A QUADRATIC CONE RELAXATION-BASED ALGORITHM FOR LINEAR PROGRAMMING

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Mutiara Sondjaja

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Mutiara Sondjaja, Ph.D.
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We present and analyze a linear programming (LP) algorithm based on replacing the non-negative orthant with larger quadratic cones. For each quadratic relaxation that has an optimal solution, there naturally arises a parameterized family of quadratic cones for which the optimal solutions create a path leading to the LP optimal solution. We show that this path can be followed efficiently, thereby resulting in an algorithm whose complexity matches the best bounds proven for interior-point methods. We then extend the algorithm to semidefinite programming (SDP).

BIOGRAPHICAL SKETCH

Tia Sondjaja was born in January 1986 in Jakarta, Indonesia. She began her graduate studies in the School of Operations Research and Information Engineering at Cornell University in August 2008, after completing her undergraduate degree in Mathematics at Harvey Mudd College in May 2008. She earned her M.S in Operations Research in January 2013 and completes her Ph.D in August 2014.

For my parents

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CHAPTER 1

INTRODUCTION

Linear programming is the problem of minimizing (or maximizing) a linear objective function over a linearly-constrained subset of $\mathbb{R}^n$. In 1947, George Dantzig formulated the simplex method, the first practical algorithm for solving linear programming problem [5]. Although the algorithm performs well in practice, the simplex method is not a polynomial-time algorithm. That is, there exist problem instances (see the Klee-Minty cube, [15]) on which the simplex method runs for a number of iterations that is exponential in the number of decision variables, for certain pivoting rules and starting points.

In [14], Khachiyan proved that the ellipsoid method—which had been developed and studied by Shor, Nemirovski, and Yudin in the context of general convex optimization problems—has a worst-case iteration-complexity bound that is polynomial in the bitlength of the input size of the problem instance, when applied to linear programming problems. However, its performance in practice is often inferior to that of the simplex method.

The next breakthrough in linear programming algorithms is due to Karmarkar [12], who formulated a polynomial-time interior-point algorithm for linear programming. It is an "interior-point" algorithm because, in contrast to the simplex method whose iterates are vertices of the linear program's polyhedral feasible region, the iterates of Karmarkar's algorithm stay in the interior of the feasible region. Karmarkar's algorithm performs well in practice, in addition to having a good iteration-complexity bound.

Roughly speaking, Karmarkar uses projective scalings in order to transform the linear programming feasible region such that the current iterate lies “in the center” of the feasible region. This allows the algorithm to take a large enough step towards the boundary of the scaled region, resulting in good progress towards reaching an optimal solution.

The work of Barnes and of Vanderbei et al. (see [2] and [36]) explores a simplified version of Karmarkar’s algorithm which dispenses with the projective aspect of the scalings. Instead, they use affine scalings that achieve the same goal of transforming the feasible region so that each iterate lies “in the center” of the transformed feasible region. It was later discovered that Dikin had formulated the same algorithm [7, 8] in 1967 (with minor differences, such as in nondegeneracy assumptions and choice of step length).

Despite its appealing simplicity and good performance in practice (see [35]), the affine-scaling algorithm in the original form formulated by Dikin has not been proven to have polynomial complexity bounds. In fact, it is widely believed that its worst case complexity is exponential in the number of decision variables (see [22]). However, variants of Dikin’s affine-scaling algorithm have been studied and proven to have polynomial-time convergence. In particular, primal-dual algorithms such as [25], [10], and [30] take Dikin’s original idea of optimizing the objective function over an inscribed ellipsoid around the current iterate. However, the ellipsoids (or the ellipsoid-like regions) lie in a space that simultaneously takes into account the primal and the dual iterate.

The cone affine-scaling algorithm studied by Sturm and Zhang in [30], in particular, is interesting because instead of using an inscribed ellipsoid, it uses a conic section, the intersection of a cone with an affine subspace that is inscribed in the linear program’s feasible region. Furthermore, this algorithm generalizes to semidefinite programming in a relatively straightforward manner (see [3]). The idea of considering conic sections had also been used by Megiddo in [21] and by Todd in [31].

In [4], Chua studies a primal-dual algorithm for semidefinite programming which could also be seen as a cone affine-scaling algorithm. The algorithm alternates between solving the primal and the dual problem over an inscribed conic section. This conic section is obtained by taking a quadratic cone that is contained in the positive semidefinite cone and intersecting it with the affine constraints of the respective problem. Chua extends this algorithm to apply to cone optimization problems over symmetric cones.

In the work that we present in this thesis, we develop a linear programming algorithm that is based on solving a sequence of quadratic cone-based relaxations of the linear programming feasible region. This algorithm can be seen as a primal affine-scaling algorithm, where the "ellipsoid-like" region which provides this relaxed feasible region is the intersection of a quadratic cone with an affine linear space. However, in contrast to most affine-scaling algorithms which deal with regions inscribed in the linear programming feasible region, our algorithm solves a sequence of relaxations, solving problems over regions that circumscribe the original feasible region of the linear programming problem. We then extend the algorithm and the analysis to semidefinite programming.

The structure of this thesis

In Chapter 2, we introduce basic ideas and results from the theory of linear programming and conic optimization that are pertinent to the development and analysis of our algorithm.

Chapter 4 introduces the reader to the class of quadratic cones that are used in the algorithm. In Chapter 5, we introduce the algorithm and carry out the analysis for the simplest case of linear programming, even though the analysis for the semidefinite programming case is similar. We do this because it is easier to build a more concrete intuition that is needed in the analysis of our algorithm in the context of linear programming. On the other hand, linear programming is sufficiently rich in structure, such that when we extend to semidefinite programming in Chapter 6, the extension follows relatively easily.

CHAPTER 2

BACKGROUND

2.1 Linear programming

Linear programming is the constrained optimization problem of minimizing a linear functional over a region that is described by linear equality and inequality constraints. We can write any linear programming problem ("linear program") in the following standard form:

$$\begin{aligned} \min \quad & \langle c, x \rangle \\ s.t. \quad & Ax = b \qquad\qquad (LP) \\ & x \in \mathbb{R}^n_+, \end{aligned}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m\times n}$ (an $m \times n$ matrix with real entries), with $m \leq n$. We use $\mathbb{R}^n_+$ ($\mathbb{R}^n_{++}$) to denote the nonnegative (positive) orthant: the set of vectors in $\mathbb{R}^n$ whose coordinates are nonnegative (positive).

We use $\langle \cdot, \cdot \rangle$ to denote an inner product on $\mathbb{R}^n$, which could be chosen to be the dot product or another inner product. For simplicity and concreteness, however, in the present section, we choose $\langle \cdot, \cdot \rangle$ to be the dot product.

The constraint "$Ax = b$" describes an affine linear subspace whose dimension is ($n-$the rank of A) = the dimension of the null space of A. We will assume that A is full-rank. (Otherwise, we can remove some of the rows of A and the corresponding rows of b to obtain an equivalent system of linear equality constraints $\hat{A}x = \hat{b}$ with $\hat{A}$ having linearly independent rows, while describing the same affine linear space as that described by the system $Ax = b$.) The constraint "$x \in \mathbb{R}^n_+$" is often referred to as the *nonnegativity constraint*.

Given a linear programming problem in the form (LP), the associated *dual* problem is

$$\begin{array}{ll} \max & b^T y \\ s.t. & A^T y + s = c \\ & s \in \mathbb{R}^n_+. \end{array} \qquad (LP)^*$$

We write $b^T y$ instead of $\langle b, y \rangle$ for denoting the inner product on $\mathbb{R}^m$, in order to reserve the notation $\langle \cdot, \cdot \rangle$ for the inner product on $\mathbb{R}^n$. In relation to $(LP)^*$, we refer to (LP) as the *primal* problem.

We refer to x, which lies in $\mathbb{R}^n$, and (s, y), which lies in $\mathbb{R}^{n+m}$, as the primal and dual *decision variables*, respectively. We refer to $\langle c, x \rangle$ and $b^T y$ as the primal and dual *objective functions*, respectively.

The set of points x that satisfy the constraints $Ax = b$ and $x \in \mathbb{R}^n_+$ is referred to as the *feasible region* of the primal problem. Similarly, the set of points (s, y) that satisfy the dual constraints is the feasible region of the dual problem.

Let

$$Int := \{x \in \mathbb{R}^n_{++} \mid Ax = b\}$$

and

$$Int^* := \{(s, y) \in \mathbb{R}^n_{++} \times \mathbb{R}^m \mid A^T y + s = c\}.$$

If Int is nonempty or (LP) is infeasible, then Int is the relative interior of the feasible region of (LP). Similarly, if Int^* is nonempty or $(LP)^*$ is infeasible, then Int^* is the relative interior of the feasible region of $(LP)^*$. If Int or Int^* are not empty, we say that the corresponding problem is *strictly feasible* and we call elements of Int or Int^* *strictly feasible solutions*.

When the feasible region is empty, we say that the problem is *infeasible*. We say that a problem is *unbounded* if it is feasible but the objective function to be minimized or maximized does not have a finite minimum or maximum value, respectively. Note that it is possible to have a bounded problem whose feasible region is an unbounded set.

Whenever (LP) (and $(LP)^*$, respectively) is infeasible, we define its optimal value as $+\infty$ ($-\infty$, respectively). Whenever (LP) (and $(LP)^*$, respectively) is unbounded, we define its optimal value as $-\infty$ ($+\infty$, respectively).

Notation

Throughout this thesis, column vectors are denoted by lowercase letters while matrices are denoted by uppercase letters. When we consider a vector and its corresponding diagonal matrix, we denote them using the lowercase and uppercase versions of the same letter. For instance, for $x \in \mathbb{R}^n$, we let $X := \operatorname{diag} x$, the $n \times n$ diagonal matrix whose diagonal entries are given by the vector x. We use $\mathbb{1}$ to denote the vector of all ones in $\mathbb{R}^n$.

We use subscripts to denote the entries of vectors and matrices. For instance, x_i is the ith component of a vector x and X_{ij} is the (i, j) entry of the matrix X.

For vectors $x, y \in \mathbb{R}^n$ and a given inner product $\langle \cdot, \cdot \rangle$, we use $\angle(x, y)$ to denote the angle between x and y with respect to this inner product.

We write $x.y$ to denote the componentwise product of the two vectors and $x./y$ to denote componentwise division. For each integer k, we write x^k to denote the vector whose ith component is the kth power of the ith component of x. For example, $x^2 = (x_1^2, \ldots, x_n^2)^T$, $x^{-1} = (1/x_1, \ldots, 1/x_n)^T$, and $x^{1/2} = \sqrt{x} = (\sqrt{x_1}, \ldots, \sqrt{x_n})^T$. Superscripts on matrices denote the usual matrix powers. For example, $X^2 = XX$, X^{-1} = the inverse of the matrix X, and $X^{-2} = X^{-1}X^{-1}$. If X is a symmetric positive semidefinite matrix, we use $X^{1/2}$ and $X^{-1/2}$ to denote the positive semidefinite square root matrices of X and X^{-1}, respectively. We use superscripts with parentheses (e.g., $x^{(k)}$ or $X^{(k)}$) to denote iterates produced by an algorithm.

For any set S, int S denotes the interior of S while ∂S denotes the boundary of S.

Duality theorems for linear programming

One motivation, or interpretation, of the dual problem is as the problem of finding lower bounds for the primal problem's objective function value. This interpretation is justified by the following fundamental result, commonly known as *weak duality*.

Theorem 2.1 (Weak duality)**.** *For any feasible solution x for (LP) and any feasible solution (s, y) for $(LP)^*$,*

$$\langle c, x\rangle \geq b^T y.$$

That is, the value of any feasible primal solution is always an upper bound for the value of any feasible dual solution.

Proof. Let x and (s, y) be feasible solutions for (LP) and $(LP)^*$, respectively. Since $Ax = b$ and $A^T y = c - s$, then

$$b^T y = (Ax)^T y = \langle x, A^T y\rangle = \langle x, c - s\rangle = \langle c, x\rangle - \langle s, x\rangle.$$

Since both x and s lie in the nonnegative orthant, we know that $\langle s, x\rangle \geq 0$, which implies that $b^T y \leq \langle c, x\rangle$. □

In fact, we can say something much stronger about the optimal values of the primal and dual linear programming problems, a result that is known as the *strong duality* theorem ([34, Theorem 5.2]).

Theorem 2.2 (Strong duality)**.** *If either the dual or the primal problem is feasible, then the optimal value of the primal problem is equal to the optimal value of the dual problem.*

These theorems motivate the notion of the *duality gap*:

Definition 2.1. Given a feasible solution x for (LP) and a feasible solution (s, y) for $(LP)^*$, the duality gap between them is

$$\langle c, x\rangle - b^T y,$$

which is also equal to $\langle x, s\rangle$ (since $s = c - A^T y$).

Clearly, by the theorems, the duality gap is a measure of how close x and (s, y) are to being optimal.

Many primal-dual interior-point algorithms (among others: [25, 10, 30, 4]) are based on reducing the duality gap at each iteration. That is, when the current duality gap is sufficiently close to zero, the current iterate is a good approximation of the optimal solution.

2.2 Cone optimization

Consider a generalization of linear programming, where the nonnegativity constraint "$x \in \mathbb{R}^n_+$" is replaced by a constraint of the form $x \in K$, for sets $K \subseteq \mathbb{R}^n$ that are closed convex cones.

Definition 2.2. Consider a subset K of a finite-dimensional vector space. Then,

- K is *convex* if for all $x, y \in K$ and for all $\lambda \in [0, 1]$, the point $x(\lambda) := \lambda x + (1-\lambda)y$ is also in K.
- K is a *cone* if for each $v \in K$, $\lambda v \in K$ and all scalars $\lambda > 0$, the vector λv is also in the cone K.
- K is a *regular* cone if it is a cone, its interior is nonempty, and it does not contain linear subspaces other than $\{0\}$.

In the rest of this work, we always assume the cones K that we consider are closed, convex, and regular, unless explicitly specified otherwise. Intuitively, regular cones "have a volume" and are "pointed" at the origin. For instance, for $n \geq 2$, a closed halfspace is a closed convex cone that is not regular because it contains a nontrivial linear subspace. A ray is not regular because its interior is empty.

By replacing the nonnegative orthant with a (regular closed convex) cone K, we then obtain a problem of minimizing a linear functional subject to a set of linear equality constraints and a cone constraint:

$$\begin{aligned} \min \quad & \langle c, x \rangle \\ s.t. \quad & Ax = b \\ & x \in K. \end{aligned} \qquad (CP)$$

We call problems of this form *(convex) cone optimization* or *conic programming* problems. As in the case of linear programming, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$, a full-rank matrix. The inner product $\langle \cdot, \cdot \rangle$ denotes an inner product on $\mathbb{R}^n$, which could be the dot product on $\mathbb{R}^n$ or another inner product.

Note that although (CP) as we explicitly state here always has decision variables that lie in $\mathbb{R}^n$, with $K \subseteq \mathbb{R}^n$, we can in fact consider cone optimization problems over any finite-dimensional Euclidean space $\mathbb{E}$. Because any n-dimensional Euclidean space $\mathbb{E}$ is essentially "the same" as $\mathbb{R}^n$, the way we state (CP) above is without loss of generality.

Definition 2.3. Given a cone $K \subseteq \mathbb{R}^n$ and an inner product $\langle \cdot, \cdot \rangle$, the *dual cone* is the set

$$K^* := \{u \in \mathbb{R}^n \mid \langle u, v \rangle \geq 0 \quad \forall v \in K\}.$$

From this, it is well-known that for any regular closed convex cone K, the dual cone K^* is also a regular closed convex cone.

The dual of (CP) is the problem

$$\begin{array}{rl} \max & b^T y \\ s.t. & A^* y + s = c \\ & s \in K^*. \end{array} \qquad (CP)^*$$

We use $A^* : \mathbb{R}^m \to \mathbb{R}^n$ to denote the *adjoint* of $A : \mathbb{R}^n \to \mathbb{R}^m$, which has the property that

$$y^T (Ax) = \langle A^* y, x \rangle$$

for all $y \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$. When the inner product $\langle \cdot, \cdot \rangle$ is just the dot product, then the adjoint of A is just A^T.

In the context of cone optimization, we reuse $Int := \{x \in \text{int}\, K \mid Ax = b\}$ to denote the interior of the feasible region of (CP) and $Int^* := \{(s, y) \mid s \in \text{int}\, K^* \text{ and } A^*y + s = c\}$ to denote the interior of the feasible region of $(CP)^*$. As before, we say that a problem is strictly feasible if the interior of its feasible region is not empty.

2.2.1 Example: semidefinite programming

Semidefinite programming (SDP) is a cone optimization problem over the vector space of symmetric $n \times n$ matrices (with real entries), where the cone constraint is the set of *symmetric positive semidefinite matrices*.

An $n \times n$ matrix U is positive semidefinite if its eigenvalues are nonnegative. It is positive definite if its eigenvalues are positive.

We use $\mathbb{S}^n$ to denote the space of $n \times n$ symmetric matrices, and $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$ to denote the sets of symmetric positive semidefinite and positive definite matrices, respectively. From here onwards, whenever we refer to a positive (semi)definite matrix, we implicitly mean that the matrix is also symmetric.

More specifically, in a semidefinite programming problem, we minimize an objective functional defined on $\mathbb{S}^n$ subject to linear constraints and the constraint that the feasible matrices must also lie in $\mathbb{S}^n_+$. The following is a semidefinite programming problem in standard form:

$$\begin{array}{rl} \min & \langle C, X \rangle \\ s.t. & \langle A_i, X \rangle = b_i, \quad \forall i \in \{1, \ldots, m\}, \qquad (SDP) \\ & X \in \mathbb{S}^n_+. \end{array}$$

Here, $C, A_1, \ldots, A_m \in \mathbb{S}^n$ and $b \in \mathbb{R}^m$. We assume that $b \neq 0$, $m \leq n(n+1)/2$, and that $C, A_1, \ldots, A_m$ form a linearly independent set. The decision variable X lies in the set of symmetric $n \times n$ matrices. The inner product $\langle \cdot, \cdot \rangle$ is chosen to be the trace inner product, which is defined on $\mathbb{R}^{n\times n}$ as follows: For $U, V \in \mathbb{R}^{n\times n}$,

$$\langle U, V \rangle := \operatorname{tr}(U^T V) = \sum_{i,j} U_{ij} V_{ij}.$$

When U, V are symmetric matrices, $\langle U, V \rangle = \operatorname{tr}(U^T V) = \operatorname{tr}(UV)$.

The dual problem of (SDP) is given by

$$\begin{aligned} \max \quad & b^T y \\ s.t. \quad & \sum_{i=1}^m A_i y_i + S = C \qquad (SDP)^* \\ & S \in \mathbb{S}^n_+. \end{aligned}$$

For a set of m fixed symmetric matrices $A_1, \ldots, A_m$, let us define the operator $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ as

$$\mathcal{A}X = (\langle A_1, X \rangle, \ldots, \langle A_m, X \rangle)^T$$

for all $X \in \mathbb{S}^n$. Let $\mathcal{A}^* : \mathbb{R}^m \to \mathbb{S}^n$, the adjoint of $\mathcal{A}$, be given by:

$$\mathcal{A}^* y = \sum_{i=1}^m A_i y_i,$$

for all $y \in \mathbb{R}^m$.

Using $\mathcal{A}$ and $\mathcal{A}^*$, we can rewrite the primal and dual semidefinite programs as

$$\begin{aligned} \min \quad & \langle C, X \rangle \\ s.t. \quad & \mathcal{A}X = b \qquad (SDP) \\ & X \in \mathbb{S}^n_+ \end{aligned}$$

and

$$
\begin{array}{lll}
\max & b^T y & \\
s.t. & \mathcal{A}^* y + S = C & \qquad (SDP)^* \\
 & S \in \mathbb{S}^n_+. &
\end{array}
$$

To conclude this example, we make explicit how semidefinite programming fits into the cone optimization "template" provided by (CP) and $(CP)^*$ above, which is described to have decision variables that are column vectors, instead of decision variables that are matrices. We do this by explicitly identifying each matrix in $\mathbb{S}^n$ with a vector in $\mathbb{R}^{n(n+1)/2}$, writing its upper triangular portion as a vector in $\mathbb{R}^{n(n+1)/2}$ by "stacking" its columns.

Consider the mapping $svec : \mathbb{S}^n \to \mathbb{R}^{n(n+1)/2}$ given by: for each $i, j = 1, \ldots, n$ with $i \leq j$

$$
(svec(X))_{i+j(j-1)/2} := \begin{cases} \sqrt{2} X_{ij}, & \text{if } i \neq j, \\ X_{ij}, & \text{if } i = j, \end{cases}
$$

for all $X \in \mathbb{S}^n$. Note that $svec$ is a linear mapping: $svec(\lambda X) = \lambda\, svec(X)$ for any scalar λ and $svec(X + Y) = svec(X) + svec(Y)$. Furthermore, for any $U, V \in \mathbb{S}^n_+$,

$$
\langle U, V \rangle = svec(U)^T svec(V).
$$

That is, using the trace inner product on $\mathbb{S}^n$ is equivalent to using the dot product (of the images under $svec$) on $\mathbb{R}^{n(n+1)/2}$. In other words, the operation $svec$ preserves the inner product. Hence, the constraint $\mathcal{A}X = b$ can also be written as:

$$
A\; svec(X) = b,
$$

where A is an $(m \times (n(n+1)/2))$ matrix given by

$$A := \begin{bmatrix} svec(A_1)^T \\ svec(A_2)^T \\ \vdots \\ svec(A_m)^T \end{bmatrix}.$$

The matrix A is full rank because we assume that $A_1, \ldots, A_m$ form a linearly independent set.

It is not difficult to see that *svec* is one-to-one and onto, and thus invertible. In fact, it is easy to see that its inverse is

$$(svec^{-1}(x))_{ij} = \begin{cases} x_{i(i+1)/2} & \text{if } i = j \\ x_{i+j(j-1)/2} & \text{if } i \neq j. \end{cases}$$

Since *svec* is linear and invertible, the map $svec^{-1}$ is continuous and maps open sets in $\mathbb{R}^n$ to open sets in $\mathbb{S}^n$. Since the complement of $\mathbb{S}^n_+$ is open, then the preimage, namely the complement of $svec\ \mathbb{S}^n_+$, is also open. Thus, $svec\ \mathbb{S}^n_+$, the image of $\mathbb{S}^n_+$ under the mapping *svec*, is closed.

Because *svec* is a linear mapping, the image of $\mathbb{S}^n_+$ under *svec* is also a cone in $\mathbb{R}^{n(n+1)/2}$: Suppose that the vector v is in $svec\ \mathbb{S}^n_+$. Then, there is some $X \in \mathbb{S}^n_{++}$ such that $v = svec(X) \in svec\ \mathbb{S}^n_+$. Then for all $\lambda \geq 0$,

$$\lambda\, v = \lambda\, svec(X) = svec(\lambda X) \in svec\ \mathbb{S}^n_+,$$

since $\lambda X \in \mathbb{S}^n_+$. Thus, $svec\ \mathbb{S}^n_+$ is a cone. It is easy to see that it is also convex: for any $x, y \in svec\ \mathbb{S}^n_+$ and $\lambda \in [0, 1]$, linearity of *svec* guarantees that $\lambda x + (1-\lambda)y = svec(\lambda X + (1-\lambda)Y) \in svec\ \mathbb{S}^n_+$ (where X and Y are matrices in $\mathbb{S}^n_+$ that maps to x and y, respectively).

We observe that that the interior of *svec* $\mathbb{S}^n_+$ is not empty, due to the continuity of $svec^{-1}$. Since *svec* is linear and we know that $\mathbb{S}^n_+$ does not contain linear subspaces other than the zero matrix, then *svec* $\mathbb{S}^n_+$ does not contain linear subspaces other than the zero vector. Thus, *svec* $\mathbb{S}^n_+$ is a regular cone.

In addition, since the *svec* operation preserves the inner product (it is an isometry), then it also preserves dual cones. In particular, the image of $\mathbb{S}^n_+$ under this operation is self-dual, like $\mathbb{S}^n_+$ itself.

2.2.2 Duality theorems

As for linear programming, we have a weak duality theorem for conic optimization problems (see [27, Chapter 3]).

Theorem 2.3 (Weak duality)**.** *For any feasible solution x for (CP) and any feasible solution (s, y) for $(CP)^*$,*

$$\langle c, x\rangle \geq b^T y.$$

That is, the value of any feasible primal solution is always an upper bound for the value of any feasible dual solution.

Unfortunately, for general convex cone optimization problems, the optimal value of the primal problem might not be equal to the optimal value of the dual problem. There are, however, certain conditions that guarantees a strong duality theorem like what we have for LP, such as the following "strong duality" result (see [27, Corollary 3.2.7, Theorem 3.2.8]).

Theorem 2.4. *Suppose that the primal problem is feasible and the dual problem is strictly feasible. Then, the primal problem has an optimal solution.*

As in the case of linear programming, given a pair of feasible solutions x and (s, y) for (CP) and $(CP)^*$, respectively, the duality gap between them is

$$\langle c, x\rangle - b^T y = \langle x, s\rangle, \tag{2.1}$$

since $s = c - A^T y$.

2.2.3 Optimality conditions

Consider nonlinear optimization problems in the following form:

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & g_i(x) \le 0, \quad \forall i \in \{1, \dots, l\}, \\ & h_j(x) = 0, \quad \forall j \in \{1, \dots, m\}, \end{aligned} \tag{2.2}$$

where $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$ are assumed to be C^1 functions (continuously differentiable).

Then, Fritz John ([11], [19]) provides a set of necessary conditions which have to be satisfied by any optimal solution to (2.2).

Theorem 2.5 (Fritz John's Necessary Optimality Conditions). *If $\bar{x}$ is an optimal solution of* (2.2)*, then there exist a vector $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_l)^T \in \mathbb{R}_+^{l+1}$ and $\mu = (\mu_1, \dots, \mu_m)^T \in \mathbb{R}^m$ such that*

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i=1}^{l} \lambda_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla h_j(\bar{x}) = 0, \tag{2.3}$$

$$\sum_{i=1}^{l} \lambda_i g_i(\bar{x}) = 0, \tag{2.4}$$

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} \ne 0. \tag{2.5}$$

Karush, and Kuhn and Tucker, provide a second set of necessary optimality conditions for nonlinear optimization problems in the form (2.2) ([13, 17], [29, Theorem 3.25]). This second set of conditions, normally referred to as the *KKT conditions*, is similar to Fritz John's, with one main exception being that KKT conditions guarantees that there exist $\lambda \in \mathbb{R}^{l+1}, \mu \in \mathbb{R}^m$ such that $\lambda_0 = 1$, whereas in the case of Fritz John's, it is possible that $\lambda_0 = 0$.

Furthermore, the KKT conditions hold only if certain constraint qualifications—a set of conditions on the constraints—hold at the optimal point $\bar{x}$. One set of such constraints qualifications is as follows: There exists a vector $\bar{y} \in \mathbb{R}^n$ such that

$$\bar{y}^T \nabla g_i(\bar{x}) < 0, \qquad \forall i \in \{i = 1, \ldots, l \mid g_i(\bar{x}) = 0\} \tag{2.6}$$

$$\bar{y}^T \nabla h_j(\bar{x}) = 0, \qquad \forall j \in \{1, \ldots, m\}, \tag{2.7}$$

and that

$$\nabla h_1(\bar{x}), \ldots, \nabla h_m(\bar{x}) \text{ are linearly independent.} \tag{2.8}$$

The conditions (2.6)-(2.8) are known as the *Mangasarian-Fromovitz Constraint Qualifications* (MFCQ).

Theorem 2.6 (Karush-Kuhn-Tucker (KKT) Necessary Optimality Conditions)**.** *Let $\bar{x}$ be a local minimum of the nonlinear optimization problem* (2.2)*.*

Assume that at $\bar{x}$, one of the following constraint qualifications is satisfied:

1. *(MFCQ) The constraint qualifications* (2.6)-(2.8)
2. *(Slater's condition) The functions g_i are convex, h_j are affine, and there exists a feasible solution $\hat{x}$ such that $g_i(\hat{x}) < 0$ for all $i \in \{1, \ldots, l\}$.*

Then, there exist multipliers $\lambda = (\lambda_1, \ldots, \lambda_l)^T \in \mathbb{R}^l_+$ *and* $\mu = (\mu_1, \ldots, \mu_m)^T \in \mathbb{R}^m$ *such that*

$$\nabla f(\bar{x}) + \sum_{i=1}^{l} \lambda_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla h_j(\bar{x}) = 0 \tag{2.9}$$

and

$$\lambda_i g_i(\bar{x}) = 0, \quad \forall i \in \{1, \ldots, l\}. \tag{2.10}$$

We are interested in applying these optimality conditions to cone optimization problems of the form (CP). That is, when the objective function and equality constraints are linear:

$$f(x) = \langle c, x \rangle$$

and

$$h(x) = Ax - b.$$

The only place in which we might have nonlinearity is in the cone constraint. Supposing that the chosen cone K can be described by l constraints (not necessarily linear):

$$K = \left\{ x \in \mathbb{R}^n \mid g_j(x) \leq 0, \forall j \in \{1, \ldots, l\} \right\},$$

where each g_j is a convex, C^1 function, then we can simplify the Fritz John and KKT optimality conditions as follows.

Corollary 2.7 (Fritz John's Necessary Optimality Conditions for (CP))**.** *If* $\bar{x}$ *is an optimal solution of* (CP)*, then there exist a vector* $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_l)^T \in \mathbb{R}^{l+1}_+$ *and* $\mu = (\mu_1, \ldots, \mu_m)^T \in \mathbb{R}^m$ *such that*

$$\lambda_0 c + \sum_{i=1}^{l} \lambda_i \nabla g_i(\bar{x}) + A^T \mu = 0, \tag{2.11}$$

$$\sum_{i=1}^{l} \lambda_i g_i(\bar{x}) = 0, \tag{2.12}$$

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} \neq 0. \tag{2.13}$$

Similarly, the KKT conditions for (CP) are also simplified:

Corollary 2.8 (Karush-Kuhn-Tucker (KKT) Necessary Optimality Conditions for (CP))**.** *Let $\bar{x}$ be a local minimum of (CP). Assume that at $\bar{x}$, one of the following constraint qualifications is satisfied:*

1. *(MFCQ) There exists a vector $\bar{y} \in \mathbb{R}^n$ such that*
$$\bar{y}^T \nabla g_i(\bar{x}) < 0, \qquad \forall i \in \{i = 1, \ldots, l \mid g_i(\bar{x}) = 0\} \tag{2.14}$$
$$A^T y = 0, \tag{2.15}$$
and that
$$A : \mathbb{R}^n \to \mathbb{R}^m \text{ has rank } m. \tag{2.16}$$
2. *(Slater's condition)The problem (CP) is strictly feasible.*

Then, there exist multipliers $\lambda = (\lambda_1, \ldots, \lambda_l)^T \in \mathbb{R}^l_+$ and $\mu = (\mu_1, \ldots, \mu_m)^T \in \mathbb{R}^m$ such that
$$c + \sum_{i=1}^{l} \lambda_i \nabla g_i(\bar{x}) + A^T \mu = 0 \tag{2.17}$$
and
$$\lambda_i g_i(\bar{x}) = 0, \quad \forall i \in \{1, \ldots, l\}. \tag{2.18}$$

When the optimization problem at hand is a minimization problem of a convex function over a convex feasible region, then the KKT conditions above are not only necessary but also *sufficient* ([29, Theorem 3.34]). That is, if the respective constraint qualification holds and if there exists $(\bar{x}, \lambda, \mu)$ which satisfies the conditions given in the respective theorems (Corollaries 2.7 and 2.8), then $\bar{x}$ is an optimal solution.

2.3 The barrier functionals and the local inner products

The *(logarithmic) barrier functional* for the nonnegative orthant is the functional $f : \mathbb{R}^n_{++} \to \mathbb{R}$ given by

$$f(x) := -\sum_{i=1}^{n} \ln x_i. \tag{2.19}$$

In the interior-point methods literature, the barrier functional is used in devising the so-called *path-following methods* for solving linear programming problems. These interior-point algorithms solve a sequence of convex optimization problems that are obtained by removing the constraint "$x \in \mathbb{R}^n_+$" from the linear program to be solved and adding a positive multiple of the barrier functional to the objective. These problems take the following form:

$$\begin{array}{lll} \min & \langle c, x \rangle - \mu \sum_{i=1}^{n} \ln x_i & \\ s.t. & Ax = b, & (BP_\mu) \end{array}$$

where $\mu > 0$ is called the *barrier parameter*. We call these problems the *barrier problems*.

We note that f forms a "barrier" on the boundary of the positive orthant: $f(x)$ approaches infinity as $x \in \mathbb{R}^n_{++}$ approaches the boundary of the positive orthant, because for at least one coordinate i, x_i approaches 0. Thus, we can think of the term $\mu f(x)$ in the objective function of (BP_μ) as a "penalty" term for being too close to the boundary of the positive orthant.

Another important property of f is that it is strictly convex on the positive orthant, which can be confirmed by taking the Hessian of f and showing that it is positive definite on $\mathbb{R}^n_{++}$. In particular, with respect to the dot product, the

Hessian of f at $x \in \mathbb{R}^n{++}$ is

$$H(x) = \begin{bmatrix} 1/x_1^2 & 0 & \cdots & 0 \\ 0 & 1/x_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/x_n^2 \end{bmatrix} = X^{-2}. \tag{2.20}$$

(Recall that we use X to denote diag x, a diagonal matrix whose ith diagonal component is given by x_i.)

Since $f(x)$ is strictly convex on $\mathbb{R}^n_{++}$ and $\langle c, x\rangle$ is linear, the objective of (BP_μ) is also strictly convex on $\mathbb{R}^n_{++}$. Thus, if (BP_μ) has an optimal solution, it must be unique. Furthermore, if we assume that (LP) and $(LP)^*$ are strictly feasible, then for each $\mu > 0$, the problems (BP_μ) has an optimal solution in the interior of (LP), and this optimal solution is unique (see [34, Theorem 17.2]).

Denoting the optimal solution to (BP_μ) with $x(\mu)$, the set of all such solutions $x(\mu)$ is called the central path:

$$\text{Central path} := \{x(\mu) \mid \mu > 0\}. \tag{2.21}$$

It can be shown that $x(\mu)$ approaches an optimal linear programming solution as μ approaches zero. Path-following interior-point methods produce iterates that converge to the optimal linear programming solution by approximately following the central path. That is, they approximate $x(\mu)$ for a decreasing sequence of μ. By decreasing μ in a particular fashion, as $\mu \downarrow 0$, the iterates which approximate the corresponding $x(\mu)$ can be shown to approach the optimal LP solution.

Since the Hessian of the barrier functional is positive definite on $\mathbb{R}^n_{++}$, it can be used to define *local inner products* (see *intrinsic inner products* in [27, Chapter

2]): For each $e \in \mathbb{R}^n_{++}$, the local inner product at e is given by

$$\langle u, v\rangle_e := \langle u, H(e)v\rangle, \tag{2.22}$$

for all $u, v \in \mathbb{R}^n$. Here, $H(e)$ denotes the Hessian of the barrier functional at e, with respect to the inner product $\langle\cdot,\cdot\rangle$. The local inner product $\langle\cdot,\cdot\rangle_e$ induces the norm

$$\|u\|_e := \sqrt{\langle u, u\rangle_e}. \tag{2.23}$$

Observe that we can interpret the product of vectors u, v under the local inner product at $e \in \mathbb{R}^n_{++}$ as just the inner product under $\langle\cdot,\cdot\rangle$ of their images under an affine scaling. For instance, when $\langle\cdot,\cdot\rangle$ is the dot product, consider the affine scaling by $(diag\ e)^{-1} = E^{-1}$. Since $H(e) = E^{-2}$, then

$$\langle u, v\rangle_e = \langle u, E^{-2}v\rangle = \langle E^{-1}u, E^{-1}v\rangle.$$

The path-following methods and the notions of the central path and a local inner product can be extended for cone optimization problems over more general convex cones K, as long as there exists a "barrier functional" with the appropriate properties, defined on the interior of the cone. In [26], Nesterov and Nemirovskii show the existence of such a barrier functional for any closed convex subset K of $\mathbb{R}^n$, which means that in theory, interior-point methods can be extended to solve cone optimization problems over general regular closed convex cones K. Among the "appropriate properties" is that the Hessian of the barrier functional is positive definite, and can be used to define a local inner product at each interior point of the cone.

In the case of semidefinite programming, the barrier functional is defined on $\mathbb{S}^n_{++}$ and is given by

$$f(\,\cdot\,) := -\ln(\det(\,\cdot\,)). \tag{2.24}$$

This barrier functional is strictly convex and has a positive definite Hessian. In particular, its Hessian with respect to the trace inner product, evaluated at $E \in \mathbb{S}^n_{++}$, is (see [27, Section 1.3]):

$$H(E)[\,\cdot\,] = E^{-1}(\,\cdot\,)E^{-1}. \tag{2.25}$$

Then, the local inner product at $E \in \mathbb{S}^n_{++}$ is:

$$\langle U, V\rangle_E := \langle U, H(E)V\rangle = \operatorname{tr}(UE^{-1}VE^{-1}), \tag{2.26}$$

for all $U, V \in \mathbb{S}^n$, where the inner product $\langle\cdot,\cdot\rangle$ is the trace inner product.

Again, we can interpret $\langle U, V\rangle_E$ as the trace product of the images of U, V under a particular scaling: Since $H(E)[\,\cdot\,] = E^{-1}(\,\cdot\,)E^{-1}$, then

$$\begin{aligned}
\langle U, V\rangle_E &= \operatorname{tr}(UE^{-1}VE^{-1}) \\
&= \operatorname{tr}((E^{-1/2}UE^{-1/2})\,(E^{-1/2}VE^{-1/2})) \\
&= \langle E^{-1/2}UE^{-1/2}, E^{-1/2}VE^{-1/2}\rangle,
\end{aligned}$$

the trace product between the images of U and V under the scaling $E^{-1/2}(\,\cdot\,)E^{-1/2}$. Here, $E^{-1/2}$ denotes the *square root* of E^{-1}, the symmetric positive definite matrix that satisfies $E^{-1} = E^{-1/2}E^{-1/2}$.

Note that here, we used the "cyclic" property of the trace product: for any symmetric matrices $A, B, C \in \mathbb{S}^n$,

$$\operatorname{tr}(ABC) = \operatorname{tr}(BCA) = \operatorname{tr}(CAB).$$

CHAPTER 3

LITERATURE REVIEW: AFFINE-SCALING ALGORITHMS

In 1967, Dikin proposed an algorithm for linear programming which is now referred to as Dikin's affine-scaling algorithm [7]. The main idea of the algorithm is as follows. Given a current iterate x that is an interior point of the linear programming feasible region, the next iterate is the optimal solution to a restriction of the linear programming problem, in which the nonnegativity constraint in the linear program is replaced by an inscribed ellipsoid centered at x.

It is not known, however, whether this algorithm has a polynomial-time convergence rate; many believe that it may indeed be an exponential algorithm in the worst case. One indication that this might be the case is a result of Megiddo and Shub [22] that the continuous version of the affine-scaling algorithm comes near each vertex of Klee-Minty's cube at least once, assuming that the initial iterate is an appropriate point.

However, many variants of Dikin's affine-scaling algorithm have been proven to have polynomial-time convergence. These affine-scaling algorithms follow a framework inspired by Dikin's idea of solving a sequence of optimization problems over an ellipsoid, or an ellipsoid-like region, that is inscribed in the feasible region of the linear program.

Our reason for presenting the existing affine-scaling algorithms is to provide a context in which to view our algorithm—which will be introduced and analyzed in Chapters 4 and 5—as an affine-scaling algorithm.

The present chapter will be a survey of several affine-scaling algorithms and will start with an exposition of Dikin's algorithm. We present a general framework which we can use to compare the various affine-scaling algorithms and examine a selection of primal-dual affine-scaling algorithms under this general framework.

3.1 Setting and notation

The algorithms we discuss in this chapter solve a linear programming problem of the following form:

$$\begin{aligned} \max \quad & \langle c, x \rangle \\ s.t. \quad & Ax = b \\ & x \in \mathbb{R}^n_+, \end{aligned}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$, together with the dual problem

$$\begin{aligned} \max \quad & b^T y \\ s.t. \quad & A^T y + s = c \\ & s \in \mathbb{R}^n_+. \end{aligned}$$

We assume that A is full rank and that c^T is linearly independent of the rows of A. Furthermore, throughout this chapter, $\langle \cdot, \cdot \rangle$ denotes the dot product on $\mathbb{R}^n$ and $\| \cdot \|$ the 2-norm.

We denote vectors with lowercase letters. For each of these vectors, in particular x and s, the corresponding uppercase letter is reserved to denote diagonal matrices with the vector as the diagonal elements. For example, $X = \operatorname{diag} x$ and $S = \operatorname{diag} s$.

3.2 Dikin's affine-scaling algorithm

We first consider Dikin's affine-scaling algorithm, as presented by Vanderbei and Lagarias in [35].

The main idea of the algorithm is relatively simple to describe. At each iteration, we have a point x that lies in the interior of the feasible region. To obtain the next iterate, we replace the nonnegativity constraint of the linear program with a particular inscribed, closed ellipsoid around x and solve this auxiliary optimization problem. This problem is feasible (since x is a feasible solution) and has a unique optimal solution because its feasible region is compact and strictly convex. We choose the optimal solution to this problem to be the next iterate.

Algorithm 3.1. Dikin's affine-scaling algorithm

Initialization Start with a solution $x^{(0)}$ in Int.

***k*th iteration** Let $x^{(k)} \in Int$ denote the current iterate.

Let $x^{(k+1)}$ denote the optimal solution to

$$\begin{aligned} \min \quad & \langle c, x \rangle \\ s.t. \quad & Ax = b \\ & x \in E_{x^{(k)}}, \end{aligned}$$

where $E_{x^{(k)}} := \left\{ z \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} \frac{(x_i^{(k)} - z_i)^2}{(x_i^{(k)})^2} \leq 1 \right\}.$

In [36], the proof of convergence relies on assuming that the problem is both primal and dual nondegenerate. On the other hand, Dikin's proof of convergence in, presented in [8], assumes only primal nondegeneracy, allowing the dual problem to be degenerate.

Search directions

In the pseudocode given above, $x^{(k+1)}$ is described simply as the solution to a particular optimization problem. We will now find the closed-form solution for the search direction for finding $x^{(k+1)}$ given $x^{(k)}$. For simplicity of notation, we will from now on denote the current iterate by x and the next iterate by $\hat{x}$.

Hence, $\hat{x}$ is the optimal solution to the following optimization problem:

$$\begin{aligned} \min \quad & \langle c, z \rangle \\ s.t. \quad & Az = b \\ & z \in E_x, \end{aligned}$$

where E_x is an ellipsoid centered at x, given by

$$\begin{aligned} E_x &= \left\{ z \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} \frac{(x_i - z_i)^2}{(x_i)^2} \leq 1 \right\} \\ &= \left\{ z \in \mathbb{R}^n \mid \|X^{-1}(x - z)\| \leq 1 \right\}. \end{aligned}$$

Note that in fact, $E_x = \{z \in \mathbb{R}^n \mid \|x - z\|_x \leq 1\}$ is a closed unit ball, centered at x, where the norm is the one induced by the local inner product at x (see (2.22) - (2.23)). It can be easily shown that $E_x \subseteq \mathbb{R}^n_+$ (see [27, Section 2.2.1]).

