THE GEOMETRY OF GENERALIZED LAMPLIGHTER GROUPS

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THE GEOMETRY OF GENERALIZED LAMPLIGHTER GROUPS

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This work examines geometric properties of generalized lamplighter groups. The thesis contains two parts. The first part gives an elementary account of Bartholdi, Neuhauser and Woess's result that the Cayley graphs of a family of metabelian groups can be realized as 1-skeleta of horocyclic products of trees, extends the result to a wider family of groups (including an infinite valence case, like $\mathbb{Z} \wr \mathbb{Z}$), and makes the translation between the algebraic and geometric descriptions explicit. The second part examines important geometric properties of Baumslag and Remeslennikov's metabelian group $\Gamma_2 = \langle a, s, t \mid [a, a^t] = 1,\ [s, t] = 1,\ a^s = aa^t \rangle$. We show that the Cayley 2-complex of a suitable presentation of Γ_2 is a horocyclic product of three infinitely branching trees. We prove that the subgroup generated by a is undistorted in Γ_2. Finally, we reduce the question of finding an upper bound on the filling length function of Γ_2 to a combinatorial question about propagating configurations on a two-dimensional rhombic grid.

BIOGRAPHICAL SKETCH

Margarita was born in 1986 in Odessa, Ukraine. She moved to the United States of America at the age of 14. Margarita received a Bachelor of Science degree in Mathematics with a minor in Computer Science from Polytechnic Institute of NYU in May of 2007. After completing her undergraduate degree, she spent a year teaching at the Mathematics department of Polytechnic Institute of NYU as an Adjunct Instructor. Margarita entered the graduate program in Mathematics at Cornell University in August of 2008.

In loving memory of my grandfather, Anatoliy Tarnopolskyy

(August 21, 1934 - April 15, 2012)

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Many thanks to Starbucks for helping me get through the writing process by providing numerous free refills ☺.

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CHAPTER 1

INTRODUCTION

Our conventions throughout will be $[a,b] = a^{-1}b^{-1}ab$ and $a^{nb} = ba^nb^{-1}$ for group elements a, b and integers n. Our group actions are on the right.

The contents of Section 1.10, Chapter 2 and Chapter 3 are based on the article [1] written in collaboration with Tim Riley.

1.1 The simplest lamplighter group

The most basic example of a lamplighter group is the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ which in this thesis we will refer to as $\Gamma_1(2)$. As an abelian group the ring $(\mathbb{Z}/2\mathbb{Z})[x, x^{-1}]$ is isomorphic to the additive group $\bigoplus_{i\in\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ of finitely supported sequences of zeros and ones. So $\Gamma_1(2)$, which by definition is $\bigoplus_{i\in\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$, can also be expressed as $(\mathbb{Z}/2\mathbb{Z})[x, x^{-1}] \rtimes \mathbb{Z}$ and this latter definition provides a convenient description of the action of the $\mathbb{Z}$-factor, namely a generator of the $\mathbb{Z}$-factor acts on $(\mathbb{Z}/2\mathbb{Z})[x, x^{-1}]$ by multiplication by x.

Elements $\left(\sum_{j\in\mathbb{Z}} f_j x^j, k\right) \in \Gamma_1(2)$ can be visualized as a street (the real line) with lamps at all integer locations, a lamplighter located by lamp k, and, for each $f_j = 1$, the lamp at j is lit. We will call this the lamplighter model for $\Gamma_1(2)$. The identity element $(0,0)$ corresponds to all lights being turned off and the lamplighter at location 0. Figure 1.1 illustrates $(x^{-4} + 1 + x + x^3, 5) \in \Gamma_1(2)$.

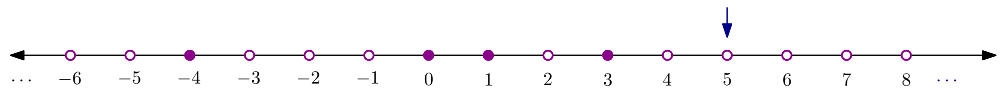

Figure 1.1: An element $(x^{-4} + 1 + x + x^3, 5)$ of $\Gamma_1(2)$. The lamps at positions $-4, 0, 1$ and 3 are turned on and the lamplighter is standing by the lamp at location 5.

As we will show in Section 2.1, one possible presentation for $\Gamma_1(2)$ is

$$\Gamma_1(2) = \left\langle \, a, t \;\middle|\; a^2 = 1, \left[a, a^{t^k}\right] = 1 \; (k \in \mathbb{Z}) \, \right\rangle.$$

Elements of $\Gamma_1(2)$ expressed using this presentation can be visualized on the lamplighter model above by starting with the model for the identity element, reading off one letter at a time: upon reading t we move the lamplighter one unit to the right (hence upon reading t^{-1} we move one unit to the left), and upon reading a we flip the switch on the lamp at which the lamplighter is currently located. For example, some ways to express the element pictured on Figure 1.1 using this presentation are $t^{-4}at^4atat^2at^2$ or $at^{-1}at^4at^{-7}at^3at^2at^4$.

1.2 Cayley graphs

The *Cayley graph* of a group G with respect to a generating set A is the graph which has elements of G as its vertex set and, for every $g \in G$ and $a \in A$, has a directed edge labeled a from g to ga. A presentation complex of a finitely presented group $G = \langle A \mid R \rangle$ denoted by P_G is a 2-dimensional cell complex which has a single vertex, one loop at the vertex for each generator of G and one 2-cell for each relation in the presentation glued along the corresponding edge-loop. The universal cover $\widetilde{P_G}$ of P_G is called the *Cayley 2-complex* of G, and the 1-skeleton of $\widetilde{P_G}$ gives the Cayley graph of G with respect to this presentation.

For example, the Cayley graph of a free group $F_2 = \langle a, b \rangle$ is an infinite tree whose vertices are 4-valent (one direction for each of the a, b, a^{-1}, b^{-1} elements). Since there are no relations between the generators, the Cayley 2-complex of F_2 is just its Cayley graph.

Another example is the Cayley graph of a free abelian group on two generators, $\mathbb{Z}^2 = \langle a, b \,|\, [a, b] = 1 \rangle$ which can be seen as a 1-skeleton of a plane tiled by squares. Figure 1.2 shows a sketch of a piece of the Cayley graph, where the arrows facing right correspond to the generator a, while the arrows facing upwards correspond to b.

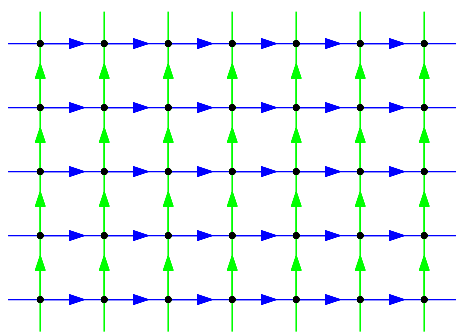

Figure 1.2: A piece of the Cayley graph of $\mathbb{Z}^2$.

The presentation 2-complex of $\mathbb{Z}^2$ is a torus, so the Cayley 2-complex is the plane tiled by squares.

1.3 A primer on horocyclic products of trees

An example of a binary tree $\mathcal{T}_{\mathbb{Z}/2\mathbb{Z}}$ with a height function h is shown on Figure 1.3. The most basic example of a horocyclic product of trees is constructed from the product of two copies of $\mathcal{T}_{\mathbb{Z}/2\mathbb{Z}}$ by taking the subset of $\mathcal{T}_{\mathbb{Z}/2\mathbb{Z}} \times \mathcal{T}_{\mathbb{Z}/2\mathbb{Z}}$:

$$\mathcal{H}_1(\mathbb{Z}/2\mathbb{Z}) := \left\{ (p_0, p_1) \in \mathcal{T}^2_{\mathbb{Z}/2\mathbb{Z}} \,\middle|\, h(p_0) + h(p_1) = 0 \right\}$$

We will generalize this construction to products of $n+1$ trees by taking the subset of $(n+1)$-tuples of points in the tree whose heights sum to zero. We will give precise definitions in Chapter 3.

