HOMOLOGICAL STABILITY FOR
DIFFEOMORPHISM GROUPS OF 3-MANIFOLDS

A Dissertation
Presented to the Faculty of the Graduate School
of Cornell University
in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy

by
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January 2015
HOMOLOGICAL STABILITY FOR DIFFEOMORPHISM GROUPS OF
3-MANIFOLDS
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Cornell University 2015

To study the homology of classifying spaces of diffeomorphism groups of compact, orientable 3-manifolds, we use the stabilization map that glues a prime 3-manifold, $P$, to an existing manifold $N$ with non-empty boundary via connected sum. The main result is that the stabilization map induces a homology isomorphism for the $i$-th homology after the number $n$ of existing $P$ summands in $N$ is greater than $2i + 2$.

We also examine stabilization by the 3-ball, $D^3$, for the case $P = S^1 \times S^2$, induces homology isomorphism in the same range as before, i.e. if the number $n$ of existing $P$ summands in $N$ is greater than $2i + 2$. This result allows us to exhibit homological stability even if $N$ does not have boundary in the case $P = S^1 \times S^2$. Along with known results about the stable homology, this gives a way to compute homology of $BDiff(\#_n S^1 \times S^2)$ in the range where homological stability holds.
BIOGRAPHICAL SKETCH

Chor Hang Lam earned his Bachelor of Art degree in Mathematics from Dartmouth College, located in New Hampshire, in 2008. Afterward, he moved to Ithaca to continue his study of Mathematics at Cornell University. He received his Master degree in 2011 and is currently finishing his dissertation, Homological stability for diffeomorphism groups of 3-manifolds, under the supervision of Allen Hatcher.
This thesis is dedicated to my classmates and my teachers at Cornell.
ACKNOWLEDGEMENTS

I want to thank my advisor, Allen Hatcher, for his guidance throughout this long journey. I am also grateful for the assistances provided by my committee members Tim Riley, Ken Brown and Karen Vogtmann, and the Director of Graduate Studies, Reyer Sjamaar, as well as Oscar Randal-Williams.
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CHAPTER 1

INTRODUCTION

Homological stability is a phenomenon that arises from studying the group homology of specific sequences of groups, where each group $G_n$ in the sequence $\{G_n\}_{n \in \mathbb{N}}$ naturally includes into the next group $G_{n+1}$. Our main result is that this holds for the diffeomorphism groups of a compact, orientable 3-manifold with non-empty boundary, glued to $n$-copies of another compact, orientable 3-manifold via connected sum.

One of the first topological examples where homological stability occurs is the sequence of mapping class groups $\pi_0(Diff(S_{n,1}, \partial S_{n,1}))$ of orientable surfaces $S_{n,1}$ of genus $n$ with 1 boundary component, where $f \in Diff(A,B)$ is a diffeomorphism $f : A \to A$ and $f|_{N_B} = id$ for some neighborhood $N_B$ of the subspace $B$ of $\partial A$. The inclusion $\pi_0(Diff(S_{n,1}, \partial S_{n,1})) \hookrightarrow \pi_0(Diff(S_{n+1,1}, \partial S_{n,1}))$ is defined by extending any given diffeomorphism of $S_{n,1}$ that fixes the boundary to the identity map in $S_{n+1,1} - S_{n,1}$. The homological stability result obtained by Harer, and later improved by Ivanov, Boldsen and Randal-Williams, is that this map induces homology isomorphisms

$$H_i(\pi_0(Diff(S_{n,1}, \partial S_{n,1})); \mathbb{Z}) \to H_i(\pi_0(Diff(S_{n+1,1}, \partial S_{n,1})); \mathbb{Z})$$

in the range $i \leq \frac{2n-2}{3}$. Another example in which homological stability holds is the sequence of mapping class groups $\pi_0(Diff(N^P_n, \partial N))$ of 3-manifolds of the form $N^P_n := N \#(\#_nP)$, where $N$ and $P$ are compact, orientable 3-manifolds, and $N$ has nonempty boundary. Hatcher and Wahl proved that

$$H_i(\pi_0(Diff(N^P_n, \partial N)); \mathbb{Z}) \to H_i(\pi_0(Diff(N^P_{n+1}, \partial N)); \mathbb{Z})$$
is an isomorphism for $i \leq \frac{n-2}{2}$. Note that we mainly work with homology with coefficients $\mathbb{Z}$, and so $\mathbb{Z}$ will be omitted from the notation from now on.

The approach first used to prove these results involves finding a simplicial complex $C_n$ on which $G_n$ acts, with properties such as the simplex stabilizers $\text{stab}_{G_n} \sigma$ being isomorphic to $G_{n-\dim \sigma - 1}$. Homological stability can then be deduced from the equivariant spectral sequence using a connectivity property of $C_n$. For example, in the case of $\pi_0(\text{Diff}(S_{n,1}, \partial S_{n,1}))$, one can use the complex of isotopy classes of arc embeddings with fixed boundary in $\partial S_{g,1}$ and with a fixed ordering for the arcs about the boundary points; see [10] for details. In the case of $\pi_0(\text{Diff}(N_n^{S^1 \times S^2}, \partial N))$, one can use the complex of isotopy classes of tethered sphere embeddings, where a tether is an arc joining a fixed point in $\partial N$ to a fixed point on the sphere; see [6] for details.

There is a more topological approach introduced by Randal-Williams in [9] that works with the usual homology $H_i(BG_n)$ of a classifying space $BG_n$ of $G_n$, which is isomorphic to the group homology $H_i(G_n)$ of $G_n$. It involves applying the spectral sequence for augmented semi-simplicial spaces to an augmented semi-simplicial space $X^n_* \to BG_n$ with certain connectivity properties. Instead of mapping class groups $\pi_0(\text{Diff}(S_{n,1}, \partial S_{n,1}))$, the full diffeomorphism group $\text{Diff}(S_{n,1}, \partial S_{n,1})$ is used, which yields the same homology because $\text{Diff}(S_{n,1}, \partial S_{n,1})$ is homotopy equivalent to $\pi_0(\text{Diff}(S_{n,1}, \partial S_{n,1}))$, because it is shown in [1] that the components of $\text{Diff}(S_{n,1}, \partial S_{n,1})$ are contractible. This allows one to recover homological stability for mapping class groups of surfaces. In [9], this approach is also used to obtain homological stability for the more general diffeomorphism groups of surfaces with tangential structure, which include the diffeomorphism groups of non-orientable surfaces and of surfaces in a simply-connected background space as special cases.
For 3-manifolds, $\text{Diff}(N^P_n, \partial N)$ does not always have contractible components, so homological stability for $\{\pi_0(\text{Diff}(N^P_n, \partial N))\}_{n \in \mathbb{N}}$ does not in general imply the same result for $\{\text{Diff}(N^P_n, \partial N)\}_{n \in \mathbb{N}}$. In this paper, we examine classifying spaces of the latter sequence of groups to obtain:

**Theorem 1.** For $i \leq \frac{n-2}{2}$,

$$H_i(B\text{Diff}(N^P_n, \partial N)) \approx H_i(B\text{Diff}(N^P_{n+1}, \partial N)),$$

where $N$ and $P$ are compact, orientable 3-manifolds and $N$ has non-empty boundary.

We focus on the case $P = S^1 \times S^2$ in Section 4 and deal with the case $P \neq S^1 \times S^2$ in Section 6. Note that we can assume from now on that $P$ is prime because the result for this case implies the result for general $P$. The case $P = S^1 \times S^2$ also admits another stabilization that induces homological isomorphisms, which we will explore in Section 5:

**Theorem 2.** For all $i \leq \frac{n-2}{2}$,

$$H_i(B\text{Diff}(N^P_n S^1 \times S^2, \partial N)) \approx H_i(B\text{Diff}((N \# D^3)_n S^1 \times S^2, \partial (N \# D^3))),$$

where $N$ is a compact, orientable 3-manifold.

Note that $N$ can have empty boundary, in which case, the diffeomorphism group $\text{Diff}(N, \partial N)$ is defined to consist only of the orientation-preserving diffeomorphisms of $N$. For $N$ with non-empty boundary, the stated homology isomorphism is induced by a stabilization via the connected sum with $D^3$, while for $N$ without boundary, the isomorphism in the theorem is induced by a ball-filling map in the reverse direction, which we will consider in Section 8. The two stated theorems
allow us to conclude that the homology $H_i(\text{BDiff}(\#_n S^1 \times S^2))$ is constant as we vary $n$ starting from $\frac{n-2}{2} \geq i$.

As mentioned, we will need connectivity properties for certain semi-simplicial spaces to apply the spectral sequence argument. Simplifications are made by Galatius and Randal-Williams in [2], relating the connectivity of the semi-simplicial space and the connectivity of the underlying semi-simplicial set, where the set of $k$-simplexes are topologized discretely for any $k \in \mathbb{N}$. This allows us to apply techniques for simplicial complexes, such as the join argument, to obtain the connectivity results in Section 7.

Before moving on, let us mention the context where these diffeomorphism groups arise. A classifying space $BDiff(M, \partial M)$ of $\text{Diff}(M, \partial M)$, for a compact manifold $M$, has the property that any smooth fiber bundle $\pi : E \to B$ over a manifold $B$ with fiber $M$ is the pullback of the universal bundle $E\text{Diff}(M, \partial M) \to B\text{Diff}(M, \partial M)$ via a map $f_\pi : B \to B\text{Diff}(M, \partial M)$. This is a one-to-one correspondence between the set of isomorphism classes of fiber bundles and homotopy classes of maps from $B$ to $B\text{Diff}(M, \partial M)$. Each $c \in H^*(B\text{Diff}(M, \partial M))$ gives a characteristic class, by associating $\pi : E \to B$ with $f_\pi^*(c) \in H^*(B)$, which helps in classifying fiber bundle over $B$ with fiber $M$. For fiber bundles where the fiber is a surface $S$, we can obtain characteristic classes from $H^*(B\text{Diff}(S, \partial S))$. For $n$-coverings, we can obtain characteristic classes from $H^*(B\Sigma_n)$, where $\Sigma_n$ is the symmetric group on $n$ elements (the homological stability result for this group is due to Nakaoka and a proof can be found in [8]).

To compute the corresponding homology $H_*(BG_n)$ in the range where homological stability holds, one can instead compute the stable homology $H_*(\bigcup_{n \in \mathbb{N}} BG_n)$. For surfaces, $H_*(B\text{Diff}(S_{n,1}, \partial S_{n,1})) = H_*(B\text{Diff}(S_{n+1,1}, \partial S_{n+1,1}))$ for $\frac{2n-2}{3} \geq *$ by
the homological stability result earlier, so one can obtain $H_*(\text{BDiff}(S_{n,1}, \partial S_{n,1}))$ for $\frac{2n-2}{3} \geq *$ by computing the stable homology $H_*(\bigcup_{n \in \mathbb{N}} \text{BDiff}(S_{n,1}, \partial S_{n,1}))$, using the technique of scanning to relate the stable homology to the homology of an infinite loop space. The result, also known as the Madsen-Weiss Theorem, is an isomorphism $H_*(\bigcup_{n \in \mathbb{N}} \text{BDiff}(S_{n,1}, \partial S_{n,1})) \approx H_*(\Omega_0^\infty AG_{\infty,2}^+)$ for $* \leq \frac{2n-2}{3}$, where $\Omega_0^k$ denotes one component of the $k$-iterated loop space and $AG_{k,d}$ is the affine Grassmannian of oriented $d$-planes in $\mathbb{R}^k$ and the superscript $+$ denotes the 1-point compactification as the distance of the $d$-plane from the origin approaches infinity. The rational cohomology $H^*(\Omega_0^\infty AG_{\infty,2}^+; \mathbb{Q})$ is a polynomial algebra $\mathbb{Q}[x_2, x_4, x_6, \ldots]$, with $x_{2i} \in H^{2i}$; for more details, see [4]. For 3 dimensions, the stable homology can be computed to be $H_*(\Omega_0^\infty S^\infty BSO(d)_\pm)$, for $d = 3$ and $d = 4$ in the cases of handlebodies and $\#_n S^1 \times S^2$, respectively; see [3]. By the homological stability results in this paper, this computation of the stable homology gives a computation for $H_*(\text{BDiff}(\#_n S^1 \times S^2))$ if $* \leq \frac{n-2}{2}$.
CHAPTER 2
BACKGROUND

We first provide some general definitions and properties of semi-simplicial spaces. The main property is the relation between the homology of the space $X_p$ of $p$-simplexes and that of the geometric realization $|X_*$|, in the form of a spectral sequence

$$E_{p,q}^1 = H_q(X_p), \quad d_{p,q}^1 = \sum_{i=0}^{p} (-1)^i H_q(d_i) \Rightarrow H_{p+q}(|X_*|),$$

given in Section 2 of [9]. The terms are defined as follows.

**Definition 3.** A semi-simplicial space $X_*$ consists of topological spaces $X_k$ for $k \in \mathbb{N}$, called the space of $k$-simplexes, and maps $d_i : X_k \to X_{k-1}$ for $0 \leq i \leq k$, called the face maps, satisfying

$$d_i \circ d_j = \begin{cases} d_j \circ d_{i-1} & \text{if } i > j \\ d_i \circ d_{i+1} & \text{if } i = j \\ d_{j+1} \circ d_i & \text{if } i < j \end{cases}.$$

The sequence of spaces in a semi-simplicial space can be glued together to produce a topological space via the geometric realization.

**Definition 4.** The geometric realization of $X_*$ is

$$|X_*| = \prod_{i \geq 0} X_i \times \Delta^i \sim,$$

where the equivalence relation $\sim$ is defined as $(d_k(x), t) \sim (x, \iota_k(t))$ for $x \in X_i$ and $t \in \Delta^i$. Here, $d_k$ is the face map and $\iota_k : \Delta^i \to \Delta^{i+1}$ is the inclusion of the standard $i$-simplex as the $(k+1)$-face of the standard $(i+1)$-simplex given by $(t_0, \ldots, t_i) \mapsto (t_0, \ldots, t_{k-1}, 0, t_{k+1}, \ldots, t_i)$. 
The semi-simplicial space we will use has a map to the classifying space of interest, and the map induces a map from the geometric realization to the target. This situation is analogous to how one augments a chain complex to have a $-1$ degree term.

**Definition.** An augmented semi-simplicial space is a semi-simplicial space $X_*$ along with a space $X_{-1}$ and $\epsilon : X_0 \to X_{-1}$, such that $\epsilon \circ d_0 = \epsilon \circ d_1 : X_1 \to X_{-1}$. The map $\epsilon$ called an augmentation.

The augmentation induces a map $|\epsilon_*| : |X_*| \to X_{-1}$, defined by $(x, t) \mapsto \epsilon \circ d_0 \circ \cdots \circ d_0(x)$ for $x \in X_i$. This is well-defined because $\epsilon \circ d_{i_0} \circ \cdots \circ d_{i_k}$, for any sequence $i_0, \ldots, i_k$, can be turned into $\epsilon \circ d_0 \circ \cdots \circ d_0$ using the face relations $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$ to lower the subscript of $d_j$ without increasing $\sum_{i=0}^k i_t$ until $i_0 \geq \cdots \geq i_k$, and if $i_0 \neq 0$ we can use $\epsilon \circ d_0 = \epsilon \circ d_1$ and continue the decrease of $\sum_{i=0}^k i_t$ until $0 = i_0 \geq \cdots \geq i_k$, after at most $\sum_{i=0}^k i_t$ steps.

With the map $|\epsilon_*|$ defined, we can consider a version of the spectral sequence for a semi-simplicial space that includes the augmentation, which is more convenient for our purpose:

$$E^1_{p,q} = H_q(X_p), \quad d^1_{p,q} = \sum_{i=0}^p (-1)^i H_q(d_i) \Rightarrow H_{p+q+1}(X_{-1}, |X_*|)$$

where $p \geq -1$, $q \geq 0$, and $H_{p+q+1}(X_{-1}, |X_*|)$ comes from $|\epsilon_*| : |X_*| \to X_{-1}$.

The task in the coming sections is to find a semi-simplicial space $X_*(N_n^P)$ that augments $BDiff(N_n^P, \partial N)$, where the map $|\epsilon_*|$ is highly connected, so that we can deduce $d^1_{p,q}$ is an isomorphism in a certain range using the spectral sequence above.
CHAPTER 3
SPACES OF EMBEDDINGS

In this section, we define a model for $BDiff(N^P_n, \partial N)$ and a semi-simplicial space that augments it. There are some boundary conditions we impose for the purpose of gluing manifolds together.

**Definition 5.** Let $N$ and $P$ be compact, orientable 3-manifolds, where $P$ is prime and $\partial N$ is non-empty, with connected components denoted by $\partial_0, \ldots, \partial_s$ for some $s \in \mathbb{N}$. Define $N^P_n$ as the connected sum $N \# (\#_{i=0}^{n-1} P)$. For each $k \in \{0, \ldots, s\}$, we fix an embedding $i_{\partial_k} : \partial_k \to [-1, 1]^\infty$ of $\partial_k$ and a collar neighborhood $N_{\partial_k}$ of $\partial_k$ in $N^P_n$ with identification $j_{N_{\partial_k}} : N_{\partial_k} \to [0, \epsilon_\partial) \times \partial_k$, where $\epsilon_\partial := \frac{1}{4}$. This gives an embedding $id_{(0, \epsilon_\partial)} \times (i_{\partial_k} + 3k\epsilon_1^2)$ of $[0, \epsilon_\partial) \times \partial_k$ in $[0, \epsilon_\partial) \times \mathbb{R}^\infty$, where $i_{\partial_k} + 3k\epsilon_1^2$ means that we shift the first coordinate of $i_{\partial_k}$ by $3k$ units, so that the embedding is disjoint from an embedding with a different value of $k$. Let $N_{\epsilon, k}$ be the neighborhood $j_{N_{\partial_k}}^{-1}((0, \epsilon) \times \partial_k)$ of $\partial_k$ in $N_{\partial_k}$.

- Let $Emb^\partial(N^P_n, [0, \infty) \times \mathbb{R}^\infty)$ be the space of embeddings $e : N^P_n \hookrightarrow [0, \infty) \times \mathbb{R}^\infty$, with the boundary condition $e|_{N_{\epsilon, k}} = (id \times i_{\partial_k} + 3k\epsilon_2^2) \circ j_{N_{\partial_k}}|_{N_{\epsilon, k}}$ for some $\epsilon > 0$. This space is topologized by the $C^\infty$ topology.
- Let $Diff(N^P_n, \partial N)$ be the group of diffeomorphisms $f : N^P_n \to N^P_n$ such that $f|_{N_{\epsilon, k}} = id$ for some $\epsilon$. This group is topologized by the $C^\infty$ topology.
- Define the space of submanifolds of the form $N^P_n$ in $[0, \infty) \times \mathbb{R}^\infty$ to be
  \[ Sub^\partial(N^P_n, [0, \infty) \times \mathbb{R}^\infty) := Emb^\partial(N^P_n, [0, \infty) \times \mathbb{R}^\infty)/Diff(N^P_n, \partial N). \]

When the target space is $[0, \infty) \times \mathbb{R}^\infty$, we abbreviate $Emb^\partial(N^P_n, [0, \infty) \times \mathbb{R}^\infty)$ and $Sub^\partial(N^P_n, [0, \infty) \times \mathbb{R}^\infty)$, by omitting $[0, \infty) \times \mathbb{R}^\infty$, as $Emb^\partial(N^P_n)$ and $Sub^\partial(N^P_n)$, respectively.
• Define an embedding \((0, 1] \times \partial_0) \# P \hookrightarrow [0, 1] \times \mathbb{R}^\infty\) of a cylinder \(i_{[0,1]} \times i_{\partial_0} : [0, 1] \times \partial_0 \hookrightarrow [0, 1] \times [-1, 1]\) and glued via connected sum to some fixed embedding of \(P\) in \([\frac{1}{3}, \frac{2}{3}] \times [0, 2]\). We denote the union of these embeddings and of the embeddings \(id_{[0,1]} \times (i_{\partial_k} + 3k\vec{e}_1)\) by \(c_0^P\).

• Define a map \(Emb^\beta(N^P_n) \to Emb^\beta(N^P_{n+1})\) by \(e \mapsto c_0^P \cup (1 \cdot \vec{e}_1 + e)\) where \(1 \cdot \vec{e}_1 + e\) means shifting the embedding \(e\) one unit in the first coordinate.

This map induces \(s^P : Sub^\beta(N^P_n) \to Sub^\beta(N^P_{n+1})\), called the stabilization map.

**Definition.** A classifying space \(BG\) of a group \(G\) is defined as follows: if \(EG\) is a weakly contractible space on which \(G\) acts freely, then \(BG := EG/G\) is a classifying space of \(G\).

Note that \(Sub^\beta(N^P_n)\) is a classifying space of \(\text{Diff}(N^P_n, \partial N)\) because it is well known that \(Emb^\beta(N^P_n)\) is weakly contractible; for a proof, see Proposition 10. We restate Theorem 1 using \(Sub^\beta(N^S^1 \times S^2)\):

**Theorem 6.** \(H_q(S^1 \times S^2) : H_q(Sub^\beta(N^S^1 \times S^2)) \to H_q(Sub^\beta(N^S^1 \times S^2))\) is an isomorphism for \(q \leq \frac{n-2}{2}\).

Next, we describe embeddings of certain submanifolds of \(N^S^1 \times S^2\) that are used to decompose \(N^S^1 \times S^2\) into smaller pieces. We use nonseparating spheres embedded in \(N^S^1 \times S^2\) that are tethered to the boundary \(\partial_0\), along with some boundary conditions that respect the boundary conditions in the previous definition of \(Emb^\beta(N^S^1 \times S^2)\):

**Definition 7.** Given \(e \in Emb^\beta(N^S^1 \times S^2)\), let \(\phi_{\partial_0} : H_{\partial_0} \to \mathbb{R}^2\) be a chart for an open ball in \(\partial_0\). Let \(N_{\partial_0} \subset N_{\partial_0}\) be a collar neighborhood of \(H_{\partial_0}\) in \(N^S^1 \times S^2\). Fix a
chart \( \phi_{NH_0} : NH_0 \to [0, \epsilon_0) \times \mathbb{R}^2 \) of \( NH_0 \), such that \( e(x) \) and \( \phi_{NH_0}(x) \) agrees in the first coordinate for all \( x \) in \( NH_0 \).

- Let \( \text{Emb}^\partial(I + S^2, N_n^{S^1 \times S^2}) \) be the space of embeddings where an embedding \( f \) is an embedding of \( I \) and of \( S^2 \) in \( N_n^{S^1 \times S^2} \), such that the images \( f(I) \) and \( f(S^2) \) intersect transversely at one point. We require that there is some \( t \in (0, \infty) \) and some \( \epsilon > 0 \) such that for any \( s \in [0, \epsilon), f|_I(s) = \phi_{NH_0}^{-1}(s, -t, 0) \) and \( f|_I(1-s) = \phi_{NH_0}^{-1}(s,t,0) \). This space is topologized by the \( C^\infty \) topology.

- Let \( \text{Emb}^\partial(I + S^2, N_n^{S^1 \times S^2}) \) be the space of embeddings where an embedding \( f \) is an embedding of \( I \times D^2 \) and of \( S^2 \times I \) in \( N_n^{S^1 \times S^2} \), such that \( f(I \times \{c\}) \) intersects \( f(S^2 \times \{b\}) \) transversely at one point for any \( b \in I \) and \( c \in D^2 \). There is a deformation retract of an embedding of \( \overline{I + S^2} \) to \( I + S^2 \) obtained by shrinking \( \overline{I} := I \times D^2 \) to \( I \times \{0\} \) and by shrinking \( \overline{S^2} := S^2 \times I \) to \( S^2 \times \{\frac{1}{2}\} \); denote this deformation retract by \( r_t : \overline{I + S^2} \to \overline{I + S^2} \), where \( t \in [0,1] \) with \( r_0 = \text{id} \) and \( \text{im}(r_1) = I + S^2 \). We require \( f \in \text{Emb}^\partial(\overline{I + S^2}, N_n^{S^1 \times S^2}) \) to satisfy the boundary condition. For some \( \epsilon > 0 \), with a chart for \( r_t^{-1}([0,\epsilon)) \) and \( r_t^{-1}((1-\epsilon,1]) \), respectively, denoted by \( \phi_0 : r_t^{-1}([0,\epsilon)) \to [0,\epsilon) \times D^2 \) and \( \phi_1 : r_t^{-1}((1-\epsilon,1]) \to [0,\epsilon) \times D^2 \), there exist \( r \in \left(0, \frac{C_0}{2}\right) \) and \( t \in \mathbb{R} \) such that for all \( s \in [0,\epsilon) \), \( f \circ \phi_0^{-1}(s,x,y) = \phi_{NH_0}^{-1}(s,rx+ty,ry) \) and \( f \circ \phi_1^{-1}(1-s,x,y) = \phi_{NH_0}^{-1}(s,rx+ty,ry+C_y) \).