Solving for $\hat{x}$ as above is equivalent to finding a step d_x, so that $\hat{x} = x + d_x$, which is given by the optimal solution of

$$\begin{aligned} \min \quad & \langle c, d \rangle \\ s.t. \quad & Ad = 0 \\ & \|X^{-1}d\| \leq 1. \end{aligned}$$

The last constraint in the above search direction problem is an ellipsoid centered at the origin. Equivalently, this constraint describes a closed unit ball with respect to the local inner product norm at x.

If we consider $\tilde{d} := X^{-1}d$ as a transformation of the search direction, scaled by the matrix X^{-1}, we obtain an optimization problem over the intersection of a linear subspace with the closed unit ball:

$$\begin{aligned} \min \quad & \langle \tilde{c}, \tilde{d} \rangle \\ s.t. \quad & \tilde{A}\tilde{d} = 0 \\ & \|\tilde{d}\| \leq 1. \end{aligned} \tag{3.1}$$

where $\tilde{A} := AX$ and $\tilde{c} := Xc$.

The optimal solution to the scaled search-direction problem (3.1) is simply the projection of $-\tilde{c}$ onto the nullspace of $\tilde{A}$, normalized to a unit length:

$$\begin{aligned} \tilde{d}_x &= -\frac{\mathrm{Proj}_{\ker AX} Xc}{\|\mathrm{Proj}_{\ker AX} Xc\|} \\ &= -\frac{(I - (AX)^T (AX^2 A^T)^{-1} AX) Xc}{\|(I - (AX)^T (AX^2 A^T)^{-1} AX) Xc\|}. \end{aligned}$$

The optimal search direction for the unscaled problem is then

$$\begin{aligned} d_x &= X\tilde{d}_x = -\frac{X(I - (AX)^T (AX^2 A^T)^{-1} AX) Xc}{\|(I - (AX)^T (AX^2 A^T)^{-1} AX) Xc\|} \\ &= -\frac{X^2 (I - A^T (AX^2 A^T)^{-1} AX^2) c}{\|X(I - A^T (AX^2 A^T)^{-1} AX^2) c\|}. \end{aligned}$$

Step length and convergence

In [7, 8], a full step towards the boundary of the ellipsoid is taken. The algorithm in [36] is very similar to Dikin's algorithm, but it takes shorter step lengths. With this particular step length, Dikin was able to prove that the algorithm converges to an optimal solution of the linear program, although no iteration-complexity bounds are proven, just by relying on a primal nondegeneracy assumption. This is in contrast to having to assume both primal and dual nondegeneracy, as is done in [36].

3.3 A general framework for affine-scaling algorithms

The variants of affine-scaling algorithms differ from one another in the choice of search direction and the choice of step length at each iteration. In this section, we focus on examining how the search directions are obtained. We consider three settings: primal-only, dual-only, and primal-dual.

3.3.1 The primal setting

Recall that the search direction in Dikin's algorithm is obtained by solving the scaled problem

$$\begin{aligned} \min \quad & \langle (Xc), \tilde{d} \rangle \\ s.t. \quad & (AX)\tilde{d} = 0 \\ & \|\tilde{d}\| \leq 1 \end{aligned}$$

and recovering the search direction for the original problem by multiplying the optimal $\tilde{d}$ by X.

We can generalize Dikin's approach by considering primal affine-scaling methods where the search direction d_x is obtained by solving the scaled search-direction problem

$$\begin{aligned} \min \quad & \langle (Dc), \tilde{d} \rangle \\ s.t. \quad & (AD)\tilde{d} = 0 \\ & \|\tilde{d}\| \leq 1, \end{aligned}$$

where D is the scaling matrix, followed by multiplying the optimal solution by D. Note that the scaling matrix D has to be a diagonal matrix with positive diagonal entries, because we want to consider only scalings that are invertible and fix the nonnegative orthant.

The search direction is then

$$d_x = -\frac{D\mathrm{Proj}_{\ker AD} Dc}{\|\mathrm{Proj}_{\ker AD} Dc\|}, \tag{3.2}$$

where for any vector $u \in \mathbb{R}^n$, the projection of u onto $\ker AD$ is

$$\mathrm{Proj}_{\ker AD} u = (I - (AD)^T (AD^2 A^T)^{-1} AD) u. \tag{3.3}$$

Hence, the primal affine-scaling search direction with scaling matrix D is

$$d_x = -\frac{D^2 (I - A^T (AD^2 A^T)^{-1} AD^2) c}{\|D(I - A^T (AD^2 A^T)^{-1} AD^2) c\|}. \tag{3.4}$$

When $D = X$, we obtain Dikin's search direction described in the previous section.

3.3.2 The dual setting

We can think of applying the affine-scaling algorithm described above to a problem in dual form. Alternatively, suppose that in addition to having a primal strictly feasible solution, we also have a dual strictly feasible solution. As we update the primal iterate as described in the above section, we can use the same scaling matrix to obtain a new strictly feasible dual solution.

As we scale the primal problem with a scaling matrix D, obtaining the problem:

$$\begin{aligned} \min \quad & \langle (Dc), \tilde{x} \rangle \\ s.t. \quad & (AD)\tilde{x} = b \\ & \tilde{x} \in \mathbb{R}^n_+, \end{aligned}$$

we obtain the corresponding scaled dual problem:

$$\begin{aligned} \max \quad & b^T \tilde{y} \\ s.t. \quad & (AD)^T \tilde{y} + \tilde{s} = Dc \\ & \tilde{s} \in \mathbb{R}^n_+. \end{aligned}$$

Note that x is feasible for the unscaled primal problem if and only if $D^{-1}x$ is feasible for the scaled version, and that (s, y) is feasible for the unscaled dual problem if and only if $(\tilde{s}, \tilde{y}) = (Ds, y)$ is feasible for the scaled version.

Given a fixed x that is feasible for the primal problem, we can replace the dual objective as follows

$$b^T y = (Ax)^T y = (D^{-1}x)^T (AD)^T y = -(D^{-1}x)^T (Ds) + \text{a constant}.$$

This allows us to write the corresponding scaled dual problem as:

$$\begin{aligned} \min \quad & \langle (D^{-1}x), \tilde{s} \rangle \\ s.t. \quad & (AD)^T \tilde{y} + \tilde{s} = Dc \\ & \tilde{s} \in \mathbb{R}^n_+. \end{aligned}$$

Then, the Dikin search direction for the dual problem is obtained by solving the following scaled dual search direction problem.

$$\begin{aligned} \min \quad & \langle (D^{-1}x), \tilde{d}_s \rangle \\ s.t. \quad & (AD)^T \tilde{d}_y + \tilde{d}_s = 0 \\ & \|\tilde{d}_s\| \leq 1. \end{aligned}$$

That is, if $(\tilde{d}_s, \tilde{d}_y)$ is the optimal solution to this problem, then the Dikin search direction is $(d_s, d_y) = (D\tilde{d}_s, \tilde{d}_y)$.

The constraint "$(AD)^T \tilde{d}_y + \tilde{d}_s = 0$" means that a feasible $\tilde{d}_s$ must lie in the space orthogonal to the kernel of AD. So, the optimal solution $\tilde{d}_s$ must be in the direction of the projection of $D^{-1}x$ onto this linear space:

$$\tilde{d}_s = -\frac{\text{Proj}_{(\ker AD)^\perp}(D^{-1}x)}{\|\text{Proj}_{(\ker AD)^\perp}(D^{-1}x)\|},$$

where for any vector $u \in \mathbb{R}^n$, the projection of u onto $(\ker AD)^\perp$ is

$$\text{Proj}_{(\ker AD)^\perp} u = (AD)^T (AD^2 A^T)^{-1} ADu. \tag{3.5}$$

Hence, the search direction in the original space is:

$$d_s = -\frac{A^T (AD^2 A^T)^{-1} b}{\|(AD)^T (AD^2 A^T)^{-1} b\|}. \tag{3.6}$$

We know that (d_s, d_y) must satisfy $A^T d_y + d_s = 0$. So,

$$d_y = \frac{(AD^2 A^T)^{-1} b}{\|(AD)^T (AD^2 A^T)^{-1} b\|} \tag{3.7}$$

produces a feasible search direction (d_s, d_y).

The "standard" dual affine-scaling method uses the scaling matrix $D = S^{-1}$. For examples of work on dual affine-scaling algorithms, see [1] and [33].

3.3.3 The primal-dual setting: the V-space

In the analysis of primal-dual interior-point methods, we must consider primal and dual solutions, $x \in Int$ and $(s, y) \in Int^*$, simultaneously. One setting for doing this is often referred to as "the V-space", a notion which was developed in [16] and [23] in the context of linear complementarity problems and in [24] in the linear programming context, among others. We introduce the V-space in this section and use it in the next section to describe a unifying framework for viewing various primal-dual affine-scaling algorithms.

The following theorem states that analyzing pairs of strictly feasible primal and dual solutions $(x, s, y) \in Int \times Int^*$ is in a sense equivalent to analyzing their image under a particular mapping to the positive orthant.

Theorem 3.1. *Assume that A is full-rank and that Int, Int^* are nonempty.*

The mapping $(x, s, y) \mapsto x.s = (x_1 s_1, \ldots, x_n s_n)^T$ is a bijection from $Int \times Int^*$ onto $\mathbb{R}^n_{++}$.

Proof. We will first show that given a vector w in $\mathbb{R}^n_{++}$, there exists at most one pair of solutions $x \in Int, (s, y) \in Int^*$ that maps to w. Then, we will prove that there must exist at least one such a pair of solutions that maps to w.

In order to prove the two parts of the theorem, we utilize the following weighted barrier problem, where $w \in \mathbb{R}^n_{++}$ is a given vector

$$\begin{array}{ll} \min & f(x) := \langle c, x\rangle - \sum_{i=1}^{n} w_i \log x_i \\ s.t. & Ax = b. \end{array} \tag{3.8}$$

We note that the domain of f is $\mathbb{R}^n_{++}$, so the feasible region of the problem (3.8) is precisely *Int*, and that f is strictly convex. The KKT conditions for this barrier problem, which are necessary and sufficient, are as follow: The solution x is optimal for the above problem if and only if there exists some $y \in \mathbb{R}^m$ such that x and y together satisfy:

$$(c - w./x) - A^T y = 0, \tag{3.9}$$

$$Ax = b. \tag{3.10}$$

We are now ready to prove the theorem. We first consider an arbitrary $w \in \mathbb{R}^n_{++}$ and show that it has a unique preimage in $Int \times Int^*$, assuming that there is a primal-dual pair of solutions that maps to w.

Thus, assume that $x \in Int$ and $(s, y) \in Int^*$ such that $w = x.s$. Therefore, $s = w./x$. Furthermore, since (s, y) is a feasible dual solution, then the first KKT condition, namely (3.9), is satisfied by x. Since x is a feasible primal solution, the second KKT condition, namely (3.10), is also satisfied. This implies that x is optimal for (3.8). Moreover, since f is strictly convex, then f has at most one minimizer; so x is a unique point in *Int* such that there is some $(s, y) \in Int^*$ that together with x maps to w. Furthermore, s is uniquely determined by x, and the corresponding y is uniquely determined by s since we assume that A is full rank. This proves uniqueness.

Next, we prove the existence of the preimage of w. Let *val* denote the optimal value of (3.8). We assume that *Int* is nonempty, so we know that (3.8) is feasible and $val < \infty$. Suppose that $\{x^{(k)}\}$ is a sequence of points in *Int* such that $\{f(x^{(k)})\}$ converges to *val*. We claim that $\{x^{(k)}\}$ has a subsequence that converges to a point in *Int*. By continuity of f, this point must be an optimal solution of (3.8). To show the existence of this subsequence, we only need to show that $\{x^{(k)}\}$ lies in a compact subset of *Int*.

Suppose that $\{x^{(k)}\}$ did not lie in a compact subset of *Int*, then $\{x^{(k)}\}$ must be either unbounded or has a limit point on the boundary of $\mathbb{R}^n_{++}$. We claim that it cannot have a limit point in the boundary of $\mathbb{R}^n_{++}$. Suppose to the contrary that it had a limit point $\hat{x}$ on the boundary, where $\hat{x}_j = 0$ for some index j. This means that $f(x^{(k)}) \to \infty$ since $\ln x_j^{(k)} \to -\infty$, resulting in a contradiction.

We now show that it cannot be unbounded: suppose there is a subsequence (for simplicity, also denoted $\{x^{(k)}\}$) such that $\|x^{(k)}\| \to \infty$.

Since Int^* is nonempty, consider any fixed $(\tilde{s}, \tilde{y}) \in Int^*$. We know that $c = \tilde{s} + A^T\tilde{y}$ and $\tilde{s} \in \mathbb{R}^n_{++}$. Therefore,

$$\begin{aligned} f(x^{(k)}) &= \langle c, x^{(k)} \rangle - \sum_{i=1}^{n} w_i \ln x_i^{(k)} \\ &= b^T\tilde{y} + \langle \tilde{s}, x^{(k)} \rangle - \sum_{i=1}^{n} w_i \ln x_i^{(k)} \\ &\geq b^T\tilde{y} + \|x^{(k)}\| \min_i \tilde{s}_i - \sum_{i=1}^{n} w_i \ln x_i^{(k)}. \end{aligned}$$

Since $\|x^{(k)}\| \to \infty$, then

$$\|x^{(k)}\| \min_i \tilde{s}_i - \sum_{i=1}^{n} w_i \ln x_i^{(k)} \to \infty,$$

which, again, results in a contradiction since $\lim_{k\to\infty} f(x^{(k)}) = val < \infty$.

So, there must be a sequence of points $\{x^{(k)}\}$ that converges to an optimal solution of (3.8), call it x. By the optimality conditions for (3.8), there exists y such that $c - w./x - A^T y = 0$. Let $s := w./x$. Clearly, $x.s = w$, and furthermore, $A^T y + s = c$. This proves the existence of a preimage (x, s, y) for each $w \in \mathbb{R}^n_{++}$, thereby proving that the mapping $(x, s, y) \mapsto x.s$ from $Int \times Int^*$ to $\mathbb{R}^n_{++}$ is a bijection. □

Viewing $\mathbb{R}^n_{++}$ as the image of the above bijection, we refer to $\mathbb{R}^n_{++}$ as the *V-space*. Finding a search direction for a primal-dual algorithm, for instance, could be done by finding a direction in the V-space instead of considering the primal and dual components of the search direction separately.

Recall our discussion on the (primal) central path in Section 2.3. For problems in the dual form $(LP)^*$, we also have the same notion of a (dual) central path, defined as follows. The dual central path consists of points $(s(\mu), y(\mu))$ for all $\mu > 0$, where for each μ, $(s(\mu), y(\mu))$ is the optimal solution to the following barrier problem:

$$\begin{array}{ll} \min & b^T y - \mu \sum_{i=1}^{n} \ln s_i \\ s.t. & A^T y + s = c. \end{array} \qquad (BP_\mu)^*$$

Similarly, the primal-dual central path is defined as the set

$$\{(x(\mu), s(\mu), y(\mu)) \mid \mu > 0\},$$

where $x(\mu)$, $(s(\mu), y(\mu))$ lie on the primal and the dual central paths, respectively.

The following theorem provides a geometric insight on the image of the primal-dual pairs of solutions that lies on the primal-dual central path under the bijection described in Theorem 3.1.

Theorem 3.2. *Assume that A is full-rank and that Int, Int^* are nonempty.*

The primal-dual central path for (LP) *is the set*

$$\{(x, s, y) \in Int \times Int^* \mid x.s = \mu\mathbb{1} \text{ for some } \mu > 0\},$$

where $\mathbb{1}$ *denotes the vector of all ones.*

Proof. Suppose $(\hat{x}, \hat{s}, \hat{y}) \in Int \times Int^*$ lies on the (primal-dual) central path. Let us first show that $\hat{x}.\hat{s} = \mu\mathbb{1}$ for some $\mu > 0$. We know that there is some $\mu > 0$ such that $\hat{x}$ is the optimal solution to

$$\begin{aligned} \min_x \quad & f_\mu(x) := \langle c, x\rangle - \mu \sum_{i=1}^n \ln x_i \\ s.t. \quad & Ax = b, \end{aligned} \tag{3.11}$$

where the domain of f_μ is $\mathbb{R}^n_{++}$, and such that $(\hat{s}, \hat{y})$ is the optimal solution to

$$\begin{aligned} \max_{s,y} \quad & f^*_\mu(s, y) := b^T y + \mu \sum_{i=1}^n \ln s_i \\ s.t. \quad & A^T y + s = c. \end{aligned} \tag{3.12}$$

The KKT conditions for (3.11) are as follow. A feasible solution x for (3.11) is optimal if and only if x together with some $y \in \mathbb{R}^m$ satisfy

$$\begin{aligned} (c - \mu x^{-1}) - A^T y &= 0 \\ Ax &= b. \end{aligned}$$

Similarly, the following are the KKT conditions for (3.12). A feasible solution (s, y) for (3.12) is optimal if and only if (s, y) together with some $x \in \mathbb{R}^m$ satisfy

$$\begin{aligned} \mu s^{-1} - x &= 0 \\ Ax &= b \\ A^T y + s &= c. \end{aligned}$$

Note that since $\hat{x}$ is optimal for (3.11), then there exists some $\bar{y} \in \mathbb{R}^m$ such that the first set of KKT conditions are satisfied. These KKT conditions imply that $(\mu\hat{x}^{-1}, \bar{y})$ is a strictly feasible dual solution. Furthermore, it is easy to see that $\hat{x}$ together with $(\mu\hat{x}^{-1}, \bar{y})$ satisfy the second set of KKT conditions, showing that in fact $(\mu\hat{x}^{-1}, \bar{y})$ is an optimal solution for (3.12). Since f^* is a strictly convex function, (3.12) must have at most one optimal solution. This means that $\bar{y} = \hat{y}$ and $\hat{s} = \mu\hat{x}^{-1}$. Thus, $\hat{x}.\hat{s} = \mu\mathbb{1}$.

We now show the converse. Suppose that $(\hat{x}, \hat{s}, \hat{y}) \in Int \times Int^*$ where $\hat{x}.\hat{s} = \mu\mathbb{1}$ for some $\mu > 0$. Then, it is easy to see that $\mu\hat{s}^{-1} - \hat{x} = 0$, showing that the second set of KKT conditions are satisfied by $(\hat{s}, \hat{y})$ together with $\hat{x}$.

Similarly, it is easy to see that $c - \mu\hat{x}^{-1} - A^T\hat{y} = 0$, showing that the first set of KKT conditions are satisfied by $\hat{x}$ together with $\hat{y}$.

These two observations show that $\hat{x}$ is optimal for (3.11) and $(\hat{s}, \hat{y})$ is optimal for (3.12) for the given value of μ. This proves that $(\hat{x}, \hat{s}, \hat{y})$ lies on the central path. □

In particular, the above two theorems imply the following easy corollary about the mapping $(x, s, y) \mapsto \sqrt{x.s}$.

Corollary 3.3. *Assume that A is full-rank and that Int, Int^* are nonempty. Then*

1. *The mapping $(x, s, y) \mapsto \sqrt{x.s}$ is a bijection from $Int \times Int^*$ onto $\mathbb{R}^n_{++}$.*
2. *The primal-dual central path for (LP) is the set*

$$\left\{(x, s, y) \in Int \times Int^* \mid \sqrt{x.s} = \nu\mathbb{1} \text{ for some } \nu > 0\right\}.$$

More specifically, this corollary is a direct consequence of Theorems 3.1 and 3.2, together with the fact that the mapping $w \mapsto \sqrt{w}$ is a bijection on $\mathbb{R}^n_{++}$.

3.3.4 The primal-dual setting: search directions

For the primal-dual case, we will introduce the general derivation of the search direction in the V-space, the scaled space on which we can simultaneously consider the primal and dual iterates. Suppose that we have a pair of strictly feasible solutions $(x, s, y) \in Int \times Int^*$. We choose a scaling matrix

$$D = S^{-1/2} X^{1/2}, \tag{3.13}$$

where $S = \text{diag } s$ and $X = \text{diag } x$, so that the primal solution x and the dual solution s are mapped to:

$$\begin{aligned} D^{-1} x &= X^{-1/2} S^{1/2} x = \sqrt{x.s} \\ Ds &= S^{-1/2} X^{1/2} s = \sqrt{x.s}. \end{aligned}$$

That is, they are mapped to the same point, namely

$$v := \sqrt{x.s}. \tag{3.14}$$

By Corollary 3.3, (x, s, y) lies on the primal-dual central path if and only if v is a multiple of the vector of all ones.

The search directions in the primal and dual spaces are denoted (d_x, d_s, d_y). However, we often will focus our attention to just d_x and d_s, treating the s-component as the "main" dual solution, because for a fixed choice of d_s, the choice of d_y is uniquely determined (under the assumption that A is full-rank). These directions correspond to the search directions p_x, p_s in the transformed space: $p_x = D^{-1}d_x$ and $p_s = Dd_s$. (The y-component of the search direction is not scaled: $d_y = p_y$.)

Because d_x and d_s always lie in orthogonal subspaces $\ker A$ and $(\ker A)^\perp$, respectively, then p_x and p_s also lie in orthogonal subpsaces, namely $\ker AD$ and $(\ker AD)^\perp$, respectively. Then we can solve for one search direction, which we will denote by p_v, then orthogonally decompose p_v into p_x and p_s:

$$\begin{aligned} p_x &= \mathrm{Proj}_{\ker AD} p_v, \\ p_s &= \mathrm{Proj}_{(\ker AD)^\perp} p_v. \end{aligned}$$

Once p_x, p_s are found, we can transform them back to obtain d_x, d_s. Therefore, in order to specify the set of possible scaled search directions p_x, p_s, it is sufficient to specify the constraints on p_v.

The various primal-dual affine-scaling algorithms that we will see differ in the choice of scaled search direction p_v. That is, in obtaining p_v, they solve optimization problems over different Dikin-ellipsoid-like constraints in the V-space. As for the objective value, minimizing the duality gap between the current strictly feasible primal and dual solutions makes sense, and this is what serves as the objective to be minimized in all primal-dual affine-scaling algorithms that we will discuss in this chapter.

To write this objective in terms of p_v, we observe that if we update our solution to $x + d_x$ and $s + d_s$, the new duality gap is

$$\langle x + d_x, s + d_s \rangle = \langle x, s \rangle + \langle x, d_s \rangle + \langle s, d_x \rangle + \langle d_x, d_s \rangle.$$

Note however, that d_x lies in the nullspace of A while s lies in its orthogonal space. So, $\langle d_x, d_s \rangle = 0$. Furthermore, rewriting the above expression in terms of the corresponding scaled points, we obtain that the duality gap is equal to

$$\langle v, v\rangle + \langle v, p_s\rangle + \langle v, p_x\rangle = \langle v, v\rangle + \langle v, (p_x + p_s)\rangle = \langle v, v\rangle + \langle v, p_v\rangle.$$

Since for fixed x and s, $\langle v, v\rangle$ is a constant, we can just minimize $\langle v, p_v\rangle$.

In fact, all three primal-dual affine-scaling algorithms that we will see obtain p_v by solving problems of the following form:

$$\begin{aligned} \min \quad & \langle v, p_v\rangle \\ s.t. \quad & p_v \in E, \end{aligned} \tag{3.15}$$

where E is a region that can be seen a primal-dual analog of the Dikin's ellipsoid. The algorithms differ in the choice of the ellipsoidal constraint E and in the step size.

3.4 Primal-dual affine-scaling algorithms

In this section, we will compare several primal-dual affine-scaling algorithms, due to Monteiro, Adler, and Resende in [25]; Jansen, Roos, and Terlaky in [10]; and Sturm and Zhang in [30]. We will use the framework just described to introduce the algorithms.

In each of the following sections, let (x, y, s) denote the current iterate and let $D = S^{-1/2}X^{1/2}$ be the scaling matrix and $v = \sqrt{x.s}$ be the current iterate in the scaled space, as specified in (3.13)-(3.14).

3.4.1 Monteiro, Adler, and Resende

The search direction

The scaled search direction p_v that is used in this algorithm is obtained by solving

$$\begin{aligned} \min \quad & \langle v, p_v \rangle \\ s.t. \quad & \|p_v\| \leq 1. \end{aligned}$$

Note that the "Dikin ellipsoid" in this case is just a closed unit ball in the V-space. Hence, the optimal solution is $p_v = -\frac{v}{\|v\|}$, which resulted in the following search directions in the unscaled primal and dual spaces (omitting the normalizing factor of $\frac{1}{\|v\|}$ to put more focus on the direction itself and less on the length):

$$\begin{aligned} d_x &= Dp_x = -D \operatorname{Proj}_{kerAD} v, \\ d_s &= D^{-1} p_s = -D^{-1} \operatorname{Proj}_{(kerAD)^{\perp}} v. \end{aligned}$$

Using (3.3) and (3.5), and substituting in $v = \sqrt{x.s} = D^{-1}x = Ds$, we obtain:

$$\begin{aligned} d_x &= -D \operatorname{Proj}_{kerAD}(Ds) \\ &= -D \operatorname{Proj}_{kerAD}(Dc) \\ &= -D(I - (AD)^T (AD^2 A^T)^{-1} AD)Dc \\ &= -D^2(I - A(AD^2 A^T)^{-1} AD^2)c, \end{aligned} \tag{3.16}$$

$$\begin{aligned} d_s &= -D^{-1}\operatorname{Proj}_{(\ker AD)^\perp}(D^{-1}x) \\ &= -D^{-1}(AD)^T(AD^2A^T)^{-1}(AD)D^{-1}x \\ &= -A^T(AD^2A^T)^{-1}Ax \\ &= -A^T(AD^2A^T)^{-1}b, \end{aligned} \tag{3.17}$$

$$d_y = (AD^2A^T)^{-1}b. \tag{3.18}$$

In the second equality for d_x above, we used the fact that $Ds = D(c - A^T y) = Dc - (AD)^T y$ and that $\operatorname{Proj}_{(\ker AD)^\perp}((AD)^T y) = 0$.

Note that the search direction (d_x, d_s, d_y) given in (3.16) - (3.18) above coincides with the primal affine-scaling search direction in (3.4) and the dual affine-scaling search direction in (3.6) - (3.7) for this particular D, when omitting the normalizing constant.

Remarks

We can also obtain this search direction by starting from a different point of view, namely by considering the barrier function problem for the primal problem, with parameter $\mu > 0$:

$$\begin{array}{ll} \min & \langle c, x\rangle - \mu \sum_{i=1}^{n} \ln x_i \\ s.t. & Ax = b. \end{array}$$

The KKT conditions for this problem is as follows: x is optimal if there exist $s \in \mathbb{R}^n, y \in \mathbb{R}^m$ such that

$$\begin{aligned} XS\mathbb{1} &= \mu\mathbb{1} \\ Ax &= b \\ A^T y + s &= c. \end{aligned}$$

Suppose that $(x, s, y) \in \mathbb{R}^{n+n+m}$ satisfies

$$\begin{aligned} Ax &= b \\ A^T y + s &= c. \end{aligned}$$

Then, we can take a Newton step (d_x, d_s, d_y) towards a point $(x + d_x, s + d_s, y + d_y)$ that satisfies the KKT conditions for the above barrier problem. This direction must satisfy

$$\begin{aligned} S d_x + X d_s &= -XS\mathbb{1} + \mu\mathbb{1} \\ A d_x &= 0 \\ A^T d_y + d_s &= 0. \end{aligned}$$

The algorithm of Monteiro et al. solves the search direction (d_x, d_s, d_y) for $\mu = 0$. That is, the search direction that satisfies:

$$\begin{aligned} S d_x + X d_s &= -XS\mathbb{1} \\ A d_x &= 0 \\ A^T d_y + d_s &= 0. \end{aligned}$$

The solution of the above system is:

$$d_x = -S^{-1}X\left(I - A^T(AS^{-1}XA^T)^{-1}AS^{-1}X\right)c, \tag{3.19}$$

$$d_s = -A^T(AS^{-1}XA^T)^{-1}b, \tag{3.20}$$

$$d_y = (AS^{-1}XA^T)^{-1}b, \tag{3.21}$$

which agrees with the direction in (3.16) - (3.18) when we explicitly write $D = S^{-1/2}X^{1/2}$.

3.4.2 Jansen, Roos, Terlaky

The search direction

The scaled search direction p_v that is used in this algorithm is obtained by solving

$$\begin{array}{rl} \min & \langle v, p_v \rangle \\ s.t. & \|V^{-1}p_v\| \leq 1. \end{array}$$

Here, the Dikin ellipsoid is the ellipsoid in the V-space described by the constraint $\|V^{-1}p_v\| \leq 1$, where $V = \operatorname{diag} v$. We can think of this region as just the closed unit ball that has been scaled by the scaling matrix V, or equivalently, as the closed unit ball with respect to the local inner product norm at v. This is in contrast to just a closed unit ball with respect to the 2-norm in the scaled search-direction problem that is used by Monteiro, Adler, and Resende's algorithm. The optimal solution of this problem is $p_v = -\frac{v^3}{\|v^2\|}$, which corresponds to the search direction (again, omitting the normalizing constant):

$$\begin{aligned} d_x &= Dp_x = -D\,\text{Proj}_{\ker AD} v^3 \\ d_s &= D^{-1}p_s = -D^{-1}\,\text{Proj}_{(\ker AD)^{\perp}} v^3. \end{aligned}$$

Using (3.3) and (3.5), and substituting in $D = S^{-1/2}X^{1/2}$ and $v^3 = S^{3/2}X^{3/2}\mathbb{1} = DS^2X\mathbb{1} = D^{-1}SX^2\mathbb{1}$, we obtain:

$$\begin{aligned} d_x &= -D(I - (AD)^T(AD^2A^T)^{-1}(AD))DS^2X\mathbb{1} \\ &= -D^2(I - A^T(AD^2A^T)^{-1}AD^2)S^2X\mathbb{1} \\ &= -S^{-1}X(I - A^T(AS^{-1}XA^T)^{-1}AS^{-1}X)S^2X\mathbb{1}, \end{aligned}$$

$$\begin{aligned} d_s &= -A^T(AD^2A^T)^{-1}(AD)D^{-1}SX^2\mathbb{1} \\ &= -A^T(AD^2A^T)^{-1}ASX^2\mathbb{1} \\ &= -A^T(AS^{-1}XA^T)^{-1}ASX^2\mathbb{1} \end{aligned}$$

$$d_y = (AS^{-1}XA^T)^{-1}ASX^2\mathbb{1}.$$

Remarks

We note that the search direction in [25] and that in [10] coincide if v and v^3 are scalar multiples of one another. This happens if v is a scalar multiple of $\mathbb{1}$, the vector of all ones. Therefore, by Corollary 3.3, the two search directions coincide whenever the current iterate (x, s, y) lies on the central path.

3.4.3 Sturm and Zhang

The search direction

The scaled search direction p_v in this algorithm is obtained by solving

$$\begin{array}{lll} \min & \langle v, p_v \rangle & \\ s.t. & \cos \angle \left(\begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix}, \begin{pmatrix} v \\ v + p_v \end{pmatrix} \right) & \geq \sqrt{\dfrac{2n-1}{2n}}. \end{array}$$

Here, the "Dikin elipsoid" is in fact a convex quadratic cone. We consider the set of all vectors $(v, v + p_v)$ whose angles with $(\mathbb{1}, \mathbb{1})$ are at most $\arccos \sqrt{(2n-1)/(2n)}$ in $\mathbb{R}^{n+n}_{++}$.

The above problem might not have a solution if the angle between $\mathbb{1}$ and v is too large. Hence, we assume that the initial iterate v lies within a neighborhood of e (i.e. the current solution lies in a neighborhood of the central path) given by:

$$\mathcal{N}(\beta) := \{v \in \mathbb{R}^n_+ \mid \sqrt{n-1} \tan \angle(\mathbb{1}, v) \leq \beta, \}$$

for some $\beta \in (0, 1)$. That is, the angle between $\mathbb{1}$ and v is at most $\arccos \sqrt{\frac{n-1}{n-1+\beta^2}}$.

The solution to the optimization problem above is given by the following theorem (see [30, Theorem 2.1]):

Theorem 3.4. *If $v \in \mathcal{N}(\beta)$ for some $\beta \in (0, 1)$ and $v \neq 0$, then the optimal solution to the above problem is given by*

$$p_v = -\frac{\xi+1}{\xi} v + \frac{\xi-1}{\xi} \frac{\langle \mathbb{1}, v \rangle}{n-1} \mathbb{1},$$

where $\xi = \sqrt{2n/(1-\delta^2) - 1}$ and $\delta = \sqrt{n-1} \tan \angle(\mathbb{1}, v)$.

The theorem asserts that, provided that the current iterate v lies in a particular neighborhood, the optimal scaled search direction p_v is a linear combination of $-v$ and $\mathbb{1}$. Thus, the search direction is

$$\begin{aligned} d_x &= -\frac{\xi+1}{\xi} D \mathrm{Proj}_{\ker(AD)} v + \frac{\xi-1}{\xi} \frac{\langle \mathbb{1}, v\rangle}{n-1} D \mathrm{Proj}_{\ker(AD)} \mathbb{1}, \\ d_s &= -\frac{\xi+1}{\xi} D^{-1} \mathrm{Proj}_{(\ker(AD))^\perp} v + \frac{\xi-1}{\xi} \frac{\langle \mathbb{1}, v\rangle}{n-1} D^{-1} \mathrm{Proj}_{(\ker(AD))^\perp} \mathbb{1}. \end{aligned}$$

Remarks

Recall that the search direction of the primal-dual affine-scaling algorithm of the search direction in [25] is $p_v = -\frac{v}{\|v\|}$. Thus, the search direction of the algorithm of Sturm and Zhang is in fact a linear combination of that in [25] with the vector of all ones.

We again observe that when v is a scalar multiple of the vector of all ones, then the search direction of Sturm and Zhang coincides with the search directions of Monteiro et al. and of Jansen et al.

In [3], this algorithm is extended to semidefinite programming. In [6], the semidefinite programming version of [25] and [10] are presented.

3.5 Other related work

Two other papers contain works that are closely related to the work that we present in this thesis. The first is the work by Chua [4] where he develops a primal-dual algorithm for semidefinite programming and general symmetric cone optimization problems. Chua's algorithm can be seen as a cone affine-scaling algorithm which alternates between solving a restriction of the primal problem (over a quadratic cone that is inscribed in the positive semidefinite cone) and solving a restriction of the dual problem (over a different quadratic cone that is inscribed in the positive semidefinite cone). The second is the work by Litvinchev [18] which develops a primal algorithm for linear programming, based on the same family of quadratic cones. Litvinchev's algorithm will be discussed thoroughly because of the similarities with our work, in the way it utilizes quadratic cone relaxations for finding search directions.

Our algorithm is related to the above two works through the use of the same family of quadratic cones, among others. We postpone our discussion of Chua's and Litvinchev's work until Section 5.8 because we will be able to present a more thorough discussion after we introduce this family of quadratic cones in the next chapter, and after we present and thoroughly analyze our algorithm in Chapter 5. In addition, Chua's and Litvinchev's algorithms are quite different from the three that we just presented and do not quite fit the V-space-based primal-dual framework under which we presented the three primal-dual affine-scaling algorithms in the preceding section.

CHAPTER 4

A FAMILY OF QUADRATIC CONES AND THE QUADRATIC-CONE RELAXATIONS OF LP

4.1 The quadratic cones

The standard second-order cone in $\mathbb{R}^n$ is the set

$$K_{SO} = \left\{ x \in \mathbb{R}^n \mid x_1 \geq \sqrt{x_2^2 + \ldots + x_n^2} \right\}.$$

Geometrically, it is the set of points whose angle with the vector $e_1 := (1, 0, \ldots, 0)^T$ is at most $\pi/4$. We can rewrite the inequality describing K_{SO} as

$$K_{SO} = \left\{ x \in \mathbb{R}^n \mid \langle e_1, x \rangle \geq \frac{1}{\sqrt{2}} \|e_1\| \, \|x\| \right\},$$

where, here and throughout this chapter, $\langle \cdot, \cdot \rangle$ denotes the dot product in $\mathbb{R}^n$ and $\| \cdot \|$ the 2-norm.

So, e_1 is the "center direction" of the second-order cone, with every other point in the cone lying within an circular slice of the cone, centered at e_1. Rearranging the inequality, we obtain

$$\frac{\langle e_1, x \rangle}{\|e_1\| \, \|x\|} \geq \frac{1}{\sqrt{2}},$$

or equivalently,

$$\angle(e_1, x) \leq \arccos \frac{1}{\sqrt{2}} = \pi/4.$$

It is now more apparent that the cone K_{SO} consists of points x whose angles with e_1 are at most $\pi/4$.

In fact, for each $e \in \mathbb{R}^n$ and each $\theta \in [0, \pi/2]$, the set

$$\{x \in \mathbb{R}^n \mid \langle e, x \rangle \geq \cos\theta \, \|e\| \, \|x\|\}$$

is a cone, consisting of points x which form angles that are at most θ with the center direction e. When θ is near zero, the corresponding cone is “thinner”, approaching a ray, while for a greater value of θ, the cone is “wider”. The cones are convex as long as θ is at most $\pi/2$, with the cone being a halfspace when θ is exactly $\pi/2$.

The family of cones that we are interested in are the cones which contain the nonnegative orthant. In particular, for each $r \in [0, 1]$, define

$$K_{\mathbb{1},r} := \{x \in \mathbb{R}^n \mid \langle \mathbb{1}, x\rangle \geq r\|x\|\}, \tag{4.1}$$

the set of points whose angle with $\mathbb{1} := (1, \ldots, 1)^T \in \mathbb{R}^n$ (the vector of all ones) are at most $\arccos \frac{r}{\sqrt{n}}$. Note that when r is near one, the resulting cone is thinner, while when r is near zero, the resulting cone is wider, approaching a halfspace. Since r does not exceed one, the cone cannot be too thin and always contains the nonnegative orthant, as we will see in the next section.

4.1.1 Properties of the cone $K_{\mathbb{1},r}$

Proposition 4.1. *For each $r \in [0, 1]$, the cone $K_{\mathbb{1},r}$ contains the nonnegative orthant.*

Proof. Suppose that $x \in \mathbb{R}^n_+$. Then, $\langle \mathbb{1}, x\rangle \geq 0$ and

$$\langle \mathbb{1}, x\rangle^2 = \left(\sum_{i=1}^n x_i\right)^2 \geq \sum_{i=1}^n x_i^2 = \|x\|^2 \geq r^2\|x\|^2,$$

showing that $x \in K_{\mathbb{1},r}$. This proves that $\mathbb{R}^n_+ \subseteq K_{\mathbb{1},r}$. □

When $r = 0$, the cone $K_{\mathbb{1},0}$ is the halfspace $\{x \in \mathbb{R}^n \mid \langle \mathbb{1}, x\rangle \geq 0\}$. Thus, $K_{\mathbb{1},0}$ is a closed convex cone whose interior is nonempty, but it is not a regular cone because it contains a nontrivial linear subspace: For any $x \in \partial K_{\mathbb{1},0}$, the linear subspace spanned by x is also contained in $\partial K_{\mathbb{1},0}$. However, for all $r \in (0, 1]$, we claim that $K_{\mathbb{1},r}$ is regular.

Note that we can extend the cones described in (4.1) for all $r \in [0, \sqrt{n}]$, but for $r > 1$, the cones $K_{\mathbb{1},r}$ do not contain $\mathbb{R}^n_+$. In particular, $K_{\mathbb{1},\sqrt{n}}$ is a single ray in the direction of the vector of all ones. (To see this, suppose that $x \in K_{\mathbb{1},\sqrt{n}}$. Then,

$$\langle \mathbb{1}, x\rangle \geq \sqrt{n}\|x\|.$$

Since $\|\mathbb{1}\| = \sqrt{n}$, then either $x = 0$ or x satisfies

$$\cos\angle(\mathbb{1}, x) = \frac{\langle \mathbb{1}, x\rangle}{\|\mathbb{1}\|\,\|x\|} \geq \frac{\sqrt{n}}{\|\mathbb{1}\|} = 1.$$

Since $\cos\angle(\mathbb{1}, x)$ cannot exceed 1, it is exactly equal to 1, which means that x must be a positive multiple of $\mathbb{1}$.)

Proposition 4.2. *For each $r \in [0, 1]$, the dual of the cone $K_{\mathbb{1},r}$ is*

$$K^*_{\mathbb{1},r} = \left\{ s \in \mathbb{R}^n \mid \langle \mathbb{1}, s\rangle \geq \sqrt{n - r^2}\|s\| \right\}.$$

In other words, $K^*_{\mathbb{1},r} = K_{\mathbb{1},\sqrt{n-r^2}}$.

Proof. First, consider the case when $r = 0$. The cone $K_{\mathbb{1},0}$ is the halfspace $\{x \in \mathbb{R}^n \mid \langle \mathbb{1}, x\rangle \geq 0\}$. Clearly, the dual consists only of positive multiples of $\mathbb{1}$, which we know from the preceding discussion to be just the cone $K_{\mathbb{1},\sqrt{n}}$. Thus, $K^*_{\mathbb{1},0} = K_{\mathbb{1},\sqrt{n}}$.

For $r \in (0, 1]$, we first show that $K_{\mathbb{1},\sqrt{n-r^2}} \subseteq K^*_{\mathbb{1},r}$. Suppose that $s \in K_{\mathbb{1},\sqrt{n-r^2}}$. Then, the angle between s and $\mathbb{1}$ satisfies

$$\angle(\mathbb{1}, s) \leq \arccos\sqrt{(n - r^2)/n}.$$

For all $x \in K_{\mathbb{1},r}$,

$$\angle(x, s) \leq \angle(\mathbb{1}, x) + \angle(\mathbb{1}, s).$$

Therefore, for all $x \in K_{\mathbb{1},r}$,

$$\begin{aligned}
\arccos \frac{\langle x, s\rangle}{\|x\|\,\|s\|} &\leq \arccos(r/\sqrt{n}) + \arccos\sqrt{(n-r^2)/n} \\
\frac{\langle x, s\rangle}{\|x\|\,\|s\|} &\geq \cos\left(\arccos(r/\sqrt{n}) + \arccos\sqrt{(n-r^2)/n}\right) \\
&= \frac{r}{\sqrt{n}}\sqrt{\frac{n-r^2}{n}} - \sin\left(\arccos\frac{r}{\sqrt{n}}\right)\sin\left(\arccos\sqrt{\frac{n-r^2}{n}}\right) \\
&= \frac{r\sqrt{n-r^2}}{n} - \sqrt{\frac{n-r^2}{n}}\,\frac{r}{\sqrt{n}} = 0,
\end{aligned}$$

showing that $\langle x, s\rangle \geq 0$ for all $x \in K_{\mathbb{1},r}$. Thus, $K_{\mathbb{1},\sqrt{n-r^2}} \subseteq K^*_{\mathbb{1},r}$.

Next, we show that $K^*_{\mathbb{1},r} \subseteq K_{\mathbb{1},\sqrt{n-r^2}}$ for $r \in (0, 1]$. Suppose that $s \notin K_{\mathbb{1},\sqrt{n-r^2}}$.

If $\langle \mathbb{1}, s\rangle \leq 0$, then there exists a point x near $\mathbb{1}$ such that $\langle x, s\rangle < 0$, showing that $s \notin K^*_{\mathbb{1},r}$. Then, consider $s \notin K_{\mathbb{1},\sqrt{n-r^2}}$ where

$$0 < \langle \mathbb{1}, s\rangle < \sqrt{n-r^2}\|s\|.$$

This means that the angle between $\mathbb{1}$ and s is large,

$$\angle(\mathbb{1}, s) > \arccos\sqrt{(n-r^2)/n},$$

and that we can construct an element of $K_{\mathbb{1},r}$ such that the angle between s and this element is greater than $\pi/2$ (hence, its inner product with s is negative).