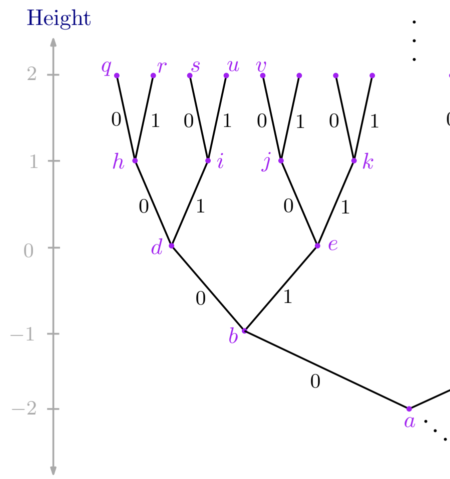

Figure 1.3: A part of an infinite binary tree with a height function. Some of the vertices in $\mathcal{H}_1(\mathbb{Z}/2\mathbb{Z})$ are $(d, d), (d, e), (d, f), (e, d), (j, c), (v, a), (a, u)$. Some of the edges in $\mathcal{H}_1(\mathbb{Z}/2\mathbb{Z})$ are $\{(d, e), (h, b)\}$, $\{(d, e), (b, k)\}$, and $\{(u, a), (i, c)\}$.

This striking generic construction turns out to give a Cayley graph of $\Gamma_1(2)$ –

Proposition 1.3.1. *The Cayley graph of* $\Gamma_1(2)$ *with respect to the generating set* $\{a, at\}$ *is* $\mathcal{H}_1(\mathbb{Z}/2\mathbb{Z})$.

This result, which is the starting point for the first half of this thesis, originates with P. Neumann and R. Möller in 2000. They noticed that, with respect to a suitable generating set, the Cayley graph of $\Gamma_1(2) = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ is a *highly-arc-transitive digraph* constructed by Möller in [37], which is the horocyclic product $\mathcal{H}_1(\mathbb{Z}/2\mathbb{Z})$ of two infinite binary trees [39] (see also [4, 15, 46] for this result). This result is a special case (with $n = 1$ and $R = \mathbb{Z}/2\mathbb{Z}$) of our Theorem 1 which identifies Cayley graphs of generalized lamplighter groups with the 1-skeleta of horocyclic products of trees (defined in detail in Chapter 3).

1.4 What are generalized lamplighter groups?

Another group we can consider is $\mathbb{Z} \wr \mathbb{Z}$ which we denote by Γ_1. Again, as an abelian group the ring $\mathbb{Z}[x, x^{-1}]$ is isomorphic to the additive group $\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$ of $\mathbb{Z}$-indexed finitely supported sequences of integers. So Γ_1 can also be expressed as $\mathbb{Z}[x, x^{-1}] \rtimes \mathbb{Z}$ where a generator of the $\mathbb{Z}$-factor acts on $\mathbb{Z}[x, x^{-1}]$ by multiplication by x. The model for Γ_1 is similar to that of $\Gamma_1(2)$, except each lamp has $\mathbb{Z}$-worth of brightness levels. A presentation for Γ_1 is $\Gamma_1 = \left\langle\, a, t \;\middle|\; \left[a, a^{t^k}\right] = 1 \ (k \in \mathbb{Z}) \,\right\rangle$, which is similar to that of $\Gamma_1(2)$ except that a has infinite order.

Similarly, for any commutative ring with unity R, we can construct a group $\Gamma_1(R) = R[x, x^{-1}] \rtimes \mathbb{Z}$ and consider the model where the lamps have $|R|$-worth of brightness levels. In this notation, $\Gamma_1(2) = \Gamma_1(\mathbb{Z}/2\mathbb{Z})$ and $\Gamma_1 = \Gamma_1(\mathbb{Z})$. The case where $n = 1$ of Theorem 1 states that the horocyclic product of two R-branching trees $\mathcal{H}_1(R)$ (defined in Section 3.1) is the Cayley graph of $\Gamma_1(R)$ with respect to a suitable generating set (proved in Section 3.3).

We can generalize these constructions further. The group Γ_2 is a celebrated example of Baumslag [5] and Remeslennikov [41]

$$\mathbb{Z}\left[x, x^{-1}, (1+x)^{-1}\right] \rtimes \mathbb{Z}^2$$

where, if the $\mathbb{Z}^2$-factor is $\langle t, s\rangle$, the actions of t and s are by multiplication by x and $(1+x)$, respectively. It was the first example of a finitely presented group with an abelian normal subgroup of infinite rank — specifically, the derived subgroup $[\Gamma_2, \Gamma_2]$. We will show in Proposition 2.1.3 that one of the presentations for Γ_2 is $\left\langle\, a, s, t \;\middle|\; [a, a^t] = 1,\ [s, t] = 1,\ a^s = aa^t \,\right\rangle$, which plays an important role in the second part of the thesis. An analogous lamplighter model for general $\Gamma_2(R) = R\left[x, x^{-1}, (1+x)^{-1}\right] \rtimes \mathbb{Z}^2$ will be discussed in Section 2.2. Restricting to the case where $n = 2$, Theorem 1 states that the 1-skeleton of the horocyclic product of three R-branching trees $\mathcal{H}_2(R)$ is the Cayley graph of $\Gamma_2(R)$ with respect to a suitable generating set (proved in Section 3.4).

We can generalize these constructions even further to obtain the family of groups $\Gamma_n(R)$ that figure in Theorem 1 defined as follows.

For $n = 1, 2, \ldots$, let $A_n(R)$ be the polynomial ring

$$R\left[x, x^{-1}, (1+x)^{-1}, \ldots, (n-1+x)^{-1}\right].$$

For $\mathbf{h} = (h_0, \ldots, h_{n-1}) \in \mathbb{Z}^n$ and $f \in A_n(R)$, define

$$f \cdot \mathbf{h} \;:=\; f x^{h_0}(1+x)^{h_1} \cdots (n-1+x)^{h_{n-1}}.$$

Then $\Gamma_n(R) := A_n(R) \rtimes \mathbb{Z}^n$ where the group operation is $(f, \mathbf{h})(\hat{f}, \hat{\mathbf{h}}) = (f + \hat{f}\cdot\mathbf{h}, \mathbf{h} + \hat{\mathbf{h}})$.

This definition can be conveniently repackaged as:

$$\Gamma_n(R) \;\cong\; \left\{ \left(\begin{array}{cc} x^{k_0}(1+x)^{k_1}\cdots(n-1+x)^{k_{n-1}} & f \\ 0 & 1 \end{array} \right) \;\middle|\; k_0, \ldots, k_{n-1} \in \mathbb{Z},\ f \in A_n(R) \right\},$$

where the matrix multiplication naturally realizes the semi-direct product structure of the group.

For brevity, define $\Gamma_n := \Gamma_n(\mathbb{Z})$ and $\Gamma_n(m) := \Gamma_n(\mathbb{Z}/m\mathbb{Z})$.

It will prove natural for us to index the coordinates of $\mathbb{Z}^n$ by $0, \ldots, n-1$. Accordingly, we use $\mathbf{e}_0, \ldots, \mathbf{e}_{n-1}$ to denote the standard basis for $\mathbb{Z}^n$.

In higher rank, the examples originate with Baumslag, Dyer, and Stammbach in [7, 8]. Bartholdi, Neuhauser and Woess [3] studied the family including $\Gamma_n(m)$ for $n = 1, 2, \ldots$ and $m \in \mathbb{N}$ such that $2, 3, \ldots, n-1$ are invertible in $\mathbb{Z}/m\mathbb{Z}$. And recently, Kropholler and Mullaney [36], building on Groves and Kochloukova [32], studied $\Gamma_n(\mathbb{Z}[1/(n-1)!]) \rtimes \mathbb{Z}$ where a generator of the $\mathbb{Z}$-factor acts as multiplication by $(n-1)!$ on the $A_n(\mathbb{Z}[1/(n-1)!])$-factor in $\Gamma_n(\mathbb{Z}[1/(n-1)!])$ and trivially on the $\mathbb{Z}^n$-factor. To put it another way, these groups are $A_n(\mathbb{Z}[1/(n-1)!]) \rtimes \mathbb{Z}^{n+1}$, defined like $\Gamma_n(\mathbb{Z}[1/(n-1)!])$, but with a generator of the additional $\mathbb{Z}$-factor acting on $A_n(\mathbb{Z}[1/(n-1)!])$ by multiplication by $(n-1)!$.

