j = I_2$. By Lemma 4.2.14 it follows that $l = l'$ and by reordering the indices if necessary,

$$\prod_{w \in J_j} \lambda_w \sim \prod_{w \in K_j} \nu_w$$

for each $1 \leq j \leq l = l'$. This in particular implies that $J_1, ..., J_l \in \{C_1, ..., C_k\}$ and $K_1, ..., K_l \in \{D_1, ..., D_k\}$. Recall that for each $1 \leq i \leq k$, C_i (and respectively D_i) can be expressed as a union of sets in $P_1 = \{C'_1, ...C'_s\}$ (and respectively $P_2 = \{D'_1, ..., D'_s\}$). This means that I_1 (and respectively I_2) can be expressed as a union of sets in $P_1 = \{C'_1, ...C'_s\}$ (and respectively $P_2 = \{D'_1, ..., D'_s\}$).

This proves our assertion and $F\zeta$ is a 0-cell of the cluster Δ_3. □

Lemma 4.2.16. *Let $\lambda_1, ..., \lambda_n$ be special forms and*

$$\{F\tau, F\lambda_1\tau\}, \{F\tau, F\lambda_2\tau\}, ..., \{F\tau, F\lambda_n\tau\}$$

be 1-cells incident to $F\tau$ such that $\{F\lambda_i\tau, F\tau\}$,$\{F\lambda_j\tau, F\tau\}$ are facial 1-cells of a 2-cluster at τ for each $i, j \in \{1, ..., n\}, i \neq j$. Then

$$\{F\lambda_1\tau, F\tau\}, \{F\lambda_2\tau, F\tau\}, ..., \{F\lambda_n\tau, F\tau\}$$

are facial 1-cells of the n-cluster parametrized with basepoint at τ and parameters $\lambda_1, ..., \lambda_n$.

Proof. We show this using induction on n. Assume this holds for the first $n-1$ edges in our list. The pairs of edges $\{F\lambda_i\tau, F\tau\}$,$\{F\lambda_n\tau, F\tau\}$ belong to a 2-cluster for each $1 \leq i < n$. This 2-cluster can be parametrized by ν_i, ψ_i based at τ where $\nu_i \sim \lambda_i$ and $\psi_i \sim \lambda_n$ are special forms such that ν_i, ψ_i are independent. By Lemma 4.1.12 it follows that λ_n is independent with all the forms in the list $\lambda_1, ..., \lambda_{n-1}$ and so our n-cluster has the parameters $\lambda_1, .., , \lambda_n$ with basepoint τ. □

For each n the group G acts on the set of n-clusters. We now study the orbits of this action.

Lemma 4.2.17. *The action of G on the set of n-clusters has precisely $2^{(2n-1)}$ orbits.*

Proof. Observe that given an n-cluster with a type 1 basepoint τ, the action of τ^{-1} moves this basepoint to $\emptyset$. Morever the subgroup of G that stabilizes the trivial coset F is F. So it suffices to show that the action of F on the set of n-clusters parametrized with type 1 basepoint $\emptyset$ has precisely $2^{(2n-1)}$ orbits.

Consider two n-clusters Δ_1, Δ_2 with a type 1 basepoint $\emptyset$ and balanced parameters $\lambda_1, ..., \lambda_n$ and $\nu_1, ..., \nu_n$, respectively.

By Lemma 4.1.17 together with the assumption about types there is an $f \in F$ such that $\Delta_1 \cdot f = \Delta_2$ (or $F\lambda_i \cdot f = F\nu_i$) if and only if all of the following conditions hold:

1. ν_i has the same type and parity as λ_i.
2. The pair λ_i, λ_{i+1} is consecutive if an only if ν_i, ν_{i+1} is consecutive.

Observe that the type of λ_1 together with the parity of $\lambda_1, ..., \lambda_n$ determines the type of $\lambda_2, ..., \lambda_n$. Since the types of λ_1, ν_1 are assumed to be the same the first condition reduces to requiring that the parity of ν_i be the same as the parity of λ_i. Our claim follows immediately by counting the possiblities. □

4.2.2 Higher dimensional cells

In this section we shall describe a method to add higher cells to each n-cluster to obtain a CW complex homeomorphic to an n-cube.

Definition 4.2.18. Let Δ be an n-cluster based at τ and parametrized by $\nu_1, ..., \nu_n$ such that the parametrization has the property that if ν_i, ν_{i+1} are consecutive special forms, then they have alternating sign. Such a parametrization is called a *proper* parametrization. Note that in particular a balanced parametrization is proper, but not vice versa. Let the above be a proper parametrization, which we call ρ. Let $A_{\Delta,\rho}$ be the following set:

$$\{B \subseteq \{1,...,n\} \mid |B| = 1 \text{ or } B = \{i, i{+}1, ..., i{+}k\}, \text{ and } \nu_i, ..., \nu_{i+k} \text{ are consecutive}\}$$

We will show that for a cluster Δ, the set $A_{\Delta,\rho}$ is independent of the choice of a proper parametrization. As a consequence we will drop the ρ in the notation and simply denote the set by A_Δ.

Lemma 4.2.19. *Let Δ be a cluster with two different proper parametrizations:*

1. *ρ_1: Based at τ_1 and with parameters $\lambda_1, ..., \lambda_n$.*
2. *ρ_2: Based at τ_2 and with parameters $\nu_1, ..., \nu_n$.*

Then $A_{\Delta,\rho_1} = A_{\Delta,\rho_2}$.

Proof. The proof comprises of three steps:

1. First we consider the case where $\tau_1 \sim \tau_2$.
2. Next, we consider the case where $(\prod_{i\in I} \lambda_i)\tau_1 = \tau_2$ for some $I \subseteq \{1, ..., n\}$.
3. Finally, we consider the general case where $(\prod_{i\in I} \lambda_i)\tau_1 \sim \tau_2$ for some $I \subseteq \{1, ..., n\}$.

Case 1: Let $\tau_1 = f\tau_2$. There are special forms ν_i' such that $\nu_i \sim \nu_i' \sim \lambda_i f$. By Lemma 4.1.4 ν_i, ν_{i+1} are consecutive and of alternating sign if and only if ν_i', ν_{i+1}' have this property. In particular by Lemma 4.1.8 this is the case if and only if λ_i, λ_{i+1} have this property. So it follows that in this case $A_{\Delta,\rho_1} = A_{\Delta,\rho_2}$.

Case 2: A list of parameters for Δ with respect to the basepoint $(\prod_{i\in I}\lambda_i)\tau$ is $\lambda'_1, ..., \lambda'_n$ where $\lambda'_i = \lambda_i$ if $i \notin I$ and $\lambda'_i = \lambda_i^{-1}$ if $i \in I$. It follows from our assumption that $\nu_i \sim \lambda'_i$. Now since this is a proper parametrization, it must be the case that if λ_i, λ_{i+1} are consecutive, then $i \in I$ if and only if $i+1 \in I$. So λ_i, λ_{i+1} is consecutive and of alternating sign if and only if $\lambda'_i, \lambda'_{i+1}$ has this property. It follows that ν_i, ν_{i+1} are consecutive and of alternating sign if and only if λ_i, λ_{i+1} has this property. Therefore in this case $A_{\Delta,\rho_1} = A_{\Delta,\rho_2}$.

Case 3: This follows from combining the first two cases. □

Consider the unit n-cube $\square^n = [0,1] \times ... \times [0,1]$ in $\mathbb{R}^n$. We shall use $(z_1, .., z_n)$ as notation for the set of coordinates of vectors in $\mathbb{R}^n$ with respect to the standard choice of orthonormal basis for $\mathbb{R}^n$. For a given cluster Δ, we now produce a cellular decomposition of $\square^n$ whose 1-skeleton is Δ.

Definition 4.2.20. Let Δ be an n-cluster. A Δ-system of equations and inequalities on $z_1, ..., z_n$ comprises of a subcollection of equalities and inequalities from the following collection:

1. The equalities
$$z_j = z_{j+1} ... = z_{j+k}$$
where $\{j, ..., j+k\} \in A_\Delta$.
2. The equalities
$$z_j = ... = z_{j+k} = b_i$$
for $b_i \in \{0,1\}$ and $\{j, ..., j+k\} \in A_\Delta$.
3. The inequalities
$$z_j = ... = z_{j+k} < z_{j+k+1} = ... z_{j+k+t}$$

and

$$z_{j+k+1} = ...z_{j+k+t} < z_j = ... = z_{j+k}$$

for $\{j, ..., j+k\}, \{j+k+1..., j+k+t\} \in A_\Delta$

4. For each z_i, the inequalities

$$z_i < 1, z_i > 0, 0 < z_i < 1$$

and equations

$$z_i = 1, z_i = 0$$

Definition 4.2.21. Given an Δ-system Ξ of inequalities and equalities, there is a convex set $Conv(\Xi) \subseteq \square^n$ (possibly empty) that is defined by the system.

The convex sets $\{Conv(\Xi) \mid \Xi \text{ is a } \Delta\text{-system}\}$ are partially ordered by inclusion. The minimal elements of this partial order form a cellular decomposition of $\square^n$ which we call $\Delta(\square^n)$.

In the next lemma we will show that the graphs $(\Delta(\square^n))^{(1)}$ are Δ are naturally isomorphic.

Lemma 4.2.22. *Let Δ be parametrized by $\lambda_1, ..., \lambda_n$ at τ, such that this is a proper parametrization. The map $f : \Delta^{(0)} \to \rho(\square^n)^{(0)}$ that maps*

$$F((\prod_{i \in I} \lambda_i)\tau) \to (\chi_I(1), ..., \chi_I(n))$$

extends to a graph isomorphism.

Proof. First we show that if for sets $I, J \subseteq \{1, ..., n\}$ the 0-cells

$$F((\prod_{i \in I} \lambda_i)\tau), F((\prod_{i \in J} \lambda_i)\tau)$$

form an edge in Δ, then the 0-cells $(\chi_I(1), ..., \chi_I(n)), (\chi_J(1), ..., \chi_J(n))$ of $\Delta(\square^n)^{(1)}$ form an edge. We know that $(\prod_{i\in I}\lambda_i)(\prod_{j\in J}\lambda_j)^{-1}$ is a special form. Now consider

$$(\prod_{i\in I}\lambda_i)(\prod_{j\in J}\lambda_j)^{-1} = (\prod_{i\in I\setminus J}\lambda_i)(\prod_{j\in J\setminus I}\lambda_j)^{-1}$$

Since this is a special form and our parametrization is proper, it must be the case that either $I \subseteq J$ or $J \subseteq I$. Let us assume that $J \subseteq I$. (The other case is similar.)

Now consider the partition of $\{1, ..., n\}$ consisting of singletons $L_j = \{j\}$ for $j \notin I$ and the sets $I \setminus J, J$. The system of equations and inequalities,

1. $x_j = 0$ if $j \notin I$.
2. $x_j = 1$ if $j \in J$.
3. $x_{i+1} = x_{i+2} = ... = x_{i+k_0}$ where $I \setminus J = \{i+1, i+2, ..., i+k_0\}$.

is a Δ-system which defines a 1-dimensional subspace of $\mathbb{R}^n$ whose intersection with $\square^n$ is precisely the 1-cell connecting the vertices

$$(\chi_I(1), ..., \chi_I(n)), (\chi_J(1), ..., \chi_J(n))$$

Now we show the other direction. Let e be a 1-cell in $\rho(\square^n)$. By definition there is an ρ-system Ξ that describes a 1-dimensional affine subspace of $\mathbb{R}^n$ whose intersection with $\square^n$ is the 1-cell e. Such a system must have the following form. There are disjoint sets $I', J' \subseteq \{1, ..., n\}$ and a partition consisting of the set I' and sets $\{x_i\}$ for each $i \notin I'$ such that the system is described by the following list of equations and inequalities.

1. $x_i = \chi_{J'}(i)$ for each $i \notin I'$.
2. $x_{j+1} = x_{j+2} = ... = x_{j+l}$ where $I' = \{j+1, j+2, ..., j+l\} \in A_\Delta$.

Then it follows that e is the image of the 1-cell in Δ connecting the 0-cells

$$F((\prod_{i\in J'}\lambda_i)\tau), F((\prod_{i\in J'\cup I'}\lambda_i)\tau)$$

This proves our assertion. □

We shall show that each n-cluster can be filled in this way such that the following holds.

1. The filling of a cluster is independent of choice of proper parametrization.
2. If Δ_1 is a subcluster of Δ_2 then the subcomplex of the filling of Δ_2 that comprises of the cells whose boundary 1-skeleton is a subset of Δ_1 is isomorphic with the filling of Δ_1 in a natural way. Since by Lemma 4.2.15 the intersection of a set of clusters is a cluster, the filling of any two clusters in $X^{(1)}$ is *compatible.*

Observe that in Lemma 4.2.22 the map depends upon the choice of base-vertex $F\tau$, i.e. the 0-cell $F\tau$ is mapped to the vector $(0,...,0)$ in $\mathbb{R}^n$. We will show that the prescribed CW-filling of Δ is independent of choice of basepoint, or the proper parametrization. This will imply (1) above.

Lemma 4.2.23. *Let Δ be a cluster with two different proper parametrizations.*

1. *ρ_1: Based at τ_1 and with parameters $\lambda_1,...,\lambda_n$.*
2. *ρ_2: Based at τ_2 and with parameters $\nu_1,...,\nu_n$*

Then the identity map $i:\Delta\to\Delta$ extends to an isomorphism of the CW complexes that are obtained from the fillings with respect to the two parameterizations.

Proof. The proof comprises of two steps:

1. We consider the case where $(\prod_{i\in I}\lambda_i)\tau_1 = \tau_2$ for some $I \subseteq \{1, ..., n\}$.
2. We consider the case where $F((\prod_{i\in I}\lambda_i)\tau_1) = F\tau_2$ or equivalently $(\prod_{i\in I}\lambda_i)\tau_1 \sim \tau_2$ for some $I \subseteq \{1, ..., n\}$.

Case (1): In this case, $\tau_2 = (\prod_{i\in I}\lambda_i)\tau_1$ for some $I \subseteq \{1, ..., n\}$ and $\nu_i = \lambda_i$ if $i \notin I$ and $\nu_i = \lambda_i^{-1}$ if $i \in I$. It follows from the definition of proper parametrization that if λ_i, λ_{i+1} are consecutive, then $i \in I$ if and only if $i + 1 \in I$.

In this case we shall consider two different coordinate systems for $\mathbb{R}^n$. These coordinate system have the same underlying $\mathbb{R}^n$, but different origin and orthonormal bases vectors. However the two coordinate systems are related by the composition of an affine transformation with orientation reversing transformations.

The first coordinate system consists of an origin $o_1 = (0, ..., 0)$ and unit basis vectors $e_1 = (1, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), ..., e_n = (0, 0, ..., 1)$. We consider the unit square $\square^n = [0, 1]^n$ in this coordinate system. We denote this system by $(\mathbb{R}^n, C_1)$.

We now consider a different, but related coordinate system denoted by $(\mathbb{R}^n, C_2)$. The origin for this system is the point $o_2 = (\chi_I(1), \chi_I(2), ..., \chi_I(n))$ in our original system. The orthonormal basis vectors $(u_1, ..., u_n)$ in $(\mathbb{R}^n, C_2)$ correspond to the 1-cells incident to $(\chi_I(1), ..., \chi_I(n))$ in $\square^n$. More formally, in $(\mathbb{R}^n, C_1)$, the vectors u_i are $u_i = e_i$ if $i \notin I$ and $u_i = -e_i$ if $i \in I$.

We claim that the set of minimal elements of the partial order of convex sets of $\mathbb{R}^n$ obtained by Δ systems in the coordinate system $(\mathbb{R}^n, C_1)$ is the same as those obtained from Δ systems with the coordinate system $(\mathbb{R}^n, C_2)$.

Given an inequality $z_i < z_j$ in a Δ system, by definition of proper parametrization and 4.2.20 we know that λ_i, λ_j must be consecutive elements so in particular $i \in I$ if and only if $j \in I$. So the inequality $z_i < z_j$ in the Δ system represents the same region of $\mathbb{R}^n$ in the system $(\mathbb{R}^n, C_2)$ as $z_j < z_i$ in the Δ system in the coordinate system $(\mathbb{R}^n, C_2)$. An equality $x_i = x_j$ produces the same hyperplane of $\mathbb{R}^n$ in each system. If $i \in I$ the equality $x_i = a$ for $a \in \{0, 1\}$ in the coordinate system $(\mathbb{R}^n, C_1)$ corresponds to the same hyperplane in $\mathbb{R}^n$ as $x_i = b$ in the coordinate system $(\mathbb{R}^n, C_2)$ for $b \in \{0, 1\} \setminus \{a\}$. If $i \notin I$, they correspond to the same hyperplane of $\mathbb{R}^n$ in each system. This proves our claim.

Case (2): This case follows from combining the first case together with Lemma 4.2.19 and Definitions 4.2.20 and 4.2.21. $\square$

This justifies introducing the following notation.

Definition 4.2.24. We denote the (unique) filling of a cluster Δ with respect to some (any) proper parametrization as $\overline{\Delta}$.

Lemma 4.2.25. *Let Δ_1 be a subcluster of Δ_2. The inclusion $\phi : \Delta_1 \hookrightarrow \Delta_2$ induces an isomorphism of the CW complex $\overline{\Delta_1}$ with the subcomplex of $\overline{\Delta_2}$ which consists of the union of cells in $\overline{\Delta_1}$ whose boundary 1-skeleton is a subgraph of Δ_1.*

Proof. The proof has two cases.

1. Δ_1 is a facial subcluster of Δ_2.
2. Δ_1 is a diagonal subcluster of Δ_2.

In the first case, we have proper parametrizations of Δ_1, Δ_2 as follows: We start with a proper parametrization ρ_2 of Δ_2 as based at τ and parametrized by

$\lambda_1, ..., \lambda_n$. Then there are disjoint sets $I, J \subseteq \{1, ..., n\}$ such that Δ_1 is based at $(\prod_{j\in I} \lambda_j)\tau$ and parametrized by $\lambda_{i_1}, ..., \lambda_{i_l}$ for the ordered set $\{i_1, ..., i_l\} = J$, for which $i_j < i_k$ if $j < k$. It follows that this is a proper parametrization for Δ_1.