Without loss of generality, assume that $\langle \mathbb{1}, s\rangle = n$. This implies that

$$n < \sqrt{n-r^2}\|s\|,$$

and hence,

$$\sqrt{\|s\|^2 - n} > \sqrt{\frac{nr^2}{n-r^2}}. \tag{4.2}$$

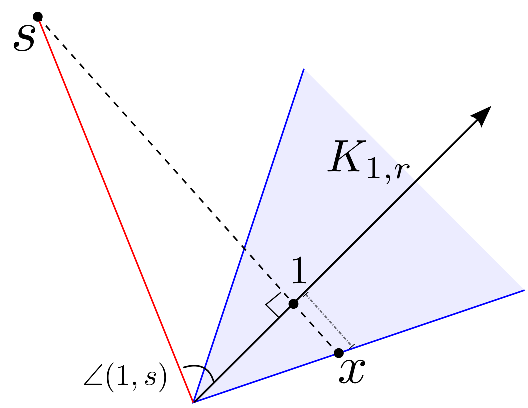
Letting

$$x := \mathbb{1} - \sqrt{\frac{n(n-r^2)}{r^2(\|s\|^2 - n)}}(s - \mathbb{1}),$$

we can easily check that

$$\langle \mathbb{1}, x\rangle = n \text{ and } \langle x, x\rangle = \frac{n^2}{r^2},$$

which implies that $\frac{\langle \mathbb{1}, x\rangle}{\|x\|\ \|\mathbb{1}\|} = \frac{r}{\sqrt{n}}$. That is, $x \in K_{\mathbb{1},r}$.



However, by (4.2),

$$\begin{aligned}
\langle x, s\rangle &= \langle \mathbb{1}, s\rangle - \sqrt{\frac{n(n-r^2)}{r^2(\|s\|^2 - n)}}\ \langle s - \mathbb{1}, s\rangle \\
&= \langle \mathbb{1}, s\rangle - \sqrt{\frac{n(n-r^2)}{r^2}}\ \sqrt{\|s\|^2 - n} \\
&< n - \sqrt{\frac{n(n-r^2)}{r^2}}\ \sqrt{\frac{nr^2}{n-r^2}} = n - n = 0,
\end{aligned}$$

showing that $s \notin K^*_{\mathbb{1},r}$. This concludes our proof. □

Corollary 4.3. *For each $r \in [0,1]$, the cone $K^*_{\mathbb{1},r} = K_{\mathbb{1},\sqrt{n-r^2}}$ is contained in the nonnegative orthant.*

Proof. From the definition of the dual cone, it is straightforward to see that if $K_1 \subseteq K_2$, then $K_1^* \supseteq K_2^*$.

Since $K_{\mathbb{1},r} \supseteq \mathbb{R}^n_+$, then $K^*_{\mathbb{1},r} \subseteq (\mathbb{R}^n_+)^* = \mathbb{R}^n_+$. $\square$

4.1.2 The cones $K_{e,r}$

For each $e \in \mathbb{R}^n_{++}$, define

$$K_{e,r} := \{x \in \mathbb{R}^n \mid \langle \mathbb{1}, x./e \rangle \geq r\|x./e\|\}, \tag{4.3}$$

where the notation "$x./e$" means "component-wise division of x by e". Note that this operation is well-defined because e does not have any zero components.

Let E denote the diagonal matrix whose diagonal components are the vector e. Since $e \in \mathbb{R}^n_{++}$, then E has an inverse, E^{-1}, and both are linear scalings on $\mathbb{R}^n$ which fix $\mathbb{R}^n_{++}$. It is not difficult to see that

$$x \in K_{e,r} \quad \text{if and only if} \quad E^{-1}x = x./e \in K_{\mathbb{1},r}. \tag{4.4}$$

In other words,

$$E^{-1}K_{e,r} = K_{\mathbb{1},r} \quad \text{or, equivalently,} \quad K_{e,r} = EK_{\mathbb{1},r}, \tag{4.5}$$

where $E^{-1}K_{e,r}$ denotes image of $K_{e,r}$ under the scaling by E^{-1}, namely $\{E^{-1}x \mid x \in K_{e,r}\}$; similarly for $EK_{\mathbb{1},r}$.

More generally, for any scaling matrix D that fixes $\mathbb{R}^n_+$, observe that

$$DK_{e,r} = K_{De,r}. \tag{4.6}$$

By any scaling matrix D, we mean any diagonal $n \times n$ matrix with positive diagonal entries. These are scalings that are invertible and fix the nonnegative orthant. Hence, any linear program that is scaled by such a matrix D remains a linear program.

It easily follows that $K_{e,r}$ inherits some of the important properties of $K_{\mathbb{1},r}$.

Corollary 4.4. *For each $e \in \mathbb{R}^n_{++}$ and $r \in [0,1]$, the cone $K_{e,r}$ contains the nonnegative orthant.*

Proof. Suppose that $x \in \mathbb{R}^n_+$. Clearly, $E^{-1}x \in \mathbb{R}^n_+ \subseteq K_{\mathbb{1},r}$ as well. Thus, by (4.4), we see that $x \in K_{e,r}$. □

We can extend the cone definition (4.3) to all values of $r \in [0, \sqrt{n}]$. Note that the cone $K_{e,0}$ is just the halfspace $\{x \in \mathbb{R}^n \mid \langle e^{-1}, x\rangle \geq 0\}$, while

$$K_{e,\sqrt{n}} = EK_{\mathbb{1},\sqrt{n}} = E\{\gamma\mathbb{1} \mid \gamma \geq 0\} = \{\gamma e \mid \gamma \geq 0\}$$

is a ray in direction e. The cones $K_{e,0}$ are closed convex cones whose interiors are nonempty, but they are not regular because their boundaries are hyperplanes, which are nontrivial linear subspaces. On the other hand, for all $r > 0$, the cones $K_{e,r}$ do not contain any nontrivial linear subspaces, and therefore are regular cones. For $r \in [0,1]$, the following proposition describes the dual of the cones $K_{e,r}$.

Corollary 4.5. *For each $r \in [0,1]$ and each $e \in \mathbb{R}^n_{++}$, the dual of $K_{e,r}$ with respect to the inner product $\langle\cdot,\cdot\rangle$, denoted $K^*_{e,r}$, is*

$$K^*_{e,r} = K_{e^{-1},\sqrt{n-r^2}}.$$

In other words,

$$K^*_{e,r} = \left\{s \in \mathbb{R}^n \mid \langle\mathbb{1}, s.e\rangle \geq \sqrt{n-r^2}\|s.e\|\right\}.$$

Furthermore, $K^*_{e,r} \subseteq \mathbb{R}^n_+$.

Proof. We have that $s \in K^*_{e,r}$ if and only if for each $x \in K_{e,r}$,

$$\langle s.e, x./e\rangle = \langle s, x\rangle \geq 0.$$

Since $x \in K_{e,r}$ if and only if $x./e \in K_{\mathbb{1},r}$, then $s \in K^*_{e,r}$ if and only if $s.e \in (K_{\mathbb{1},r})^*$, if and only if $s \in E^{-1}K_{\mathbb{1},\sqrt{n-r^2}}$. From this, and using Proposition 4.2 and (4.5),

$$K^*_{e,r} = E^{-1}K^*_{\mathbb{1},r} = E^{-1}K_{\mathbb{1},\sqrt{n-r^2}} = K_{e^{-1},\sqrt{n-r^2}}.$$

Furthermore, we know from Corollary 4.3 that $K^*_{\mathbb{1},r} \subseteq \mathbb{R}^n_+$. Since E^{-1} is a diagonal matrix with only positive components in its diagonal, then it maps $\mathbb{R}^n_{++}$ to $\mathbb{R}^n_{++}$. Therefore, $K^*_{e,r} = E^{-1}K^*_{\mathbb{1},r} \subseteq \mathbb{R}^n_+$. □

4.2 Quadratic-cone relaxations of *LP*

Given $r \in [0,1]$ and $e \in \mathbb{R}^n_{++}$, consider the following pair of cone optimization problems, where the conic constraints are given by $K_{e,r}$ and its dual:

$$\begin{array}{rll} \min & \langle c, x\rangle & \\ s.t. & Ax = b & (QP_{e,r}) \\ & x \in K_{e,r}, & \end{array}$$

$$\begin{array}{rll} \max & b^T y & \\ s.t. & A^T y + s = c & (QP_{e,r})^* \\ & x \in K^*_{e,r}. & \end{array}$$

We are interested in $(QP_{e,r})$ and $(QP_{e,r})^*$ in relation to the pair of linear programming problems

$$\begin{aligned} \min \quad & \langle c, x \rangle \\ s.t. \quad & Ax = b \qquad (LP) \\ & x \in \mathbb{R}^n_+, \end{aligned}$$

$$\begin{aligned} \max \quad & b^T y \\ s.t. \quad & A^T y + s = c \qquad (LP)^* \\ & s \in \mathbb{R}^n_+. \end{aligned}$$

We assume that c^T and the rows of A are linearly independent (in particular, A is full rank) and that $b \neq 0$. Thus, the zero vector is not a feasible primal solution and the set of optimal solutions of (LP) is not the entire feasible region.

We note that since $K_{e,r} \supseteq \mathbb{R}^n_+$ and $K^*_{e,r} \subseteq \mathbb{R}^n_+$, then $(QP_{e,r})$ is a relaxation of (LP) while $(QP_{e,r})^*$ is a restriction of $(LP)^*$.

We make the standard assumption that (LP) and $(LP)^*$ are strictly feasible, which implies that each has an optimal solution. However, whether $(QP_{e,r})$ has an optimal solution and $(QP_{e,r})^*$ is feasible depend on the choices of r and e.

Since $(QP_{e,r})$ is a relaxation of (LP), then $(QP_{e,r})$ is always feasible. However, it might not have an optimal solution; that is, it might have unbounded objective. On the other hand, since $(QP_{e,r})^*$ is a restriction of $(LP)^*$, which is assumed to have an optimal solution, then $(QP_{e,r})^*$ would never have unbounded objective, but it could have an empty feasible region.

For a fixed choice of r, we will give a name to the set of vectors e for which $(QP_{e,r})$ has an optimal solution.

Definition 4.1. Given (LP) and $r \in [0, 1]$, let Swath(r) denote the set of points $e \in Int$ such that $(QP_{e,r})$ has an optimal solution.

We will later show that, when $(QP_{e,r})$ has an optimal solution, the dual problem $(QP_{e,r})^*$ does too (Section 5.1).

Proposition 4.6. *Assume that* (LP) *and* $(LP)^*$ *are strictly feasible, with* c^T *and the rows of* A *linearly independent. Then,*

1. *Swath*(0) *is the central path for* (LP)
2. *For* $0 \leq r_1 \leq r_2 \leq 1$*, Swath*$(r_1) \subseteq$ *Swath*(r_2)*.*

In particular, this implies that if (LP) *and* $(LP)^*$ *are strictly feasible, then for each* $r \in [0, 1]$*, Swath*(r) *is nonempty.*

Proof. A point $e \in Int$ lies on the central path if and only if it is the optimal solution to a barrier problem (BP_μ) for some $\mu > 0$:

$$\begin{array}{ll} \min & \langle c, x \rangle - \mu \sum_{i=1}^{n} \ln x_i \\ s.t. & Ax = b. \end{array}$$

That is, e lies on the central path if and only if there exist some $\mu > 0$ and $y \in \mathbb{R}^m$ such that the following KKT conditions are satisfied

$$c - \mu(e^{-1}) - A^T y = 0 \tag{4.7}$$

$$Ae = b \tag{4.8}$$

$$e \in \mathbb{R}^n_{++}. \tag{4.9}$$

On the other hand, a point e lies in Swath(0) if and only if $e \in Int$ and the optimization problem $(QP_{e,0})$ below has an optimal solution

$$\begin{aligned} \min \quad & \langle c, x \rangle \\ s.t. \quad & Ax = b \\ & \langle \mathbb{1}, x./e \rangle \geq 0. \end{aligned}$$

That is, $e \in$ Swath(0) if and only if $e \in Int$ and there exist some $x \in \mathbb{R}^n_+$, $\mu \geq 0$, and $y \in \mathbb{R}^m$ such that the following KKT conditions are satisfied

$$c - \mu(e^{-1}) - A^T y = 0 \tag{4.10}$$

$$Ax = b \tag{4.11}$$

$$\langle \mathbb{1}, x./e \rangle \geq 0 \tag{4.12}$$

$$\mu(\langle \mathbb{1}, x./e \rangle) = 0. \tag{4.13}$$

Since we assume that c^T and the rows of A are linearly independent, then c does not lie in the range of A^T. This implies that e^{-1} also must not lie in the range of A^T and that μ must be strictly positive. Furthermore, since $\mu > 0$, then $\langle \mathbb{1}, x./e \rangle = 0$, or equivalently, x lies on the boundary of $K_{e,0}$.

Thus, the second set of KKT conditions, (4.10) - (4.13), can be summarized as follows:

1. There exists some $\mu > 0, y \in \mathbb{R}^m$ such that $c - \mu(e^{-1}) - A^T y = 0$, and
2. There exists some x that satisfies: $Ax = b$ and $\langle \mathbb{1}, x./e \rangle = 0$.

We claim that the existence of such an x that satisfies condition #2 is guaranteed whenever for the given e, condition #1 above is satisfied. To see this: The point e satisfies $Ae = b$ and $\langle \mathbb{1}, e./e \rangle > 0$. Thus, there exists a solution x that meets the above condition if and only if there exists a direction $v \in \ker A$ such that $\langle \mathbb{1}, v./e \rangle \neq 0$. Such a direction v exists as long as e^{-1} is not orthogonal to $\ker A$. That is, as long as there exists some $\mu > 0, y \in \mathbb{R}^m$ such that $c - \mu(e^{-1}) - A^T y = 0$.

Therefore, the KKT conditions (4.10) - (4.13) for $e \in Int$ are exactly the same as the KKT conditions (4.7) - (4.9): $e \in Int$ and there exist some $\mu > 0$ and $y \in \mathbb{R}^m$, $c - \mu(e^{-1}) - A^T y = 0$.

To prove the second assertion, we begin by noting that for each $e \in Int$ and cone-width parameters $r_1, r_2 \in [0, 1]$ with $r_1 \leq r_2$, $K_{e,r_1} \supseteq K_{e,r_2}$.

Suppose that $e \in \text{Swath}(r_1)$, then (QP_{e,r_1}) has an optimal solution. Then (QP_{e,r_2}) also has an optimal solution since it is feasible and (QP_{e,r_1}), which is its relaxation, has an optimal solution. This shows that e is also in $\text{Swath}(r_2)$, proving the second assertion.

In Section 2.3, we saw that if (LP) and $(LP)^*$ are strictly feasible, then the central path is nonempty. This, together with the previous two assertions, implies that for all $r \in [0, 1]$, the set $\text{Swath}(r)$ is nonempty. □

Furthermore, we can make the following claim when $e \in \text{Swath}(r)$.

Proposition 4.7. *Suppose that c^T and the rows of the constraint matrix A are linearly independent and $b \neq 0$. Fix $r \in (0, 1]$. If $e \in Swath(r)$, then $(QP_{e,r})$ has a unique optimal solution.*

Proof. If $e \in \text{Swath}(r)$, then $(QP_{e,r})$ has an optimal solution. We wish to show that there is in fact exactly one optimal solution. Suppose that x^* and $\hat{x}$ are distinct optimal solutions for $(QP_{e,r})$. Letting $v := x^* - \hat{x}$, we see that

$$Av = 0 \text{ and } \langle c, v \rangle = 0.$$

Then, for all $\lambda \in [0,1]$, $x(\lambda) := \hat{x} + \lambda v$ is optimal as well. Furthermore, since the objective function is linear and c^T is assumed to be linearly independent of the rows of A, each $x(\lambda)$ lies on the boundary of the feasible region. That is,

$$0 = \langle \mathbb{1}, x(\lambda)./e) \rangle^2 - r^2 \|x(\lambda)./e\|^2.$$

Expanding the right-hand side,

$$\begin{aligned} 0 \quad = \quad & \left(\langle \mathbb{1}, \hat{x}/e \rangle^2 - r^2 \langle \|\hat{x}\|^2 \right) + 2\lambda \left(\langle \mathbb{1}, \hat{x}./e \rangle \langle \mathbb{1}, v./e \rangle - r^2 \langle \hat{x}./e, v./e \rangle \right) \\ & + \lambda^2 \left(\langle \mathbb{1}, v./e \rangle^2 - r^2 \|v./e\|^2 \right). \end{aligned}$$

Since $\hat{x}$ is on the boundary of the cone, then $\langle \mathbb{1}, \hat{x}/e \rangle^2 - r^2 \|\hat{x}\|^2 = 0$. So,

$$0 \quad = \quad 2 \left(\langle \mathbb{1}, \hat{x}./e \rangle \langle \mathbb{1}, v./e \rangle - r^2 \langle \hat{x}./e, v./e \rangle \right) + \lambda \left(\langle \mathbb{1}, v./e \rangle^2 - r^2 \|v./e\|^2 \right).$$

Since this has to hold for all $\lambda \in [0,1]$, then it must be the case that

$$\langle \mathbb{1}, v./e \rangle^2 - r^2 \|v./e\|^2 = 0$$

and

$$\langle \mathbb{1}, \hat{x}./e \rangle \langle \mathbb{1}, v./e \rangle - r^2 \langle \hat{x}./e, v./e \rangle = 0.$$

The first equality implies that v is on the boundary of the cone. From this, and by the Cauchy-Schwartz inequality, we have that

$$\begin{aligned} \langle \hat{x}./e, v./e \rangle \quad & \leq \quad \|\hat{x}./e\| \, \|v./e\| \\ & = \quad \frac{1}{r^2} \langle \mathbb{1}, \hat{x}./e \rangle \langle \mathbb{1}, v./e \rangle. \end{aligned}$$

So, we know that $r^2 \langle \hat{x}./e, v./e \rangle \leq \langle \mathbb{1}, \hat{x}./e \rangle \langle \mathbb{1}, v./e \rangle$. Furthermore, we know that equality holds if and only if v is a scalar multiple of $\hat{x}$.

Hence, the second equality implies that $\hat{x}$ and x^* are scalar multiples of one another: $\hat{x} = \gamma x^*$ for some $\gamma > 0$. Since $x^*, \hat{x}$ are feasible, $b = Ax^* = A\hat{x} = \gamma A x^*$, which implies that $b = 0$, contradicting our assumption.

Thus, $(QP_{e,r})$ has at most one optimal solution. □

4.3 Solving $(QP_{e,r})$

That $(QP_{e,r})$ can be solved easily is important in Chapter 5 for showing that we have an algorithm that could be practical computationally. We describe how we can find the optimal solution for $(QP_{e,r})$ when $e \in \text{Swath}(r)$. Furthermore, we can identify the case when $e \notin \text{Swath}(r)$ by detecting when $(QP_{e,r})$ has unbounded objective (note that $(QP_{e,r})$ is feasible since we assume that (LP) is). We do this by solving the Fritz John optimality conditions for $(QP_{e,r})$.

Fix $e \in \mathbb{R}^n_{++}$ and consider $r \in (0, 1)$. We start by noting that the cone $K_{e,r}$ is described by the inequality

$$\langle \mathbb{1}, x./e \rangle \geq r \| x./e \|,$$

which is equivalent to the following two inequalities:

$$\langle \mathbb{1}, x./e \rangle^2 - r^2 \| x./e \|^2 \geq 0 \;\text{ and }\; \langle \mathbb{1}, x./e \rangle \geq 0.$$

Note that the first inequality describes the cone $K_{e,r}$ and its "mirror image", $-K_{e,r}$. The second inequality is only satisfied by vectors in $K_{e,r}$.

We will consider the Fritz John optimality condition for a relaxed version of $(QP_{e,r})$:

$$\begin{array}{lc} \min & \langle c, x \rangle \\ s.t. & Ax = b \\ & -(\langle \mathbb{1}, x./e \rangle^2 - r \| x./e \|^2) \leq 0, \end{array} \tag{4.14}$$

where we omit the constraint $\langle \mathbb{1}, x./e \rangle \geq 0$. For simplification purposes, we will solve the Fritz John conditions for this relaxed system. Among the solutions which satisfy the optimality conditions for (4.14), we can check whether it is truly feasible (and optimal) for $(QP_{e,r})$ by checking if it satisfies the omitted constraint $\langle \mathbb{1}, x./e\rangle \geq 0$.

The Fritz John optimality condition for the relaxed problem (4.14) is as follows: The solution $x \in \mathbb{R}^n$ is optimal for $(QP_{e,r})$ if there is $0 \neq (y, \mu, \lambda) \in \mathbb{R}^{m+1}$ and $\mu, \lambda \geq 0$ such that (x, y, μ, λ) satisfies

$$\begin{array}{rcl} -\lambda\Big(\langle \mathbb{1}, x./e\rangle(1./e) - r^2(x./e^2)\Big) \; - \; A^T y \; + \; \mu c & = & 0, \\ Ax & = & b, \\ -(\langle \mathbb{1}, x./e\rangle^2 - r\|x./e\|^2) & \leq & 0 \\ \lambda(\langle \mathbb{1}, x./e\rangle^2 - r^2\|x./e\|^2) & = & 0. \end{array}$$

We know that $\lambda \neq 0$. Otherwise, $\mu c - A^T y = 0$ where $(y, \mu) \neq 0$, which contradicts our assumption that c^T and the rows of A are linearly independent. Thus, we can assume that $\lambda = 1$. We know that an optimal solution x must lie on the boundary of $K_{e,r}$, so the last equality is satisfied.

Simplifying, we need only solve the following conditions for (x, y, μ):

$$\begin{array}{rcll} -\Big(\langle \mathbb{1}, x./e\rangle(1./e) - r^2(x./e^2)\Big) \; - \; A^T y \; + \; \mu c & = & 0, & \text{(a)} \\ Ax & = & b, & \text{(b)} \\ \langle \mathbb{1}, x./e\rangle^2 - r^2\|x./e\|^2 & = & 0. & \text{(c)} \end{array}$$

If such a solution exists and if $\langle \mathbb{1}, x./e\rangle \geq 0$, then x is the optimal solution to $(QP_{e,r})$ and $e \in \text{Swath}(r)$. On the other hand, if such a solution does not exist, then $(QP_{e,r})$ is unbounded and $e \notin \text{Swath}(r)$.

The first two equations are linear in x, thus the set of points (x, y, μ) that satisfy (a) and (b) are solutions to:

$$\begin{bmatrix} -((e^{-1})(e^{-1})^T - r^2 E^{-2}) & -A^T & c \\ A & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \tag{4.15}$$

which is an underdetermined system, with $m+n+1$ variables and $m+n$ equations. Since we assume that c and the rows of A are linearly independent, it is easy to see that the $(m+n) \times (m+n+1)$ matrix on the left hand side above is also full rank. Thus, the above system has a one-dimensional solution space.

We first solve for a solution $z^{(0)} = (x^{(0)}, y^{(0)}, \mu^{(0)})$ satisfying (4.15) and a direction $v = (v_x, v_y, v_\mu)$, a vector in the nullspace, which together describe the solution set for (4.15):

$$\left\{ z^{(0)} + \lambda v \mid \lambda \in \mathbb{R} \right\}.$$

We can then solve for a solution which also satisfies (c) by solving for λ such that

$$\langle \mathbb{1}, (x^{(0)} + \lambda v_x)./e \rangle^2 - r^2 \langle (x^{(0)} + \lambda v_x)./e, (x^{(0)} + \lambda v_x)./e \rangle = 0. \tag{4.16}$$

Note that (4.16) is a quadratic in λ, so we can solve for its roots: $\lambda^{(1)}, \lambda^{(2)}$. In the case that the roots are not real-valued, then we can conclude that the problem has no optimal solution, or equivalently, that e is not in Swath(r). Otherwise, we have two candidates for the optimal solution for $(QP_{e,r})$, namely

$$x^{(1)} = x^{(0)} + \lambda^{(1)} v_x \text{ and } x^{(2)} = x^{(0)} + \lambda_{(2)} v_x,$$

together with $y^{(i)} = y^{(0)} + \lambda^{(i)} v_y$ and $\mu^{(i)} = \mu^{(0)} + \lambda^{(i)} v_\mu$, for each $i = 1, 2$.

Then, for each $i = 1, 2$, we need to check that $\langle \mathbb{1}, x^{(i)}./e \rangle \geq 0$ in order to determine whether $x^{(i)}$ is feasible for $(QP_{e,r})$. If $\langle \mathbb{1}, x^{(i)}./e \rangle < 0$, then $x^{(i)}$ is not feasible for $(QP_{e,r})$ since it lies not in $K_{e,r}$ but in $-K_{e,r}$.

For each $x^{(i)}$ that is feasible, we need to check that it is indeed the minimizing solution—not one that maximizes $\langle c, \cdot \rangle$—by checking that the corresponding "μ", namely

$$\mu^{(i)} = \mu^{(0)} + \lambda^{(i)} v_\mu,$$

is positive.

Thus, the optimal solution for $(QP_{e,r})$ is the feasible $x^{(i)}$ whose corresponding $\mu^{(i)}$ is positive. Furthermore, we know that there cannot be two such feasible solutions which correspond to optimal solutions (Proposition 4.7).

If no such an $x^{(i)}$ exists, we can conclude that $(QP_{e,r})$ does not have an optimal solution and that e is not in Swath(r).

4.4 Local inner product viewpoint

Thus far, we have described the cones $K_{e,r}$ in terms of the inner product $\langle \cdot, \cdot \rangle$ which at the beginning of this chapter we chose to be the dot product. Our discussions on the dual cones and dual quadratic-cone optimization problems have also been done with respect to $\langle \cdot, \cdot \rangle$.

However, in the analysis that follows (in Chapter 5), there are settings in which it is most natural to consider dual cones with respect to a different inner product. In particular, for each $e \in \mathbb{R}^n_{++}$ and each $r \in [0, 1]$, the cone $K_{e,r}$ can be described most naturally in terms of the local inner product at e.

To prepare ourselves for this type of analysis in Chapter 5, we use this section to understand this viewpoint, switching from the dot product to the local inner products.

4.4.1 Norms and angles with respect to the local inner product

Consider a fixed $e \in \mathbb{R}^n_{++}$. Recall from (2.22) that *the local inner product at e* is defined using the Hessian of the standard log-barrier function for the positive orthant, evaluated at the point e under consideration:

$$\langle u, v\rangle_e := \langle u, H(e)^2 v\rangle = \langle u, E^{-2} v\rangle = \langle u./e, v./e\rangle,$$

where $E = \operatorname{diag} e$, the diagonal matrix whose diagonal entries are the components of e. This local inner product induces the norm $\|\cdot\|_e$ which is given by $\|u\|_e = \sqrt{\langle u, u\rangle_e}$.

We can think of using a different inner product as 'changing the "geometry"' of $\mathbb{R}^n$, in the sense that lengths of vectors and angles between vectors are defined, or measured, differently (in comparison to their lengths and angles when we use the inner product $\langle\cdot,\cdot\rangle$). Relating the local inner product at e with the inner product $\langle\cdot,\cdot\rangle$, lengths of vectors are changed in the following manner:

$$\|u\|_e := \sqrt{\langle u, u\rangle_e} = \sqrt{\langle u./e, u./e\rangle} = \|u./e\|.$$

That is, we can think of the length of u with respect to the norm $\|\cdot\|_e$ as the length of the image of u under the affine scaling by the diagonal matrix E^{-1} (mapping u to $E^{-1}u = u./e$) with respect to the norm $\|\cdot\|$.

For instance, note that for any $e \in \mathbb{R}^n_{++}$, the length of e in terms of the norm induced by the local inner product at e is always $\sqrt{n}$:

$$\|e\|_e^2 = \langle e, e\rangle_e = \langle e./e, e./e\rangle = \langle \mathbb{1}, \mathbb{1}\rangle = n,$$

because under the image of e under the affine scaling E^{-1} is just $\mathbb{1}$, whose length with respect to the norm $\|\cdot\|$ is $\sqrt{n}$.

Angles between vectors are also changed in the same manner. In the point of view of the inner product $\langle\cdot,\cdot\rangle$, given two vectors u and v in $\mathbb{R}^n$, the cosine of the angle between them is

$$\cos(\angle(u,v)) = \frac{\langle u,v\rangle}{\|u\|\,\|v\|}.$$

Replacing the inner products and norms from $\langle\cdot,\cdot\rangle$ to $\langle\cdot,\cdot\rangle_e$ and $\|\cdot\|_e$, we define the angle between u and v with respect to this local inner product, which we denote $\angle_e(u,v)$, to satisfy

$$\cos(\angle_e(u,v)) = \frac{\langle u,v\rangle_e}{\|u\|_e\,\|v\|_e}.$$

That is, $\angle_e(u,v)$ is defined as

$$\angle_e(u,v) := \arccos\left(\frac{\langle u,v\rangle_e}{\|u\|_e\,\|v\|_e}\right). \tag{4.17}$$

Again, we can interpret $\angle_e(u,v)$ as the angle between the images of u and v under the affine scaling E^{-1}, with respect to the inner product $\langle\cdot,\cdot\rangle$.

Recall the cone $K_{e,r}$ which we introduced in Section 4.1. For each $e \in \mathbb{R}^n_{++}$, we can re-express the cone $K_{e,r}$ in terms of the local inner product at e:

$$\begin{aligned} K_{e,r} &= \{x \in \mathbb{R}^n \mid \langle \mathbb{1}, x./e\rangle \geq r\|x./e\|\} \\ &= \{x \in \mathbb{R}^n \mid \langle e, x\rangle_e \geq r\|x\|_e\}. \end{aligned} \tag{4.18}$$

Rearranging the inequality in the second line, we see that $K_{e,r}$ is the set of points x which satisfy

$$\frac{\langle e,x\rangle_e}{\|x\|_e\,\|e\|_e} \geq \frac{r}{\sqrt{n}}.$$

That is, $x \in K_{e,r}$ if and only if

$$\cos(\angle_e(e,x)) \geq \frac{r}{\sqrt{n}},$$

or equivalently,

$$\angle_e(e,x) \leq \arccos\left(\frac{r}{\sqrt{n}}\right).$$

4.4.2 Dual cones with respect to the local inner product

In Corollary 4.5, we compute the dual of the cone $K_{e,r}$ with respect to the dot product, $\langle \cdot, \cdot \rangle$:

$$K_{e,r}^* = K_{e^{-1},\sqrt{n-r^2}}.$$

Recall that the notion of the dual of a set depends on the inner product being used. We will now investigate what the dual cones with respect to the local inner products look like.

Fix two vectors $e, \hat{e} \in \mathbb{R}^n_{++}$, and let us consider the dual cone of $K_{e,r}$ with respect to the local inner product at $\hat{e}$, $\langle \cdot, \cdot \rangle_{\hat{e}}$, which we will denote "$K_{e,r}^{*_{\hat{e}}}$":

$$\begin{aligned} K_{e,r}^{*_{\hat{e}}} &= \{\tilde{s} \mid \langle x, \tilde{s} \rangle_{\hat{e}} \geq 0, \ \forall x \in K_{e,r}\} \\ &= \{\tilde{s} \mid \langle x./\hat{e}, \tilde{s}./\hat{e} \rangle \geq 0, \ \forall x \in K_{e,r}\} \\ &= \{\hat{E}^2 s \mid \langle x, \hat{E}^{-2}\hat{E}^2 s \rangle \geq 0, \ \forall x \in K_{e,r}\} \\ &= \{\hat{E}^2 s \mid \langle x, s \rangle \geq 0, \ \forall x \in K_{e,r}\} \\ &= \hat{E}^2 K_{e,r}^*. \end{aligned} \tag{4.19}$$

Thus, for any local inner product $\langle \cdot, \cdot \rangle_{\hat{e}}$, the dual cone of $K_{e,r}$ with respect to this local inner product is

$$\begin{aligned} K_{e,r}^{*_{\hat{e}}} &= \hat{E}^2 K_{e,r}^* = \hat{E}^2 K_{e^{-1},\sqrt{n-r^2}} \\ &= K_{\hat{E}^2 e^{-1},\sqrt{n-r^2}}, \end{aligned} \tag{4.20}$$

where the last equality follows from (4.6).

The setting that we will be particularly interested in is when $\hat{e} = e$. Then, $\hat{E}^2 e^{-1} = E^2 e^{-1} = e$, and

$$K_{e,r}^{*_e} = K_{e,\sqrt{n-r^2}}. \tag{4.21}$$

Note that when $e = \mathbb{1}$, (4.21) reduces to the assertion of Proposition 4.2, namely that $K^*_{\mathbb{1},r} = K_{\mathbb{1},\sqrt{n-r^2}}$ where the dual is with respect to $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathbb{1}}$, the dot product.

4.4.3 Primal-dual pairs of quadratic problems with respect to the local inner product

In Section 4.2, we considered the primal-dual pair of quadratic cone optimization problems

$$\begin{aligned} \min \quad & \langle c, x\rangle \\ s.t. \quad & Ax = b \qquad\qquad (QP_{e,r}) \\ & x \in K_{e,r}, \end{aligned}$$

$$\begin{aligned} \max \quad & b^T y \\ s.t. \quad & A^T y + s = c \qquad\qquad (QP_{e,r})^* \\ & s \in K^*_{e,r}, \end{aligned}$$

specified for each $r \in (0,1)$ and $e \in \mathbb{R}^n_{++}$. The dual cone in the constraint of $(QP_{e,r})^*$ is taken with respect to the dot product, $\langle\cdot,\cdot\rangle$.

We can also consider the dual of quadratic-cone optimization problems with respect to these local inner products.

Consider some $\hat{e} \in \mathbb{R}^n_{++}$. Writing the primal problem using the local inner product at $\hat{e}$,

$$\begin{array}{ll} \min & \langle \hat{E}^2 c, x\rangle_{\hat{e}} \\ s.t. & Ax = b \qquad\qquad (QP_{e,r}) \\ & x \in K_{e,r}, \end{array}$$

the dual problem with respect to the local inner product at $\hat{e}$ is

$$\begin{array}{ll} \max & b^T \tilde{y} \\ s.t. & \hat{E}^2 A^T \tilde{y} + \tilde{s} = \hat{E}^2 c \qquad\qquad (QP_{e,r})^{*_{\hat{e}}} \\ & \tilde{s} \in K_{e,r}^{*_{\hat{e}}}. \end{array}$$

We note that here, $\hat{E}^2 A^T : \mathbb{R}^m \to \mathbb{R}^n$ is the adjoint of $A : \mathbb{R}^n \to \mathbb{R}^m$ with respect to $\langle\cdot,\cdot\rangle_{\hat{e}}$: for each $\tilde{y} \in \mathbb{R}^m$ and each $w \in \mathbb{R}^n$,

$$\begin{aligned} y^T(Aw) &= \langle A^T y, w\rangle \\ &= \langle (\hat{E}^2 A)^T y, \hat{E}^{-2} w\rangle \\ &= \langle (\hat{E}^2 A)^T y, w\rangle_{\hat{e}}. \end{aligned}$$

Again, for the quadratic-cone optimization problem $(QP_{e,r})$, we are particularly interested in the dual with respect to $\langle\cdot,\cdot\rangle_e$, the local inner product at the same point e:

$$\begin{array}{ll} \max & b^T \tilde{y} \\ s.t. & E^2 A^T \tilde{y} + \tilde{s} = E^2 c \qquad\qquad (QP_{e,r})^{*_e} \\ & \tilde{s} \in K_{e,r}^{*_e}. \end{array}$$

Consider the linear constraint $E^2 A^T \tilde{y} + \tilde{s} = E^2 c$ in the description of $(QP_{e,r})^{*_e}$ above. That $(\tilde{s}, \tilde{y})$ satisfies this constraint can be equivalently written as

$$\tilde{s} \in E^2 c + \mathcal{L}^{\perp_e},. \tag{4.22}$$

where $\mathcal{L} := \{x \mid Ax = 0\} = \ker A$, the kernel of A, and $\mathcal{L}^{\perp_e}$ is its orthogonal complement with respect to the local inner product at e. Indeed, since $\mathcal{L}^{\perp} = \{A^T y \mid y \in \mathbb{R}^m\}$, it suffices to show that $E^2\mathcal{L}^{\perp} = \mathcal{L}^{\perp_e}$.

However, note that $v \in \mathcal{L}^{\perp_e}$ if and only if

$$\langle v, x\rangle_e = 0$$

for all $x \in \mathcal{L}$. That is, if and only if

$$0 \quad = \quad \langle v, x\rangle_e = \langle v, E^{-2}x\rangle = \langle E^{-2}v, x\rangle$$

for all $x \in \mathcal{L}$. So, $v \in \mathcal{L}^{\perp_e}$ if and only if $E^{-2}v \in \mathcal{L}^{\perp}$. Thus, $\mathcal{L}^{\perp_e} = E^2\mathcal{L}^{\perp}$, proving (4.22).

4.5 The tangent space

Recall that for a continuously differentiable function $p : \mathbb{R}^n \to \mathbb{R}$ and a point $\bar{x}$ satisfying $p(\bar{x}) = 0$ and $\nabla p(\bar{x}) \neq 0$, the tangent space at $\bar{x}$ to the set $\{x \mid p(x) = 0\}$ consists precisely of the vectors v satisfying $\nabla p(\bar{x})^T v = 0$ (see [20, Chapter 6]).

Fix $e \in Int$ and $r \in (0, 1)$. Suppose that $\bar{x}$ lies on the boundary of the cone $K_{e,r}$, then $p_{e,r}(\bar{x}) = 0$ where $p_{e,r} : \mathbb{R}^n \to \mathbb{R}$ is given by

$$p_{e,r}(u) := \langle \mathbb{1}, u./e\rangle^2 - r^2\langle u./e, u./e\rangle.$$

Thus, assuming that $\bar{x} \neq 0$ and $\nabla p_{e,r}(\bar{x}) \neq 0$, the tangent space at $\bar{x}$ to the set

$$\{x \mid p_{e,r}(x) = 0\} = \partial K_{e,r}$$

consists of vectors v such that $\langle \nabla p_{e,r}(\bar{x}), v\rangle = 0$.

Denote this tangent space $\mathcal{T}_{e,r,\bar{x}}$. Since

$$\nabla p_{e,r}(\bar{x}) = \langle \mathbb{1}, \bar{x}./e \rangle (e^{-1}) - r^2(\bar{x}./e^2),$$

we see that $\nabla p_{e,r}(\bar{x}) \neq 0$ for all $\bar{x} \neq 0$. Thus, $\mathcal{T}_{e,r,\bar{x}}$ consists of vectors v such that

$$\langle \mathbb{1}, \bar{x}./e \rangle \langle \mathbb{1}, v./e \rangle - r^2 \langle \bar{x}./e, v./e \rangle = 0. \tag{4.23}$$

If x is the optimal solution for $(QP_{e,r})$ then x lies in $\partial K_{e,r}$, and furthermore, it satisfies the Fritz John conditions which we state in Section 4.3. In particular, there exist $y \in \mathbb{R}^m$ and $\mu > 0$ such that

$$\langle \mathbb{1}, x./e \rangle (e^{-1}) - r^2(x./e^2) = A^T y - \mu c.$$

Therefore, if x is optimal for $(QP_{e,r})$, then $\mathcal{T}_{e,r,x}$ consists of all vectors v which satisfy:

$$0 = \langle (A^T y - \mu c), v \rangle = y^T (Av) - \mu \langle c, v \rangle. \tag{4.24}$$

We are mainly interested in tangent directions v that are also in $\mathcal{L}$ (recall that $\mathcal{L} = \ker A$) because we are interested in examining points $x + \lambda v$ which remain feasible for $(QP_{e,r})$ for infinitesimal λ.

Thus, following (4.24), the space $\mathcal{T}_{e,r,x} \cap \mathcal{L}$ consists of vectors v where the inner product between c and v must be zero (since $\mu > 0$):

$$\mathcal{T}_{e,r,x} \cap \mathcal{L} = \{v \in \mathbb{R}^n \mid Av = 0, \langle c, v \rangle = 0\}.$$

In particular, we note that the description of the vectors v that belong to $\mathcal{T}_{e,r,x} \cap \mathcal{L}$ above is independent of e, r, or x. This means that for any $\hat{e} \in \text{Swath}(r)$ and $\hat{x}$ that is optimal for $(QP_{\hat{e},r})$, the set

$$\{v \in \mathbb{R}^n \mid Av = 0, \ \langle c, v \rangle = 0\}$$

is tangent to $K_{\hat{e},r}$ at $\hat{x}$.

Thus, from here onwards, we eliminate the subscripts from "$\mathcal{T}_{e,r,x} \cap \mathcal{L}$" and let

$$\mathcal{T} := \mathcal{T}_{e,r,x} \cap \mathcal{L} = \{v \in \mathbb{R}^n \mid Av = 0,\ \langle c, v \rangle = 0\}. \tag{4.25}$$

We remark that $\mathcal{T}$ is a subspace of $\mathcal{L}$ of codimension 1. In particular, the vector c (alternatively, the vector $e - x$) and vectors in $\mathcal{T}$ together span $\mathcal{L}$.

We close this section with the following fact about vectors in $\mathcal{T}$. This proposition will be used in proving Lemma 5.9, a result about the projection of an optimal solution of a quadratic cone relaxation to the space orthogonal to $\mathcal{L}$ with respect to the local inner product at the current iterate.

Proposition 4.8. *Suppose x is optimal for ($QP_{e,r}$), and $\mathcal{T}$ is as given in* (4.25)*. Suppose that $v \in \mathcal{T}$ with $\langle v, x \rangle_e \neq 0$. Then, there exist $w \in \mathbb{R}^n$ and $\lambda > 0$ such that*

$$\lambda v = x + w$$

and

$$\langle x, w \rangle_e = \langle e, w \rangle_e = 0.$$

Proof. Let $w(\lambda) := \lambda v - x$. Then,

$$\langle w(\lambda), x \rangle_e = \lambda \langle v, x \rangle_e - \|x\|_e^2.$$

Choose $\bar{\lambda} := \frac{\|x\|_e^2}{\langle v, x \rangle_e}$ so that $\langle w(\bar{\lambda}), x \rangle_e = 0$. Then, using (4.23) and the fact that $\langle e, x \rangle_e = r\|x\|_e$ (since x lies on the boundary of $K_{e,r}$), we note that

$$\bar{\lambda} = \frac{\|x\|_e^2}{\langle v, x \rangle_e} = \frac{r^2\|x\|_e^2}{\langle e, x \rangle_e \langle e, v \rangle_e} = \frac{r\|x\|_e}{\langle e, v \rangle_e}.$$

Hence, again using the fact that $\langle e, x\rangle_e = r\|x\|_e$,

$$\begin{aligned}
\langle e, w(\bar{\lambda})\rangle_e &= \bar{\lambda}\langle e, v\rangle_e - \langle e, x\rangle_e \\
&= \frac{r\|x\|_e}{\langle e, v\rangle_e}\langle e, v\rangle_e - \langle e, x\rangle_e \\
&= r\|x\|_e - \langle e, x\rangle_e = 0.
\end{aligned}$$

□

CHAPTER 5

A LINEAR PROGRAMMING ALGORITHM

We consider a linear programming problem in standard form:

$$\begin{aligned} \min \quad & \langle c, x \rangle \\ s.t. \quad & Ax = b \qquad\qquad (LP) \\ & x \in \mathbb{R}^n_+, \end{aligned}$$

whose dual problem is

$$\begin{aligned} \max \quad & b^T y \\ s.t. \quad & A^T y + s = c \qquad\qquad (LP)^* \\ & s \in \mathbb{R}^n_+. \end{aligned}$$

Here, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. We assume that $b \neq 0$ and that c^T and the rows of A are linearly independent. Throughout this chapter, $\langle \cdot, \cdot \rangle$ is chosen to be the dot product on $\mathbb{R}^n$ and $\| \cdot \|$ the 2-norm.