1.5 Why are lamplighter groups metabelian?

By definition a group is metabelian if its commutator subgroup is abelian. Another definition is that a group G is metablelian if and only if it has an abelian normal subgroup H such that the quotient group G/H is abelian. The groups that we are working with are clearly metabelian since for $G = \Gamma_n(R) = R[x, x^{-1}, (1+x)^{-1}, \ldots, (n-1+x)^{-1}] \rtimes \mathbb{Z}^n$ we can let H be the first factor, which is an abelian normal subgroup of G and considering the homomorphism that kills off the first factor, by the first isomorphism theorem we get that $G/H \cong \mathbb{Z}^n$.

1.6 Basic definitions

1.6.1 Geometric action

A group acts *geometrically* on a metric space if the action is cocompact, by isometries, and properly discontinuous (that is, every two points have neighborhoods such that only finitely many group elements translate one neighborhood in such a way that it intersects the other). For example, the action of a group G on itself by right-multiplication naturally extends to a geometric action on a Cayley graph that is defined using a finite generating set.

1.6.2 Equivalence relation of functions

Throughout this thesis we will discuss functions such as filling length and subgroup distortion. These functions are studied up to an equivalence relation. For functions $f, g : \mathbb{N} \to [0, \infty)$, we say $f \preccurlyeq g$ if there exists $C > 0$ such that $f(n) \leq C \cdot g(Cn + C) + Cn + n$ for all $n \in \mathbb{N}$. If $f \preccurlyeq g$ and $g \preccurlyeq f$, we say that f and g are equivalent and write $f \simeq g$.

1.6.3 Quasi-isometry

We will often study functions that correspond to various group invariants up to quasi-isometry. For finitely presented groups, knowing that the $\simeq$-equivalence class of functions is invariant up to quasi-isometry allows us to define group invariants that do not depend on the group presentation.

Definition 1.6.1. *Given two metric spaces (M_1, d_1) and (M_2, d_2) a function f is a* ***quasi-isomtery*** *from (M_1, d_1) to (M_2, d_2) if there exist constants $C \geq 1$ and $C' \geq 0$ such that both of the following are satisfied*

- $\forall x, x' \in M_1, \frac{1}{C}d_1(x, x') - C' \leq d_2(f(x), f(x')) \leq Cd_1(x, x') + C'$,
- $\forall y \in M_2, \exists x \in M_1$ *such that* $d_2(y, f(x)) \leq C'$.

If there exists a quasi-isometry from (M_1, d_1) to (M_2, d_2), then we say that these two metric spaces are quasi-isometric. Quasi-isometry is an equivalence relation on metric spaces. For a finitely presented group, its Cayley graphs under different generating sets viewed as metric spaces (where the length of each edge is 1) are quasi-isometric to each other. Hence, the quasi-isometry class of the Cayley graph is a group invariant.

1.7 Examples of group invariants

One of the major developments in group theory was the realization that groups are often best thought of as geometric objects. Geometric Group Theory emerged as a result of the transformative work of Gromov and his ensuing program of understanding discrete groups up to quasi-isometry. A major theme in this development is the study of properties invariant under quasi-isometries (and hence independent of the presentation used for a group). The Dehn function of a finitely presented group, the number of ends of a group, asymptotic cones, finiteness properties and the solvability of the word problem are some such invariants that have received much attention. Here are some results that are known for these invariants.

Let $G = \langle A \mid R \rangle$ be a finitely presented group. For w representing identity in G define Area(w) to be the minimal N such that $w = \prod_{i=1}^{N} u_i^{-1} r_i u_i$ in $F(A)$ for some words $u_i \in G$ and $r_i \in R^{\pm 1}$. Then define the *Dehn function* for a finitely presented group G, Dehn : $\mathbb{N} \to \mathbb{N}$ by

$$\mathrm{Dehn}(n) := \max \{ \mathrm{Area}(w) \mid w = 1 \text{ in } G \text{ and } |w| \leq n \} .$$

A free nilpotent group of class c on 2 letters has a Dehn function n^{c+1}. The Dehn function of a group is bounded above by linear function if and only if the group is hyperbolic. And if a group has subquadratic Dehn function, then in fact the Dehn function for that group must be bounded above by a linear function. The word problem for a finitely presented group is solvable if and only if the Dehn function of the group is recursive [29].

The number of ends of a finitely generated group G is the supremum over the number of infinite connected components remaining when any finite collection of edges is removed from a Cayley graph of G. The ends of finitely generated groups are well-understood. A finitely generated group can have $\{0, 1, 2, \infty\}$ ends, where the group has zero ends if and only if it is finite. Stallings' theorem states that a finitely generated group has more than one end if and only if the group admits a nontrivial decomposition as an amalgamated free product or an HNN-extension over a finite subgroup.

There are many results about solvability of the word problem in groups. Novikov (1955) showed that there exists a finitely presented group with undecidable word problem. Automatic, combable and 1-relator groups are known to have solvable word problem.

1.8 What is distortion?

Consider a finitely generated group G, let C_G be its Cayley graph and d_G the word metric. Let H be a finitely generated subgroup of G with Cayley graph C_H and word metric d_H.

Definition 1.8.1. *The* ***distortion function of*** *H* ***in*** *G is given by*

$$\delta_H^G(n) := \max\left\{ d_H(1_G, h) \,\middle|\, h \in H, d_G(1_G, h) \leq n \right\}.$$

Subgroup distortion function compares the size of an element in a Cayley graph of G with its size in a Cayley graph of H.

If H is undistorted in G, then $\delta_H^G(n) \simeq n$. Subgroup distortion is an invariant in a sense that up to $\simeq$, $\delta_H^G(n)$ is independent of the choice of the finite generating sets for H and G. Distortion has proved to be an important invariant. For example, if H is an undistorted subgroup of a hyperbolic group G, then H is itself hyperbolic.

1.9 What is filling length?

While the above mentioned invariants have been studied extensively, the filling length function of a group, which is a natural "space" analog of the Dehn function, has not been studied to the same extent.

To simplify notation, we will frequently use the following terminology:

Definition 1.9.1. *For a group $G = \langle A \mid R \rangle$, we will say that $w \in G$ is a **trivial word** if w is a word on $A^{\pm 1}$ representing the identity in G. We will use ϵ to denote the empty word.*

The *filling length of a trivial word* w in a finitely presented group $G = \langle A \mid R \rangle$ is the minimal integer L such that w can be converted to the empty word ϵ through words of length at most L by applying defining relations and freely reducing/expanding. More precisely, $\mathrm{FL}(w)$ is the minimal L such that there exists a finite sequence $\{w_i\}$ starting with w and ending in ϵ in which each w_{i+1} is obtained from w_i by either

- applying a relator to $w_i = xuy$ to get $w_{i+1} = xvy$, where a cyclic conjugate of $uv^{-1} \in R^{\pm 1}$ and $x, y \in G$
- freely reducing $w_i = xaa^{-1}y$ to get $w_{i+1} = xy$, where $a \in A^{\pm 1}$ and $x, y \in G$
- freely expanding $w_i = xy$ to get $w_{i+1} = xaa^{-1}y$, where $a \in A^{\pm 1}$ and $x, y \in G$

and $|w_i| \leq L$ for all i.