Let Ξ be a Δ_2-system describing a cell of Δ_2 whose boundary 1-skeleton is a subgraph of Δ_1. It follows that the following equations must hold in Ξ.

1. $z_j = 0$ for each $j \in \{1, ..., n\} \setminus (I \cup J)$.
2. $z_j = 1$ for each $j \in I$.

Observe that any such system naturally corresponds to the cell of Δ_1 that is represented by the Δ_1-system Ξ' which consists of the same equations and inequalities in Ξ except that the equalities or inequalities that contain z_j are removed for each $j \notin J$.

Now consider a cell e in the filling of Δ_1 given by an Δ_1 system Ξ'. This naturally corresponds to an Δ_2 system Ξ obtained by replacing z_j with z_{i_j} in Ξ' together with the equations $z_j = 0$ if $j \notin I \cup J$ and $z_j = 1$ if $j \in I$.

This system represents the cell $\phi(e)$ in Δ_2. Therefore ϕ is an isomorphism of $\overline{\Delta_1}$ with the image $\phi(\overline{\Delta_1})$ in $\overline{\Delta_2}$.

Now we consider the second case. Let Δ_2 be parametrized by ρ_2 as above. Assume that Δ_1 is not a diagonal subcluster of a proper facial subcluster of Δ_2, since otherwise from the first case of our proof we can replace Δ_2 with that facial subcluster in the statement of the Lemma. By Lemma 4.2.10 we can parametrize Δ_1 in the following way: It is based at τ and the parameters are $\nu_1, ..., \nu_k$ such that:

1. There is a partition $I_1, ..., I_k$ of $\{1, ..., n\}$ such that each $\prod_{i \in I_j} \lambda_i$ is a special form, and if $i < j$ and $p \in I_i, q \in I_j$ then $p < q$.
2. $\nu_j = \prod_{i \in I_j} \lambda_j$.

We call this parametrization ρ_1.

Let Ξ be an Δ_1 system that represents a cell of $\overline{\Delta_1}$. We produce a Δ_2 system Ξ' that respresents the cell $\phi(e)$. This system comprises of the following set of inequalities and equalities:

1. For each $1 \leq l \leq k$ and each $i, j \in I_l$ an equation $z_i = z_j$.
2. We choose representatives $j_i \in I_i$ for each $1 \leq i \leq k$. For each inequality/equation μ in Ξ, we include the inequality/equation μ' obtained by replacing the symbol z_i (that represents a coordinate in $\mathbb{R}^k$) with the symbol z_{j_i} (that represents a coordinate in $\mathbb{R}^n$) and include it in Ξ'.

The system Ξ' defines the cell $\phi(e)$ in Δ_2.

Next we show that every Δ_2 system that defines a cell whose boundary 1-skeleton is in $\Delta_1 \subset \Delta_2$ is the ϕ-image of a cell represented by a corresponding Δ_1 system.

Given such a cell e, by convexity it follows that e is contained in the convex hull of the boundary vertices of e which are all in $\Delta_1(\square^k)^{(1)} \subseteq \Delta_2(\square^n)^{(1)}$. In fact, the subcomplex of $\bar{\Delta}_2$ with 1-skeleton Δ_1 is convex in $\mathbb{R}^n$.

So in particular it follows that if Ξ is a Δ_2-system defining e then the following equations must hold in Ξ. (possibly in addition to other equations and inequalities.) The equations $z_i = z_j$ for each $1 \leq i, j \leq n$ such that $i, j \in I_l$ for some $1 \leq l \leq k$.

Now we construct an Δ_1 admissible system Ξ' as follows. For any occurence of z_i in an equation or inequality of Ξ, we replace the symbol z_i by z_j where $i \in I_j$. The resulting Δ_2 system is Ξ', and it defines a cell e' in $\overline{\Delta_1}$ such that $\phi(e') = e$. $\square$

By Lemma 4.2.15 that since the intersection of a set of clusters is a cluster. So the filling is naturally well defined under intersection. Consider clusters $\Delta_1, ..., \Delta_n$ and $\Delta = \cap_{1 \leq i \leq n} \Delta_n$. since Δ is a subcluster of every cluster $\Delta_1, ..., \Delta_n$, the filling of Δ agrees with the filling inherited by Δ from each cluster Δ_i. Now we can finally define our complex X.

Definition 4.2.26. The complex X is the union of filled clusters in the set

$$\{\bar{\Delta} \mid \Delta \text{ is a cluster of } X^{(1)}\}$$

The following lemma follows from the definitions.

Lemma 4.2.27. *Let Δ be a cluster in $X^{(1)}$. Then $\bar{\Delta}$ is a topological n-cube. If Δ' is a k-facial subcluster of Δ, then $\bar{\Delta}'$ is an k cube which is a face of $\bar{\Delta}$. Moreover, if F is an k-subcube of $\bar{\Delta}$ then $F^{(1)}$ is a k-facial subcluster of Δ.*

Now we show that every n-cell e of X is contained in the filling of an n-cluster.

Lemma 4.2.28. *Let e be an n-cell of X. Then e is contained in $\bar{\Delta}$ where Δ is an n-cluster of X.*

Proof. Let e be the n-cell of a k-cluster Δ that is based at τ and parametrized by $\lambda_1, ..., \lambda_k$. Assume that $k > n$. We will show that e is contained in a subcluster of Δ of dimension less than k. Let ρ be a proper parametrization of Δ.

Recall that by definition 4.2.20 we know that e corresponds to a convex subset of $\mathbb{R}^n$, in particular a cell of $\Delta(\square^n)$. This is represented by an Δ-system of equations and inequalities, which we denote as Ξ. We claim that there is an equation of the form $z_i = z_j$, $z_i = 0$ or $z_i = 1$ holds for the system Ξ.

If there are no equations of the above type, then the convex region is described by $H \cap \square^n$, where H is an intersection of open half spaces of $\mathbb{R}^n$. Such a region must be either empty or k-dimensional. Since e is n dimensional and we assumed that $n < k$, it follows that there must be an equation of the above type. There are three cases to consider.

1. If $z_i = z_j$ for $i < j$, by definition 4.2.23 the equation $z_i = z_{i+1} = ...z_{i+l}$ (where $j = i+l$) is in Ξ. So in fact Ξ describes a cell of a subcluster properly parametrized with basepoint τ, and parameters

$$\lambda_1, ..., \lambda_i\lambda_{i+1}...\lambda_{i+l}, \lambda_{i+l+1}, ...\lambda_k$$

2. If $z_i = 1$ (or $z_i = 0$), then the cell e lies in the subcluster properly parametrized with basepoint $\lambda_i\tau$ (or respectively τ) and parameters

$$\lambda_1, ..., \lambda_{i-1}, \lambda_{i+1}, ..., \lambda_k$$

This proves our assertion. $\square$

We end this section by showing that an n-cell of an n-cluster is incident to the "extreme points" of a proper parametrization.

Lemma 4.2.29. *Let e be an n-cell in $\bar{\Delta}$, where Δ is an n-cluster with a proper parametrization based at τ and with parameters $\lambda_1, ..., \lambda_n$. Then the 0-cells*

$$F\tau, F((\prod_{i\in\{1,...,n\}} \lambda_i)\tau)$$

are incident to e.

Proof. By definition 4.2.20 the n-cell e is precisely the intersection of n-halfspaces of the form $z_i > 0, z_i < 1$ or $z_i < z_j$. The 0-cells $F\tau, F((\prod_{i\in\{1,...,n\}} \lambda_i)\tau)$ lie in the boundary of every such half space, and we claim that as a consequence they must lie in the boundary of the intersection if the intersection is non-trivial. The intersection of two closed convex sets is closed and convex. Also, the intersection of open half spaces in $\mathbb{R}^n$, if nonempty, is n-dimensional. So assuming this intersection is nonempty, the intersection of the corresponding closed half spaces must contain our 0-cells as a limit point of a sequence of points in the intersection of the open half spaces. □

Remark 4.2.30. For the rest of the chapter, we shall denote a filled cluster $\bar{\Delta}$ as simply Δ.

4.2.3 The action of G on X

In this section it will be our goal to show that the stabilizer of each cell of X in G is of type F_∞. The following lemma is a crucial first step in understanding the stabilizer of cells.

Lemma 4.2.31. *Let e be an n-cell in X contained in Δ where Δ is an n-cluster with a proper parametrization ρ with basepoint $\emptyset$ and parameters $\lambda_1, ..., \lambda_n$. If $\nu \in Stab_G(e)$ then $\nu \in F$.*

Proof. First note that by Lemma 4.2.29 we know that the 0-cells $F, F((\prod_{1\leq i\leq n} \lambda_i))$ are incident to e. Define

$$P = \{I \subseteq \{1, ..., n\} \mid F(\prod_{i\in I} \lambda_i) \text{ is a 0-cell incident to } e\}$$

By our observation above we know that $\{\emptyset\}, \{1, ..., n\} \in P$

Let $\nu \in G$ be an element such that $e \cdot \nu = e$. Since $e \cdot \nu = e$ it follows that the action of ν induces a permutation of the set of 0-cells $V(e) = \{F(\prod_{i\in I} \lambda_i) \mid I \in P\}$. So in particular, it induces a permutation of the set P.

Let $I_1 \in P$ be the set with the property that $F \cdot \nu = F(\prod_{i\in I_1} \lambda_i)$. This means that $\nu = f(\prod_{i\in I_1} \lambda_i)$ for some $f \in F$. We claim that $I_1 = \emptyset$. Assume that this is not the case and let $I_2 \in P$ be the set with the property that

$$F(\prod_{i\in\{1,...,n\}} \lambda_i) \cdot f(\prod_{i\in I_1} \lambda_i) = F(\prod_{i\in I_2} \lambda_i)$$

Let λ_i' be special forms such that $\lambda_i' \sim \lambda_i \cdot f$. It follows that

$$F(\prod_{i\in\{1,...,n\}} \lambda_i) \cdot f(\prod_{i\in I_1} \lambda_i) = F(\prod_{i\in\{1,...,n\}} \lambda_i')(\prod_{i\in I_1} \lambda_i)$$

Let $I_2 \in P$ be the set with the property that

$$F(\prod_{i\in I_2} \lambda_i) \cdot (f \prod_{i\in I_1} \lambda_i) = F(\prod_{i\in I_2} \lambda_i')(\prod_{i\in I_1} \lambda_i) = F(\prod_{i\in\{1,...,n\}} \lambda_i)$$

It follows that

$$F(\prod_{i\in I_2} \lambda_i') = F(\prod_{i\in I_1^c} \lambda_i)$$

We claim that $I_2 = \{1, ..., n\}$. Assume that this is not the case. Consider the smallest $k \in \{1, ..., n\}$ such that either $k \in I_2, k+1 \notin I_2$ or $k \notin I_2, k+1 \in I_2$. Let us assume the former case. (The latter case is similar.) We claim that this implies that

$$F(\prod_{i\in\{1,...,n\}} \lambda_i) \cdot (f \prod_{j\in I_1} \lambda_i)$$

is not a 0-cell in e.

Now we have

$$F(\prod_{i\in\{1,...,n\}} \lambda_i) \cdot (f \prod_{i\in I_1}) = F(\prod_{i\in\{1,...,n\}} \lambda_i')(\prod_{i\in I_1} \lambda_i) = F(\prod_{i\in I_2^c} \lambda_i')(\prod_{i\in I_2} \lambda_i')(\prod_{i\in I_1} \lambda_i)$$

There is a $j \in I_1$ such that

1. λ_j, λ'_k are consecutive special forms of alternating type for some $k \in I_2$.
2. λ_j, λ_{k-1} have subwords τ_1, τ_2 and ν_1, ν_2 such that $\lambda_j = \tau_1\tau_2$, $\lambda'_k = \nu_1\nu_2$ and τ_2, ν_2 are equivalent special forms.

This means that the subword $\lambda'_{k-1}\lambda_j$ of

$$F(\prod_{i\in I_2^c}\lambda'_i)(\prod_{i\in I_2}\lambda'_i)(\prod_{i\in I_1}\lambda_i)$$

contains a calculation of exponent 2. So by Lemma 4.1.16 it follows that the word

$$(\prod_{i\in I_1^c}\lambda_i)(\prod_{i\in I_1}\lambda'_i)(\prod_{i\in I_1}\lambda_i)$$

contains a calculation with no potential cancellations and two occurences of y, y^{-1}. This contradicts the fact that

$$F(\prod_{i\in I_1^c}\lambda'_i)(\prod_{i\in I_1}\lambda'_i)(\prod_{i\in I_1}\lambda_i)$$

is a 0-cell of e.

So it must be the case that $I_2 = \{1, ..., n\}$. So we have

$$F(\prod_{i\in\{1,\ldots,n\}}\lambda_i)(f\prod_{i\in I_1}\lambda_i) = F(\prod_{i\in\{1,\ldots,n\}}\lambda_i)$$

This implies that

$$F(\prod_{i\in\{1,\ldots,n\}}\lambda'_i) = F(\prod_{i\in I_1^c}\lambda_i)$$

But this means that for any $I \in P$

$$F(\prod_{i\in I}\lambda_i)\cdot(f\prod_{i\in I_1}\lambda_i)) = F(\prod_{i\in I}\lambda'_i)(\prod_{i\in I_1}\lambda_i) \neq F$$

This means that there is no 0-cell u that gets mapped to F under this action, which is a contradiction. Therefore, we must have that I_1 is empty and hence $\nu \in F$. ◻

Lemma 4.2.32. *Let e be an n-cell of X. Every element of G that stabilizes e must stabilize every 0-cell of e.*

Proof. Let Δ be an n-cluster in X containing e. We can assume without loss of generality that Δ is properly parametrized as follows. It is based at $\emptyset$, and has parameters $\lambda_1, ..., \lambda_n$. This is because if Δ is properly parametrized and based at τ, then the stabilizer of cells of Δ in G is conjugate to the stabilizer of corresponding cells of the cluster $\Delta \cdot \tau^{-1}$ in G. By the previous Lemma, we know that $Stab_G(e) = Stab_F(e)$.

Just as in the proof of the previous Lemma, we define

$$P = \{I \subseteq \{1, ..., n\} \mid F(\prod_{i \in I} \lambda_i) \text{ is a 0-cell incident to } e\}$$

By our assumption above we know that $\{\emptyset\}, \{1, ..., n\} \in P$.

We now define a linear order on the 0-cells incident to e, as follows. Let $I_1, I_2 \in P$ and $F(\prod_{i \in I_1} \lambda_i), F(\prod_{i \in I_2} \lambda_i)$ be 0-cells incident to e. Then

$$F(\prod_{i \in I_1} \lambda_i) < F(\prod_{i \in I_2} \lambda_i)$$

if either $I_1 \subseteq I_2$ or $inf(I_2 \setminus I_1) < inf(I_1 \setminus I_2)$. This is easily seen to be transitive and it follows immediately that this is a linear order.

We claim that the action of $Stab_F(e)$ preserves this linear order. Let $I_1, I_2 \in P$ such that $F(\prod_{i \in I_1} \lambda_i) < F(\prod_{i \in I_2} \lambda_i)$ and let $F(\prod_{i \in I_1} \lambda_i) \cdot f = F(\prod_{i \in I'_1} \lambda_i)$ and $F(\prod_{i \in I_2} \lambda_i) \cdot f = F(\prod_{i \in I'_2} \lambda_i)$. We claim that $F(\prod_{i \in I'_1} \lambda_i) < F(\prod_{i \in I'_2} \lambda_i)$. Observe that $F(\prod_{i\in I_1}\lambda_i) < F(\prod_{i\in I_2}\lambda_i)$ if and only if the word

$$(\prod_{i\in I_1}\lambda_i)(\prod_{i\in I_2}\lambda_i)^{-1} = (\prod_{i\in I_2\setminus I_1}\lambda_i^{-1})(\prod_{i\in I_1\setminus I_2}\lambda_i)$$

expressed in standard form as $\lambda_{s_1}^{t_1}...\lambda_{s_n}^{t_n}$ with $t_i \in \{1,-1\}, s_i \in \{1,...,n\}$ and $s_i <_{\text{lex}} s_j$ if $i < j$ has the property that $t_1 = -1$. It is easily checked by considering the action of F that this property also holds for

$$(\prod_{i\in I'_1}\lambda_i)(\prod_{i\in I'_2}\lambda_i)^{-1} = (\prod_{i\in I'_2\setminus I'_1}\lambda_i^{-1})(\prod_{i\in I'_1\setminus I'_2}\lambda_i)$$

expressed in standard form as $\lambda_{p_1}^{q_1}...\lambda_{p_m}^{q_m}$ just as above. That is, $q_1 = -1$.

Since an order preserving bijection of a finite totally ordered set is in fact the identity map, our claim follows. □

Proposition 4.2.33. *The action of G on the complex X satsfies the following properties:*

1. *X/G has finitely many cells in each dimension.*
2. *The stabilizer of each cell is of type F_∞.*

Proof. We know by Lemma 4.2.17 that the action of G on the set of n-clusters has precisely $2^{(2n-1)}$ orbits. Moreover, if e is an n-dimensional cell, then by Lemma 4.2.28 it is contained in the filling of an n-cluster. So the first statement holds.

We now prove the second statement. Let e be an n-cell. Let Δ be the n-cluster containing e, which without loss of generality is assumed to be properly parametrized as based at $\emptyset$ and with parameters $\lambda_1, ..., \lambda_n$. We know from the previous two Lemmas that $Stab_G(e) = Stab_F(e)$ and elements of $Stab_F(e)$ stabilize each vertex of e.

As in the previous two Lemmas, let

$$P = \{I \subseteq \{1, ..., n\} \mid F(\prod_{i \in I} \lambda_i) \text{ is a 0-cell incident to } e\}$$

We first observe a general fact before proceeding to prove the lemma. Let $I_1, I_2 \subseteq \{1, ..., n\}$ be sets such that $I_1 = \{i, i+1, ..., i+l\}, I_2 = \{j, j+1, ..., j+m\}$. If $f \in F$ stabilizes the cosets $F(\prod_{i \in I_1} \lambda_i), F(\prod_{i \in I_2} \lambda_i)$ then in fact f stabilizes the cosets $F(\prod_{i \in I_1 \cap I_2} \lambda_i), F(\prod_{i \in I_1 \setminus I_2} \lambda_i), F(\prod_{i \in I_2 \setminus I_1} \lambda_i)$. We remark that $\prod_{i \in I_1 \cap I_2} \lambda_i, \prod_{i \in I_1 \setminus I_2} \lambda_i, \prod_{i \in I_2 \setminus I_2} \lambda_i$ are special forms by definition unless they are empty.