In Chapter 4, we introduced the quadratic cones $K_{e,r}$. Recall that, for each $r \in (0, 1)$ and $e \in \mathbb{R}^n_{++}$, the quadratic-cone optimization problem

$$\begin{aligned} \min \quad & \langle c, x \rangle \\ s.t. \quad & Ax = b \qquad\qquad (QP_{e,r}) \\ & x \in K_{e,r}, \end{aligned}$$

is a relaxation of (LP). Furthermore, because $K^*_{e,r} \subseteq (\mathbb{R}^n_+)^* = \mathbb{R}^n_+$, then the dual quadratic-cone optimization problem

$$\begin{aligned} \max \quad & b^T y \\ s.t. \quad & A^T y + s = c \qquad\qquad (QP_{e,r})^* \\ & s \in K^*_{e,r}, \end{aligned}$$

is a restriction of $(LP)^*$. Although in Chapter 4 we considered cones $K_{e,r}$ for all $r \in [0,1]$, we consider only cones with parameter $r \in (0,1)$ throughout the present chapter.

Consider a fixed $r \in (0,1)$ and a vector $e \in \mathbb{R}^n_{++}$ that is a feasible solution for (LP). If $(QP_{e,r})$ has an optimal solution x, then x is not a feasible solution for (LP) because it lies outside of the nonnegative orthant. However, the objective value at x is a lower bound for the optimal value of (LP). Furthermore, since e is strictly feasible for (LP), the objective value at e is an upper bound for the optimal value of (LP). Recall that for each $r \in (0,1)$, we define $\mathrm{Swath}(r)$ as the set of vectors e that are strictly feasible for (LP) and for which $(QP_{e,r})$ has an optimal solution. Then, given $e \in \mathrm{Swath}(r)$, we can obtain another strictly feasible solution with better (smaller) objective value by choosing a point that lies in the line segment between e and x.

This observation inspires an algorithm for solving linear programs of the form (LP), which we sketch below.

Algorithm 5.1. Fix $r \in (0,1)$.

Initialization: Find an initial direction $e^{(0)} \in \mathrm{Swath}(r)$.

Iteration k**:** As long as the stopping criterion has not been achieved:

Find an optimal solution, $x^{(k)}$, of $(QP_{e^{(k)},r})$.

Choose a step size $t^{(k)} > 0$ and let $e^{(k+1)} = \frac{1}{1+t^{(k)}}(e^{(k)} + t^{(k)}x^{(k)})$.

In this chapter, we will make this algorithm more precise and prove the polynomial-time convergence towards an optimal solution for (LP).

Throughout the chapter, we consider r as a fixed parameter, taking a value in the open interval $(0, 1)$. We also make the following additional assumptions about (LP) and $(LP)^*$:

- $b \neq 0$, which implies that $x = 0$ is not a feasible solution.
- The rows of A and c^T are linearly independent, which means that optimal solutions lie on the boundary of the feasible region.
- (LP) and $(LP)^*$ are both strictly feasible.

In particular, these assumptions imply that

- For each $r \in (0, 1)$, the set $\text{Swath}(r)$ is not empty (see Proposition 4.6). Therefore, there exists an initial iterate $e^{(0)} \in \text{Swath}(r)$ for the algorithm.
- If $e \in \text{Swath}(r)$, then $(QP_{e,r})$ has a unique optimal solution (see Proposition 4.7). This ensures that given an iterate $e^{(k)}$, the next iterate is well-defined since $x^{(k)}$ is unique once $t^{(k)}$ is chosen.

We will show that at each iteration, there exists a choice of the step size $t^{(k)}$—which depends on the current iterate $(x^{(k)}, e^{(k)})$—which guarantees that the algorithm terminates in polynomial time, with the sequence $\{e^{(k)}\}$ converging to the optimal solution of (LP).

In particular, we will show that choosing $t^{(k)} = \frac{r}{2\|x^{(k)}./e^{(k)}\|}$ produces an iteration complexity of $O(\sqrt{n}\log(\gamma_0/\epsilon))$ for reducing the duality gap from γ_0 to ϵ, matching the best complexity bounds for interior-point methods in the literature. Here, the constant hidden in $O(\sqrt{n}\log(\gamma_0/\epsilon))$ depends on r and will be explicitly computed at the end of Section 5.5.

5.1 The dual solution and the duality gap

Even though our algorithm does not explicitly involve solving the dual quadratic-cone optimization problem, almost all parts of our analysis rely on examining the feasible solutions of the dual problem. We start by discussing the solution to the dual problem and by making explicit the stopping criterion hinted in Algorithm 5.1.

The dual solution

Given a point $e \in \text{Swath}(r)$ and a corresponding x $(= x(e))$ optimal for $(QP_{e,r})$, we can find an optimal solution to the dual quadratic programming problem easily, from the KKT conditions that must be satisfied by x. That is, for some $y \in \mathbb{R}^m$ and $\lambda \geq 0$,

$$
\begin{aligned}
c - A^T y - \lambda(\langle \mathbb{1}, x./e\rangle(e^{-1}) - r^2(x./e^2)) &= 0 \qquad (5.1)\\
\langle \mathbb{1}, x./e\rangle - r\|x./e\| &= 0, \qquad (5.2)\\
Ax - b &= 0. \qquad (5.3)
\end{aligned}
$$

Recall that for vectors $u, v \in \mathbb{R}^n$ and integer k, we use $u.v$ to denote component-wise multiplication, $u./v$ to denote component-wise division, and u^k to denote component-wise exponentiation, raising each component of u to the kth power. We use this notation frequently throughout the chapter.

It is easy to see that y together with $s := \lambda(\langle \mathbb{1}, x./e\rangle(e^{-1}) - r^2(x./e^2))$ satisfy the linear equality constraint of $(QP_{e,r})^*$. Taking the inner product of (5.1) with $e - x$, which we know to lie in the null space of A, we can find the actual value of λ:

$$\begin{aligned} 0 &= \langle c, e-x\rangle - \langle A^T y, e-x\rangle \\ &\quad -\lambda\Big(\langle \mathbb{1}, x./e\rangle\langle e^{-1}, e-x\rangle - r^2\langle x./e^2, e-x\rangle\Big) \\ &= \langle c, e-x\rangle - \lambda(n-r^2)\langle \mathbb{1}, x./e\rangle. \end{aligned}$$

This gives us

$$\lambda = \frac{\langle c, e-x\rangle_e}{(n-r^2)\langle \mathbb{1}, x./e\rangle} = \frac{\langle c, e-x\rangle}{(n-r^2)r\|x./e\|}$$

(since $\langle \mathbb{1}, x./e\rangle = r\|x./e\|$ for optimal x, by (5.2)). Therefore,

$$\begin{aligned} s &= \frac{\langle c, e-x\rangle}{(n-r^2)r\|x./e\|}(\langle \mathbb{1}, x./e\rangle(e^{-1}) - r^2(x./e^2)) \\ &= \frac{\langle c, e-x\rangle}{(n-r^2)r\|x./e\|}(r\|x./e\|(e^{-1}) - r^2(x./e^2)), \end{aligned}$$

that is,

$$s = \frac{\langle c, e-x\rangle}{n-r^2}\left(e^{-1} - r\frac{x./e^2}{\|x./e\|}\right). \tag{5.4}$$

Using (5.4), we obtain

$$\begin{aligned} \langle \mathbb{1}, s.e\rangle &= \langle c, e-x\rangle \geq 0, \\ \langle s.e, s.e\rangle &= \frac{\langle c, e-x\rangle^2}{n-r^2}, \end{aligned}$$

which implies that s satisfies the inequality that defines the quadratic cone $K^*_{e,r}$ tightly:

$$\langle \mathbb{1}, s.e\rangle - \sqrt{n-r^2}\|s.e\| = 0.$$

This shows that s lies on the boundary of the dual cone $K^*_{e,r}$. Thus, (s, y) is feasible for $(QP_{e,r})^*$.

To see that (s, y) is in fact optimal for $(QP_{e,r})^*$, we need only show that the duality gap between x and (s, y) is zero. Taking the inner product of x and s

gives this duality gap (see (2.1))

$$\begin{aligned}\langle x, s\rangle &= \frac{\langle c, e-x\rangle}{n-r^2}\left(\langle x, e^{-1}\rangle - r\frac{\langle x, x./e^2\rangle}{\|x./e\|}\right)\\ &= \frac{\langle c, e-x\rangle}{n-r^2}\left(r\|x./e\| - r\frac{\|x./e\|^2}{\|x./e\|}\right) = 0,\end{aligned}$$

which concludes our proof that (s, y) is optimal for $(QP_{e,r})^*$.

Thus, given our assumptions on (LP) and $(LP)^*$, having e in Swath(r) implies that both $(QP_{e,r})$ and $(QP_{e,r})^*$ have optimal solutions, and that strong duality holds between them.

The duality gap and the stopping criterion

The duality gap between x and (s, y), where s is given by (5.4), is always zero which tells us that this choice of s always gives an optimal solution for the dual quadratic-cone optimization problem. However, thus far, this does not tell us much about how close e is to an optimal (LP) solution.

Fortunately, (s, y) is always a feasible solution to the dual linear programming problem, due to the observation that the dual quadratic cone $K^*_{e,r}$ is always contained in the nonnegative orthant. Thus, e and (s, y) is a pair of feasible solutions to the primal-dual linear programming pair, with a duality gap of $\langle e, s\rangle$. Computing this quantity using the choice of s from (5.4),

$$\langle e, s\rangle \;=\; \langle \mathbb{1}, s.e\rangle = \langle c, e-x\rangle,$$

showing that we can compute the duality gap just from “primal information”— the center direction e together with x, the optimal solution to $(QP_{e,r})$— and using dual solutions only for the analysis.

Many primal-dual linear programming algorithms use the duality gap as a measure of proximity to optimality and as a stopping criterion (see [25, 10, 30, 4], among others). Although our algorithm is not a primal-dual algorithm, a strictly feasible dual solution for the linear programming looms in the background at each iteration, and we can easily compute the duality gap between our current iterate e with this dual solution.

Thus, we shall use the duality gap between this pair of strictly feasible solutions as a measure of proximity to optimality and as a stopping criterion. That is for a chosen $\epsilon > 0$, we stop only when $\langle c, e^{(k)} - x^{(k)} \rangle < \epsilon$.

5.2 Keeping the iterates in Swath(r)

Given a current iterate $e \in \text{Swath}(r)$ and x that is optimal for $(QP_{e,r})$, let us denote the next iterate as a function of the step size $t > 0$:

$$e(t) := \frac{1}{1+t}(e + tx). \tag{5.5}$$

From here onwards, we will use e and $e(t)$ instead of $e^{(k)}$ and $e^{(k+1)}$ for denoting consecutive iterates, to keep the notation simple and to emphasize our focus in the next few sections on analyzing the best choice of step size t.

In choosing the value of t, we must ensure that $e(t)$ is strictly feasible for (LP). Furthermore, we must make sure that $e(t)$ lies in Swath(r) to guarantee the existence of an optimal solution for $(QP_{e(t),r})^*$, for otherwise the algorithm will be "stuck" at the next iteration.

Fortunately, an upper bound for t that guarantees this condition is not terribly hard to find. Furthermore, the ideas behind how we might find such an upper bound, using duality theory for conic optimization, are esthetically appealing.

Lemma 5.1. *Suppose that* $e \in Swath(r)$, *x optimal for* $(QP_{e,r})$, *and* (s, y) *is optimal for* $(QP_{e,r})^*$. *Let* $e(t) = \frac{1}{1+t}(e + tx)$. *As long as* $0 < t < \frac{r}{\|x./e\|}$, *then* $e(t) > 0$ *and s lies in the interior of* $K^*_{e(t),r}$.

Proof. First, we show that $e(t) > 0$ for $0 < t < \frac{r}{\|x./e\|}$. Observe that since $r < 1$ and since $\|x./e\| \geq \|x./e\|_\infty$, then

$$\frac{r}{\|x./e\|} \quad < \quad \frac{1}{\|x./e\|_\infty} = \frac{1}{\max_i |x_i/e_i|}.$$

Then, for all index i, $t < \left|\frac{e_i}{x_i}\right|$. In particular, for each index i where $x_i < 0$,

$$e_i + tx_i > e_i + \left|\frac{e_i}{x_i}\right| x_i > 0.$$

Thus, $e(t) = \frac{1}{1+t}(e + tx) > 0$, as desired.

Next, we wish to show that for $0 < t < \frac{r}{\|x./e\|}$, s satisfies:

$$\langle \mathbb{1}, s.e(t)\rangle \geq 0 \quad \text{and}$$

$$\langle \mathbb{1}, s.e(t)\rangle^2 - (n - r^2)\|s.e(t)\|^2 > 0. \tag{5.6}$$

We first note that

$$(1 + t)\langle \mathbb{1}, s.e(t)\rangle \quad = \quad \langle \mathbb{1}, s.e\rangle + t\langle \mathbb{1}, s.x\rangle = \langle \mathbb{1}, s.e\rangle = \langle c, e - x\rangle,$$

and

$$\begin{aligned}(1 + t)^2\langle s.e(t), s.e(t)\rangle \quad &= \quad \|s.e\|^2 + 2t\langle s.e, s.x\rangle + t^2\langle s.x, s.x\rangle \\ &= \quad \frac{\langle \mathbb{1}, s.e\rangle^2}{n - r^2} + 2t\langle s.e, s.x\rangle + t^2\langle s.x, s.x\rangle\end{aligned}$$

Therefore, (5.6) is satisfied if and only if

$$2t\langle s.e, s.x\rangle + t^2\langle s.x, s.x\rangle < 0.$$

Using s as given by (5.4) and the fact that $\langle \mathbb{1}, x./e\rangle = r\|x./e\|$ for x optimal for $(QP_{e,r})$, we have that

$$\langle s.e, s.x\rangle \;=\; -\frac{\langle c, e-x\rangle^2}{(n-r^2)^2} r\|x./e\| \left(1 - \frac{r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3}\right), \tag{5.7}$$

$$\begin{aligned}\langle s.x, s.x\rangle \;=\; & \frac{\langle c, e-x\rangle^2}{(n-r^2)^2}\|x./e\|^2 \\ & \left(1 - \frac{2r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3} + \frac{r^2\langle (x./e)^2, (x./e)^2\rangle}{\|x./e\|^4}\right).\end{aligned} \tag{5.8}$$

Since t must be positive, we need only to choose t that satisfies

$$t < -2\frac{\langle s.e, s.x\rangle}{\langle s.x, s.x\rangle} = 2\gamma \frac{r}{\|x./e\|},$$

where

$$\gamma := \frac{1 - \frac{r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3}}{1 - \frac{2r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3} + \frac{r^2\langle (x./e)^2, (x./e)^2\rangle}{\|x./e\|^4}}.$$

It is easy to see that γ is nonnegative: the denominator is a positive multiple of $\|s.x\|^2$ where $s.x \neq 0$, so it is positive, while the numerator is positive because

$$\langle x./e, (x./e)^2\rangle = \sum_{i=1}^{n}\left(\frac{x_i}{e_i}\right)^3 \leq \|x./e\|_3^3 \leq \|x./e\|^3.$$

Since

$$\frac{r^2\langle (x./e)^2, (x./e)^2\rangle}{\|x./e\|^4} = \frac{r^2}{\|x./e\|^4}\sum_{i=1}^{n}\left(\frac{x_i}{e_i}\right)^4 \leq 1,$$

then

$$\frac{r^2}{\|x./e\|^4}\langle (x./e)^2, (x./e)^2\rangle - \frac{r}{\|x./e\|^3}\langle x./e, (x./e)^2\rangle \leq 1 - \frac{r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3},$$

which implies that

$$1 - \frac{2r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3} + \frac{r^2\langle (x./e)^2, (x./e)^2\rangle}{\|x./e\|^4} \leq 2\left(1 - \frac{r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3}\right).$$

Dividing both sides by the left-hand side, we see that $1 \leq 2\gamma$, or equivalently, that $\gamma \geq \frac{1}{2}$.

Thus, assuming that $e(t) \in \mathbb{R}^n_{++}$, choosing $0 < t < \frac{r}{\|x./e\|}$ is sufficient to guarantee that s is in the interior of $K^*_{e(t),r}$, as desired. □

Proposition 5.2. *Suppose that $e \in Swath(r)$ and x optimal for $(QP_{e,r})$. Let $e(t) = \frac{1}{1+t}(e + tx)$. As long as $0 < t < \frac{r}{\|x./e\|}$, then $e(t) \in Swath(r)$.*

Proof. First note that since $Ax = b$ and $Ae = b$, then $Ae(t) = b$. Furthermore, Lemma 5.1 shows that $e(t) > 0$ for $0 < t < \frac{r}{\|x./e\|}$. To show that $e(t) \in$ Swath(r), it remains to show that $(QP_{e(t),r})$ has an optimal solution.

Theorem 2.4 states that if the primal problem of a pair of primal-dual conic optimization problems is feasible and the dual problem is strictly feasible, then the primal problem has an optimal solution.

We know that $(QP_{e(t),r})$ is feasible: $e(t)$ is a (strictly) feasible solution. Therefore, if we can show that $(QP_{e(t),r})^*$ has a strictly feasible solution, Theorem 2.4 then implies that $(QP_{e(t),r})$ has an optimal solution, as desired.

To show that $(QP_{e(t),r})^*$ has a strictly feasible solution, we only need to produce one point that lies in the interior of its feasible region. Our candidate for this point is none other than (s, y), the optimal solution for $(QP_{e,r})^*$, where s is defined in (5.4).

We know that (s, y) satisfies the linear constraint $A^T y + s = c$. Furthermore, by Lemma 5.1, s lies in the interior of $K^*_{e(t),r}$, showing that (s, y) is indeed strictly feasible for $(QP_{e(t),r})^*$.

This concludes our proof that that choosing $0 < t < \frac{r}{\|x./e\|}$ is sufficient to guarantee that $(QP_{e(t),r})$ has an optimal solution, and thereby concluding the proof of the proposition. $\square$

Lemma 5.1 is essential not only in proving the above proposition, but also in the proof of other results in this chapter.

5.3 Primal and dual monotonicity

Given $e \in \text{Swath}(r)$ and x that is optimal for $(QP_{e,r})$, let $e(t) := \frac{1}{1+t}(e + tx)$ denote the next iterate. Assuming that $0 < t < \frac{r}{\|x./e\|}$, so that $e(t) \in \text{Swath}(r)$, let $x(e(t))$ denote the optimal solution to $(QP_{e(t),r})$ and $(s(e(t)), y(e(t)))$ the optimal solution to $(QP_{e(t),r})^*$.

It may not be immediately evident why "moving e towards x" always reduces the duality gap. That is, why

$$\langle c, e(t) - x(e(t))\rangle \leq \langle c, e - x\rangle. \tag{5.9}$$

Note that for the duality gap to be monotonically decreasing, it is sufficient to have that the objective value of the primal iterates be monotonically decreasing:

$$\langle c, e(t)\rangle \leq \langle c, e\rangle \tag{5.10}$$

and to have the objective value of the corresponding quadratic cone relaxation optimal solutions be monotonically increasing:

$$\langle c, x(e(t))\rangle \geq \langle c, x\rangle. \tag{5.11}$$

While it is easy to see why (5.10) must hold (that is, because $e(t)$ is a convex combination of e and x, where the objective value of x is smaller than that of e), it may not be as immediately obvious why (5.11) holds.

To see why (5.11) must hold, we again appeal to the dual problem. From Lemma 5.1, we see that for all $0 < t < \frac{r}{\|x./e\|}$, the optimal dual solution (s, y) of $(QP_{e,r})^*$ is a strictly feasible solution to $(QP_{e(t),r})^*$. This means that the optimal value of $(s(e(t)), y(e(t)))$ is at least as large as the optimal value of (s, y). Since strong duality holds for both pairs of quadratic-cone optimization problems, then

$$\langle c, x \rangle = b^T y \leq b^T y(e(t)) = \langle c, x(e(t)) \rangle,$$

as desired.

Hence, it makes sense to think of the property (5.10) as "primal monotonicity" and of (5.11) as "dual monotonicity", noting that the sequence $\{e^{(k)}\}$ of strictly feasible primal solutions has monotonic decreasing objective values while the sequence $\{(s^{(k)}, y^{(k)})\}$ of strictly feasible dual solutions has monotonic increasing objective values.

In the following sections, we will show that the duality gap is not only decreasing, but that it is reduced by a significant multiplicative factor at least at every other iteration. In one case (Proposition 5.3), the large reduction in duality gap is due to a significant decrease of primal objective value while in another case (Proposition 5.5), it is due to a significant increase of dual objective value.

5.4 A simple bound on the duality gap reduction

To ensure that the sequence of points $\{e^{(k)}\}$ converges to an optimal (LP) solution in polynomial time, the inequality (5.9) is not sufficient. We would need to be able to say that $\langle c, e(t) - x(e(t)) \rangle$ is not just smaller than $\langle c, e - x \rangle$, but that it is quite significantly smaller.

Fortunately, in the case when $\|x./e\|$ is not too large, it is not hard to observe that the duality gap reduction that resulted from taking a step of order $r/\|x./e\|$ is quite significant.

Proposition 5.3. *Suppose that $e \in Swath(r)$, x is optimal for $(QP_{e,r})$, with $\|x./e\| \leq \sqrt{n}$. We let $e(t) = \frac{1}{1+t}(e + tx)$. Then, taking a step of size $t = \eta \frac{r}{\|x./e\|}$, for some $\eta \in (0, 1)$, reduces the duality gap by a factor of $\left(1 - \frac{\eta r}{2\sqrt{n}}\right)$.*

Proof. Since we choose $t = \eta \frac{r}{\|x./e\|}$,

$$\begin{aligned}
\langle c, e(t) - x(e(t)) \rangle &= \frac{1}{1+t}\langle c, e\rangle + \frac{t}{1+t}\langle c, x\rangle - \langle c, x(e(t))\rangle \\
&\leq \frac{1}{1+t}\langle c, e\rangle + \frac{t}{1+t}\langle c, x\rangle - \langle c, x\rangle \\
&= \left(1 - \frac{t}{1+t}\right)\langle c, e - x\rangle \\
&\leq \left(1 - \frac{\eta r}{2\sqrt{n}}\right)\langle c, e - x\rangle,
\end{aligned}$$

where the first inequality is due to dual monotonicity, (5.11), and the second inequality is due to $\|x./e\| \leq \sqrt{n}$. □

Hence, choosing $\eta = \frac{1}{2}$, for instance, will reduce the duality gap by a factor of $\left(1 - \frac{r}{4\sqrt{n}}\right)$.

Note that the significant reduction of duality gap in the case that is covered by Proposition 5.3 is due to a large decrease of the primal objective value.

5.5 Further bounds on the duality gap reduction

In the proof of Proposition 5.3 above, the significant bound on the duality gap reduction relies on the relatively large distance between e and $e(t)$ compared to the distance between e and x, which is controlled by the step size t. Naturally, when $\|x./e\|$ is small, t is large, and we achieve a large duality gap reduction.

However, when $\|x./e\|$ is large, the step size t is small, and the distance between e and $e(t)$ is only a small fraction of the distance between e and x. Thus, in order to obtain a better bound on the duality gap reduction, we need to show that not only is $\langle c, x(e(t))\rangle$ greater than $\langle c, x\rangle$, but also that it is quite significantly greater. That is, we desire a good lower bound for the quantity $\langle c, x(e(t)) - x\rangle$. The results that we will mention in this section in fact show that $\langle c, x(e(t)) - x\rangle$ is large when $\|x(e(t))./e(t)\|$ is large. This result shows that a good improvement is achieved at least at every two iterations.

Letting (s, y) be the optimal solution for $(QP_{e,r})^*$ and $(s(e(t)), y(e(t)))$ the optimal solution for $(QP_{e(t),r})^*$, we can rewrite $\langle c, x(e(t)) - x\rangle$ in terms of the dual solutions as follows

$$\begin{aligned}\langle c, x(e(t)) - x\rangle &= b^T(y(e(t)) - y) \\ &= (Ax(e(t)))^T(y(e(t)) - y) \\ &= \langle x(e(t)), A^T(y(e(t)) - y)\rangle \\ &= \langle x(e(t)), s - s(e(t))\rangle. \qquad (5.12)\end{aligned}$$

That is, the inner product between $x(e(t))$ and $s - s(e(t))$ provides the amount of change of objective value. We will not be solving for $s(e(t))$, but we can come up with a different feasible solution (but not necessarily an optimal solution) for $(QP_{e(t),r})^*$ relatively easily.

Of course, (s, y) is feasible for $(QP_{e(t),r})^*$ due to our choice of t (see proof of Proposition 5.2). However, we would like to produce another solution, call it (s', y'), that better approximates $(s(e(t)), y(e(t)))$ in the sense that the objective value of (s', y') is quite significantly larger than that of (s, y).

Supposing that such a solution (s', y') could be found, then

$$\langle c, x(e(t)) - x\rangle \geq \langle x(e(t)), s - s'\rangle.$$

Using this idea, we find the following lower bound for $\langle c, x(e(t)) - x\rangle$ for a particular choice of step size t.

Recall that we use $\mathcal{L}$ to denote the nullspace of A, the constraint matrix specifying the affine linear constraint of (LP) and of its quadratic-cone relaxations.

Note that the lower bound for $\langle c, x(e(t)) - x\rangle$ below is expressed in terms of the norm $\|\cdot\|_{e(t)}$, induced by the local inner product at $e(t)$, and in terms of a projection onto $\mathcal{L}^{\perp_{e(t)}}$, the orthogonal complement of $\mathcal{L}$ with respect to the local inner product at $e(t)$.

Proposition 5.4. *Suppose that $e \in Swath(r)$, x is optimal for $(QP_{e,r})$, and (s, y) is optimal for $(QP_{e,r})^*$. Let $e(t) := \frac{1}{1+t}(e + tx)$.*

Then, taking $t = \frac{r}{2\|x./e\|}$ and letting $x(e(t))$ denote the optimal solution for $(QP_{e(t),r})$, we have that

$$\langle c, x(e(t)) - x\rangle \geq R\, \|Proj_{\mathcal{L}^{\perp_{e(t)}}} x(e(t))\|_{e(t)},$$

where $R := \frac{\|s.e(t)\|}{n}\left(r\sqrt{n - \beta^2} - \beta\sqrt{n - r^2}\right)$ and $\beta := r\sqrt{\frac{1+r}{2}}$.

We do this switch of inner products because computations with respect to this local inner product turn out to be the simplest and most natural, as we shall see when we prove this proposition in Section 5.6.

This result implies that the new duality gap has the following upper bound

$$\begin{aligned}\langle c, e(t) - x(e(t))\rangle &= \langle c, e(t)\rangle - \langle c, x\rangle - \langle c, x(e(t)) - x\rangle \\ &\leq \frac{1}{1+t}\langle c, e - x\rangle \\ &\quad - R\, \|\mathrm{Proj}_{\mathcal{L}^{\perp_{e(t)}}}(x(e(t)))\|_{e(t)}. \end{aligned} \tag{5.13}$$

By explicitly computing a lower bound for $\|\mathrm{Proj}_{\mathcal{L}^{\perp_{e(t)}}}(x(e(t)))\|_{e(t)}$, we obtain the following bound on the duality gap reduction.

Proposition 5.5. *Suppose that $e \in Swath(r)$, x is optimal for $(QP_{e,r})$, and (s, y) is optimal for $(QP_{e,r})^*$. As usual, we let $e(t) = \frac{1}{1+t}(e + tx)$. Assume that $n \geq 4$.*

Choose $t = \frac{r}{2\|x./e\|}$ and let $x(e(t))$ denote the optimal solution for $(QP_{e(t),r})$. If $\|x(e(t))./e(t)\| > \sqrt{n}$, then

$$\langle c, e(t) - x(e(t))\rangle \leq \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)\langle c, e - x\rangle.$$

While the duality gap reduction in Proposition 5.3 relies on primal-objective improvement, the duality gap reduction in Proposition 5.5 relies on showing improvement in the dual objective. This idea, that an improvement can be made in either the primal or the dual, can be found elsewhere in the linear programming interior-point method literature, such as in [37] and [9].

From Proposition 5.3 and Proposition 5.5, we obtain our main result.

Theorem 5.6. *Fix $r \in (0, 1)$ and assume that $n \geq 4$. Suppose that we have an initial iterate $e^{(0)}$ in $Swath(r)$. For each $k = 0, 1, \ldots$, given the current iterate $e^{(k)}$, let $x^{(k)}$ denote the optimal solution for $(QP_{e^{(k)},r})$ and define the next iterate as follows*

$$e^{(k+1)} := \frac{1}{1+t^{(k)}}(e^{(k)} + t^{(k)}x^{(k)}),$$

with $t^{(k)} = \frac{r}{2\|x^{(k)}./e^{(k)}\|}$.

Then, for each k, at least one of the following holds:

$$\langle c, e^{(k+1)} - x^{(k+1)} \rangle \leq \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right) \langle c, e^{(k)} - x^{(k)} \rangle, \tag{5.14}$$

$$\langle c, e^{(k+2)} - x^{(k+2)} \rangle \leq \left(1 - \frac{r}{4\sqrt{n}}\right) \langle c, e^{(k+1)} - x^{(k+1)} \rangle. \tag{5.15}$$

Proof of Theorem 5.6. We do this by induction on k. Starting at $e^{(0)}$, we know from Proposition 5.2 that with $t^{(0)} = \frac{r}{2\|x^{(0)}./e^{(0)}\|}$, then $e^{(1)} \in \text{Swath}(r)$.

Assuming that that $e^{(k)} \in \text{Swath}(r)$, we consider four cases:

1. If $\|x^{(k)}./e^{(k)}\| \leq \sqrt{n}$ and $\|x^{(k+1)}./e^{(k+1)}\| \leq \sqrt{n}$, then we can apply Proposition 5.3 with $\eta = \frac{1}{2}$ to conclude that

$$\begin{aligned} \langle c, e^{(k+1)} - x^{(k+1)} \rangle &\leq \left(1 - \frac{r}{4\sqrt{n}}\right) \langle c, e^{(k)} - x^{(k)} \rangle \\ &\leq \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right) \langle c, e^{(k)} - x^{(k)} \rangle, \end{aligned}$$

   since $\frac{r}{4} \geq \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}$ for all $r \in (0, 1)$.

2. If $\|x^{(k)}./e^{(k)}\| \leq \sqrt{n}$ and $\|x^{(k+1)}./e^{(k+1)}\| > \sqrt{n}$, then

$$\begin{aligned} \langle c, e^{(k+1)} - x^{(k+1)} \rangle &= \left(1 - \frac{t}{1+t}\right) \langle c, e^{(k)} - x^{(k)} \rangle - \langle c, x^{(k+1)} - x^{(k)} \rangle \\ &\leq \left(1 - \frac{r}{4\sqrt{n}}\right) \langle c, e^{(k)} - x^{(k)} \rangle - \langle c, x^{(k+1)} - x^{(k)} \rangle \\ &\leq \left(1 - \frac{r}{4\sqrt{n}} - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right) \langle c, e^{(k)} - x^{(k)} \rangle \\ &\leq \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right) \langle c, e^{(k)} - x^{(k)} \rangle, \end{aligned}$$

where the first and second inequalities are due to Proposition 5.3 and Proposition 5.5, respectively.

3. On the other hand, if $\|x^{(k)}./e^{(k)}\| > \sqrt{n}$ but $\|x^{(k+1)}./e^{(k+1)}\| \leq \sqrt{n}$, then Proposition 5.3 again implies that

$$\langle c, e^{(k+2)} - x^{(k+2)} \rangle \leq \left(1 - \frac{r}{4\sqrt{n}}\right) \langle c, e^{(k+1)} - x^{(k+1)} \rangle.$$

4. The remaining case is when $\|x^{(k)}./e^{(k)}\| > \sqrt{n}$ and $\|x^{(k+1)}./e^{(k+1)}\| > \sqrt{n}$. In this case, Proposition 5.5 implies that

$$\langle c, e^{(k+1)} - x^{(k+1)} \rangle \leq \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right) \langle c, e^{(k)} - x^{(k)} \rangle.$$

□

This theorem ensures a significant duality gap between the kth and the $(k+1)$th iterates if $\|x^{(k)}./e^{(k)}\| \leq \sqrt{n}$ or if both $\|x^{(k)}./e^{(k)}\| > \sqrt{n}$ and $\|x^{(k+1)}./e^{(k+1)}\| > \sqrt{n}$. In the case that $\|x^{(k)}./e^{(k)}\| > \sqrt{n}$ but $\|x^{(k+1)}./e^{(k+1)}\| \leq \sqrt{n}$, the theorem does not provide a bound on the duality gap reduction between the kth and the $(k+1)$th iterates. That is, the duality gap is guaranteed to be reduced by the given factors only at every other iteration.

Recall that $\frac{r}{4} \geq \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}$ for all $r \in (0, 1)$. So, considering the worst case that the duality gap is reduced significantly only every other iteration, by the weaker bound of $1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}$,

$$\langle c, e^{(k+2)} - x^{(k+2)} \rangle \leq \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right) \langle c, e^{(k)} - x^{(k)} \rangle,$$

which means that after $\bar{k}$ iterations,

$$\langle c, e^{(\bar{k})} - x^{(\bar{k})} \rangle \leq \gamma_0 \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}} \right)^{\bar{k}/2},$$

where $\gamma_0 := \langle c, e^{(0)} - x^{(0)} \rangle$, the initial duality gap.

In order to achieve a final duality gap of ϵ, for some $\epsilon > 0$, it is then sufficient to satisfy

$$\gamma_0 \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}} \right)^{\bar{k}/2} \leq \epsilon,$$

or equivalently

$$\log \gamma_0 + \frac{\bar{k}}{2} \log \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}} \right) \leq \log \epsilon.$$

Since $\log(1+\delta) \leq \delta$ for all $\delta > -1$ [1], then we want $\bar{k}$ such that

$$\log \gamma_0 + \frac{\bar{k}}{2} \left(- \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}} \right) \leq \log \epsilon,$$

or equivalently,

$$\bar{k} \geq \frac{8\sqrt{3n}}{r\left(1 - \sqrt{\frac{1+r}{2}}\right)} \log(\gamma_0/\epsilon).$$

Thus, $\bar{k} = O\left(\frac{\sqrt{n}}{r(1 - \sqrt{(1+r)/2})} \log(\gamma_0/\epsilon) \right)$

Note that, treating r as a fixed parameter, the iteration complexity is $O(\sqrt{n} \log(\gamma_0/\epsilon))$. As a function of r, however, we see that the choice of r which give the best bound is when r is bounded away from both 0 and 1.

We conclude this section by noting that the final conclusion of Theorem 5.6 can be strengthened slightly: after $\bar{k}$ iterations, the duality gap is reduced by a

[1] To see this, observe that the function $F(\delta) := \log(1+\delta)$ is concave and $F'(0) = 1$.

factor of

$$\left(1-\frac{r\left(1-\sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)^{\bar{k}-1}.$$

To see why this is the case, consider the four cases that we listed in the proof of the theorem. If for some iteration k, we are in the second case, where $\|x^{(k)}./e^{(k)}\| \leq \sqrt{n}$ and $\|x^{(k+1)}./e^{(k+1)}\| > \sqrt{n}$, then

$$\begin{aligned}
\langle c, e^{(k+1)} - x^{(k+1)}\rangle &\leq \left(1-\frac{r}{4\sqrt{n}}-\frac{r\left(1-\sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)\langle c, e^{(k)} - x^{(k)}\rangle \\
&\leq \left(1-\frac{r}{4\sqrt{n}}\right)\left(1-\frac{r\left(1-\sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)\langle c, e^{(k)} - x^{(k)}\rangle \\
&\leq \left(1-\frac{r\left(1-\sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)^2\langle c, e^{(k)} - x^{(k)}\rangle,
\end{aligned}$$

resulting in twice the multiplicative factor of reduction, compared to the duality gap reduction that is guaranteed in the first and fourth cases.

On the other hand, if for some iteration k, we are in the third case, where $\|x^{(k)}./e^{(k)}\| > \sqrt{n}$ and $\|x^{(k+1)}./e^{(k+1)}\| \leq \sqrt{n}$, we cannot say much about the duality gap $\langle c, e^{(k+1)} - x^{(k+1)}\rangle$ other than that it is less than $\langle c, e^{(k)} - x^{(k)}\rangle$. Note however, that if among any $\bar{k}$ iterations, this "bad" case occurs κ times, the previous "very good" case occurs at least $\kappa - 1$ times. Thus, we can in fact assert that for any number of iterations $\bar{k}$,

$$\langle c, e^{(\bar{k})} - x^{(\bar{k})}\rangle \leq \left(1-\frac{r\left(1-\sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)^{\bar{k}-1}\langle c, e^{(0)} - x^{(0)}\rangle.$$

5.6 Proving Proposition 5.4

In the next two sections, we consider a fixed step size of $t = \frac{r}{2\|x./e\|}$.

Recall that we would like to obtain a solution (s', y') that is feasible for $(QP_{e(t),r})^*$ and whose objective value is significantly better than (s, y). Such a solution (s', y') provides the following lower bound (see (5.12))

$$\langle c, x(e(t)) - x \rangle \geq \langle x(e(t)), s - s' \rangle.$$

We can rewrite $\langle x(e(t)), s - s' \rangle$ in terms of the local inner product at $e(t)$ as follows

$$\langle x(e(t)), s - s' \rangle = \langle x(e(t)), E(t)^2(s - s') \rangle_{e(t)}.$$

We switch inner products here because, as we will see more concretely in the proofs of the results which follow later, it is most natural to carry out our analysis using the local inner product at $e(t)$, since the cones $K_{e(t),r}$, $K^*_{e(t),r}$, and $K^{*_{e(t)}}_{e(t),r}$ are more naturally expressed in terms of $\langle \cdot, \cdot \rangle_{e(t)}$.

We remark that any (s', y') is a feasible solution for $(QP_{e(t),r})^*$ if and only if $(E(t)^2 s', y')$ is a feasible solution for

$$\begin{aligned} \max \quad & b^T y \\ s.t. \quad & E(t)^2 A^T \tilde{y} + \tilde{s} = E(t)^2 c \\ & s \in E(t)^2 K^*_{e(t),r} = K^{*_{e(t)}}_{e(t),r}. \end{aligned}$$

This problem is just $(QP_{e(t),r})^*$ scaled by $E(t)^2$, or equivalently, the dual of $(QP_{e(t),r})$ with respect to the local inner product at $e(t)$, which we discussed in greater length in Section 4.4.3. We denote this problem as $(QP_{e(t),r})^{*_{e(t)}}$.

Therefore, we will instead focus on finding a point $(\bar{s}', \bar{y}')$ that is feasible for $(QP_{e(t),r})^{*e(t)}$ and whose objective value is significantly better than that of

$$(\bar{s}, \bar{y}) := (E(t)^2 s, y),$$

providing the lower bound

$$\langle c, x(e(t)) - x \rangle \geq \langle x(e(t)), \bar{s} - \bar{s}' \rangle_{e(t)}.$$

The problem of finding such an $\bar{s}'$ can be rephrased as the problem of finding a vector $\bar{v}$ of unit norm (with respect to $\|\cdot\|_{e(t)}$) and a positive scalar R which satisfy:

1. $\langle x(e(t)), \bar{v} \rangle_{e(t)} > 0$,
2. $\bar{v} \in \mathcal{L}^{\perp_{e(t)}}$ (where $\mathcal{L}$ is the nullspace of A), and
3. The $\|\cdot\|_{e(t)}$-ball centered at $\bar{s}$ of radius R is contained in $K^{*e(t)}_{e(t),r}$.

With R and $\bar{v}$ that satisfy the above conditions, choose

$$\bar{s}' := \bar{s} - R\bar{v}.$$

The second of the above conditions ensures the existence an $\bar{y}' \in \mathbb{R}^m$ such that $E(t)^2 A^T \bar{y}' + \bar{s}' = c$ ($\bar{y}'$ is uniquely determined by the choice of $\bar{s}'$ since A is full rank) while the third condition ensures that $\bar{s}' \in K^{*e(t)}_{e(t),r}$. Thus, $(\bar{s}', \bar{y}')$ is feasible for $(QP_{e(t),r})^{*e(t)}$.

Furthermore, the first condition ensures that

$$\langle c, x(e(t)) - x \rangle \geq \langle x(e(t)), \bar{s} - \bar{s}' \rangle_{e(t)} = R \langle x(e(t)), \bar{v} \rangle_{e(t)} > 0.$$

Therefore, we will first compute a radius R that guarantees that $B_{e(t)}(\bar{s}, R) \subseteq K^{*e(t)}_{e(t),r}$, where $B_{e(t)}(\bar{s}, R)$ denotes the ball of radius R centered at $\bar{s}$, with respect to $\|\cdot\|_{e(t)}$.

Lemma 5.7. *Let $\beta \in (0, r]$. Suppose that $e(t) = \frac{1}{1+t}(e + tx) \in \mathit{Swath}(\beta)$. If $\hat{s} \in K^{*e(t)}_{e(t),\beta}$, then $K^{*e(t)}_{e(t),r}$ contains the ball $B_{e(t)}(\hat{s}, R)$ where*

$$R = \frac{\|\hat{s}\|_{e(t)}}{n}\left(r\sqrt{n-\beta^2} - \beta\sqrt{n-r^2}\right).$$

Proof. By (4.21),

$$K^{*e(t)}_{e(t),r} = K_{e(t),\sqrt{n-r^2}}$$

and

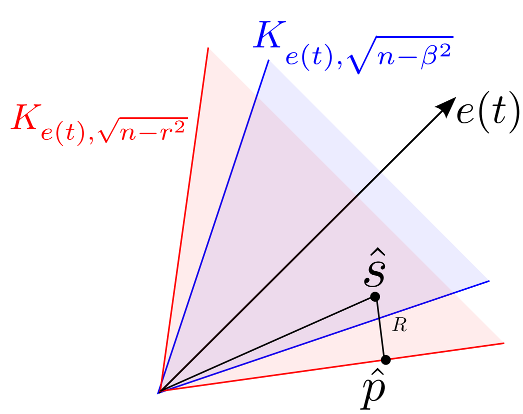
$$K^{*e(t)}_{e(t),\beta} = K_{e(t),\sqrt{n-\beta^2}}.$$

A nonzero vector $\hat{s}$ is in $K_{e(t),\sqrt{n-\beta^2}}$ if and only if its angle with $e(t)$ does not exceed $\arccos\sqrt{(n-\beta^2)/n}$, or equivalently, if and only if

$$\cos \angle_{e(t)}(\hat{s}, e(t)) \geq \sqrt{(n-\beta^2)/n}. \tag{5.16}$$

(Note that in this proof, we consider angles, lengths, and projections with respect to the local inner product at $e(t)$.) Moreover, for $\hat{s} \in K_{e(t),\sqrt{n-\beta^2}} \subseteq K_{e(t),\sqrt{n-r^2}}$, the point in the boundary of the cone $K_{e(t),\sqrt{n-r^2}}$ closest to $\hat{s}$ is the projection of $\hat{s}$ onto the ray whose angle with $\hat{s}$ is smallest among all rays whose angle with $e(t)$ is $\arccos\sqrt{(n-r^2)/n}$. Call the projection of $\hat{s}$ to this ray $\hat{p}$. Clearly, the angle between $\hat{s}$ and $\hat{p}$ is

$$\angle_{e(t)}(\hat{s}, \hat{p}) = \arccos\sqrt{(n-r^2)/n} - \angle_{e(t)}(\hat{s}, e(t)).$$



Therefore, the distance from $\hat{s}$ to the boundary of $K_{e(t),\gamma}$ is given by

$$\begin{aligned}
\|\hat{s}-\hat{p}\|_{e(t)} &= \|\hat{s}\|_{e(t)} \sin \angle_{e(t)}(\hat{s},\hat{p}) \\
&= \|\hat{s}\|_{e(t)} \ \sin\Big(\arccos\ \sqrt{(n-r^2)/n} - \angle_{e(t)}(\hat{s},e(t))\Big) \\
&= \|\hat{s}\|_{e(t)} \ \Big(\sin \arccos\ \sqrt{(n-r^2)/n}\ \cos \angle_{e(t)}(\hat{s},e(t)) \\
&\qquad - \sin \angle_{e(t)}(\hat{s},e(t))\ \cos(\arccos\ \sqrt{(n-r^2)/n})\Big).
\end{aligned}$$

Using (5.16),

$$\begin{aligned}
\|\hat{s}-\hat{p}\|_{e(t)} &\geq \|\hat{s}\|_{e(t)} \ \Big(\sqrt{1-(n-r^2)/n}\ \sqrt{(n-\beta^2)/n} \\
&\qquad - \sqrt{1-(n-\beta^2)/n}\ \sqrt{(n-r^2)/n}\Big) \\
&= \frac{\|\hat{s}\|_{e(t)}}{n}\Big(r\sqrt{n-\beta^2} - \beta\sqrt{n-r^2}\Big),
\end{aligned}$$

as desired. □

Lemma 5.7 provides the desired lower bound R of the distance from $\bar{s}$ to the boundary of the cone $K^{*e(t)}_{e(t),\beta}$, under the assumption that $e(t)$ is in Swath(β) (in addition to being in Swath(r)) for a smaller cone-width parameter β. This is a nontrivial assumption, because Swath(β) $\subseteq$ Swath(r), for $\beta \leq r$.