- Let \( \text{Emb}_k(\overline{I + S^2}, N_n^{S^1 \times S^2}) \) be the space of \((k+1)\)-tuples \((f_0, \ldots, f_k)\) of disjoint embeddings \( f_i \in \text{Emb}^\partial(\overline{I + S^2}, N_n^{S^1 \times S^2}) \) where the complement \( N_n^{S^1 \times S^2} - \bigsqcup_{i=0}^k \text{im}(f_i) \) is connected. We require that \( f_i \circ \phi_0(s,x,y) = \phi_{NH_0}^{-1}(s,r^ix+t_i,r^iy) \) and \( f_i \circ \phi_1(s,x,y) = \phi_{NH_0}^{-1}(s,r^ix+t_i,r^iy+C_y) \) for some \( r^i \in \left(0, \frac{C_y}{2}\right) \) and \( t_i \in \mathbb{R} \), that \( t_0 < t_1 < \cdots < t_k \), and that \( I \) intersects \( S^2 \times \{a\} \) transversely, for \( S^2 \times \{a\} \subset S^2 \times I \), i.e. for each parallel slice in the thickened
spherical part of $I + S^2$. We topologize $Emb_k(I + S^2, N_n^{S^1 \times S^2})$ by the $C^\infty$ topology. $Emb_k(I + S^2, N_n^{S^1 \times S^2})$ can be defined similarly.

- $Emb^\partial(I + S^2, [0, \infty) \times \mathbb{R}^\infty)$ is defined as the space of elements $f \circ g$ where $g \in Emb^\partial(I + S^2, N_n^{S^1 \times S^2})$ and $f \in Emb^\partial(N_n^{S^1 \times S^2})$.

Essentially, an embedding of $I + S^2$, which we will call a tethered sphere, is a neighborhood of $I + S^2$ in $N_n^{S^1 \times S^2}$, along with boundary conditions that allows one to extend the $I$ part when gluing another manifold on $\partial_0$. Finally, we define the semi-simplicial space that is used to augment $B\text{Diff}(N_n^{S^1 \times S^2}, \partial N)$.

**Definition 8.** Let $Emb_*(I + S^2, N_n^{S^1 \times S^2})$ be a semi-simplicial space such that the space of $k$-simplexes is $Emb_k(I + S^2, N_n^{S^1 \times S^2})$. Define the face maps to be the forgetful maps: $Emb_k(I + S^2, N_n^{S^1 \times S^2}) \xrightarrow{d_j} Emb_{k-1}(I + S^2, N_n^{S^1 \times S^2})$ given by $(f_0, \ldots, f_k) \mapsto (f_0, \ldots, \hat{f}_j, \ldots, f_k)$.

Define $X_*(N_n^{S^1 \times S^2})$ as the balanced product

$$Emb^\partial(N_n^{S^1 \times S^2}) \times \text{Diff}_{(N_n^{S^1 \times S^2}, \partial N)} Emb_*(I + S^2, N_n^{S^1 \times S^2}) := Emb^\partial(N_n^{S^1 \times S^2}) \times Emb_*(I + S^2, N_n^{S^1 \times S^2}) / \sim$$

where $(e, (f_0, \ldots, f_k)) \sim (e', (f'_0, \ldots, f'_k))$ if $e' = e \circ h$ and $f'_i = h^{-1} \circ f_i$ for some $h \in \text{Diff}(N_n^{S^1 \times S^2}, \partial N)$. We denote such an equivalence class by $[e, (f_0, \ldots, f_k)] \in X_k(N_n^{S^1 \times S^2})$. The face maps are again the forgetful maps. Define $X_{-1}(N_n^{S^1 \times S^2}) := \text{Sub}^\partial(N_n^{S^1 \times S^2})$ and define an augmentation map $X_0(N_n^{S^1 \times S^2}) \to X_{-1}(N_n^{S^1 \times S^2})$ by $[e, (f_0)] \mapsto [e]$.

Another way to view an element $[e, (f_0, \ldots, f_k)] \in X_k(N_n^{S^1 \times S^2})$ is as a pair $(\text{im}(e), (e \circ f_0, \ldots, e \circ f_k))$, i.e. a submanifold $\text{im}(e)$ marked by $k + 1$ tethered sphere embeddings $e \circ f_0, \ldots, e \circ f_k$ in $\text{im}(e)$. We will use this in the next section.
CHAPTER 4

STABILIZATION BY $S^1 \times S^2$

In this section, we relate the semi-simplicial space $X_*(N^{S^1 \times S^2}_n)$ and its face maps to $\{BDiff(N^{S^1 \times S^2}_n, \partial N)\}_{n \in \mathbb{N}}$ and the associated stabilization map in order to obtain homological stability. The approach taken here is similar to Section 5 of [2]. The idea is to use the spectral sequence for augmented semi-simplicial spaces and prove various properties of $X_*$ so that the homology group $H_q(BDiff(N^{S^1 \times S^2}_{n-p-1}, \partial N))$ is the entry $E^1_{p,q}$ and the $d^1_{p,q}$ are precisely the maps induced by $s^{S^1 \times S^2}$.

First, we discuss a few properties of the space of embeddings into $[0, \infty) \times \mathbb{R}^\infty$.

**Definition 9.** Let $M$ be a compact manifold and let $\partial_0'$ be a subspace of $\partial M$. Let $Emb^\partial(M, [0, \infty) \times \mathbb{R}^\infty, \partial_0')$ be the space of embeddings of $M$ in $[0, \infty) \times \mathbb{R}^\infty$ such that each embedding $e \in Emb^\partial(M, [0, \infty) \times \mathbb{R}^\infty, \partial_0')$ agrees in some collar neighborhood $N_{\partial_0'}$ of $\partial_0'$ with an embedding $e_0 : M \to [0, \infty) \times \mathbb{R}^\infty$ that we fixed a priori.

**Proposition 10.** $Emb^\partial(M, [0, \infty) \times \mathbb{R}^\infty, \partial_0')$ is contractible.

**Proof.** Given an embedding $e' \in Emb^\partial(M, [0, \infty) \times \mathbb{R}^\infty, \partial_0')$ and $s \in M$, denote $e'(s)$ by $(x_1(s), x_2(s), \ldots)$, and after performing a homotopy on $e'$, we will denote the result by $e'$ as well. For $s \in \partial_0'$, note that for some $k$, $x_i(s) = 0$ for all $i > k$, where $k$ depends on the number of dimensions it takes to embed $M$ in $[0, \infty) \times \mathbb{R}^\infty$. We need to deform all such embeddings to one particular embedding. First, choose an embedding $a^0$ in $Emb^\partial(M, [0, \infty) \times \mathbb{R}^\infty, \partial_0')$ with

$$a^0(s) = (e_1(s), \ldots, e_k(s), 0, 0, \ldots)$$
for $s \in M$. Deform $e'$ in $[0, \infty) \times \mathbb{R}^\infty$ via the straight-line homotopy

$$(x_1, \ldots x_k, (1-t)x_{k+1}, \ldots (1-t)x_{2k}, tx_{k+1} + (1-t)x_{2k+1}, \ldots)$$

for $t \in [0, 1]$, ending in $(x_1, \ldots, x_k, 0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots)$. This keeps any embedding in $Emb^\partial(M, [0, \infty) \times \mathbb{R}^\infty, \partial_0')$ an embedding because if 2 points $(x_1, x_2, \ldots)$ and $(x_1', x_2', \ldots)$ become the same for some $t$, then $x_i = x_i'$ for $i \leq 2k$, which would imply the same for $i > 2k$, so that they are the same point to start with. The boundary conditions still hold because they only concern the first $k$ coordinates, which are unchanged.

Next, pick a function $h : M \to [0, \infty)$ that maps to 0 for the closure of the collar neighborhood $N_{\partial_0'}$ of $\partial_0'$ in $M$ specified in the boundary condition of the definition of $Emb^\partial$, and to some nonzero value otherwise; for instance, one can use the distance from the collar neighborhood, via a metric on $M$, to obtain $h$. Now, perform a straight-line homotopy on $e'$, sending $(x_1, \ldots, x_k, 0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots)$ to

$$(x_1, \ldots, x_k, h(s)e_1(s), \ldots, h(s)e_k(s), x_{k+1}, x_{k+2}, \ldots).$$

This remains an embedding because different points have different $x_i$, for some $i$. This homotopy keeps the collar neighborhood of $\partial M$ fixed because of how we defined $h$.

Next, use a straight-line homotopy to deform $e'$ to

$$(x_1, \ldots, x_k, h(s)e_1(s), \ldots, h(s)e_k(s), 0, 0, \ldots),$$

which remains an embedding because the first $k$ coordinates distinguish different points in the collar neighborhood while the next $k$ coordinates distinguish points outside the collar neighborhood and are not all 0, which distinguishes these from the points in the collar neighborhood.
Finally, perform a straight-line homotopy that ends with

\[(e_1(s), \ldots, e_k(s), h(s)e_1(s), \ldots, h(s)e_k(s), 0, 0, \ldots),\]

which keeps the collar neighborhood fixed because \(a^0\) agrees with \(e'\). This finishes the process of homotoping any embedding to a specific embedding, and so we are done.

**Proposition 11.** Let \(M\) be a compact 3-manifold, with \(M = M'_1 \cup M'_2\), where \(M'_1\) and \(M'_2\) are compact 3-manifolds that intersect in a compact 2-manifold in \(\partial M'_1 \cap \partial M'_2\). Then \(\text{Emb}^0(M'_1, [0, \infty) \times \mathbb{R}^\infty - e(\text{int}(M'_2)), \partial M'_1)\) is weakly contractible, for any \(e \in \text{Emb}^0(M'_2, [0, \infty) \times \mathbb{R}^\infty, \emptyset)\).

**Proof.** Let \(r : \text{Emb}^0(M'_1 \cup M'_2, [0, \infty) \times \mathbb{R}^\infty, \partial M'_1 - \partial M'_2) \to \text{Emb}^0(M'_2, [0, \infty) \times \mathbb{R}^\infty, \emptyset)\) be defined by \(f \mapsto f|_{M'_2}\). It is a fibration by isotopy extension, and both spaces are contractible by Proposition 10, so the fiber \(r^{-1}(e)\), where \(e \in \text{Emb}^0(M'_2, [0, \infty) \times \mathbb{R}^\infty, \emptyset)\), is weakly contractible by the short exact sequence of homotopy groups for a fibration. Since \(r^{-1}(e) = \text{Emb}^0(M'_1, [0, \infty) \times \mathbb{R}^\infty - e(\text{int}(M'_2)), \partial M'_1)\), we are done.

**Proposition 12.** \(X_k(N_n^{S^1 \times S^2})\) is a classifying space of \(\text{Diff}^0(N_n^{S^1 \times S^2}, \partial N)\) for \(n \geq k + 1\).

**Proof.** First, let’s see that \(X_0(N_n^{S^1 \times S^2})\) is homeomorphic to

\[X'_0 := \{((\text{im}(f), (h_0, \ldots, h_k)) \in \text{Sub}^0(N_n^{S^1 \times S^2}) \times \text{Emb}_k(T + S^2, [0, \infty) \times \mathbb{R}^\infty) : \text{im}(f) \supset \text{im}(h_0)\}.

The map from \(X_0(N_n^{S^1 \times S^2})\) to \(X'_0\) is defined by \([f, g] \mapsto (\text{im}(f), (f \circ g))\), which is well-defined because for \(u \in \text{Diff}(N_n^{S^1 \times S^2}, \partial N)\), \([f \circ u, u^{-1} \circ g]\) is mapped to
$(\text{im}(f), (f \circ g))$ as well. The inverse is defined by $(\text{im}(f), (h_0)) \mapsto [f, f^{-1} \circ h_0]$, where $f^{-1}$ is the inverse of the diffeomorphism $f : N_n^{S^1 \times S^2} \to \text{im}(f)$, which has domain \(\text{im}(f) \supset \text{im}(h_0)\); so it is well-defined again because for \(u \in \text{Diff}(N_n^{S^1 \times S^2}, \partial N)\), $\text{im}(f \circ u, h_0)$ is mapped to $[f, f^{-1} \circ h_0]$.

Consider the projection $o : X'_0 \to \text{Emb}^\beta(\overline{T + S^2}, [0, \infty) \times \mathbb{R}^\infty)$, defined by $(\text{im}(f), (h_0)) \mapsto h_0$, which is a Serre fibration by the isotopy extension theorem. The base space is contractible by Proposition 10, so $X'_0$ is weakly homotopy equivalent to the fiber $o^{-1}(h_0)$ for some $h_0 \in \text{Emb}^\beta(\overline{T + S^2}, [0, \infty) \times \mathbb{R}^\infty)$. The fiber $o^{-1}(h_0)$ is $\{(\text{im}(f), (g_0)) \in \text{Sub}^\beta(N_n^{S^1 \times S^2}) \times \{h_0\} : \text{im}(f) \supset \text{im}(h_0)\}$, which we will see is homeomorphic to $\text{Sub}^\beta(M_n - f_0(\text{int}(\overline{T + S^2})), [0, \infty) \times \mathbb{R}^\infty - h_0(\text{int}(\overline{T + S^2})))$ for some fixed $(f_0) \in \text{Emb}_0(\overline{T + S^2}, N_n^{S^1 \times S^2})$. First, given $(f'(0)) \in \text{Sub}^\beta(N_n^{S^1 \times S^2} - f_0(\text{int}(\overline{T + S^2})), [0, \infty) \times \mathbb{R}^\infty - h_0(\text{int}(\overline{T + S^2})))$, we can extend $f'$ to $f : N_n^{S^1 \times S^2} \to [0, \infty) \times \mathbb{R}^\infty$ by gluing to $N_n^{S^1 \times S^2} - f_0(\text{int}(\overline{T + S^2}))$ the embedding $h_0 \circ f_0^{-1} : f_0(\overline{T + S^2}) \to [0, \infty) \times \mathbb{R}^\infty$, so $\text{im}(f) \mapsto \text{im}(f'(0))$ is a well-defined map to $o^{-1}(h_0)$. In the other direction we can define $(\text{im}(f), (h_0)) \mapsto \text{im}(f) - h_0(\text{int}(\overline{T + S^2})).$ This is well-defined because we can find a diffeomorphism $u_f \in \text{Diff}^\beta(N_n^{S^1 \times S^2})$ such that $f \circ u_f$ agrees with $h_0 \circ f_0^{-1}$ in $f_0(\overline{T + S^2})$, by extending the embedding $f^{-1} \circ h_0 \circ f_0^{-1} : f_0(\overline{T + S^2}) \to N_n^{S^1 \times S^2}$, and $\text{im}(f) - \text{im}(h_0)$ is the same as $\text{im} \left( f \circ u_f |_{N_n^{S^1 \times S^2} - f_0(\text{int}(\overline{T + S^2}))} \right)$, which is an element in $\text{Sub}^\beta(N_n^{S^1 \times S^2} - f_0(\text{int}(\overline{T + S^2})), [0, \infty) \times \mathbb{R}^\infty - h_0(\text{int}(\overline{T + S^2})))$.

The maps are inverses of each other because they involve adding and removing $h_0(\text{int}(\overline{T + S^2}))$ to and from the existing submanifold. Thus,

$X_0(N_n^{S^1 \times S^2}) \approx \text{Sub}^\beta \left( N_n^{S^1 \times S^2} - f_0(\text{int}(\overline{T + S^2})), [0, \infty) \times \mathbb{R}^\infty - h_0(\text{int}(\overline{T + S^2})) \right),$

so it is a classifying space of $\text{Diff}^\beta(N_n^{S^1 \times S^2})$ because $N_n^{S^1 \times S^2} - \text{int}(f_0(\overline{T + S^2}))$ is $N_{n-1}^{S^1 \times S^2}$ and $\text{Emb}^\beta \left( N_{n-1}^{S^1 \times S^2}, [0, \infty) \times \mathbb{R}^\infty - h_0(\text{int}(\overline{T + S^2})) \right)$ is weakly contractible.
by Proposition 11.

We can similarly prove the general case with \( n \geq k + 1 \), using

\[
X'_k := \{(im(f), (h_0, \ldots, h_k)) \in Sub^\partial(N_n^{S^1 \times S^2}) \times \text{Emb}_k(\overline{I + S^2}, [0, \infty) \times \mathbb{R}^\infty) : im(f) \supset im(h_i) \text{ for } i = 0, \ldots, k\}
\]

as well as the projection \( o : X'_k \to \text{Emb}_k(\overline{I + S^2}, [0, \infty) \times \mathbb{R}^\infty) \) onto the second component, \( (im(f), (h_0, \ldots, h_k)) \mapsto (h_0, \ldots, h_k) \). Again, the base is contractible by Proposition 10, and the fiber \( o^{-1}((h_0, \ldots, h_k)) \) can be seen to be \( Sub^\partial(N_n^{S^1 \times S^2} - \coprod_{i=0}^k f_i(int(\overline{I + S^2})), [0, \infty) \times \mathbb{R}^\infty - \coprod_{i=0}^k h_i(int(\overline{I + S^2}))) \) by picking \( k + 1 \) disjoint, nonseparating embeddings \( (f_0, \ldots, f_k) \in \text{Emb}_k(\overline{I + S^2}, N_n^{S^1 \times S^2}) \), and so \( X_k(N_n^{S^1 \times S^2}) \) is a classifying space of \( \text{Diff}(N_{n-k-1}, \partial N) \).

We now construct a weak homotopy equivalence between \( Sub^\partial(N_{n-k-1}^{S^1 \times S^2}) \) and \( X_k(N_n^{S^1 \times S^2}) \) that is compatible with the stabilization map \( Sub^\partial(N_n^{S^1 \times S^2}) \to Sub^\partial(N_{n+1}^{S^1 \times S^2}) \). As in the previous proof, we represent elements in \( X_0(N_n^{S^1 \times S^2}) \) with elements in \( X'_0 \).

**Definition 13.** Recall that \( c_0^{S^1 \times S^2} : (I \times \partial_0) \# (S^1 \times S^2) \to [0, 1] \times \mathbb{R}^\infty \) is the embedding used to define the stabilization map \( s^{S^1 \times S^2} : Sub^\partial(N_n^{S^1 \times S^2}) \to Sub^\partial(N_{n+1}^{S^1 \times S^2}) \). Define the map

\[
a_0 : Sub^\partial(N_{n-1}^{S^1 \times S^2}) \to X_0(N_n^{S^1 \times S^2})
\]

\[
\iota(N_{n-1}^{S^1 \times S^2}) \mapsto \left( im(c_0^{S^1 \times S^2}) \cup \left( \overline{c_1 + \iota(N_{n-1}^{S^1 \times S^2})} \right), (g_0) \right),
\]

where \( g_0 : \overline{I + S^2} \to im(c_0^{S^1 \times S^2}) \) is a fixed embedding of \( \overline{I + S^2} \): the \( S^2 \) part of \( g_0 \) is chosen to be a slice \( \{p\} \times S^2 \) in the copy \( S^1 \times S^2 \) of \( im(c_0^{S^1 \times S^2}) \).
More generally, define

\[ a_k : Sub^B(N^{S^1 \times S^2}_{n-k-1}) \rightarrow X_k(N^{S^1 \times S^2}_n) \]

\[ \iota(N^{S^1 \times S^2}_{n-k-1}) \mapsto \left( \text{im}(c_k^{S^1 \times S^2}) \cup \left( (k+1) \cdot \vec{e}_1 + \iota(N^{S^1 \times S^2}_{n-k-1}) \right), (g_0, \ldots, g_k) \right), \]

where \( c_k^{S^1 \times S^2} : = \cup_{m=0}^k (m \cdot \vec{e}_1 + c_0) \) is an embedding obtained by stacking \( k+1 \) copies of \( \text{im}(c_0^{S^1 \times S^2}) \) together, and where \( g_m \) is an embedding of \( I + S^2 \) inside \( \text{im}(c_m) \) for \( m \in \{1, \ldots, k\} \). The \( S^2 \) part of \( g_m : I + S^2 \rightarrow \text{im}(c_m^{S^1 \times S^2}) \) is chosen to be a slice \( \{p\} \times S^2 \) in the copy \( S^1 \times S^2 \) in \( \text{im}(m \cdot \vec{e}_1 + c_0) \). Similar to how \( \phi_{c^{S^1 \times S^2}_0} \) is defined, we have a chart \( \phi_{c^{S^1 \times S^2}_m} : \left( c_m^{S^1 \times S^2} \right)^{-1}(0, m+1] \times \partial_{\vec{e}_1}(H_{\partial_{\vec{e}_1}}) \rightarrow [0, m+1] \times \mathbb{R}^2 \). We require that the \( I \) part of \( g_m : I + S^2 \rightarrow \text{im}(c_m^{S^1 \times S^2}) \) starts out horizontally until it moves beyond a neighborhood of \([0, m] \times \mathbb{R}^\infty \). By horizontal, we mean that as \( s \) varies, only the first coordinate changes; for instance, we can have \( g_m \circ r_1(s, x, y) = (10 \cdot m \cdot s, r^m x + t_m, r^m y, 0, 0, \ldots) \) and \( g_m \circ r_1(1-s, x, y) = (10 \cdot m \cdot s, r^m x + t_m, r^m y, 0, 0, \ldots) \) for \( s \in [0, 0.1] \).

From now on, we make the definition of \( c_0^{S^1 \times S^2} \) more specific. We also need the analogous function for other prime, compact, orientable 3-manifolds in Section 6, so we define \( c_0^P \) in this more general setting as well.

**Definition 14.** \( c_0^P : (I \times \partial_0 P) \coprod \left( \coprod_{l=1}^s I \times \partial_l \right) \rightarrow [0, 1] \times \mathbb{R}^\infty \) is defined to be consisted of the embeddings \( (id_{[0,1]} \times \partial_{l} + 3l \cdot \vec{e}_2) \circ \iota_{N_{D_l}} \) for \( l = 0, \ldots, s \) and an embedding of \( P - D^3 \) that is identified with the embedding of \( I \times \partial_0 - D^3 \) at \( \partial D^3 \).

We require the removed \( D^3 \) in \( I \times \partial_0 \) to be in some neighborhood \( N_D \) of \( I \times \partial_0 \) that is outside of \( I \times \partial_{\vec{e}_1}(H_{\partial_{\vec{e}_1}}) \), and that \( c_0^P(N_D) \) is \( U^3 \times \{0\}^\infty \) for some open cube \( U^3 \). By assuming that the diameter of the removed embedding \( D^3 \) of \( I \times \partial_0 \) in \( U^3 \times \{0\}^\infty \) is small enough, we can shift the \( D^3 \) in \( c_0^P(I \times \partial_0) \) by some amount \( C_3 \) in the third coordinate so that the shifted \( D^3 \) is disjoint from the original \( D^3 \), and it is mapped inside \( U^3 \times \{0\}^\infty \) by \( id_I \times \partial_{\vec{e}_1} \) throughout the shifting. At the same
time, we can shift the embedding $P - D^3$ by $C_3$ in the third coordinate, resulting in an embedding that is disjoint from the original embedding of $P - D^3$ by assuming that its third-coordinate width is inside the third-coordinate width of the removed $D^3$ in the embedding of $I \times \partial_0$. The shifting of the embedding of $P - D^3$ is disjoint from the embedding of $I \times \partial_0 - N_D$ by assuming that $I \times \partial_0$ embeds with nonzero coordinates up to the $k$-th coordinate, for some $k$, while the embedding of $P - D^3$ has nonzero $(k + 1)$-th coordinate except in a neighborhood of $\partial D^3$ that is in the region occupied by the removed $D^3$ of the embedding of $I \times \partial_0$.