Definition 1.9.2. *The **filling length function** $\mathrm{FL}_G \colon \mathbb{N} \to \mathbb{N}$ for a finitely presented group G is defined by*

$$\mathrm{FL}_G(n) := \max\left\{ \mathrm{FL}(w) \,\middle|\, w = 1 \text{ in } G \text{ and } |w| \leq n \right\}.$$

A *van Kampen diagram D* for a trivial word in a finitely presented group $\langle A \mid R \rangle$ is a connected and simply connected planar finite cell complex with each one-cell directed and labeled by a letter in A, one zero-cell on the boundary specified as a base-vertex, and such that the boundary of each two-cell reads off a word which is a cyclic permutation of an element of $R^{\pm 1}$.

The filling length function for a trivial word w can be described using van Kampen diagrams by considering elementary homotopies that reduce the boundary word w to the empty word ϵ at the base point [by successively collapsing either 1-cells (free reduction) or 2-cells (applying a cyclic permutation of an element in $R^{\pm 1}$)] and picking the one that keeps the maximal perimeter of the diagrams as small as possible. The details are in [12].

The filling length is one of several group invariants of finitely presented groups that can be defined via van Kampen diagrams – some others are Dehn function (number of 2-cells in D), diameter (maximal distance between vertices in 1-skeleton of D), and gallery length (diameter of the dual graph of the 1-skeleton of D), where D is a van Kampen diagram for trivial words in the group [12]. These invariants came out of consideration of the word problem for finitely presented groups since van Kampen diagrams illustrate geometrically how the trivial word can be decomposed into relations given in the presentation of the group.

Filling length of a trivial word w can also be defined topologically as the minimal length L such that there is a basepoint-preserving combinatorial null-homotopy of the boundary of a van Kampen diagram for w through loops of length at most L. Some variants of filling length function include free filling length (FFL) function (which is defined similarly, but the null-homotopy is not required to be basepoint-preserving), and free and fragmenting filling length (FFFL) (which further allows the loops to bifurcate).

In [13], Bridson and Riley constructed a finitely presented group with the property that for any N, there exists a trivial word w of length N for which $\mathrm{FL}(w)$ and $\mathrm{FFL}(w)$ differ dramatically. They also proved that the filling length function as well as the two generalizations (FFL and FFFL) are quasi-isometry invariants.

On the level of Turing machines we can think of filling length as a space function, while Dehn function corresponds to time. For details see [12].

In [43], Sapir, Birget and Rips showed that every Dehn function of a finitely presented group is equivalent to the time function of some (not necessarily deterministic) Turing machine. They also showed that for most Turing machines their time complexity function is equivalent to the Dehn function of some finitely presented group. More precisely, if D_4 is the set of all Dehn functions of finitely presented groups that are at least quartic, T_4 is set of functions which are equivalent to time functions of Turing machines, and T^4 is the set of superadditive functions which are fourth powers of time functions, then $T^4 \subseteq D_4 \subseteq T_4$.

In a recent paper [40], Olshanskii showed that the space complexity of an arbitrary deterministic Turing machine is equivalent to the free and fragmenting filling length function of some finitely presented group (hence providing a space analog of the Sapir, Birget and Rips result). So, there exist examples of groups with a wide variety of filling length functions. However, an understanding of filling length for standard classes of groups such as finitely presentable metabelian, polycyclic or solvable groups, remains elusive.

It is known that groups having at most quadratic Dehn function (for example, CAT(0) groups) necessarily have linear filling length [42]. All combable groups and nilpotent groups have linear filling length [12].

In 1981, Kharlampovich [35] constructed an example of a finitely presented solvable group with an unsolvable word problem. Therefore, the filling length function of this group must be huge; that is, it is not bounded above by any recursive function.

There are many groups for which the filling length function is not known. It is possible to construct examples of groups with various filling length functions, but for commonly encountered groups with known filling length function, it is either linear, exponential or super-exponential.

My goal is to find a well-known group whose filling length function grows faster than linear, but slower than exponential. An example that seems promising in this context is Baumslag and Remeslennikov's metabelian group, Γ_2 – a 2-dimensional version of the lamplighter group. If we allow an extra torsion relation, we obtain the group $\Gamma_2(m)$, which is known to have quadratic Dehn function, and so it has linear filling length [21]. The Dehn function for the non-torsion version of Baumslag and Remeslennikov's group, Γ_2, was recently proved to be exponential by Kassabov and Riley [34]. This implies that FL_{Γ_2} is at most exponential. We conjecture that the filling length function for Γ_2 is quadratic.

Conjecture 1. *The filling length function of* Γ_2 *is quadratic.*

In Chapter 5, we work toward proving a quadratic upper bound on the filling length function for Γ_2.

1.10 Significance of lamplighters

Here are some of the applications, properties, and cousins of the groups $\Gamma_n(R)$.

Instances of the family $\Gamma_n(R)$ and the related horocyclic products have featured in some major breakthroughs. Baumslag and Remeslennikov's construction of Γ_2 precipitated their theorem that every finitely generated metabelian group embeds in a finitely presented metabelian group [6, 41].

Grigorchuk, Linnell, Schick, and Żuk showed that the L^2-Betti numbers of Riemannian manifold with torsion-free fundamental group need not be integers (answering a strong version of a question of Atiyah [2]) by constructing a 7-dimensional such manifold with fundamental group $\Gamma_2(2)$ and third L^2-Betti number 1/3 in [31].

Diestel and Leader in [22] put forward the horocyclic product of an infinite 2-branching and an infinite 3-branching tree as a candidate to answer a question of Woess as to whether there is a vertex-transitive graph not quasi-isometric to a Cayley graph. Eskin, Fisher and Whyte [27] verified this. (Accordingly, the 1-skeleta of $\mathcal{H}_n(\mathbb{Z}/m\mathbb{Z})$ of Section 3.2 are termed *Diestel–Leader graphs* in [3].) Woess recently wrote an account of this breakthrough and its history [47].

Eskin, Fisher and Whyte [27] also classified lamplighter groups up to quasi-isometry. Dymarz [23] used lamplighter examples to show that quasi-isometric finitely generated groups need not be bilipshitz equivalent. In both cases, the horocyclic product view-point was essential to their analyses.

A number of properties of these groups have been identified.

Bartholdi and Woess [4] studied the asymptotic behaviour of the N-step return probabilities of a simple random walk on a horocyclic product of two regular (finitely) branching trees. Woess [46] described positive harmonic functions in terms of the boundaries of the two trees. Bartholdi, Neuhauser and Woess [3] identified the ℓ^2-spectrum of the simple random walk operator and studied the Poisson boundary for a large class of group-invariant random walks on horocyclic products of trees.

A group G is *of type* $\mathcal{F}_n$ if there exists a $K(G, 1)$ (an Eilenberg–Maclane space — a CW-complex whose fundamental group is G and which has contractible universal cover) with finite n-skeleton. All groups are $\mathcal{F}_0$, being finitely generated is equivalent to $\mathcal{F}_1$, and being finitely presentable is equivalent to $\mathcal{F}_2$. Bartholdi, Neuhauser and Woess [3] showed that $\mathcal{H}_n(\mathbb{Z}/m\mathbb{Z})$ is $(n-1)$-connected but not n-connected and deduced that $\Gamma_n(m)$ is of type $\mathcal{F}_n$ but not of type $\mathcal{F}_{n+1}$ when $1, \ldots, n-1$ are invertible in $\mathbb{Z}/m\mathbb{Z}$. Kropholler and Mullaney [36] used Bieri–Neumann–Strebel invariants to prove that $\Gamma_n(\mathbb{Z}[1/(n-1)!]) \rtimes \mathbb{Z}$ (as defined in

Section 1.1) is of type $\mathcal{F}_n$ but not of type $\mathcal{F}_{n+1}$. The Bieri–Stallings groups [11, 44] exhibit the same finiteness properties, and bear close comparison with the family $\Gamma_n(2)$ in that both are level sets in products of trees (just the height functions concerned differ).