In the lemma below, we will choose a particular value for β, namely $\beta := r\sqrt{\frac{1+r}{2}}$, and show that for the step of length $t = \frac{r}{2\|x./e\|}$, it is guaranteed that $e(t) \in$ Swath(β).

Lemma 5.8. *Let $\beta = r\sqrt{\frac{1+r}{2}}$. Suppose that $e \in$ Swath(r), x is the optimal solution for ($QP_{e,r}$), and (s,y) is the optimal solution for $(QP_{e,r})^*$. Let $e(t) = \frac{1}{1+t}(e+tx)$. Then, taking $t = \frac{r}{2\|x./e\|}$ implies that $s \in K^*_{e(t),\beta}$ and $e(t) \in$ Swath(β).*

Proof. We wish to show that for $t = \frac{r}{2\|x./e\|}$, the dual solution s lies in the interior of $K^*_{e(t),\beta}$, or equivalently, that the following inequality is satisfied,

$$\langle \mathbb{1}, s.e(t)\rangle^2 - (n-\beta^2)\langle s.e(t), s.e(t)\rangle > 0 \tag{5.17}$$

and that $\langle \mathbb{1}, s.e(t)\rangle \geq 0$. Recall that

$$(1+t)\langle \mathbb{1}, s.e(t)\rangle \;=\; \langle \mathbb{1}, s.e\rangle + t\langle \mathbb{1}, s.x\rangle = \langle \mathbb{1}, s.e\rangle = \langle c, e-x\rangle \geq 0,$$

and

$$(1+t)^2\langle s.e(t), s.e(t)\rangle \;=\; \frac{\langle \mathbb{1}, s.e\rangle^2}{n-r^2} + 2t\langle s.e, s.x\rangle + t^2\langle s.x, s.x\rangle.$$

Therefore, $s \in K^*_{e(t),\beta}$ if and only if

$$-\frac{r^2(1-(\beta/r)^2)}{n-r^2}\langle c, e-x\rangle^2 - (n-\beta^2)\left(2t\langle s.e, s.x\rangle + t^2\langle s.x, s.x\rangle\right) > 0.$$

In (5.7) and (5.8), we computed $\langle s.e, s.x\rangle$ and $\langle s.x, s.x\rangle$, reproduced below:

$$\begin{aligned}
\langle s.e, s.x\rangle &= -\frac{\langle c, e-x\rangle^2}{(n-r^2)^2} r\|x./e\|\left(1 - \frac{r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3}\right),\\
\langle s.x, s.x\rangle &= \frac{\langle c, e-x\rangle^2}{(n-r^2)^2}\|x./e\|^2\left(1 - \frac{2r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3} + \frac{r^2\langle (x./e)^2, (x./e)^2\rangle}{\|x./e\|^4}\right).
\end{aligned}$$

Letting $\eta := \frac{t\|x./e\|}{r}$, to simplify the notation, and using the values of $\langle s.e, s.x\rangle$ and $\langle s.x, s.x\rangle$ above, the inequality (5.17) is equivalent to

$$\frac{\langle c, e-x\rangle^2 r^2}{n-r^2}\left(-(1-(\beta/r)^2 + \frac{n-\beta^2}{n-r^2}\left(1 - \frac{r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3}\right)\left(2\eta - \eta^2\frac{1}{\gamma}\right)\right) > 0,$$

where

$$\gamma := \frac{1 - \frac{r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3}}{1 - \frac{2r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3} + \frac{r^2\langle (x./e)^2, (x./e)^2\rangle}{\|x./e\|^4}}.$$

From the proof of Lemma 5.1, we know that $\gamma \geq \frac{1}{2}$. Using this, and since $\frac{\langle c, e-x\rangle^2 r^2}{n-r^2} > 0$ and $\frac{\langle x./e, (x./e)^2\rangle}{\|x./e\|^3} \leq 1$, we see that in order to satisfy (5.17), it is sufficient to choose η such that we satisfy:

$$-\left(1-(\beta/r)^2\right) + 2\frac{n-\beta^2}{n-r^2}(1-r)\eta\,(1-\eta) > 0$$

or equivalently,

$$(1-r)\eta\,(1-\eta) > \frac{1-(\beta/r)^2}{2}\frac{n-r^2}{n-\beta^2}.$$

Substituting for $\beta = r\sqrt{\frac{1+r}{2}}$, then $1 - (\beta/r)^2 = \frac{1-r}{2}$. Furthermore, using

$$\frac{n - r^2}{n - \beta^2} < 1,$$

it is then sufficient to satisfy

$$\eta(1-\eta) \geq \frac{1-r}{2}\frac{1}{2(1-r)} = \frac{1}{4}.$$

Hence, choosing $\eta = \frac{1}{2}$, or equivalently, choosing $t = \frac{r}{2\|x./e\|}$, guarantees that $s \in \operatorname{int} K^*_{e(t),\beta}$. Furthermore, since $(QP_{e(t),\beta})^*$ has a strictly feasible solution, then for this choice of t, we know that $e(t) \in \text{Swath}(\beta)$. □

We conclude this section with the proof of Proposition 5.4, whose statement we reproduce below.

Proposition (Proposition 5.4)**.** *Suppose that $e \in Swath(r)$, x is optimal for $(QP_{e,r})$, and (s, y) is optimal for $(QP_{e,r})^*$. As usual, we let $e(t) = \frac{1}{1+t}(e + tx)$.*

Then, taking $t = \frac{r}{2\|x./e\|}$ and letting $x(e(t))$ denote the optimal solution for $(QP_{e(t),r})$, we have that

$$\langle c, x(e(t)) - x\rangle \geq R\ \|Proj_{\mathcal{L}^{\perp_{e(t)}}} x(e(t))\|_{e(t)},$$

where $R := \frac{\|s.e(t)\|}{n}\left(r\sqrt{n-\beta^2} - \beta\sqrt{n-r^2}\right)$ and $\beta := r\sqrt{\frac{1+r}{2}}$.

Proof. Recall, from the exposition at the start of the present section, that we would like to find a unit vector $\bar{v}$ and a positive scalar R which satisfy:

1. $\langle x(e(t)), \bar{v}\rangle_{e(t)} > 0$,
2. $\bar{v} \in \mathcal{L}^{\perp_{e(t)}}$ (where $\mathcal{L}$ is the nullspace of A), and
3. The unit ball centered at $\bar{s}$ of radius R is contained in $K^{*e(t)}_{e(t),r}$.

With R and $\bar{v}$ that satisfy the above conditions, letting $\bar{s}' := \bar{s} - Rv$, we have the following lower bound on the increase of the dual solution objective value:

$$\langle c, x(e(t)) - x \rangle \geq \langle x(e(t)), \bar{s} - \bar{s}' \rangle_{e(t)} = R \langle x(e(t)), \bar{v} \rangle_{e(t)} > 0.$$

By Lemma 5.8, taking $t = \frac{r}{2\|x./e\|}$, we have that $e(t) \in \text{Swath}(\beta)$ for $\beta = r\sqrt{\frac{1+r}{2}}$. Hence, we can apply Lemma 5.7 which shows that

$$R := \frac{\|\bar{s}\|_{e(t)}}{n} \left(r\sqrt{n - \beta^2} - \beta\sqrt{n - r^2} \right)$$

works, where $\|\bar{s}\|_{e(t)} = \|s.e(t)\|$ since $\bar{s} = s.e(t)^2$.

As for the choice of $\bar{v}$, note that the projection of $x(e(t))$ onto $\mathcal{L}^{\perp_{e(t)}}$, denoted $\text{Proj}_{\mathcal{L}^{\perp_{e(t)}}} x(e(t))$, clearly satisfies the second of the above conditions. Furthermore, it has a positive inner product with $x(e(t))$, satisfying the first condition. In particular, choosing

$$\bar{v} := \frac{\text{Proj}_{\mathcal{L}^{\perp_{e(t)}}} x(e(t))}{\|\text{Proj}_{\mathcal{L}^{\perp_{e(t)}}} x(e(t))\|_{e(t)}},$$

we see that

$$\begin{aligned} \langle x(e(t)), \bar{v} \rangle_{e(t)} &= \left\langle x(e(t)), \frac{\text{Proj}_{\mathcal{L}^{\perp_{e(t)}}} x(e(t))}{\|\text{Proj}_{\mathcal{L}^{\perp_{e(t)}}} x(e(t))\|_{e(t)}} \right\rangle_{e(t)} \\ &= \|\text{Proj}_{\mathcal{L}^{\perp_{e(t)}}} x(e(t))\|_{e(t)}. \end{aligned}$$

Therefore,

$$\langle c, x(e(t)) - x \rangle \geq R\, \|\text{Proj}_{\mathcal{L}^{\perp_{e(t)}}} x(e(t))\|_{e(t)},$$

where $R := \frac{\|s.e(t)\|}{n} \left(r\sqrt{n - \beta^2} - \beta\sqrt{n - r^2} \right)$ and $\beta := r\sqrt{\frac{1+r}{2}}$, as desired. □

5.7 Proving Proposition 5.5

Proposition 5.4 provides an expression for a good lower bound for $\langle c, x(e(t)) - x \rangle$ in terms of a multiple of $\|\text{Proj}_{\mathcal{L}^{\perp_{e(t)}}} x(e(t))\|_{e(t)}$. To render this bound useful, we need to compute an expression for the length of this projection.

Throughout this section, we rely on observations about the tangent spaces to the quadratic cones, which we discussed more thoroughly in Section 4.5. In particular, we will consider the set $\mathcal{T}$, which is defined in (4.25) as the intersection between $\mathcal{L}$ and the space tangent to the cone $K_{e(t),r}$ at the optimal solution $x(e(t))$ of $(QP_{e(t),r})$.

Lemma 5.9. *Suppose that $e \in Swath(r)$, x is optimal for $(QP_{e,r})$, and (s, y) is optimal for $(QP_{e,r})^*$. As usual, we let $e(t) = \frac{1}{1+t}(e + tx)$ and let $x(e(t))$ be the optimal solution for $(QP_{e(t),r})$. Then, for all $0 < t < \frac{r}{\|x./e\|}$,*

$$\|Proj_{\mathcal{L}^{\perp_{e(t)}}} x(e(t))\|_{e(t)}^2 = \frac{\|x(e(t))\|_{e(t)}^2 \sin^2\theta}{1 + \frac{(\|x(e(t))\|_{e(t)} - r)^2}{n - r^2} \sin^2\theta},$$

where θ denotes the angle between $x(e(t))$ and its projection onto $\mathcal{T}$.

Proof. Choose $v \in \mathcal{T}$ which minimizes $\sin^2 \angle_{e(t)}(x(e(t)), v)$ among all vectors in $\mathcal{T}$. That is, v is a scalar multiple of the projection of $x(e(t))$ onto $\mathcal{T}$, with respect to the inner product $\langle \cdot, \cdot \rangle_{e(t)}$. Let $\theta := \angle_{e(t)}(x(e(t)), v)$, the angle between $x(e(t))$ and v with respect to $\langle \cdot, \cdot \rangle_{e(t)}$.

We first consider the case when $\sin^2\theta < 1$, which means that $\langle x(e(t)), v \rangle_{e(t)} \neq 0$. By Proposition 4.8, we can write such a vector v in the form $v = x(e(t)) + w$ (possibly after some scaling), for some w where $\langle x(e(t)), w \rangle_{e(t)} = \langle e(t), w \rangle_{e(t)} = 0$. Writing $v = x(e(t)) + w$, we obtain the following equalities:

$$\langle x(e(t)), v \rangle_{e(t)} = \|x(e(t))\|_{e(t)}^2, \tag{5.18}$$

$$\begin{aligned} \langle v, e(t) - x(e(t)) \rangle_{e(t)} &= \langle x(e(t)), e(t) - x(e(t)) \rangle_{e(t)} & (5.19) \\ &= \|x(e(t))\|_{e(t)} \left(r - \|x(e(t))\|_{e(t)} \right), & (5.20) \end{aligned}$$

and

$$|\cos\theta| = \left|\frac{\langle x(e(t)), v\rangle_{e(t)}}{\|x(e(t))\|_{e(t)}\ \|v\|_{e(t)}}\right| = \left|\frac{\|x(e(t))\|_{e(t)}}{\|v\|_{e(t)}}\right| \tag{5.21}$$

Suppose that p denotes the projection of $x(e(t))$ onto $\mathcal{L}^{\perp_{e(t)}}$ and q denotes the projection of $x(e(t))$ onto $\mathcal{L}$, with respect to the inner product $\langle\cdot,\cdot\rangle_{e(t)}$.

Recall from Section 4.5 that $\mathcal{T}$ is a subspace of codimension 1 in $\mathcal{L}$. Therefore, we can write q as a linear combination of v and another vector in $\mathcal{L}$ that is orthogonal to v. In particular, note that

$$u := \frac{\|v\|^2_{e(t)}}{\langle v, e(t) - x(e(t))\rangle_{e(t)}}(e(t) - x(e(t))) - v \tag{5.22}$$

lies in $\mathcal{L}$ and satisfies $\langle v, u\rangle_{e(t)} = 0$:

$$\begin{aligned} Au &= \frac{\|v\|^2_{e(t)}}{\langle v, e(t) - x(e(t))\rangle_{e(t)}}A(e(t) - x(e(t))) - Av = 0, \\ \langle v, u\rangle_{e(t)} &= \frac{\|v\|^2_{e(t)}}{\langle v, e(t) - x(e(t))\rangle_{e(t)}}\langle v, e(t) - x(e(t))\rangle_{e(t)} - \langle v, v\rangle_{e(t)} \\ &= \|v\|^2_{e(t)} - \|v\|^2_{e(t)} = 0. \end{aligned}$$

Then, the projection of $x(e(t))$ onto v is

$$\frac{\langle x(e(t)), v\rangle_{e(t)}}{\|v\|^2_{e(t)}}v = \frac{\|x(e(t))\|^2_{e(t)}}{\|v\|^2_{e(t)}}v = (\cos^2\theta)v,$$

where the first equality is due to (5.18) and the second is due to (5.21). The projection of $x(e(t))$ onto u is

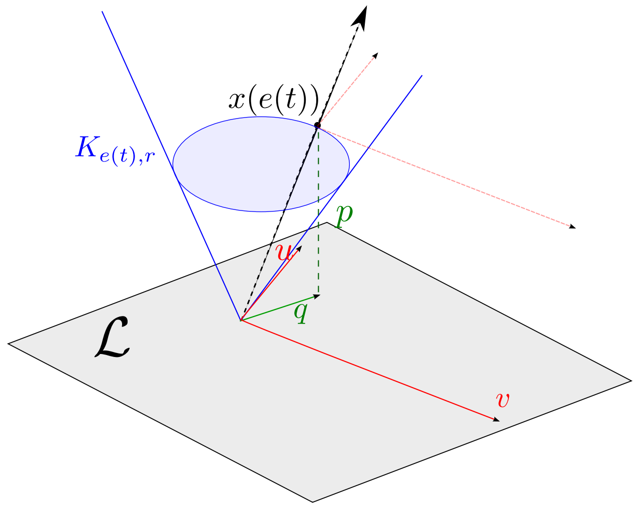
$$\frac{\langle x(e(t)), u\rangle_{e(t)}}{\langle u, u\rangle_{e(t)}}u.$$

Therefore, in terms of u and v, the projection of $x(e(t))$ onto $\mathcal{L}$ is

$$q = \frac{\|x(e(t))\|^2_{e(t)}}{\|v\|^2_{e(t)}}v + \frac{\langle x(e(t)), u\rangle_{e(t)}}{\langle u, u\rangle_{e(t)}}u,$$

whose squared norm is

$$\|q\|^2_{e(t)} = \|x(e(t))\|^2_{e(t)}\cos^2\theta + \frac{\langle x(e(t)), u\rangle^2_{e(t)}}{\langle u, u\rangle_{e(t)}}.$$



Using the expression for u given in (5.22), the equalities (5.18)-(5.21) about the tangent vector v, and the fact that $\|e(t)\|_{e(t)} = \sqrt{n}$, we note that

$$\begin{aligned}
\langle x(e(t)), u\rangle_{e(t)} &= \frac{\|v\|^2_{e(t)}}{\langle v, e(t) - x(e(t))\rangle_{e(t)}}\langle x(e(t)), e(t) - x(e(t))\rangle_{e(t)} - \langle x(e(t)), v\rangle_{e(t)} \\
&= \|v\|^2_{e(t)} - \|x(e(t))\|^2_{e(t)} = \|v\|^2_{e(t)} \sin^2\theta
\end{aligned}$$

and that

$$\begin{aligned}
\langle u, u\rangle_{e(t)} &= \left(\frac{\|v\|^2_{e(t)}}{\langle v, e(t) - x(e(t))\rangle_{e(t)}}\right)^2 \|e(t) - x(e(t))\|^2 \\
&\quad - 2\frac{\|v\|^2_{e(t)}}{\langle v, e(t) - x(e(t))\rangle_{e(t)}}\langle v, e(t) - x(e(t))\rangle_{e(t)} + \|v\|^2_{e(t)} \\
&= \left(\frac{\|v\|^2_{e(t)}}{\langle v, e(t) - x(e(t))\rangle_{e(t)}}\right)^2 \|e(t) - x(e(t))\|^2_{e(t)} - \|v\|^2_{e(t)} \\
&= \|v\|^4_{e(t)}\left(\frac{\|e(t) - x(e(t))\|^2_{e(t)}}{\langle v, e(t) - x(e(t))\rangle^2_{e(t)}} - \frac{1}{\|v\|^2_{e(t)}}\right) \\
&= \frac{\|v\|^4_{e(t)}}{\|x(e(t))\|^2_{e(t)}}\left(\frac{\|e(t) - x(e(t))\|^2_{e(t)}}{(r - \|x(e(t))\|_{e(t)})^2} - \cos^2\theta\right).
\end{aligned}$$

These imply that

$$\frac{\langle x(e(t)), u\rangle^2_{e(t)}}{\langle u, u\rangle_{e(t)}} = \frac{\|x(e(t))\|^2_{e(t)}\ \sin^4\theta}{\frac{\|e(t)-x(e(t))\|^2_{e(t)}}{(r-\|x(e(t))\|_{e(t)})^2} - \cos^2\theta}.$$

Hence,

$$\|q\|_{e(t)}^2 = \|x(e(t))\|_{e(t)}^2 \left(\cos^2\theta + \frac{\sin^4\theta}{\frac{\|e(t)-x(e(t))\|_{e(t)}^2}{(r-\|x(e(t))\|_{e(t)})^2} - \cos^2\theta} \right).$$

So, the (squared) length of the projection of $x(e(t))$ onto $\mathcal{L}^{\perp_{e(t)}}$ is

$$\begin{aligned}
\|p\|_{e(t)}^2 &= \|x(e(t))\|_{e(t)}^2 - \|q\|_{e(t)}^2 \\
&= \|x(e(t))\|_{e(t)}^2 \left(1 - \cos^2\theta - \frac{\sin^4\theta}{\frac{\|e(t)-x(e(t))\|_{e(t)}^2}{(r-\|x(e(t))\|_{e(t)})^2} - \cos^2\theta} \right) \\
&= \|x(e(t))\|_{e(t)}^2 \sin^2\theta \left(1 - \frac{\sin^2\theta}{\frac{\|e(t)-x(e(t))\|_{e(t)}^2}{(r-\|x(e(t))\|_{e(t)})^2} - \cos^2\theta} \right).
\end{aligned}$$

Since $\|e(t)-x(e(t))\|_{e(t)}^2 = n - 2r\|x(e(t))\|_{e(t)} + \|x(e(t))\|_{e(t)}^2$, then

$$\begin{aligned}
1 - \frac{\sin^2\theta}{\frac{\|e(t)-x(e(t))\|_{e(t)}^2}{(r-\|x(e(t))\|_{e(t)})^2} - \cos^2\theta} &= \left(1 + \frac{\sin^2\theta}{\frac{\|e(t)-x(e(t))\|_{e(t)}^2}{(r-\|x(e(t))\|_{e(t)})^2} - 1} \right)^{-1} \\
&= \left(1 + \frac{\sin^2\theta}{\frac{n-2r\|x(e(t))\|_{e(t)}+\|x(e(t))\|_{e(t)}^2}{r^2-2r\|x(e(t))\|_{e(t)}+\|x(e(t))\|_{e(t)}^2} - 1} \right)^{-1} \\
&= \left(1 + \frac{(r-\|x(e(t))\|_{e(t)})^2}{n-r^2} \sin^2\theta \right)^{-1}.
\end{aligned}$$

Thus,

$$\|p\|_{e(t)}^2 = \frac{\|x(e(t))\|_{e(t)}^2 \sin^2\theta}{1 + \frac{(r-\|x(e(t))\|_{e(t)})^2}{n-r^2}\sin^2\theta}, \tag{5.23}$$

when $\sin^2\theta < 1$.

In the case that $\sin^2\theta = 1$, then $\langle x(e(t)), v\rangle_{e(t)} = 0$ for all $v \in \mathcal{T}$. That is, $x(e(t))$ is orthogonal to all vectors in $\mathcal{T}$. Since the $\mathcal{T}$ together with $e(t) - x(e(t))$ spans $\mathcal{L}$, then it must be the case that the projection of $x(e(t))$ onto $\mathcal{L}$ is a multiple of $e(t) - x(e(t))$, namely:

$$q = \frac{\langle x(e(t)), e(t) - x(e(t))\rangle}{\|e(t) - x(e(t))\|_{e(t)}^2}(e(t) - x(e(t))).$$

Hence,

$$\begin{aligned}
\|q\|_{e(t)}^2 &= \frac{\langle x(e(t)), e(t) - x(e(t))\rangle^2}{\|e(t) - x(e(t))\|_{e(t)}^2} \\
&= \frac{\|x(e(t))\|_{e(t)}^2 (r - \|x(e(t))\|_{e(t)})^2}{\|e(t) - x(e(t))\|_{e(t)}^2}.
\end{aligned}$$

Since $\|e(t) - x(e(t))\|_{e(t)}^2 = n - 2r\|x(e(t))\|_{e(t)} + \|x(e(t))\|_{e(t)}^2$, then

$$\begin{aligned}
\|p\|_{e(t)}^2 &= \|x(e(t))\|_{e(t)}^2 - \|q\|_{e(t)}^2 \\
&= \|x(e(t))\|_{e(t)}^2 \left(1 - \frac{(r - \|x(e(t))\|_{e(t)})^2}{\|e(t) - x(e(t))\|_{e(t)}^2}\right) \\
&= \|x(e(t))\|_{e(t)}^2 \left(1 + \frac{1}{\frac{\|e(t)-x(e(t))\|_{e(t)}^2}{(r-\|x(e(t))\|_{e(t)})^2} - 1}\right)^{-1} \\
&= \frac{\|x(e(t))\|_{e(t)}^2}{1 + \frac{(r-\|x(e(t))\|_{e(t)})^2}{n-r^2}}.
\end{aligned}$$

Thus, (5.23) in fact also holds when $\sin^2\theta = 1$, proving our claim. □

When we take a step of length $t = \frac{r}{2\|x./e\|}$, we can obtain an explicit bound on the angle between $x(e(t))$ and vectors in $\mathcal{T}$.

Lemma 5.10. *Let $\beta = r\sqrt{\frac{1+r}{2}}$. Let $e \in Swath(r)$ and let x be the optimal solution for $(QP_{e,r})$. Choose $t = \frac{r}{2\|x./e\|}$ and let $e(t) = \frac{1}{1+t}(e + tx)$, with $x(e(t))$ the optimal solution for $(QP_{e(t),r})$.*

Then, for any $v \in \mathcal{T}$,

$$\cos^2(\angle_{e(t)}(x(e(t)), v)) \leq \frac{\beta^2}{r^2}.$$

Proof. Note that by Lemma 5.8, with $t = \frac{r}{2\|x./e\|}$, we know that $e(t) \in \text{Swath}(\beta) \subseteq \text{Swath}(r)$ for $\beta = r\sqrt{\frac{1+r}{2}}$. Thus, we can talk about $x(e(t))$, the optimal solution to $(QP_{e(t),r})$ and also of the optimal solution to $(QP_{e(t),\beta})$ which we will call x'.

Hence, restricting ourselves to tangent vectors that are also in $\mathcal{L}$, the set of tangent vectors to $K_{e(t),r}$ at $x(e(t))$ is the same as the set of tangent vectors to $K_{e(t),\beta}$ at x' (again, see discussion in Section 4.5), namely:

$$\mathcal{T} = \{v \in \mathbb{R}^n \mid Av = 0,\ \langle c, v\rangle = 0\}.$$

Consider an arbitrary vector v from $\mathcal{T}$. Since $x' \neq 0$ is a point in the boundary of $K_{e(t),\beta}$ and v is tangent to the cone $K_{e(t),\beta}$, then the line $\{x' + tv \mid t \in \mathbb{R}\}$ does not intersect the interior of $K_{e(t),\beta}$, neither v or $-v$ is contained in the interior, and thus $\theta' := \angle_{e(t)}(e(t), v)$, the angle between v and $e(t)$, satisfies

$$\cos^2(\theta') \leq \frac{\beta^2}{n},$$

because $K_{e(t),\beta}$ consists of the vectors whose angle with $e(t)$ does not exceed $\arccos(\beta/\sqrt{n})$.

Also recall from (4.23) that v satisfies

$$\langle e(t), x(e(t))\rangle_{e(t)}\ \langle e(t), v\rangle_{e(t)} = r^2\langle x(e(t)), v\rangle_{e(t)}.$$

Furthermore, since $\langle e(t), x(t)\rangle_{e(t)} = r\|x(e(t))\|_{e(t)}$, then

$$\frac{\langle x(e(t)), v\rangle_{e(t)}}{\|x(e(t))\|_{e(t)}} = \frac{\langle e(t), v\rangle_{e(t)}}{r}.$$

Let $\theta := \angle_{e(t)}(x(e(t)), v)$, the angle between $x(e(t))$ and v. Since $v \in \mathcal{T}$, then θ satisfies

$$\cos^2(\theta) = \frac{\langle x(e(t)), v\rangle^2_{e(t)}}{\|x(e(t))\|^2_{e(t)}\ \|v\|^2_{e(t)}} = \frac{\langle e(t), v\rangle^2_{e(t)}}{r^2\|v\|^2_{e(t)}} = \frac{\|e(t)\|^2_{e(t)}}{r^2}\cos^2(\theta') \leq \frac{\beta^2}{r^2},$$

where the inequality is due to $\cos^2(\theta') \leq \frac{\beta^2}{n}$ and $\|e(t)\|_{e(t)} = \sqrt{n}$. □

Combining the results of the above two lemmas, we obtain the following easy corollary.

Corollary 5.11. *Suppose that $e \in Swath(r)$, x is optimal for $(QP_{e,r})$, and (s, y) is optimal for $(QP_{e,r})^*$. As usual, we let $e(t) = \frac{1}{1+t}(e + tx)$ and $x(e(t))$ the optimal solution for $(QP_{e(t),r})$. Then, for $t = \frac{r}{2\|x./e\|}$,*

$$\|Proj_{\mathcal{L}^{\perp_{e(t)}}} x(e(t))\|_{e(t)}^2 \geq \frac{\|x(e(t))\|_{e(t)}^2(1 - (\beta/r)^2)}{1 + \frac{(\|x(e(t))\|_{e(t)} - r)^2}{n - r^2}(1 - (\beta/r)^2)},$$

where $\beta = r\sqrt{\frac{1+r}{2}}$.

Proof. By Lemma 5.8, the choice $t = \frac{r}{2\|x./e\|}$ guarantees that $e(t) \in \text{Swath}(\beta)$. This means that $(QP_{e(t),\beta})$ has an optimal solution.

Lemma 5.10 implies that for each $v \in \mathcal{T}$, $\sin^2(\angle_{e(t)}(x(e(t)), v)) \geq 1 - \frac{\beta^2}{r^2}$. Thus, combining this bound with the result of Lemma (5.9), we see that

$$\|\text{Proj}_{\mathcal{L}^{\perp_{e(t)}}} x(e(t))\|_{e(t)}^2 \geq \frac{\|x(e(t))\|_{e(t)}^2(1 - \frac{\beta^2}{r^2})}{1 + \frac{(r - \|x(e(t))\|_{e(t)})^2}{n - r^2}(1 - \frac{\beta^2}{r^2})},$$

proving our claim. □

We are now ready to prove Proposition 5.5, which is reproduced below.

Proposition (Proposition 5.5)**.** *Suppose that $e \in Swath(r)$, x is optimal for $(QP_{e,r})$, and s is optimal for $(QP_{e,r})^*$. As usual, we let $e(t) = \frac{1}{1+t}(e + tx)$. Assume that $n \geq 4$.*

Choose $t = \frac{r}{2\|x./e\|}$ and let $x(e(t))$ denote the optimal solution for $(QP_{e(t),r})^*$. If $\|x(t)./e(t)\| > \sqrt{n}$, then

$$\langle c, e(t) - x(e(t))\rangle \leq \langle c, e - x\rangle \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right).$$

Proof. Choose $t = \frac{r}{2\|x./e\|}$. From (5.13) and Proposition 5.4 we have that

$$\langle c, e(t) - x(e(t))\rangle \leq \frac{1}{1+t}\langle c, e - x\rangle - R\,\|\text{Proj}_{\mathcal{L}^{\perp_{e(t)}}} x(e(t))\|_{e(t)},$$

where $R := \frac{\|s.e(t)\|}{n}\left(r\sqrt{n-\beta^2} - \beta\sqrt{n-r^2}\right)$ and $\beta := r\sqrt{\frac{1+r}{2}}$.

Applying the lower bound for $\|\text{Proj}_{\mathcal{L}^{\perp_{e(t)}}}(x(e(t)))\|_{e(t)}$ that we obtain in Corollary 5.11,

$$\begin{aligned}\langle c, e(t) - x(e(t))\rangle &\leq \frac{1}{1+t}\langle c, e - x\rangle - R\left(\frac{\|x(e(t))./e(t)\|^2(1-(\beta/r)^2)}{1 + \frac{(\|x(e(t))./e(t)\|-r)^2}{n-r^2}(1-(\beta/r)^2)}\right)^{1/2} \\ &= \frac{1}{1+t}\langle c, e - x\rangle - \|s.e(t)\|\, C_1 C_2,\end{aligned}$$

where

$$C_1 := \frac{1}{n}\left(r\sqrt{n-\beta^2} - \beta\sqrt{n-r^2}\right)$$

and

$$C_2 := \left(\frac{\|x(e(t))\|_{e(t)}^2(1-(\beta/r)^2)}{1 + \frac{(\|x(e(t))\|_{e(t)}-r)^2}{n-r^2}(1-(\beta/r)^2)}\right)^{1/2}.$$

Since $\beta \leq r < 1$, then

$$C_1 \geq \frac{1}{n}(r-\beta)\sqrt{n-r^2} \geq \frac{\sqrt{n-1}}{n}(r-\beta).$$

Also, since $0 < r < 1$ and using $\beta = r\sqrt{\frac{1+r}{2}}$,

$$\begin{aligned}C_2 &\geq \left(\frac{(n-1)\|x(e(t))\|_{e(t)}^2(1-(\beta/r)^2)}{(n-1) + (\|x(e(t))\|_{e(t)} - r)^2(1-(\beta/r)^2)}\right)^{1/2} \\ &\geq \sqrt{n-1}\left(\frac{\|x(e(t))\|_{e(t)}^2(1-(\beta/r)^2)}{n + \|x(e(t))\|_{e(t)}^2(1-(\beta/r)^2)}\right)^{1/2} \\ &= \sqrt{n-1}\left(\frac{\|x(e(t))\|_{e(t)}^2\frac{1+r}{2}}{n + \|x(e(t))\|_{e(t)}^2\frac{1+r}{2}}\right)^{1/2},\end{aligned}$$

where the second inequality is due to $\|x(e(t))\| \geq \sqrt{n} \geq r$. Hence,

$$C_1C_2 \geq \frac{n-1}{n} r\left(1 - \sqrt{\frac{1+r}{2}}\right)\left(\frac{\|x(e(t))\|_{e(t)}^2\frac{1+r}{2}}{n + \|x(e(t))\|_{e(t)}^2\frac{1+r}{2}}\right)^{1/2}.$$

We assumed that $\|x(e(t))./e(t)\| = \|x(e(t))\|_{e(t)} > \sqrt{n}$, so

$$\begin{aligned} C_1C_2 &\geq \frac{n-1}{n} r \left(1 - \sqrt{\frac{1+r}{2}}\right) \left(\frac{n^{\frac{1+r}{2}}}{n + n^{\frac{1+r}{2}}}\right)^{1/2} \\ &= \frac{n-1}{n} r \left(1 - \sqrt{\frac{1+r}{2}}\right) \left(\frac{\frac{1+r}{2}}{1 + \frac{1+r}{2}}\right)^{1/2}. \end{aligned}$$

Recall that

$$(1+t)^2 \|s.e(t)\|^2 = \frac{\langle c, e-x\rangle^2}{n-r^2} + 2t\langle s.e, s.x\rangle + t^2 \langle s.x, s.x\rangle,$$

where (from (5.7) and (5.8)),

$$\begin{aligned} \langle s.e, s.x\rangle &= -\frac{\langle c, e-x\rangle^2}{(n-r^2)^2} r\|x./e\| \left(1 - \frac{r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3}\right), \\ \langle s.x, s.x\rangle &= \frac{\langle c, e-x\rangle^2}{(n-r^2)^2} \|x./e\|^2 \left(1 - \frac{2r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3} + \frac{r^2\langle (x./e)^2, (x./e)^2\rangle}{\|x./e\|^4}\right). \end{aligned}$$

Using $t = \frac{r}{2\|x./e\|}$, then $(1+t)^2\|s.e(t)\|^2 = \frac{\langle c, e-x\rangle^2}{(n-r^2)^2}\kappa$, where

$$\kappa := (n-r^2) - r^2\left(1 - \frac{r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3}\right) + \frac{r^2}{4}\left(1 - \frac{2r\langle x./e, (x./e)^2\rangle}{\|x./e\|^3} + \frac{r^2\langle (x./e)^2, (x./e)^2\rangle}{\|x./e\|^4}\right).$$

Simplifying, we see that

$$\begin{aligned} \kappa &\geq (n-r^2) - r^2\left(1 - \frac{r\langle x./e, (x./e)^2\rangle}{2\|x./e\|^3}\right) \\ &\geq n - 3r^2 \geq n - 3. \end{aligned}$$

Thus,

$$\|s.e(t)\| \geq \frac{\langle c, e-x\rangle}{(n-r^2)(1+t)} \sqrt{n-3} \geq \frac{\sqrt{n-3}}{n} \frac{\langle c, e-x\rangle}{1+t}.$$

Therefore,

$$C_1C_2\|s.e(t)\| \geq \frac{\langle c, e-x\rangle}{1+t} \frac{r(n-1)\sqrt{n-3}}{n^2} \left(1 - \sqrt{\frac{1+r}{2}}\right)\left(\frac{\frac{1+r}{2}}{1 + \frac{1+r}{2}}\right)^{1/2}$$

and

$$\begin{aligned}\langle c, e(t) - x(e(t))\rangle &\le \frac{\langle c, e-x\rangle}{1+t}\left(1 - \frac{r(n-1)\sqrt{n-3}}{n^2}\left(1-\sqrt{\frac{1+r}{2}}\right)\left(\frac{\frac{1+r}{2}}{1+\frac{1+r}{2}}\right)^{1/2}\right)\\ &\le \frac{\langle c, e-x\rangle}{1+t}\left(1 - \frac{r\left(1-\sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)\\ &\le \langle c, e-x\rangle\left(1 - \frac{r\left(1-\sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right),\end{aligned}$$

where the second to the last inequality is due to the following bounds. For all $n \ge 4$, $\sqrt{n-3} \ge \sqrt{n/4}$ and $\frac{n-1}{n^2} \ge \frac{1}{2n}$. So,

$$\frac{(n-1)\sqrt{n-3}}{n^2} \ge \sqrt{\frac{n}{4}}\frac{1}{2n} = \frac{1}{4\sqrt{n}}.$$

Since $r \ge 0$, then

$$\frac{\frac{1+r}{2}}{1+\frac{1+r}{2}} = \frac{1}{1+\frac{2}{1+r}} \ge \frac{1}{3}.$$

□

5.8 Related work

5.8.1 Interpretation as an affine-scaling algorithm

We can view our algorithm as an affine-scaling algorithm. As we described in Chapter 3, in Dikin's affine-scaling algorithm, the feasible region of the linear program to be solved is replaced by an ellipsoid centered at the current iterate. If $x \in Int$ denotes the current iterate, the ellipsoid is

$$E_x := \{w \in \mathbb{R}^n \mid \|X^{-1}(w-x)\| \le 1\},$$

where $X = \text{diag } x$. We note that this ellipsoid is in fact the unit ball centered at x with respect to the norm induced by the local inner product at x:

$$E_x = \{w \in \mathbb{R}^n \mid \|w - x\|_x \leq 1\}.$$

That is, the current iterate x provides not just the center of the ellipsoid, but also the affine scaling that determines the shape of the ellipsoid.

For each $x \in \mathbb{R}^n_{++}$, E_x is guaranteed to be contained in the positive orthant (see [27]). That is, the scaling by X ensures that the shape of E_x "conforms" to the boundary of the feasible region near x, from the inside. The algorithm then finds the optimal solution on the intersection of this ellipsoid with the affine linear constraint of the linear program, which provides the next iterate.

Our algorithm uses the cone $K_{e,r}$ which is centered at the current iterate e, and whose shape is determined by the affine scaling given by the inverse of $E = \text{diag } e$. That is, similar to Dikin's algorithm, each of our iterates determines the center and the shape of the space that determines the feasible region of the optimization problems that determines the search directions.

However, our cones $K_{e,r}$ circumscribe the linear programming feasible region, while Dikin's ellipsoids are inscribed within it. Still, the scaling by E^{-1} ensures that the shape of the cone in a sense conforms to the boundary of the linear programming feasible region near e, but from the outside.

While Dikin's feasible region is the intersection of an ellipsoid with an affine subset of $\mathbb{R}^n$, our feasible region is a conic section: the intersection of a cone with the affine subset. Among the primal-dual affine-scaling algorithms which we discussed in Chapter 3, the cone affine-scaling algorithm of Sturm and Zhang [30] generalizes Dikin's approach by considering an inscribed conic section instead of an ellipsoid.

5.8.2 Chua's primal-dual quadratic-cone based algorithm for symmetric cones

Our use of the quadratic cone relaxations in this algorithm is inspired by Chua's algorithm in [4]. In this work, he describes an algorithm which at each iteration starts with a primal and dual pair of solutions $x^{(k)}$ and $(s^{(k)}, y^{(k)})$ that are strictly feasible for the linear program (that is, they lie in $\mathbb{R}^n_{++}$), with $s^{(k)}$ assumed to lie in the cone $K^*_{x^{(k)},r}$. We obtain the next iterate, call it $(x^{(k+1)}, s^{(k+1)}, y^{(k+1)})$, in the following way: $(s^{(k+1)}, y^{(k+1)})$ is chosen to be the optimal solution to the problem $(D^{(k)})$, the "dual-form" problem in the following pair:

$$\begin{array}{rll} \min & \langle c, x\rangle & \\ s.t. & Ax = b & (P^{(k)}) \\ & x \in K_{x^{(k)},r}, & \end{array}$$

$$\begin{array}{rll} \max & b^T y & \\ s.t. & A^T y + s = c & (D^{(k)}) \\ & s \in K^*_{x^{(k)},r}. & \end{array}$$

We know that an optimal solution $(s^{(k+1)}, y^{(k+1)})$ exists because $(P^{(k)})$ is strictly feasible (for instance, $x^{(k)}$ is a strictly feasible solution) and $(D^{(k)})$ is feasible since we assume that $s^{(k)}$ lies in the cone $K^*_{x^{(k)},r}$ (see Theorem 2.4, [27, Theorem 3.2.8]).

Then, $x^{(k+1)}$ is chosen to be the optimal solution to the problem $(\tilde{P}^{(k+1)})$, the "primal-form" problem in the following pair:

$$\begin{array}{rll} \min & \langle c, x\rangle & \\ s.t. & Ax = b & (\tilde{P}^{(k+1)}) \\ & x \in K^*_{s^{(k+1)},r}, & \end{array}$$

$$\begin{array}{rll} \max & b^T y & \\ s.t. & A^T y + s = c & (\tilde{D}^{(k+1)}) \\ & s \in K_{s^{(k+1)},r}. & \end{array}$$

Note that $(\tilde{D}^{(k+1)})$ is strictly feasible because we know that $s^{(k)}$ lies in $K^*_{x^{(k)},r}$ and $K^*_{x^{(k)},r} \subseteq \mathbb{R}^n_+ \subseteq \text{int } K_{s^{(k+1)},r}$. The primal-form problem $(\tilde{P}^{(k+1)})$ is feasible: since $s^{(k+1)} \in K^*_{x^{(k)},r}$, then

$$\langle \mathbb{1}, s^{(k+1)}.x^{(k)}\rangle^2 \geq \sqrt{n-r^2}\|s^{(k+1)}.x^{(k)}\|,$$

which means that $x^{(k)} \in K^*_{s^{(k+1)},r}$. Since $(\tilde{P}^{(k+1)})$ is feasible and $(\tilde{D}^{(k+1)})$ is strictly feasible, then we know that an optimal solution $x^{(k+1)}$ for $(\tilde{P}^{(k+1)})$ exists.