Let P' be the smallest collection of subsets of $\{1, ..., n\}$ such that

1. $P \subseteq P'$.
2. P' is closed under the operations of taking intersection and set difference.

The elements of P' are ordered by inclusion. Let $I_1, ..., I_k$ be the minimal elements of P' in this ordering. Observe that,

1. $I_1, ..., I_k$ forms a partition of $\{1, ..., n\}$.
2. $\nu_i = \prod_{j \in I_i} \lambda_j$ is a special form for each $1 \leq i \leq k$.
3. ν_i, ν_j are independent if $i \neq j$.

It follows that $Stab_F(e) = \cap_{1 \leq i \leq k} Stab_F(F(\prod_{j \in I_i} \lambda_j))$. By Lemma 4.1.13 this is a group of type F_∞. $\square$

4.3 X is simply connected.

In this section we will show that X is simply connected. This is shown by providing an explicit homotopy for every loop with a trivial loop. Given a loop l in X, we can assume that l is a path along the 1-skeleton that is parametrized by a sequence of 0-cells

$$F\tau, F(\nu_k\tau), F(\nu_{k-1}\nu_k\tau), ..., F(\nu_1...\nu_k\tau), F\tau$$

where τ is a Y-word, $\nu_1, ..., \nu_k$ are special forms, and the Y-word $\prod_{1\leq i\leq k}\nu_i$ is equivalent to a special form. (Note that it may not be the case that $\nu_1...\nu_k$ is itself a special form, or that pairs ν_i, ν_j are independent. Since the last two vertices in the sequence are connected by an edge we deduce that $F(\nu_1...\nu_k) \in \Gamma$ and in particular $\nu_1...\nu_k = f\psi$ where $f \in F$ and ψ is a special form.)

There are some basic *homotopies* that correspond to moves in the analysis of standard forms. We list them below:

1. The expansion substitution $y_\sigma \to x_\sigma y_{\sigma 0} y_{\sigma 10}^{-1} y_{\sigma 11}$ corresponds to homotopic paths $F\tau, F(y_\sigma\tau)$ and

   $$F\tau, F(y_{\sigma 11}\tau), F(y_{\sigma 10}^{-1} y_{\sigma 11}\tau), F(y_{\sigma 0} y_{\sigma 10}^{-1} y_{\sigma 11}\tau)$$

   Similarly the expansion substitution $y_\sigma^{-1} \to x_\sigma^{-1} y_{\sigma 00}^{-1} y_{\sigma 10} y_{\sigma 1}^{-1}$ corresponds to a homotopy between paths $F\tau, F(y_\sigma\tau)$ and

   $$F\tau, F(y_{\sigma 1}^{-1}\tau), F(y_{\sigma 10} y_{\sigma 1}^{-1}\tau), F(y_{\sigma 00}^{-1} y_{\sigma 10} y_{\sigma 1}^{-1}\tau)$$

   (These paths are homotopic in X because they are homotopic in a 3-cluster of X.)

2. Consider the path $F\tau, F(y_s^t f\tau)$. Performing expansion substitutions on y_s^t we obtain special form $y_{s_1}^{t_1}...y_{s_n}^{t_n} \sim y_s^t$ such that f acts on $s_1, ..., s_n$. Our path is homotopic to the path described by the sequence

$$F\tau, F(y_{s_n}^{t_n} f\tau), F(y_{s_{n-1}}^{t_{n-1}} y_{s_n}^{t_n} f\tau), ..., F(y_{s_1}^{t_1}...y_{s_n}^{t_n} f\tau)$$

Applying the rearranging substitution

$$y_{s_1}^{t_1}...y_{s_n}^{t_n} f = f y_{f(s_1)}^{t_1}...y_{f(s_n)}^{t_n} = y_{r_1}^{t_1}...y_{r_n}^{t_n}$$

produces a reparametrization of this path as

$$F, F(y_{r_n}^{t_n}\tau), F(y_{r_{n-1}}^{t_{n-1}} y_{r_n}^{t_n}\tau), ..., F(y_{r_1}^{t_1}...y_{r_n}^{t_n}\tau)$$

3. A commuting substitution $y_s^t y_p^q = y_p^q y_s^t$ for $t, q \in \{-1, 1\}$ and s, p incompatible, corresponds to a homotopy between paths of the form $F\tau, F(y_s^t\tau), F(y_p^q y_s^t\tau)$ and $F\tau, F(y_p^q\tau), F(y_s^t y_p^q\tau)$, for any $\tau \in G$. This can be performed in the complex since these paths are homotopic in a 2-cluster.

4. A cancellation substitution $y_s^t y_s^{-t} \to \emptyset$ corresponds to shrinking a path of the form $F\tau, F(y_s^{-t}\tau), F(y_s^t y_s^{-t}\tau)$ to a trivial path.

Lemma 4.3.1. *Given a loop*

$$L = \{F\tau, F(\lambda_n\tau), F(\lambda_{n-1}\lambda_n\tau), ..., F(\lambda_1...\lambda_n\tau)\}$$

where $\lambda_1, ..., \lambda_n$ are special forms, it is homotopic to a loop of the form

$$F\tau, F(y_{s_m}^{t_m}\tau), F(y_{s_{m-1}}^{t_{m-1}} y_{s_m}^{t_m}\tau), ..., F(y_{s_1}^{t_1}...y_{s_m}^{t_m}\tau)$$

Proof. We show this by induction on n. For $n = 1$, let $\lambda_1 = y_{p_1}^{q_1}...y_{p_k}^{q_k}$. Now the 1-cell $\{F\tau, F(\lambda_1\tau)\}$ is the cross diagonal 1-cell of the k-cluster at τ parametrized by the special forms $y_{p_1}^{q_1}, ..., y_{p_k}^{q_k}$. It follows that this 1-cell is homotopic to the path

$$F\tau, F(y_{p_k}^{q_k}\tau), F(y_{p_{k-1}}^{q_{k-1}} y_{p_k}^{q_k}\tau), ..., F(y_{p_1}^{q_1}...y_{p_k}^{q_k}\tau)$$

The inductive step is essentially the same as the base case, since by the inductive hypothesis we replace the path

$$\{F\tau, F(\lambda_n\tau), F(\lambda_{n-1}\lambda_n\tau), ..., F(\lambda_2...\lambda_n\tau), F(\lambda_1...\lambda_n\tau)$$

by a path

$$F\tau, F(y_{u_l}^{v_l}\tau), F(y_{u_{l-1}}^{v_{l-1}}y_{u_l}^{v_l}\tau), ..., F(\lambda_1 y_{u_1}^{v_1}...y_{u_l}^{v_l}\tau)$$

and then argue the last edge

$$F(y_{u_1}^{v_1}...y_{u_l}^{v_l}\tau), F(\lambda_1 y_{u_1}^{v_1}...y_{u_l}^{v_l}\tau)$$

traversed in the path is homotopic to path of a suitable sequence of edges as in the base case.

□

Proposition 4.3.2. *The complex X is simply connected.*

Proof. Let L be a loop described as a path in $X^{(1)}$. By considering the group action we can assume that the loop L begins and ends at F. By the previous Lemma this loop is homotopic to a loop of the form

$$F, F(y_{s_n}^{t_n}), F(y_{s_{n-1}}^{t_{n-1}}y_{s_n}^{t_n}), ..., F(y_{s_1}^{t_1}...y_{s_n}^{t_n})$$

Since the 0-cells $F = F(y_{s_1}^{t_1}...y_{s_n}^{t_n})$ it follows that the word $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ represents an element of F. From Chapter 3 we know that this word can be reduced to an X-word by applying the expansion, commuting and rearranging substitutions. Since an application of each substitution produces a loop homotopic to L, we observe that this process provides an explicit homotopy between L and the trivial loop. □

4.4 A relation on the 1-cells of X.

The rest of the chapter will be devoted to proving that the higher homotopy groups of X are trivial. In particular we will show that for any finite subcomplex Y of X, there is a subcomplex Y' of X such that $Y \subseteq Y'$ and Y' is homeomorphic to a nonpositively curved cube complex.

In this section we will define an equivalence relation on the 1-cells of X. This will play a crucial role in the proof.

Definition 4.4.1. Let e_1, e_2 be 1-cells in X that admit parametrizations of the form $e_1 = \{F(\lambda\tau_1), F\tau_1\}, e_2 = \{F(\lambda\tau_2), F(\tau_2)\}$. Such parametrizations are called *common parametrizations* for e_1, e_2.

Given a common parametrization as above, if λ' is a special form of the same type and parity as λ then we can find a common parametrization

$$e_1 = \{F(\lambda'\tau_1'), F(\tau_1')\}, e_2 = \{F(\lambda'\tau_2'), F(\tau_2')\}$$

as follows. Let $f \in F$ such that $\lambda \cdot f \sim \lambda'$. Then $F(\lambda\tau_i) = F(\lambda' f^{-1}\tau_i)$, and so we get $\tau_i' = f^{-1}\tau_i$.

Now we define the notion of *percolating support* of a normal form, which will be useful in this section.

Definition 4.4.2. Given a word λ in the infinite generating set, recall that the support of λ in $\mathbb{R}$, or $supp(\lambda)$, is the closure of the set of points in $\mathbb{R}$ that are not fixed by λ. Given a normal form $fy_{s_1}^{t_1}...y_{s_n}^{t_n}$ we denote the percolating support of $fy_{s_1}^{t_1}...y_{s_n}^{t_n}$ as $supp_Y(fy_{s_1}^{t_1}...y_{s_n}^{t_n})$, and this is defined as $supp_Y(fy_{s_1}^{t_1}...y_{s_n}^{t_n}) = supp(y_{s_1}^{t_1}...y_{s_n}^{t_n})$. So if an element $\lambda \in G$ has a normal form $f\nu$ for $f \in F$ and a Y-normal form ν, then we define $supp_Y(\lambda) = supp_Y(f\nu) = supp(\nu)$. By our definition it follows that if $\psi \in G$ and $f \in F$, then $supp_Y(\psi) = supp_Y(f\psi)$.

The following lemmas are important technical properties of the notion of a percolating support. The first three are corollaries of Theorem 3.3.8 and we leave proofs of these to the reader.

Lemma 4.4.3. *Let $gy_{s_1}^{t_1}...y_{s_n}^{t_n}$ be a normal form where $g \in F$ and $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ is a Y-normal form. Let $I \subseteq \mathbb{R}$ be the set of reals for which the action of $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ does not preserve tail equivalence. Then*

$$supp_Y(gy_{s_1}^{t_1}...y_{s_n}^{t_n}) = supp(y_{s_1}^{t_1}...y_{s_n}^{t_n}) = \bar{I}$$

Lemma 4.4.4. *Let $f \in F$ and let $gy_{s_1}^{t_1}...y_{s_n}^{t_n}$ be a normal form. Let $hy_{u_1}^{v_1}...y_{u_m}^{v_m}$ be a normal form for the group element described by $f^{-1}(gy_{s_1}^{t_1}...y_{s_n}^{t_n})f$. Let $\bar{I} = supp_Y(gy_{s_1}^{t_1}...y_{s_n}^{t_n})$ Then $supp_Y(hy_{u_1}^{v_1}...y_{u_m}^{v_m}) = f(\bar{I})$.*

Lemma 4.4.5. *Let $f \in F$ and let $\nu \in G$ be such that $supp(f) \subseteq supp_Y(\nu)$. Then it follows that $supp_Y(\nu) = supp_Y(f\nu f^{-1})$.*

Lemma 4.4.6. *Let $\nu_1, \nu_2 \in G$ and let $I \subseteq \mathbb{R}$. If $I \subseteq supp_Y(\nu_1)$ and $I \cap supp(\nu_2)$ is a nullset, then it follows that $I \subseteq supp_Y(\nu_1\nu_2)$.*

Proof. Let $fy_{s_1}^{t_1}...y_{s_n}^{t_n}, gy_{u_1}^{v_1}...y_{u_m}^{v_m}$ be normal forms for the elements ν_1, ν_2 respectively such that $f, g \in F$.

First observe that by our assumption it follows that in fact $I \cap supp(g), I \cap supp(y_{u_1}^{v_1}...y_{u_m}^{v_m})$ are nullsets. The set $I \cap supp(g)$ is a nullset because otherwise there is an interval $I_1 \subseteq (I \cap supp(g))$ with the property that $I_1 \cap supp(gy_{u_1}^{v_1}...y_{u_m}^{v_m})$ is a nullset. It follows that for any $r \in I_1$, $(gy_{u_1}^{v_1}...y_{u_m}^{v_m})(r) = r$. This is impossible by Theorem 3.3.8. It also follows that $I \cap supp(y_{u_1}^{v_1}...y_{u_m}^{v_m})$ is a nullset.

Now let $hy_{p_1}^{q_1}...y_{p_l}^{q_l}$ be a standard form such that

1. $hy_{p_1}^{q_1}...y_{p_l}^{q_l} = y_{s_1}^{t_1}...y_{s_n}^{t_n}$.
2. $hy_{p_1}^{q_1}...y_{p_l}^{q_l}$ is obtained by performing expansion and rearranging substitutions on percolating elements of $y_{s_1}^{t_1}...y_{s_n}^{t_n}$.
3. g acts on $p_1, ..., p_n$, i.e. it acts on $y_{p_1}^{q_1}...y_{p_l}^{q_l}$.
4. For each $1 \leq i \leq l, 1 \leq j \leq m$ we have $|g(p_i)| > |u_j|$.

Note that it follows that $y_{p_1}^{q_1}...y_{p_l}^{q_l}$ has no potential cancellations and that $supp(y_{p_1}^{q_1}...y_{p_l}^{q_l}) = supp(y_{s_1}^{t_1}...y_{s_n}^{t_n})$.

Now we have

$$\nu_1\nu_2 = (fhy_{p_1}^{q_1}...y_{p_l}^{q_l})(gy_{u_1}^{v_1}...y_{u_m}^{v_m})$$

$$= (fhg)(y_{g(p_1)}^{q_1}...y_{g(p_l)}^{q_l}y_{u_1}^{v_1}...y_{u_m}^{v_m})$$

Now converting the word

$$(fhg)(y_{g(p_1)}^{q_1}...y_{g(p_l)}^{q_l}y_{u_1}^{v_1}...y_{u_m}^{v_m})$$

into a normal form does not involve any cancellations along infinite binary sequences $u \in I$ since $I \cap supp(g)$ and $I \cap supp(y_{u_1}^{v_1}...y_{u_m}^{v_m})$ are null sets, it follows that the resulting normal form $f'\lambda$ (with $f' \in F$ and λ a Y-normal form), has the property that $I \subseteq supp(\lambda)$. This proves our assertion. □

Now we shall describe the equivalence relation.

Definition 4.4.7. Let e_1, e_2 be two 1-cells of X. We say that $e_1 \doteq e_2$ if there is a common parametrization

$$e_1 = \{F\tau_1, F(\lambda\tau_1)\}, e_2 = \{F\tau_2, F(\lambda\tau_2)\}$$

such that $supp(\lambda) \cap supp(\tau_2\tau_1^{-1})$ is a nullset.

If this holds, consider the parametrizations

$$e_1 = \{F(\lambda\tau_1), F(\lambda^{-1}(\lambda\tau_1))\}, e_2 = \{F(\lambda\tau_2), F(\lambda^{-1}(\lambda\tau_2))\}$$

(with basepoints $\lambda\tau_1, \lambda\tau_2$ and parameter λ^{-1}). Since $supp(\lambda) \cap supp(\tau_2\tau_1^{-1})$ is a nullset, we have that

$$\lambda\tau_1(\lambda\tau_2)^{-1} = \lambda\tau_1\tau_2^{-1}\lambda^{-1} = \tau_1\tau_2^{-1}$$

It follows that

$$supp(\lambda^{-1}) \cap supp(\lambda\tau_2(\lambda\tau_1)^{-1}) = supp(\lambda) \cap supp(\tau_2\tau_1^{-1})$$

is a nullset. So this parametrization also satisfies the definition of equivalence. This motivates defining the following notion. In the above the pairs of 0-cells $F\tau_1, F\tau_2$ and $F(\lambda\tau_1), F(\lambda\tau_2)$ are defined to be *associated pairs* for the equivalence $e_1 \doteq e_2$.

We remark that in the above, the orientations $e_1 = \{F(\lambda\tau_1), F(\lambda^{-1}(\lambda\tau_1)\}$ (with parameter λ^{-1}) $e_2 = \{F(\tau_2), F(\lambda\tau_2)\}$ (with parameter λ) do not satisfy the condition of the definition. This is made precise in the Lemma below.

Lemma 4.4.8. *Let $e_1 = \{u_1 = F(\lambda\tau_1), u_2 = F(\tau_1)\}, e_2 = \{v_1 = F(\lambda\tau_2), v_2 = F\tau_2\}$ be 1-cells such that $e_1 \doteq e_2$ and the given parametrization satisfies the definition of equivalence. Then u_i, v_i are associated pairs for this equivalence but there is no parametrization for which u_1, v_2 or u_2, v_1 are associated pairs for this equivalence.*

Proof. Let $e_1 = \{F(\psi\eta_1), F\eta_1\}, e_2 = \{F(\psi\eta_2), F\eta_2\}$ be a parametrization satisfying $e_1 \doteq e_2$ and so that $F(\psi\eta_1) = u_2, F\eta_2 = v_2$. Note that since $F(\psi\eta_2) = F(\lambda\tau_2), F\eta_2 = F\tau_2$, it follows that the type of ψ, λ must be the same. But since $F(\psi\eta_1) = F\tau_1, F\eta_2 = F(\lambda\tau_1)$, it follows that the type of ψ, λ is different. This is a contradiction. □

Lemma 4.4.9. *The relation $\doteq$ is an equivalence relation.*

Proof. We will show that if $e_1 \doteq e_2$, $e_2 \doteq e_3$ then $e_1 \doteq e_3$. Let

$$e_1 = \{F\tau_1, F(\lambda\tau_1)\}, e_2 = \{F\tau_2, F(\lambda\tau_2)\}$$

be parametrizations such that $supp(\tau_2\tau_1^{-1}) \cap supp(\lambda)$ is a nullset. Also, let

$$e_2 = \{F\psi_1, F(\nu\psi_1)\}, e_3 = \{F\psi_2, F(\nu\psi_2)\}$$

be parametrizations such that $supp(\nu) \cap supp(\psi_1\psi_2^{-1})$ is a nullset and $F\psi_1 = F\tau_2$.