Cleary and Taback [20] showed that, with respect to a standard generating set, $\Gamma_1(2)$ has *unbounded dead-end depth*: there is no $L > 0$ such that for every group element g, there is a group element further from the identity than g that is within a distance less than L from g. (Cf. Question 8.4 in [10], which Erschler observed can be resolved using $\Gamma_1(2)$.) Cleary and Riley [19] exhibited $\Gamma_2(2)$ as the first finitely presentable group known to have the same property. By finding a combinatorial formula for the word metric, Stein and Taback [45] showed that, with respect to generating sets for which the Cayley graphs are horocyclic products, $\Gamma_n(m)$ have no regular language of geodesics and have unbounded dead-end depth. We understand that Cleary has unpublished work and Davids and Taback have work in progress on whether or not almost convexity holds for $\Gamma_2(2)$ with respect to certain generating sets.

De Cornulier and Tessera showed that the Dehn function of $\Gamma_2(2)$ grows quadratically [21], and Kassabov and Riley [34] showed that the Dehn function of Γ_2 grows exponentially.

The horocyclic product construction can be applied to any family of spaces with height functions. A fruitful alternative to $\mathcal{T}_{\mathbb{Z}/m\mathbb{Z}}$ (as defined in Section 3.1) is the hyperbolic plane $\mathbb{H}^2$, viewed as the upper half of the complex plane, with height function given by $\log_q(\operatorname{Im} z)$ for some fixed $q > 1$. The horocyclic product of n copies of $\mathbb{H}^2$ (each with the same $q > 1$) is a manifold Sol_{2n-1}. (Varying

q is a dilation.) The horocyclic product of $\mathcal{T}_{\mathbb{Z}/p\mathbb{Z}}$ and $\mathbb{H}^2$ with parameter q is termed *treebolic space* in [9]. When $p = q$ it is shown to be a model space for the Baumslag–Solitar group $\langle a, b \mid b^{-1}ab = a^p \rangle$ — that is, the group acts on the space cocompactly by isometries.

These constructions and their parallels have been pursued particularly by Woess and his coauthors [3, 4, 9, 14, 16, 46], focusing on stochastic processes, harmonic maps, and boundaries. He gives an introduction in [47]. Additionally, the boundaries of these various horocyclic products admit similar analyses, which is why the work of Eskin, Fisher and Whyte [25, 27, 26, 28] encompasses both Sol_3 and lamplighter groups. Dymarz [24] also exploits the parallels.

Strikingly, most $\Gamma_n(m)$ are automata groups as set out in [3, Remark 4.9] (building on the $n = 1$ case in [38]).

1.11 Outline of the results

The main theorem which we prove in full generality in Section 3.6 is:

Theorem 1. *For $n = 1, 2, \ldots$, if $2, \ldots, n-1$ are invertible in R, then the 1-skeleton of $\mathcal{H}_n(R)$ is the Cayley graph of $\Gamma_n(R)$ with respect to the generating set*

$$\left\{ (r, \mathbf{e}_j),\ (r, \mathbf{e}_j)(r, \mathbf{e}_k)^{-1} \;\middle|\; r \in R,\ 0 \le j, k \le n-1 \text{ and } j < k \right\}.$$

In particular, if $|R| < \infty$, then $\Gamma_n(R)$ acts geometrically on $\mathcal{H}_n(R)$.

For R finite, this theorem is due to Bartholdi, Neuhauser and Woess [3]. (Instead of working with $A_n(R)$ and insisting that $2, \ldots, n-1$ are invertible in R, they work more generally with polynomials $R[x, (\ell_0 + x)^{-1}, \dots, (\ell_{n-1} + x)^{-1}]$ such that the pairwise differences $\ell_i - \ell_j$ are all invertible. Our treatment could be extended to this generality if desired.) We aim here to give as elementary, explicit and transparent a proof as possible for general $\Gamma_n(R)$. The proof in [3] proceeds via manipulations of formal Laurent series. We will work with 'lamplighter models' as far as possible — the cases $n = 1$ and $n = 2$ — and use these models to illuminate a proof in the general case which involves suitably manipulating polynomials.

In Section 2.1, we discuss various presentations for Γ_1 and Γ_2, including the one which reflects the horocyclic product structure. That presentation then features in this embellishment of an $n = 2$ case of Theorem 1 with $R = \mathbb{Z}$ (which we prove in Section 3.5):

Theorem 2. *$\mathcal{H}_2(\mathbb{Z})$ is the Cayley 2-complex with respect to this presentation of Γ_2:*

$$\left\langle\ \lambda_i, \mu_i, \nu_i\ (i \in \mathbb{Z})\ \middle|\ \lambda_i = \nu_i\mu_i,\ \lambda_{i+j} = \mu_i\nu_j\ (i, j \in \mathbb{Z})\ \right\rangle.$$

In Section 2.1, we also show that another way to present Γ_2 is

$$\left\langle\ a, s, t\ \middle|\ [a, a^t] = 1,\ [s, t] = 1,\ a^s = aa^t\ \right\rangle.$$

And in Chapter 4 we prove —

Theorem 3. *The subgroup $\langle a \rangle$ is undistorted in Γ_2.*

In Chapter 5, we work toward an upper bound on the filling length function of Γ_2. We conjecture that the filling length is quadratic and reduce the question of finding an upper bound on the filling length of Γ_2 to a combinatorial question about configurations (Theorem 5 and Open Question 5.1.3).

CHAPTER 2

BACKGROUND ON RANK-1 AND RANK-2 LAMPLIGHTERS

2.1 Presentations

In this section we give some presentations of Γ_1, $\Gamma_1(m)$ and Γ_2 including the one that reflects their descriptions as horocyclic products of trees. Our presentations for Γ_2 include the one which we will prove in Section 3.5 to have Cayley 2-complex $\mathcal{H}_2(\mathbb{Z})$.

Recall that our conventions are that $[a, b] = a^{-1}b^{-1}ab$ and $a^{nb} = ba^n b^{-1}$ for group elements a, b and integers n. Our group actions are on the right.

Proposition 2.1.1. *Presentations for the group*

$$\Gamma_1 \; = \; \mathbb{Z} \wr \mathbb{Z} \; \cong \; \mathbb{Z}[x, x^{-1}] \rtimes \mathbb{Z} \; = \; \left\{ \left(\begin{array}{cc} x^k & f \\ 0 & 1 \end{array} \right) \middle| \; k \in \mathbb{Z}, \; f \in \mathbb{Z}[x, x^{-1}] \right\}$$

include

(i) $\left\langle a, t \;\middle|\; \left[a, a^{t^k}\right] = 1 \; (k \in \mathbb{Z}) \right\rangle$,

(ii) $\left\langle \lambda, \mu \;\middle|\; \lambda^k \left(\lambda^{-1}\mu\lambda^{-1}\right)^k = \mu^k \lambda^{-k} \; (k \in \mathbb{Z}) \right\rangle$,

(iii) $\left\langle \lambda_i \; (i \in \mathbb{Z}) \;\middle|\; {\lambda_i}^k {\lambda_j}^{-k} = {\lambda_{-j}}^k {\lambda_{-i}}^{-k} \; (i, j, k \in \mathbb{Z}) \right\rangle$.

These are related via $\lambda = t$, $\mu = at$, and $\lambda_i = a^i t$.

Proof. As an abelian group,

$$\mathbb{Z}[x, x^{-1}] \; = \; \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} \; = \; \langle \, a_i \; (i \in \mathbb{Z}) \mid [a_i, a_j] = 1 \; \forall i, j \, \rangle.$$

So $\mathbb{Z}[x, x^{-1}] \rtimes \mathbb{Z} = \langle t, a_i \ (i \in \mathbb{Z}) \mid t a_i t^{-1} = a_{i+1},\ [a_i, a_j] = 1 \ \forall i, j \rangle$, which simplifies with $a = a_0$ to give *(i)*.