Therefore, the new iterate, $(x^{(k+1)}, s^{(k+1)}, y^{(k+1)})$, satisfies: $x^{(k+1)}$ is strictly feasible for the primal linear program, $(s^{(k+1)}, y^{(k+1)})$ strictly feasible for the dual linear program, and $s^{(k+1)} \in K^*_{x^{(k+1)},r}$. We can then repeat the process to obtain the next iterate until we achieve a sufficiently small duality gap.

Thus, the algorithm alternately solves a quadratic-cone restriction of the primal and the dual linear program, stopping when the duality gap of the current pair of iterates has been reduced to a chosen factor of the initial duality gap.

We note that although we describe Chua's algorithm specifically for solving linear programs, it was presented and analyzed in the context of semidefinite programming, then extended to the more general setting of cone optimization problems over symmetric cones.

5.8.3 Litvinchev's quadratic-cone relaxation based algorithm

We recently found that in [18], Litvinchev has formulated a primal interior-point algorithm for linear programming that is based on solving quadratic-cone relaxations of the linear program. This algorithm is similar to the algorithm we describe in this chapter, in that the same quadratic-cone relaxations are used, resulting in the same search direction when certain conditions are satisfied. There are, however, a number of nontrivial differences in the algorithm, analysis, and in the resulting complexity bound.

- The most fundamental difference is the fact that Litvinchev's algorithm does not constrain the iterates $e^{(k)}$ to stay within Swath(r), for the chosen cone-width parameter r, nor does he define such a region as Swath(r). Thus, it is possible that during some iteration k, the problem $(QP_{e^{(k)},r})$ is unbounded. To get around this issue, it is assumed at the start of the algorithm that an initial lower bound $z^{(0)}$ of the optimal value of the linear program is known. This lower bound is to be updated at each iteration.

  Then, in the case when $(QP_{e^{(k)},r})$ is unbounded, the search direction is given by $x^{(k)} - e^{(k)}$ where $x^{(k)}$ is any feasible solution for $(QP_{e^{(k)},r})$ whose objective value is less than the current lower bound $z^{(k)}$. In this case, the lower bound stays the same for the next iteration: $z^{(k+1)} = z^{(k)}$.

  In the case when $(QP_{e^{(k)},r})$ has an optimal solution, then the search direction is the same as our algorithm's, namely $x^{(k)} - e^{(k)}$ where $x^{(k)}$ is the optimal solution to $(QP_{e^{(k)},r})$. In this case, the lower bound for the next iteration is $z^{(k+1)} = \max\{z^{(k)}, \langle c, x^{(k)}\rangle\}$.

- The step size used by Litvinchev is $t^{(k)} = \frac{r}{2\langle \mathbb{1}, x^{(k)}./e^{(k)}\rangle}$. If $x^{(k)}$ lies on the boundary of the cone, then this step size is $t^{(k)} = \frac{1}{2\|x^{(k)}./e^{(k)}\|}$, which is a little larger than our step size of $t^{(k)} = \frac{r}{2\|x^{(k)}./e^{(k)}\|}$.
- The cone-width parameter r could be adjusted at each iteration. Under a particular condition, $r = 1$ is chosen, while when this condition is not satisfied, r can be freely chosen from the interval $(0.5, 1)$. In contrast, we fix $r \in (0, 1)$ at the start and leave it unchanged, although we could extend our results to allow varying r.
- The stopping criterion is the gap between the current objective value and the current lower bound: $\langle c, e^{(k)}\rangle - z^{(k)}$. Note that if the current iterate $e^{(k)}$ is in Swath(r), then this stopping criterion is upper bounded by the duality gap $\langle c, e^{(k)} - x^{(k)}\rangle$.

Furthermore, the convergence proof uses a very different route. The approach used by Litvinchev is based on Karmakar's potential function and is strictly primal in nature, in contrast to the dual-based approach that we take in our analysis. That is, he shows that at each iteration, the following potential function is reduced by a positive constant

$$\phi(e^{(k)}, z^{(k)}) = \sum_{i=1}^{n} \ln \frac{\langle c, e^{(k)}\rangle - z^{(k)}}{(e^{(k)})_i}.$$

The resulting iteration complexity is of order $O(n \ln(1/\epsilon))$ for obtaining a gap $\langle c^{(k)}, e^{(k)}\rangle - z^{(k)}$ of size at most $\epsilon > 0$. The constant hidden in this bound also depends on the particular linear program instance, as in [32, Corollary 3.1]. Note that our complexity bound, which grows as $\sqrt{n}$, is better than the bound obtained in Litvinchev's paper. On the other hand, this iteration complexity is also proven for when a conceptualized version of the algorithm applied to optimization problems of the form

$$\begin{array}{rl} \min & f(x) \\ s.t. & x \in \Omega \\ & x \in \mathbb{R}^n_+ \end{array}$$

where f is a convex function that is not necessarily linear and Ω is a convex subset of $\mathbb{R}^n$.

5.9 Concluding remarks

In this chapter, we describe a novel algorithm for linear programming. The algorithm considers a sequence of relaxations that are based on quadratic cones that circumscribe the nonnegative orthant. This algorithm is shown to have an iteration complexity of $O(\sqrt{n}\ln(\gamma_0/\epsilon))$ for reducing the duality gap from γ_0 to ϵ, for some chosen value of $\epsilon > 0$, matching the best bounds among interior-point methods for linear programming.

Although it is a primal algorithm, the analysis is almost exclusively based on analyzing the corresponding dual quadratic cone optimization problems. This approach provides concrete geometric insights into the convergence proof.

This algorithm can be seen as an affine-scaling algorithm, with each iterate providing an affine scaling that transforms each quadratic cone into a circular cone that is centered at the vector of all ones. We compare our algorithm to other affine-scaling algorithms in literature, in particular, the cone affine-scaling algorithm of Sturm and Zhang. We also compare our work to Chua's primal-dual algorithm and Litvinchev's algorithm, which use the same family of quadratic cones.

Litvinchev's algorithm is remarkably similar to ours in the way in which the quadratic cones are used to derive search directions. However, we highlight the nontrivial differences in the algorithm, analysis, and the final complexity result, which is weaker than what we obtain.

The cone affine-scaling linear programming algorithm by Sturm and Zhang was extended to apply to semidefinite programming in [3]. Chua's algorithm, which was described in the context of semidefinite programming, was extended in the same paper, [4], to apply to cone optimization problems over symmetric cones.

In the next chapter, we extend our linear programming algorithm to apply to semidefinite programming.

CHAPTER 6

EXTENSION TO SEMIDEFINITE PROGRAMMING

In Chapter 5, we described and analyzed an algorithm for linear programming that is based on relaxations given by a particular family of quadratic cones. The quadratic cones can be described in terms of affine scalings, or equivalently, in terms of the local inner products given by the iterates, which are both based on the Hessian of the barrier functional for the nonnegative orthant. These notions can be naturally extended to the space of symmetric matrices by considering the barrier functional for the cone of positive semidefinite matrices.

In this chapter, we describe the natural analog of these quadratic cone relaxations for the cone of positive semidefinite matrices. Then, using this family of quadratic cones, we extend the linear programming algorithm from the previous chapter to semidefinite programming. The complexity analysis in the previous chapter was also done in terms of the scalings and the local inner products, through which the complexity analysis from the previous chapter extends to the semidefinite programming setting.

6.1 Setting and notation

We consider the semidefinite programming problem in standard form

$$\begin{aligned} \min \quad & \langle C, X\rangle \\ s.t. \quad & \mathcal{A}X = b \qquad\qquad (SDP) \\ & X \in \mathbb{S}^n_+, \end{aligned}$$

together with its dual

$$\begin{aligned} \max \quad & b^T y \\ s.t. \quad & \mathcal{A}^* y + S = C \qquad (SDP)^* \\ & S \in \mathbb{S}^n_+. \end{aligned}$$

We use $\mathbb{S}^n$ to denote the set of $n \times n$ symmetric matrices, $\mathbb{S}^n_+$ to denote the set of $n \times n$ symmetric positive semidefinite matrices, and $\mathbb{S}^n_{++}$ to denote the positive definite matrices. Here, $C \in \mathbb{S}^n$ and $b \in \mathbb{R}^m$. The operator $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ is given by

$$\mathcal{A}X := (\langle A_1, X\rangle, \ldots, \langle A_m, X\rangle)^T,$$

where $A_i \in \mathbb{S}^n$ for each $i = 1, \ldots, m$, and we assume that the A_i's form a linearly independent set. In this chapter, we choose the inner product $\langle\cdot,\cdot\rangle$ to be the trace inner product. For $y \in \mathbb{R}^m$, we let $\mathcal{A}^* : \mathbb{R}^m \to \mathbb{S}^n$ denote the adjoint of $\mathcal{A}$ with respect to this inner product, given by

$$\mathcal{A}^* y = \sum_{i=1}^m A_i y_i.$$

We redefine Int and Int^* to denote the relative interiors of the feasible regions of (SDP) and $(SDP)^*$, respectively:

$$Int := \{X \in \mathbb{S}^n_{++} \mid \mathcal{A}X = b\},$$

$$Int^* := \{(S, y) \in \mathbb{S}^n_{++} \times \mathbb{R}^m \mid \mathcal{A}^* y + S = C\}.$$

Recall from (2.24) - (2.25) that in the case of positive semidefinite cones, the barrier functional is

$$f(E) := -\log(\det E),$$

whose Hessian is

$$H(E)[\,\cdot\,] := E^{-1}(\,\cdot\,)E^{-1}.$$

Then, given $\mathbb{S}^n$ with the trace inner product $\langle \cdot, \cdot \rangle$ and $E \in \mathbb{S}^n_{++}$, *the local inner product at* E was defined in (2.26) as follows: for each $U, V \in \mathbb{S}^n$,

$$\langle U, V \rangle_E = \langle U, H(E)V \rangle = \langle U, E^{-1} V E^{-1} \rangle$$

$$= \langle E^{-1/2} U E^{-1/2}, E^{-1/2} V E^{-1/2} \rangle.$$

That is, $\langle U, V \rangle_E$ can be thought of as the trace inner product between the images of U and of V under the linear scaling $\mathcal{E}^{-1}$ given by

$$\mathcal{E}^{-1}[\,\cdot\,] := E^{-1/2}(\,\cdot\,)E^{-1/2}. \tag{6.1}$$

More generally, for any $E \in \mathbb{S}^n_{++}$, we can let $\mathcal{E}$ denote the operator

$$\mathcal{E}[\,\cdot\,] := E^{1/2}(\,\cdot\,)E^{1/2}, \tag{6.2}$$

and let $\mathcal{E}^2$ and $\mathcal{E}^{-2}$ denote the operators $E(\,\cdot\,)E$ and $E^{-1}(\,\cdot\,)E^{-1}$, respectively. It is easy to see that $\mathcal{E}$ and $\mathcal{E}^{-1}$ are inverses of one another and that applying $\mathcal{E}^2$ is the same as applying $\mathcal{E}$ twice.

With this notation, then $H(E) = \mathcal{E}^{-2}$, which notationally resembles the Hessian for the barrier functional for LP at $e \in \mathbb{R}^n_{++}$, namely $H(e) = E^{-2}$ where $E = \operatorname{diag} e$.

As we will observe, virtually all proofs in Chapter 5 can be extended word-by-word to proofs for the SDP case, by replacing all "scalings given by componentwise division by e" with "scalings given by $\mathcal{E}^{-1}$" and by replacing the local inner products accordingly.

6.2 Quadratic cones and quadratic cone relaxations of SDP

Recall the family of quadratic cones $K_{e,r}$ which are introduced in Chapter 4. For each $e \in \mathbb{R}^n_{++}$ and each $r \in [0,1]$, the cone $K_{e,r}$ was defined as follows

$$\begin{aligned} K_{e,r} &:= \{x \in \mathbb{R}^n \mid \langle \mathbb{1}, x./e \rangle \geq r\|x./e\|\} \\ &= \{x \in \mathbb{R}^n \mid \langle e, x \rangle_e \geq r\|x\|_e\}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_e$ denotes the local inner product at e and $\|\cdot\|_e$ the induced norm. The second description for $K_{e,r}$ above, in terms of the local inner products, emphasizes that $K_{e,r}$ is the set of all points $x \in \mathbb{R}^n$ whose angles with the center direction e are at most $\arccos \frac{r}{\sqrt{n}}$, with respect to $\langle \cdot, \cdot \rangle_e$.

Since the notion of local inner product extends naturally to $\mathbb{S}^n$, we can extend the quadratic cones to $\mathbb{S}^n$ as well: For a fixed choice of $E \in \mathbb{S}^n_{++}$ and $r \in [0,1]$, let $K_{E,r}$ denote the set of points $X \in \mathbb{S}^n$ whose angles with E are at most $\arccos \frac{r}{\sqrt{n}}$, where the angles are with respect to the local inner product at E. That is,

$$K_{E,r} := \{X \in \mathbb{S}^n \mid \langle I, \mathcal{E}^{-1}X \rangle \geq r\|\mathcal{E}^{-1}X\|\} \quad (6.3)$$

$$= \{X \in \mathbb{S}^n \mid \langle E, X \rangle_E \geq r\|X\|_E\}. \quad (6.4)$$

Note that (6.3) highlights the interpretation of $K_{E,r}$ as the set of points $X \in \mathbb{S}^n$ whose image under the scaling $\mathcal{E}^{-1}$ forms an angle of at most $\arccos \frac{r}{\sqrt{n}}$ with the identity matrix. This parallels the interpretation of $K_{e,r}$ as the set of points $x \in \mathbb{R}^n$ whose image under the scaling E^{-1} (where $E = \operatorname{diag} e$) forms an angle of at most $\arccos \frac{r}{\sqrt{n}}$ with the vector of all ones, with respect to the trace inner product. Hence, it is not surprising that the cones $K_{E,r}$ as defined in (6.3)-(6.4) above inherit virtually all of the properties of the cones $K_{e,r}$ which we described in Chapter 4.

Our goal in this section is to summarize the important properties of $K_{E,r}$ which we use when extending the complexity bound of the algorithm to the semidefinite programming case. That is, we present the semidefinite programming analog of Chapter 4. As we prove these properties, we highlight the similarities of these proofs to the proofs of the linear programming version.

From here onwards, $E \in \mathbb{S}^n_{++}$ and $r \in [0,1]$ unless explicitly specified otherwise.

6.2.1 Basic properties of $K_{E,r}$

Following the definition of $\mathcal{E}$ in (6.2), for any $n \times n$ symmetric positive definite matrix D, we can consider an affine scaling $\mathcal{D} : \mathbb{S}^n \to \mathbb{S}^n$, given by

$$\mathcal{D}[\,\cdot\,] := D^{1/2}(\,\cdot\,)D^{1/2}.$$

Then, we claim that for all such matrices $\mathcal{D}$,

$$\mathcal{D}K_{E,r} = K_{\mathcal{D}E,r}, \tag{6.5}$$

where the notation $\mathcal{D}K_{E,r}$ means the set $\{\mathcal{D}X \mid X \in K_{E,r}\}$.

To see why (6.5) is true, let $F := (\mathcal{D}E)^{-1} = D^{-1/2}E^{-1}D^{-1/2}$ (which is positive definite since $E^{-1} \in \mathbb{S}^n_{++}$ and $\mathcal{D}$ fixes $\mathbb{S}^n_{++}$), and let $F^{1/2}$ denote its square root matrix. Define $\mathcal{F}[\,\cdot\,] := F^{1/2}(\,\cdot\,)F^{1/2}$. Then, note that

$$\begin{aligned}\langle I, \mathcal{E}^{-1}(\mathcal{D}^{-1}X)\rangle &= \operatorname{tr}(I(E^{-1/2}D^{-1/2}XD^{-1/2}E^{-1/2})) \\ &= \operatorname{tr}((D^{-1/2}E^{-1}D^{-1/2})X) \\ &= \operatorname{tr}((F^{1/2}F^{1/2})X) \\ &= \operatorname{tr}(I(F^{1/2}XF^{1/2})) = \langle I, \mathcal{F}X\rangle\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{E}^{-1}(\mathcal{D}^{-1}X)\|^2 &= \langle \mathcal{E}^{-1}(\mathcal{D}^{-1}X), \mathcal{E}^{-1}(\mathcal{D}^{-1}X)\rangle \\
&= \mathrm{tr}((E^{-1/2}D^{-1/2}XD^{1/2}E^{-1/2})(E^{-1/2}D^{-1/2}XD^{1/2}E^{-1/2})) \\
&= \mathrm{tr}((D^{-1/2}E^{-1}D^{-1/2})X(D^{-1/2}E^{-1}D^{-1/2})X) \\
&= \mathrm{tr}((F^{1/2}F^{1/2})X(F^{1/2}F^{1/2})X) \\
&= \mathrm{tr}((F^{1/2}XF^{1/2})(F^{1/2}XF^{1/2})) \\
&= \langle \mathcal{F}X, \mathcal{F}X\rangle = \|\mathcal{F}X\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{D}K_{E,r} &= \{\mathcal{D}X \in \mathbb{S}^n \mid \langle I, \mathcal{E}^{-1}X\rangle \geq r\|\mathcal{E}^{-1}X\|\} \\
&= \{\tilde{X} \in \mathbb{S}^n \mid \langle I, \mathcal{E}^{-1}(\mathcal{D}^{-1}\tilde{X})\rangle \geq r\|\mathcal{E}^{-1}(\mathcal{D}^{-1}\tilde{X})\|\} \\
&= \{\tilde{X} \in \mathbb{S}^n \mid \langle I, \mathcal{F}\tilde{X}\rangle \geq r\|\mathcal{F}\tilde{X}\|\} \\
&= K_{F^{-1},r} = K_{\mathcal{D}E,r}.
\end{aligned}$$

In particular, when $\mathcal{D} = \mathcal{E}^{-1}$, then (6.5) implies that

$$\mathcal{E}^{-1}K_{E,r} = K_{I,r} \quad \text{or equivalently,} \quad K_{E,r} = \mathcal{E}K_{I,r}. \tag{6.6}$$

Proposition 6.1. *For each $E \in \mathbb{S}^n_{++}$ and $r \in [0,1]$,*

1. $K_{E,r} \supseteq \mathbb{S}^n_+$.
2. *The dual of $K_{E,r}$ with respect to the inner product $\langle\cdot,\cdot\rangle$ is*

$$K^*_{E,r} = K_{E^{-1},\sqrt{n-r^2}} = \{S \in \mathbb{S}^n \mid \langle I, \mathcal{E}S\rangle \geq \sqrt{n-r^2}\|\mathcal{E}S\|\}.$$

3. $K^*_{E,r} \subseteq \mathbb{S}^n_+$.

Proof. We begin by proving the first two assertions for when $E = I$, the $n \times n$ identity matrix.

Suppose that $X \in \mathbb{S}^n_+$, then

$$\langle I, X\rangle = \mathrm{tr}X = \sum_{i=1}^{n} \hat{x}_i \geq 0,$$

where $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ denotes the vector of the eigenvalues of X, which are all nonnegative. Furthermore,

$$\langle I, X\rangle^2 = (\mathrm{tr}X)^2 = \left(\sum_{i=1}^{n} \hat{x}_i\right)^2 \geq \sum_{i=1}^{n} \hat{x}_i^2 = \mathrm{tr}(X^2) = \|X\|^2.$$

Thus, for all $r \in [0, 1]$, $\langle I, X\rangle \geq r\|X\|$, proving that $\mathbb{S}^n_+ \subseteq K_{I,r}$, for all $r \in [0, 1]$.

The proof that $K_{I,\sqrt{n-r^2}}$ is the dual of $K_{I,r}$ is the same as the proof that $K_{\mathbb{1},\sqrt{n-r^2}}$ is the dual of $K_{\mathbb{1},r}$ (Proposition 4.2), after replacing $\mathbb{1}$ with I and the dot product with the trace inner product.

We are now ready to prove the three assertions of this proposition for general $E \in \mathbb{S}^n_{++}$. By (6.6), $K_{E,r} = \mathcal{E}K_{I,r}$. Note that $\mathcal{E}$ fixes $\mathbb{S}^n_+$ and is invertible. Thus, $X \in \mathbb{S}^n_+$ if and only if $\mathcal{E}X \in \mathbb{S}^n_+$. This shows that $\mathbb{S}^n_+ = \mathcal{E}\mathbb{S}^n_+ \subseteq \mathcal{E}K_{I,r} = K_{E,r}$, which proves the first assertion.

By the definition of the dual cone, $S \in K^*_{E,r}$ if and only if for each $X \in K_{E,r}$,

$$0 \leq \langle X, S\rangle.$$

Equivalently, using (6.6), $S \in K^*_{E,r}$ if and only if for each $\tilde{X} \in K_{I,r}$,

$$0 \leq \langle \mathcal{E}X, S\rangle = \langle X, \mathcal{E}S\rangle.$$

That is, $S \in K^*_{E,r}$ if and only if $\mathcal{E}S \in K^*_{I,r}$. Therefore, $\mathcal{E}K^*_{E,r} = K^*_{I,r}$, or equivalently,

$$K^*_{E,r} = \mathcal{E}^{-1}K^*_{I,r} = \mathcal{E}^{-1}K_{I,\sqrt{n-r^2}} = K_{E^{-1},\sqrt{n-r^2}},$$

where the last equality is due to (6.6).

Finally, the third assertion is an easy consequence of the fact that $K_{E,r} \supseteq \mathbb{S}^n_+$ and the fact that $\mathbb{S}^n_+$ is self-dual:

$$K^*_{E,r} \subseteq (\mathbb{S}^n_+)^* = \mathbb{S}^n_+.$$

□

6.2.2 The quadratic cone relaxations

Consider $E \in \mathbb{S}^n_{++}$ and $r \in [0, 1]$. Since $K_{E,r} \supseteq \mathbb{S}^n_+$, then by replacing the constraint $X \in \mathbb{S}^n_+$ in (SDP) with $X \in K_{E,r}$, we have a relaxed problem:

$$\begin{array}{rll} \min & \langle C, X\rangle & \\ s.t. & \mathcal{A}X = b & \qquad (QP_{E,r}) \\ & X \in K_{E,r}. & \end{array}$$

The dual problem is

$$\begin{array}{rll} \max & b^T y & \\ s.t. & \mathcal{A}^* y + S = C & \qquad (QP_{E,r})^* \\ & S \in K^*_{E,r}, & \end{array}$$

whose feasible region is contained in the feasible region of $(SDP)^*$.

We define the set Swath(r) similarly to that in the setting of linear programming: For each $r \in [0, 1]$,

$$\text{Swath}(r) := \{E \in Int \mid (QP_{E,r}) \text{ has an optimal solution}\}.$$

Proposition 6.2. *Assume that* (SDP) *and* $(SDP)^*$ *are strictly feasible, with* $C, A_1, \ldots, A_n$ *linearly independent. Then,*

1. *Swath*(0) = *the central path for* (SDP)
2. *For* $0 \le r_1 \le r_2 \le 1$, *Swath*$(r_1) \subseteq$ *Swath*(r_2).

In particular, this implies that if (SDP) *and* $(SDP)^*$ *are strictly feasible, then for each* $r \in [0,1]$, *Swath*(r) *is nonempty.*

Proof. A point $E \in Int$ lies on the central path if it is the optimal solution to a barrier problem (BP_μ) for some $\mu > 0$:

$$\begin{array}{ll} \min & \langle C, X\rangle - \mu \ln\det X \\ s.t. & \mathcal{A}X = b. \end{array}$$

That is, if and only if there exist some $\mu > 0$ and $y \in \mathbb{R}^m$ such that the following KKT conditions are satisfied by (E, y, μ):

$$C - \mu E^{-1} - \mathcal{A}^* y = 0 \tag{6.7}$$

$$\mathcal{A}E = b \tag{6.8}$$

$$E \in \mathbb{S}^n_{++}. \tag{6.9}$$

On the other hand, a point $E \in Int$ lies in Swath(0) if the optimization problem $(QP_{E,0})$ below has an optimal solution

$$\begin{array}{ll} \min & \langle C, X\rangle \\ s.t. & \mathcal{A}X = b \\ & \langle I, \mathcal{E}^{-1}X\rangle \ge 0. \end{array}$$

Therefore, $E \in \text{Swath}(0)$ if and only if there exist some $X \in \mathbb{S}^n_+$, $\mu \geq 0$, and $y \in \mathbb{R}^m$ such that the following KKT conditions are satisfied

$$
\begin{aligned}
C - \mu E^{-1} - \mathcal{A}^* y &= 0 && (6.10)\\
\mathcal{A}X &= b && (6.11)\\
\langle I, \mathcal{E}^{-1}X \rangle &\geq 0 && (6.12)\\
\mu \langle I, \mathcal{E}^{-1}X \rangle &= 0. && (6.13)
\end{aligned}
$$

Since we assume that $C, A_1, \ldots, A_n$ are linearly independent, then C does not lie in the range of $\mathcal{A}^*$. This implies that E^{-1} also must not lie in the range of $\mathcal{A}^*$ and that μ must be strictly positive. Furthermore, since $\mu > 0$, then $\langle I, \mathcal{E}^{-1}X \rangle = 0$, or equivalently, X lies on the boundary of $K_{E,0}$.

Thus, the second set of KKT conditions, (6.10) - (6.13), can be summarized as follows:

1. There exists some $\mu > 0, y \in \mathbb{R}^m$ such that $C - \mu E^{-1} - \mathcal{A}^T y = 0$, and
2. There exists some X that satisfies: $\mathcal{A}X = b$ and $\langle I, \mathcal{E}^{-1}X \rangle = 0$.

We claim that the existence of such an X is guaranteed whenever the first requirement above is satisfied:

The point E satisfies $\mathcal{A}X = b$ and $\langle \mathbb{1}, \mathcal{E}^{-1}E \rangle > 0$. Thus, there exists a solution X that meets the above condition if and only if there exists a direction $V \in \ker \mathcal{A}$ such that $\langle I, \mathcal{E}^{-1}V \rangle \neq 0$. Such a direction V exists as long as E^{-1} is not orthogonal to $\ker \mathcal{A}$. That is, as long as there exists some $\mu > 0, y \in \mathbb{R}^m$ such that $C - \mu E^{-1} - \mathcal{A}^T y = 0$.

Therefore, the KKT conditions (6.10) - (6.13) for $E \in Int$ are exactly the same as the KKT conditions (6.7) - (6.9): $E \in Int$ and there exist some $\mu > 0$ and $y \in \mathbb{R}^m$, $C - \mu(E^{-1}) - \mathcal{A}^* y = 0$.

To prove the second assertion, we begin by noting that for each $E \in Int$ and cone-width parameters $r_1, r_2 \in [0, 1]$ with $r_1 \leq r_2$, $K_{E,r_1} \supseteq K_{E,r_2}$. Suppose that $E \in$ Swath(r_1), which means that (QP_{E,r_1}) has an optimal solution. This means that (QP_{E,r_2}) also has an optimal solution, since it is feasible and since (QP_{E,r_1}) which is its relaxation has an optimal solution. This shows that E is also in Swath(r_2), proving the second assertion.

If (SDP) and $(SDP)^*$ are strictly feasible, then the central path is nonempty. This, together with the previous two assertions, implies that for all $r \in [0, 1]$, the set Swath(r) is nonempty. □

Proposition 6.3. *Suppose that $C, A_1, \ldots, A_m$ form a linearly independent set, and $b \neq 0$. Fix $r \in (0, 1]$. If $E \in$ Swath(r), then $(QP_{E,r})$ has a unique optimal solution.*

Intuitively, this proposition asserts that when the set of optimal solutions of the problem $(QP_{E,r})$ is nonempty and is not the entire feasible region, then the optimal solution is in fact unique because the feasible region of $(QP_{E,r})$ is strictly convex.

The detailed proof is exactly the same as that of Proposition 4.7, with the following replacements:

- the dot product on $\mathbb{R}^n$ is replaced by the trace inner product on $\mathbb{S}^n$, and
- componentwise division by e (or, the affine scaling by the matrix E^{-1}) is replaced by the affine scaling by $\mathcal{E}^{-1}$.

Furthermore, we also replace the following linear programming objects with the semidefinite programming version:

- the $m \times n$ matrix A (as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$) is replaced by the operator $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$,
- vectors in $\mathbb{R}^n$ are replaced by matrices in $\mathbb{S}^n$ (for example, $c \in \mathbb{R}^n$ is replaced by $C \in \mathbb{S}^n$ and $v \in \mathbb{R}^n$ is replaced by $V \in \mathbb{S}^n$),
- vectors in $\mathbb{R}^n_{++}$ are replaced by matrices in $\mathbb{S}^n_{++}$, and
- the cone $K_{e,r} \subseteq \mathbb{R}^n$ is replaced by $K_{E,r} \subseteq \mathbb{S}^n$.

In fact, these "replacements" (some of which could seem purely notational on the surface) are essentially the basis of the extension of our results from linear programming to semidefinite programming. There are indeed a few nontrivial details that must be shown in the semidefinite programming version that were not done in the linear programming version, which we will fully present. However, the key ideas and insights that are essential for proving the results in this chapter have all been introduced in the linear programming context.

6.2.3 Solving $(QP_{E,r})$

We consider $E \in Int$ and $r \in (0, 1)$. As we did in Section 4.3, the problem $(QP_{E,r})$ can be solved by considering its Fritz John optimality conditions.

The analogous condition of the Fritz John conditions in Section 4.3 for $(QP_{E,r})$ is as follows. A solution X is optimal for $(QP_{E,r})$ if and only if $\langle I, \mathcal{E}^{-1}X\rangle \geq 0$, and there exist $y \in \mathbb{R}^m$ and $\mu \geq 0$ such that

$$\begin{array}{rcll} -\left(\langle I, \mathcal{E}^{-1}X\rangle E^{-1} - r^2\mathcal{E}^{-2}X\right) - \mathcal{A}^*y - \mu C & = & 0, & (a) \\ \mathcal{A}X & = & b, & (b) \\ \langle I, \mathcal{E}^{-1}X\rangle^2 - r^2 \| \mathcal{E}^{-1}X\|^2 & = & 0. & (c) \end{array}$$

However, this system is a bit more tricky, because in (a) and (c), X is multiplied from the left and the right by $E^{-1/2}$. We can transform (a) - (c) to an equivalent system where the variable "X" to be solved is only multiplied from the left. Let

$$\tilde{X} := \mathcal{E}^{-1}X = E^{-1/2}XE^{-1/2},$$

$$\tilde{C} := \mathcal{E}C = E^{1/2}CE^{1/2},$$

$$\tilde{A}_i := \mathcal{E}A_i = E^{1/2}A_iE^{1/2},$$

and let $\tilde{\mathcal{A}} : \mathbb{S}^n \to \mathbb{R}^m$ be defined by

$$\tilde{\mathcal{A}}(\cdot) := (\langle \tilde{A}_1, \cdot\rangle, \ldots, \langle \tilde{A}_m, \cdot\rangle)^T.$$

Then, (a) - (c) is equivalent to

$$\begin{array}{rcll} -\left(\langle I, \tilde{X}\rangle I - r^2\tilde{X}\right) - \tilde{\mathcal{A}}^*y - \mu\tilde{C} & = & 0, & (\tilde{a}) \\ \tilde{\mathcal{A}}\tilde{X} & = & b, & (\tilde{b}) \\ \langle I, \tilde{X}\rangle^2 - r^2\langle \tilde{X}, \tilde{X}\rangle & = & 0. & (\tilde{c}) \end{array}$$

Since $\tilde{X}$ is an $n\times n$ symmetric matrix, y is an m-vector, and μ is a scalar, then the number of variables in this system of equations is $n(n+1)/2+m+1$. Considering just the equations given by $(\tilde{a})$ - $(\tilde{b})$, we have a system of linear equations in $\tilde{X}$ involving $n(n+1)/2+m$ equations. Since we assume that $C, A_1, \ldots, A_m$ are linearly independent, then the solution space for just $(\tilde{a})$ - $(\tilde{b})$ is one-dimensional: $\{\tilde{Z}^{(0)} + \lambda\tilde{V} \mid \lambda \in \mathbb{R}\}$ for some $\tilde{Z}^{(0)} = (\tilde{X}^{(0)}, \tilde{y}^{(0)}, \tilde{\mu}^{(0)})$ that satisfies $(\tilde{a})$ - $(\tilde{b})$ and $\tilde{V} = (\tilde{V}_X, \tilde{V}_y, \tilde{V}_\mu)$ that lies in the null space of the linear operator on the left-hand side of ($\tilde{a}$) - ($\tilde{b}$). To find $\tilde{X}$ that also satisfies ($\tilde{c}$) in addition to ($\tilde{a}$) - ($\tilde{b}$), we solve the following quadratic equation in λ:

$$\langle I, (\tilde{X}^{(0)} + \lambda \tilde{V}_X) \rangle^2 - r^2 \langle (\tilde{X}^{(0)} + \lambda \tilde{V}_X), (\tilde{X}^{(0)} + \lambda \tilde{V}_X) \rangle = 0. \tag{6.14}$$

The quadratic equation above has two roots, call them $\lambda^{(1)}$ and $\lambda^{(2)}$. If the roots are not real-valued, we can conclude that E is not in Swath(r). Otherwise, we have two candidate solutions for the original system (a) - (c), namely

$$X^{(i)} = \mathcal{E}\left(\tilde{X}^{(0)} + \lambda^{(i)} \tilde{V}_X\right),$$

for $i = 1, 2$, whose corresponding Lagrangian multipliers "y" and "μ" are

$$y^{(i)} = \tilde{y}_0 + \lambda^{(i)} V_y$$

and

$$\mu^{(i)} = \tilde{\mu}_0 + \lambda^{(i)} V_\mu.$$

We first need to check if $X^{(i)}$ is feasible for $(QP_{E,r})$) by checking if $\langle I, \mathcal{E}X^{(i)} \rangle \geq 0$. If both $X^{(i)}$ are feasible, then the minimizer is the one with $\mu^{(i)} \geq 0$ (by Proposition 6.3, it cannot be the case that both are minimizers). If only one $X^{(i)}$ is feasible, then it is the minimizer if $\mu^{(i)} \geq 0$. Otherwise, then there is no optimal solution; that is, E must not have been in Swath(r).

To actually solve for $\tilde{Z}^{(0)}$ and $\tilde{V}$, we can translate the system ($\tilde{a}$) - ($\tilde{b}$) into an equation in the form $Mz = d$ where M is an $(n(n+1)/2 + m) \times (n(n+1)/2 + m + 1)$ matrix and z is a vector in $R^{n(n+1)/2}$ which corresponds to solutions Z to the system ($\tilde{a}$) - ($\tilde{b}$). We do this by mapping matrices in $\mathbb{S}^n$ to vectors in $R^{n(n+1)/2}$ using the mapping which in Section 2.2 we call *svec*: for $X \in \mathbb{S}^n$,

$$svec(X)_{i+j(j-1)/2} := \begin{cases} \sqrt{2} X_{ij}, & \text{if } i \neq j, \\ X_{ij}, & \text{if } i = j, \end{cases}$$

for each $i, j = 1, \ldots, n$ with $i \leq j$.

Thus, letting

$$\tilde{A} := \begin{bmatrix} svec(\tilde{A}_1)^T \\ svec(\tilde{A}_2)^T \\ \vdots \\ svec(\tilde{A}_m)^T \end{bmatrix}$$

and

$$M := \begin{bmatrix} -(svec(I)svec(I)^T - r^2 I_{n(n+1)/2}) & -\tilde{A}^T & -C \\ \tilde{A} & 0 & 0 \end{bmatrix},$$

with

$$d = \begin{pmatrix} 0 \\ b \end{pmatrix}, \; z = \begin{pmatrix} x \\ y \\ \mu \end{pmatrix},$$

we can proceed to solve for $z = (x, y, \mu)$ in $Mz = d$. Then, $\tilde{Z}^{(0)} = (svec(x), y, \mu)$ and $\tilde{X}^{(0)} = svec(x)$. Similarly, if $v = (v_x, v_y, v_\mu)$ is a vector in the nullspace of M, then $\tilde{V} = (svec(v_x), v_y, v_\mu)$ and $\tilde{V}_X = svec(v_x)$.

6.2.4 Dual cones and dual problems with respect to the local inner product

In Proposition 6.1, we computed the dual of the cone $K_{E,r}$ with respect to the trace inner product $\langle \cdot, \cdot \rangle$:

$$K_{E,r}^* = K_{E^{-1}, \sqrt{n-r^2}}.$$

Recall that the notion of the dual cone depends on the inner product being used. We will now investigate what the dual cones with respect to the local inner products look like.

Fix two matrices $E, \hat{E} \in \mathbb{S}^n_{++}$ and let us consider the dual cone of $K_{E,r}$ with respect to the local inner product $\langle \cdot, \cdot \rangle_{\hat{E}}$, which we will denote $K^{*\hat{E}}_{E,r}$:

$$\begin{aligned} K^{*\hat{E}}_{E,r} &= \{\tilde{S} \mid \langle X, \tilde{S} \rangle_{\hat{E}} \geq 0, \forall X \in K_{E,r}\} \\ &= \{\tilde{S} \mid \langle X, \hat{\mathcal{E}}^{-2} \tilde{S} \rangle \geq 0, \forall X \in K_{E,r}\} \\ &= \{\hat{\mathcal{E}}^2 S \mid \langle X, S \rangle \geq 0, \forall X \in K_{E,r}\} \\ &= \hat{\mathcal{E}}^2 K^*_{E,r}. \end{aligned}$$

Since $K^*_{E,r} = K_{E^{-1}, \sqrt{n-r^2}}$ (by Proposition 6.1), then

$$\begin{aligned} K^{*\hat{E}}_{E,r} &= \hat{\mathcal{E}}^2 K_{E^{-1}, \sqrt{n-r^2}} \\ &= K_{\hat{\mathcal{E}}^2 E^{-1}, \sqrt{n-r^2}}, \end{aligned}$$

where the last equality follows from (6.5).

In particular, the dual of $K_{E,r}$ with respect to the local inner product at E is

$$K^{*E}_{E,r} = K_{E, \sqrt{n-r^2}}. \tag{6.15}$$

We can also talk about dual problems when the dual is taken with respect to $\langle \cdot, \cdot \rangle_E$. In particular, writing the primal objective using $\langle \cdot, \cdot \rangle_E$, the local inner product at E,

$$\begin{aligned} \min \quad & \langle \mathcal{E}^2 C, X \rangle_E \\ s.t. \quad & \mathcal{A} X = b \qquad (QP_{E,r}) \\ & X \in K_{E,r}, \end{aligned}$$

the dual problem with respect to $\langle \cdot, \cdot \rangle_E$ is

$$\begin{array}{lll} \min & b^T \tilde{y} & \\ s.t. & \mathcal{E}^2(\mathcal{A}^* \tilde{y}) + \tilde{S} = \mathcal{E}^2 C & (QP_{E,r})^{*_E} \\ & \tilde{S} \in K_{E,r}^{*_E}, & \end{array}$$

where, $\mathcal{E}^2(\mathcal{A}^*\tilde{y}) = \sum_{i=1}^m (\mathcal{E}^2 A_i)\tilde{y}_i$. We note that $\mathcal{E}^2\mathcal{A}^* : \mathbb{R}^m \to \mathbb{S}^n$ is the adjoint of $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ with respect to $\langle \cdot, \cdot \rangle_E$: for each $\tilde{y} \in \mathbb{R}^m$ and each $W \in \mathbb{S}^n$,

$$\begin{aligned} y^T(\mathcal{A}W) &= \langle \mathcal{A}^* y, W \rangle \\ &= \langle \mathcal{E}^2(\mathcal{A}^* y), \mathcal{E}^{-2} W \rangle \\ &= \langle \mathcal{E}^2(\mathcal{A}^* y), W \rangle_E. \end{aligned}$$

Consider the linear constraint $\mathcal{E}^2(\mathcal{A}^*\tilde{y}) + \tilde{S} = \mathcal{E}^2 C$ in the description of $(QP_{E,r})^{*_E}$ above. That $(\tilde{S}, \tilde{y})$ satisfies this constraint can be equivalently written as follows

$$\tilde{S} \in \mathcal{E}^2 C + \mathcal{L}^{\perp_E}, \qquad (6.16)$$

where $\mathcal{L} := \ker A$, the nullspace of A, and $\mathcal{L}^{\perp_E}$ is its orthogonal complement with respect to the local inner product at E. Indeed, since $\mathcal{L}^\perp = \{\mathcal{A}^* y \mid y \in \mathbb{R}^m\}$, it suffices to show that $\mathcal{E}^2 \mathcal{L}^\perp = \mathcal{L}^{\perp_E}$.

However, note that $V \in \mathcal{L}^{\perp_E}$ if and only if

$$\langle V, X \rangle_E = 0$$

for all $X \in \mathcal{L}$. That is, if and only if

$$0 = \langle V, X \rangle_E = \langle V, \mathcal{E}^{-2} X \rangle = \langle \mathcal{E}^{-2} V, X \rangle$$

for all $X \in \mathcal{L}$. So, $V \in \mathcal{L}^{\perp_E}$ if and only if $\mathcal{E}^{-2} V \in \mathcal{L}^\perp$. Thus, $\mathcal{L}^{\perp_E} = \mathcal{E}^2 \mathcal{L}^\perp$, proving (6.16).

6.2.5 The tangent space

Fix $E \in Int$ and $r \in (0,1)$. Suppose that $\bar{X} \neq 0$ lies on the boundary of the cone $K_{E,r}$, then $p_{E,r}(\bar{X}) = 0$ where $p_{E,r} : \mathbb{S}^n \to \mathbb{R}$ is given by

$$p_{E,r}(U) := \langle I, \mathcal{E}^{-1}U\rangle^2 - r^2\langle \mathcal{E}^{-1}U, \mathcal{E}^{-1}U\rangle,$$

for each $U \in \mathbb{S}^n$.

Thus, the tangent space at $\bar{X}$ to the set

$$\{U \in \mathbb{S}^n \mid p_{E,r}(U) = 0,\ \langle I, \mathcal{E}^{-1}U\rangle \geq 0\} = \partial K_{E,r}$$

consists of vectors V such that $\langle \nabla p_{E,r}(\bar{X}), V\rangle = 0$.

Denote this tangent space $\mathcal{T}_{E,r,\bar{X}}$. Since

$$\nabla p_{E,r}(\bar{X}) = \langle I, \mathcal{E}^{-1}\bar{X}\rangle(E^{-1}) - r^2\mathcal{E}^{-2}\bar{X},$$

then the tangent space $\mathcal{T}_{E,r,\bar{X}}$ consists of vectors V such that

$$\langle I, \mathcal{E}^{-1}\bar{X}\rangle\langle I, \mathcal{E}^{-1}V\rangle - r^2\langle \mathcal{E}^{-1}\bar{X}, \mathcal{E}^{-1}V\rangle = 0. \tag{6.17}$$

If X is the optimal solution for $(QP_{E,r})$ then X lies in $\partial K_{E,r}$, and furthermore, it satisfies the Fritz John conditions which we state in Section 6.2.3. In particular, there exist $y \in \mathbb{R}^m$ and $\mu > 0$ such that

$$\langle I, \mathcal{E}^{-1}X\rangle E^{-1} - r^2\mathcal{E}^{-2}X = \mathcal{A}^*y - \mu C.$$

Therefore, if X is optimal for $(QP_{E,r})$, then $\mathcal{T}_{E,r,X}$ consists of all vectors V which satisfy:

$$0 = \langle(\mathcal{A}^*y - \mu C), V\rangle = y^T(\mathcal{A}V) - \mu\langle C, V\rangle. \tag{6.18}$$

We are mainly interested in tangent directions V that are also in $\mathcal{L}$ (recall that $\mathcal{L} = \ker \mathcal{A}$) because we are interested in examining points $X + \lambda V$ which remain feasible for $(QP_{E,r})$ for infinitesimal λ.