**Proposition 15.** $a_k : Sub^\partial(N_{n-k-1}^{S^1 \times S^2}) \to X_k(N_n^{S^1 \times S^2})$ is a weak homotopy equivalence.

**Proof.** Recall that $(\text{id} \times i_{\partial_0}) \circ j_{N_{\partial_0}}$ is the fixed embedding of the collar neighborhood of $\partial_0$ in $N_n^{S^1 \times S^2}$. $\text{im}(c_k^{S^1 \times S^2})$ deformation retracts to $\text{im}((\text{id} \times i_{\partial_0}) \circ j_{N_{\partial_0}}) \cup (\bigcup_{i=0}^{k} \text{im}(g_i))$ relative to the boundary in $\{0\} \times \mathbb{R}^\infty$, since the complement of the latter in $\text{im}(c_k^{S^1 \times S^2})$ is $D^3$. By an isotopy of $[0, \infty) \times \mathbb{R}^\infty$, $g_i$ are modified so that $\text{im}((\text{id} \times i_{\partial_0}) \circ j_{N_{\partial_0}}) \cup (\bigcup_{i=0}^{k} \text{im}(g_i)) = \text{im}(c_k^{S^1 \times S^2})$ and we can think of $\text{im}(a_k)$ as the subspace of $o^{-1}((g_0, \ldots, g_k))$ consisting of $(\iota(N_n^{S^1 \times S^2}), (g_0, \ldots, g_k))$ where $\iota : N_n^{S^1 \times S^2} \to [0, \infty) \times \mathbb{R}^\infty$ is any embedding such that $\iota(N_n^{S^1 \times S^2}) - \text{im}(c_k^{S^1 \times S^2})$ lies in $[k+1, \infty) \times \mathbb{R}^\infty$. To show that the subspace $\text{im}(a_k)$ is weakly homotopy equivalent to $o^{-1}((g_0, \ldots, g_k))$, we will homotope $f : (D^k, \partial D^k) \to (o^{-1}((g_0, \ldots, g_k)), \text{im}(a_k))$ through maps with codomain $(X_k(N_n^{S^1 \times S^2}), \text{im}(a_k))$ to a map with codomain $\text{im}(a_k)$; since $o^{-1}((g_0, \ldots, g_k))$ is a weakly homotopy equivalent subspace of $X_k(N_n^{S^1 \times S^2})$, it is possible to work in the latter, bigger space. We will modify each element $f(x) = (\iota(N_n^{S^1 \times S^2}), (g_0, \ldots, g_k))$ by specifying how we modify $\iota(N_n^{S^1 \times S^2})$ and $c_k^{S^1 \times S^2}$.

First, for each $f(x) = (\iota(N_n^{S^1 \times S^2}), (g_0, \ldots, g_k))$, translate the first coordinate of $\iota(N_n^{S^1 \times S^2})$ and $c_k^{S^1 \times S^2}$ by $r_x$ unit to the right and the part of $c_k^{S^1 \times S^2}$ with image in
\{0\} \times \mathbb{R}^\infty will be stretched into a horizontal cylinder in \([0, r_x] \times \mathbb{R}^\infty\), (by horizontal, we mean the axis of the cylinder is in the first coordinate direction) where \(r_x\) is some amount needed to move \(\iota(N_s^{S^1 \times S^2}) - \text{im}(c_k^{S^1 \times S^2})\) out of \([0, k + 1] \times \mathbb{R}^\infty\), and for \(x \in \partial D^k\), we can set \(r_x\) to 0. We keep denoting the modified embedding and submanifold by \(c_k^{S^1 \times S^2}\) and \(\iota(N_s^{S^1 \times S^2})\), respectively. Let \(R_x := \max_{x \in D^k} r_x\). We then stretch out the submanifold of \(\iota(N_s^{S^1 \times S^2})\) in \([k + 1 + r_x, k + 1 + R_x] \times \mathbb{R}^\infty\) to occupy a horizontal cylinder \([k + 1 + r_x, k + 1 + R_x] \times \mathbb{R}^\infty\) without modifying the embedding \(c_k^{S^1 \times S^2}\). Denote by \(c'\) the embedding \(c_k^{S^1 \times S^2}\) along with the stretched horizontal cylinder embedding in \([k + 1 + r_x, k + 1 + R_x] \times \mathbb{R}^\infty\), and denote by \(f'\) the end of this homotopy from \(f\). Next, we homotope \(f'\) so that the part \(\iota(N_s^{S^1 \times S^2}) - \text{im}(c')\) of \(f'(x)\) becomes constant with respect to \(x \in D^k\): deform \(\iota(N_s^{S^1 \times S^2}) - \text{im}(c')\) using the \(f'\)-image of the contraction of \(D^k\) to a point; by isotopy extension, we can extend this to an isotopy of \(\iota(N_s^{S^1 \times S^2})\), fixing \([0, k + 1] \times \mathbb{R}^\infty\) as well as the horizontal cylinder of \(\text{im}(c')\). Now, we can apply Proposition 11 to deform \(c'\), relative to \(\iota(N_s^{S^1 \times S^2}) - \text{im}(c')\), to agree with the \(c'\) in \(f'\partial D^k\). The resulting homotopy of \(c_k^{S^1 \times S^2}\) and \(\iota(N_s^{S^1 \times S^2})\) gives an element in \(\text{im}(a_k)\), and we are done.

**Proposition 16.** The following diagram

\[
\begin{array}{ccc}
\text{Sub}^\beta(N_s^{S^1 \times S^2}) & \xrightarrow{s^{S^1 \times S^2}} & \text{Sub}^\beta(N_{n+1}^{S^1 \times S^2}) \\
\downarrow a_k & & \downarrow a_{k-1} \\
X_k(N_{n+k+1}^{S^1 \times S^2}) & \xrightarrow{d_k} & X_{k-1}(N_{n+k+1}^{S^1 \times S^2})
\end{array}
\]

commutes.

**Proof.** For any \(\iota(N_s^{S^1 \times S^2}) \in \text{Sub}^\beta(N_s^{S^1 \times S^2})\), we check that \(d_k \circ a_k(\iota(N_s^{S^1 \times S^2}))\) and
\( a_{k-1} \circ S^1 \times S^2 (\iota(N_n^{S^1 \times S^2})) \) are the same; the first quantity is

\[
d_k \left( \left( \text{im}(c_k) \cup (k+1) \cdot \vec{e}_1 + \iota(N_n^{S^1 \times S^2}) \right), (g_0, \ldots, g_k) \right)
\]

\[
= \left( \text{im}(c_k) \cup (k+1) \cdot \vec{e}_1 + \iota(N_n^{S^1 \times S^2}) \right), (g_0, \ldots, g_{k-1})
\]

while the second quantity is

\[
a_{k-1} \left( \text{im}(c_0) \cup (\vec{e}_1 + \iota(N_n^{S^1 \times S^2})) \right)
\]

\[
= \left( \text{im}(c_{k-1}) \cup (k \cdot \vec{e}_1 + \text{im}(c_0)) \cup (k+1) \cdot \vec{e}_1 + \iota(N_n^{S^1 \times S^2}) \right), (g_0, \ldots, g_{k-1})
\]

Because \( c_k = c_{k-1} \cup (k \cdot \vec{e}_1 + c_0) \), the two quantities are the same. \qed

Thus, to prove Theorem 6, we can work with \( H_q(d_k) \), instead of \( H_q(s) \), and show that \( H_q(d_k) \) is an isomorphism in the appropriate range.

**Proposition 17.** For \( l, j \in \{0, \ldots, k\} \), \( d_i, d_j : X_k(N_n^{S^1 \times S^2}) \to X_{k-1}(N_n^{S^1 \times S^2}) \) are homotopic.

**Proof.** It suffices to show that \( d_i \) and \( d_{i+1} \) are homotopic. Assume \( i = 0 \). As \( a_k \left( \text{Sub}^\beta(N_{n-k-1}^{S^1 \times S^2}) \right) \hookrightarrow X_k(N_n^{S^1 \times S^2}) \) is a weak homotopy equivalence by Proposition 15, we may assume that the given embedding of \( N_n^{S^1 \times S^2} \) is of the form

\[
\left( \text{im}(c_k^{S^1 \times S^2}) \cup (k+1) \cdot \vec{e}_1 + \iota(N_{n-k-1}^{S^1 \times S^2}) \right), (g_0, \ldots, g_k)
\]

as in Definition 13. First, we isotope \( [0, 2] \times \mathbb{R}^\infty \), fixing the boundary, taking \( g_0 \) to \( g_1 \) except the part with image in \( [0, 2] \times i_{\text{th}}(H_{\text{th}}) \). We can do that by moving the copy of \( S^1 \times S^2 - D^3 \) containing \( g_0(S^2) \) to the other copy by shifting the first coordinate 1 unit to the right, and at the same time shifting the first coordinate of the second copy 1 unit to the left; but to prevent these two \( S^1 \times S^2 - D^3 \) from bumping, we shift the second copy in the third coordinate by \( C_3 \) beforehand and shift by \(-C_3\) afterward, so that \( \text{im}(c_1^{S^1 \times S^2}) \) is not changed after the isotopy. The
tethers of \( g_0 \) will be dragged so that at the end, the part outside of \([0, 2] \times i_{\partial_1}(H_{\partial_1})\) will agree with the tethers of \( g_1 \) (while the tethers of \( g_1 \) just need to avoid \( g_0 \)). This gives the desired isotopy of \([0, 2] \times \mathbb{R}^\infty\) by isotopy extension. To get the rest of the \( g_0 \) to match up with \( g_1 \), we just need to translate the horizontal part of \( g_0 \) in the second coordinate from \( t_0 \) to \( t_1 \), and moving \( g_1 \) up and shrinking the thickness \( r^1 \) of the tether if necessary.

To show that \( d_i \) is homotopic to \( d_{i+1} \) for general \( i \), we can use a similar homotopy in \([i, i + 2] \times \mathbb{R}^\infty\) for switching the two copies of \( S^1 \times S^2 - D^3 \). \( \square \)

As mentioned earlier, it is more convenient to work with augmented semi-simplicial spaces for the spectral sequence argument. Let’s augment \( X_*(N_n^{S^1 \times S^2}) \) by

\[
X_{-1}(N_n^{S^1 \times S^2}) := \text{Emb}^\partial(N_n^{S^1 \times S^2}) \times \text{Diff}(N_n^{S^1 \times S^2}, \partial N) \{pt\} = \text{Sub}^\partial(N_n^{S^1 \times S^2}),
\]

where the augmentation \( \epsilon_0 : X_0(N_n^{S^1 \times S^2}) \to X_{-1}(N_n^{S^1 \times S^2}) \) is the forgetful map defined by \((i(N_n^{S^1 \times S^2}, (f_0)) \mapsto i(N_n^{S^1 \times S^2})\). Let \( \epsilon_k : X_k(N_n^{S^1 \times S^2}) \to \text{Sub}^\partial(N_n^{S^1 \times S^2}) \) be \((i(N_n^{S^1 \times S^2}), (f_0, \ldots, f_k)) \mapsto i(N_n^{S^1 \times S^2}), \) which induces a map \(|\epsilon_*| : |X_*(N_n^{S^1 \times S^2})| \to X_{-1}(N_n^{S^1 \times S^2})\). The connectivity property we need of this map is the following:

**Lemma 18.** \( H_q(X_{-1}(N_n^{S^1 \times S^2}), |X_*(N_n^{S^1 \times S^2})|) = 0 \) if \( q \leq \frac{n-1}{2} \).

We will postpone the proof of this lemma until the Section 7. Assuming it, we now prove the main result.

**Proposition 19.** \( H_q(S^{S^1 \times S^2}) : H_q(\text{Sub}^\partial(N_n^{S^1 \times S^2}) \to H_q(\text{Sub}^\partial(N_n^{S^1 \times S^2})) \) is an isomorphism for \( q \leq \frac{n-3}{2} \).

**Proof.** We will prove this by inducting on \( q \). For \( q = 0 \), the statement is true because \( \text{Sub}^\partial(N_n^{S^1 \times S^2}) \) is connected for any \( n \), since \( \text{Emb}^\partial(N_n^{S^1 \times S^2}) \) is connected
by Proposition 10. For the general case, we will use the spectral sequence for 
an augmented semi-simplicial spaces, where $E^1_{p,r} = H_r(X_p(N_n^{S^1 \times S^2}))$ for $p \geq -1$ with 
differentials $d^1_{p,r} = \sum_{i=0}^{p} (-1)^i H_r(d_i)$ for $p \geq 0$ (where $d_0$ is $\epsilon$ if $p = 0$ to be precise). By Proposition 17, $H_r(d_i) = H_r(d_j)$, so $d^1_{p,r} = 0$ when $p$ is odd and 
d$H_r(d_p)$ when $p$ is even. So, it suffices to prove that $d^1_{0,q} : H_q(X_0(N_n^{S^1 \times S^2})) \to 
H_q(X_{-1}(N_{n-1}^{S^1 \times S^2}))$ is an isomorphism since this would imply that $H_q(s^{S^1 \times S^2}) : 
H_q(Sub^\beta(N_n^{S^1 \times S^2})) \to H_q(Sub^\beta(N_{n-1}^{S^1 \times S^2}))$ is an isomorphism as well by Propositions 
15 and 16. Suppose that $d^1_{0,q}$ is not an isomorphism, then $E^{2}_{0,q}$ or $E^{2}_{-1,q}$ is nonzero. 
We will prove that for $r \geq 2$, $E^{r+1}_{0,q} = E^{r}_{0,q}$ and $E^{r+1}_{-1,q} = E^{r}_{-1,q}$, which would 
mean $E^{\infty}_{0,q}$ or $E^{\infty}_{-1,q}$ is nonzero, i.e. $H_s(X_{-1}(N_n^{S^1 \times S^2}), |X_s(N_n^{S^1 \times S^2})|$ is nonzero 
for $s = q + 1$ or $s = q$, respectively, giving a contradiction to Lemma 18, because $q + 1 \leq \frac{n-3}{2} + 1 = \frac{n-1}{2}$.

It remains to prove that $E^{r}_{b,q} = E^{r+1}_{b,q}$, for $b \in \{0, -1\}$ and for $r \geq 2$. The 
target of $d^1_{b,q} : E^{r}_{b,q} \to E^{r}_{b-r,q+r-1}$ is 0 since $b - r \leq -2$, and it remains to show 
that the domain of $E^{r}_{b+r,q-r+1} : E^{r}_{b+r,q-r+1} \to E^{r}_{b,q}$ is 0 as well; if $q < r - 1$, then 
$E^{1}_{b+r,q-r+1}$ is already 0 so $E^{r}_{b+r,q-r+1} = 0$. So let’s assume $q \geq r - 1$, which means 
n \geq 2q + 3 \geq 2r + 1$. If $r$ is even, then 

$d^1_{b+r,q-r+1} : H_{q-r+1}(X_{b+r}(N_n^{S^1 \times S^2})) \to H_{q-r+1}(X_{b+r-1}(N_n^{S^1 \times S^2}))$

is an isomorphism if we can show that 

$H_{q-r+1}(s^{S^1 \times S^2}) : H_{q-r+1}(Sub^\beta(N_{n-b-r-1}^{S^1 \times S^2})) \to H_{q-r+1}(Sub^\beta(N_{n-b-r}^{S^1 \times S^2}))$

is an isomorphism, by $n - b - r - 1 \geq r > 0$ and Proposition 16. As $r \geq 2$, 
$q - r + 1 \leq \frac{n-3-2r+2}{2} \leq \frac{(n-b-r)-1}{2}$, so it is indeed an isomorphism by the inductive 
hypothesis. So $E^{2}_{b+r,q-r+1} = 0$ and $E^{2}_{b+r,q-r+1} = 0$ for all $r \geq 2$. If $r$ is odd, then 

$d^1_{b+r+1,q-r+1} : H_{q-r+1}(X_{b+r+1}(N_n^{S^1 \times S^2})) \to H_{q-r+1}(X_{b+r}(N_n^{S^1 \times S^2}))$
is an isomorphism because

$$H_{q-r+1}(S^{S_1\times S^2}) : H_{q-r+1}(Sub^\partial(N^{S_1\times S^2}_{n-b-r-2})) \to H_{q-r+1}(Sub^\partial(N^{S_1\times S^2}_{n-b-r-1}))$$

is an isomorphism because $n - b - r - 2 \geq r - 1 > 0$ and the inductive hypothesis applies because $q - r + 1 \leq \frac{n-3-2r+2}{2} \leq \frac{(n-b-r-1)-1}{2}$, using $r \geq 3$. This means $E^{2}_{b+r,q-r+1} = 0$, and so $E^{r}_{b+r,q-r+1} = 0$ for all $r \geq 2$. Thus, $E^{r+1}_{0,q} = E^{r}_{0,q}$ and $E^{r+1}_{1,q} = E^{r}_{1,q}$ as claimed. \hfill \Box
CHAPTER 5

STABILIZATION BY $D^3$

In this section, we show that stabilizing $BDiff(N_n^{S^1 \times S^2}, \partial N)$ by $D^3$ induces homological stability in the same range as $s^{S^1 \times S^2}$. The stabilization map $s^{S^1 \times S^2}$ identifies the boundary component $\partial_0$ of $N_n^{S^1 \times S^2}$ with the $\{0\} \times \partial_0$ part of $S^1 \times S^2 \# (I \times \partial_0)$, so we can think of the map $s^{S^1 \times S^2}$ as the composition $s_{-D^3} \circ s_{D^3}$; the map $s_{D^3} : \text{Sub}^\partial (N_n^{S^1 \times S^2}) \rightarrow \text{Sub}^\partial ((N \# D^3)_n^{S^1 \times S^2})$ is induced by identifying the $\partial_0$ part of $N_n^{S^1 \times S^2}$ with the $\{0\} \times \partial_0$ part of $(I \times \partial_0) \# D^3$, while the map $s_{-D^3} : \text{Sub}^\partial ((N \# D^3)_n^{S^1 \times S^2}) \rightarrow \text{Sub}^\partial (N_{n+1}^{S^1 \times S^2})$ is induced by identifying the $\partial_0$ part and $\partial D^3$ part of $(N \# D^3)_n^{S^1 \times S^2}$ with the $\{0\} \times \partial_0$ part and $\partial D^3$ part of $(I \times \partial_0) \# D^3$ respectively. Thus, $H_\ast (s_{D^3})$ is injective in the range that $H_\ast (s^{S^1 \times S^2})$ is bijective.

To obtain surjectivity, we will show that $s' := s_{D^3} \circ s_{-D^3} : \text{Sub}^\partial ((N \# D^3)_n^{S^1 \times S^2}) \rightarrow \text{Sub}^\partial ((N \# D^3)_{n+1}^{S^1 \times S^2})$ induces homological stability in the same range. To simplify notations, we use $N$ in place of $N \# D^3$.

**Definition 20.** Let $s' : \text{Sub}^\partial (N_n^{S^1 \times S^2}) \rightarrow \text{Sub}^\partial (N_{n+1}^{S^1 \times S^2})$ be the stabilization map defined above, where $N$ has a boundary component $\partial_0$ and another boundary component $\partial_1 \approx S^2$.

- Let $\text{Emb}^\partial (\overline{T + S^2}, N_n^{S^1 \times S^2}, \partial_0, \partial_1)$ be the space of tethered sphere embeddings $f$ such that one end of the tether goes to $\partial_0$ and the other end goes to $\partial_1$, satisfying similar boundary conditions as before: there exist $t \in \mathbb{R}$ and $\epsilon > 0$ such that for all $s \in [0, \epsilon)$, $f \circ \phi^{-1}_0 (s, x, y) = \phi^{-1}_{\partial_0} (s, rx + t, y)$ and $f \circ \phi^{-1}_1 (1 - s, x, y) = \phi^{-1}_{\partial_1} (s, rx + t, y)$, where $\phi_0 : r^{-1}_1 ([0, \epsilon]) \rightarrow [0, \epsilon) \times D^2$ and $\phi_1 : r^{-1}_1 ((1 - \epsilon, 1]) \rightarrow [0, \epsilon) \times D^2$ are charts for tether’s ends and $\phi_{\partial_0} : N_{\partial_0} \rightarrow [0, \epsilon_0) \times \mathbb{R}^2$ is the chart of $N_{\partial_0}$ defined in Definition 7.
• Let $\text{Emb}_k(\overline{I + S^2}, N_n^{S^1 \times S^2}, \partial_0, \partial_1)$ be the space of $(k + 1)$-tuples $(f_0, \ldots, f_k)$ of tethered sphere embeddings such that for $i = 0, \ldots, k$, there exist $t_0 < \cdots < t_k$, $r^i > 0$ and $\epsilon > 0$ such that for all $s \in [0, \epsilon)$, $f_i \circ \phi_0^{-1}(s, x, y) = \phi^{-1}_{N_\partial} (s, r^i x + t_i, r^i y)$ and $f_i \circ \phi_1^{-1}(1 - s, x, y) = \phi^{-1}_{N_\partial} (s, r^i x + t_i, r^i y)$.

• Let $X_*(N_n^{S^1 \times S^2}, \partial_0, \partial_1)$ be the semi-simplicial space

$$\text{Emb}^\beta(N_n^{S^1 \times S^2}) \times \text{Diff}(N_n^{S^1 \times S^2}, \partial N) \text{Emb}_k(\overline{I + S^2}, N_n^{S^1 \times S^2}, \partial_0, \partial_1).$$

**Proposition 21.** $X_k(N_n^{S^1 \times S^2}, \partial_0, \partial_1)$ is a classifying space of $\text{Diff}(N_{n-k-1}, \partial N)$.

**Proof.** This is similar to Proposition 12, where we express an element $[f, (g_0, \ldots, g_k)]$ of $X_k(N_n^{S^1 \times S^2}, \partial_0, \partial_1)$ as $(\text{im}(f), (f \circ g_0, \ldots, f \circ g_k))$. Define the map

$$o : X_k(N_n^{S^1 \times S^2}, \partial_0, \partial_1) \to \text{Emb}_k(\overline{I + S^2}, [0, \infty) \times \mathbb{R}^\infty, \partial_0, \partial_1)$$

$$(\text{im}(f), (h_0, \ldots, h_k)) \mapsto (h_0, \ldots, h_k),$$

which is a fibration by isotopy extension, and has contractible base by 10. The fiber is homeomorphic to $\text{Sub}^\beta(N_n^{S^1 \times S^2} - \coprod_{i=0}^k g_i(\text{int}(\overline{I + S^2})), [0, \infty) \times \mathbb{R}^\infty - \coprod_{i=0}^k f \circ g_i(\text{int}(\overline{I + S^2})))$, where $N_n^{S^1 \times S^2} - \coprod_{i=0}^k g_i(\text{int}(\overline{I + S^2}))$ is diffeomorphic to $N_n^{S^1 \times S^2}$.

Since $\text{Emb}^\beta\left(N_{n-k-1}^{S^1 \times S^2}, [0, \infty) \times \mathbb{R}^\infty - \coprod_{i=0}^k f \circ g_i(\text{int}(\overline{I + S^2})))\right)$ is weakly contractible by Proposition 11, the space in the previous sentence is a classifying space of $\text{Diff}(N_{n-k-1}^{S^1 \times S^2}, \partial N)$. \qed

We need to construct a weak homotopy equivalence $a'_k : \text{Sub}^\beta(N_{n-k-1}^{S^1 \times S^2}) \to X_k(N_n^{S^1 \times S^2}, \partial_0, \partial_1)$ that is compatible with the stabilization map $s' : \text{Sub}^\beta(N_n^{S^1 \times S^2}) \to \text{Sub}^\beta(N_{n+1}^{S^1 \times S^2})$.