For *(ii)*, it suffices to show that $\left\langle\ a, t \ \middle|\ \left[a, a^{t^k}\right] = 1 \ (k \in \mathbb{Z}) \ \right\rangle$ can be re–expressed as

$$\left\langle\ a, t \ \middle|\ t^k (t^{-1} a)^k = (at)^k t^{-k} \ (k \in \mathbb{Z}) \ \right\rangle,$$

since the latter becomes *(ii)* via $\lambda = t$ and $\mu = at$. Well, $t^k(t^{-1}a)^k$ and $(at)^k t^{-k}$ freely equal $(t^{k-1} a t^{-(k-1)}) \ldots (tat^{-1})\, a$ and $a\, (tat^{-1}) \ldots (t^{k-1} a t^{-(k-1)})$, respectively, and a straight–forward induction shows that the family $\left\{a^{t^k} a = a a^{t^k}\right\}_{k \in \mathbb{Z}}$ is equivalent to

$$\left\{ a^{t^k} \cdots a^t a = a a^t \cdots a^{t^k},\ a^{t^{-k}} \cdots a^{t^{-1}} a = a a^{t^{-1}} \cdots a^{t^{-k}} \right\}_{k > 0}.$$

Finally we establish *(iii)*. If $\lambda_i = a^i t$ then λ_i must correspond to $\begin{pmatrix} x & i \\ 0 & 1 \end{pmatrix}$ and so $\lambda_i{}^k$ to $\begin{pmatrix} x^k & i(1 + \cdots + x^{k-1}) \\ 0 & 1 \end{pmatrix}$ and $\lambda_i{}^{-k}$ to $\begin{pmatrix} x^{-k} & -i(x^{-k} + \cdots + x^{-1}) \\ 0 & 1 \end{pmatrix}$. From there it is easy to check that the relations $\lambda_i{}^k \lambda_j{}^{-k} = \lambda_{-j}{}^k \lambda_{-i}{}^{-k}$ correspond to valid matrix identities

$$\begin{pmatrix} x^k & i(1 + \cdots + x^{k-1}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{-k} & -j(x^{-k} + \cdots + x^{-1}) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (i-j)(1 + \cdots + x^{k-1}) \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} x^k & -j(1 + \cdots + x^{k-1}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{-k} & i(x^{-k} + \cdots + x^{-1}) \\ 0 & 1 \end{pmatrix},$$

and so must be consequences of the relations $\left[a, a^{t^k}\right] = 1 \ (k \in \mathbb{Z})$.

Conversely, given that $\lambda_0 = \lambda = t$ and $\lambda_1 = \mu = at$, we find that $\lambda_{-1} = a^{-1} t = \lambda \mu^{-1} \lambda$, and so the relations $\lambda^k \left(\lambda^{-1} \mu \lambda^{-1}\right)^k = \mu^k \lambda^{-k}$ of *(ii)* are $\lambda_i{}^k \lambda_j{}^{-k} = \lambda_{-j}{}^k \lambda_{-i}{}^{-k}$ in the case $i = 0$ and $j = -1$. $\square$

On introducing torsion, adding the relation $a^m = 1$ to presentation *(i)* of Proposition 2.1.1, we get presentations for $\Gamma_1(m)$. These can be reorganized in the manner of presentations *(ii)* and *(iii)*, and in the case $m = 2$ can be simplified significantly:

Proposition 2.1.2.

$$\begin{aligned}
\Gamma_1(2) &= (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z} = \left\langle \lambda, \mu \;\middle|\; \left(\lambda^k \mu^{-k}\right)^2 = 1 \ (k \in \mathbb{Z}) \right\rangle, \\
\Gamma_1(m) &= (\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z} \\
&= \left\langle \lambda_0, \ldots, \lambda_{m-1} \;\middle|\; {\lambda_i}^k {\lambda_j}^{-k} = {\lambda_{-j}}^k {\lambda_{-i}}^{-k},\ \left({\lambda_i}^k {\lambda_j}^{-k}\right)^m = 1 \ (i, j \in \mathbb{Z}/m\mathbb{Z},\ k \in \mathbb{Z}) \right\rangle,
\end{aligned}$$

where $m \geq 2$, $\lambda = t$, $\mu = at$, and $\lambda_i = a^i t$.

Proof. The presentation for $\Gamma_1(2)$ comes from simplifying presentation *(ii)* of Proposition 2.1.1 using the relation $a^2 = 1$, which is equivalent to $\lambda^{-1}\mu\lambda^{-1} = \mu^{-1}$. The family $\lambda^k(\lambda^{-1}\mu\lambda^{-1})^k = \mu^k\lambda^{-k}$ becomes the family $(\lambda^k\mu^{-k})^2 = 1$. The case $k = 1$ provides the relation $a^2 = 1$.

For $\Gamma_1(m)$, consider adding the family of relations $\left({\lambda_i}^k {\lambda_j}^{-k}\right)^m = 1$ for all $i, j, k \in \mathbb{Z}$ to presentation *(iii)* of Proposition 2.1.1. In particular this adds the relation $a^m = 1$, which is the case: $\left(\lambda_1 {\lambda_0}^{-1}\right)^m = \left(at\,t^{-1}\right)^m = 1$. In the resulting group $\lambda_i = \lambda_j$ when $i = j$ modulo m since then $a^i t = a^j t$ because $a^m = 1$. This group must be $\Gamma_1(m)$ because all the remaining added relations hold in $\Gamma_1(m)$, after all when $k > 0$ (and similarly when $k < 0$),

$$\begin{aligned}
\left({\lambda_i}^k {\lambda_j}^{-k}\right)^m &= \left((a^i t)^k (a^j t)^{-k}\right)^m \\
&= \left(a^i (t a^i t^{-1}) \cdots (t^{k-2} a^i t^{-(k-2)})(t^{k-1} a^{i-j} t^{-(k-1)})(t^{k-2} a^{-j} t^{-(k-2)}) \cdots (t a^{-j} t^{-1}) a^{-j}\right)^m
\end{aligned}$$

which is 1 because $\left(a^{t^p}\right)^m = 1$ and a^{t^p} and a^{t^q} commute in $\Gamma_1(m)$ for all $p, q \in \mathbb{Z}$. $\square$

Proposition 2.1.3. *Presentations of*

$$\Gamma_2 = \left\{ \left(\begin{array}{cc} x^k(1+x)^l & f \\ 0 & 1 \end{array} \right) \middle| \; k,l \in \mathbb{Z}, \; f \in \mathbb{Z}\left[x, x^{-1}, (1+x)^{-1}\right] \right\}$$

include

(i) $\left\langle \, a, s, t \;\middle|\; [a, a^t] = 1, \; [s,t] = 1, \; a^s = aa^t \, \right\rangle$,

(ii) $\left\langle \, \mu, \nu, c, d \;\middle|\; [\mu, \nu] = 1, \; \mu^{-1}c^2\nu = c, \; \nu^{-1}d^2\mu = d \, \right\rangle$,

(iii) $\left\langle \, \lambda_i, \mu_i, \nu_i \; (i \in \mathbb{Z}) \;\middle|\; \lambda_i = \nu_i\mu_i, \; \lambda_{i+j} = \mu_i\nu_j \; (i, j \in \mathbb{Z}) \, \right\rangle$.

These are related by

$$a \mapsto \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), \qquad t \mapsto \left(\begin{array}{cc} x & 0 \\ 0 & 1 \end{array} \right), \qquad s \mapsto \left(\begin{array}{cc} 1+x & 0 \\ 0 & 1 \end{array} \right),$$

$\mu = s$, $\nu = t^{-1}s$, $c = at$, $d = t^{-1}a$, and $\lambda_i = a^it$, $\mu_i = a^is$ (and hence $\nu_i = \lambda_i{\mu_i}^{-1} = a^its^{-1}a^{-i}$).

The generators λ_i, μ_i, and $\nu_i := \lambda_i{\mu_i}^{-1}$ agree with those employed in Section 3.4. After all,

$$\lambda_i = a^it \mapsto \left(\begin{array}{cc} x & i \\ 0 & 1 \end{array} \right) \qquad \text{and} \qquad \mu_i = a^is \mapsto \left(\begin{array}{cc} 1+x & i \\ 0 & 1 \end{array} \right),$$

which are alternative ways of expressing $(i, \mathbf{e}_0)$ and $(i, \mathbf{e}_1)$.

Presentation *(i)* and the given matrix representation are due to Baumslag in [5] and our proof below that they agree is an embellishment of the argument in his paper. Presentation *(ii)* is striking as it shows that Γ_2 maps onto a free–product with amalgamation of two $\mathrm{BS}(1,2)$ groups (via identifying μ and ν).