Now there is a $g \in F$ such that $\psi_1 = g\tau_2$. By replacing ν with an equivalent special form if necessary, we can assume that g acts on ν. Moreover, since $F(\nu g\tau_2) = F(\lambda\tau_2)$ it follows that $F(\nu g) = F\lambda$ and $e_2 = \{F(\lambda g^{-1}\tau_2), F(g^{-1}\tau_2)\}$. We also obtain that $e_3 = \{F(\lambda g^{-1}\psi_2), F(g^{-1}\psi_2)\}$.

Now since $supp(g^{-1}\nu g) = supp(\lambda)$ and $supp(g^{-1}\nu g) \cap supp(g^{-1}\psi_2\psi_1^{-1}g)$ is a nullset, it follows that $supp(\lambda) \cap supp(g^{-1}\psi_2\tau_2^{-1})$ is a nullset.

Therefore

$$supp(\lambda) \cap supp(((g^{-1}\psi_2)\tau_2^{-1})(\tau_2\tau_1^{-1}))$$

and

$$supp(\lambda) \cap supp(g^{-1}\psi_2\tau_1^{-1})$$

are nullsets. This proves our claim. □

From the proof of the previous lemma it follows that for 1-cells $e_1, ..., e_n$ in X, such that $e_i \doteq e_j$ for each $1 \leq i, j \leq n$, there are elements $\tau_1, ..., \tau_n \in G$ and a special form λ such that the following holds.

1. $e_i = \{F\tau_i, F(\lambda\tau_i)\}$.

2. For every $1 \leq i, j \leq n, i \neq j$, $supp(\lambda) \cap supp(\tau_i \tau_j^{-1})$ are nullsets.

Such a description will be useful in the proofs to follow. Now we show that any two distinct 1-cells e_1, e_2 that share a 0-cell are inequivalent.

Lemma 4.4.10. *Let e_1, e_2 be two distinct 1-cells of X that are incident to the same 0-cell. Then e_1, e_2 are inequivalent.*

Proof. Let $e_1 = \{F\tau, F(\lambda_1 \tau)\}, e_2 = \{F\tau, F(\lambda_2 \tau)\}$. Note that if λ_1, λ_2 are of a different parity, then it is impossible to have a common parametrization for e_1, e_2 and hence they must be inequivalent. So we assume that λ_1, λ_2 have the same parity.

There are two cases to consider.

1. λ_1, λ_2 have the same type.
2. λ_1, λ_2 have different types.

Case (1): There is a common parametrization

$$e_1 = \{F(\psi(f\tau)), F(f\tau)\}, e_2 = \{F(\psi(g\tau)), F(g\tau)\}$$

such that $f, g \in F$ and $supp(\psi) \cap supp((f\tau)(g\tau)^{-1})$ is a nullset. By replacing ψ by an equivalent special form if necessary we can assume that f, g act on ψ.

Let ψ_1, ψ_2 be special forms such that $\psi \cdot f = \psi_1, \psi \cdot g = \psi_2$. Then $F(\psi f) = F(\psi_1) = F(\lambda_1), F(\psi g) = F(\psi_2) = F(\lambda_2)$. Since $supp(\psi) \cap supp(fg^{-1})$ is a nullset, it follows that $(fg^{-1})\psi(gf^{-1}) = \psi$, and so $f^{-1}\psi f = g^{-1}\psi g$. Therefore $F\psi_1 = F\psi_2$ and hence $F\lambda_1 = F\lambda_2$. This contradicts or assumption that e_1, e_2 are distinct.

Case (2): There is a common parametrization

$$e_1 = \{F(\psi(\psi^{-1} f\tau)), F(\psi^{-1} f\tau)\}, e_2 = \{F(\psi(g\tau)), F(g\tau)\}$$

with parameter ψ such that $f, g \in F$ and $supp(\psi) \cap supp((\psi^{-1} f\tau)(g\tau)^{-1})$ is a nullset. A consequence of fact that $fg^{-1} \in F$, together with the statement about tail equivalence in Theorem 3.3.8 is that $supp(\psi) \subseteq supp(\psi^{-1} fg^{-1})$. This is a contradiction. □

We obtain the following Corollary immediately from the previous lemma together with the definition of equivalence.

Corollary 4.4.11. *Let Δ be a cluster based at τ and parametrized by $\lambda_1, ..., \lambda_n$. Let*

$$e_1 = \{F((\prod_{i \in I} \lambda_i)\tau), F((\lambda_j(\prod_{i \in I} \lambda_i)\tau)\}$$

and

$$e_2 = \{F((\prod_{i \in J} \lambda_i)\tau), F(\lambda_k(\prod_{i \in J} \lambda_i)\tau)\}$$

for $I, J \subseteq \{1, ..., n\}$ and $j \notin I, k \notin J$ be two facial 1-cells. If $e_1 \doteq e_2$ then it follows that e_1, e_2 are "parallel" facial 1-cells of the cluster, i.e. $j = k$.

We now define two (related) notions called *disparate 1* and *disparate 2*, which will be crucial in determining when two 1-cells are inequivalent.

Definition 4.4.12. (Disparate 1) Consider a pair of a parametrizations

$$e = \{F(\lambda\tau_1), F\tau_1\}, u = F\tau_2$$

1. We say that the pair of parametrizations is *disparate with associated pair* $F\tau_1, F\tau_2$ if $supp(\lambda) \subseteq supp_Y(\tau_2\tau_1^{-1})$.

2. We say that the above pair of parametrizations is *disparate with associated pair* $F(\lambda\tau_1), F\tau_2$ if $supp(\lambda^{-1}) \subseteq supp_Y(\tau_2\tau_1^{-1}\lambda^{-1})$.

If both the conditions hold for the given parametrizations, then we simply say that the pair of parametrizations is *disparate.*

Now we show that the above definition is independent of parametrization. Therefore the property of being disparate (with associated pairs or otherwise) is a property of the pair e, u, rather than just a property of the parametrizations. This is done in the next few lemmas. We will also show that if e, u is disparate, then there is no e' incident to u such that $e \doteq e'$.

Lemma 4.4.13. *Let* $e = \{F(\lambda\tau_1), F\tau_1\}$ *and* $u = F\tau_2$ *be such that the parametrizations satisfy the definition of disparate with associated pair* $F\tau_1, F\tau_2$*. Then if* $\lambda' \sim \lambda$ *is a special form, the parametrization* $e = \{F(\lambda'\tau_1), F\tau_1\}$ *and* $u = F\tau_2$ *also satisfy the definition of disparate with the associated pair* $F\tau_1, F\tau_2$*. Moreover, we can use any special form* $\tau_2' \sim \tau_2$ *as the representative for the coset* $F\tau_2$ *to satisfy the definition of disparate.*

Proof. It is clear that $supp(\lambda') = supp(\lambda)$, whenever λ', λ are equivalent special forms. So it follows that $supp(\lambda') \subseteq supp_Y(\tau_2\tau_1^{-1})$. The last claim of the Lemma follows from the fact that $supp_Y(f\tau_2\tau_1^{-1}) = supp_Y(\tau_2\tau_1^{-1})$ for any $f \in F$. □

Lemma 4.4.14. *Let* $e = \{F(\lambda_1\tau_1), F\tau_1\} = \{F(\lambda_2\tau_2), F\tau_2\}$ *and* $u = F\psi$ *be a pair such that* $F\tau_1 = F\tau_2$ *and* e, u *is disparate with respect to the parametrization* $e = \{F(\lambda_1\tau_1), F\tau_1\}$ *with associated pair* $F\tau_1, F\psi$*. Then it is disparate with respect to the parametrization* $\{F(\lambda_2\tau_2), F\tau_2\}$ *with associated pair* $F\tau_2, F\psi$*.*

Moreover, if $e = \{F(\lambda_1\tau_1), F\tau_1\}$ *and* $u = F\psi$ *are disparate, the parametrization* $e = \{F(\lambda_2\tau_2), F\tau_2\}$ *and* $u = F\psi$ *also satisfies the definition of disparate.*

Proof. First we shall prove the first claim of our lemma. We know that $supp(\lambda_1) \subseteq supp_Y(\psi\tau_1^{-1})$. Now $\tau_1 = f\tau_2$ for some $f \in F$. We can assume that f acts on λ_1 by replacing λ_1 with an equivalent special form if necessary. Let λ_1' be a special form such that $\lambda_1' = f\lambda_1 f^{-1}$. It follows that $\lambda_2 \sim \lambda_1'$. So for the rest of the proof we can assume that $\lambda_2 = f^{-1}\lambda_1 f$. (By Lemma 4.4.13 this does not change the fact that the definition of disparate is satisfied.) We also note that $supp(\lambda_2) = supp(f\lambda_1 f^{-1})$.

We know that $\psi\tau_2^{-1} = \psi\tau_1^{-1} f$ and so

$$supp_Y(\psi\tau_2^{-1}) = supp_Y(\psi\tau_1^{-1} f) = supp_Y(f^{-1}\psi\tau_1^{-1} f)$$

Since $supp(f^{-1}\lambda_1 f) \subseteq supp_Y(f^{-1}\psi\tau_1^{-1} f)$ we conclude that $supp(\lambda_2) \subseteq supp_Y(\psi\tau_2^{-1})$. So the pair is disparate (for the given associated pair) with the new parametrization.

For the second claim, our assumption says that in addition to the above, it is also true that $supp(\lambda_1) \subseteq supp_Y(\psi\tau_1^{-1}\lambda_1^{-1})$. It suffices to show that $supp(\lambda_2) \subseteq supp_Y(\psi\tau_2^{-1}\lambda_2^{-1})$. Now

$$supp_Y(\psi\tau_2^{-1}\lambda_2^{-1}) = supp_Y(f^{-1}\psi\tau_2^{-1}\lambda_2^{-1})$$

$$= supp_Y(f^{-1}\psi\tau_1^{-1} f(f^{-1}\lambda_1^{-1} f)) = supp_Y(f^{-1}\psi\tau_1^{-1}\lambda_1^{-1} f)$$

Now we know that since $supp(\lambda_1) \subseteq supp_Y(\psi\tau_1^{-1}\lambda_1^{-1})$, it follows that

$$supp(f^{-1}\lambda_1^{-1} f) \subseteq supp_Y(f^{-1}\psi\tau_1^{-1}\lambda_1^{-1} f)$$

This implies that

$$supp(\lambda_2) \subseteq supp_Y(\psi\tau_1^{-1}\lambda_1^{-1} f) = supp_Y(\psi\tau_2^{-1}\lambda_2^{-1})$$

proving our assertion. □

We now show that the notion of disparate with respect to a fixed 0-cell is actually a property of equivalence classes of 1-cells. We make this precise in the next lemma.

Lemma 4.4.15. *Let* $e_1 = \{F(\lambda\tau_1), F\tau_1\}, e_2 = \{F(\lambda\tau_2), F\tau_2\}$ *be* 1*-cells such that* $e_1 \doteq e_2$ *and the given parametrizations satisfy the definition of equivalence with associated pairs* $F(\lambda\tau_1), F(\lambda\tau_2)$ *and* $F\tau_1, f\tau_2$*. Let* $u = F\tau_3$ *be a* 0*-cell such that* e_1, u *are disparate with associated pair* $F\tau_1, F\tau_3$*. Then it follows that* e_2, u *are disparate with associated pair* $F\tau_2, F\tau_3$*.*

Proof. From the definitions of disparate and the equivalence relation, it follows that:

1. $supp(\lambda) \cap supp(\tau_1\tau_2^{-1})$ are nullsets.
2. $supp(\lambda) \subseteq supp_Y(\tau_3\tau_1^{-1})$.

It follows from Lemma 4.4.6 that

$$supp(\lambda) \subseteq supp_Y((\tau_3\tau_1^{-1})(\tau_1\tau_2^{-1})) = supp_Y(\tau_3\tau_2^{-1})$$

□

Lemma 4.4.16. *If the pair* e, u *is disparate, and* $e \doteq e'$*, then* e', u *is disparate. If* e, u *is disparate, then there is no* 1*-cell* e' *incident to* u *such that* $e \doteq e'$*.*

Proof. The first statement follows from the Lemmas 4.4.15 and 4.4.14. For the second statement, assume by way of contradiction that there are parametrizations

$$e = \{F(\lambda\tau_2), F\tau_2\}, e' = \{F(\lambda\tau_2), F\tau_2\}, u = F\tau_2$$

such that $e \doteq e'$ with associated pair $F\tau_1, F\tau_2$. Then it follows that $supp(\lambda) \cap supp(\tau_2\tau_1^{-1})$ is a nullset. This contradicts the fact that e, u are disparate. □

The following criterion allows us to detect equivalent 1-cells.

Lemma 4.4.17. *Let* $e_1 = \{F(\lambda\tau_1), F\tau_1\}$ *be a* 1*-cell and* $u = F\tau_2$ *be a* 0*-cell such that* $supp(\lambda) \cap supp_Y(\tau_2\tau_1^{-1})$ *is a nullset. Then there is a* 1*-cell* $e_1 = \{F(\lambda\tau_3), F\tau_3\}$ *such that* $e_1 \doteq e_2$ *with associated pairs* $F\tau_1, F\tau_3$*,* $F(\lambda\tau_1), F(\lambda\tau_3)$*, and so that* $F\tau_3 = F\tau_2$*.*

Proof. Let $f\nu$ be a normal form such that $f \in F$, ν is a Y-normal form and $f\nu = \tau_2\tau_1^{-1}$. By our assumption it follows that

$$supp(\lambda) \cap supp(\nu) = supp(\lambda) \cap supp(\lambda) \cap supp(f^{-1}\tau_2\tau_1^{-1})$$

is a nullset.

It follows that the 1-cell $e_1 = \{F(\lambda\tau_3), F\tau_3\}$ where $\tau_3 = f^{-1}\tau_2$ has the required property. □

Now we will define a second notion of disparate. First we need to define a relation on pairs of elements of Ω.

Definition 4.4.18. (Overlay for elements of Ω) Let $F\lambda_1, F\lambda_2 \in \Omega$. We say that the pair $F\lambda_1, F\lambda_2$ contains an *overlay* if there are special forms $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ and $y_{u_1}^{v_1}...y_{u_m}^{v_m}$ such that $Fy_{s_1}^{t_1}...y_{s_n}^{t_n} = F\lambda_1$, $Fy_{u_1}^{v_1}...y_{u_m}^{v_m} = F\lambda_2$, and

$$\{y_{u_1}^{v_1}, ..., y_{u_m}^{v_m}\} \cap \{y_{s_1}^{t_1}, ..., y_{s_n}^{t_n}\} \neq \emptyset$$

(Cancellation free.) Let $F\lambda_1, F\lambda_2 \in \Omega$. The pair λ_1, λ_2 is said to be *cancellation free* if $F\lambda_1, F\lambda_2$ does not contain an overlay.

The following is an immediate consequence of the above definition and Lemma 5.9 in [24].

Lemma 4.4.19. *Let λ_1, λ_2 be special forms such that $F\lambda_1, F\lambda_2$ are cancellation free. Let $f \in F$ be such that f acts on λ_1, λ_2, and let ν_1, ν_2 be special forms such that $\nu_i = f^{-1}\lambda_i f$. Then the pair $F\nu_1, F\nu_2$ is cancellation free.*

We can now introduce the second notion of disparate which is defined for pairs of 1-cells that share an incident 0-cell.

Definition 4.4.20. (Disparate 2)Let $e_1 = \{F(\lambda_1\tau), F\tau\}, e_2 = \{F(\lambda_2\tau), F\tau\}$ be parametrizations of 1-cells in X. Then the parametrizations are said to be disparate if $F\lambda_1, F\lambda_2$ is cancellation free. Otherwise the parametrizations are said to be *coupled*. (Orthogonal 1-cells) Let $e_1 = \{F(\lambda_1\tau), F\tau\}, e_2 = \{F(\lambda_2\tau), F\tau\}$ be parametrizations of 1-cells such that $supp(\lambda_1) \cap supp(\lambda_2)$ is a nullset. Then the parametrizations are said to be *orthogonal*. We remark that if the parametrizations are orthogonal then they are also disparate.

We remark that we shall use the word *disparate* for both definitions and the specific definition that is being used will be clear from the context. We will now show that these notions of disparate and orthogonal are both properties of 1-cells that are independent of parametrizations. We leave the proof of the following lemmas to the reader. (The proofs are a routine application of Lemma 4.4.19.)

Lemma 4.4.21. *Let $e_1 = \{F(\lambda_1\tau), F\tau\}, e_2 = \{F(\lambda_2\tau), F\tau\}$ be two 1-cells in X such that e_1, e_2 are disparate in the given parametrization. Let $e_1 = \{F(\nu_1\psi), F\psi\}, e_2 = \{F(\nu_2\psi), F\psi\}$ be another parametrization such that $F\psi = F\tau$. Then e_1, e_2 are disparate with respect to this parametrization as well.*

Lemma 4.4.22. *Let $e_1 = \{F(\lambda_1\tau), F\tau\}, e_2 = \{F(\lambda_2\tau), F\tau\}$ be two 1-cells in X such that e_1, e_2 are orthogonal in the given parametrization. Let $e_1 = \{F(\nu_1\psi), F\psi\}, e_2 = \{F(\nu_2\psi), F\psi\}$ be another parametrization such that $F\psi = F\tau$. Then e_1, e_2 are orthogonal with respect to this parametrization as well.*

4.5 Expansions and systems of cells.

In this section we will describe two procedures that take as input a finite set of 1-cells of X and a finite set of 0-cells incident to them, and produce a new set of 1-cells and 0-cells. These procedures will be used in the proof that the higher homotopy groups of X are trivial, which will be shown at the end of this chapter. In particular we will start with the 1-skeleton of a finite subcomplex Y of X, and perform a sequence of these procedures to obtain a set of 0-cells and 1-cells in X, and we shall use these as a building block to build a subcomplex Y' of X that is homeomorphic to a nonpositively curved cube complex and contains Y.