**Definition 22.** Let $c'_0 : (I \times \partial_0 \# I \times \partial_1) \bigcup_{i=2}^{s-1} (I \times \partial_i) \to [0, 1] \times \mathbb{R}^\infty$ be an embedding, where for $k \geq 2$, $c'_0|_{I \times \partial_k}$ is $id_I \times (t_{\partial_k} + 3k \cdot \vec{e}_1)$; To define $c'_0|_{I \times \partial_0 \# I \times \partial_1}$, \ldots
we modify $id_I \times \iota_{\partial_0}$ and $id_I \times (\iota_{\partial_1} + 3 \cdot \vec{e}_1)$, 2 cylinders whose axes are in the first coordinate direction:

Join the two cylinders together using a cylinder/handle $I \times S^2$ with axis in the second coordinate direction, by removing a copy of $D^3$ from each cylinder. Assume that, for each $j \in \{0, 1\}$, the corresponding removed $D^3$ is in some neighborhoods $N_{D_j}$ outside of $I \times (\iota_{\partial_j} + 3j \cdot \vec{e}_1)(H_{\partial_j})$, and that $\iota'_0(N_{D_j})$ is $U^3_j \times \{0\}$ for some open cube $U^3_j$. By assuming that for each $j \in \{0, 1\}$, the diameter of the removed embedding of $D^3$ in $U^3_j \times \{0\}$, is small enough, we can shift the $D^3$ in $I \times \partial_j$ in a way that corresponds to shifting the image by some amount $C_3$ in the third coordinate so that this shifted $D^3$ is disjoint from the original $D^3$ and is still mapped inside $U^3_j \times \{0\}$ by $id_I \times \iota_{\partial_j}$ throughout the shifting. At the same time, we shift the embedded handle $I \times S^2$ by $C_3$ in the third coordinate, resulting in an embedded handle that is disjoint from the original embedded handle by assuming that its third-coordinate width is inside the third-coordinate width of the two removed $D^3$. The shifting of the embedded handle is disjoint from the embedding of $I \times \partial_j - N_{D_j}$ for each $j \in \{0, 1\}$, by assuming that $I \times \partial_j$ embeds with nonzero coordinates up to the $k$-th coordinate, for some $k \in \mathbb{N}$, while the embedding of the handle has nonzero $(k + 1)$-th coordinate except in a neighborhood of the two $\partial D^3$ that are in the region occupied by removed $D^3$ of the embeddings of $I \times \partial_0$ and $I \times \partial_1$.

Define the map

$$a'_0 : \text{Sub}^3(N_{n-1}^{S^1 \times S^2}) \to X_0(N_n^{S^1 \times S^2}, \partial_0, \partial_1)$$

$$\iota(N_n^{S^1 \times S^2}) \mapsto \left(\text{im}(\iota'_0) \cup \left(\vec{e}_1 + \iota(N_n^{S^1 \times S^2})\right), (g'_0)\right),$$

where $g'_0$ is an embedding of $I + S^2$ that maps the $S^2$ part of $g'_0$ to where we join the 2 cylinders, $\text{im}(id_{[0,1]} \times \iota_{S^2})$ and $\text{im}(id_{[0,1]} \times (\iota_{S^2} + 3 \cdot \vec{e}_1))$. For the $I$ part, recall that there is the boundary condition imposed by $\text{Emb}_0(I + S^2, N_n^{S^1 \times S^2}, \partial_0, \partial_1)$: for
some $\epsilon > 0$, $f|_{T}(1 - s, x, y) = \phi_{N_{u_{0}}}^{-1} (s, x, y)$ and $f|_{T}(s, x, y) = \phi_{N_{u_{0}}}^{-1} (s, x, y)$ for all $s \in [0, \epsilon)$.

For general $k$, define

$$a_k' : Sub^\beta(N_{n-k-1}^{S^1 \times S^2}) \rightarrow X_k(N_n^{S^1 \times S^2}, \partial_0, \partial_1)$$

$$\iota(N_n^{S^1 \times S^2}) \mapsto \left(\text{im}(c_k') \cup \left((k + 1) \cdot e_1 + \iota(N_n^{S^1 \times S^2})\right), (g_0', \ldots, g_k')\right),$$

where $c_k' := \bigcup_{m=0}^{k} (m \cdot e_1 + e_0')$ is an embedding by stacking $k + 1$ copies of $\text{im}(c'_0)$ together, and where $g_m'$ defined in a similar fashion as $g_0'$, map the $S^2$ part of $g_m' : \Omega + S^2 \rightarrow \text{im}(c_m)$ to where we join the 2 cylinders, $\text{im}(\iota_{[m, m+1]} \times \partial_0)$ and $\text{im}(\iota_{[m, m+1]} \times (\partial_0 + 3 \cdot e_1))$ in $[m, m + 1] \times \mathbb{R}^\infty$. For the $\Omega$ part, we require that it stays inside $[0, m + 1] \times \mathbb{R}^\infty$ and starts out horizontal until it moves out of a neighborhood of $[0, m] \times \mathbb{R}^\infty$.

**Proposition 23.** $a_k' : Sub^\beta(N_{n-k-1}^{S^1 \times S^2}) \rightarrow X_k(N_n^{S^1 \times S^2})$ is a weak homotopy equivalence.

**Proof.** This is similar to the proof of Proposition 15. $\text{im}(c_k')$ can be deformation retracted to $\text{im}((\iota \times \iota_0) \circ j_{N_0}) \cup \text{im}((\iota \times \iota_1) \circ j_{N_1}) \cup \left(\bigcup_{i=0}^{k} \text{im}(g_i')\right)$ relative to the boundary in $\{0\} \times \mathbb{R}^\infty$, since the complement of the latter in $\text{im}(c_k')$ is $D^\beta \bigcup D^\beta$. By an isotopy of $[0, \infty) \times \mathbb{R}^\infty$, $g_i'$ are modified so that $\text{im}((\iota \times \iota_0) \circ j_{N_0}) \cup \text{im}((\iota \times \iota_1) \circ j_{N_1}) \cup \left(\bigcup_{i=0}^{k} \text{im}(g_i')\right) = \text{im}(c_k')$ and we can think of $\text{im}(a_k')$ as the subspace of $\iota^{-1}((g_0', \ldots, g_k'))$ consisting of $(\iota(N_n^{S^1 \times S^2}), (g_0', \ldots, g_k'))$ where $\iota : N_n^{S^1 \times S^2} \rightarrow [0, \infty) \times \mathbb{R}^\infty$ is any embedding such that $\iota(N_n^{S^1 \times S^2}) - \text{im}(c_k')$ lies in $[k + 1, \infty) \times \mathbb{R}^\infty$. To show that the subspace $\text{im}(a_k)$ is weakly homotopy equivalent to $\iota^{-1}((g_0', \ldots, g_k'))$, we will homotope $f : (D^k, \partial D^k) \rightarrow (\iota^{-1}((g_0', \ldots, g_k')), \text{im}(a_k'))$ through maps with codomain $(X_k(N_n^{S^1 \times S^2}, \partial_0, \partial_1), \text{im}(a_k))$ to a map with codomain $\text{im}(a_k)$; by Proposition 21, $\iota^{-1}((g_0', \ldots, g_k'))$ is a weakly homotopy equivalent sub-
space of $X_k(N_s^{1 \times S^2}, \partial_0, \partial_1)$, so it’s possible to work in the latter, bigger space. We will modify each element $f(x) = (\iota(N_s^{1 \times S^2}), (g_0', \ldots, g_k'))$ by specifying how we modify $\iota(N_s^{1 \times S^2})$ and $c_k^{S^1 \times S^2}$.  

First, for each $f(x) = (\iota(N_s^{1 \times S^2}), (g_0', \ldots, g_k'))$, translate the first coordinate of $\iota(N_s^{1 \times S^2})$ and $c_k'$ by $r_x$ unit to the right and the part of $c_k'$ with image in $\{0\} \times \mathbb{R}^\infty$ will be stretched into a horizontal cylinder in $[0, r_x] \times \mathbb{R}^\infty$. Here, $r_x$ is some amount needed to move $\iota(N_s^{1 \times S^2}) - im(c_k')$ out of $[0, k + 1] \times \mathbb{R}^\infty$, and for $x \in \partial D^k$, we can set $r_x$ to 0. We keep denoting the modified embedding and submanifold by $c_k'$ and $\iota(N_s^{1 \times S^2})$, respectively. Let $R_x := \max_{x \in D^k} r_x$. We then stretch out the submanifold of $\iota(N_s^{1 \times S^2})$ in $[k + 1 + r_x, k + 1 + r_x + \epsilon_\partial] \times \mathbb{R}^\infty$ to occupy a horizontal cylinder $[k + 1 + r_x, k + 1 + R_x] \times \mathbb{R}^\infty$ without modifying the embedding $c_k'$. Denote by $c'$ the embedding $c_k'$ along with the stretched horizontal cylinder embedding in $[k + 1 + r_x, k + 1 + R_x] \times \mathbb{R}^\infty$, and denote by $f'$ the end of this homotopy from $f$. Next, we homotop $f'$ so that the part $\iota(N_s^{1 \times S^2}) - im(c')$ of $f'(x)$ become constant with respect to $x \in D^k$: deform $\iota(N_s^{1 \times S^2}) - im(c')$ using the $f'$-image of the contraction of $D^k$ to a point; by isotopy extension, we can extend this to an isotopy of $\iota(N_s^{1 \times S^2})$, fixing $[0, k + 1] \times \mathbb{R}^\infty$ as well as the horizontal cylinder of $im(c')$. Now, we can apply Proposition 11 to deform $c'$, relative to $\iota(N_s^{1 \times S^2}) - im(c')$, to agree with the $c'$ in $f'((\partial D^k)$. The resulting homotopy of $c_k'$ and $\iota(N_s^{1 \times S^2})$ gives an element in $im(a_k')$.  

**Proposition 24.** The following diagram  

$$
\begin{array}{ccc}
\text{Sub}^\partial(N_s^{1 \times S^2}) & \xrightarrow{a_k'} & \text{Sub}^\partial(N_{s+1}^{1 \times S^2}) \\
\downarrow a'_{k-1} & & \downarrow a'_{k-1} \\
X_k(N_{n+k+1}^{1 \times S^2}, \partial_0, \partial_1) & \xrightarrow{d_k} & X_{k-1}(N_{n+k+1}^{1 \times S^2}, \partial_0, \partial_1)
\end{array}
$$

commutes.
Proof. The proof is essentially the same as Proposition 16. For any \( \iota(N_n^{S^1 \times S^2}) \in \text{Sub}^\partial(N_n^{S^1 \times S^2}) \), we check that \( d_k \circ a'_k(\iota(N_n^{S^1 \times S^2})) \) and \( a'_{k-1} \circ \iota'(\iota(N_n^{S^1 \times S^2})) \) are the same; the first quantity is
\[
d_k\left( \text{im}(c'_k) \cup (k+1) \cdot (N_n^{S^1 \times S^2}) \right), (g_0, \ldots, g_k)
\]
while the second quantity is
\[
a_{k-1}\left( \text{im}(c'_0) \cup (N_n^{S^1 \times S^2}) \right)
\]
Because \( c'_k = c'_{k-1} \cup (k \cdot \bar{e}_1 + c'_0) \), the 2 quantities are the same.

Proposition 25. \( d_j : X_k(N_n^{S^1 \times S^2}, \partial_0, \partial_1) \to X_{k-1}(N_n^{S^1 \times S^2}, \partial_0, \partial_1) \) is homotopic to \( d_l \) for \( l, j \in \{0, \ldots, k\} \).

Proof. This is similar to Proposition 17. First, we show that \( d_0 \) and \( d_1 \) are homotopic. As \( a'_k(\text{Sub}^\partial(N_n^{S^1 \times S^2})) \hookrightarrow X_k(N_n^{S^1 \times S^2}) \) is a weak homotopy equivalence by Proposition 15, we may assume that the given embedding of \( M_{n,a} \) is of the form
\[
\left( \text{im}(c'_{k,s}) \cup (k+1) \cdot (N_n^{S^1 \times S^2}) \right), (g_0, \ldots, g_k)
\]
as in Definition 13. First, we perform an isotopy of \([0,2] \times \mathbb{R}^\infty \) relative to the boundary that takes \( g'_0 \) to \( g'_1 \) except the part in the image of \([0,2] \times H_{\partial_0} \) and \([0,2] \times H_{\partial_1} \). We can do that by moving the handle containing \( g_0(S^2) \) to the other handle by shifting it by 1 unit in the first coordinate, and at the same time shifting the first coordinate of the second handle by \(-1\) unit; but to avoid these two handles from bumping, we shift the second one in the third coordinate by \( C_3 \) beforehand and shift by \(-C_3 \) afterward, so that at the end of this isotopy is the same as the original \( \text{im}(c'_{k,s}) \). The tethers of \( g'_0 \) will be dragged so that at the end, the part
outside of $[0, 2] \times H'_0$ and $[0, 2] \times H'_1$, will agrees with the tethers of $g'_1$ (while the tethers of $g'_0$ just need to avoid $g'_0$). This gives the desired isotopy of $[0, 2] \times \mathbb{R}^\infty$ by isotopy extension. To get the rest of the $g'_0$ to match up with $g'_1$, we simply translate the horizontal part of $g'_0$ in the second coordinate from $t_0$ to $t_1$ and move $g'_1$ up and shrinking the thickness $r^1$ of the tether if necessary to prevent $g'_0$ from bumping into $g'_1$.

To show that $d_i$ is homotopic to $d_{i+1}$ for general $i$, we can use a similar homotopy in $[i, i+2] \times \mathbb{R}^\infty$ for switching the two vertical handles.

\[ \square \]

The analog of Lemma 18 holds:

**Lemma 26.** $H_q(X_{-1}(N_n^{S^1 \times S^2}, \partial_0, \partial_1), |X_*(N_n^{S^1 \times S^2}, \partial_0, \partial_1)|) = 0$ if $q \leq \frac{n-1}{2}$.

We will prove the lemma in the Section 7.

**Proposition 27.** $H_q(Sub^\partial(N^p_{n-1}^{S^1 \times S^2})) \xrightarrow{H_q(s')} H_q(Sub^\partial(N^p_n^{S^1 \times S^2}))$ is an isomorphism for $q \leq \frac{n-3}{2}$.

**Proof.** The proof is similar to that of Proposition 19 using induction on $q$. For $q = 0$, it is true because $Sub^\partial(N^p_n^{S^1 \times S^2})$ is connected. For general $q$, we apply the spectral sequence for augmented semi-simplicial spaces to $\epsilon_* : X_* (N^p_n^{S^1 \times S^2}, \partial_0, \partial_1) \rightarrow X_{-1}(N^p_n^{S^1 \times S^2}, \partial_0, \partial_1)$. The differential $d^1_{p,q}$ of the $E^1_{p,q}$ page is homotopic to $H_q(s') : H_q(Sub^\partial(N^p_n^{S^1 \times S^2})) \rightarrow H_q(Sub^\partial(N^p_{n-1}^{S^1 \times S^2}))$ for even $p$ and the zero map for odd $p$ by Propositions 23, 24 and 25, so it suffices to show that $d^1_{0,q}$ is an isomorphism. If it isn't, then $E^2_{0,q}$ or $E^2_{1,q}$ is nonzero. But by Lemma 26, $E^r_{0,q}$ and $E^r_{1,q}$ converge to zero since $q + 1 \leq \frac{n-1}{2}$, and one can show that $E^r_{0,q}$ and $E^r_{1,q}$ have already converged when $r = 2$ because the differentials with $E^r_{0,q}$ and $E^r_{1,q}$ as domain or
codomain have the zero space as the codomain or domain, respectively, when \( r \geq 2 \) as shown in Proposition 19, using the inductive hypothesis.

**Corollary 28.** Let \( N \) be a compact, orientable 3-manifold with at least 1 boundary sphere. Then \( H_q(s_{D^3}) : H_q(Sub^\partial(N_n^{S^1 \times S^2})) \to H_q\left(Sub^\partial((N \# D^3)_n^{S^1 \times S^2}) \right) \) is an isomorphism for \( q \leq \frac{n-2}{2} \).

**Proof.** The map \( H_q(s^{S^1 \times S^2}) : H_q(Sub^\partial(N_n^{S^1 \times S^2})) \to H_q(Sub^\partial(N_{n+1}^{S^1 \times S^2})) \) is an isomorphism for \( q \leq \frac{n-2}{2} \) by Proposition 19, so because \( s^{S^1 \times S^2} = s_{-D^3} \circ s_{D^3} \), the map \( H_q(s_{D^3}) : H_q(Sub^\partial(N_n^{S^1 \times S^2})) \to H_q\left(Sub^\partial((N \# D^3)_n^{S^1 \times S^2}) \right) \) is injective for \( q \leq \frac{n-2}{2} \). Similarly, because \( H_q(s') : H_q(Sub^\partial(N_n^{S^1 \times S^2})) \to H_q(Sub^\partial(N_{n+1}^{S^1 \times S^2})) \) is an isomorphism for \( q \leq \frac{n-2}{2} \) by Proposition 27, and \( s' = s_{D^3} \circ s_{-D^3} \), so the map \( H_q(s_{D^3}) : H_q(Sub^\partial(N_n^{S^1 \times S^2})) \to H_q\left(Sub^\partial((N \# D^3)_n^{S^1 \times S^2}) \right) \) is surjective for \( q \leq \frac{n-2}{2} \).

We postpone the case where \( N \) has no boundary component until Section 8.
CHAPTER 6

STABILIZATION BY PRIME SUMMANDS $P \neq S^1 \times S^2$

In this section, we prove homological stability for the stabilization map $s^P : BDiff(N^P, \partial N) \to BDiff(N^{P+1}, \partial N)$ with $P \neq S^1 \times S^2$. We need to modify the complex $X_*(N^{S^1 \times S^2})$ to obtain properties for $s^P$ similar to those for $s^{S^1 \times S^2}$ and to obtain similar connectivity results in Section 7.

**Definition 29.** Restrict $P$ to be a prime 3-manifold that is not $S^1 \times S^2$, and let $P'$ denote $P - D^3$ and $S_P$ denote the boundary sphere corresponding to $D^3$. Recall that $N$ is assumed to have at least 1 boundary component, denoted by $\partial_0$, while the rest of the boundary is denoted by $\partial_1, \ldots, \partial_s$, with fixed embeddings $i_{\partial_i} : \partial_i \to [-1, 1] \to \partial N_{\partial_i}$ and with fixed collar neighborhoods $N_{\partial_i}$ and an embedding $(id_{[0, \epsilon_0]} \times i_{\partial_0}) \circ j_{N_{\partial_0}}$ of it to $[0, \infty) \times \mathbb{R}^\infty$.

Let $Emb^\beta(I \setminus P', [0, \infty) \times \mathbb{R}^\infty)$ be the $C^\infty$ space of embeddings $f$ of $P'$ in $(0, \infty) \times \mathbb{R}^\infty$ and of $I := I \times D^2$, such that $\{1\} \times D^2$ is identified with a disk in $S_P$, and at the other end, we require that $f|_{[0, \epsilon) \times D^2}$ agrees with $(id_{[0, \epsilon_0]} \times i_{\partial_0}) \circ j_{N_{\partial_0}}$ in some collar subneighborhood $N_{\epsilon}$ of $N_{\partial_0}$.

Let $Emb^\beta(I \setminus P', N^n_P)$ be the $C^\infty$ space of embeddings $f$ of $P'$ in $N^n_P$ and of $I := I \times D^2$, such that $\{1\} \times D^2$ is identified with a disk in $S_P$. At the other end, we require that there is some $\epsilon > 0$ so that in $[0, \epsilon) \times D^2 \subset \overline{I}$, $f(s, x, y) = \phi_{N^n_{H_{\partial_0}}}^{-1}(s, rx + t, ry)$ for some $r > 0$ and $t \in \mathbb{R}$ where $\phi_{N^n_{H_{\partial_0}}} : N_{H_{\partial_0}} \to [0, \epsilon_0) \times \mathbb{R}^2$ is the chart defined in the definition of $Emb^\beta(I + S^2, N^n_{S^1 \times S^2})$.

Let $Emb_k(I \setminus P', N^n_P)$ be the semi-simplicial space where the space of $k$-simplexes consists of $(k+1)$-tuples $(f_0, \ldots, f_k)$ of disjoint embeddings from $Emb^\beta(I \setminus P', N^n_P)$, where the boundary condition is $f_i(s, x, y) = \phi_{N^n_{H_{\partial_0}}}^{-1}(s, r_ix + t_i, r_iy)$ for some
$$r_i > 0 \text{ and } t_0 < \cdots < t_k.$$  

Define the semi-simplicial space $Y_\ast(N^n_P)$ by

$$Y_k(N^n_P) := Emb^\partial(N^n_P, [0, \infty) \times \mathbb{R}^\infty) \times \text{Diff}^\partial(N^n_P) \times Emb_k(\bar{T} \lor P', N^n_P)$$

with elements denoted by $[f, (g_0, \ldots, g_k)]$, which is a semi-simplicial space that augments $\text{Sub}^\partial(N^n_P, [0, \infty) \times \mathbb{R}^\infty)$ by the map $\epsilon : [f, (g_0)] \mapsto \text{im}(f)$.

We will denote elements in $Y_k(N^n_P)$ by $(\text{im}(f), (f \circ g_0, \ldots, f \circ g_k))$ because $Y_k(N^n_P)$ is homeomorphic to

$$\{(\text{im}(f), (h_0, \ldots, h_k)) \in \text{Sub}^\partial (N^n_P, [0, \infty) \times \mathbb{R}^\infty) \times Emb_k(\bar{T} \lor P', [0, \infty) \times \mathbb{R}^\infty) : \text{im}(f) \supset \text{im}(h_i) \text{ for } i = 0, \ldots, k\}.$$  

**Proposition 30.** $Y_k(N^n_P)$ is a classifying space of $\text{Diff}^\partial(N^n_{n-1-k})$.

**Proof.** Assume $k = 0$, since the proof is similar for general $k$. The map $o : Y_0(N^n_P) \to Emb_0(\bar{T} \lor P', [0, \infty) \times \mathbb{R}^\infty)$ given by $[f, (g_0)] \mapsto f \circ g_0$ is a Serre fibration by isotopy extension, and the base space is contractible by Proposition 11. The fiber is homeomorphic to $\text{Sub}^\partial(N^n_P - g_0(\bar{T} \lor P'), [0, \infty) \times \mathbb{R}^\infty - f \circ g_0(\bar{T} \lor P'))$. $N^n_P - g_0(\bar{T} \lor P')$ is diffeomorphic to $N^n_{n-1}$ by the prime decomposition of compact, orientable 3-manifolds. Since $Emb^\partial(N^n_{n-1}, [0, \infty) \times \mathbb{R}^\infty - f \circ g_0(\bar{T} \lor P'))$ is a weakly contractible by Proposition 11, we have a classifying space of $\text{Diff}(N^n_{n-1}, \partial N)$. 

**Definition 31.** Define the map

$$a_0 : \text{Sub}^\partial(N^n_{n-1}) \to Y_0(N^n_P)$$

$$\iota(N^n_{n-1}) \mapsto \left( c_0^P(P_{1,2}) \cup (\bar{e}_1 + \iota(N^n_{n-1})), (g_0^P) \right),$$

where $g_0^P$ is a fixed embedding of $\bar{T} \lor P'$: the $P'$ is exactly the copy glued to $I \times \partial_0$ when defining $c_0^P$, and the $\bar{T}$ part is defined so that it stays in $\text{im}(c_0^P)$ and when it
enters the cylinder $I \times i_{\partial_b}(H_{\partial_b})$, its second coordinate stays constant at $t_0 = 0$ as you vary the first component of $\mathcal{T} := I \times D^2$.

For general $k$, let $c_k^P := \bigcup_{m=0}^k (m \cdot \vec{c}_1 + c_k^P)$ and define

$$a_k^P : \text{Sub}^\partial(N_{n-k-1}^P) \to Y_k(N_n^P),$$

$$\iota(N_{n-k-1}^P) \mapsto (c_k^P(P_{k,2}) \cup ((k + 1) \cdot \vec{c}_1 + \iota(N_{n-k-1}^P)), (g_0^P, \ldots, g_k^P)),$$

where $g_i^P$ is obtained from $g_0^P$ by modifying the tethers to have second coordinate centered at $t_i = i$ and extending it to $[0, i] \times \mathbb{R}^\infty$ by only varying the first coordinate.