The generators of presentation *(iii)* are those we will use in Sections 3.4 and 3.6 to relate Γ_2 to a horocyclic product of trees. In Section 4 of [33], presentations of similar matrix groups are given (e.g. in Section 4.3.1) using techniques that are similar to those that follow and are based on ideas in [5] and [17].

In the course of proving Proposition 2.1.3 we will also establish:

Lemma 2.1.4 (*Normal form*)**.** *Elements g in Γ_2, presented as (i), are represented by a unique word*

$$w_g \;=\; a^{m_1 t^{k_1}} \cdots a^{m_K t^{k_K}} \; a^{n_1 s^{l_1}} \cdots a^{n_L s^{l_L}} \; s^l t^k \tag{2.1}$$

with $k_1, \ldots, k_K, l_1, \ldots, l_L, l, k, L, K \in \mathbb{Z}$ and $m_1, \ldots, m_K, n_1, \ldots, n_L \in \mathbb{Z} \smallsetminus \{0\}$ satisfying $k_1 < \cdots < k_K$ and $l_1 < \ldots < l_L < 0$.

Proof of Proposition 2.1.3 and Lemma 2.1.4. Let us establish the existence part of Lemma 2.1.4. Suppose w is any word on a, s, t representing g. First convert w to a word of the form $\prod_i a^{s^{p_i} t^{q_i}} s^l t^k$ by inserting suitable words on $\{s^{\pm 1}, t^{\pm 1}\}$ after each a and then using the relation $[s, t] = 1$. Then eliminate all the positive p_i by expressing $a^{s^{p_i}}$ as a product of terms like a^{t^j} using the relation $a^s = aa^t$. In Γ_2, $[a, a^{t^n}] = 1$ for all $n \geq 0$ as can be seen by an induction via

$$1 \;=\; \left[a, a^{t^n}\right]^s \;=\; \left[a^s, (a^s)^{t^n}\right] \;=\; \left[aa^t, a^{t^n} a^{t^{n+1}}\right] \;=\; \left[a, a^{t^{n+1}}\right].$$

(We see here that the relation $a^s = aa^t$, which Baumslag calls *mitosis*, is the key to coding the infinite family of defining relators $\left[a, a^{t^n}\right] = 1$ ($n \in \mathbb{Z}$) in a finite presentation.) So, as a^{s^i} can be expressed as a product of terms of the form a^{t^j} ($j \in \mathbb{Z}$), elements of the set $\left\{a^{s^i}, a^{t^j}, \mid i, j \in \mathbb{Z}\right\}$ pairwise commute in Γ_2. So we can rearrange terms to get the form of w_g.

Next we observe that the map ϕ from the group presented by *(i)* to the given matrix group, defined for a, s and t as indicated in the proposition, is well-defined and is a homomorphism: the defining relations correspond to identities which hold in the matrix group. It maps a group element g represented by the word w_g of Lemma 2.1.4 to

$$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x^k(1+x)^l & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^k(1+x)^l & f \\ 0 & 1 \end{pmatrix}, \tag{2.2}$$

where

$$f \;=\; m_1 x^{k_1} + \cdots + m_K x^{k_K} + n_1(1+x)^{l_1} + \cdots + n_L(1+x)^{l_L}.$$

So ϕ is surjective. We will show in Remark 2.3.4 of Section 2.2 that $\left\{x^i, (1+x)^j \mid i, j \in \mathbb{Z},\ j < 0\right\}$ is a basis for $\mathbb{Z}[x, x^{-1}, (1+x)^{-1}]$. So ϕ is also injective and the normal form words of Lemma 2.1.4 each represent different group elements. So *(i)* is a presentation of Γ_2.

The translation between presentations *(i)* and *(ii)* comes from that the relations $[s,t] = 1$ and $[\mu, \nu] = 1$ are equivalent, and, in the presence of that commutator, $\mu^{-1}c^2\nu = c$ and $\nu^{-1}d^2\mu = d$ are equivalent to $a^s = aa^t$ and $a^s = a^t a$, respectively.

Presentations *(i)* and *(iii)* agree as follows. When $i = j = 0$, the relation $\lambda_{i+j} = \mu_i \nu_j$ becomes $[s,t] = 1$, and, in the presence of $[s,t] = 1$, when $i = -j = 1$, it gives $a^s = a^t a$, and when $-i = j = 1$, it gives $a^s = aa^t$. Moreover, in terms of a, s, t the relation $\lambda_{i+j} = \mu_i \nu_j$ is $a^{i+j}t = a^i s a^j t s^{-1} a^{-j}$, which holds in Γ_2 because $a^i s a^j t s^{-1} a^{-j} t^{-1} a^{-i-j} = a^i (sas^{-1})^j t a^{-j} t^{-1} a^{-i-j} = a^i (aa^t)^j a^{-jt} a^{-i-j} = a^{i+j} a^{jt} a^{-jt} a^{-i-j} = 1$. □

For presentations of the groups $\Gamma_n(m)$ in general see Theorem 4.7 in [3].

2.2 A lamplighter model for $\Gamma_2(R)$

Recall that

$$\Gamma_2(R) \;=\; R\left[x, x^{-1}, (1+x)^{-1}\right] \rtimes \mathbb{Z}^2$$

where, if the $\mathbb{Z}^2$-factor is $\langle t, s\rangle$, the actions of t and s are multiplication by x and $1+x$, respectively.

We will use a *lamplighter description* of Γ_2 developed from [3] and [19]. A lamplighter is located at a *lattice point* in a skewed rhombic $\mathbb{Z}^2 = \langle t, s\rangle$ grid, as in Figure 2.1. (The lattice points are the vertices of the tessellation of the plane by unit equilateral triangles.) Each vertex has six closest neighbors — one in each of what we will call the s-, s^{-1}-, t-, t^{-1}-, st^{-1}- and $s^{-1}t$-directions — and can be specified using t- and s-coordinates. A *configuration* $\mathcal{K}$ is a finitely supported assignment of an element of R to each lattice point.

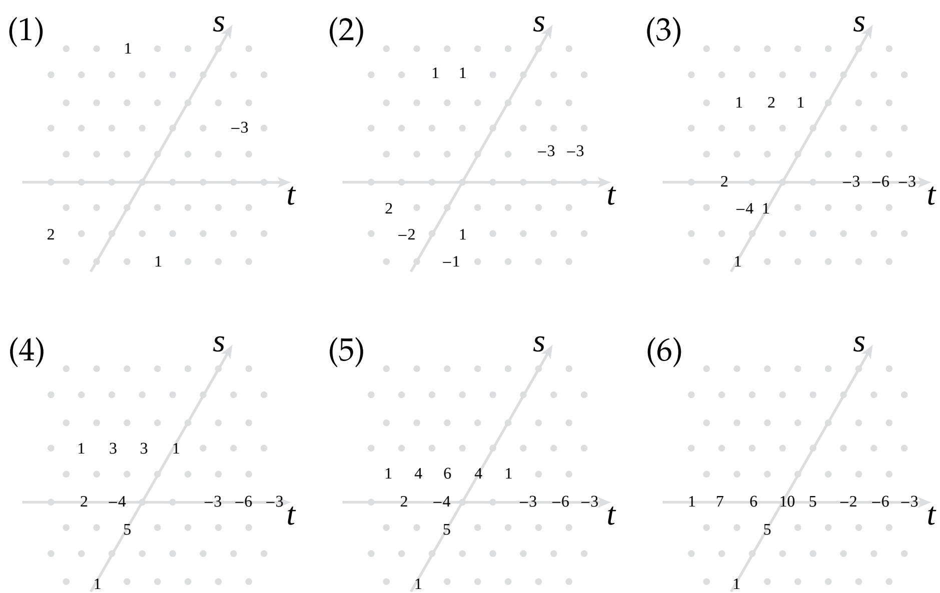

Figure 2.1: An example of propagation to a configuration supported on $L_{0,0}$ (namely, the t-axis and the negative half of s-axis).