Thus, following (6.18), the space $\mathcal{T}_{E,r,X} \cap \mathcal{L}$ consists of matrices V where the inner product between C and V must be zero (since $\mu > 0$):

$$\mathcal{T}_{E,r,X} \cap \mathcal{L} = \{V \in \mathbb{S}^n \mid \mathcal{A}V = 0, \langle C, V\rangle = 0\}.$$

In particular, we note that the description of the matrices V that belong to $\mathcal{T}_{E,r,X} \cap \mathcal{L}$ above is independent of E, r, or X. This means that for any $\hat{E} \in \mathrm{Swath}(r)$ and $\hat{X}$ that is optimal for $(QP_{\hat{E},r})$, the set

$$\{V \in \mathbb{S}^n \mid \mathcal{A}V = 0,\ \langle C, V\rangle = 0\}$$

is tangent to $K_{\hat{E},r}$ at $\hat{X}$.

Thus, from here onwards, we eliminate the subscripts from "$\mathcal{T}_{E,r,X} \cap \mathcal{L}$" and let

$$\mathcal{T} := \mathcal{T}_{E,r,X} \cap \mathcal{L} = \{V \in \mathbb{S}^n \mid \mathcal{A}V = 0,\ \langle C, V\rangle = 0\}. \tag{6.19}$$

We remark that $\mathcal{T}$ is a subspace of $\mathcal{L}$ of codimension 1. In particular, C (alternatively, $E - X$) and the vectors in $\mathcal{T}$ together span $\mathcal{L}$.

We close this section with the following fact about vectors in $\mathcal{T}$. This proposition will be used in proving Lemma 6.13, a result about the projection of an optimal solution of a quadratic cone relaxation to the space orthogonal to $\mathcal{L}$ with respect to the local inner product at the current iterate.

Proposition 6.4. *Let $E \in \mathbb{S}^n_{++}$ and $r \in [0, 1]$. Suppose X is optimal for $(QP_{E,r})$, and $\mathcal{T}$ is as given in* (6.19)*. For each $V \in \mathcal{T}$ where $\langle V, X\rangle_E \neq 0$, there exist $W \in \mathbb{S}^n$ and $\lambda > 0$*

such that

$$\lambda V = X + W$$

and

$$\langle X, W\rangle_E = \langle E, W\rangle_E = 0.$$

Proof. Let $W(\lambda) := \lambda V - X$. Then,

$$\langle W(\lambda), X\rangle_E = \lambda\langle V, X\rangle_E - \|X\|_E^2.$$

Choose $\bar{\lambda} := \frac{\|X\|_E^2}{\langle V,X\rangle_E}$ so that $\langle W(\bar{\lambda}), X\rangle_E = 0$. Then, using (6.17) and the fact that $\langle E, X\rangle_E = r\|X\|_E$, we note that

$$\bar{\lambda} = \frac{\|X\|_E^2}{\langle V, X\rangle_E} = \frac{r^2\|X\|_E^2}{\langle E, X\rangle_E\,\langle E, V\rangle_E} = \frac{r\|X\|_E}{\langle E, V\rangle_E},$$

Hence, using the fact that $\langle E, X\rangle_E = r\|X\|_E$ (since X lies on the boundary of $K_{E,r}$),

$$\begin{aligned}\langle E, W(\bar{\lambda})\rangle_E &= \bar{\lambda}\langle E, V\rangle_E - \langle E, X\rangle_E \\ &= \frac{r\|X\|_E}{\langle E, V\rangle_E}\langle E, V\rangle_E - \langle E, X\rangle_E \\ &= r\|X\|_E - \langle E, X\rangle_E = 0.\end{aligned}$$

□

6.3 The algorithm and the main result

We assume that the following conditions are satisfied by (SDP) and $(SDP)^*$:

- $b \neq 0$, which implies that $X = 0$ is not a feasible solution
- $C, A_1, \ldots, A_m$ are linearly independent, which means that the optimal solutions lie on the boundary of the feasible region

- (SDP) and $(SDP)^*$ are both strictly feasible

Furthermore, these assumptions imply that

- For each $r \in (0, 1)$, the set Swath(r) is not empty (see Proposition 6.2). Therefore, there exists an initial iterate $E^{(0)} \in$ Swath(r) as assumed by the theorem.
- If $E \in$ Swath(r), then $(QP_{E,r})$ has a unique optimal solution (see Proposition 6.3). This means that each iterate of our algorithm is well-defined.

Under these assumptions on the pair of problems (SDP) and $(SDP)^*$, we extend the algorithm to semidefinite programming.

Algorithm 6.1. Fix $\epsilon > 0$, $r \in (0, 1)$.

Initialization: Find an initial direction $E^{(0)} \in$ Swath(r).

Iteration k**:** As long as $\langle C, E^{(k)} - X^{(k)} \rangle > \epsilon$,

Find an optimal solution, $X^{(k)}$, of $(QP_{E^{(k)},r})$.

Let $E^{(k+1)} = \frac{1}{1+t^{(k)}}(E^{(k)} + t^{(k)} X^{(k)})$, with $t^{(k)} = \frac{r}{2\|(\mathcal{E}^{(k)})^{-1} X^{(k)}\|}$

As we can see, Algorithm 6.1 is essentially the same as the linear programming version, including in the choice of step size of $t^{(k)} = \frac{r}{2\|(\mathcal{E}^{(k)})^{-1} X^{(k)}\|}$.

Our main goal in this chapter is to prove the polynomial-time convergence of this algorithm with the above choice of step size. In particular, we prove the following analog of Theorem 5.6 under the above assumptions.

Theorem 6.5. *Fix $r \in (0, 1)$ and assume that $n \geq 4$. Suppose that $E^{(0)} \in$ Swath(r). For* *each* $k = 0, 1, 2, \ldots$*, let* $X^{(k)}$ *denote the optimal solution for* $(QP_{E^{(k)},r})$*, where*

$$E^{(k+1)} := \frac{1}{1+t^{(k)}}\left(E^{(k)} + t^{(k)}X^{(k)}\right),$$

with $t^{(k)} = \frac{r}{2\|(\mathcal{E}^{(k)})^{-1}X^{(k)}\|}$.

Then, for each k, at least one of the following holds:

$$\langle C, E^{(k+1)} - X^{(k+1)}\rangle \leq \left(1 - \frac{r\left(1-\sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)\langle C, E^{(k)} - X^{(k)}\rangle, \tag{6.20}$$

$$\langle C, E^{(k+2)} - X^{(k+2)}\rangle \leq \left(1 - \frac{r}{4\sqrt{n}}\right)\langle C, E^{(k+1)} - X^{(k+1)}\rangle. \tag{6.21}$$

Therefore, in $O(\sqrt{n}\log(\gamma_0/\epsilon))$ *iterations, we obtain a duality gap of at most* ϵ*, assuming that the initial duality gap is no more than* γ_0*.*

Main ideas

As we saw in Section 6.2, the proofs of the semidefinite programming version of the results closely resemble the proofs for the corresponding linear programming version. The main ideas, and most of the detail, involved in proving Theorem 6.5 are the same as those involved in proving the linear programming version. We list the main ideas and steps as follows, with the more detailed explanation postponed to latter sections.

1. **Dual solution**

   Given $E \in \text{Swath}(r)$ with X being the optimal solution to $(QP_{E,r})$, the optimal dual solution is (S, y) where S is given by

$$S = \frac{\langle C, E - X\rangle}{n - r^2}\left(E^{-1} - r\frac{\mathcal{E}^{-2}X}{\|\mathcal{E}^{-1}X\|}\right)$$

and y is uniquely determined by S (since $A_1, \ldots, A_n$ are linearly independent). We can easily check that the duality gap between X and (S, y) is zero.

2. **Duality gap**

   We let E, X denote the current iterate and the corresponding quadratic cone problem optimal solution.

   The duality gap between E and (S, y) satisfies

   $$b^T y - \langle C, E \rangle = \langle C, E - X \rangle.$$

3. **Keeping the iterates in Swath(r)**

   Letting E, X denote the current iterate and the corresponding quadratic cone problem optimal solution as before, let the next iterate be given by

   $$E(t) := \frac{1}{1+t}(E + tX),$$

   where $t > 0$ is the step size. We obtain a bound on t that guarantees that $E(t)$ stays in Swath(r).

   Proposition 6.6. *Given $E \in$ Swath(r) and X optimal for ($QP_{E,r}$), let $E(t) = \frac{1}{1+t}(E + tX)$. As long as $0 < t < \frac{r}{\|\mathcal{E}^{-1}X\|}$, then $E(t) \in$ Swath(r).*

4. **Primal and dual monotonicity**

   We can show that

   $$\langle C, E(t) \rangle \leq \langle C, E \rangle \qquad \text{primal monotonicity,}$$
   $$\langle C, X(E(t)) \rangle \geq \langle C, X \rangle \qquad \text{dual monotonicity.}$$

   The first inequality shows that the primal iterates have monotonically decreasing objective values. The second inequality shows that the corresponding optimal solutions to the quadratic relaxation have monotonically increasing objective values. We will explain later why we call the second inequality"dual monotonicity".

5. **A simple duality gap reduction guarantee**

   We obtain a straightforward duality gap reduction guarantee for the case when $\|\mathcal{E}^{-1}X\| \leq \sqrt{n}$.

   Proposition 6.7. *Suppose that $E \in Swath(r)$, X is optimal for $(QP_{E,r})$, with $\|\mathcal{E}^{-1}X\| \leq \sqrt{n}$. Let $E(t) = \frac{1}{1+t}(E + tX)$. Then, taking a step of size $t = \eta \frac{r}{\|\mathcal{E}^{-1}X\|}$ for some $\eta \in (0, 1)$ reduces the duality gap by a factor of $\left(1 - \frac{\eta r}{2\sqrt{n}}\right)$.*

6. **Further duality gap reduction guarantee**

   First, we obtain a lower bound for $\langle C, X(E(t)) - X\rangle$ in terms of a positive multiple of $\|\text{Proj}_{\mathcal{L}^{\perp E(t)}} X(E(t))\|_{E(t)}$.

   Proposition 6.8. *Suppose that $E \in Swath(r)$, X optimal for $(QP_{E,r})$ and (S, y) is optimal for $(QP_{E,r})^*$. Let $E(t) = \frac{1}{1+t}(E + tX)$.*

   Then, taking $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$ and letting $X(E(t))$ denote the optimal solution for $(QP_{E(t),r})$, we have that

   $$\langle C, X(E(t)) - X\rangle \geq R\|Proj_{\mathcal{L}^{\perp E(t)}} X(E(t))\|_{E(t)},$$

   where $R := \frac{\|\mathcal{E}(t)S\|}{n}\left(r\sqrt{n-\beta^2} - \beta\sqrt{n - r^2}\right)$ and $\beta = r\sqrt{\frac{1+r}{2}}$.

   Then, we use Proposition 6.8 to obtain a duality gap reduction guarantee for the case when $\|\mathcal{E}^{-1}X\| > \sqrt{n}$ and $\|\mathcal{E}(t)^{-1}X(E(t))\| > \sqrt{n}$.

   Proposition 6.9. *Suppose that $E \in Swath(r)$, X is optimal for $(QP_{E,r})$, and (S, y) is optimal for $(QP_{E,r})^*$. Let $E(t) = \frac{1}{1+t}(E + tX)$. Assume that $n \geq 4$.*

Choose $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$ *and let* $X(E(t))$ *denote the optimal solution for* $(QP_{E(t),r})$. *If* $\|\mathcal{E}^{-1}X\| > \sqrt{n}$ *and* $\|\mathcal{E}(t)^{-1}X(E(t))\| > \sqrt{n}$, *then*

$$\langle C, E(t) - X(E(t))\rangle \leq \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)\langle C, E - X\rangle.$$

7. **The main result**

The main result, Theorem 6.5, is a straightforward consequence of Proposition 6.7 and Proposition (6.9). It guarantees a bound on the duality gap reduction at least every other iteration.

We emphasize that the proofs of the propositions that lead to the main theorem rely heavily on examining the dual problems. In particular, we rely on the following fact: The problem $(QP_{E(t),r})$ has an optimal solution if $(QP_{E(t),r})^*$ is strictly feasible. The strict feasibility of $(QP_{E(t),r})^*$ is shown by proving that the S-component of the old dual solution (S, y) lies in the interior of the new dual cone $K^*_{E(t),r}$. Furthermore, we prove that the duality gap is reduced significantly by deriving a strong lower bound on the distance of S to the boundary of this new dual cone and then making use of Proposition 6.8.

In the next sections, we provide complete proofs of the above results. However, these proofs are essentially the same as the proofs for the corresponding results in Chapter 5. The main differences come from the barrier functional and its Hessian, which means that we use different scalings.

6.4 The dual solution and the duality gap

Given a point $E \in \text{Swath}(r)$ and X that is optimal for $(QP_{E,r})$, we can find an optimal solution to the dual quadratic programming problem, $(QP_{E,r})^*$, from the KKT conditions that must be satisfied by X. That is, for some $y \in \mathbb{R}^m$ and $\lambda \geq 0$, the following conditions are satisfied by X:

$$C - \mathcal{A}^* y - \lambda\left(\langle I, \mathcal{E}^{-1}X\rangle E^{-1} - r^2\mathcal{E}^{-2}X\right) = 0, \tag{6.22}$$

$$\langle I, \mathcal{E}^{-1}X\rangle^2 - r^2\|\mathcal{E}^{-1}X\|^2 = 0, \tag{6.23}$$

$$\mathcal{A}X - b = 0. \tag{6.24}$$

Letting $S := \lambda\left(\langle I, \mathcal{E}^{-1}X\rangle E^{-1} - r^2\mathcal{E}^{-2}X\right)$, it is not hard to see that (S, y) satisfies the linear equality constraint of $(QP_{E,r})^*$. We can solve for λ in terms of E and X by taking the inner product of (6.22) with $E - X$, which we know to lie in the null space of $\mathcal{A}$:

$$\begin{aligned} 0 &= \langle C, E - X\rangle - \langle \mathcal{A}^* y, E - X\rangle \\ &\quad -\lambda\left(\langle I, \mathcal{E}^{-1}X\rangle\langle E^{-1}, E - X\rangle - r^2\langle \mathcal{E}^{-2}X, E - X\rangle\right) \\ &= \langle C, E - X\rangle - \lambda(n - r^2)\langle I, \mathcal{E}^{-1}X\rangle. \end{aligned}$$

Solving for λ,

$$\lambda = \frac{\langle C, E - X\rangle}{(n - r^2)\langle I, \mathcal{E}^{-1}X\rangle} = \frac{\langle C, E - X\rangle}{(n - r^2)r\|\mathcal{E}^{-1}X\|}$$

since $\langle I, \mathcal{E}^{-1}X\rangle^2 - r^2\|\mathcal{E}^{-1}X\|^2 = 0$. Therefore,

$$S = \frac{\langle C, E - X\rangle}{(n - r^2)r\|\mathcal{E}^{-1}X\|}\left(\langle I, \mathcal{E}^{-1}X\rangle E^{-1} - r^2\mathcal{E}^{-2}X\right),$$

that is,

$$S = \frac{\langle C, E - X\rangle}{n - r^2}\left(E^{-1} - r\frac{\mathcal{E}^{-2}X}{\|\mathcal{E}^{-1}X\|}\right). \tag{6.25}$$

Using (6.25),

$$\begin{aligned}\langle I, \mathcal{E}S\rangle &= \langle E, S\rangle = \langle C, E - X\rangle \geq 0, &(6.26)\\ \langle \mathcal{E}S, \mathcal{E}S\rangle &= \frac{\langle C, E - X\rangle^2}{n - r^2}, &(6.27)\end{aligned}$$

which implies that S satisfies the inequality that defines the quadratic cone $K^*_{E,r}$ tightly,

$$\langle I, \mathcal{E}S\rangle^2 - (n - r^2)\|\mathcal{E}S\|^2 = 0,$$

showing that S lies on the boundary of the dual cone $K^*_{E,r}$. Thus, (S, y) is feasible for $(QP_{E,r})^*$.

To see that in fact (S, y) is optimal for $(QP_{E,r})^*$, we need only show that the duality gap between X and (S, y) is zero. Taking the inner product of X and S gives this duality gap (see (2.1))

$$\begin{aligned}\langle X, S\rangle &= \frac{\langle C, E - X\rangle}{n - r^2}\left(\langle X, E^{-1}\rangle - r\frac{\langle X, \mathcal{E}^{-2}X\rangle}{\|\mathcal{E}^{-1}X\|}\right)\\ &= \frac{\langle C, E - X\rangle}{n - r^2}\left(r\|\mathcal{E}^{-1}X\| - r\frac{\|\mathcal{E}^{-1}X\|^2}{\|\mathcal{E}^{-1}X\|}\right) = 0,\end{aligned}$$

which concludes our proof that (S, y) is optimal for $(QP_{E,r})^*$.

The duality gap between X and (S, y), where S is given by (6.25), is always zero, which tells us that this choice of S always gives an optimal solution for the dual quadratic cone optimization problem.

(In particular, note that whenever $E \in \text{Swath}(r)$, both $(QP_{E,r})$ and $(QP_{E,r})^*$ have an optimal solution, and strong duality holds between them.)

We see that (S, y) is always a feasible solution to the dual semidefinite programming problem because the dual quadratic cone, $K^*_{E,r}$, is always contained in the positive semidefinite cone (Proposition 6.1). Thus, E and (S, y) is a pair of feasible solutions to the primal-dual semidefinite programming problem pair, with a duality gap of $\langle E, S \rangle$.

Computing this quantity using the choice of S from (6.25),

$$\langle E, S \rangle = \langle I, \mathcal{E}S \rangle = \langle C, E - X \rangle,$$

showing that we can compute this duality gap just from "primal information"—the center direction E together with X, the optimal solution to $(QP_{E,r})$—and using the dual solutions explicitly only for the analysis.

6.5 Keeping the iterates in Swath(r)

Next, we prove a simple upper bound on the step size t which guarantees that the new iterate, $E(t)$, stays in Swath(r). The following results are analogs of the linear programming version (Lemma 5.1 and Proposition 5.2).

Lemma 6.10. *Suppose that $E \in$ Swath(r), X optimal for $(QP_{E,r})$, and (S, y) is optimal for $(QP_{E,r})^*$. Let $E(t) = \frac{1}{1+t}(E + tX)$. As long as $0 < t < \frac{r}{\|\mathcal{E}^{-1}X\|}$, then $E(t) \in \mathbb{S}^n_{++}$ and S lies in the interior of $K^*_{E(t),r}$.*

Proof. We first show that for this choice of step size t, $E(t)$ is a positive definite matrix. Since $E(t)$ is just a scalar multiple of $E + tX$, it is sufficient to show that the eigenvalues of $E + tX$ are strictly positive.

We know that E is positive definite and that the mapping $\mathcal{E}^{-1} = E^{-1/2}(\cdot)E^{-1/2}$ fixes the cone of positive definite matrices. Thus, $E + tX$ is positive definite if and only if

$$I + t\mathcal{E}^{-1}X = I + tE^{-1/2}XE^{-1/2}$$

is positive definite. Suppose that $\mathcal{E}^{-1}X = E^{-1/2}XE^{-1/2} = Q\hat{X}Q^T$ is the eigenvalue decomposition of the matrix $\mathcal{E}^{-1}X$. That is, Q is an orthogonal matrix and $\hat{X}$ is a diagonal matrix whose diagonal entries are the eigenvalues of $\mathcal{E}^{-1}X$:

$$\hat{X} = \operatorname{diag} \hat{x},$$

where $\hat{x} := (\hat{x}_1, \ldots, \hat{x}_n)^T$ is the vector of eigenvalues of $\mathcal{E}^{-1}X$.

The eigenvalues of $I + t\mathcal{E}^{-1}X$ are the same as the eigenvalues of

$$Q^T(I + t\mathcal{E}^{-1}X)Q = I + t\hat{X}.$$

Therefore, we only need to show that for all $t < \frac{r}{\|\mathcal{E}^{-1}X\|}$, the eigenvalues of $I + t\hat{X}$ are strictly positive.

Note that the trace norm of the matrix $\mathcal{E}^{-1}X$ is the same as the 2-norm of the vector of its eigenvalues. Since $\|\cdot\|_2 \geq \|\cdot\|_\infty$, then:

$$\|\mathcal{E}^{-1}X\| = \|\hat{x}\|_2 \geq \|\hat{x}\|_\infty = \max_i |\hat{x}_i|.$$

Hence,

$$t < \frac{r}{\|\mathcal{E}^{-1}X\|} \leq \frac{1}{\max_i |\hat{x}_i|}.$$

In particular, for each index i where the eigenvalue $\hat{x}_i$ is negative, the ith eigenvalue of $I + t\hat{X}$ is

$$1 + t\hat{x}_i > 1 + \frac{1}{|\hat{x}_i|}\hat{x}_i > 0.$$

This shows that all eigenvalues of $I + t\hat{X}$ are strictly positive. Hence, the matrix $E(t) = \frac{1}{1+t}(E + tX)$ is positive definite.

Next, we show that $S \in K^*_{E(t),r}$. That is, we wish to show that the following conditions are satisfied:

$$\langle I, \mathcal{E}(t)S \rangle \geq 0, \text{ and}$$

$$\langle I, \mathcal{E}(t)S\rangle^2 - (n - r^2)\|\mathcal{E}(t)S\|^2 > 0. \tag{6.28}$$

We first note that

$$\begin{aligned}
(1+t)\langle I, \mathcal{E}(t)S\rangle &= \operatorname{tr}((E + tX)S) \\
&= \operatorname{tr}(ES) + t\operatorname{tr}(XS) \\
&= \langle I, \mathcal{E}S\rangle + t\langle I, XS\rangle \\
&= \langle I, \mathcal{E}S\rangle \quad (\text{since } \langle I, XS\rangle = 0) \\
&= \langle E, S\rangle \\
&= \langle C, E - X\rangle \geq 0
\end{aligned}$$

and

$$\begin{aligned}
(1+t)^2\|\mathcal{E}(t)S\|^2 &= \|(E + tX)S\|^2 \\
&= \operatorname{tr}(ESES) + 2t\operatorname{tr}(ESXS) + t^2\operatorname{tr}(XSXS) \\
&= \|\mathcal{E}S\|^2 + 2t\langle \mathcal{E}S, XS\rangle + t^2\langle SX, XS\rangle \\
&= \frac{\langle I, \mathcal{E}S\rangle^2}{n - r^2} + 2t\langle \mathcal{E}S, XS\rangle + t^2\langle SX, XS\rangle.
\end{aligned}$$

Note that the matrix product XS which appears above is in general not a symmetric matrix. We could write

- $\langle I, S^{1/2}XS^{1/2}\rangle$ instead of $\langle I, XS\rangle$,
- $\langle S^{1/2}ES^{1/2}, S^{1/2}XS^{1/2}\rangle$ instead of $\langle \mathcal{E}S, XS\rangle$, and
- $\|S^{1/2}XS^{1/2}\|^2$ instead of $\langle SX, XS\rangle$

in order to obtain expressions that are in terms of symmetric matrices only. However, we will keep our current expression of terms involving "XS" in their nonsymmetric forms for notational simplicity. We also do this because it is easier to manipulate expressions involving S in which we must use the explicit expression for S as given in (6.25). (We only need to keep in mind that for nonsymmetric matrices $U, V \in \mathbb{R}^{n\times n}$, the trace product is $\langle U, V\rangle = \operatorname{tr} U^T V \neq \operatorname{tr} UV$.)

Therefore, (6.28) is satisfied if and only if

$$2t\langle \mathcal{E}S, XS\rangle + t^2\langle SX, XS\rangle < 0.$$

Using S as specified in (6.25) and the fact that $\langle I, \mathcal{E}^{-1}X\rangle = r\|\mathcal{E}^{-1}X\|$ for the solution X that is optimal for $(QP_{E,r})$, we have that

$$\begin{aligned}\langle \mathcal{E}S, XS\rangle &= -\frac{\langle C, E-X\rangle^2}{(n-r^2)^2} r\|\mathcal{E}^{-1}X\| \\ &\quad \left(1 - \frac{r\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^3}\right) \quad (6.29)\end{aligned}$$

$$\begin{aligned}\langle SX, XS\rangle &= \frac{\langle C, E-X\rangle^2}{(n-r^2)^2}\|\mathcal{E}^{-1}X\|^2 \\ &\quad \left(1 - \frac{2r\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^3} + \frac{r^2\|(\mathcal{E}^{-1}X)^2\|^2}{\|\mathcal{E}^{-1}X\|^4}\right). \quad (6.30)\end{aligned}$$

Since t must be nonnegative, we need only to choose t that satisfies

$$t < -2\frac{\langle \mathcal{E}S, XS\rangle}{\|XS\|^2} = 2\gamma\frac{r}{\|\mathcal{E}^{-1}X\|},$$

where

$$\gamma := \frac{1 - \frac{r\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^3}}{1 - \frac{2r\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^3} + \frac{r^2\|(\mathcal{E}^{-1}X)^2\|^2}{\|\mathcal{E}^{-1}X\|^4}}.$$

It is easy to see that γ is nonnegative, because the denominator is a positive multiple of $\|XS\|^2$ where $XS \neq 0$, while the numerator is nonnegative because

$$\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2\rangle \leq \|\mathcal{E}^{-1}X\|^3. \quad (6.31)$$

To see why this is the case, let $\hat{x} \in \mathbb{R}^n$ denote the vector of the eigenvalues of X and and note that

$$\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2 \rangle = \mathrm{tr}((\mathcal{E}^{-1}X)^3) = \sum_{i=1}^{n} \hat{x}_i^3 = \|\hat{x}\|_3^3$$

and that

$$\|\mathcal{E}^{-1}X\|^3 = \left(\sum_{i=1}^{n} \hat{x}_i^2 \right)^{3/2} = \|\hat{x}\|_2^3.$$

Since

$$\|\hat{x}\|_3 \leq \|\hat{x}\|_2,$$

then (6.31) must hold. In a similar manner, we can also show that

$$\langle (\mathcal{E}^{-1}X)^2, (\mathcal{E}^{-1}X)^2 \rangle \leq \|\mathcal{E}^{-1}X\|^4. \tag{6.32}$$

Since,

$$\frac{r^2 \|(\mathcal{E}^{-1}X)^2\|^2}{\|\mathcal{E}^{-1}X\|^4} \leq 1,$$

then

$$1 - \frac{2r\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2 \rangle}{\|\mathcal{E}^{-1}X\|^3} + \frac{r^2 \|(\mathcal{E}^{-1}X)^2\|^2}{\|\mathcal{E}^{-1}X\|^4} \leq 2\left(1 - \frac{r\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2 \rangle}{\|\mathcal{E}^{-1}X\|^3} \right).$$

Since the left hand side is the denominator of γ and the right hand side twice its numerator, we see that $\gamma \geq \frac{1}{2}$.

This implies that letting $0 < t < \frac{r}{\|\mathcal{E}^{-1}X\|}$ is sufficient to guarantee that S lies in the interior of $K^*_{E(t),r}$. □

Proposition (Proposition 6.6)**.** *Given $E \in Swath(r)$ and X optimal for $(QP_{E,r})$, let $E(t) = \frac{1}{1+t}(E + tX)$. As long as $0 < t < \frac{r}{\|\mathcal{E}^{-1}X\|}$, then $E(t) \in Swath(r)$.*

In the proof of this proposition and subsequent results, we will use scalings defined using $E(t)$ which we will denote with $\mathcal{E}(t)$:

$$\mathcal{E}(t)U := E(t)^{1/2} U E(t)^{1/2}.$$

Furthermore, $\mathcal{E}(t)^{-1}, \mathcal{E}(t)^{-2}, \mathcal{E}(t)^2$ are defined analogously as $\mathcal{E}^{-1}, \mathcal{E}^{-2}$, and $\mathcal{E}^2$.

Proof. Since $E \in \text{Swath}(r)$, we know that it is feasible for (SDP). In particular, both X and E satisfies the linear equality constraint: $\mathcal{A}X = b$ and $\mathcal{A}E = b$. Then, it is easy to see that $\mathcal{A}E(t) = b$ as well, for all $t \in \mathbb{R}$. Furthermore, Lemma 6.10 shows that for $0 < t < \frac{r}{\|\mathcal{E}^{-1}X\|}$, $E(t) \in \mathbb{S}^n_{++}$.

To show that $E(t) \in \text{Swath}(r)$, it remains to show that $(QP_{E(t),r})$ has an optimal solution. By Theorem 2.4, we only need to assert that the dual solution (S, y) where S is given in (6.25) is strictly feasible for $(QP_{E,r})^*$. We already know that (S, y) satisfies the affine linear constraint $\mathcal{A}^*y + S = C$. Furthermore, Lemma 6.10 shows that for each $t < \frac{r}{\|\mathcal{E}^{-1}X\|}$, S lies in the interior of the dual cone $K^*_{E(t),r}$. Thus, (S, y) is a strictly feasible solution for $(QP_{E(t),r})$, which conclude the proof of this proposition. ◻

We remark that the proof of Lemma 6.10 is the same as that of the linear programming version, after replacing the inner products and the scalings appropriately. The proof of Proposition 6.6 includes showing that $E(t)$ lies in the positive definite cone, which is similar but a tad more involved than showing that $E(t)$ lies in the positive orthant in Proposition 5.2, for the chosen range of step size t.

6.6 Primal and dual monotonicity

Let $E \in \text{Swath}(r)$, X be the optimal solution of $(QP_{E,r})$, and $E(t) = \frac{1}{1+t}(E + tX)$ the next iterate. Assuming that $0 < t < \frac{r}{\|\mathcal{E}^{-1}X\|}$ such that $E(t) \in \text{Swath}(r)$, let $X(E(t))$

denote the optimal solution to $(QP_{E(t),r})$ and $(S(E(t)), y(E(t)))$ the optimal solution to $(QP_{E(t),r})^*$.

It may not be immediately evident why the duality gap is reduced at each iteration:

$$\langle C, X\rangle = b^T y \leq b^T y(E(t)) = \langle C, X(E(t))\rangle. \tag{6.33}$$

Note that for the duality gap to be monotonically decreasing, it is sufficient to have the objective value of the primal iterates be monotonically decreasing:

$$\langle C, E(t)\rangle \leq \langle C, E\rangle \tag{6.34}$$

and to have the objective value of the corresponding quadratic cone relaxation optimal solutions be monotonically increasing:

$$\langle C, X(E(t))\rangle \geq \langle C, X\rangle. \tag{6.35}$$

While it is easy to see why (6.34) must hold (that is, because $E(t)$ is a convex combination of E and X, where the objective value of X is smaller than that of E), it may not be as immediately obvious why (6.35) holds.

To see why (6.35) must hold, we again appeal to the dual problem. From Lemma 6.10, we see that for all $t < \frac{r}{\|\mathcal{E}^{-1}X\|}$, the optimal dual solution (S, y) of $(QP_{E,r})^*$ is a strictly feasible solution to $(QP_{E(t),r})^*$. This means that the optimal value of $(S(E(t)), y(E(t)))$ is at least as large as the optimal value of (S, y). Since strong duality holds for both pairs of quadratic-cone optimization problems, then

$$\langle C, E(t) - X(E(t))\rangle \leq \langle C, E - X\rangle,$$

as desired.

Hence, it makes sense to think of the property (6.34) as "primal monotonicity" and to (6.35) as "dual monotonicity", noting that the sequence $\{E^{(k)}\}$ of strictly feasible primal solutions has monotonic decreasing objective values while the sequence $\{(S^{(k)}, y^{(k)})\}$ of strictly feasible dual solutions has monotonic increasing objective values.

This concludes our proof that for all $t < \frac{r}{\|\mathcal{E}^{-1}X\|}$,

$$\langle C, E(t) - X(E(t))\rangle \leq \langle C, E - X\rangle. \tag{6.36}$$

In the following sections, we will show that the duality gap can be reduced by a significant multiplicative factor at least at every other iteration. In one case (Proposition 6.7), the large reduction in duality gap is due to a significant decrease of primal objective value while in another case (Proposition 6.9), it is due to a significant increase of dual objective value.

6.7 Proof of Proposition 6.7

Proposition (Proposition 6.7). *Suppose that $E \in Swath(r)$, X is optimal for $(QP_{E,r})$, with $\|\mathcal{E}^{-1}X\| \leq \sqrt{n}$. Let $E(t) = \frac{1}{1+t}(E + tX)$. Then, taking a step of size $t = \eta\frac{r}{\|\mathcal{E}^{-1}X\|}$ for some $\eta \in (0,1)$ reduces the duality gap by a factor of $\left(1 - \frac{\eta r}{2\sqrt{n}}\right)$.*

Proof. Since we choose $t = \eta \frac{r}{\|\mathcal{E}^{-1}X\|}$,

$$\begin{aligned}
\langle C, E(t) - X(E(t))\rangle &= \frac{1}{1+t}\langle C, E\rangle + \frac{t}{1+t}\langle C, X\rangle - \langle C, X(E(t))\rangle \\
&\leq \frac{1}{1+t}\langle C, E\rangle + \frac{t}{1+t}\langle C, X\rangle - \langle C, X\rangle \\
&= \left(1 - \frac{t}{1+t}\right)\langle C, E - X\rangle \\
&\leq \left(1 - \frac{\eta r}{2\sqrt{n}}\right)\langle C, E - X\rangle,
\end{aligned}$$

where the first inequality is due to dual monotonicity, (6.35), and the second inequality is due to $\|\mathcal{E}^{-1}X\| \leq \sqrt{n}$. □

6.8 Proof of Proposition 6.8

In the next two sections, we consider a fixed step size of $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$.

In the proof of Proposition 6.7 above, the significant bound on the duality gap reduction relies on the relatively large distance between E and $E(t)$ compared to the distance between E and X, which is controlled by the step size t. Naturally, when $\|\mathcal{E}^{-1}X\|$ is small, t is large, and we achieve a large duality gap reduction.

However, when $\|\mathcal{E}^{-1}X\|$ is large, the step size t is small, and the distance between E and $E(t)$ is only a small fraction of the distance between E and X. Thus, in order to obtain a better bound on the duality gap reduction, we need to show that not only is $\langle C, X(E(t))\rangle$ greater than $\langle C, X\rangle$, but also that it is quite significantly greater. That is, we desire a good lower bound for the quantity $\langle C, X(E(t)) - X\rangle$.

Letting (S, y) be the optimal solution for $(QP_{E,r})^*$ and $(S(E(t)), y(E(t)))$ the optimal solution for $(QP_{E(t),r})^*$, we can rewrite $\langle C, X(E(t)) - x\rangle$ in terms of the dual solutions as follows

$$\begin{aligned} \langle C, X(E(t)) - X\rangle &= b^T(y(E(t)) - y) \\ &= (\mathcal{A}X(E(t)))^*(y(E(t)) - y) \\ &= \langle X(E(t)), \mathcal{A}^*(y(E(t)) - y)\rangle \\ &= \langle X(E(t)), S - S(E(t))\rangle. \end{aligned} \tag{6.37}$$

That is, the inner product between $X(E(t))$ and $S - S(E(t))$ provides the amount of change of objective value. We will not be solving for $S(E(t))$, but we can come up with a different feasible solution (but not necessarily an optimal solution) for $(QP_{E(t),r})^*$ relatively easily.

Of course, (S, y) is feasible for $(QP_{E(t),r})^*$ due to our choice of t (see proof of Proposition 6.6). However, we would like to produce another solution, call it (S', y'), that better approximates $(S(E(t)), y(E(t)))$ in the sense that the objective value of (S', y') is quite significantly larger than that of (S, y).

Supposing that such a solution (S', y') could be found, then

$$\langle C, X(E(t)) - X\rangle \geq \langle X(E(t)), S - S'\rangle.$$

Using this idea, we find the following lower bound for $\langle C, X(e(t)) - X\rangle$ for a particular choice of step size t.

We can rewrite $\langle X(E(t)), S - S'\rangle$ in terms of the local inner product at $E(t)$ as follows

$$\langle X(E(t)), S - S'\rangle = \langle X(E(t)), \mathcal{E}(t)^2(S - S')\rangle_{E(t)}.$$

As in the previous chapter, we switch inner products because it is most natural to carry out our analysis using the local inner product at $E(t)$, since the cones $K_{E(t),r}$, $K^*_{E(t),r}$, and $K^{*E(t)}_{E(t),r}$ are more naturally expressed in terms of $\langle \cdot, \cdot \rangle_{E(t)}$.

We remark that any (S', y') is a feasible solution for $(QP_{E(t),r})^*$ if and only if $(\mathcal{E}(t)^2 S', y')$ is a feasible solution for

$$\begin{array}{rl} \max & b^T y \\ s.t. & \mathcal{E}(t)^2(\mathcal{A}^* \tilde{y}) + \tilde{S} = \mathcal{E}(t)^2 C \\ & S \in \mathcal{E}(t)^2 K^*_{E(t),r} = K^{*E(t)}_{E(t),r}. \end{array}$$

This problem is just $(QP_{E(t),r})^*$ scaled by $\mathcal{E}(t)^2$, or equivalently, the dual of $(QP_{E(t),r})$ with respect to the local inner product at $E(t)$, which we discussed in greater length in Section 6.2.4. We denote this problem as $(QP_{E(t),r})^{*E(t)}$.

Therefore, we will instead focus on finding a point $(\bar{S}', \bar{y}')$ that is feasible for $(QP_{E(t),r})^{*E(t)}$ and whose objective value is significantly better than that of

$$(\bar{S}, \bar{y}) := (\mathcal{E}(t)^2 S, y),$$

providing the lower bound

$$\langle C, X(E(t)) - X \rangle \geq \langle X(E(t)), \bar{S} - \bar{S}' \rangle_{E(t)}.$$

The problem of finding such an $\bar{S}'$ can be rephrased as the problem of finding a unit vector $\bar{V}$ and a positive scalar R which satisfy:

1. $\langle X(E(t)), \bar{V} \rangle_{E(t)} > 0$,
2. $\bar{V} \in \mathcal{L}^{\perp E(t)}$ (where $\mathcal{L}$ is the nullspace of $\mathcal{A}$), and
3. The $\| \cdot \|_{E(t)}$-ball centered at $\bar{S}$ of radius R is contained in $K^{*E(t)}_{E(t),r}$.

With R and $\bar{V}$ that satisfy the above conditions, choose

$$\bar{S}' := \bar{S} - R\bar{V}.$$

The second of the above conditions ensures the existence an $\bar{y}' \in \mathbb{R}^m$ such that $\mathcal{E}(t)^2(\mathcal{A}^*\bar{y}') + \bar{S}' = C$ (note that $\bar{y}'$ is uniquely determined by the choice of $\bar{S}'$ since $A_1, \ldots, A_m$ are linearly independent) while the third condition ensures that $\bar{S}' \in K^{*E(t)}_{E(t),r}$. Thus, $(\bar{S}', \bar{y}')$ is feasible for $(QP_{E(t),r})^{*E(t)}$.

Furthermore, the first condition ensures that

$$\langle C, X(E(t)) - X\rangle \geq \langle X(E(t)), \bar{S} - \bar{S}'\rangle_{E(t)} = R\langle X(E(t)), \bar{V}\rangle_{E(t)} > 0.$$

Therefore, we will first compute a sufficiently large radius R that still guarantees that

$$B_{E(t)}(\bar{S}, R) \subseteq K^{*E(t)}_{E(t),r},$$

where $B_{E(t)}(\bar{S}, R)$ denotes the ball of radius R centered at $\bar{S}$, with respect to $\|\cdot\|_{E(t)}$.

Lemma 6.11. *Let $\beta \in (0, r]$. Suppose that $E(t) = \frac{1}{1+t}(E + tX) \in Swath(\beta)$. If $\hat{S} \in K^{*E(t)}_{E(t),\beta}$, then $K^{*E(t)}_{E(t),r}$ contains the ball $B_{E(t)}(\hat{S}, R)$ where*

$$R = \frac{\|\hat{S}\|_{E(t)}}{n}\left(r\sqrt{n - \beta^2} - \beta\sqrt{n - r^2}\right).$$

Proof. By (6.15),

$$K^{*E(t)}_{E(t),r} = K_{E(t),\sqrt{n-r^2}}$$

and

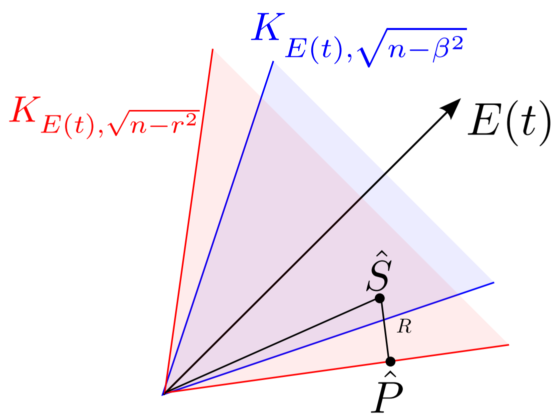
$$K^{*E(t)}_{E(t),\beta} = K_{E(t),\sqrt{n-\beta^2}}.$$

A nonzero vector $\hat{S}$ is in $K_{E(t),\sqrt{n-\beta^2}}$ if and only if its angle with $E(t)$ does not exceed $\arccos\sqrt{(n - \beta^2)/n}$, or equivalently, if and only if

$$\cos \angle_{E(t)}(\hat{S}, E(t)) \geq \sqrt{(n - \beta^2)/n}. \tag{6.38}$$

(Note that in this proof, we consider angles with respect to the local inner product at $E(t)$.) Moreover, for $\hat{S} \in K_{E(t),\sqrt{n-\beta^2}} \subseteq K_{E(t),\sqrt{n-r^2}}$, the point in the boundary of the cone $K_{E(t),\sqrt{n-r^2}}$ closest to $\hat{S}$ is the projection of $\hat{S}$ onto the ray whose angle with $\hat{S}$ is smallest among all rays whose angle with $E(t)$ is $\arccos\sqrt{(n-r^2)/n}$. Call the projection of $\hat{S}$ to this ray $\hat{P}$. Clearly, the angle between $\hat{S}$ and $\hat{P}$ is

$$\angle_{E(t)}(\hat{S},\hat{P}) = \arccos\sqrt{(n-r^2)/n} - \angle_{E(t)}(\hat{S},E(t)).$$



Therefore, the distance from $\hat{S}$ to the boundary of $K_{E(t),\gamma}$ is given by

$$\begin{aligned}
\|\hat{S}-\hat{P}\|_{E(t)} &= \|\hat{S}\|_{E(t)} \sin\angle_{E(t)}(\hat{S},\hat{P}) \\
&= \|\hat{S}\|_{E(t)} \sin\left(\arccos\sqrt{(n-r^2)/n} - \angle_{E(t)}(\hat{S},E(t))\right) \\
&= \|\hat{S}\|_{E(t)} \left(\sin\arccos\sqrt{(n-r^2)/n}\,\cos\angle_{E(t)}(\hat{S},E(t))\right. \\
&\qquad \left. - \sin\angle_{E(t)}(\hat{S},E(t))\,\cos(\arccos\sqrt{(n-r^2)/n})\right).
\end{aligned}$$

Using (6.38),

$$\begin{aligned}
\|\hat{S}-\hat{P}\|_{E(t)} &\geq \|\hat{S}\|_{E(t)} \left(\sqrt{1-(n-r^2)/n}\,\sqrt{(n-\beta^2)/n}\right. \\
&\qquad \left. - \sqrt{1-(n-\beta^2)/n}\,\sqrt{(n-r^2)/n}\right) \\
&= \frac{\|\hat{S}\|_{E(t)}}{n}\left(r\sqrt{n-\beta^2} - \beta\sqrt{n-r^2}\right),
\end{aligned}$$

as desired. □

Lemma 6.11 provides the desired lower bound R of the distance from $\bar{S}$, the scaled version of the optimal solution for $(QP_{E,r})^*$, to the boundary of the cone $K^{*E(t)}_{E(t),r}$, under the assumption that $E(t)$ is in Swath(β) (in addition to being in Swath(r)) for a cone-width parameter $\beta \le r$. This is a nontrivial assumption, because Swath(β) $\subseteq$ Swath(r), for $\beta \le r$.