We will now define an operation that takes as input a parametrized 1-cell e of X incident to a 0-cell u of X, and produces a set of 1-cells $e_1, .., e_n$ incident to u.

Definition 4.5.1. (Decomposition) Let λ be a special form and $\nu_1, ..., \nu_n$ be a sorted list of pairwise independent special forms such that $\lambda \sim \nu_1...\nu_n$. Then the formal expression $\nu_1...\nu_n$ is said to be a *decomposition of* λ *as a product of special forms*, or simply a *decomposition*. We also say $\lambda \sim \nu_1...\nu_n$ is a decomposition.

Definition 4.5.2. (Expansion) Input: A 1-cell $e = \{F(\lambda\tau), F\tau\}$, a decomposition $\nu_1...\nu_n$ of λ, and a 0-cell $F\tau$ incident to e.

Output: The expansion operation *at* $F\tau$ for this decomposition produces the following 1-cells as the output:

$$e_1 = \{F(\nu_1\tau), F\tau\}, e_2 = \{F(\nu_2\tau), F\tau\}, ..., e_n = \{F(\nu_n\tau), F\tau\}$$

Definition 4.5.3. (op-expansion) An expansion of $e = \{F(\lambda\tau), F\tau\}$ with respect to the decomposition $\nu_1...\nu_n$ of λ at $F\lambda\tau$ is the defined to be the expansion at $F((\nu_1..\nu_n)\tau) = F\lambda\tau$ with respect to the parametrization

$$\{F((\nu_1..., \nu_n)^{-1}(\nu_1...\nu_n\tau)), F(\nu_1...\nu_n\tau)\}$$

and decomposition $\nu_1^{-1}...\nu_n^{-1}$ of $(\nu_1...\nu_n)^{-1}$. (Note that ν_i, ν_j commute, so $\nu_1^{-1}...\nu_n^{-1} = \nu_n^{-1}...\nu_1^{-1}$.) This produces the 1-cells

$$e_1' = \{F(\nu_1^{-1}(\nu_1...\nu_n\tau)), F(\nu_1...\nu_n\tau)\}, ..., e_n' = \{F(\nu_n^{-1}(\nu_1...\nu_n\tau)), F(\nu_1...\nu_n\tau)\}$$

We will call this second type of expansion an *op-expansion*, since this is an analogous expansion for the opposite orientation of the parameters.

Finally, we can combine the two notions of expansion above to obtain a *two sided expansion*.

Definition 4.5.4. (Two sided expansion) A two sided expansion of $e = \{F(\lambda\tau), F\tau\}$ with the above decompositon produces both families of 1-cells

$$e_1 = \{F(\nu_1\tau), F\tau\}, ..., e_n = \{F(\nu_n\tau), F\tau\}$$

$$e_1' = \{F(\nu_1^{-1}(\nu_1...\nu_n\tau)), F((\nu_1...\nu_n)\tau)\}, ..., e_n' = \{F(\nu_n^{-1}(\nu_1...\nu_n)\tau), F((\nu_1...\nu_n)\tau)\}$$

We remark that this can be viewed as performing expansions at both incident 0-cells of e using analogous decompositions.

For the above expansions consider the n-cluster Δ based at τ and parametrized by $\nu_1, ..., \nu_n$. It follows that the 1-cells $e_1, ..., e_n$ of the above expansion of e at $F\tau$ are precisely the facial 1-cells of Δ incident to $F\tau$. And the 1-cells $e_1', ..., e_n'$ of the op-expansion are precisely the facial 1-cells of Δ incident to $F(\nu_1...\nu_n\tau)$.

In particular it follows that $e_i \doteq e_i'$ for each $1 \leq i \leq n$ and the associated pairs for each equivalence are $F\tau, F(\nu_i^{-1}(\nu_1...\nu_n)\tau))$ and $F(\nu_i\tau), F((\nu_1...\nu_n)\tau)$.

Definition 4.5.5. (Offspring and parent) For the above expansions we say that the cells $e_1, ..., e_n$ (or $e_1', ..., e_n'$) are *offsprings* of e, and e is the *parent*. If we perform a sequence of expansions on e and its offsprings (at the same 0-cell each time), then an offspring of an offspring is also considered an offspring of e.

The proof of the following lemma is left to the reader.

Lemma 4.5.6. *(Independence of choice of parameters) Let* $e = \{F(\lambda\tau), F\tau\}$ *be a* 1*-cell and* $\nu_1 ... \nu_n$ *be a decomposition of* λ*. Let* $e = \{F(\lambda'(f\tau)), F(f\tau)\}$ *be another parametrization, with the decomposition* $\nu'_1 ... \nu'_n$ *of* λ' *where* $\nu'_i \sim \nu_i \cdot f^{-1}$*. Both associated expansions of* e *at* $F\tau$ *produce the same* 1*-cells. It also follows that both associated expansions of* e *at* $F(\lambda\tau)$ *produce the same* 1*-cells.*

(Sequence of expansions) If e *is a* 1*-cell, and if* $e_1, ..., e_n$ *are* 1*-cells that are produced after performing a sequence of expansions on* e *and its offsprings, all at the same* 0*-cell incident to* e*, then we can in fact obtain* $e_1, ..., e_n$ *from a single expansion of* e *at that* 0*-cell.*

Definition 4.5.7. Let e be a 1-cell incident to a 0-cell u, We define $E(e, u)$ to be the set of expansions of e at u. The set of expansions $E(e, u)$ is endowed with a natural partial order defined in the following manner. Let $e = \{F(\lambda\tau), F\tau\}, u = F\tau$. (By Lemma 4.5.6 we can choose any set of parameters for the sake of the definition.) Let $\alpha, \beta \in E(e, u)$ be expansions and let $\nu_1 ... \nu_n \sim \lambda$ and $\psi_1 ... \psi_m \sim \lambda$ be the respective decompositions of λ associated with α, β. Then $\alpha < \beta$ if there is a partition $\{i_0 = 1, 2, ..., i_1 - 1\}, \{i_1, ..., i_2 - 1\}, ...\{i_{m-1}, i_{m-1} + 1, ..., i_m - 1 = n\}$ of $\{1, ..., n\}$ such that $\nu_{i_{k-1}} \nu_{i_{k-1}+1} ... \nu_{i_k - 1} \sim \psi_k$ for each $1 \leq k \leq m$. In particular it holds that the expansion α is a composition of the expansion β followed by expansions on the offsprings of e in β.

The proofs of the following lemmas are straightforward and are left to the reader.

Lemma 4.5.8. *Let* e *be a* 1*-cell incident to a* 0*-cell* u*. For each* $\alpha, \beta \in E(e, u)$ *there is a* $\gamma \in E(e, u)$ *such that* $\gamma < \alpha$ *and* $\gamma < \beta$*.*

Lemma 4.5.9. *Let* $e_1 = \{F(\lambda_1\tau), F\tau\}, e_2 = \{F(\lambda_2\tau), F\tau\}$ *be* 1*-cells such that* e_1, e_2 *are disparate. Let* $\lambda_1 = \psi_1...\psi_n, \lambda_2 = \nu_1...\nu_m$ *be decompositions. Let* $e_{1,1}, ..., e_{1,n}$ *and* $e_{2,1}, ..., e_{2,m}$ *be the* 1*-cells obtained from the respective expansions of* e_1, e_2 *at* $F\tau$*. Then each pair of* 1*-cells in the set* $\{e_{1,1}, ..., e_{1,n}, e_{2,1}, ..., e_{2,m}\}$ *is disparate.*

Lemma 4.5.10. *Let* $e = \{v_1, v_2\}$ *be a* 1*-cell and* u *be a* 0*-cell such that* e, u *is disparate with associated pair* v_1, u*. If* $e_1, ..., e_n$ *are offsprings of an expansion of* e *at* v_1*, then for each* $1 \leq j \leq n$ e_j, u *is disparate with associated pair* v_1, u*. Moreover from this and Lemma 4.4.15 it follows that if* e, u *is disparate, then* e_i, u *is disparate for each* $1 \leq i \leq n$*.*

Lemma 4.5.11. *Let* e_1, e_2 *be* 1*-cells incident to* 0*-cells* u, v *respectively such that* $e_1 \doteq e_2$ *and* u, v *is an associated pair for this equivalence. Let* e_1', e_2' *be offsprings of expansions of* e_1, e_2 *respectively at* u, v *such that* $e_1' \doteq e_2'$*. Then it follows that* u, v *is an associated pair for* $e_1' \doteq e_2'$*.*

Recall that if e_1, e_2 are 1-cells incident to a 0-cell u have the property that they are not disparate, then they are said to be coupled. We shall define a similar notion for a collection of 1-cells that are incident to a given 0-cell.

Definition 4.5.12. Let $I = \{e_1, ..., e_n\}$ be a set of 1-cells that are incident to a 0-cell u. This set of 1-cells is said to be *coupled* if for any pair e_i, e_j, there are 1-cells $e_i = e_{i_1}, ..., e_{i_k} = e_j \in I$ such that the pair $e_{i_l}, e_{i_{l+1}}$ is coupled for each $1 \leq l \leq k-1$.

This motivates us to define the notion of a *coupling graph.*

Definition 4.5.13. (Coupling graph) Let $I = \{e_1, ..., e_n\}$ be 1-cells in X and let u be a 0-cell. The *coupling graph* $\Gamma_u(I)$ is defined as follows. The vertices are elements $\{e_{i_1}, , , ., e_{i_k}\}$ of I that are incident to u. Two vertices are connected by an edge if the corresponding pair of 1-cells is coupled at u. For each cell $e \in \{e_{i_1},,,., e_{i_k}\}$ we define $\Gamma_{u,e}(I)$ as the connected component containing e in $\Gamma_u(I)$.

Definition 4.5.14. (Equivariant isomorphism for coupling graphs.) Let I be a finite set of 1-cells in X and let u, v be 0-cells in X. Let $\{e_1, ..., e_k\}, \{e'_1, ..., e'_k\}$ be the vertices of connected components $\Gamma_{u,e_1}(I), \Gamma_{v,e'_1}(I)$ respectively. We say that $\Gamma_{u,e_1}(I), \Gamma_{v,e'_1}(I)$ are *equivariantly isomorphic* if there is a graph isomorphism $\phi : \Gamma_{u,e_1} \to \Gamma_{v,e'_1}$ such that $e_l \doteq \phi(e_l)$ for each $1 \leq l \leq k$ with associated pair u, v.

We say that $\Gamma_{u,e_1}(I), \Gamma_{v,e'_1}(I)$ are *op-equivariantly isomorphic* if there is a graph isomorphism $\phi : \Gamma_{u,e_1} \to \Gamma_{v,e'_1}$ such that $e_l \doteq \phi(e_l)$ for each $1 \leq l \leq k$ such that u, v is not an associated pair for the equivalence.

Definition 4.5.15. (The decoupling problem for a 0-cell.) Given a 0-cell u and 1-cells $e_1, ..., e_n$ incident to u, the decoupling problem requires one to perform expansions on $e_1, ..., e_n$ at u to obtain 1-cells $e'_1, ..., e'_m$ incident to u such that any pair e'_i, e'_j of distinct 1-cells is disparate.

Now we will describe a *decoupling procedure* that solves the decoupling problem at a 0-cell. First this is described for the case of two 1-cells.

Lemma 4.5.16. *(Decoupling lemma 1) Let e_1, e_2 be 1-cells incident to a 0-cell u. Then we can perform expansions on e_1, e_2 at u such that the union of the set of outputs of the expansions has the property that any distinct pair of 1-cells in the set is disparate.*

Proof. Let $e_1 = \{F(\lambda_1\tau), F\tau\}, e_2 = \{F(\lambda_2\tau), F\tau\}$. By considering the right action of the group, we can assume without loss of generality that $\tau = \emptyset$. There is a special form $\lambda_1 \sim \lambda_1'$ such that $\lambda_1'\lambda_2^{-1}$ is a standard form and so $\lambda_1\lambda_2^{-1} = f\lambda_1'\lambda_2^{-1}$ for some $f \in F$.

We know from Lemma 3.3.3 that $f\lambda_1'\lambda_2^{-1}$ can be further refined into a standard form with no potential cancellations. But since our word $\lambda_1'\lambda_2^{-1}$ is a standard form that is a product of two special forms, this can be done in the following way. For every percolating element y_s^t of λ_1^{-1}, either there is a percolating element y_u^v in λ_2^{-1} such that the pair y_s^t, y_u^v is a neighboring pair with a potential cancellation, or there is no such percolating element of λ_2^{-1} with this property. It follows that can find a special form $y_{s_1}^{t_1}...y_{s_n}^{t_n} \sim \lambda_1'$ such that there is a partition $I_1 \cup I_2$ of $\{1, ..., n\}$ such that if $i \in I_1$ then $y_{s_i}^{t_i}$ faces potential cancellation from a neighboring percolating element y_u^v of λ_2^{-1} and if $i \in I_2$ then $y_{s_i}^{t_i}$ does not face such a potential cancellation. Moreover, we can find such a word that satisfies that for any $i \in I_1, j \in I_2$ $|s_i| < |s_j|$.

We apply expansion and rearranging substitutions on percolating elements of $y_{s_1}^{t_1}...y_{s_n}^{t_n}\lambda_2^{-1}$ until all the potential cancellations for neighboring pairs have been performed. Therefore there is a word $y_{u_1}^{v_1}..y_{u_m}^{v_m} \sim \lambda_2$ such that for each $1 \leq i \leq n, 1 \leq j \leq m$ either $y_{s_i}^{t_i} = y_{u_j}^{v_j}$ or $y_{s_i}^{t_i}, y_{u_j}^{v_j}$ are cancellation free. It follows that the expansions associated with the decompositions $\lambda_1 \sim y_{s_1}^{t_1}...y_{s_n}^{t_n}, \lambda_2 \sim y_{u_1}^{v_1}...y_{u_m}^{v_m}$ satisfy the required property. □

In the proof of the previous Lemma, we have actually proved something stronger.

Corollary 4.5.17. *Let $\nu_1, ..., \nu_n$ be special forms that are pairwise independent, or equivalently that the supports supp$(\nu_1), ...,$ supp(ν_n) are pairwise disjoint. Let*

$$e_1 = \{F(\nu_1\tau), F\tau\}, ..., e_n = \{F(\nu_n\tau), F\tau\}$$

and $e = \{F(\lambda\tau), F\tau\}$ be 1-cells for some 0-cell $F\tau$. Then we can perform expan-

sions on the 1*-cells* $e_1, ..., e_n, e$ *to obtain* 1*-cells* $e'_1, ..., e'_n$ *that are pairwise disparate.*

Proof. In the proof of the previous lemma replace λ_1 with $\nu_1 ... \nu_n$. (This is not a special form, but the method of reducing $\nu_1 ... \nu_n \lambda^{-1}$ to a standard form with no potential cancellations can be applied here.) □

Lemma 4.5.18. *Let* $e_1, ..., e_n$ *be* 1*-cells incident to a* 0*-cell* u *that are pairwise disparate. Let* e *be a* 1*-cell incident to* u*. Then we can perform expansions on* $e_1, ..., e_n, e$ *to obtain* 1*-cells* $e'_1, ..., e'_m$ *that are pairwise disparate.*

Proof. We prove this by induction on n. The case $n = 1$ is simply the statement of Lemma 4.5.16. For the general case, assume that we have performed expansions on $e_1, ..., e_{n-1}, e$ to obtain 1-cells $e'_1, ..., e'_m$ that are pairwise disparate. Let e'_i be an offspring of e at the end of the sequence of expansions that were performed on e in the process. We know from Lemma 4.5.9 that if e_i is also an offspring of e_l for some $1 \leq l \leq n-1$, then e_i, e_n are disparate. Let

$$\{e'_{i_1}, ..., e'_{i_k}\} := \{e'_j \mid 1 \leq j \leq m, e'_j \text{ is an offspring of } e \text{ but not of } e_1, ..., e_{n-1}\}$$

Let

$$e'_{i_1} = \{F(\nu_1 \tau), F\tau\}, e'_{i_2} = \{F(\nu_2 \tau), F\tau\}, ..., e'_{i_k} = \{F(\nu_k \tau), F\tau\}$$

and $e_n = \{F(\lambda\tau), F\tau\}$. Since $e'_{i_1}, ..., e'_{i_k}$ are offsprings of e it follows that the supports $supp(\nu_1), ..., supp(\nu_k)$ are pairwise disjoint. Now from Corollary 4.5.17 it follows that we can perform expansions on $e'_{i_1}, ..., e'_{i_k}, e_n$ to obtain the required set of 1-cells. □

Lemma 4.5.19. *(Decoupling lemma* 2*) Let* $e_1, ..., e_n$ *be* 1*-cells in* X *that are incident to the same* 0*-cell. Then we can perform expansions on* $e_1, ..., e_n$ *at the* *given 0-cell to produce 1-cells $e'_1, ..., e'_m$ such that any pair e'_i, e'_j of distinct 1-cells is disparate.*

Proof. We proceed by induction on n. The case $n = 2$ was shown in Lemma 4.5.16.

By the inductive hypothesis we assume that this has been shown for $n - 1$. Consider a set of 1-cells $e_1, ..., e_n$ that are incident to the same 0-cell. By our inductive hypothesis we perform expansions on $e_1, ..., e_{n-1}$ at the 0-cell to obtain 1-cells $e'_1, ..., e'_m$ that are pairwise disparate. Now by Lemma 4.5.18 we can perform expansions on $e'_1, ..., e'_m, e_n$ at the given 0-cell to obtain the required set of 1-cells. □

Now we shall define a notion of simultaneous expansion for equivalent 1-cells.

Definition 4.5.20. (Equivariant expansion) Let

$$e_1 = \{F(\lambda\tau_1), F\tau_1\}, ..., e_n = \{F(\lambda\tau_n), F\tau_n\}$$

be 1-cells such that

$$e_1 \doteq e_2 \doteq ... \doteq e_n$$

and that the given set of parameters satisfies the definition of the equivalence, i.e.

$$supp(\lambda) \cap supp(\tau_i\tau_j^{-1}) = \emptyset$$

for each $1 \leq i, j \leq n$. Then a decomposition $\lambda = \nu_1...\nu_m$ can be used to perform an *equivariant expansion* to produce 1-cells

$$e_{i,1} = \{F(\nu_1\tau_i), F\tau_i\}, ..., e_{i,m} = \{F(\nu_m\tau_i), F\tau_i\}$$

for each $1 \leq i \leq n$.