**Proposition 32.** $a_k^P : \text{Sub}^\partial(N_{n-k-1}^P) \to Y_k(N_n^P)$ is a weak homotopy equivalence.

**Proof.** This is similar to the proof of Proposition 15. $\text{im}(c_k^P)$ can be deformation retracted to $\text{im}((id \times i_{\partial_b}) \circ j_{\partial_b}) \cup (\bigcup_{i=0}^k \text{im}(g_i^P))$ relative to the boundary in $\{0\} \times \mathbb{R}^\infty$, since the complement of the latter in $\text{im}(c_k^P)$ is $D^3$, where $(id \times i_{\partial_b}) \circ j_{\partial_b}$ is the fixed embedding of the collar neighborhood of $\partial_0$ in $N_{n-k-1}^P$. By an isotopy of $[0, \infty) \times \mathbb{R}^\infty$, $g_i^P$ are modified so that $\text{im}((id \times i_{\partial_b}) \circ j_{\partial_b}) \cup (\bigcup_{i=0}^k \text{im}(g_i^P)) = \text{im}(c_k^P)$ and we can think of $\text{im}(a_k^P)$ as the subspace of $o^{-1}((g_0^P, \ldots, g_k^P))$ consisting of $(\iota(N_n^P), (g_0^P, \ldots, g_k^P))$ where $\iota : N_n^P \to [0, \infty) \times \mathbb{R}^\infty$ is any embedding such that $\iota(N_n^P - P_{k,2})$ lies in $[k + 1, \infty) \times \mathbb{R}^\infty$. To show that the subspace $\text{im}(a_k^P)$ is weakly homotopy equivalent to $o^{-1}((g_0^P, \ldots, g_k^P))$, we will homotope $f : (D^k, \partial D^k) \to o^{-1}((g_0^P, \ldots, g_k^P), \text{im}(a_k^P))$ through maps with codomain $(Y_k(N_n^P), \text{im}(a_k^P))$ to a map with codomain $\text{im}(a_k^P)$; by Proposition 30, $o^{-1}((g_0^P, \ldots, g_k^P))$ is a weakly homotopy equivalent subspace of $Y_k(N_n^P)$, so it’s possible to work in the latter, bigger space. We will modify each element $f(x) = (\iota(N_n^P), (g_0^P, \ldots, g_k^P))$ by specifying how we modify $\iota(N_n^P)$ and $c_k^P$.

First, for each $f(x) = (\iota(N_n^P), (g_0^P, \ldots, g_k^P))$, translate the first coordinate of $\iota(N_n^P)$ and $c_k^P$ by $r_x$ unit to the right and the part of $c_k^P$ with image in $\{0\} \times \mathbb{R}^\infty$
will be stretched into a horizontal cylinder in $[0, r_x] \times \mathbb{R}^\infty$. Here, $r_x$ is some amount needed to move $\iota(N_n^P) - im(c_k^P)$ out of $[0, k + 1] \times \mathbb{R}^\infty$, and for $x \in \partial D^k$, we can set $r_x$ to 0. We keep denoting the modified embedding and submanifold by $c_k^P$ and $\iota(N_n^P)$, respectively. Let $R_x := \max_{x \in D^k} r_x$. We then stretch out the submanifold of $\iota(N_n^P)$ in $[k + 1 + r_x, k + 1 + r_x + \epsilon_\partial] \times \mathbb{R}^\infty$ to occupy a horizontal cylinder $[k + 1 + r_x, k + 1 + R_x] \times \mathbb{R}^\infty$ without modifying the embedding $c_k^P$. Denote by $c'$ the embedding $c_k^P$ along with the stretched horizontal cylinder embedding in $[k + 1 + r_x, k + 1 + R_x] \times \mathbb{R}^\infty$, and denote by $f'$ the end of this homotopy of $f$.

Next, we homotope $f'$ so that the part $\iota(N_n^P) - im(c')$ of $f'(x)$ become constant with respect to $x \in D^k$: deform $\iota(N_n^P) - im(c')$ using the $f'$-image of the contraction of $D^k$ to a point; by isotopy extension, we can extend this to an isotopy of $\iota(N_n^P)$, fixing $[0, k + 1] \times \mathbb{R}^\infty$ as well as the horizontal cylinder of $im(c')$. Now, we can apply Proposition 11 to deform $c'$, relative to $\iota(N_n^P) - im(c')$, to agree with the $c'$ given in $f'(\partial D^k)$, keeping $c'$ for $x \in \partial D^k$ fixed. The resulting homotopy of $c_k^P$ and $\iota(N_n^P)$ gives an element in $im(a_k^P)$, and we are done. \hfill \Box

**Proposition 33.** The diagram

\[
\begin{array}{ccc}
Sub^0(N_n^P) & \xrightarrow{s^P} & Sub^0(N_n^{P+1}) \\
\downarrow a_k^P & & \downarrow a_{k-1}^P \\
X_k(N_{n+k+1}) & \xrightarrow{d_k} & X_{k-1}(N_{n+k+1})
\end{array}
\]

commutes.

**Proof.** The proof is essentially the same as that of Proposition 16. For any $\iota(N_n^P) \in Sub^0(N_n^P)$, we check that $d_k \circ a_k^P(\iota(N_n^P))$ and $a_{k-1}^P \circ s^P(\iota(N_n^P))$ are the same; the
The first quantity is
\[ d_k \left( (\text{im}(c_k^P) \cup ((k + 1) \cdot \vec{e}_1 + \iota(N_n^P)), (g_0^P, \ldots, g_k^P)) \right) \]
\[ = \left( \text{im}(c_k^P) \cup ((k + 1) \cdot \vec{e}_1 + \iota(N_n^P)), (g_0^P, \ldots, g_k^P) \right), \]
while the second quantity is
\[ a_{k-1}^P \left( \text{im}(c_0^P) \cup (\vec{e}_1 + \iota(N_n^P)) \right) \]
\[ = \left( \text{im}(c_{k-1}^P) \cup (k \cdot \vec{e}_1 + \text{im}(c_0^P)) \cup ((k + 1) \cdot \vec{e}_1 + \iota(N_n^P)), (g_0^P, \ldots, g_{k-1}^P) \right). \]

Because \( c_k^P = c_{k-1}^P \cup (k \cdot \vec{e}_1 + c_0^P) \), the 2 quantities are the same.

**Proposition 34.** For \( l, j \in \{0, \ldots k\} \), \( d_l, d_j : X_k(N_n^{S^1 \times S^2}) \to X_{k-1}(N_n^{S^1 \times S^2}) \) are homotopic.

**Proof.** This is similar to the proof of Proposition 17. First, we show that \( d_0 \) and \( d_1 \) are homotopic. As \( a_k^P \left( \text{Sub}_\beta(N_{n-k}^P) \right) \hookrightarrow Y_k(N_n^P) \) is a weak homotopy equivalence, we may assume that the given embedding of \( N_n^P \) is of the form
\[ \left( \text{im}(c_k^P) \cup ((k + 1) \cdot \vec{e}_1 + \iota(N_{n-k}^P)), (g_0^P, \ldots, g_k^P) \right) \]
as in Definition 13. First, we need to isotope \([0, 2] \times \mathbb{R}^\infty\), fixing the boundary, taking \( g_0^P \) to \( g_1^P \) except the part with image in \([0, 2] \times i_{\partial_0}(H_{\partial_0})\). We can do that by moving the copy of \( P - D^3 \) containing \( \text{im}(g_0^P) \) to the other copy of \( P - D^3 \) by shifting the first coordinate 1 unit to the right, and at the same time shifting the first coordinate of the second copy 1 unit to the left; but to avoid these two copies from bumping, we shift the second copy in the second coordinate by \( C_2 \) beforehand and shift by \(-C_2\) afterward, so that \( \text{im}(c_1) \) is not changed after the isotopy. The tethers of \( g_0^P \) will be dragged so that at the end, the part outside of \([0, 2] \times i_{\partial_0}(H_{\partial_0})\) will agrees with the tethers of \( g_1^P \) (while the tethers of \( g_1^P \) just need to avoid \( g_0^P \)). This gives the desired isotopy of \([0, 2] \times \mathbb{R}^\infty\) by isotopy extension. To get the rest
of the $g_0^P$ to match up with $g_1^P$, we just need to translate the horizontal part of $g_0^P$ in the second coordinate from $t_0$ to $t_1$, and moving $g_1^P$ up and shrinking the thickness $r_1$ of the tether if necessary.

To show that $d_i$ is homotopic to $d_{i+1}$ for general $i$, we can use a similar homotopy in $[i, i + 2] \times \mathbb{R}^\infty$ for switching the two copies of $P - D^3$ in it.

\[\text{Lemma 35. } H_q(Y_{-1}(N_n^P), |Y_*(N_n^P)|) = 0 \text{ if } q \leq \frac{n-1}{2}.\]

Assuming this lemma, which we will prove in Section 7, we can obtain homological stability for $s^P$.

\[\text{Proposition 36. } H_q(s^P): H_q(B\text{Diff}(N_{n-1}^P, \partial N)) \to H_q(B\text{Diff}(N_n^P, \partial N)) \text{ is an isomorphism for } q \leq \frac{n-3}{2}.\]

\[\text{Proof.}\] The proof is similar to that of Proposition 19 using induction on $q$. For $q = 0$, it is true because $\text{Sub}^\partial(N_n^P)$ is connected. For general $q$, we apply the spectral sequence for augmented semi-simplicial space to $\epsilon_*: Y_*(N_n^P) \to Y_{-1}(N_n^P)$. The differential $d_{p,q}^{1}$ of the $E_{p,q}^1$ page is homotopic to $H_q(s^P) : H_q(\text{Sub}^\partial(N_{n-p}^P)) \to H_q(\text{Sub}^\partial(N_{n-p-1}^P))$ for even $p$ and the zero map for odd $p$ by Proposition 34, 32 and 33, so it suffices to show that $d_{0,q}^1$ is an isomorphism. If it isn’t, then $E_{0,q}^2$ or $E_{-1,q}^2$ is nonzero. But by Lemma 35, $E_{0,q}^r$ and $E_{-1,q}^r$ converges to zero since $q + 1 \leq \frac{n-1}{2}$, and one can show that $E_{0,q}^r$ and $E_{-1,q}^r$ has already converged when $r = 2$ because the differentials with $E_{0,q}^r$ and $E_{-1,q}^r$ as domain or codomain has the zero space as the codomain or domain, respectively, when $r \geq 2$ as shown in Proposition 19, using the inductive hypothesis. \[\square\]
To prove Lemma 18, we compute the homotopy fiber of $|\epsilon_*| : |X_*(N_n^{S^1 \times S^2})| \to X_{-1}(N_n^{S^1 \times S^2})$, showing that for $q \leq \frac{n-3}{2}$, the $q$-th homotopy group of the homotopy fiber is trivial, which implies that $\pi_q(|\epsilon_*|)$ is bijective and $\pi_{q+1}(|\epsilon_*|)$ is surjective by the exact sequence for homotopy groups of a fibration. So for $q \leq \frac{n-1}{2}$, $\pi_q(X_{-1}(N_n^{S^1 \times S^2}), |X_*(N_n^{S^1 \times S^2})|)$ is trivial by the long exact sequence of homotopy groups. By the Hurewicz theorem, the homology $H_q(X_{-1}(N_n^{S^1 \times S^2}), |X_*(N_n^{S^1 \times S^2})|)$ is trivial as well in this range, as claimed in Lemma 18. First, we need to show that $|\epsilon_*|$ is a quasibibration, which would imply that the homotopy fiber of $|\epsilon_*|$ is weakly homotopy equivalent to the fiber of $|\epsilon_*|$, and we can work with the latter instead.

**Definition 37.** $p : E \to B$ is a quasibibration if $p_* : \pi_i(E, p^{-1}(b), x_0) \to \pi_i(B, b)$ is an isomorphism for any $b \in B$ and $x_0 \in p^{-1}(b)$.

**Proposition 38.** Let $\pi : X \to X/G$ be a principal $G$-bundle and $Y$ is a $G$-space. The map $\epsilon : X \times_G Y \to X/G$ defined by $[x, y] \mapsto [x]$ is a Serre fibration. In particular, the augmentation $\epsilon_k : X_k(N_n^{S^1 \times S^2}) \to X_{-1}(N_n^{S^1 \times S^2})$ is a Serre fibration with fiber $\text{Emb}_k(I + S^2, N_n^{S^1 \times S^2})$, $\epsilon'_k : X_k(N_n^{S^1 \times S^2}, \partial_0, \partial_1) \to X_{-1}(N_n^{S^1 \times S^2}, \partial_0, \partial_1)$ is a Serre fibration with fiber $\text{Emb}_k(I + S^2, M, \partial_0, \partial_1)$, and $\rho'_k : Y_k(N_n^P) \to Y_{-1}(N_n^P)$ is a Serre fibration with fiber $\text{Emb}_k(I \lor P', N_n^P)$.

**Proof.** To show that $\epsilon$ is a Serre fibration, we need to find a map $F$ that makes the diagram

\[
\begin{array}{ccc}
I^n \times \{0\} & \xrightarrow{f} & X \times_G Y \\
\downarrow{\iota} & \nearrow{F} & \\
I^n \times [0, 1] & \xrightarrow{g} & X/G
\end{array}
\]
commute for any \( n \in \mathbb{N} \). Given \( f : I^n \times \{0\} \to X \times_G Y \), we can assume \( f = [f_1, f_2] \) for \( f_1 : I^n \times \{0\} \to X \) and \( f_2 : I^n \times \{0\} \to Y \) because \( \pi \) being a principal \( G \)-bundle implies that \( X \times Y \to X \times_G Y \) is a principal \( G \)-bundle and hence a Serre fibration. Because \( \pi \) is a Serre fibration, we can lift \( g : I^n \times I \to X/G \) to \( g' : I^n \times I \to X \) starting with \( f_1 : I^n \times \{0\} \to X \). We can obtain the lift \( F \) of \( g \) by defining \( F(x,t) = [g'(x,t), f_2(x,0)] \). Because \( g'(x,0) = f_1(x,0) \), this makes the diagram commute.

Because \( \text{Emb}^\partial(N_n^P) \to \text{Sub}^\partial(N_n^P) \) is a principal \( \text{Diff}(N_n^P, \partial N) \)-bundle, the first statement of the proposition can be applied to \( \epsilon_k, \epsilon_k' \) and \( \epsilon_k'' \). The fiber \( \epsilon_k^{-1}(e) \) for \( e \in X_{-1}(N_n^{S^1 \times S^2}) \) is

\[
\{[e', (f_0, \ldots, f_k)] \in \text{Emb}^\partial(N_n^{S^1 \times S^2}) \times \text{Diff}(N_n^{S^1 \times S^2}, \partial N) \text{Emb}_k(T + S^2, N_n^{S^1 \times S^2}) : e' = e \circ h_e \text{ for some } h_e \in \text{Diff}(N_n^{S^1 \times S^2}, \partial N)\},
\]

which is homeomorphic to \( \text{Emb}_k(T + S^2, N_n^{S^1 \times S^2}) \) by the map \( [e', (f_0, \ldots, f_k)] \mapsto (h_e \circ f_0, \ldots, h_e \circ f_k) \), with inverse \( (f_0, \ldots, f_k) \mapsto [e, (f_0, \ldots, f_k)] \). Similarly, \( \epsilon_k''^{-1}(e_1) \) for \( e_1 \in X_{-1}(N_n^{S^1 \times S^2}, \partial_0, \partial_1) \) is homeomorphic to \( \text{Emb}_k(T + S^2, N_n^{S^1 \times S^2}, \partial_0, \partial_1) \), and \( \epsilon_k''''^{-1}(e_2) \) for \( e_2 \in Y_{-1}(N_n^P) \) is homeomorphic to \( \text{Emb}_k(T \vee P', N_n^P) \).

**Lemma 39.** If all maps \( \epsilon_k : X_k \to X_{-1} \) are Serre fibrations, then \( |\epsilon_*| : |X_*| \to X_{-1} \) is a quasifibration.

**Proof.** Let \( x \in X_{-1} \) and let \( U_x \) be a contractible neighborhood of \( x \). We will show that the inclusion \( |\epsilon_*|^{-1}(x) \to |\epsilon_*|^{-1}(U_x) \) is a weak equivalence, so that \( \pi_1(|\epsilon_*|^{-1}(U_x), |\epsilon_*|^{-1}(x)) = \pi_1(U_x) \), which means that \( |\epsilon_*| : |\epsilon_*|^{-1}(U_x) \to U_x \) is a quasifibration. We would be done if we can cover \( X_{-1} \) by \( \{U_x\}_{x \in X_{-1}} \) and with the assumption that all intersections of any of the \( U_x \) are contractible, because then, each of these pieces is a quasifibration, which implies that \( |\epsilon_*| \) is also a...
quasibration by Lemma 4K.3(a) and (b) of [5] (4K.3a says that the union of 2 quasibrations, whose intersection is a quasibration, is a quasibration. 4K.3b says that the union of an increasing sequence of quasibrations is a quasibration if given a compact subset, there is an element in the increasing sequence that contains it.). The assumption of contractible intersection will be removed at the end of the proof as this assumption may not be true for a general space $X_{-1}$.

As $\epsilon_k$ is a Serre fibration, $\epsilon_k^{-1}(x) \rightarrow \epsilon_k^{-1}(U_x)$ is a weak equivalence. For surjectivity, note that any map $i : S^l \rightarrow \epsilon_k^{-1}(U_x)$ can be deformed to $S^l \rightarrow \epsilon_k^{-1}(x)$, where the deformation $S^l \times [0, 1] \rightarrow \epsilon_k^{-1}(U_x)$ comes from applying the homotopy lifting property

\[
\begin{align*}
S^l & \xrightarrow{j} \epsilon_k^{-1}(U_x) \\
\downarrow & \hspace{1cm} \downarrow \epsilon_k \\
S^l \times [0, 1] & \xrightarrow{j} U_x,
\end{align*}
\]

with $j$ being picked to satisfy $j(s, 0) = \epsilon_k \circ i(s)$ and $j(s, 1) = x$, which is possible as $U_x$ is contractible. For injectivity, note that any map $i : (D^l, \partial D^l) \rightarrow (\epsilon_k^{-1}(U_x), \epsilon_k^{-1}(x))$ can be deformed to have image entirely in $\epsilon_k^{-1}(x)$ where the deformation $D^l \times [0, 1] \rightarrow \epsilon_k^{-1}(U_x)$ comes from applying the homotopy lifting property as before with $S^l$ replaced by $D^l$.

To go from $\epsilon_k^{-1}(x) \rightarrow \epsilon_k^{-1}(U_x)$ being a weak equivalence to $|\epsilon_*(-)\cdot|^{-1}(x) \rightarrow |\epsilon_*(-)\cdot|^{-1}(U_x)$ being a weak equivalence, we will need the following result (Corollary 4K.2 of [5]):

Given $f : X \rightarrow Y$ and open covers $\{U_i\}$ of $X$ and $\{V_i\}$ of $Y$ with $f(U_i) \subset V_i$ for all $i$, then if each restriction $f : U_i \rightarrow V_i$ and more generally each $f : U_{i_1} \cap \cdots \cap U_{i_n} \rightarrow V_{i_1} \cap \cdots \cap V_{i_n}$ is a weak homotopy equivalence, so is $f : X \rightarrow Y$.

Consider an open covering of $\Delta^k$ (as illustrated by Figure 7.0.1 for the case
where we have one contractible open subset $V_i$ of $\Delta^k$ for each subsimplex $\Delta_i$, covering only part of $\Delta_i$ and excluding the proper subsimplexes of $\Delta_i$ (denoted by $\partial \Delta_i$), and $\cap_{i \in I} V_i$ is a contractible open subset covering only part of the subsimplex $\cap_{i \in I} \Delta_i$ but excluding $\partial (\cap_{i \in I} \Delta_i)$. Consider the action of the permutation group $\Sigma_k$ permuting the coordinates of $\Delta^k$. For a permutation that takes $\Delta_i$ to $\Delta_j$, we assume that the corresponding diffeomorphism of $\Delta^k$ maps $V_i$ to $V_j$.

Represent points in $|X_*|$ by $[y, (t_0, \ldots, t_n)] \in \coprod_{i=0}^{\dim X_i} X_i \times \Delta^i / \sim$, where $y \in X_n$ and $(t_0, \ldots, t_n) \in \Delta^n$. For each $U_x$, cover $|\epsilon_*|^{-1}(U_x)$ with the subsets of the form

$$U_{x,V_i} := \{ [y, (t_0, \ldots, t_i)] \in X_i \times \Delta^i : y \in U_x, (t_0, \ldots, t_i) \in V_i \},$$

one for each $V_i$, the neighborhood of $\Delta_i$ that was defined above. Each $U_{x,V_i}$ deformation retracts to $\epsilon^{-1}_{\dim \Delta_i}(U_x)$ by deforming $V_i$ to $\Delta_i$ and then to a point in $\Delta_i$. Similarly, any intersection $\cap_{i \in I} \Delta_i = \Delta_l$ for some $l$, means $\cap_{i \in I} U_{x,V_i}$ will deformation retract to $\epsilon^{-1}_{\dim \Delta_l}(U_x)$. So the restriction to any intersection is a quasifibration and so by Corollary 4K.2 of [5], $|\epsilon_*|^{-1}(x) \to |\epsilon_*|^{-1}(U_x)$ is a weak equivalence as well.

The contractibility restriction is not required, as we will now show. Recall that
what we need to show is that

$$\pi_j(|\epsilon_\ast|) : \pi_j\left(|X_\ast|, |\epsilon_\ast|^{-1}(x)\right) \to \pi_j(X_{-1})$$

is an isomorphism. Suppose $$\pi_j(|\epsilon_\ast|)$$ is not injective, then there is a map

$$g : (D^j, \partial D^j, \{pt_0\}) \to (|X_\ast|, |\epsilon_\ast|^{-1}(x), \{g(pt_0)\})$$

that is not nullhomotopic while $$|\epsilon_\ast| \circ g : (D^j, \partial D^j, \{pt_0\}) \to (X_{-1}, \{x\}, \{x\})$$ is nullhomotopic; let’s say that $$\Gamma_t$$ is a nullhomotopy of $$|\epsilon_\ast| \circ g$$. The pullback

$$\Gamma_t^*\left(|\epsilon_\ast|\right) : \Gamma_t^*\left(|X_\ast|\right) \to D^j \times I$$

of $$|\epsilon_\ast| : |X_\ast| \to X_{-1}$$ via $$\Gamma_t$$ is a quasifibration because the base space of $$\Gamma_t^*\left(|\epsilon_\ast|\right)$$ is $$D^j \times I$$ and we can apply the argument in the previous paragraphs. $$|\epsilon_\ast| \circ g : (D^j, \partial D^j, \{pt_0\}) \to (X_{-1}, \{x\}, \{x\})$$ factors through $$f : (D^j, \partial D^j, \{pt_0\}) \to (\Gamma_t^*\left(|\epsilon_\ast|\right)|^{-1}(X_\ast), \Gamma_t^*\left(|\epsilon_\ast|\right)|^{-1}(pt_1), pt_2)$$ for some $$pt_1$$ and $$pt_2$$ by the universal property of pullback. But then, $$f$$ is nullhomotopic because the inclusion $$\Gamma_t^*\left(|\epsilon_\ast|\right)|^{-1}(pt_1) \to \Gamma_t^*\left(|\epsilon_\ast|\right)|^{-1}(X_\ast)$$ is a weak equivalence. So $$g$$ is nullhomotopic and we have a contradiction. Suppose $$\pi_j(|\epsilon_\ast|)$$ is not surjective, so that there is a map $$G : (D^j, \partial D^j, \{pt_0\}) \to (X_{-1}, \{x\}, \{x\})$$ whose homotopy class is not in $$im(\pi_j(|\epsilon_\ast|))$$. The pullback $$G^*\left(|\epsilon_\ast|\right)$$ is a quasifibration by the previous paragraphs and so we can lift the nullhomotopy of $$id : D^j \to D^j$$ to $$G^*\left(|X_\ast|\right)$$, and mapping it back to $$|X_\ast|$$ gives us the homotopy that contradicts the non-surjectivity supposition.