In the lemma below, we will choose a particular value for β, namely $\beta = r\sqrt{\frac{1+r}{2}}$, and show that for a step of length $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$, it is guaranteed that $E(t) \in$ Swath(β).

Lemma 6.12. *Let $\beta = r\sqrt{\frac{1+r}{2}}$. Suppose that $E \in$ Swath(r), X is the optimal solution for $(QP_{E,r})$ and (S, y) is the optimal solution for $(QP_{E,r})^*$. Let $E(t) = \frac{1}{1+t}(E + tX)$. Then, taking $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$ implies that $S \in$ int $K^*_{E(t),\beta}$ and $E(t) \in$ Swath(β).*

Proof. We wish to show that for $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$, the dual solution S lies in the interior of $K^*_{E(t),\beta}$, or equivalently, that the following inequality is satisfied,

$$\langle I, \mathcal{E}(t)S\rangle^2 - (n - \beta^2)\langle \mathcal{E}(t)S, \mathcal{E}(t)S\rangle > 0 \tag{6.39}$$

and $\langle I, \mathcal{E}(t)S\rangle \ge 0$. Recall that

$$(1+t)\langle I, \mathcal{E}(t)S\rangle = \langle I, \mathcal{E}S\rangle + t\langle I, XS\rangle = \langle I, \mathcal{E}S\rangle \ge 0,$$

and

$$(1+t)^2\langle \mathcal{E}(t)S, \mathcal{E}(t)S\rangle = \frac{\langle I, \mathcal{E}S\rangle^2}{n-r^2} + 2t\langle \mathcal{E}S, XS\rangle + t^2\langle SX, XS\rangle.$$

Therefore, $S \in K^*_{E(t),\beta}$ if and only if

$$-\frac{r^2(1-(\beta/r)^2)}{n-r^2}\langle C, E-X\rangle^2 - (n-\beta^2)\left(2t\langle \mathcal{E}S, XS\rangle + t^2\langle SX, XS\rangle\right) > 0.$$

In (6.29) and (6.30), we computed $\langle \mathcal{E}S, XS\rangle$ and $\langle XS, XS\rangle$, reproduced below:

$$\langle \mathcal{E}S, XS\rangle = -\frac{\langle C, E-X\rangle^2}{(n-r^2)^2} r\|\mathcal{E}^{-1}X\|\left(1 - \frac{r\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^3}\right),$$

$$\langle SX, XS\rangle = \frac{\langle C, E-X\rangle^2}{(n-r^2)^2}\|\mathcal{E}^{-1}X\|^2\left(1 - \frac{2r\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^3} + \frac{r^2\langle (\mathcal{E}^{-1}X)^2, (\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^4}\right).$$

Letting $\eta := \frac{t\|\mathcal{E}^{-1}X\|}{r}$, to simplify the notation, and using the values of $\langle \mathcal{E}S, XS\rangle$ and $\langle SX, XS\rangle$ above, the inequality (6.39) is equivalent to

$$\frac{\langle C, E-X\rangle^2 r^2}{n-r^2}\left(-(1-(\beta/r)^2 + \frac{n-\beta^2}{n-r^2}\left(1-\frac{r\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^3}\right)\left(2\eta - \eta^2\frac{1}{\gamma}\right)\right) > 0,$$

where

$$\gamma := \frac{1-\frac{r\langle \mathcal{E}^{-1}X,(\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^3}}{1-\frac{2r\langle \mathcal{E}^{-1}X,(\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^3}+\frac{r^2\langle (\mathcal{E}^{-1}X)^2,(\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^4}}.$$

From the proof of Lemma 6.10, we know that $\gamma \geq \frac{1}{2}$. Using this, and since $\frac{\langle C,E-X\rangle^2 r^2}{n-r^2} > 0$ and $\frac{\langle \mathcal{E}^{-1}X,(\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^3} \leq 1$, we see that in order to satisfy (6.39), it is sufficient to choose η such that we satisfy:

$$-\left(1-(\beta/r)^2\right) + 2\frac{n-\beta^2}{n-r^2}(1-r)\eta\,(1-\eta) > 0$$

or equivalently,

$$(1-r)\eta\,(1-\eta) > \frac{1-(\beta/r)^2}{2}\frac{n-r^2}{n-\beta^2}.$$

Substituting for $\beta = r\sqrt{\frac{1+r}{2}}$, then $1-(\beta/r)^2 = \frac{1-r}{2}$. Furthermore, using

$$\frac{n-r^2}{n-\beta^2} < 1,$$

it is then sufficient to satisfy

$$\eta(1-\eta) \geq \frac{1-r}{2}\frac{1}{2(1-r)} = \frac{1}{4}.$$

Hence, choosing $\eta = \frac{1}{2}$, or equivalently, choosing $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$, guarantees that $S \in \operatorname{int} K^*_{E(t),\beta}$. Furthermore, since $(QP_{E(t),\beta})^*$ has a strictly feasible solution, then for this choice of t, we know that $E(t) \in \mathrm{Swath}(\beta)$. □

We conclude this section with the proof of Proposition 6.8, whose statement we reproduce below.

Proposition (Proposition 6.8). *Suppose that $E \in Swath(r)$, X optimal for $(QP_{E,r})$ and (S, y) is optimal for $(QP_{E,r})^*$. Let $E(t) = \frac{1}{1+t}(E + tX)$.*

Then, taking $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$ and letting $X(E(t))$ denote the optimal solution for $(QP_{E(t),r})$, we have that

$$\langle C, X(E(t)) - X\rangle \geq R\|Proj_{\mathcal{L}^{\perp_{E(t)}}} X(E(t))\|_{E(t)},$$

where $R := \frac{\|\mathcal{E}(t)S\|}{n}\left(r\sqrt{n - \beta^2} - \beta\sqrt{n - r^2}\right)$ and $\beta = r\sqrt{\frac{1+r}{2}}$.

Proof. Recall that we would like to obtain a solution (S', y') that is feasible for $(QP_{E,r})^*$, whose objective value is significantly better than (S, y). Such a solution (S', y') provides the following lower bound

$$\langle C, X(E(t)) - X\rangle \geq \langle X(E(t)), S - S'\rangle = \langle X(E(t)), \bar{S} - \bar{S}'\rangle_{E(t)},$$

where $\langle U, V\rangle_{E(t)} = \langle \mathcal{E}^{-1}U, \mathcal{E}^{-1}V\rangle$ and $\bar{S} := \mathcal{E}^2 S$ and $\bar{S}' := \mathcal{E}^2 S'$.

Since S lies in the interior of the cone $K^*_{E(t),r}$, then $\bar{S}$ lies in the interior of the cone $\mathcal{E}(t)^2 K^*_{E(t),r} = K^{*E(t)}_{E(t),r}$. Consider the ball centered at $\bar{S}$ of radius $R > 0$ with respect to the norm induced by the inner product $\langle\cdot,\cdot\rangle_{E(t)}$, call it $B_{E(t)}(\bar{S}, R)$.

If we can show that $B_{E(t)}(\bar{S}, R) \subseteq K^{*E(t)}_{E(t),r}$, then for any vector $V \in \mathcal{E}(t)^2\mathcal{L}^{\perp} = \mathcal{L}^{\perp_{E(t)}}$ with unit norm, the point

$$\bar{S}' = \bar{S} - RV$$

is feasible for the scaled version of $(QP_{E(t),r})^*$ (scaled by $\mathcal{E}(t)^2$).

In particular, if we take V to be the direction of the projection of $X(E(t))$ onto $\mathcal{L}^{\perp_{E(t)}}$, namely

$$V := \frac{\text{Proj}_{\mathcal{L}^{\perp_{E(t)}}} X(E(t))}{\|\text{Proj}_{\mathcal{L}^{\perp_{E(t)}}} X(E(t))\|_{E(t)}},$$

then

$$\begin{aligned}
\langle C, X(E(t)) - X \rangle &\geq \langle X(E(t)), \bar{S} - \bar{S}' \rangle_{E(t)} \\
&= R \left\langle X(E(t)), \frac{\text{Proj}_{\mathcal{L}^{\perp E(t)}} X(E(t))}{\|\text{Proj}_{\mathcal{L}^{\perp E(t)}} X(E(t))\|_{E(t)}} \right\rangle \\
&= R \|\text{Proj}_{\mathcal{L}^{\perp E(t)}} X(E(t))\|_{E(t)}.
\end{aligned}$$

By Lemma 6.12, taking $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$, we have that $E(t) \in \text{Swath}(\beta)$ for $\beta = r\sqrt{\frac{1+r}{2}}$. Hence, we can apply Lemma 6.11 and obtain that

$$R = \frac{\|\bar{S}\|_{E(t)}}{n} \left(r\sqrt{n - \beta^2} - \beta\sqrt{n - r^2} \right)$$

works. Recall that $\bar{S} = \mathcal{E}(t)^2 S$. Thus, $\|\bar{S}\|_{E(t)} = \|\mathcal{E}(t) S\|$. □

6.9 Proving Proposition 6.9

Proposition 6.8 provides an expression for a good lower bound for $\langle C, X(E(t)) - X \rangle$ in terms of a multiple of $\|\text{Proj}_{\mathcal{L}^{\perp E(t)}} X(E(t)\|_{E(t)}$. However, we still need to compute an expression for the length of this projection.

Throughout this section, we rely on observations about the tangent spaces to the quadratic cones, which we discussed more thoroughly in Section 6.2.5. In particular, we will consider the set $\mathcal{T}$, which is defined in (6.19) as the intersection between $\mathcal{L}$ and the space that is tangent to the cone $K_{E(t),r}$ at the optimal solution $X(E(t))$ of $(QP_{E(t),r})$.

Lemma 6.13. *Suppose that $E \in Swath(r)$, X is optimal for $(QP_{E,r})$, and (S, y) is optimal for $(QP_{E,r})^*$. As usual, we let $E(t) = \frac{1}{1+t}(E + tX)$ and $X(E(t))$ the optimal solution for $(QP_{E(t),r})$. Then, for all $t < \frac{r}{\|\mathcal{E}^{-1}X\|}$,*

$$\|Proj_{\mathcal{L}^{\perp E(t)}} X(E(t))\|^2_{E(t)} = \frac{\|X(E(t))\|^2_{E(t)} \sin^2\theta}{1 + \frac{(\|X(E(t))\|_{E(t)} - r)^2}{n - r^2} \sin^2\theta},$$

where θ denotes the angle between $X(E(t))$ and its projection onto $\mathcal{T}$.

Proof. Choose $V \in \mathcal{T}$ which minimizes $\sin^2 \angle_{E(t)}(X(E(t)), V)$ among all vectors in $\mathcal{T}$. That is, V is a scalar multiple of the projection of $X(E(t))$ onto $\mathcal{T}$, with respect to the inner product $\langle \cdot, \cdot \rangle_{E(t)}$. Let $\theta := \angle_{E(t)}(X(E(t)), V)$, the angle between $X(E(t))$ and V with respect to $\langle \cdot, \cdot \rangle_{E(t)}$.

If $\sin^2 \theta < 1$, then we know that $\langle X, V \rangle_E \neq 0$. By Proposition 6.4, we can write any such a vector V in the form $V = X(E(t)) + W$ (possibly after some scaling), where $\langle X(E(t)), W \rangle_{E(t)} = \langle E(t), W \rangle_{E(t)} = 0$.

Thus, from (6.17),

$$\langle X(E(t)), V \rangle_{E(t)} = \|X(E(t))\|^2_{E(t)}, \tag{6.40}$$

$$\begin{aligned} \langle X(E(t)), E(t) - X(E(t)) \rangle_{E(t)} &= \langle V, E(t) - X(E(t)) \rangle_{E(t)} && (6.41) \\ &= \|X(E(t))\|_{E(t)} \left(r - \|X(E(t))\|_{E(t)} \right), && (6.42) \end{aligned}$$

and

$$|\cos \theta| = \left| \frac{\langle X(E(t)), V \rangle_{E(t)}}{\|X(E(t))\|_{E(t)} \, \|V\|_{E(t)}} \right| = \left| \frac{\|X(E(t))\|_{E(t)}}{\|V\|_{E(t)}} \right|. \tag{6.43}$$

Let P denote the projection of $X(E(t))$ onto $L^{\perp_{E(t)}}$ and Q denote the projection of $X(E(t))$ onto $\mathcal{L}$, with respect to the inner product $\langle \cdot, \cdot \rangle_{E(t)}$. Then, we can write Q as a linear combination of V and another vector $U \in \mathcal{L}$ that is orthogonal to V since $\mathcal{T}$ is a subspace of $\mathcal{L}$ of codimension 1. In particular, note that

$$U := \frac{\|V\|^2_{E(t)}}{\langle V, E(t) - X(E(t)) \rangle_{E(t)}} (E(t) - X(E(t))) - V \tag{6.44}$$

lies in $\mathcal{L}$ and satisfies $\langle V, U\rangle_{E(t)} = 0$:

$$\begin{aligned}
\mathcal{A}U &= \frac{\|V\|^2_{E(t)}}{\langle V, E(t) - X(E(t))\rangle_{E(t)}}\mathcal{A}(E(t) - X(E(t))) - \mathcal{A}V = 0, \\
\langle V, U\rangle_{E(t)} &= \frac{\|V\|^2_{E(t)}}{\langle V, E(t) - X(E(t))\rangle_{E(t)}}\langle V, E(t) - X(E(t))\rangle_{E(t)} - \langle V, V\rangle_{E(t)} \\
&= \|V\|^2_{E(t)} - \|V\|^2_{E(t)} = 0.
\end{aligned}$$

Then, the projection of $X(E(t))$ onto V is

$$\frac{\langle X(E(t)), V\rangle_{E(t)}}{\|V\|^2_{E(t)}}V = \frac{\|X(E(t))\|^2_{E(t)}}{\|V\|^2_{E(t)}}V = \left(\cos^2_{E(t)}\theta\right)V,$$

where the first equality is due to (6.40) and the second is due to (6.43). The projection of $X(E(t))$ onto U is

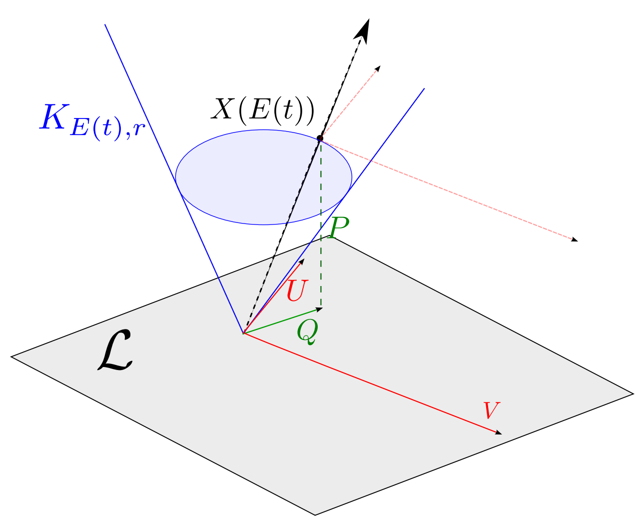
$$\frac{\langle X(E(t)), U\rangle_{E(t)}}{\langle U, U\rangle_{E(t)}}U.$$

Therefore, the projection of $X(E(t))$ onto $\mathcal{L}$ is

$$Q = \frac{\|X(E(t))\|^2_{E(t)}}{\|V\|^2_{E(t)}}V + \frac{\langle X, U\rangle_{E(t)}}{\langle U, U\rangle_{E(t)}}U,$$

whose squared norm is

$$\|Q\|^2_{E(t)} = \|X(E(t))\|^2_{E(t)}\cos^2\theta + \frac{\langle X(E(t)), U\rangle^2_{E(t)}}{\langle U, U\rangle_{E(t)}}.$$



Using the expression for U given in (6.44), the equalities (6.40)-(6.43) about the tangent vector V, and $\|E(t)\|_{E(t)} = \sqrt{n}$, we note that

$$\begin{aligned}
\langle X(E(t)), U\rangle_{E(t)} &= \frac{\|V\|^2_{E(t)}}{\langle V, E(t) - X(E(t))\rangle_{E(t)}}\langle X(E(t)), E(t) - X(E(t))\rangle_{E(t)} \\
&\quad - \langle X(E(t)), V\rangle_{E(t)} \\
&= \|V\|^2_{E(t)} - \|X(E(t))\|^2_{E(t)} = \|V\|^2 \sin^2\theta,
\end{aligned}$$

and that

$$\begin{aligned}
\langle U, U\rangle_{E(t)} &= \left(\frac{\|V\|^2_{E(t)}}{\langle V, E(t) - X(E(t))\rangle_{E(t)}}\right)^2 \|E(t) - X(E(t))\|^2 \\
&\quad - 2\frac{\|V\|^2_{E(t)}}{\langle V, E(t) - X(E(t))\rangle_{E(t)}}\langle V, E(t) - X(E(t))\rangle_{E(t)} + \|V\|^2_{E(t)} \\
&= \left(\frac{\|V\|^2_{E(t)}}{\langle V, E(t) - X(E(t))\rangle_{E(t)}}\right)^2 \|E(t) - X(E(t))\|^2_{E(t)} - \|V\|^2_{E(t)} \\
&= \|V\|^4_{E(t)}\left(\frac{\|E(t) - X(E(t))\|^2_{E(t)}}{\langle V, E(t) - X(E(t))\rangle^2_{E(t)}} - \frac{1}{\|V\|^2_{E(t)}}\right) \\
&= \frac{\|V\|^4_{E(t)}}{\|X(E(t))\|^2_{E(t)}}\left(\frac{\|E(t) - X(E(t))\|^2_{E(t)}}{(r - \|X(E(t))\|_{E(t)})^2} - \cos^2\theta\right).
\end{aligned}$$

These imply that

$$\frac{\langle X(E(t)), U\rangle^2_{E(t)}}{\langle U, U\rangle_{E(t)}} = \frac{\|X(E(t))\|^2_{E(t)}\sin^4\theta}{\frac{\|E(t)-X(E(t))\|^2_{E(t)}}{(r-\|X(E(t))\|_{E(t)})^2} - \cos^2\theta}.$$

Hence,

$$\|Q\|^2_{E(t)} = \|X(E(t))\|^2_{E(t)}\left(\cos^2\theta + \frac{\sin^4\theta}{\frac{\|E(t)-X(E(t))\|^2_{E(t)}}{(r-\|X(E(t))\|_{E(t)})^2} - \cos^2\theta}\right).$$

So, the squared length of the projection of $X(E(t))$ onto $\mathcal{L}^{\perp_{E(t)}}$ is

$$\begin{aligned}
\|P\|^2_{E(t)} &= \|X(E(t))\|^2_{E(t)} - \|Q\|^2_{E(t)} \\
&= \|X(E(t))\|^2_{E(t)}\left(1 - \cos^2\theta - \frac{\sin^4\theta}{\frac{\|E(t)-X(E(t))\|^2_{E(t)}}{(r-\|X(E(t))\|_{E(t)})^2} - \cos^2\theta}\right) \\
&= \|X(E(t))\|^2_{E(t)}\sin^2\theta\left(1 - \frac{\sin^2_{E(t)}\theta}{\frac{\|E(t)-X(E(t))\|^2_{E(t)}}{(r-\|X(E(t))\|_{E(t)})^2} - \cos^2\theta}\right).
\end{aligned}$$

Since $\|E(t) - X(E(t))\|_{E(t)}^2 = n - 2r\|X(E(t))\|_{E(t)} + \|X(E(t))\|_{E(t)}^2$, then

$$\|P\|_{E(t)}^2 = \frac{\|X(E(t))\|_{E(t)}^2 \sin^2\theta}{1 + \frac{(r - \|X(E(t))\|_{E(t)})^2}{n - r^2}\sin^2\theta} \tag{6.45}$$

when $\sin^2\theta < 1$.

In the case that $\sin^2\theta = 1$, then $\langle X(E(t)), V\rangle_{E(t)} = 0$ for all $V \in \mathcal{T}$. That is, $X(E(t))$ is orthogonal to all vectors in $\mathcal{T}$. Since the $\mathcal{T}$ together with $E(t) - X(E(t))$ spans $\mathcal{L}$, then it must be the case that the projection of $X(E(t))$ onto $\mathcal{L}$ is a multiple of $E(t) - X(E(t))$, namely:

$$Q = \frac{\langle X(E(t)), E(t) - X(E(t))\rangle}{\|E(t) - X(E(t))\|_{E(t)}^2}(E(t) - X(E(t))).$$

Hence,

$$\|Q\|_{E(t)}^2 = \frac{\langle X(E(t)), E(t) - X(E(t))\rangle^2}{\|E(t) - X(E(t))\|_{E(t)}^2}.$$

and

$$\begin{aligned}
\|P\|_{E(t)}^2 &= \|X(E(t))\|_{E(t)}^2 - \|Q\|_{E(t)}^2 \\
&= \|X(E(t))\|_{E(t)}^2\left(1 - \frac{(r - \|X(E(t))\|_{E(t)})^2}{\|E(t) - X(E(t))\|_{E(t)}^2}\right) \\
&= \frac{\|X(E(t))\|_{E(t)}^2}{1 + \left(\frac{\|E(t) - X(E(t))\|_{E(t)}^2}{(r - \|X(E(t))\|_{E(t)})^2} - 1\right)^{-1}} \\
&= \frac{\|X(E(t))\|_{E(t)}^2}{1 + \frac{(r - \|X(E(t))\|_{E(t)})^2}{n - r^2}}.
\end{aligned}$$

Thus, (6.45) in fact also holds for when $\sin^2\theta = 1$, proving our claim. □

When we take a step of size $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$, we obtain an explicit bound on the angle between $X(E(t))$ and vectors in $\mathcal{T}$.

Lemma 6.14. *Let $\beta = r\sqrt{\frac{1+r}{2}}$. Let $E \in \mathit{Swath}(r)$ and let X be the optimal solution for $(QP_{E,r})$. Choose $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$ and let $E(t) = \frac{1}{1+t}(E + tX)$, with $X(E(t))$ the optimal solution for $(QP_{E(t),r})$.*

Then, for any $V \in \mathcal{T}$,

$$\cos^2(\angle_{E(t)}(X(E(t)), V)) \leq \frac{\beta^2}{r^2}.$$

Proof. In this lemma, angles are with respect to the inner product $\langle \cdot, \cdot \rangle_{E(t)}$.

Note that by Lemma 6.12, with $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$, we know that $E(t) \in \mathrm{Swath}(\beta) \subseteq \mathrm{Swath}(r)$ for $\beta = r\sqrt{\frac{1+r}{2}}$. Thus, we can talk about $X(E(t))$, the optimal solution to $(QP_{E(t),r})$ and also of the optimal solution to $(QP_{E(t),\beta})$ which we will call X'.

Hence, restricting ourselves to tangent vectors that also lie in $\mathcal{L}$, the set of tangent vectors to $K_{E(t),r}$ at $X(E(t))$ is the same as the set of tangent vectors to $K_{E(t),\beta}$ at X' (see Section 6.2.5), namely:

$$\mathcal{T} = \{V \in \mathbb{S}^n \mid \mathcal{A}V = 0,\ \langle C, V \rangle = 0\}.$$

Consider an arbitrary vector V from $\mathcal{T}$. Since $X' \neq 0$ is a point in the boundary of $K_{E(t),\beta}$ and V is tangent to the cone $K_{E(t),\beta}$ at X', then the line $\{X' + tV \mid t \in \mathbb{R}\}$ does not intersect the interior of $K_{E(t),\beta}$, hence neither V or $-V$ is contained in the interior, and thus $\theta' := \angle_{E(t)}(E(t), V)$, the angle between V and $E(t)$, satisfies

$$\cos^2(\theta') \leq \frac{\beta^2}{n},$$

because $K_{E(t),\beta}$ consists of the vectors whose angle with $E(t)$ does not exceed $\arccos(\beta/\sqrt{n})$.

Recall from (6.2.5) that

$$\langle E(t), X(E(t)) \rangle_{E(t)}\ \langle E(t), V \rangle_{E(t)} = r^2 \langle X(E(t)), V \rangle_{E(t)}.$$

Furthermore, since $\langle E(t), X(E(t))\rangle_{E(t)} = r\|X(E(t))\|_{E(t)}$, then

$$\frac{\langle X(E(t)), V\rangle_{E(t)}}{\|X(E(t))\|_{E(t)}} = \frac{\langle E(t), V\rangle_{E(t)}}{r}.$$

Let $\theta := \angle_{E(t)}(X(E(t)), V)$, the angle between $X(E(t))$ and V. Then, θ satisfies

$$\cos^2(\theta) = \frac{\langle X(E(t)), V\rangle^2_{E(t)}}{\|X(E(t))\|^2_{E(t)}\ \|V\|^2_{E(t)}} = \frac{\langle E(t), V\rangle^2_{E(t)}}{r^2\|V\|^2_{E(t)}} = \frac{\|E(t)\|^2_{E(t)}}{r^2}\cos^2(\theta') \leq \frac{\beta^2}{r^2},$$

where the inequality is due to $\cos^2(\theta') \leq \frac{\beta^2}{n}$ and $\|E(t)\|_{E(t)} = \sqrt{n}$. □

Combining the results of the above two lemmas, we obtain the following easy corollary.

Corollary 6.15. *Suppose that $E \in$ Swath(r), X is optimal for $(QP_{E,r})$, and (S, y) is optimal for $(QP_{E,r})^*$. As usual, we let $E(t) = \frac{1}{1+t}(E + tX)$ and $X(E(t))$ the optimal solution for $(QP_{E(t),r})$. Then, taking $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$,*

$$\|Proj_{\mathcal{L}^{\perp_{E(t)}}} X(E(t))\|^2_{E(t)} \geq \frac{\|X(E(t))\|^2_{E(t)}(1 - (\beta/r)^2)}{1 + \frac{(\|X(E(t))\|_{E(t)} - r)^2}{n - r^2}(1 - (\beta/r)^2)},$$

where $\beta = r\sqrt{\frac{1+r}{2}}$.

Proof. The choice $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$ guarantees that $E(t) \in$ Swath(β). This means that $(QP_{E(t),\beta})$ has an optimal solution. Moreover, the vector V is also tangent to the cone $K_{E(t),\beta}$ at the optimal solution of $(QP_{E(t),\beta})$ (see Section 6.2.5), and we can apply Lemma 6.14.

Lemma 6.14 implies that $\cos^2\theta \leq \frac{\beta^2}{r^2}$. So, $\sin^2\theta \geq 1 - \frac{\beta^2}{r^2}$, and

$$\|P\|^2_{E(t)} \geq \frac{\|X(E(t))\|^2_{E(t)}(1 - \frac{\beta^2}{r^2})}{1 + \frac{(r - \|X(E(t))\|_{E(t)})^2}{n - r^2}(1 - \frac{\beta^2}{r^2})},$$

proving our claim. □

We are now ready to prove Proposition 6.9, which is reproduced below

Proposition (Proposition 6.9)**.** *Suppose that $E \in Swath(r)$, X is optimal for $(QP_{E,r})$, and (S, y) is optimal for $(QP_{E,r})^*$. Let $E(t) = \frac{1}{1+t}(E + tX)$. Assume that $n \geq 4$.*

Choose $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$ and let $X(E(t))$ denote the optimal solution for $(QP_{E(t),r})$. If $\|\mathcal{E}^{-1}X\| > \sqrt{n}$ and $\|\mathcal{E}(t)^{-1}X(E(t))\| > \sqrt{n}$, then

$$\langle C, E(t) - X(E(t))\rangle \leq \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)\langle C, E - X\rangle.$$

Proof. Choose $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$. From Proposition 6.8 we have that

$$\langle C, E(t) - X(E(t))\rangle \leq \frac{1}{1+t}\langle C, E - X\rangle - R\,\|\text{Proj}_{\mathcal{L}^{\perp E(t)}}(X(E(t)))\|_{E(t)},$$

where $R := \frac{\|\mathcal{E}(t)S\|}{n}\left(r\sqrt{n-\beta^2} - \beta\sqrt{n-r^2}\right)$ and $\beta := r\sqrt{\frac{1+r}{2}}$.

Applying the lower bound for $\|\text{Proj}_{\mathcal{L}^{\perp E(t)}}X(E(t))\|_{E(t)}$ from Lemma 6.13 for $\beta = r\sqrt{\frac{1+r}{2}}$,

$$\begin{aligned}\langle C, E(t) - X(E(t))\rangle &\leq \frac{1}{1+t}\langle C, E - X\rangle - R\left(\frac{\|\mathcal{E}(t)^{-1}X(E(t))\|^2(1-(\beta/r)^2)}{1 + \frac{(\|\mathcal{E}(t)^{-1}X(E(t))\|-r)^2}{n-r^2}(1-(\beta/r)^2)}\right)^{1/2}\\ &= \frac{1}{1+t}\langle C, E - X\rangle - \|\mathcal{E}(t)S\|\, C_1 C_2,\end{aligned}$$

where

$$C_1 := \frac{1}{n}\left(r\sqrt{n-\beta^2} - \beta\sqrt{n-r^2}\right)$$

and

$$C_2 := \left(\frac{\|X(E(t))\|^2_{E(t)}(1-(\beta/r)^2)}{1 + \frac{(\|X(E(t))\|_{E(t)}-r)^2}{n-r^2}(1-(\beta/r)^2)}\right)^{1/2}.$$

Since $\beta \leq r < 1$, then

$$\begin{aligned}C_1 &\geq \frac{1}{n}(r-\beta)\sqrt{n-r^2} \geq \frac{\sqrt{n-1}}{n}(r-\beta)\\ &= \frac{\sqrt{n-1}}{n}r\left(1 - \sqrt{\frac{1+r}{2}}\right).\end{aligned}$$

Also, since $0 < r < 1$ and using $\beta = r\sqrt{\frac{1+r}{2}}$,

$$\begin{aligned}
C_2 &\geq \left(\frac{(n-1)\|X(E(t))\|_{E(t)}^2(1-(\beta/r)^2)}{(n-1)+(\|X(E(t))\|_{E(t)}-r)^2(1-(\beta/r)^2)}\right)^{1/2} \\
&\geq \sqrt{n-1}\left(\frac{\|X(E(t))\|_{E(t)}^2(1-(\beta/r)^2)}{n+\|X(E(t))\|_{E(t)}^2(1-(\beta/r)^2)}\right)^{1/2} \\
&= \sqrt{n-1}\left(\frac{\|X(E(t))\|_{E(t)}^2\frac{1+r}{2}}{n+\|X(E(t))\|_{E(t)}^2\frac{1+r}{2}}\right)^{1/2}.
\end{aligned}$$

so that

$$C_1C_2 \geq \frac{n-1}{n}r\left(1-\sqrt{\frac{1+r}{2}}\right)\left(\frac{\|X(E(t))\|_{E(t)}^2\frac{1+r}{2}}{n+\|X(E(t))\|_{E(t)}^2\frac{1+r}{2}}\right)^{1/2}.$$

We assumed that $\|\mathcal{E}(t)^{-1}X(E(t))\| = \|X(E(t))\|_{E(t)} > \sqrt{n}$, so

$$\begin{aligned}
C_1C_2 &\geq \frac{n-1}{n}r\left(1-\sqrt{\frac{1+r}{2}}\right)\left(\frac{n\frac{1+r}{2}}{n+n\frac{1+r}{2}}\right)^{1/2} \\
&= \frac{n-1}{n}r\left(1-\sqrt{\frac{1+r}{2}}\right)\left(\frac{\frac{1+r}{2}}{1+\frac{1+r}{2}}\right)^{1/2}.
\end{aligned}$$

Recall that

$$(1+t)^2\|\mathcal{E}(t)S\|^2 = \frac{\langle C, E-X\rangle^2}{n-r^2} + 2t\langle \mathcal{E}S, XS\rangle + t^2\langle SX, XS\rangle,$$

where (from (6.29) and (6.30)),

$$\begin{aligned}
\langle \mathcal{E}S, XS\rangle &= -\frac{\langle C, E-X\rangle^2}{(n-r^2)^2}r\|\mathcal{E}^{-1}X\|\left(1-\frac{r\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^3}\right), \\
\langle SX, XS\rangle &= \frac{\langle C, E-X\rangle^2}{(n-r^2)^2}\|\mathcal{E}^{-1}X\|^2\left(1-\frac{2r\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^3}+\frac{r^2\langle (\mathcal{E}^{-1}X)^2, (\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^4}\right).
\end{aligned}$$

Using $t = \frac{r}{2\|\mathcal{E}^{-1}X\|}$, then $(1+t)^2\|\mathcal{E}(t)S\|^2 = \frac{\langle C,E-X\rangle}{(n-r^2)^2}\kappa$, where

$$\kappa := (n-r^2)-r^2\left(1-\frac{r\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^3}\right)+\frac{r^2}{4}\left(1-\frac{2r\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^3}+\frac{r^2\langle (\mathcal{E}^{-1}X)^2, (\mathcal{E}^{-1}X)^2\rangle}{\|\mathcal{E}^{-1}X\|^4}\right).$$

Simplifying, we see that

$$\begin{aligned}
\kappa &\geq (n-r^2)-r^2\left(1-\frac{r\langle \mathcal{E}^{-1}X, (\mathcal{E}^{-1}X)^2\rangle}{2\|\mathcal{E}^{-1}X\|^3}\right) \\
&\geq n-3r^2 \geq n-3.
\end{aligned}$$

Thus,

$$\|\mathcal{E}(t)S\| \geq \frac{\langle C, E - X\rangle}{(n - r^2)(1 + t)} \sqrt{n-3} \geq \frac{\sqrt{n-3}}{n} \frac{\langle C, E - X\rangle}{1 + t}.$$

Therefore,

$$C_1 C_2 \|\mathcal{E}(t)S\| \geq \frac{\langle C, E - X\rangle}{1 + t} \frac{(n-1)\sqrt{n-3}}{n^2} r \left(1 - \sqrt{\frac{1+r}{2}}\right) \left(\frac{\frac{1+r}{2}}{1 + \frac{1+r}{2}}\right)^{1/2}$$

and

$$\begin{aligned}
\langle C, E(t) - X(E(t))\rangle &\leq \frac{\langle C, E - X\rangle}{1 + t} \left(1 - \frac{r(n-1)\sqrt{n-3}}{n^2} \left(1 - \sqrt{\frac{1+r}{2}}\right) \left(\frac{\frac{1+r}{2}}{1 + \frac{1+r}{2}}\right)^{1/2}\right) \\
&\leq \frac{\langle C, E - X\rangle}{1 + t} \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right) \\
&\leq \langle C, E - X\rangle \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right),
\end{aligned}$$

where the second to the last inequality is due to the following bounds: for all $n \geq 4$, $\sqrt{n-3}$ is greater than $\sqrt{n/4}$ and $\frac{n-1}{n^2} \geq \frac{1}{2n}$. So,

$$\frac{(n-1)\sqrt{n-3}}{n^2} \geq \sqrt{\frac{n}{4}} \frac{1}{2n} = \frac{1}{4\sqrt{n}}.$$

Since $r \geq 0$, then

$$\frac{\frac{1+r}{2}}{1 + \frac{1+r}{2}} = \frac{1}{1 + \frac{2}{1+r}} \geq \frac{1}{3}.$$

□

6.10 Proof of Theorem 6.5

Proof. We do this by induction on k. Starting at $E^{(0)}$, we know from Proposition 6.6 that with $t^{(0)} = \frac{r}{2\|(\mathcal{E}^{(0)})^{-1} X^{(0)}\|}$, then $E^{(1)} \in \text{Swath}(r)$.

Assuming that that $E^{(k)} \in \text{Swath}(r)$, we consider four cases:

1. If $\|(\mathcal{E}^{(k)})^{-1}X^{(k)}\| \leq \sqrt{n}$ and $\|(\mathcal{E}^{(k+)})^{-1}X^{(k+)}\| \leq \sqrt{n}$, then we can apply Proposition 6.7 with $\eta = \frac{1}{2}$ to conclude that

$$\begin{aligned}\langle C, E^{(k+1)} - X^{(k+1)}\rangle &\leq \left(1 - \frac{r}{4\sqrt{n}}\right)\langle C, E^{(k)} - X^{(k)}\rangle \\ &\leq \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)\langle C, E^{(k)} - X^{(k)}\rangle,\end{aligned}$$

since for all $r \in (0,1)$, $\frac{r}{4\sqrt{n}} \geq \frac{r\left(1-\sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}$.

2. If $\|(\mathcal{E}^{(k)})^{-1}X^{(k)}\| \leq \sqrt{n}$ and $\|(\mathcal{E}^{(k+)})^{-1}X^{(k+)}\| > \sqrt{n}$, then

$$\begin{aligned}\langle C, E^{(k+1)} - X^{(k+1)}\rangle &= \left(1 - \frac{t}{1+t}\right)\langle C, E^{(k)} - X^{(k)}\rangle - \langle C, X^{(k+1)} - X^{(k)}\rangle \\ &\leq \left(1 - \frac{r}{4\sqrt{n}}\right)\langle C, E^{(k)} - X^{(k)}\rangle - \langle C, X^{(k+1)} - X^{(k)}\rangle \\ &\leq \left(1 - \frac{r}{4\sqrt{n}} - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)\langle C, E^{(k)} - X^{(k)}\rangle \\ &\leq \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)\langle C, E^{(k)} - X^{(k)}\rangle,\end{aligned}$$

where the first and second inequalities are due to Proposition 6.7 and Proposition 6.9.

3. On the other hand, if $\|(\mathcal{E}^{(k)})^{-1}X^{(k)}\| > \sqrt{n}$ but $\|(\mathcal{E}^{(k)})^{-1}X^{(k+1)}\| \leq \sqrt{n}$, then Proposition 6.7 again implies that

$$\langle C, E^{(k+2)} - X^{(k+2)}\rangle \leq \left(1 - \frac{r}{4\sqrt{n}}\right)\langle C, E^{(k+1)} - X^{(k+1)}\rangle.$$

4. The remaining case is when $\|(\mathcal{E}^{(k)})^{-1}X^{(k)}\| > \sqrt{n}$ and $\|(\mathcal{E}^{(k+1)})^{-1}X^{(k+1)}\| > \sqrt{n}$. In this case, Proposition 6.9 implies that

$$\langle C, E^{(k+1)} - X^{(k+1)}\rangle \leq \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)\langle C, E^{(k)} - X^{(k)}\rangle.$$

□

Recall that $\frac{r}{4} \geq \frac{r\left(1-\sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}$ for all $r \in (0,1)$. So, taking the worse case that the duality gap is reduced significantly only every other iteration, by the larger factor $1 - \frac{r\left(1-\sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}$,

$$\langle C, E^{(k+2)} - X^{(k+2)} \rangle \leq \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}} \right) \langle c, E^{(k)} - X^{(k)} \rangle,$$

which means that after $\bar{k}$ iterations,

$$\langle C, E^{(\bar{k})} - X^{(\bar{k})} \rangle \leq \gamma_0 \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}} \right)^{\bar{k}/2},$$

where $\gamma_0 := \langle c, E^{(0)} - X^{(0)} \rangle$.

In order to achieve a final duality gap of ϵ, for some $\epsilon > 0$, it is then sufficient to satisfy

$$\gamma_0 \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}} \right)^{\bar{k}/2} \leq \epsilon,$$

or equivalently

$$\log \gamma_0 + \frac{\bar{k}}{2} \log \left(1 - \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}} \right) \leq \log \epsilon.$$

Since $\log(1+\delta) \leq \delta$ for all $\delta > -1$, then we want $\bar{k}$ such that

$$\log \gamma_0 + \frac{\bar{k}}{2} \left(- \frac{r\left(1 - \sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}} \right) \leq \log \epsilon,$$

or equivalently,

$$\bar{k} \geq \frac{8\sqrt{3n}}{r\left(1 - \sqrt{\frac{1+r}{2}}\right)} \log(\gamma_0/\epsilon).$$

Thus, $\bar{k} = O\left(\frac{\sqrt{n}}{r(1-\sqrt{(1+r)/2})}\log(\gamma_0/\epsilon)\right)$.

Note that, treating r as a fixed parameter, the iteration complexity is $O(\sqrt{n}\log(\gamma_0/\epsilon))$. As a function of r, however, we see that the choice of r which give the best bound is when r is bounded away from both 0 and 1.

At the end of Section 5.5, we showed that the significant duality gap reduction for the linear programming case is in fact guaranteed to take place more than just at every other iterations. The same is also true in the semidefinite programming case. That is, after $\bar{k}$ iterations,

$$\langle C, E^{(\bar{k})} - X^{(\bar{k})}\rangle \leq \left(1 - \frac{r\left(1-\sqrt{\frac{1+r}{2}}\right)}{4\sqrt{3n}}\right)^{\bar{k}-1} \langle c, E^{(0)} - X^{(0)}\rangle.$$

This fact can be shown using the same arguments as those for the linear programming case, in Section 5.5.

6.11 Future Directions

In this thesis, we have presented an algorithm for linear programming, whose iteration complexity matches the best bounds among interior-point algorithms in the literature. In this chapter, we extend our results to semidefinite programming.

We outline several remaining questions which we would like to investigate in the future.

1. Varying the parameter r.

In the present algorithm, the cone-width parameter r takes a value in the interval $(0, 1)$ and this value is chosen and fixed at the start of the algorithm. It will be interesting to study whether the algorithm can be improved by letting r vary. Furthermore, we might consider larger values of r, allowing the cones $K_{e,r}$ to be smaller.

2. Larger step sizes.

   We prove the iteration complexity of $O(\sqrt{n}\log(\gamma_0/\epsilon))$ for reducing the duality gap from γ_0 to ϵ, using the step size of $t^{(k)} = \frac{r}{2\|x^{(k)}./e^{(k)}\|}$. In practice, however, taking longer step sizes might improve the overall performance of the algorithm, even if we might not be able to prove the same iteration-complexity bound. We are interested in exploring whether taking a longer step (for instance, a fixed fraction towards the boundary of the linear programming feasible region) would improve the algorithm in practice.

3. Hyperbolic programming relaxation of large-scale semidefinite programming problems.

   In [28], the algorithm presented in this thesis is extended to general hyperbolic programming problems. For a hyperbolic programming problem whose hyperbolicity cone arises from a degree-d polynomial, the iteration complexity of the algorithm is $O(\sqrt{d}\log(\gamma_0/\epsilon))$ for reducing the duality gap from γ_0 to ϵ. One interesting question is to explore whether for large-scale semidefinite programming problems (that is, semidefinite programming problems on $\mathbb{S}^n_+$ where n is large) that arise from specific applications, there is a low-degree hyperbolic programming relaxation whose optimal solution provides a good approximation to the optimal solution of the original problem. If that is the case, then applying this algorithm (or another interior-point algorithm with similar complexity) could help solve large-scale semidefinite programming problems, for which interior-point algorithms are not computationally good in practice.

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