(Mixed Equivariant expansion) Let $e_1, ..., e_n$ be as above. Let $I, J \subseteq= \{1, ..., n\}$ such that $I \cap J = \emptyset, I \cup J = \{1, ..., n\}$. We can use the decomposition as above to perform the *mixed equivariant expansion* that produces 1-cells

$$e_{i,1} = \{F(\nu_1 \tau_i), F\tau_i\}, ..., e_{i,m} = \{F(\nu_m \tau_i), F\tau_i\}$$

for each $i \in I$ and

$$e_{j,1} = \{F(\nu_1^{-1}(\nu_1 ... \nu_n)\tau_j), F(\nu_1 ... \nu_n \tau_j)\}, ..., e_{j,m} = \{F(\nu_m^{-1}(\nu_1 ... \nu_m \tau_j)), F(\nu_1 ... \nu_m \tau_j)\}$$

for each $j \in J$.

This notion of equivariant expansion is useful in making the following important observation.

Lemma 4.5.21. *Let e_1, e_2 be 1-cells incident to u and e_1', e_2' be 1-cells incident to v such that $e_1 \doteq e_1'$ and $e_2 \doteq e_2'$. Then if e_1, e_2 are disparate, it also follows that e_1', e_2' are disparate.*

Proof. Assume that e_1', e_2' are not disparate. Then there are expansions of e_1', e_2' at v with a common offspring e_3'. Following the expansion of e_i', we perform the analogous equivariant expansions (or mixed equivariant expansions) of e_1, e_2 at u and we obtain a common offspring $e_3 \doteq e_3'$ incident to u. This contradicts the assumption that e_1, e_2 are disparate. □

Now we are ready to define the notion of *a system of cells.*

Definition 4.5.22. (Systems of cells in $X^{(1)}$) Let I be a set of 1-cells in $X^{(1)}$, and let J be a set of the 0-cells in $X^{(1)}$ incident to 1-cells in I (Not necessarily all the 0-cells incident to 1-cells in I) such that the following holds. For each $e = \{u, v\} \in I$ there are 1-cells $e_1 = \{u_1, u_2\}, e_2 = \{v_1, v_2\}$ such that $u_1, v_1 \in J$ and $e \doteq e_1$ with associated pair u, u_1 and $e \doteq e_2$ with associated pair v, v_1. We call such a pair (I, J) *a system of cells* in $X^{(1)}$.

(Balanced system) We say that a system (I, J) is *balanced* if for each $u_i, u_j \in J, e_l \in I$ such that e_l is incident to u_i, either e_l, u_j is disparate, or there is a $e_k \in I$ incident to u_j such that $e_l \doteq e_k$, or equivalently (By Lemmas 4.5.21 and 4.5.11) Γ_{u_i,e_l} is either equivariantly isomorphic to Γ_{u_j,e_k} or op-equivariantly isomorphic to Γ_{u_j,e_k}.

(Free system) We say that a system (I, J) is *free* if it is balanced and for each $u \in J$ the graph $\Gamma_u(I)$ has no edges, i.e. for any pair of 1-cells $e_1, e_2 \in I$ incident to u, e_1, e_2 is disparate.

We define a partial order on systems as follows.

Definition 4.5.23. Let $(I_1, J), (I_2, J)$ be two systems. We say that $(I_1, J) \leq (I_2, J)$ if for each $e \in I_1$ either $e \in I_2$, or for some $n \geq 2$ there are 1-cells $e_1, ..., e_n \in I_2$ and a 0-cell $u \in J$ incident to e, such that $e_1, ..., e_n$ is the set of 1-cells produced by performing an expansion of e at u.

The remainder of the section is structured as follows.

- Let Y be a finite subcomplex of $X^{(1)}$, I be the set of 1-cells of Y and J be the set of 0-cells of Y. First we shall describe a procedure called the *Separation procedure* that takes the system (I, J) as input, and by performing expansions on elements of I, produces a balanced system (I', J) such that $(I, J) \leq (I', J)$.
- Then we describe an *Equivariant decoupling procedure* that takes the balanced system (I', J) as input and produces a free system (I'', J) such that $(I', J) \leq (I'', J)$ and hence $(I, J) \leq (I'', J)$.

4.5.1 Separation procedure

We shall first describe the case for a single pair, and then we shall describe a general serparation procedure. This procedure takes as input a 1-cell e and a 0-cell u, and produces a two-sided expansion of e producing 1-cells $e_1, ..., e_n$ such that for each $1 \leq i \leq n$ either there is a 1-cell e'_i incident to u such that $e'_i \doteq e_i$ or e_i, u are disparate. To do this we first need to prove the following lemma.

Lemma 4.5.24. *Let λ be a special form and let $f\nu$ be a normal form. There is a special form $y_{s_1}^{t_1}...y_{s_n}^{t_n} \sim \lambda$ such that for each $1 \leq i \leq n$ either $[s_i\bar{0}, s_i\bar{1}] \cap supp_Y(h\nu)$ is a nullset or $[s_i\bar{0}, s_i\bar{1}] \subseteq supp_Y(h\nu)$ where $supp(y_{s_i}^{t_i}) = [s_i\bar{0}, s_i\bar{1}] \subset 2^{\mathbb{N}}$.*

Proof. Given any special form $f\nu$, by Theorem 3.3.8 it follows that $supp_Y(f\nu)$ is always of the form $\bigcup_{1\leq i\leq m}[u_i\bar{0}, v_i\bar{1}]$ for finite binary sequences $u_1, ..., u_m, v_1, ..., v_m$ such that $u_i <_{\text{lex}} v_i$ and $v_i <_{\text{lex}} u_{i+1}$. Given any such set and a special form λ, any special form $y_{u_1}^{t_1}...y_{s_n}^{t_n} \sim \lambda$ of depth greater than $sup\{|v_1|, ..., |v_m|, |u_1|, ..., |u_m|\}$ satisfies the property that if $1 \leq i \leq n$ either $supp(y_{s_i}^{t_i}) \cap \bigcup_{1\leq j\leq m}[u_j\bar{0}, v_j\bar{1}]$ is a nullset or $supp(y_{s_i}^{t_i}) \subseteq \bigcup_{1\leq j\leq m}[u_j\bar{0}, v_j\bar{1}]$. □

Lemma 4.5.25. *Let $e = \{F(\lambda\tau_1), F\tau_1\}$ be a 1-cell and $u = F\tau_2$ be a 0-cell such that e, u is* not *disparate. Then we can perform a two sided equivariant expansion of e to obtain 1-cells $e_1, ..., e_n$ incident to $F\tau_1$ and $e'_1, ..., e'_n$ incident to $F(\lambda\tau_1)$ such that for each $1 \leq i \leq n$ either the pairs e_i, u and e'_i, u are both disparate, or there is a 1-cell e''_i incident to u such that $e_i \doteq e'_i \doteq e''_i$.*

Proof. Let $F\tau_1 = v_1, F(\lambda\tau_1) = v_2$. By Lemma 4.5.24 there is an expansion α with an associated decomposition $\lambda \sim y_{s_1}^{t_1}...y_{s_n}^{t_n}$ such that for each $1 \leq i \leq n$ either $supp(y_{s_i}^{t_i}) \cap supp_Y(\tau_2\tau_1^{-1})$ is a nullset or $supp(y_{s_i}^{t_i}) \subseteq supp_Y(\tau_2\tau_1^{-1})$

In particular there is an expansion $\alpha \in E(e, v_1)$ such that the offsprings $e_1, .., e_n$ of e satisfy the property that for each $1 \leq i \leq n$ either e_i, u is disparate with associated pair v_1 or there is a 1-cell e' incident to u such that $e_i \doteq e'$ with associated pair v_1, u.

Also by Lemma 4.5.24 and an argument similar to the first paragraph of our proof, it follows that there is an expansion $\beta \in E(e, v_2)$ for which the offsprings $e'_1, .., e'_m$ of e satisfy the property that for each $1 \leq i \leq m$ either e_i, u is disparate with associated pair v_2, u or there is a 1-cell e' incident to u such that $e'_i \doteq e'$ with associated pair v_2, u.

Let the op-expansion of $\beta \in E(e, v_2)$ be the expansion $\beta' \in E(e, v_1)$. By Lemma 4.5.8 it follows that there is an expansion $\gamma \in E(e, v_1)$ such that $\gamma < \beta'$ and $\gamma < \alpha$. The corresponding op-expansion $\gamma' \in E(e, v_2)$ has the property that $\gamma' < \beta$.

By Lemma 4.5.10 the two sided expansion of e given by γ, γ' produces the desired outcome. □

Now we outline the next step towards defining the separation procedure.

Lemma 4.5.26. *(Procedure A) Let $e_1, ..., e_n$ be 1-cells and $u_1, ..., u_k$ be the set of 0-cells incident to them. Then we can perform two sided equivariant expansions on $e_1, ..., e_n$ to produce 1-cells $e'_1, ..., e'_m$ such that for each pair $1 \leq i \leq m, 1 \leq j \leq k$ either e'_i, u_j are disparate or there is a 1-cell $e_{i,j}$ incident to u_j such that $e'_i \doteq e_{i,j}$.*

Proof. For a fixed $1 \leq i \leq n$, let $e = \{u_s, u_t\}$. For each $1 \leq l \leq k$, let $\alpha_l \in E(e_i, u_s), \alpha'_l \in E(e_i, u_t)$ be two sided expansions that perform the separation procedure for the pair e_i, u_l.

By Lemma 4.5.8 there is are dual expansions $\alpha \in E(e_i, u_s), \alpha' \in E(e_i, u_t)$ such that for each $1 \leq l \leq k$ we have that $\alpha < \alpha_l, \alpha' < \alpha'_l$.

It follows that the associated two sided expansion of e_i has the property that if e is an offspring of this expansion and $1 \leq j \leq k$ then the pair e, u_j satisfies the conclusion of the lemma. We perform these two sided expansions for each $e_1, ..., e_n$ to finish the procedure. (We remark that the expansions produce new 0-cells incident to the offsprings, but in the statement of the lemma the set of 0-cells $\{u_1, .., u_k\}$ is fixed.) □

Definition 4.5.27. Let (U, V) be a system of cells in $X^{(1)}$. Let I be the set of 1-cells e in X for which there are cells $e' \in U, v \in V$ such that e is incident to v and $e \doteq e'$. (In particular, $U \subseteq I$.) The system (I, V) is called the *equivariant closure* of the system (U, V).

We are now ready to define the separation procedure.

Definition 4.5.28. (Separation Procedure) Input: A system (U, V) of cells in $X^{(1)}$. Let (U_1, V) be the system obtained by performing Procedure A from Lemma 4.5.26.

Output: The system (U_2, V), where (U_2, V) is the equivariant closure of (U_1, V).

Now our work in this section culminates in this important proposition.

Proposition 4.5.29. *Let Y be a finite subcomplex of $X^{(1)}$. Let U be the sets of 1-cells of Y and let V be the set of 0-cells of Y. Let (U_1, V) be the system obtained after performing Procedure A on the system (U, V) and let (U_2, V) be the equivariant closure of (U_1, V). In particular, (U_2, V) is the system obtained after performing the separation procedure on (U, V). Then $(U, V) \leq (U_2, V)$ and (U_2, V) is balanced.*

Proof. The claim that $(U, V) \leq (U_2, V)$ is obvious from the definitions of procedure A and the equivariant closure. We will show that (U_2, V) is balanced.

It suffices to check the following.

1. For each $e \in U_3, u \in V$ either e, u is disparate, or there is a $e' \in U_3$ incident to u such that $e \doteq e'$.
2. Let $e_1, e_2, e_1' \in U_3, u_1, u_2 \in V$ such that:
   (a) e_1, e_2 are incident to u_1.
   (b) e_1, e_2 are not disparate.
   (c) e_1' is incident to u_2 such that $e_1 \doteq e_1'$.

   Then there is a $e_2' \in U_3$ incident to u_2 such that $e_2 \doteq e_2'$. Moreover, u_1, u_2 is an associated pair for the equivalence $e_1 \doteq e_1'$ if and only if it is the associated pair for the equivalence $e_2 \doteq e_2'$.

Proof of (1): By the definition of equivariant closure there is a 1-cell $e_1 \in U_2$ such that $e \doteq e_1$. By Procedure A either e_1, u is disparate (and hence by Lemma 4.4.15 e, u is disparate), or else there is a 1-cell $e' \in U_2$ incident to u such that $e_1 \doteq e'$ (so $e \doteq e'$ and $e' \in U_3$). The 1-cell e' is in U_3 by definition of the equivariant closure.

Proof of (2): By the definitions of Procedure A and equivariant closure there is a 1-cell $e_3 \in U_2$ incident to some 0-cell $u_3 \in V$ such that $e_3 \doteq e_2$ and u_1, u_3 are associated pairs for the equivalence. Now since e_1, e_2 are not disparate it follows that there is a common offspring e_4 of expansions of e_1, e_2 at u_1. It follows that the equivariant expansion of e_1' at u_2 has an offspring e_4' such that $e_4 \doteq e_4'$. Similarly there is an equivariant expansion of e_3 at u_3 with an offspring e_5 such that

$e_5 \doteq e_4 \doteq e'_4$. But then it follows that e_5, u_2 are not disparate, and in particular by Lemma 4.5.26 this means that e_3, u_2 are not disparate and there is a $e'_2 \in U_2$ such that $e_2 \doteq e_3 \doteq e'_2$. The last part of (2) is a consequence of Lemmas 4.5.11 and 4.4.8. □

4.5.2 The Equivariant decoupling procedure

Now we define a procedure that takes an an input a balanced system (U, V) and produces a *free* system (U', V) with the property that $(U, V) \leq (U', V)$. We shall adapt the proof from Lemma 4.5.19 to show the following.

Lemma 4.5.30. *(Equivariant decoupling procedure) Let (U, V) be a balanced system and let $V = \{v_1, ..., v_n\}$. We can perform mixed equivariant expansions of* 1-*cells in U at $\{v_1, ..., v_n\}$ to produce a set of* 1-*cells U_1 such that the system (U_1, V) is free. We remark that since we are only performing expansions at elements of V we have that $(U, V) \leq (U_1, V)$.*

Proof. Let $v_i, v_j \in V$ let $\{e_1, ..., e_n\}, \{e'_1, ..., e'_m\}$ be 1-cells incident to v_i, v_j respectively such that the following holds.

1. $\{e_1, ..., e_n\}$ is the set of vertices of a connected component of $\Gamma_{v_i}(U)$ and $\{e'_1, ..., e'_m\}$ is the set of vertices of a connected component pf $\Gamma_{v_j}(U)$.
2. $e_1 \doteq e'_1$.

By the definition of balanced it follows that in fact $m = n$ and by reordering the indices if necessary, $e_i \doteq e'_i$ and the map sending e_i to e'_i is a graph isomorphism between these components. Moreover either v_1, v_2 is an associated pair for *each* equivalence, or v_1, v_2 is not an associated pair for *each* equivalence.

For v_i by Lemma 4.5.19 we can perform expansions on the 1-cells $e_1, ..., e_k$ incident to to solve the decoupling problem for this component at u_i. Now given the expansion for e_i we perform the corresponding equivariant (or op-equivariant) expansion on each e'_i at u_j, and by Lemma 4.5.21 it follows that these expansions solve the decoupling problem for the respective components of $\Gamma_{v_1}(I), \Gamma_{v_2}(I)$. By an application of the above observation for all pairs of elements of V, and by Lemma 4.5.18, we can easily generalize this to all vertices in V. □

4.6 Cube complexes and higher homotopy groups of X

In this section it will be our goal to show that $\pi_i(X)$ is trivial for each $i > 1$. We shall do so by showing that for each finite subcomplex Y of X, there is a subcomplex Y' of X such that $Y \subseteq Y'$ and Y' is homeomorphic to a nonpositively curved cube complex.

First we will introduce a certain class of subcomplexes of X, which we call cluster-cube complexes. These are subcomplexes that are naturally homeomorphic to cube complexes.

Definition 4.6.1. Let $\{\Delta_i \mid i \in I\}$ be a set of clusters in X. Let Y be a subcomplex of X such that $Y = \cup_{i \in I} \Delta_i$. We say that Y is a *cluster-cube complex* if for each $i, j \in I$ the cluster $\Delta_i \cap \Delta_j$ is a facial subcluster of both Δ_i and Δ_j.

We observe that upon replacing each cluster in the set $\{\Delta_i \mid i \in I\}$ of Y by a Euclidean cube of the same dimension, we obtain a cube complex $C(Y)$ which is homeomorphic to the complex Y. The clusters Δ_i naturally correspond to cubes in $C(Y)$. Given any facial subcluster Δ_i' of Δ_i, Δ_i' naturally corresponds to a subcube of the cube Δ_i in $C(Y)$. Any cluster Δ in Y that corresponds to a cube in $C(Y)$ is called a *cubical cluster* of Y. We call the complex Y a *cluster-cube complex*, but formally it is a complex of clusters in X.

We now describe a procedure that takes as an input a free system (U, V), and produces a cluster-cube complex Y such that $C(Y)$ is nonpositively curved.

Lemma 4.6.2. *(The cubulation procedure) Let (U, V) be a free system. Let $\Upsilon(U, V)$ be the set of 1-cells e in X for which there is an $e_i \in U$ such that $e \doteq e_i$. Let Y be the subcomplex of X described in the following way. Y is a union of clusters Δ in X such that all the facial 1-cells of Δ are in $\Upsilon(U, V)$. Then Y is homeomorphic to a nonpositively curved cube complex.*

Proof. First we claim that any pair of 1-cells $e_1, e_2 \in \Upsilon(U, V)$ incident to a 0-cell u of X satisfy that e_1, e_2 are disparate. Assume that this is not the case. Let e_1', e_2' be 1-cells in U incident to $v_1, v_2 \in V$ respectively such that $e_i' \doteq e_i$. It follows that we can perform an expansion of e_1', e_2' at v_1, v_2 respectively such that there are equivalent offsprings e_1'', e_2''. This means that the pairs e_1', v_2 and e_2', v_1 are not disparate. Since a free system is balanced it follows that there is a 1-cell $e_2'' \in U$ such that e_2'' is incident to v_1 and $e_2'' \doteq e_2'$. But this means that $e_1', e_2'' \in U$ are not disparate, contradicting the definition of a free system. Our claim follows.