To prove Lemma 18, it suffices to show that the fiber $$|Emb_\ast(T + S^2, N_n^{S^1 \times S^2})|$$ of $$|\epsilon_\ast| : |X_\ast(N_n^{S^1 \times S^2})| \to X_{-1}(N_n^{S^1 \times S^2})$$ is $$\frac{n-3}{2}$$-connected. Similarly, to prove Lemma 35, it suffices to show that $$|Emb_\ast(T \vee P', N_n^P)|$$ is $$\frac{n-3}{2}$$-connected.

**Lemma 40.** $$|Emb_\ast(T + S^2, N_n^{S^1 \times S^2})|$$ and $$|Emb_\ast(T \vee P', N_n^P)|$$ are $$\frac{n-3}{2}$$-connected.

This would imply Lemma 26 as well, since $$|Emb_\ast(T + S^2, N_n^{S^1 \times S^2}, \partial_0, \partial_1)|$$ is weakly homotopy equivalent to $$|Emb_\ast(T + S^2, N_n^{S^1 \times S^2})|$$, since the different bound-
ary conditions of the 2 spaces can be made to coincide via a diffeomorphism of $N_n^{S^1 \times S^2}$.

To prove Lemma 40, we need to work with some intermediate spaces with higher connectivity properties. We start with sphere complexes and add certain restrictions until we arrive at the spaces given in the lemma.

The notation $\sum_{i=0}^n w_i \cdot v_i$ and $(\langle v_0, \ldots, v_n \rangle, w_0, \ldots, w_n)$ will be used interchangeably to denote an element in $|X|$, with $w_i \in [0, 1]$ such that $\sum_{i=0}^n w_i = 1$, and $\langle v_0, \ldots, v_n \rangle$ is a simplex in $X$.

**Definition 41.** Let $S(N_n^P)$ be the simplicial complex where an $i$-simplex is a set of $i + 1$ disjoint, nontrivial $S^2$ embeddings in $N_n^P$. A nontrivial sphere means it is either a nonseparating sphere or neither of the two separated components is $\#_k D^3$ for some $k \in \mathbb{N}$.

**Proposition 42.** $|S(N_n^P)|$ is contractible if $n \geq 1$ and $P = S^1 \times S^2$ or if $n \geq 2$ and $P \neq D^3$.

**Proof.** $S(N_n^{S^1 \times S^2})$ is nonempty because there is a nonseparating sphere inside a copy of $S^1 \times S^2 - D^3$ in $N_n^{S^1 \times S^2}$. If $n \geq 2$, $S(N_n^P)$ is nonempty because there is a sphere that separates $N_n^P$ into $P \# D^3$ and $N_{n-1}^P \# D^3$, where neither component bounds $\#_k D^3$, since $P$ is a prime manifold that is not $D^3$.

For any $k > 0$ and given a map $f : (D^k, \partial D^k) \to (|S(N_n^{S^1 \times S^2})|, \{1 \cdot e\})$, we need to homotope $f$ rel $\partial D^k$ to the constant map. We can assume that $f$ is simplicial by simplicial approximation. There are finitely many spheres $\{e_i\}_{i \in I}$ in $im(f)$, so we can pick a sphere embedding $e'$ parallel to $e$ that intersects all the spheres in $im(f)$ transversely; parallel means that we pick a tubular neighborhood of $e$ beforehand and vary the normal coordinate, which offers an uncountable number
of sphere embeddings to choose from. Because transversality is stable under perturbation, we can assume that there is a tubular neighborhood of $e'$, denoted by $\text{im}(e') \times (-\infty, \infty)$, such that each sphere in it that is parallel to $e'$ intersects $\{e_i\}_{i \in I}$ transversely. Similarly, we can pick a tubular neighborhood for each $e_i$, denoted by $\text{im}(e_i) \times (-\infty, \infty)$, that remains disjoint from $\text{im}(e_j) \times (-\infty, \infty)$ if $\text{im}(e_i)$ is disjoint from $\text{im}(e_j)$, such that each sphere that is parallel to $e_i$ transversely intersects any sphere in $\text{im}(e') \times (-\infty, \infty)$ that is parallel to $e'$.

We will use the sphere $\text{im}(e')$ to perform surgery. Pick a point $p$ in $\text{im}(e')$ that’s disjoint from the circles created from intersecting with the spheres $\{\text{im}(e_i)\}_{i \in I}$. This point is used to determine for each circle, which one of the two discs in $\text{im}(e')$ bordering the circle we should be using as the surgering disk; let’s use the disk that does not contain $p$. Among all circles created from intersecting $e'$ with the spheres in $\text{im}(f)$, we start the surgery using one of the innermost circles, say $e_0$, i.e. those circles with surgering disks that does not contain another circle entirely in its interior; note that there can be another circle intersecting this circle, but that would be coming from another sphere, and these 2 spheres, say $f(v_0)$ and $f(v_1)$, are originating from vertices $v_0$ and $v_1$ that are not neighbors in the simplicial complex $D^k$.

The surgery involves changing $\text{im}(e_0)$ by adding in 2 discs parallel to the surgering disk using the tubular neighborhood $\text{im}(e') \times (-\infty, \infty)$ of $e'$ that we will use for all surgeries. The 2 discs are chosen to lie in $\text{im}(e') \times \{\frac{1}{A}\}$ and $\text{im}(e') \times \{-\frac{1}{A}\}$ respectively, where $A$ is the area of the surgering disk; this ensures that another surgering disk that contains the current surgering disk, which will have a bigger area, will surger and add discs that don’t intersect any existing spheres at the end of the current surgery. We also pick a parallel copy of $e_0$ in $\text{im}(e_0) \times (-\infty, \infty)$,
specifically, $\text{im}(e_0) \times \{-\frac{1}{A}\}$ or $\text{im}(e_0) \times \{\frac{1}{A}\}$, depending on which one is on the inside, i.e. the one intersecting the surgering disk; this ensures that the resulting spheres are disjoint from $e_0$. Then remove the part of $e_0$ in $\text{im}(e') \times \{-\frac{1}{A}, \frac{1}{A}\}$.

This finishes the surgery for the first circle of intersection. At least 1 of the 2 resulting spheres, $e'_0$ or $e''_0$, will be nontrivial, because otherwise, say $e'_0$ bounds $M_{0,k_1} \approx \#_{k_1}D^3$ and $e''_0$ bounds $M_{0,k_2} \approx \#_{k_2}D^3$; if $M_{0,k_1}$ is contained in $M_{0,k_2}$ (or vice versa), then $e_0$ bounds $M_{0,k_2} - M_{0,k_1} \approx M_{0,k_2} - k_1 + 1$, and if neither are contained in each other, then $e_0$ bounds $M_{0,k_1} \cup M_{0,k_2}$, which is $M_{0,k_1+k_2-1}$, which contradicts the fact $e_0$ is nontrivial. So, we can assume $e'_0$ is nontrivial.

We can now start the homotopy of $f$ by transferring the weight from the original $e_0$ to $e'_0$. This is a homotopy of $f$ because the spheres in $\text{im}(f)$ that are disjoint from $e_0$ will remain disjoint from $e'_0$, as the surgery happens within a tubular neighborhood of $e_0$ and a neighborhood of $e'$, where the disks that are added won’t intersect other neighboring spheres by how we pick the tubular neighborhoods. Explicitly, the homotopy in each maximal simplex containing the vertex we are changing will have image like $w_0 \cdot e_0 + \sum_{i \in I-\{0\}} w_i \cdot e_i$ at the beginning, to $(1 - t) w_0 \cdot e_0 + t w_0 \cdot e'_0 + \sum_{i \in I-\{0\}} w_i \cdot e_i$ as we vary from $t = 0$ to $t = 1$.

Continuing in this manner, we can perform surgery on the next innermost circle and homotope $f$ by transferring weights to the surgered sphere embedding, which is disjoint from $\text{im}(e')$. Each such surgery decreases the number of circles of intersection by 1, so after finitely many surgeries and corresponding homotopies, the image of the resulting map has embeddings that are all disjoint from $\text{im}(e')$. Then, we can perform a homotopy on $f$, one vertex at a time, that shifts the weight to $e'$ if that vertex is not the embedding $e$ or $e'$. Finally, we can homotope $f$ by shifting the weight for each vertex that maps to $e'$ to $e$, one vertex at a time. This
gives the homotopy from \( f \) to the constant map rel \( \partial D^k \).

**Definition 43.** Let \( T(N_n^{S^1 \times S^2}) \), the thickened version of \( S(N_n^{S^1 \times S^2}) \), be a simplicial complex where an \( i \)-simplex is a set of \( i + 1 \) nontrivial \( S^2 \times I \) embeddings in \( N_n^{S^1 \times S^2} \) such that the restriction to \( S^2 \times \{ \frac{1}{2} \} \) of each embedding gives a disjoint sphere system.

**Proposition 44.** \( T(N_n^{S^1 \times S^2}) \) is contractible.

**Proof.** We can repeat the surgery process in Proposition 42. The differences are that we use the \( S^2 \times \{ \frac{1}{2} \} \) part of each \( S^2 \times I \) embedding for the process, and we give the surgered spheres a fixed thickness using the tubular neighborhoods that is involved in the surgery. Since only the \( S^2 \times \{ \frac{1}{2} \} \) has to be disjoint from each other, surgery will guarantee this disjointness condition as before.

**Definition 45.** Let \( T^C(N_n^{S^1 \times S^2}) \) be the simplicial complex where an \( i \)-simplex is a set of embeddings of \( S^2 \times I \) in \( N_n^{S^1 \times S^2} \), say \( \{e_0, \ldots, e_i\} \), such that the restriction to \( S^2 \times \{ \frac{1}{2} \} \) of each embedding gives a disjoint sphere system and the complement \( N_n^{S^1 \times S^2} - \bigcup_{0 \leq j \leq i} e_j \left( S^2 \times \{ \frac{1}{2} \} \right) \) is connected. If \( M \) is not connected, say \( M = \coprod_{l \in L} (N_l)_{n_l}^{S^1 \times S^2} \), then let \( T^C(M) \) be the join \( *_{l \in L} T^C \left( (N_l)_{n_l}^{S^1 \times S^2} \right) \).

We will prove that \( T^C(N_n^{S^1 \times S^2}) \) has a stronger property that is needed to obtain the desired connectivity for complexes of tethered spheres.

**Definition 46.** A simplicial complex \( C \) is weakly Cohen-Macaulay, abbreviated by wCM, of dimension \( n \) if \( |C| \) is \( (n - 1) \)-connected with \( \dim C \geq n \) and for each simplex \( \tau \), \( |L_k C, \tau| \) is \( (n - 2 - \dim \tau) \)-connected with \( \dim L_k C, \tau \geq n - 1 - \dim \tau \).

**Proposition 47.** \( T^C(N_n^{S^1 \times S^2}) \) is wCM of dimension \( n - 1 \).
Proof. We will prove this by inducting on \( n \). For the \( n = 1 \) case, it is \((-1)\)-connected because there is a nonseparating sphere in the copy of \( S^1 \times S^2 - D^3 \) in \( N_{S^1 \times S^2}^1 \).

For the induction of the \((n-2)\)-connected case, for any map \( f : \partial D^k \rightarrow |T^C(N_{S^1 \times S^2}^n)| \) for \( k \leq n-1 \), we need to extend \( f \) to \( D^k \). By the Proposition 42, we can extend \( f \) to \( f : D^k \rightarrow |T^C(N_{S^1 \times S^2}^n)| \), which we assume to be simplicial by simplicial approximation, for some triangulation of \( D^k \). We need to homotope \( f \) to have image inside \( |T^C(N_{S^1 \times S^2}^n)| \).

We say that a simplex \( \sigma \) in \( D^k \) is bad if \( f(\sigma) \) consists of sphere embeddings \( \{e_j\}_{j \in J} \) such that for any \( l \in J \), \( N_{S^1 \times S^2}^n - \bigcup_{j \in J \setminus \{l\}} \text{im}(e_j) \) has more components than \( N_{S^1 \times S^2}^n - \bigcup_{j \in J \setminus \{l\}} \text{im}(e_j) \). If there exists a system of separating spheres, we can remove the spheres one by one and stop right before the step where removing another sphere (no matter which one) would cause the system to not separate; this gives a system that corresponds to a bad simplex. Thus, having no bad simplex in \( D^k \) means that \( \text{im}(f) \) is contained entirely in \( T^C(N_{S^1 \times S^2}^n) \), and we would be done.

Let \( \sigma \) be a maximal bad simplex. The image of \( f|_{Lk_D^k(\sigma)} \) is contained in \( T^C \left( N_{S^1 \times S^2}^n - \bigcup_{j \in J \setminus \text{im}(e_j)} \right) \): if a sphere in \( f(Lk_D^k(\sigma)) \) is trivial in \( N_{S^1 \times S^2}^n - \bigcup_{j \in J \setminus \text{im}(e_j)} \), then it should be part of \( \sigma \), contradicting maximality, and if there is a separating sphere system \( f(\tau) \) in \( N_{S^1 \times S^2}^n - \bigcup_{j \in J \setminus \text{im}(e_j)} \), then \( \sigma \ast \tau \), being the join of two bad simplexes, is also a bad simplex in \( D^k \); this is because if \( N_{S^1 \times S^2}^n - \bigcup_{j \in J \setminus \text{im}(e_j)} \) has more components than \( N_{S^1 \times S^2}^n - \bigcup_{j \in J \setminus \{l\}} \text{im}(e_j) \), then \( N_{S^1 \times S^2}^n - \bigcup_{j \in K \setminus \{l\}} \text{im}(e_j) \) also has more components than \( N_{S^1 \times S^2}^n - \bigcup_{j \in J \setminus \{l\}} \text{im}(e_j) \) for a set \( K \) containing \( J \). To see the last statement, we can split \( N_{S^1 \times S^2}^n - \bigcup_{j \in J \setminus \{l\}} \text{im}(e_j) \) and \( N_{S^1 \times S^2}^n - \bigcup_{j \in J \setminus \text{im}(e_j)} \) by \( \{S_j\}_{j \in K \setminus J} \), one sphere at a time and see that if a sphere splits the former into one more component, it also splits the latter into one.
more component. Note that $\sigma$ is not contained in $\partial D^k$ because otherwise, $f(\sigma)$ would separate $N_n^{S^1 \times S^2}$, contradicting the fact that $f(\partial D^k)$ is in $T^C(N_n^{S^1 \times S^2})$.

$$N_n^{S^1 \times S^2} - f(\sigma) = \bigsqcup_{l \in L} (N_l)_{nt_l}^{S^1 \times S^2}$$
for some $nt_1 \in N$ and $nt_l$, and so we can express $T^C(N_n^{S^1 \times S^2} - f(\sigma))$ as $\pi_{l \in L} T^C((N_l)_{nt_l}^{S^1 \times S^2})$. We claim that $nt_l < n$ for all $l \in L$, which will allow us to apply the inductive hypothesis to $T^C((N_l)_{nt_l}^{S^1 \times S^2})$. The operation of splitting along one of the spheres transforms $N_n^{S^1 \times S^2}$ into $(N\#D^3)_{n-1}^{S^1 \times S^2}$, if it is non-separating, and into $(N_1')_{n_1'}^{S^1 \times S^2}$ and $(N_2')_{n_2'}^{S^1 \times S^2}$, if it is separating, where $n_1' + n_2' = n$; further, $n_1'$ and $n_2'$ are positive for the first sphere we split because it is nontrivial. Thus, after splitting with the first sphere, either the eventual sum satisfies $\sum_{l \in L} nt_l \leq n - 1$, or that $L = L_1 \bigsqcup L_2$ with $\sum_{l \in L_1} nt_l = n_1' < n$ and $\sum_{l \in L_2} nt_l = n_2' < n$, which means $nt_l < n$ for all $l \in L$. By induction, each term in the join, i.e. $T^C((N_l)_{nt_l}^{S^1 \times S^2})$ is $(n_l - 2)$-connected. So the join is of connectivity $(\sum_{l \in L} nt_l) - 2$ (by Lemma 2.3 of [7]). $\sum_{l \in L} nt_l$ is the number of $S^1 \times S^2$ left in $N_n^{S^1 \times S^2} - (\cup_{j \in J} im(e_j))$, which is greater than $n - \dim \sigma - 1$, because each sphere can decrease $\sum_{l \in L} nt_l$ by at most 1 and the fact that it is separating means that not all $\dim \sigma + 1$ of the spheres in $f(\sigma)$ are decreasing the number of $S^1 \times S^2$ in $N_n^{S^1 \times S^2}$.

So the connectivity of the codomain is at least $n - \dim \sigma - 2 \geq k - \dim \sigma - 1$, so we can extend $f|_{lk(\sigma)}$ to $\tilde{f} : D^{k-\dim \sigma} \to T^C \left(N_n^{S^1 \times S^2} - (\cup_{j \in J} im(e_j))\right)$, as $|lk_{D^k}(\sigma)|$ is a $(k - \dim \sigma - 1)$-sphere. This map remains simplicial for some triangulation of $D^{k-\dim \sigma}$ by simplicial approximation. Note that $f|_{St(\sigma)}$ agrees with $f|_{\partial \sigma} \ast \tilde{f}$ on $\partial St(\sigma) = \partial \sigma \ast \partial D^{k-\dim \sigma}$, so we can modify $f$ by replacing $f|_{St(\sigma)}$ by $f|_{\partial \sigma} \ast \tilde{f}$. The new simplexes formed in $\partial \sigma \ast D^{k-\dim \sigma}$ are not bad, so the number of maximal bad simplexes decreases by 1. Continuing in a similar fashion, the finitely many remaining bad simplexes will be removed. Because $\partial D^k$ was never modified in the process, this gives the desired extension of $f$ to $D^k \to T^C(N_n^{S^1 \times S^2})$. 

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The wCM dimension now follows: for any \( k \) and any \( k \)-simplex \( \sigma \) in \( T^C(N_n^{S^1 \times S^2}) \), \( Lk_{F^C(M_n,\sigma)}(\sigma) \approx T^C((N')_{n-k-1}^{S^1 \times S^2}) \), for some \( N' \), since each nonseparating sphere decreases the number of \( S^1 \times S^2 \) summands by 1 and increases the number of boundary spheres by 2, and from what we have proved, \( T^C((N')_{n-k-1}^{S^1 \times S^2}) \) is \((n-k-3)\)-connected, which is what’s needed for the claimed wCM dimension to hold.

The next step would be to consider the complex \( T^T(N_n^{S^1 \times S^2}) \) of tethered spheres and its connectivity. As the argument is similar to the case of \( P \neq S^1 \times S^2 \), we postpone it until we finish defining and computing the connectivity of the complex \( T^C(N_n^P) \) for this case.

**Definition 48.** For irreducible \( P \), let \( T(P',N_n^P) \) be the simplicial complex where each \( i \)-simplex consists of \( i+1 \) embeddings \( f_0, \ldots, f_i \) of \( P' \) in \( N_n^P - \partial N \) such that for \( j, l \in \{0, \ldots, i\} \), \( f_j(S_{P'}) \) and \( f_l(S_{P'}) \) coincide or are disjoint. Let \( T^C(N_n^P) \) be the subcomplex of \( T(P',N_n^P) \) where the embeddings in a simplex are pairwise disjoint.

**Proposition 49.** Let \( M \) be a compact, connected, orientable 3-manifold. \( |T(P',M)| \) is \((n_P - 2)\)-connected if \( M \) contains \( n_P \) copies of the irreducible manifold \( P \neq D^3 \).

**Proof.** This is analogous to Proposition 4.1 of [6]. We induct on the complexity of \( M \), which is defined as the maximum number of disjoint, nontrivial sphere embeddings in a sphere system with pairwise nonisotopic spheres.

The base case is when \( M \) contains 1 copy of \( P \), and this gives an embedding \( f_0 : P' \to M \) so that \(|T(P',M)|\) is non-empty.

For \( n_P \geq 2 \), we have a well-defined simplicial map, \( \pi : T(P',M) \to S(M) \), defined on vertices by \( \langle f_0 \rangle \mapsto \langle f_0|_{S_{P'}} \rangle \). For each vertex \( \langle g_0 \rangle \) in \( im(\pi) \), \( im(g_0) \) separates a copy of \( P' \) and we can pick an embedding \( g'_0 : P' \to M \) with that
copy of $P'$ as the image. If $M \neq P \# P$, the copy of $P'$ is unique, and for a simplex $\langle g_0, \ldots, g_i \rangle$ in $\text{im}(\pi)$, we can lift it to $\langle g'_0, \ldots, g'_i \rangle$, which is a simplex in $T(P', M)$ by the definition of $T(P', M)$. For the case $M = P \# P$, there are 2 choices of the submanifold $P'$ in $M$ whose boundary is $\text{im}(g_0)$. We fix one isotopy class $P'_{g_0}$ in a consistent way: if $g_0$ is isotopic to $g_1$, then $P'_{g_0}$ is isotopic to $P'_{g_1}$. We require that when lifting $g_0$ to an embedding $f_0 : P' \to M$, $f_0$ has image $P'_{g_0}$.

Given a map $f : S^j \to |T(P', M)|$, for any $j \leq n_P - 2$, we can assume that it is a simplicial map by simplicial approximation. We can extend $\pi \circ f$ to $F : D^{j+1} \to |S(M)|$ by Proposition 42, and we can assume that it is simplicial by simplicial approximation. We would be done after deforming $F$ in the interior of $D^{j+1}$ to have image in $\pi(T(P', M))$ and lifting the interior vertices as described in the previous paragraph to give a map $D^{j+1} \to |T(P', M)|$ that extends $f$. We call a simplex $\sigma$ in $D^{j+1}$ bad if each of the spheres in $F(\sigma)$ is nonseparating or separates $M$ into two components, neither of which is diffeomorphic to $P'$. For a maximal bad simplex $\sigma$, the map $F|_{Lk_\sigma}$ has image in $\ast_{i=0}^l T(P', M'_i)$, where $M'_i$ has lower complexity. Let $n_{M'_i}$ denote the number of copies of $P'$ contained in $M'_i$. The sum $\sum_{i=0}^l n_{M'_i} = n_P$ because the spheres in $\sigma$ is either nonseparating, which does not decrease the number of $P$ summands, or that it separates $M$ into two components not diffeomorphic to $P'$ and does not decrease the number of $P$ summands, due to the fact that $P$ is irreducible. So the join $\ast_{i=0}^l T(P', M'_i)$ is $(n_P - 2)$-connected by induction. Note that $Lk_{D^k} \sigma = S^k$ for some $k \leq n_P - 2$ because $\sigma$ is not contained in the boundary by the definition of a bad simplex. So there is an extension $\tilde{F}$ of $F|_{Lk_\sigma} : S^k \to \ast_{i=0}^l T(P', M'_i)$ to $D^{k+1} \to \ast_{i=0}^l T(P', M'_i)$, and we can modify $F$ in $\sigma$ by $F|_{Lk_\sigma} \ast \tilde{F}$ leaving $\partial \sigma$ unchanged. The bad part of $\sigma$ only remains in $\partial \sigma$ and the new simplexes are not bad because they are in $\ast_{i=0}^l T(P', M'_i)$. After finitely many such modifications of $F$, all bad simplexes will
For the case $M = P\#P$, we need to make sure that the interior vertices map to spheres whose image does not coincide with those of the vertices on the boundary, because how the boundary vertices are lifted is pre-determined and the way we lift interior vertices may conflict with the boundary ones if the spheres coincide. One interior vertex at a time, obtain a sphere parallel to the original sphere in a tubular neighborhood, and close enough that it does not bump into other neighboring spheres. This gives a homotopy of $F$ by transferring the weight from the original sphere to the new sphere. Continuing one vertex at a time, we have the desired map that can be lifted to $D^{j+1} \to |T(P', M)|$ where the boundary agrees with the original $f : S^j \to |T(P', M)|$. \hfill $\square$

For the case $P = D^3$, we have a similar result.