Let T be the set of clusters satisfying the hypothesis of the lemma. From our characterization of subclusters and the fact that the intersection of two clusters is a cluster, it follows that if two clusters Δ_1, Δ_2 in T have the property that $\Delta_1 \cap \Delta_2$ is a diagonal subcluster of Δ_1, then we can find facial 1-cells e_1, e_2 in Δ_1, Δ_2 respectively such that e_2 is a diagonal 1-cell of Δ_1 and e_1 is an offspring of e_2. In particular e_1, e_2 are incident to a common 0-cell and are not disparate. Since they are facial 1-cells of Δ_1, Δ_2 in T, $e_1, e_2 \in \Upsilon(U, V)$. This contradicts our observations above.

It follows that any pair of clusters Δ_1, Δ_2 in T with a nonempty intersection must intersect in a facial subcluster. It follows that Y is homeomorphic to a cube complex.

Now we will show that this is nonpostively curved. Let u be a 0-cell in Y such that $Lk(u, C(Y))$ has an empty n-simplex. This empty n-simplex corresponds to 1-cells $e_1, ..., e_l$ in Y incident to u such that each pair e_i, e_j satisfy the property that there is a 2-cluster $\Delta_{i,j}$ which is a cluster in T and has e_i, e_j as its facial 1-cells. In particular it follows that $e_1, ..., e_n \in \Upsilon(U, V)$. By Lemma 4.2.16 it follows that in fact $e_1, ..., e_n$ are facial 1-cells of an n-cluster Δ which must be in T, since there are n equivalence classes of the facial 1-cells of Δ with representatives $e_1, ..., e_n$, and so all the facial 1-cells of Δ are in fact in $\Upsilon(U, V)$. This means that this simplex in the link is not empty. By Gromov's characterization of nonpositively curved cube complexes, we conclude that $C(Y)$ is nonpositively curved. □

We are now ready to prove the main result of this section.

Proposition 4.6.3. *Let Y be a finite subcomplex of X. Then there is a subcomplex Y' of X such that $Y \subseteq Y'$ and Y' is homeomorphic to a nonpositively curved cube complex. In particular, it follows that $\pi_i(X)$ is trivial for $i > 1$.*

Proof. Let U_1 be the set of 1-cells of Y and V be the set of 0-cells of Y. For the system (U_1, V), following Proposition 4.5.29, we first perform the separation procedure to obtain a system (U_2, V) which is balanced. Now we peform the equivariant decoupling procedure for the balanced system (U_2, V) to obtain a free system (U_3, V). Let Y' be the nonpositively curved cube complex obtained from performing the cubulation procedure on the free system (U_3, V). We know by Lemma 4.6.2 that Y' is homeomorphic to a nonpositively curved cube complex. It remains to check that $Y \subseteq Y'$.

Let Δ be a cluster in Y that is based at τ and parametrized by $\lambda_1, ..., \lambda_n$. Let $e_1 = \{F(\lambda_1\tau), F\tau\}, ..., e_n = \{F(\lambda_n\tau), F\tau\}$. We know that $F\tau \in V$, and since $(U, V) \leq (U_3, V)$, we know that there are 1-cells $e'_1, ..., e'_m \in U_3$ such that the set $\{e'_1, ..., e'_m\}$ is precisely the set of offsprings obtained after performing some expansions on $e_1, ..., e_n$ at τ.

Now $e'_1, ..., e'_m \in \Upsilon(U_3, V)$ and since they are offsprings of 1-cells $e_1, ..., e_n$ it follows that there is an m-cluster Δ' whose facial 1-cells are $e'_1, ..., e'_m$, which are all in $\Upsilon(U_3, V)$ Clearly, Δ is a diagonal subcluster of Δ' and $\Delta' \subseteq Y'$. Therefore $Y \subseteq Y'$. □

Now we are ready to prove the main theorem.

Theorem 4.6.4. *The group G acts on a cell complex X by cell permuting homeomorphisms such that the following holds.*

1. *X is contractible.*
2. *The quotient X/G has finitely many cells in each dimension.*
3. *The stabilizers of each cell are of type F_∞.*

Since G is finitely presented, it follows from the above that G is of type F_∞.

Proof. Claim 1 follows from propositions 4.6.3 and 4.3.2. Claims 2 and 3 follow from proposition 4.2.33. □

CHAPTER 5

SUBGROUPS OF HYPERBOLIC GROUPS: AN EXAMPLE

5.1 Introduction

Hyperbolic groups were introduced by Gromov [20] as a generalization of fundamental groups of negatively curved manifolds and of finitely generated free groups. A finitely generated group is hyperbolic if its Cayley graph is hyperbolic as a metric space. (See [8] for more details.) Hyperbolic groups are finitely presented, and have solvable word and conjugacy problems. Torsion-free hyperbolic groups are of type F_∞. This follows from the following result of Rips that appears in [31].

Theorem 5.1.1. *(Rips) Let H be a hyperbolic group. Then there exists a locally finite, simply connected, finite dimensional simplicial complex on which H acts faithfully, properly, simplicially and cocompactly. In particular, if H is torsion free, then the action is free and the quotient of this complex by H is a finite Eilenberg-Maclane complex $K(H, 1)$.*

The property of being hyperbolic is not inherited by subgroups. For instance finitely generated free groups have infinitely generated subgroups which cannot be hyperbolic since hyperbolic groups are finitely presentable. It is natural to then ask if the property of being hyperbolic is inherited by finitely generated subgroups. Rips constructed the first examples of finitely generated subgroups of hyperbolic groups that are not finitely presentable [31]. So then one asks whether finitely presented groups inherit hyperbolicity.

Gromov gave what appeared to be an example of a non-hyperbolic finitely presented subgroup of a hyperbolic group. It was later discovered by Bestvina that the ambient group of this example is not hyperbolic [6]. In 1999 Noel Brady constructed the first example of a hyperbolic group with a finitely presented subgroup that is not hyperbolic[6]— his subgroup is of type F_3 and therefore cannot be hyperbolic since hyperbolic groups are of finite type (see 2.2). Brady's example was constructed as the fundamental group of a nonpositively curved cube complex whose universal cover admits a hyperbolic metric. The subgroup emerges as the kernel of a homomorphism to $\mathbb{Z}$. The cube complex X is a branched covering of a product of graphs, and the homomorphism to $\mathbb{Z}$ is the induced homomorphism of a map $X \to S^1$ which lifts to a Morse function $\widetilde{X} \to \mathbb{R}$. Bestvina–Brady Morse theory [5] is then used to prove the finiteness properties of the kernel. Brady asks if there are examples of hyperbolic groups with subgroups of type F_n but not type F_{n+1} for all $n \in \mathbb{N}$ [6].

We shall construct some new examples using Bestvina–Brady Morse theory. In Section 5.3 we present a novel example of a finitely generated but not finitely presentable subgroup of a CAT(-1) group. Reading this construction will prepare the reader for the construction in Section 5.4. In Section 5.4, we shall construct a hyperbolic group with a finitely presented subgroup which is not of type F_3. Our cube complexes will emerge as subcomplexes of products of graphs, and do not require branched coverings.

5.2 Bestvina-Brady Morse theory

Here we shall sketch the main tool used in this chapter. Bestvina–Brady Morse theory was introduced in [5] to study finiteness properties of subgroups of certain right angled Artin groups. Morse theory is defined more generally for affine cell complexes, but we shall only discuss the special case of piecewise euclidean cube complexes.

Let X be a piecewise Euclidean cube complex, and let G act freely, properly, cocompactly, cellularly and by isometries on X. Let $\mathbb{Z}$ act on $\mathbb{R}$ in the usual way. Let $\phi : G \to \mathbb{Z}$ be a homomorphism.

Definition 5.2.1. A ϕ-equivariant Morse function is a map $f : X \to \mathbb{R}$ that satisfies:

1. For any cell F of X (with the characteristic function $\chi_F : \square^n \to F$), the composition $f \circ \chi_F$ is the restriction of a nonsingular affine map $\mathbb{R}^n \to \mathbb{R}$.
2. The image of the 0-skeleton of X is a discrete subset of $\mathbb{R}$.
3. f is G-equivariant, i.e for all $g \in G, x \in X$, $f(g \cdot x) = \phi(g) \cdot f(x)$.

One can think of a Morse function as a height function on X. The kernel $H = Ker(\phi)$ acts on the level sets of X, i.e. inverse images $f^{-1}(x)$ for $x \in \mathbb{R}$. Topological properties of level sets can be used to deduce the finiteness properties of H. In [5] it was shown that the topological properties of the level sets are determined by the topology of links of vertices. We make this precise below.

Definition 5.2.2. Given a vertex v of X, the ascending link $Lk^{\uparrow}(v, X)$ is defined as

$$Lk^{\uparrow}(v, X) = \bigcup \{Lk(v, F) \mid v \text{ is the minima of } f \circ \chi_F\}$$

Similarly, the descending link $Lk^{\downarrow}$ is define as

$$Lk^{\downarrow}(v, X) = \bigcup \{Lk(v, F) \mid v \text{ is the maxima of } f \circ \chi_F\}$$

We now summarize the main result of Bestvina–Brady Morse theory. For the details of the proof see [5] or [6].

Theorem 5.2.3. *(Bestvina, Brady) Let $n \geq 1$ be a fixed natural number. Let X be a cube complex, G a group acting freely, properly, cocompactly and by isometries on X and $\phi : G \to \mathbb{Z}$ a homomorphism. Let $f : X \to \mathbb{R}$ be a ϕ-equivariant Morse function with the following property. For all vertices v of X:*

1. *$Lk^{\uparrow}(v, X), Lk^{\downarrow}(v, X)$ are simply connected.*
2. *$\widetilde{H}_k(Lk^{\uparrow}(v, X)), \widetilde{H}_k(Lk^{\downarrow}(v, X)) = 0$ for $1 \leq k \leq n-1$.*
3. *$\widetilde{H}_n(Lk^{\uparrow}(v, X), \mathbb{Z}), \widetilde{H}_n(Lk^{\downarrow}(v, X), \mathbb{Z}) \neq 0$.*

Then the kernel of the induced map on the fundamental groups $\mathrm{Ker}(\phi_* : \pi_1(X) \to \pi_1(S^1))$ *is of type F_n but not of type F_{n+1}.*

5.3 A CAT(-1) example

In this section we produce a square complex X with a map $f : X \to S^1$ which lifts to an f_*-equivariant Morse function $\widetilde{f} : \widetilde{X} \to \mathbb{R}$ with the properties:

1. The link of each vertex is a finite graph with no 3 or 4 cycles,
2. The ascending and descending links of all vertices are homeomorphic to S^1.

First we define a bipartite graph Γ which will be an ingredient in both constructions.

Definition 5.3.1. Γ is the following graph:

The vertex set: $V(\Gamma) = A_+ \cup A_- \cup B_- \cup B_+$ where,

1. $A_+ = \{a_0^+, a_1^+, ..., a_{10}^+\}$.
2. $A_- = \{a_0^-, a_1^- ..., a_{10}^-\}$.
3. $B_+ = \{b_0^+, b_1^+, ..., b_{10}^+\}$.
4. $B_- = \{b_0^-, b_1^-, ..., b_{10}^-\}$.

The edge set: $E(\Gamma) = E_1 \cup E_2 \cup E_3$ where

1. E_1 consists of the edges $\{a_i^s, b_j^s\}$ for $s \in \{+, -\}$ and $i = j$ or $j = i+1 \pmod{11}$.
2. E_2 consists of the edges $\{a_i^+, b_j^-\}$ for $j = i+3 \pmod{11}$ or $j = i+5 \pmod{11}$.
3. E_3 consists of the edges $\{a_i^-, b_j^+\}$ for $i = j$ or $j = i+2 \pmod{11}$.

Lemma 5.3.2. Γ *satisfies the following:*

1. *The subgraphs spanned by the vertex sets $A_+ \cup A_-$ and $B_+ \cup B_-$ have no edges, and $A_+ \cup B_+$, $A_+ \cup B_-$, $A_- \cup B_+$, $A_- \cup B_-$ span subgraphs that are each a cycle.*
2. *There are no 3-cycles or 4-cycles.*

Proof. Γ is a bipartite graph so there are no 3-cycles. Property (1) in the statement of the lemma is easily verified. We claim that Γ has no 4-cycles. Assume there is a 4-cycle C. There are five cases to check. (We write a cycle as a set of edges.)

1. C lies in the subgraph spanned by $A_+ \cup A_- \cup B_-$.

   So C is of the form $\{\{a_i^+, b_j^-\}, \{a_k^-, b_j^-\}, \{a_k^-, b_l^-\}, \{a_i^+, b_l^-\}\}$.

2. C lies in the subgraph spanned by $A_+ \cup A_- \cup B_+$.

   So C is of the form $\{\{a_i^+, b_j^+\}, \{a_k^-, b_j^+\}, \{a_k^-, b_l^+\}, \{a_i^+, b_l^+\}\}$.

3. C lies in the subgraph spanned by $B_+ \cup B_- \cup A_-$.

   So C is of the form $\{\{a_j^-, b_i^+\}, \{a_j^-, b_k^-\}, \{a_l^-, b_k^-\}, \{a_l^-, b_i^+\}\}$.

4. C lies in the subgraph spanned by $B_+ \cup B_- \cup A_+$.

   So C is of the form $\{\{a_j^+, b_i^+\}, \{a_j^+, b_k^-\}, \{a_l^+, b_k^-\}, \{a_l^+, b_i^+\}\}$.

5. The vertices of C $\{v_1, ..., v_4\}$ satisfy $v_1 \in A_+, v_2 \in B_+, v_3 \in A_-, v_4 \in B_-$.

   So C is of the form $\{\{a_i^+, b_j^+\}, \{a_k^-, b_j^+\}, \{a_k^-, b_l^-\}, \{a_i^+, b_l^-\}\}$.

We treat cases (1) and (5). Cases $(2), (3), (4)$ are similar to (1).

Case (1): Since the distinct vertices b_l^-, b_j^- share a neighbor a_k^-, we can assume without the loss of generality that $l = k$ and $j = k + 1 (\text{mod } 11)$. So $|i - j| = \pm 1 (\text{mod } 11)$. Now either $l = i + 3 (\text{mod } 11)$ and $j = i + 5 (\text{mod } 11)$, or $l = i + 5 (\text{mod } 11)$ and $j = i + 3 (\text{mod } 11)$. In either case we conclude that $|i - j| = \pm 2 (\text{mod } 11)$. This is a contradiction. Therefore such a 4-cycle cannot exist.

Case (5): By construction we observe,

1. $j = i$ or $j = i + 1 (\text{mod } 11)$.
2. $j = k$ or $j = k + 2 (\text{mod } 11)$.
3. $l = k$ or $l = k + 1 (\text{mod } 11)$.
4. $l = i + 3 (\text{mod } 11)$ or $l = i + 5 (\text{mod } 11)$.

From $(1), (2), (3)$ above we deduce that if $|l - i| \cong n (\text{mod } 11)$, then $n \in \{-2, -1, 0, 1, 2\}$. From (4) on the other hand $|l - i| \cong 3$ or $5 (\text{mod } 11)$. This is a contradiction. Therefore we conclude that there are no 4-cycles in Γ. $\square$

Now we will define our square complex X. Let Θ_1 be the directed graph with vertices v_1, v_2 and the edge set $E(\Theta_1) = A_- \cup A_+ \subseteq V(\Gamma)$, such that each edge meets v_1 and v_2. The A_- edges are oriented in the direction of v_2 (i.e. the arrow points to v_2) and the A_+ edges are oriented in the direction of v_1. Similarly, let Θ_2 be a graph with vertices v_3, v_4, and the edge set $E(\Theta_2) = B_- \cup B_+ \subseteq V(\Gamma)$. The B_- edges are oriented in the direction of v_4 and the B_+ edges are oriented in the direction of v_3.

Consider the 2-complex $J = \Theta_1 \times \Theta_2$. The squares of J are ordered pairs (a, b) where $a \in A_+ \cup A_-, b \in B_+ \cup B_-$. We now define a subcomplex X of J in the following manner. Let $X^{(1)} = J^{(1)}$. Glue a 2-face along the boundary of every square in $X^{(1)}$ that has the property that the corresponding pair $\{a, b\}$ is an edge in E. The resulting square complex is X.

Identify S^1 with $\mathbb{R}/\mathbb{Z}$. The orientations on the edges of Θ_1, Θ_2 determine maps $l_i : \Theta_i \to S^1$. Under this map each vertex maps to $[0]$, and the map on each edge is as follows: Identify the edge with the unit interval $[0, 1]$ in a way that the edge is oriented towards the vertex identified with 1. Then define the map l_i on an edge as $x \mapsto [x]$. Now identify each square C of X isometrically with the unit square $[0, 1]^2$, where the edges $\{v\} \times [0, 1]$ and $[0, 1] \times \{v\}$ are oriented towards $(v, 1)$ and $(1, v)$ respectively for each $v \in \{0, 1\}$. The map f on C is now defined as $(x, y) \mapsto [x + y]$.

The map $f_* : \pi_1(X) \to \pi_1(S^1)$ is the induced map on the fundamental groups, and $\pi_1(X)$ acts on the universal cover $\widetilde{X}$ by deck transformations. f lifts to a map $\widetilde{f} : \widetilde{H} \to \mathbb{Z}$, which is a f_*-equivariant Morse function on $\widetilde{X}$. Conditions $(1), (2), (3)$ of the definition of a Morse function are apparent, and condition (4) follows from the definition of the lift $\widetilde{f}$.

Theorem 5.3.3. *The square complex X has the following properties.*

1. $\widetilde{X}$ *admits a* $\mathrm{CAT}(-1)$ *metric.*
2. $\mathrm{Ker}(f_* : \pi_1(X) \to \pi_1(S^1))$ *is finitely generated but not finitely presented.*

Proof. By construction, the link of each vertex is the graph Γ. Since Γ is a flag simplicial complex with no empty-squares, (1) follows from Theorem 2.3.5. The ascending and descending links of each vertex are homeomorphic to S^1. Therefore (2) follows from Theorem 5.2.3. □

5.4 A subgroup of type F_2 but not type F_3

We shall construct a three-dimensional cube complex Δ and a Piecewise Linear function $g : \Delta \to S^1$ that lifts to a g_* equivariant Morse function $\widetilde{g} : \widetilde{\Delta} \to \mathbb{R}$ satisfying the hypothesis of Theorem 5.2.3 with $n = 2$. The cube complex Δ will be constructed as a subcomplex of a product of finite graphs.

We first define graphs U, V, W, each of which is isomorphic to $K_{22,22}$, the complete bipartite graph with 22 vertices in each "part". Let the parts of U, V, W be U_1, U_2, V_1, V_2 and W_1, W_2 respectively. The vertices of U_1, V_1, W_1 are $\{a_0^+, ..., a_{10}^+, a_0^-, ..., a_{10}^-\}$ and the vertices of U_2, V_2, W_2 are $\{b_0^+, ..., b_{10}^+, b_0^-, ..., b_{10}^-\}$.

For each of the graphs U, V, W, we fix the following orientations on edges. Given an edge $\{a_n^s, b_m^t\}$ for $0 \leq m, n \leq 10, s, t \in \{+, -\}$, if $s = t$ then the edge is oriented toward a_n^s otherwise the edge is oriented toward b_m^t. So any vertex of U, V, W has 11 incoming and 11 outgoing edges. Declare each edge of U, V, W to be isometrically identified with the unit interval $[0, 1]$ in such a way that the edge is oriented towards the vertex identified with 1. Let $U \times V \times W$ be the product cube complex. We define a cube subcomplex Δ as follows.

Definition 5.4.1. Let (u, v, w) be a vertex in $U \times V \times W$. Then $(u, v, w) \in \Delta$ if one of the following holds. (Recall that Γ is the graph defined in section 3.)

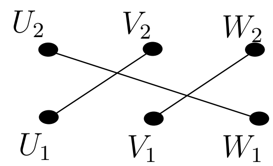
1. For some $i \in \{1, 2\}$, $u \in U_i, v \in V_i, w \in W_i$.
2. $u \in U_1, v \in V_2$ and $\{u, v\}$ is an edge in Γ.
3. $v \in V_1, w \in W_2$ and $\{v, w\}$ is an edge in Γ.
4. $u \in U_2, w \in W_1$ and $\{u, w\}$ is an edge in Γ.

We declare a cell of $U \times V \times W$ to be in Δ if all its incident vertices are in Δ.

It follows immediately that Δ is a piecewise Euclidean cube complex. Vertices in $U \times V \times W$ that satisfy (1) above are said to be *type 1* vertices, and vertices that satisfy either (2), (3) or (4) are said to be *type 2* vertices. It is an easy exercise to show that any two vertices in Δ are connected by a path in $\Delta^{(1)}$. (Check this for type 1 vertices first.) We conclude that Δ is connected.

The graph Ω of Figure 5.4 serves as a tool for determining when a given vertex is in Δ. The edges of the graph encode the conditions of the definition above as follows: Given a vertex $\tau = (u, v, w)$ such that $u \in U_i, v \in V_j, w \in W_k$, either τ is a type 1 vertex (and hence is in Δ), or there is an edge connecting two of the three nodes U_i, V_j, W_k in the graph Ω. Then $\tau \in \Delta$ if and only if the corresponding pair from u, v, w forms an edge in Γ.

One observes that the map $U \times V \times W \to U \times V \times W; (u, v, w) \mapsto (w, u, v)$ and its iterates are cell permuting isometries whose restriction to Δ induces a map



$\Delta \to \Delta$ that is a cell permuting isometry. These symmetries of our construction will be invoked in arguments that follow.

The orientations on the edges induce a PL function $f : \Delta \to S^1$, which is $x \to [x]$ on each edge, and on the product it is defined as $(x, y, z) \to [x + y + z]$. (Recall the identification of each edge with $[0, 1]$ that was described above, and the identification of S^1 with $\mathbb{R}/\mathbb{Z}$.) The map f lifts as a PL Morse function between the universal covers $\widetilde{f} : \widetilde{\Delta} \to \mathbb{R}$. Now we prove two key lemmas, the first of which examines the links of vertices.

Lemma 5.4.2. *Let $\tau \in \Delta$ be a vertex. Then the following holds. (Here $\star$ denotes the topological join and X is a discrete set of four points.)*

1. *If τ is a type 1 vertex then $Lk(\tau, \Delta)$ is homeomorphic to $X \star X \star X$.*
2. *If τ is a type 2 vertex then $Lk(\tau, \Delta)$ is homeomorphic to $\Gamma \star X$.*

In particular, Δ is nonpositively curved. Furthermore, in either case $Lk^{\uparrow}(\tau, \widetilde{\Delta}), Lk^{\downarrow}(\tau, \widetilde{\Delta})$ are both homeomorphic to S^2.

Proof. Let $\tau = (u, v, w)$ be a type 1 vertex. Assume that $u \in U_1, v \in V_1, w \in W_1$. Recall that $u, v, w \in \{a_0^+, ..., a_{10}^+, a_0^-, ..., a_{10}^-\} \subseteq V(\Gamma)$. Let the neighbors of u, v, w in Γ be

$$\{b_{k_1}^+, b_{k_2}^+, b_{k_3}^-, b_{k_4}^-\}, \{b_{l_1}^+, b_{l_2}^+, b_{l_3}^-, b_{l_4}^-\}, \{b_{j_1}^+, b_{j_2}^+, b_{j_3}^-, b_{j_4}^-\}$$

respectively. The following are the 1-cells adjacent to τ in Δ:

1. $[u, b^+_{j_i}] \times v \times w$ for $1 \leq i \leq 2$ and $[u, b^-_{j_i}] \times v \times w$ for $3 \leq i \leq 4$.
2. $u \times [v, b^+_{k_i}] \times w$ for $1 \leq i \leq 2$ and $u \times [v, b^-_{k_i}] \times w$ for $3 \leq i \leq 4$.
3. $u \times v \times [w, b^+_{l_i}]$ for $1 \leq i \leq 2$ and $u \times v \times [w, b^-_{l_i}]$ for $3 \leq i \leq 4$.

It follows that $Lk(\tau, \Delta \cap (U \times v \times w))$, $Lk(\tau, \Delta \cap (u \times V \times w))$, $Lk(\tau, \Delta \cap (u \times v \times W))$ are all discrete sets of four points each. Furthermore, it follows from the definition of Δ that $[u, p] \times [v, q] \times [w, r]$ is a cube in Δ for each $p \in \{b^+_{j_1}, b^+_{j_2}, b^-_{j_3}, b^-_{j_4}\}, q \in \{b^+_{k_1}, b^+_{k_2}, b^-_{k_3}, b^-_{k_4}\}, r \in \{b^+_{l_1}, b^+_{l_2}, b^-_{l_3}, b^-_{l_4}\}$. So $Lk(\tau, \Delta)$ is the topological join of these sets. Observe that exactly two of the four 1-cells in each of (1), (2), (3) are oriented away from τ and the remaining are oriented towards τ. So it follows immediately that $Lk^{\uparrow}(\tau, \widetilde{\Delta}), Lk^{\downarrow}(\tau, \widetilde{\Delta})$ are homeomorphic to S^2. The case where $u \in U_2, v \in V_2, w \in W_2$ is similar.

Now consider the case where $\tau = (u, v, w) \in \Delta$ is a type 2 vertex. Assume that $u \in U_2, v \in V_1, w \in W_1$. The 1-cells incident to τ in Δ are:

1. $[u, a^s_i] \times v \times w$ for $0 \leq i \leq 10$, $s \in \{+, -\}$.
2. $u \times [v, b^s_i] \times w$ for $0 \leq i \leq 10$, $s \in \{+, -\}$.
3. $u \times v \times [w, p]$, for $p \in \{b^+_{n_1}, b^+_{n_2}, b^-_{n_3}, b^-_{n_4}\}$ where $\{b^+_{n_1}, b^+_{n_2}, b^-_{n_3}, b^-_{n_4}\}$ are the four neighbors of v in Γ.

Now given 1-cells $[u, a^s_i] \times v \times w$, $u \times [v, b^t_j] \times w$, observe that there is a square $[u, a^s_i] \times [v, b^t_j] \times w$ in Δ if and only if a^s_i, b^t_j are connected in Γ by an edge. This means that $Lk(\tau, \Delta \cap (U \times V \times w)) \cong \Gamma$. Furthermore, the definition of Δ implies that $[u, a^s_i] \times [v, b^t_j] \times [w, b^r_k]$ is a cube in Δ if and only if $[u, a^s_i] \times [v, b^t_j] \times w$ is a square in Δ and $u \times v \times [w, b^r_k]$ is a 1-cell in Δ. This means that $Lk(\tau, \Delta)$ is the topological join of Γ and the discrete set of four points, $Lk(\tau, \Delta \cap (u \times v \times W))$.

Now $Lk^{\uparrow}(\tau, \Delta \cap (U \times V \times w))$, $Lk^{\downarrow}(\tau, \Delta \cap (U \times V \times w))$ are both cycles, and $Lk^{\uparrow}(\tau, \Delta \cap (u \times v \times W))$, $Lk^{\downarrow}(\tau, \Delta \cap (u \times v \times W))$ are both discrete sets of two points each. This means that $Lk^{\uparrow}(\tau, \widetilde{\Delta}) \cong S^2$ and $Lk^{\downarrow}(\tau, \widetilde{\Delta}) \cong S^2$. For any other vertex τ of type 2, the analysis is similar by symmetry of the construction. □

So far we have shown the following.

1. Δ is a nonpositively curved cube complex.
2. By Lemma 5.4.2 and Theorem 5.2.3 it follows that
   $Ker(f_* : \pi_1(\Delta) \to \pi_1(S^1))$ is finitely presented but not of type F_3.

Now we will show that $\widetilde{\Delta}$ is a hyperbolic metric space and hence $\pi_1(\Delta)$ is a hyperbolic group. We have already established that $\widetilde{\Delta}$ is a $CAT(0)$ space, and so by Theorem 2.3.6 it suffices to show that $\widetilde{\Delta}$ does not contain isometrically embedded flat planes.

Let $\tau = (u, v, w)$ be a type 2 vertex in Δ. From the previous lemma $Lk(\tau, \Delta) \cong \Gamma \star X$, where X is a discrete set of four points. So $Lk(\tau, \Delta^{(1)})$ is naturally identified with $V(\Gamma) \cup X$.

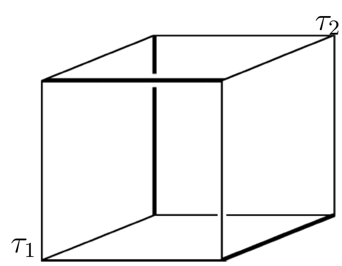
Lemma 5.4.3. *Let τ, τ' be type 2 vertices in Δ such that $[\tau, \tau']$ is a 1-cell in Δ. The $Lk(\tau, [\tau, \tau']) \in X$ if and only if $Lk(\tau', [\tau, \tau']) \in X$.*

Proof. We assume that $\tau = (u, v, w), \tau' = (u', v, w)$. Let $u \in U_i, u' \in U_j$ where $\{i, j\} = \{1, 2\}$. Also, let $v \in V_k, w \in W_l$. Assume that $Lk(\tau, [\tau, \tau']) \in X$. It follows that $Lk(\tau, \Delta \cap (U \times v \times w)) = X$. So U_j is connected by an edge in Ω with either V_k or W_l. We will show that U_i is connected by an edge with either V_k or W_l in Ω, and hence $Lk(\tau', [\tau, \tau']) \in X$.

Assume that this is not the case. Then since τ' is a type 2 vertex it must be the case that V_k, W_l are connected by an edge in Ω. This cannot be true since U_j is connected by an edge with either V_k or W_l. This proves our assertion. By symmetry of our construction this follows for any arbitrary type 2 vertex in Δ. $\square$

Definition 5.4.4. A 1-cell $[\tau, \tau']$ in Δ satisfying the statement of Lemma 5.4.3 i.e. $Lk(\tau, [\tau, \tau']) \in X$ and $Lk(\tau', [\tau, \tau']) \in X$ is called a *special 1-cell*. Denote the union of all special 1-cells in Δ by L. A lift of a special 1-cell in $\widetilde{\Delta}$ is a special 1-cell in $\widetilde{\Delta}$ and $\widetilde{L}$ is the union of all special 1-cells in $\widetilde{\Delta}$.

Figure 5.4 depicts a cube in Δ. The three bold 1-cells are the special 1-cells, the vertices τ_1, τ_2 are the type 1 vertices and the remaining vertices are of type 2.



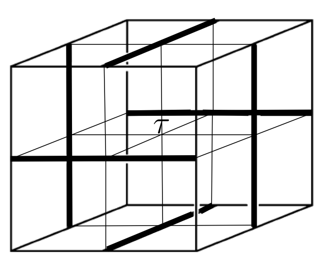
Lemma 5.4.5. $\widetilde{\Delta}$ *does not contain isometrically embedded flat planes, and hence is a hyperbolic metric space.*

Proof. The proof is similar to the proof of 6.1(3) in [6], and we claim no originality here. We adapt that argument to our construction. Let us assume that there is an isometric embedding $i : \mathbb{R}^2 \to \widetilde{\Delta}$. We say that a point $x \in i(\mathbb{R}^2)$ is a *transverse intersection point* if there is a neighborhood U of x in $i(\mathbb{R}^2)$ such that the intersection of U with $\widetilde{L}$ is the point x. Following [6], our proof is divided into two steps. In *step* 1 we will show that $i(\mathbb{R}^2)$ has a transverse intersection point. In *step* 2 we will show that the angle around the transverse intersection point in $i(\mathbb{R}^2)$ is greater than 2π, contradicting the fact that this is an isometric embedding.

Step 1: Let C be a cube in $\widetilde{\Delta}$ such that $i(\mathbb{R}^2) \cap C$ is nonempty and two dimensional. (Such cubes must exist in $\widetilde{\Delta}$ since $i(\mathbb{R}^2)$ is an isometric embedding.)There are four cases to consider.

In the first case, $i(\mathbb{R}^2)$ intersects a special 1-cell e of C in a vertex p. Now $i(\mathbb{R}^2)$ must also intersect a neighboring cube C' of C that shares a 2-face with C and contains a special 1-cell e' incident to p. Then either $i(\mathbb{R}^2)$ contains e' or p is a transverse intersection point. Since $i(\mathbb{R}^2)$ is an isometric embedding, it cannot contain e' or else it would also contain e. In the second case, $i(\mathbb{R}^2)$ intersects a special 1-cell of C in an interior point, in which case it is clear that this is a transverse intersection point. In the third case, $i(\mathbb{R}^2)$ contains a special 1-cell of C. In this case it transversely intersects a different special 1-cell of C. (Recall that $i(\mathbb{R}^2) \cap C$ is two dimensional and see Figure 5.4).

Finally, consider the case where $i(\mathbb{R}^2)$ does not intersect a special 1-cell of C. Then $i(\mathbb{R}^2)$ intersects a 1-cell incident to a type 1 vertex τ in C. Let J be the subcomplex of $\widetilde{\Delta}$ consisting of cubes in $\widetilde{\Delta}$ that have a nonempty intersection with $i(\mathbb{R}^2)$. Recall that $Lk(\tau, \widetilde{\Delta}) \cong X \star X \star X$ where X is a discrete set of four points. The set of cubes incident to τ in J is a subcomplex of a stack of 8 cubes depicted in Figure 5.4. (C is one of the 8 cubes.) Here the bold 1-cells are the special 1-cells.



Now by figure 5.4 it must be the case that $i(\mathbb{R}^2)$ transversely intersects a special 1-cell in a neighboring cube of C.

Step 2: Now we demonstrate a contradiction to our assumption that $i(\mathbb{R}^2)$ is an isometrically embedded flat plane in $\widetilde{\Delta}$. This involves computing the angle in $i(\mathbb{R}^2)$ around a transverse intersection point p.

Let Y be the subcomplex consisting of cubes in $\widetilde{\Delta}$ that contain p and for which $C \cap i(\mathbb{R}^2)$ is 2-dimensional. For each cube C in Y, $i^{-1}(i(\mathbb{R}^2) \cap C)$ is a polygon, and $i^{-1}(i(\mathbb{R}^2) \cap Y)$ is a union of polygons in $\mathbb{R}^2$ that are incident to $i^{-1}(p)$ such that the sum of the angles at $i^{-1}(p)$ at each polygon is 2π.

Let C, C' be cubes in Y such that $i^{-1}(i(\mathbb{R}^2) \cap C), i^{-1}(i(\mathbb{R}^2) \cap C')$ are adjacent polygons. Then $i(\mathbb{R}^2) \cap (C \cup C')$ is locally the intersection of $i(\mathbb{R}^2)$ with a Euclidean half space. This means that the angle sum of any two consecutive polygons in $i^{-1}(i(\mathbb{R}^2) \cap Y)$ is π. To establish a contradiction, it suffices to show that the number of polygons in $i^{-1}(i(\mathbb{R}^2) \cap Y)$ is greater than 4. Let $e = [\tau, \tau']$ be a special 1-cell containing p. Recall that since τ is a type 2 vertex, $Lk(\tau, \widetilde{\Delta})$ is naturally identified with $\Gamma \star X$. Now $Lk(\tau, Y)$ is a subcomplex of $\Gamma \star X$ and by Lemma 5.4.3 we know that $Lk(\tau, [\tau, \tau']) \in X$. So there is a natural bijection between the aforementioned set of polygons and the set of edges of a cycle in Γ. Since all cycles in Γ have more than four edges, we have established that the angle around p in $i(\mathbb{R}^2)$ is greater than 2π contradicting the fact that this is an isometric embedding. □

5.5 Concluding remarks

At this point we do not have a concrete way of distinguishing our example in Section 4 from Brady's example in [6], other than the method of construction. In fact, there is a striking similarity between the two examples, even though the methods of construction are entirely different. Nevertheless, we do believe that our approach is less abstract.

This construction does not seem to have a natural generalization in higher dimensions. It seems likely that any natural generalization in dimensions four and higher always produces flat planes in the universal cover. As a result, it is not clear whether such an approach can be used to construct hyperbolic groups with subgroups that are of type F_n but not of type F_{n+1} for $n > 2$.

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