**Proposition 50.** If $P = D^3$, $|T(P', N^n_P)|$ is contractible if $n \geq 1$.

*Proof.* For $n \geq 1$, $|T(P', N^n_P)|$ is non-empty by the same argument as before. We need to deform $f : S^j \to |T(P', N^n_P)|$ to a constant map. Pick a boundary sphere in $N^n_P$ and a collar neighborhood of the boundary sphere. By shrinking the collar neighborhood, we can assume that it is contained in each of the collar neighborhoods of that boundary sphere in $\text{im}(f)$, as there are only finitely many such collar neighborhoods. Thus, we can deform $f$ by transferring the weights of all embeddings to a sphere $e_0$ in the small collar neighborhood. \hfill $\square$

**Proposition 51.** $|T^C(N^n_P)|$ is $wCM$ of dimension $n − 1$.

*Proof.* We will induct on $n$. For the $n = 1$ case, $|T^C(N^n_P)|$ is non-empty by embedding $P'$ into the copy of $P$ that is glued to $N$. 51
For the inductive step, we need to extend a map \( f : S^j \to |T^C(N^n_P)| \) to \( D^{j+1} \to |T^C(N^n_P)| \), for any \( j \leq n - 2 \). We can extend \( f \) to \( F : (D^{j+1}, S^{j+1}) \to (|T(P', N^n_P)|, |T^C(N^n_P)|) \) by Proposition 49 and 50, where \( f \) and \( F \) can be assumed to be simplicial under some triangulation of \( D^{j+1} \) by simplicial approximation.

Call a simplex \( \sigma \) in \( D^{j+1} \) bad if each embedding of \( P' \) in \( F(\sigma) \) is not disjoint from some other embedding in \( F(\sigma) \). If there were no bad simplex, then \( F \) maps into \( |T^C(N^n_P)| \) and we would be done.

Let \( \sigma \) be a maximal bad simplex of dimension \( k \) and with \( k' \) many distinct isotopy classes of embeddings of \( P' \). \( F|_{Lk\sigma} : Lk_{D^{j+1}}\sigma \to T^C(N^n_P - F(\sigma)) \), where the embeddings are disjoint because of maximality. Note that \( Lk_{D^{j+1}}\sigma \approx S^{j-k-1} \) because \( \sigma \) is not contained in \( \partial D^{j+1} \), as \( F(\partial D^{j+1}) \) would not be contained in \( T^C(N^n_P) \) otherwise. \( N^n_P - F(\sigma) \) is \( N^n_{n-k'} \) because the embeddings that are isotopic are nested, so their complement is the complement of the biggest embedding. By induction, \( F \) can be extended to \( \tilde{F} : D^{j-k} \to T^C(N^n_{n-k'}) \) because \( |T^C(N^n_{n-k'})| \) has connectivity \( n - k' - 2 > n - k - 2 \geq j - k \). We can replace \( F \) at \( \sigma \) by \( F|_{Lk\sigma} \ast \tilde{F} \) to get rid of 1 bad simplex without creating any new bad simplex. This modifies \( F \) in the interior of \( \sigma \) and repeating the process for all maximal bad simplexes produces the desired extension of \( f \).

The wCM dimension follows because \( Lk_{T^C(N^n_P)}(f_0, \ldots, f_i) \) is \( T^C(N^n_{n-i-1}) \), which is \( (n - i - 3) \)-connected, as required.

**Definition 52.** For \( P \neq S^1 \times S^2 \), let \( T^T(N^n_P) \) be the simplicial complex where the set of \((i + 1)\)-simplexes consists of a set of embeddings \((f_0, \ldots, f_i)\) such that for some permutation \( \sigma \in \Sigma_{i+1} \), \((f_{\sigma(0)}, \ldots, f_{\sigma(i)})\) is an \((i + 1)\)-simplex in \( Emb_i(T \vee P', [0, \infty) \times \mathbb{R}^\infty) \).

Let \( T^T(N^n_{S^1 \times S^2}) \) be the simplicial complex where the set of \( i \)-simplexes consists
of \( i + 1 \) thickened tethered sphere embeddings \( \langle f_0, \ldots, f_i \rangle \), where the core parts of the embeddings form a disjoint, nonseparating system. By core, we mean the slice \( S^2 \times \{ \frac{1}{2} \} \subseteq S^2 \times I \) along with the entire tether \( I \times D^2 \). We require the same boundary condition as in \( \text{Emb}_*(T + S^2; [0, \infty) \times \mathbb{R}^\infty) \), that \( f_i \circ r_0(s, x, y) = \phi^{-1}_{N_{H_0}}(s, r_i x + t_i, y) \) and \( f_i \circ r_1(s, x, y) = \phi^{-1}_{N_{H_0}}(s, r_i x + t_i, r_i y + C_y) \) for some \( t_i \in \mathbb{R} \) and \( r_i \in \left( 0, \frac{C_y}{2} \right) \).

We now work simultaneously with both cases \( P = S^1 \times S^2 \) and \( P \neq S^1 \times S^2 \). Let \( \pi : T^T(N^P_n) \to T^C(N^P_n) \) be the simplicial map that restricts a tethered embedding to the part of the embedding without tethers. This map will allow us to relate the connectivity of the two complexes.

We will follow the proof of Theorem 3.6 in [6] concerning the connectivity of a join complex \( \pi : Y \to X \). Recall that \( \pi \) is a join complex if

1. \( \pi \) is surjective,
2. \( \pi \) is injective on individual simplexes, and
3. for each \( p \)-simplex \( \sigma = \langle x_0, \ldots, x_p \rangle \) of \( X \), the subcomplex \( Y(\sigma) \) of \( Y \) consisting of all the \( p \)-simplexes that project to \( \sigma \) is the join \( Y_{x_0}(\sigma) \ast \cdots \ast Y_{x_p}(\sigma) \) of the vertex sets \( Y_{x_i}(\sigma) = Y(\sigma) \cap \pi^{-1}(x_i) \).

The map \( \pi : T^T(N^P_n) \to T^C(N^P_n) \) does not satisfy the last condition because the tether from an element \( Y_{x_i}(\sigma) \) can intersect the tether from an element in \( Y_{x_j}(\sigma) \). But this is a minor issue and we can obtain the desired connectivity in a similar fashion, using what’s known as the Coloring lemma.

**Lemma 53.** (Lemma 3.1 of [6]) Let a triangulation of \( S^k \) be given with its vertices labeled by elements of a set \( E \) having at least \( k + 2 \) elements. Then this labeled triangulation can be extended to a labeled triangulation of \( D^{k+1} \) whose only bad
simplices lie in \( S^k \), and with the triangulation of \( S^k \) as a full subcomplex. The labels on the interior vertices of \( D^{k+1} \) can be chosen to lie in any subset \( E_0 \subset E \) with at least \( k + 2 \) elements.

Here, a bad simplex means that each label of a vertex in the simplex is also the label of another vertex in the simplex.

The approach is to use the embeddings in \( T^C(N_n^P) \) as colorings. To have enough colorings, we will work with a subcomplex of the barycentric subdivision of \( T^C(N_n^P) \).

**Definition 54.** For a simplicial complex \( X \), let \( X_k \) be the simplicial complex where an \( i \)-simplex is a chain of simplex inclusions of \( i + 1 \) simplexes in \( X \) such that the smallest dimension of the simplexes is at least \( k - 1 \). The \( i + 1 \) faces of a simplex in \( X_k \) is obtained by removing a simplex from the chain of simplex inclusions. Let’s denote a simplex in \( X_k \) by \([\sigma_0, \ldots, \sigma_n] \) where \( \sigma_i \) is a simplex in \( \sigma_j \) in \( X \) if \( i < j \).

Note that \( X_1 \) is known as the barycentric subdivision of a simplicial complex, and so \( |X_1| \) is homeomorphic to \( |X| \).

**Proposition 55.** \( |T^C(N_n^P)_k| \) is \((n - 1 - k)\)-connected.

**Proof.** Since \( T^C(N_n^P) \) is wCM of dimension \( n - 1 \), the statement follows by Lemma 3.8 of [6], which says that if \( X \) is wCM of dimension \( n \), then \( |X_m| \) is \((n - m)\)-connected. \( \square \)

**Proposition 56.** \( |T^T(N_n^{S^1 \times S^2})| \) is \( \frac{n-3}{2} \)-connected.

**Proof.** We prove this by induction on \( n \). For \( n = 1 \), \( |T^T(N_n^{S^1 \times S^2})| \) is non-empty because there is a non-separating sphere in the copy of \( S^1 \times S^2 \) and because it
is non-separating, we can tether that sphere to the boundary $N_n$. Given a map $F : S^k \rightarrow |T^T(N^P_n)|$ with $k \leq \frac{n-3}{2}$ and $n \geq 2$, we need to extend $F$ to $F : D^{k+1} \rightarrow |T^T(N^P_n)|$. We can assume that $F$ is a simplicial map by simplicial approximation.

Let $f = \pi \circ F$ and let $f_1 : S^k_1 \rightarrow T^C(N^P_n)_1$ be the barycentrically subdivided map of $f$. First, we will define $g : S^k_1 \rightarrow T^C(N^P_n)_{k+2}$ for some subdivision $S^k_{1'}$ of $S^k_1$.

Given a vertex $[\sigma]$ of $S^k_1$, where $\sigma$ is a $p$-simplex in $S^k$, define $g([\sigma])$ to be $f(\sigma) \ast \tau$, for some simplex $\tau$ in $T^C(N^P_n)$ with dimension big enough that $f(\sigma) \ast \tau$ becomes a $(k+1)$-simplex. This is possible because $Lk(\sigma)$ is $(n-1-p-1)$-connected as $T^C(N^P_n)$ is wCM of dimension $n-1$, where $n-2-p \geq n-2-(\frac{n-3}{2}+1) = \frac{n-3}{2} \geq -1$.

Given a 1-simplex $e = [\sigma_0, \sigma_1]$ of $S^k_1$, let $q_j = \dim \sigma_j$. We can view $g|_{\partial e}$ as $f(\sigma_0) \ast g_e$ where $g_e : \partial e \rightarrow T^C\left(N^P_n - (\bigcup_{i=0}^{q_0}im(e_i))\right)_{k+2-q_0}$, where $\langle e_0, \ldots, e_{q_0} \rangle$ is the simplex $F(\sigma_0)$. By induction, $|T^C\left(N^P_n - (\bigcup_{i=0}^{q_0}im(e_i))\right)_{k+2-q_0}|$ has connectivity

$$n - q_0 - 1 - (k + 2 - q_0) \geq n - k - 3 \geq \frac{n - 3}{2} \geq k,$$

so we can extend $g_e$ to a map $g_e : e \rightarrow T^C\left(N^P_n - (\bigcup_{i=0}^{q_0}im(e_i))\right)_{k+2-q_0}$, which can be assumed to be simplicial under some subdivision of $e$ by simplicial approximation. This allows us to extend $g|_{\partial e}$ to $g|_e := f(\sigma_0) \ast g_e$. Once we finish this procedure for all 1-simplices in $S^k_1$, we can continue this procedure for $i$-simplices $e = [\sigma_0, \ldots, \sigma_i]$, starting from $i = 2$ to $i = k$, because we can view $g|_{\partial e}$ as $f(\sigma_0) \ast g_e$ where $g_e : \partial e \rightarrow T^C\left(N^P_n - (\bigcup_{i=0}^{q_0}im(e_i))\right)_{k+2-q_0}$ because of how we defined $g|_{\partial e}$ in the previous step. We can extend $g|_{\partial e}$ by obtaining $g_e$ as before because $n - q_0 - 1 - (k + 2 - q_0) \geq k$ and define $g|_e := f(\sigma_0) \ast g_e$. This finishes the definition of $g : S^k_{1'} \rightarrow T^C(N^P_n)_{k+2}$. Because $T^C(N^P_n)_{k+2}$ is $(n-1-k-2)$-connected, and $n - k - 3 \geq \frac{n-3}{2} \geq k$, we can extend $g$ to a simplicial map $g : D^{k+1}_{1'} \rightarrow T^C(N^P_n)_{k+2}$.

Next, we define $g'$ for each vertex $v = [\sigma]$ of $D^{k+1}_{1'}$, which specifies how to lift $g([\sigma])$. First, if $v$ is on the boundary of $D^{k+1}_{1'}$, then $v$ is a vertex of $S^k_{1'}$ as $D^{k+1}_{1'}$ restricts to the subdivision $S^k_{1'}$ on the boundary, so $v$ is in the interior of a
unique simplex of $S^1_k$, denoted by $[\sigma_0, \ldots, \sigma_i]$. This means $g(v) = [f(\sigma_0) * \tau]$ for some simplex $\tau$. Lift $f(\sigma_0) * \tau$ by using $F(\sigma_0)$ to lift the $f(\sigma_0)$ part of $f(\sigma_0) * \tau$, which is possible because when we define $g$ in $[\sigma_0, \ldots, \sigma_i]$, we use spheres from $T^C (N^P_n - F(\sigma_0))$. Lift $\tau$ by picking a tether for each embedding in $\tau$, making sure that the tether is disjoint from all the embeddings and disjoint from tethers already defined and satisfies the boundary condition. Call this lift $g'(v)$. For a vertex in the interior of $D^{k+1}_{i'}$, we perform the lift by picking tethers, without any requirement of using $F$. When picking a tether, we can assume that it doesn’t intersect the tethers corresponding to boundary vertices of $v$, and that it doesn’t intersect any of the previously chosen tethers $\bigsqcup_{i=0}^j I_i$ from other vertices in the interior of $D^{k+1}_{i'}$ by making the radius of the tether small enough, since in the limit, it is an arc joining two boundaries in $N^P_n - \bigsqcup_{i=0}^j I_i$. This finishes the construction of $g'$.

We now define a simplicial map $G : D^{k+1}_{i''} \to T^T(N^P_n)$, where $D^{k+1}_{i''}$ will be a subdivision of $D^{k+1}_{i'}$, for each $i$-skeleton of $D^{k+1}_{i'}$ starting from $i = 0$ to $i = k + 1$. For a vertex $v$ in $D^{k+1}_{i'}$, define $G(v)$ to be a vertex from the simplex $g'(v)$. To have $G|_{\partial D^{k+1}}$ be homotopic to $F$ later on, we need to require that $G(v)$ for a vertex $v$ of $S^k_i$, which is in the interior of a unique simplex $[\sigma_0, \ldots, \sigma_i]$ of $S^k_i$ be defined by picking a vertex from the simplex $F(\sigma_0)$ in $g'(v)$. This finishes the definition of $G$ for the 0-skeleton of $D^{k+1}_{i'}$.

For each edge $e = \langle v_0, v_1 \rangle$ of $D^{k+1}_{i'}$, with $g(v_0) \subset g(v_1)$ (i.e. $g(v_0) = [\sigma_0]$ and $g(v_1) = [\sigma_1]$, where $\sigma_0$ is a face of $\sigma_1$), apply the Coloring lemma to $\pi \circ G|_{\partial e} : \partial e \to T^C(N^P_n)$ to get a subdivision of $e$ whose interior vertices are colored by $g(v_0)$ and whose boundary vertices remain the same from $\pi \circ G|_{\partial e}$. Lift the coloring using $g'(v_1)$ to get a map from each vertex in the interior of $e$ to a vertex in $T^T(N^P_n)$. This extends to a simplicial map in the interior of $e$ because $G$ maps all interior vertices
to vertices in the simplex $g'(v_1)$. If one vertex is in the boundary of $e$ and another is in the interior, then the two are lifted differently, one using $g'(v_0)$ and the other using $g'(v_1)$. The tether of the resulting tethered embeddings will not intersect other tethered embeddings because those embeddings without tethers come from vertices in $g(v_0)$, which the tether does avoid, and it avoids the other tethers by construction. The embeddings without tethers will not intersect each other because they are picked from distinct vertices in $g(v_0)$ by the Coloring lemma; note that it is important that they are distinct because coinciding embeddings without tethers can have different tethers, because of the procedure we use to pick tethers for each sphere. Thus $G$ is a well-defined simplicial map for $e$.

Repeat this procedure for the $i$-skeleton in $D_{i+1}^k$ starting from $i = 2$ to $i = k + 1$ as follows: For each $e = \langle v_0, \ldots, v_i \rangle$ in $D_{i+1}^k$, with $g(v_0) \subset \cdots \subset g(v_i)$, apply the Coloring lemma to $\pi \circ G|_{\partial e} : \partial e \to T^C(N^F_n)$ to get a subdivision of $e$ whose interior vertices are colored by $g(v_0)$ and whose boundary vertices remains the same from $\pi \circ G|_{\partial e}$. Lift the coloring using $g'(v_i)$ to get a map from each vertex in the interior of $e$ to a vertex in $T^T(N^F_n)$. The number of colors required by the Coloring lemma is $k + 2$, so using $g(v_0)$ gives enough colors. The argument that $G$ is simplicial in the $i$-simplex $e$ is similar to the $i = 1$ case, where the interior vertices all lie in $g'(v_i)$, so that $G$ is simplicial in the interior. For a simplex with part of the vertices lying in the boundary of $e$, the tethers of all the tethered embeddings will not intersect other embeddings because the embeddings without tethers come from vertices in $g(v_0)$ and the tethers are picked to avoid these spheres, as well as to avoid other tethers; the spheres will not intersect each other because either they are picked from distinct vertices in $g(v_0)$ or they coincide but are coming from vertices in $\partial D_{i+1}^k$ by the Coloring lemma; in the latter case, their tethers coincide as well by how we define $G$ on the boundary. Thus $G$ is a well-defined simplicial
To see that $G|_{S^k_i}$ is homotopic to $F$, note that the boundary of $D^{k+1}$ is $S^k_i$, with no extra subdivision when using the Coloring lemma. Each vertex $v$ that is in $S^k_i$ is contained in the interior of a unique simplex $[\sigma_0, \ldots, \sigma_i]$ in $S^k_i$ and is mapped by $G$ to a vertex $F(w_v)$ for some vertex $w_v$ in $\sigma_0$. The homotopy that keeps $G$ simplicial but moves each vertex $v$ in $S^k_i$ linearly to $w_v$ in $S^k$ (the triangulated sphere before barycentric subdivision is applied) would give the desired homotopy.

We relate the connectivity of $|T^T(N^{S^1 \times S^2})|$ to that of $|Emb_*(\overline{I} + S^2, N^{S^1 \times S^2})|$. We will use two intermediate semi-simplicial sets to go between these two spaces.

**Definition 57.** For $P \neq S^1 \times S^2$, let $E_*(N^P)$ be an abbreviation for $Emb_*(\overline{I} \vee P', N^P)$. Let $E_*(N^{S^1 \times S^2})$ be the semi-simplicial space $Emb_*(\overline{I} + S^2, N^{S^1 \times S^2})$, but only the core part of $\overline{I} + S^2$ is considered in the requirement that the system is disjoint and nonseparating. To include the case $Emb_*(\overline{I} + S^2, N^{S^1 \times S^2}, \partial_0, \partial_1)$, we use $E_*\left(N^{S^1 \times S^2}, \partial_0, \partial_1\right)$ to denote the semi-simplicial space $Emb_*(\overline{I} + S^2, N^{S^1 \times S^2}, \partial_0, \partial_1)$ but loosening the requirement of the system being disjoint and nonseparating to apply for only the core part of $\overline{I} + S^2$.

In all 3 cases, let $E^\delta_*(N^P)$ be the semi-simplicial space $E_*(N^P)$, but the space of $i$-simplexes is topologized as a discrete set for all $i$.

**Proposition 58.** $|E_*(N^{S^1 \times S^2})|$ is weakly equivalent to $|Emb_*(\overline{I} + S^2, N^{S^1 \times S^2})|$. 

**Proof.** Given a map $f : (D^k, \partial D^k) \to \left(|E_*(N^{S^1 \times S^2})|, |Emb_*(\overline{I} + S^2, N^{S^1 \times S^2})|\right)$, we can modify the embeddings by shrinking the $S^2 \times I$ part to $S^2 \times [\frac{1}{2} - w(t)(1 - r), \frac{1}{2} + w(t)(1 - r)]$, where $r$ is the distance from the origin of the unit disk $D^k$ and $w$ is a function with $w(0) := \frac{1}{2}$ and approaches 0 as $t$ approaches $\infty$. After $t$
grows beyond a finite $T$, this map will have image in $|\text{Emb}_*(\overline{T + S^2}, N_n^{S^1 \times S^2})|$, by the compactness of $D^k$.

Rephrasing the connectivity property for the complex $T^T(N_n^P)$ gives:

**Proposition 59.** $|E_*^\delta(N_n^P)|$ is $(\frac{n-3}{2})$-connected.

**Proof.** For each simplex $(f_0, \ldots, f_i)$ in $\text{Emb}_\delta^\ast(\overline{T + S^2}, N_n^{S^1 \times S^2})$, we have the simplex $(f_0, \ldots, f_i)$ in $T^T(N_n^{S^1 \times S^2})$ and for each simplex $(f_0, \ldots, f_i)$ in $T^T(N_n^{S^1 \times S^2})$, there is a unique permutation $\rho$ so that the positive coordinate of the tethers at $\partial N$ is in ascending order of $(f_\rho(0), \ldots, f_\rho(i))$. This map of simplexes induces a map between the geometric realization of the 2 spaces, mapping $\{(f_0, \ldots, f_i)\} \times \Delta^i$ to $\{(f_0, \ldots, f_i)\} \times \Delta^i$ via the identity; this map commutes with the face maps for the 2 spaces, and gives a well-defined homeomorphism. So by Proposition 56, $|\text{Emb}_\delta^\ast(\overline{T + S^2}, N_n^{S^1 \times S^2})|$ is $\frac{n-3}{2}$-connected.

Similarly for $P \neq S^1 \times S^2$, $|E_*^\delta(N_n^P)|$ is homeomorphic to $|T^T(N_n^P)|$, so Proposition 56 implies that it is $\frac{n-3}{2}$-connected.

We now show that $|E_*(N_n^P)|$ has the same connectivity as $|E_*^\delta(N_n^P)|$ in a similar fashion as Theorem 4.6 of [2] and this will finish the proof of Lemma 40.

**Definition 60.** Let $D_{*,*}$ be the bi-semisimplicial space defined as follows: let $D_{p,q}$ be the set $E_{p+q+1}^\delta(N_n^P)$ topologized as a subspace of $E_p^\delta(N_n^P) \times E_q(N_n^P)$. The face maps $d^1_i : D_{p,q} \to D_{p-1,q}$ and $d^2_i : D_{p,q} \to D_{p,q-1}$, are defined as forgetting the $(i+1)$-th component and the $(p+1+i+1)$-th component of the tuple in $E_{p+q+1}^\delta(N_n^P)$, respectively.

Let $\epsilon : D_{p,q} \to E_q(N_n^P)$ be the map $(x_p, y_q) \mapsto y_q$ that forgets all the embeddings from the $E_*^\delta(N_n^P)$ part of $D_{p,q}$, and let $\delta : E_{p,q} \to E_p^\delta(N_n^P)$ be $(x_p, y_q) \mapsto x_p$. Define
the geometric realization $|D_{*,*}|$ by $\coprod_{p,q \in \mathbb{N}} D_{p,q} \times \Delta^p \times \Delta^q / \sim$ where $\sim$ consists of

$$(x_p, y_q, (s_0, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_p), (t_0, \ldots, t_q)) \sim (d^1_i((x_p, y_q)), (s_0, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p), (t_0, \ldots, t_q))$$

and

$$(x_p, y_q, (s_0, \ldots, s_p), (t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_q)) \sim (d^2_i((x_p, y_q)), (s_0, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p), (t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_q)).$$

For a fixed $p$, $D_{p,*}$ is a semi-simplicial space with face maps $d^2_i$. Let $\epsilon_q : |D_{*,q}| \to E_q(N^P_n)$ be defined by

$$(x_p, y_q, (s_0, \ldots, s_p), (t_0, \ldots, t_q)) \mapsto y_q.$$  

$|\epsilon| : |D_{*,*}| \to |E_*(N^P_n)|$ is defined by

$$(x_p, y_q, (s_0, \ldots, s_p), (t_0, \ldots, t_q)) \mapsto (y_q, (t_0, \ldots, t_q)),$$

which is the same as the induced map $|\epsilon_*|$ of the map $\epsilon_q$ over the geometric realization. Similarly, $|\delta| : |D_{*,*}| \to |E^3_*(N^P_n)|$ is defined by

$$(x_p, y_q, (s_0, \ldots, s_p), (t_0, \ldots, t_q)) \mapsto (x_p, (s_0, \ldots, s_p)).$$

Define $\iota_* : E^3_*(N^P_n) \to E_*(N^P_n)$ by having $\iota_p$ be the identity map. We will show that $|\iota_*|$ is highly connected.

By Lemma 4.8 of [2], we know that $|\iota_*| \circ |\delta|$ is homotopic to $|\epsilon|$; the explicit homotopy $h_t : |D_{*,*}| \to |E_*(N^P_n)|$ is defined by

$$h_t((x_p, y_q, (s_0, \ldots, s_p), (t_0, \ldots, t_q)) = ((t_p(x_p), y_q), ((1-t)s_0, \ldots, (1-t)s_p, tt_0, \ldots, tt_q)).$$

We also need Corollary 2.8 in [2]:
Proposition 61. Let $Y_\ast$ be a semi-simplicial set and $Z$ be a Hausdorff space. Let $X_\ast \subset Y_\ast \times Z$ be a sub-semisimplicial space which in each degree is an open subset. Given $z \in Z$, let $X_\ast(z) \subset Y_\ast$ be the semisimplicial subset defined by $X_\ast \cap (Y_\ast \times \{z\}) = X_\ast(z) \times \{z\}$. If $|X_\ast(z)|$ is $m$-connected for all $z \in Z$, then the map $\pi : |X_\ast| \to Z$, induced from the projection $Y_\ast \times Z \to Z$, is $(m + 1)$-connected.

Now we finish proving Lemma 18 by showing:

Lemma 62. $|E_\ast(N_n^P)|$ is $(\frac{n-3}{2})$-connected.

Proof. We will apply Proposition 61 with $Y_\ast = E_\ast(N_n^P)$, $Z = E_p(N_n^P)$ and $X_\ast = D_{p,\ast}$ for $0 \leq p \leq \dim(E_\ast(N_n^P))$. $Z$ is Hausdorff because an embedding of $\overline{I} + S^2$ (or $\overline{I} \vee P$) is bounded away from any other embedding for some small distance by compactness; $X_q \subset Y_q \times Z$ is open for the same reason. For $z = (f_0, \ldots, f_p)$, $|X_\ast(z)|$ is homeomorphic to $|E_\ast(N_{n-p-1}^P)|$, which is homeomorphic to $|T^T(N_{n-p-1}^P)|$, so it is $\frac{n-p-4}{2}$-connected by Proposition 56. By Proposition 61, $\epsilon_p : |D_{\ast,p}| \to E_p(N_n^P)$ is $\frac{n-p-2}{2}$-connected, and so $|\epsilon| : |D_{\ast,\ast}| \to |E_\ast(N_n^P)|$ is $\frac{n-2}{2}$-connected by Proposition 2.6 in [2], which we state below in Lemma 63. Thus, because $|\iota_\ast| \circ |\delta|$ is homotopic to $|\epsilon|$, $|D_{\ast,\ast}| \xrightarrow{|\delta|} |E_\ast(N_n^P)| \xrightarrow{|\iota_\ast|} |E_\ast(N_n^P)|$ is $\frac{n-2}{2}$-connected as well, so $\pi_i(|\iota_\ast| \circ |\delta|)$ is surjective for $i \leq \frac{n-2}{2}$. Since $|E_\ast(N_n^P)|$ is $\frac{n-3}{2}$-connected, $\pi_i(|\iota_\ast| \circ |\delta|)$ is the trivial map for $i \leq \frac{n-3}{2}$, so $|E_\ast(N_n^P)|$ is $\frac{n-3}{2}$-connected.

To finish the proof of Lemma 62, we prove the following lemma, with an argument suggested by Randal-Williams.

Lemma 63. Let $f_\ast : X_\ast \to Y_\ast$ be a map of semisimplicial spaces. If $f_p : X_p \to Y_p$ is $(n-p)$-connected for $p = 0, \ldots, n + 1$, then $|f_\ast| : |X_\ast| \to |Y_\ast|$ is $n$-connected.
Proof. To prove this, we will prove something stronger, that \(|f_{q}^{\leq q}| : |X_{\leq q}| \rightarrow |Y_{\leq q}|\) is \(n\)-connected for any \(q\), by inducting on \(q\), where \(X_{\leq q}\) is the semi-simplicial subspace defined by

\[
X_{\leq q}^{i} := \begin{cases} 
X_{i} & \text{if } i \leq q \\
\emptyset & \text{otherwise}
\end{cases}
\]

with face maps being those of \(X_{*}\). \(f_{i}^{\leq q}\) is defined as \(f_{i}\) for \(i \leq q\).

For \(q = 0\), this is immediate, because \(|f_{*}^{\leq 0}|\) is the same as \(f_{0}\), which is \(n\)-connected by assumption. We can assume that for all \(p\), \(f_{p}\) is an inclusion for \(f\) because one can replace \(Y_{*}\) with the mapping cylinder \(M_{f} = X_{*} \times I \cup |Y_{*}|(x,1) \sim f(x)\), with face maps given by \(d_{i}^{M_{p}} := d_{i}^{X_{*}} \times id_{I}\), because \(|Y_{*}^{\leq q}| \hookrightarrow |M_{f}^{\leq q}|\) is a weak homotopy equivalence; it is a weak homotopy equivalence because the realization of the levelwise weak homotopy equivalence is a weak homotopy equivalence (see the proof of Lemma 39). Next, we can assume that \(f_{q}\) is a relative CW complex because one can replace \((Y_{q}, X_{q})\) by its CW approximation \(\iota : (Y_{q}^{\text{CW}}, X_{q}^{\text{CW}}) \rightarrow (Y_{q}, X_{q})\) and define the face map by \(d_{i}^{Y_{q}^{\text{CW}}} := d_{i}^{Y_{q}} \circ \iota\) for \(Y_{q}^{\text{CW}}\), while all other face maps remain the same, to obtain \(X_{*}^{\text{CW}}\) and \(Y_{*}^{\text{CW}}\); because \((|Y_{*}^{\leq q}|, |Y_{*}^{\text{CW}}|)\) and \((|X_{*}^{\leq q}|, |X_{*}^{\text{CW}}|)\) are weakly homotopy equivalent, it is enough to see that \((|Y_{*}^{\text{CW}}|, |X_{*}^{\text{CW}}|)\) is \(n\)-connected to obtain the result. Thus, we can assume that \((Y_{q}, X_{q})\) is a relative CW complex. This line of reasoning inductively shows that we can assume \((|Y_{*}^{\leq q-1}|, |X_{*}^{\leq q-1}|)\) to be a relative CW complex, as well. In addition, we can assume that \(Y_{q} - X_{q}\) consists of cells of dimension greater than \(n - q\); \(f_{q}\) is \((n - q)\)-connected implies that there exist \((Z_{q}, X_{q})\) and a homotopy equivalence \(h_{q} : (Z_{q}, X_{q}) \rightarrow (Y_{q}, X_{q})\) rel \(X_{q}\) by Proposition 4.15 of [5] such that the cells of \(Z_{q} - X_{q}\) have dimension greater
than $n - q$. So the semi-simplicial space $Z_*$ defined by

$$Z_i = \begin{cases} 
Y_i & \text{if } i < q \\
Z_q & \text{if } i = q \\
\emptyset & \text{otherwise}
\end{cases}$$

with face maps defined by $d_i^{Z_q} := d_i^{Y_q} \circ g_q$, has geometric realization $|Z_*|$ that is weakly homotopy equivalent to $|Y_{\leq q}|$. $(|Y_{\leq q}|, |X_{\leq q}|)$ involves gluing cells of the form $e_z \times \Delta^q$ to $(|Y_{\leq q-1}|, |X_{\leq q-1}|)$, where $e_z$ is a cell in the CW complex $Y_q - X_q$. $e_z \times \Delta^q$ is a cell of dimension greater than $n - q + q$, and $(|Y_{\leq q-1}|, |X_{\leq q-1}|)$ is $n$-connected by induction, so $(|Y_{\leq q}|, |X_{\leq q}|)$ is $n$-connected as well by Corollary 4.12 of [3].
CHAPTER 8
CLOSING THE LAST BOUNDARY

The techniques so far depend on $N$ having a boundary component so that we can tether embeddings to it to build various semi-simplicial spaces. For the case $N_n^{S^1 \times S^2}$, we can show that the homology $H_q(B\text{Diff}(N_n^{S^1 \times S^2}, \partial N))$ is the same for $N$ without boundary as that with $N$ replaced by $N - D^3$ in the same range $q \leq \frac{n-4}{2}$, via a different approach, from Section 10 of [9].

Suppose that $N$ has possibly empty boundary. Let us denote $N_{(i+1)} := N - (\bigcup_{k=0}^i D^3)$. The idea is to embed $i + 1$ disjoint disks in $N_n^{S^1 \times S^2}$, and varying $i \in \mathbb{N}$ gives a semi-simplicial space that augments $B\text{Diff}(N_n^{S^1 \times S^2}, \partial N)$ where each term is a classifying space of $\text{Diff}\left((N_{(i+1)})_n^{S^1 \times S^2}, \partial (N_{(i+1)})\right)$. Then, we can compare it with a similar semi-simplicial space that augments $B\text{Diff}\left((N_{(1)})_n^{S^1 \times S^2}, \partial (N_{(1)})\right)$, using $s_{D^3}$, which does induce homology isomorphisms in a certain range.

**Definition 64.** Let $D_*(M)$ be the semi-simplicial space where the space of $k$-simplexes is

$$D_k(M) := \text{Emb}^\partial(M) \times \text{Diff}(M, \partial M) \times \text{Emb}(\bigcup_{i=0}^k D^3, \text{int}(M)),$$

and the elements will be denoted by $[f, (g_0, \ldots, g_k)]$. The face maps are defined by $d_i : [f, (g_0, \ldots, g_k)] \mapsto [f, (g_0, \ldots, \hat{g}_i, \ldots, g_k)]$. It is an augmented semi-simplicial space with $D_{-1}(M) := \text{Emb}^\partial(M)/\text{Diff}(M, \partial M)$ and the augmentation defined by $\epsilon_0 : [f, (g_0)] \mapsto \text{im}(f)$.

As before, the space $D_0(M)$ (and similarly for $D_k(M)$) can be thought of as

$$D'_0 := \{(\text{im}(f), h) \in \text{Sub}^\partial(M, [0, \infty) \times \mathbb{R}^\infty) \times \text{Emb}(D^3, \text{int}(M)) : \text{im}(f) \supset \text{im}(h)\},$$

which we will use implicitly.
**Proposition 65.** $D_k(M)$ is a classifying space of $\text{Diff}(M_{(k+1)}, \partial(M_{(k+1)}))$.

**Proof.** We start with $k = 0$. Let $o : D_0(M) \to \text{Emb}(D^3, (0, \infty) \times \mathbb{R}^\infty)$ be defined by $[f, g] \mapsto f \circ g$. This is a Serre fibration by isotopy extension. The fiber $o^{-1}(f \circ g)$ viewed as a subspace of $D_0'$ is $\text{Sub}^\partial(M - g(D^3), [0, \infty) \times \mathbb{R}^\infty - f \circ g(D^3)) \times \{f \circ g\}$. $\text{Emb}^\partial(M - g(D^3), [0, \infty) \times \mathbb{R}^\infty - f \circ g(D^3))$ is contractible by Proposition 11, so the fiber is a classifying space of $\text{Diff}^\partial(M_{(1)}, \partial(M_{(1)}))$.

For general $k$, the argument can be repeated using $[f, (g_0, \ldots, g_k)] \mapsto f \circ \left( \bigcup_{i=0}^k g_i \right)$.

We construct a more explicit weak homotopy equivalence.

**Definition 66.** Let $a_k^D : \text{Sub}^\partial(M_{(k+1)}) \to D_k(M)$ be defined by

$$\iota(M_{(k+1)}) \mapsto ((\iota + \tilde{e}_1)(M_{(k+1)}) \cup \iota'(I \times \partial M) \cup (\bigcup_{i=0}^k g_i^D(D^3)), (g_0^D, \ldots, g_k^D)),$$

where $g_i^D$ is a fixed embedding of $D^3$ in $(0, 1] \times \mathbb{R}^\infty$ that matches up with the embedding $\iota + \tilde{e}_1$ of the $(i+1)$-th boundary sphere of $M_{(k+1)}$ as imposed by $\text{Sub}^\partial(M_{(k+1)})$, and where $\iota'$ extends the embedding of $(\iota + \tilde{e}_1)|_{\partial M}$ in $\{1\} \times \mathbb{R}^\infty$ to an embedding $I \times \partial M \to [0, 1] \times \mathbb{R}^\infty$.

**Proposition 67.** $a_k^D : \text{Sub}^\partial(M_{(k+1)}) \to D_k(M)$ is a weak homotopy equivalence.

**Proof.** For the $k = 0$ case, we need to homotope $f : (D^j, \partial D^j) \to (D_0(M), \text{im}(a_0^D))$ to have image in $\text{im}(a_0^D)$. For $f(x) = (\iota_x(M), (h_x))$, where $h_x : D^3 \to (0, \infty) \times \mathbb{R}^\infty$ is an embedding. Recall that we defined $o(\iota_x(M), (h_x)) := h_x$, and $o \circ f : (D^j, \partial D^j) \to (\text{Emb}(D^3, (0, \infty) \times \mathbb{R}^\infty - \iota'(I \times \partial M)), \{g_0^D\})$ can be deformed to have image in $\{g_0^D\}$ because $\text{Emb}(D^3, (0, \infty) \times \mathbb{R}^\infty - \iota'(I \times \partial M))$ is weakly contractible. By isotopy extension, this gives a homotopy of $\iota_x(M)$ as well, which is fixed if $x \in \partial D^j$, and
this gives us the desired homotopy of $f$. For general $k$, the same argument applies because $Emb(\coprod_{i=0}^{k} D^3, (0, \infty) \times \mathbb{R}^\infty - \iota'(I \times \partial M))$ is weakly contractible.

**Definition 68.** Let $\gamma : Sub^\beta(M_{(k+1)}) \rightarrow Sub^\beta(M_{(k)})$ be defined by $\iota(M_{(k+1)}) \mapsto (\iota + e_1)(M_{(k+1)}) \cup \iota'(I \times \partial M) \cup g_k^D(D^3)$.

**Proposition 69.** The following diagram

$$
\begin{array}{ccc}
Sub^\beta(M_{(k+1)}) & \xrightarrow{\gamma} & Sub^\beta(M_{(k)}) \\
\downarrow a_k^D & & \downarrow a_{k-1}^D \\
D_k(M) & \xrightarrow{d_k} & D_{k-1}(M)
\end{array}
$$

commutes.

**Proof.** By definition, $d_k \circ a_k^D(\iota(M_{(k+1)}))$ is

$$d_k((\iota + e_1)(M_{(k+1)}) \cup \iota'(I \times \partial M) \cup (\cup_{i=0}^{k} g_i^D(D^3)), (g_0^D, \ldots, g_k^D)))$$

$$= (\iota + e_1)(M_{(k+1)}) \cup \iota'(I \times \partial M) \cup (\cup_{i=0}^{k} g_i^D(D^3)), (g_0^D, \ldots, g_k^D)),
$$

while $a_{k-1}^D \circ \gamma(\iota(M_{(k+1)}))$ is

$$a_{k-1}^D ((\iota + e_1)(M_{(k+1)}) \cup \iota'(I \times \partial M) \cup g_k^D(D^3))$$

$$= ((\iota + e_1)(M_{(k+1)}) \cup \iota'(I \times \partial M) \cup (\cup_{i=0}^{k} g_i^D(D^3)), (g_0^D, \ldots, g_{k-1}^D)).$$

As both yields the same expression, we are done.

The connectivity property needed for the spectral sequence argument is the following:

**Proposition 70.** $|Emb(\coprod_{i=0}^{\infty} D^3, \text{int}(M))|$ is contractible.
Proof. A proof is given in Proposition 10.1 in [9]. Here is an alternate proof. We need to deform the space \( |\text{Emb}(\coprod_{i=0}^{k} D^3, \text{int}(M))| \) to some point \((1 \cdot e_N)\) specified as follows. Put a metric on \( M \) and fix an embedding \( e_S \) of a disk with center \( p_0 \) of radius \( \epsilon_0 \) for some \( p_0 \in M \) and \( \epsilon_0 > 0 \) such that the disk with center \( p_0 \) of radius \( 2\epsilon_0 \) does not cover the entire \( M \). Let \( e_N \) be a disk embedding that is disjoint from the disk with center \( p_0 \) of radius \( 2\epsilon_0 \).

Given an element \((w_0 \cdot e_0, \ldots, w_k \cdot e_k)\), where \( \sum_{i=0}^{k} w_i = 1 \) and \( e_0, \ldots, e_k \) are disjoint disk embeddings, we describe how the deformation of \( |D_*(M)| \) moves this element. First, shrink each \( e_i \) at the same rate, by restricting the domain of \( e_i : D^3 \to M \) to \( t \cdot D^3 \) via a decrease of \( t > 0 \) starting from \( t = 1 \), until all of them have diameter less than or equal to \( \epsilon_0 \). We denote the modified embedding of \( e_i \) by \( e_i \) again.

We now transfer weights from embeddings that are \( \epsilon_0 \)-far from \( p_0 \) to those that are \( \frac{\epsilon_0}{2} \)-close to \( p_0 \); an embedding \( \frac{\epsilon_0}{2} \)-close to \( p_0 \) may not exist, in which case we would need to create it as follows. Let \( \delta_i \) be the distance between the embedding \( e_i \) and the point \( p_0 \), and let \( \delta := \min_{i \in \{0, \ldots, k\}} \delta_i \). Let \( e'_S \) be a disk embedding with center \( p_0 \) and radius \( \min(\frac{\delta}{2}, \epsilon_0) \) obtained by scaling \( e_S \). Starting with \( (w_0 \cdot e_0, \ldots, w_k \cdot e_k, 0 \cdot e'_S) \), transfer the fraction \( \min(1, \frac{\delta - \epsilon_0}{\epsilon_0}) \) of the weight of the embeddings \( e_i \) with \( 2\delta_i \geq \epsilon_0 \) to \( e'_S \) and all the embeddings \( e_j \) with \( 2\delta_j \leq \epsilon_0 \), where the transfer to \( e'_S \) is weighted by \( \frac{\min(\delta - C, 0)}{\min(\delta - C, 0) + \sum_{i=0}^{k} \max(\epsilon_0 - 2\delta_i, 0)} \) and the transfer to each of the other eligible \( e_j \) is weighted by \( \frac{\epsilon_0 - 2\delta_j}{\min(\delta - C, 0) + \sum_{i=0}^{k} \max(\epsilon_0 - 2\delta_i, 0)} \), where \( C \in (0, \epsilon_0) \) is some small constant, say \( C = \frac{\epsilon_0}{2} \). This offset by \( C \) guarantees that \( e'_S \) has weight 0 if \( \delta \leq C \). The deformation is continuous: for a fixed \( j \), as \( \delta_j \) approaches \( \frac{\epsilon_0}{2} \) from either side, the resulting weight of \( e_j \) approaches being unmodified, and as \( \delta_j \) approaches \( \epsilon_0 \), the resulting weight \( e_j \) approaches 0. Thus,
the resulting embeddings are disjoint from \( \text{im}(e'_S) \) because their diameter is less than \( \epsilon_0 \). Finally, we shift the weight of all the resulting embeddings to \( e_N \), which is well-defined because they are all disjoint from \( e_N \). This finishes the deformation of \( |\text{Emb}(\coprod_{i=0}^n D^3, \text{int}(M))| \) to \( \{(1 \cdot e_N)\} \).

**Proposition 71.** \( H_q(D_{-1}(M), |D_s(M)|) = 0 \) for any \( q \in \mathbb{N} \).

**Proof.** \( \epsilon_k : D_k(M) \rightarrow D_{-1}(M) \) is a Serre fibration, as we have shown in Proposition 38, with fiber \( \text{Emb}(\coprod_{i=0}^k D^3, \text{int}(M)) \), so by Proposition 39, \( |\epsilon_*| \) is a quasifibration, which is \( q \)-connected for all \( q \in \mathbb{N} \) because \( |\text{Emb}(\coprod_{i=0}^n D^3, \text{int}(M))| \) is contractible by Proposition 70. \( \square \)

**Definition 72.** Let \( \gamma_* : D_*(M_{(1)}) \rightarrow D_*(M) \) be the augmented semi-simplicial map that fills in the extra boundary sphere. Specifically, \( \gamma_k : (\text{im}(f), (g_0, \ldots, g_k)) \mapsto (\text{im}((f + \vec{e}_1) \cup f'), (g_0, \ldots, g_k)) \), where \( f' \) is a fixed embedding of \( D^3 \) in \([\frac{1}{2}, 1] \times \mathbb{R}^\infty \) with the boundary matching up with the extra boundary sphere of \( M_{(1)} \) in \( \{1\} \times \mathbb{R}^\infty \), and some extra embeddings that extend other boundary components of \( M_{(1)} \) across \([0, 1] \times \mathbb{R}^\infty \).

We will use the relative version of the spectral sequence for augmented semi-simplicial spaces found in Section 2 of [RW]: Let \( f_* : (X_* \xrightarrow{f} X_{-1}) \rightarrow (Y_* \xrightarrow{f} Y_{-1}) \) be a map of augmented semi-simplicial spaces. Then a spectral sequence is defined for \( s \geq -1 \),

\[
E_{s,t}^1 = H_{s+t}(X_s, X_{s-t}) \Rightarrow H_{s+t+1}(C_{|\epsilon_Y|}, C_{|\epsilon_X|}),
\]

where \( C_{|\epsilon|} \) is the mapping cone of \( |\epsilon| \).

**Theorem 73.** Suppose that \( N \) has possibly empty boundary. Then

\[
H_q(\gamma) : H_q(B\text{Diff}((N(1))_{(n)}^{S^1 \times S^2}, \partial(N(1)))) \rightarrow H_q(B\text{Diff}(N_{(n)}^{S^1 \times S^2}, \partial N))
\]

is an isomorphism for \( q \leq \frac{n-2}{2} \).
Proof. We will induct on \( q \). For \( q = 0 \), this is true because \( \text{BDiff}(N^{S^1 \times S^2}_n, \partial N) \) and \( \text{BDiff}((N(1))^{S^1 \times S^2}_n, \partial(N(1))) \) are connected. For \( q = q' \), we apply the relative spectral sequence to \( \gamma_* : D_*((N(1))^{S^1 \times S^2}_n) \to D_*((N(1))^{S^1 \times S^2}_n) \), which for \( p \geq -1 \) gives

\[
E^1_{p,q} = H_q(D_p(N^{S^1 \times S^2}_n), D_p((N(1))^{S^1 \times S^2}_n)) \Rightarrow 0,
\]

because \( H_{p+q+1}(D_p(N^{S^1 \times S^2}_n)) \) and \( H_{p+q+1}(D_p((N(1))^{S^1 \times S^2}_n)) \) are trivial by Proposition 71.

We need to show that \( H_{q'}(D_{-1}(N^{S^1 \times S^2}_n), D_{-1}((N(1))^{S^1 \times S^2}_n)) = 0 \), the term corresponding to \( E^1_{-1,q'} \), so suppose instead that \( E^1_{-1,q'} \neq 0 \). We claim that \( E^r_{-1,q'-r+1} = 0 \) for \( r \geq 1 \), so \( E^r_{-1,q'} \) does not change from \( r = 1 \), which contradicts \( E^\infty_{-1,q'} = 0 \), and we would be done. To see that \( E^r_{-1,q'-r+1} = 0 \), it suffices to show that \( E^1_{r-1,q'-r+1} = 0 \). \( E^1_{r-1,q'-r+1} = H_{q'-r+1}(D_{r-1}(N^{S^1 \times S^2}_n), D_{r-1}((N(1))^{S^1 \times S^2}_n)) \) is \( H_{q'-r+1}(\text{BDiff}((N(r))^{S^1 \times S^2}_n, \partial(N(r))), \text{BDiff}((N(r+1))^{S^1 \times S^2}_n, \partial(N(r+1)))) \) by Proposition 69, which is 0 by Corollary 28 because \( q' - r + 1 \leq q' \leq \frac{n-2}{2} \). \(\square\)
BIBLIOGRAPHY


http://www.math.cornell.edu/~hatcher/Papers/luminytalk.pdf