Figure 2.1 shows six examples of configurations where $R = \mathbb{Z}$. Vertices where no element of R is shown should be understood to be assigned zeros. As an example of the terminology in action, the integer at $(-2, 1)$ in grid (5) is 4 and its neighbors in the s-, s^{-1}-, t-, t^{-1}-, st^{-1}- and $s^{-1}t$-directions are 0, 2, 6, 1, 0, and -4, respectively.

We define an equivalence relation $\sim$ on configurations by setting $\mathcal{K} \sim \mathcal{K}'$ when there is a finite sequence of configurations starting with $\mathcal{K}$ and ending with $\mathcal{K}'$ in which each configuration differs from the next only in one triangle of adjacent ring elements which is ${}_b{}^a{}_c$ in one and is ${}_{b+r}{}^{a-r}{}_{c+r}$ for some $r \in R$ in the other. The six integer-configurations shown in Figure 2.1 are all equivalent, for example. A *trivial configuration* is a configuration that is equivalent to the all-zero configuration $\mathcal{K}_\epsilon$.

An element $f = \sum_{i,j \in \mathbb{Z}} n_{i,j} x^i (1+x)^j$ of $R\left[x, x^{-1}, (1+x)^{-1}\right]$ corresponds to the configuration which has $n_{i,j}$ at (i, j) for all $i, j \in \mathbb{Z}$. A motivating result for these definitions is—

Lemma 2.2.1. *Two such polynomials represent the same element of* $R\left[x, x^{-1}, (1+x)^{-1}\right]$ *if and only if their corresponding configurations are equivalent.*

Proof. The relations in $R\left[x, x^{-1}, (1+x)^{-1}\right]$ are generated by $(1+x)$ being the sum of the terms 1 and x in a manner that corresponds to the relations between configurations being generated by altering triangles of entries. Indeed, multiplying $(1+x) = 1 + x$ through by $rx^i(1+x)^j$ gives $rx^i(1+x)^{j+1} = rx^i(1+x)^j + rx^{i+1}(1+x)^j$, which corresponds to ${}_b{}^{a+r}{}_c \sim {}_{b+r}{}^a{}_{c+r}$ at a suitably located triangle of entries in a configuration. $\square$

The element $g = (f, (k, l)) \in \Gamma_2(R)$ corresponds to the lamplighter being located at (k, l) and the configuration being that associated to f.

A word on a, s, t as per presentation *(i)* of Proposition 2.1.3 for Γ_2 represents a group element whose lamplighter description can be found as follows. Start with the lamplighter located at $(0, 0)$ and the configuration entirely zeros. Working through w from left to right, increment the integer at the lamplighter's location by ± 1 on reading an $a^{\pm 1}$, move the lamplighter one step to the right or left (the t- or t^{-1}-direction) on reading a t or t^{-1}, respectively, and move the lamplighter one step to the adjacent vertex in the s- or s^{-1}-direction on reading an s or s^{-1}, respectively. We will denote the resulting configuration $\mathcal{K}_w$ and say that $\mathcal{K}_w$ is the *configuration associated to* the word $w \in \Gamma_2$.

The normal-form words of Lemma 2.1.4 read off lamplighter descriptions of group elements in which the configurations are supported on $L_{0,0}$ (that is, the t-axis and the negative half of the s-axis). If a group element g positions the lamplighter far from $L_{0,0}$, then the configuration supported on $L_{0,0}$ representing g will differ dramatically from that representing $ga^{\pm 1}$, since the effect of propagating ± 1 toward $L_{0,0}$ compounds in the manner of Pascal's triangle.

An appealing feature of this model is how it elucidates the way in which $\Gamma_1(R)$ sits inside $\Gamma_2(R)$ (for example, $\mathbb{Z} \wr \mathbb{Z}$ sits inside Γ_2) as the elements for which the lamplighter is on the t-axis and the configuration is equivalent to one that is supported on the t-axis.

2.2.1 An invariant when $R = \mathbb{Z}$

It is worth pointing out that the way that the lamplighter model and the equivalence of configurations are defined leads to a natural invariant on the configurations. We considered this invariant as a potential tool for resolving the open questions of Section 5.1 (but without success).

For a configuration $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ (that is, a finitely supported assignment of integers to the vertices of the rhombic grid), define

$S(f) := \sum_{j\in\mathbb{Z}} 2^j \sum_{i\in\mathbb{Z}} f(i, j).$

In other words: sum the entries in each row, then sum the results for the rows after multiplying by $2^{\text{row height}}$.

This is an invariant in that if f and g are equivalent configurations, then $S(f) = S(g)$, the reason being that subtracting 1 from somewhere and adding 1 to each of the two numbers immediately below does not change the quantity.

2.2.2 Norm of a configuration

Definition 2.2.2. *The* ***norm of a configuration*** $\mathcal{K}$ *is the sum of the absolute values of the entries in* $\mathcal{K}$*, denoted by* $|\mathcal{K}| = \sum_{i,j\in\mathbb{Z}} |f(i, j)|$.

This is not an invariant: two equivalent configuration may have different norms.

2.3 Half-planes and propagation

We now give a few definitions and some properties which we will use in Section 3.4 to prove Theorem 1 when $n = 2$.

Definition 2.3.1. *Using t- and s-coordinates, define the* ***half-planes***

$$\begin{aligned} H_m^\infty &:= \{(p,q) \mid p + q \geq m\}\,, \\ H_m^0 &:= \{(p,q) \mid p \leq m\}\,, \\ H_m^1 &:= \{(p,q) \mid q \leq m\}\,. \end{aligned}$$

For example, Figure 2.2 displays $H_{h_0+h_1}^\infty$, $H_{h_0-1}^0$ and $H_{h_1-1}^1$.

Our analyses will involve finding opportune representatives in the equivalence classes of given configurations. Indeed, we will in some instances (in Section 3.4) be concerned only with the part of a configuration in some half-plane. The following definition will then be useful.

Propagating to level ℓ in H_m^∞ means converting a configuration to an equivalent configuration such that the only non-zero entries in H_m^∞ are on the line with s-coordinate ℓ. This can always be done by moving the entries in H_m^∞ that are above that line by using ${}_b{}^a{}_c \sim {}_{a+b}{}^0{}_{a+c}$ and moving those below by using ${}_b{}^a{}_c \sim {}_{b-c}{}^{a+c}{}_0$. *Propagating to level ℓ in H_m^0* means converting to an equivalent configuration such that the only non-zero entries in H_m^0 are on the line with s-coordinate ℓ. This can be done using ${}_b{}^a{}_c \sim {}_{a+b}{}^0{}_{a+c}$ and ${}_b{}^a{}_c \sim {}_0{}^{a+b}{}_{c-b}$ for entries above and below the line, respectively. And *propagating to level ℓ in H_m^1* means converting to an equivalent configuration such that the only non-zero entries in H_m^1 are on the line with t-coordinate ℓ. This can be done using ${}_b{}^a{}_c \sim {}_0{}^{a+b}{}_{c-b}$ and

${}_b{}^a{}_c \sim {}_{b-c}{}^{a+c}{}_0$ for entries on the left and the right of the line, respectively.

In each case, propagation produces a finitely supported sequence, namely the entries on level ℓ of the half-plane concerned. For example, in Figure 2.1 propagating the integer-configuration (1) to level 0 in H^∞_0, H^0_{-1} and H^1_{-1} yields configurations which can be read off (6), specifically, $10, 5, -2, -6, -3, 0, 0, \ldots$ in H^∞_0, $6, 7, 1, 0, 0, \ldots$ in H^0_{-1}, and $5, 0, 1, 0, 0, \ldots$ in H^1_{-1}. And in Figure 3.5, the configuration in the centre grid propagated to level 0 yields $5, 3, 4, 2, 0, 0, \ldots$ in H^∞_3, $18, 5, 1, 0, 0, \ldots$ in H^0_0, and $2, 3, 0, 1, 0, 0, \ldots$ in H^1_1.

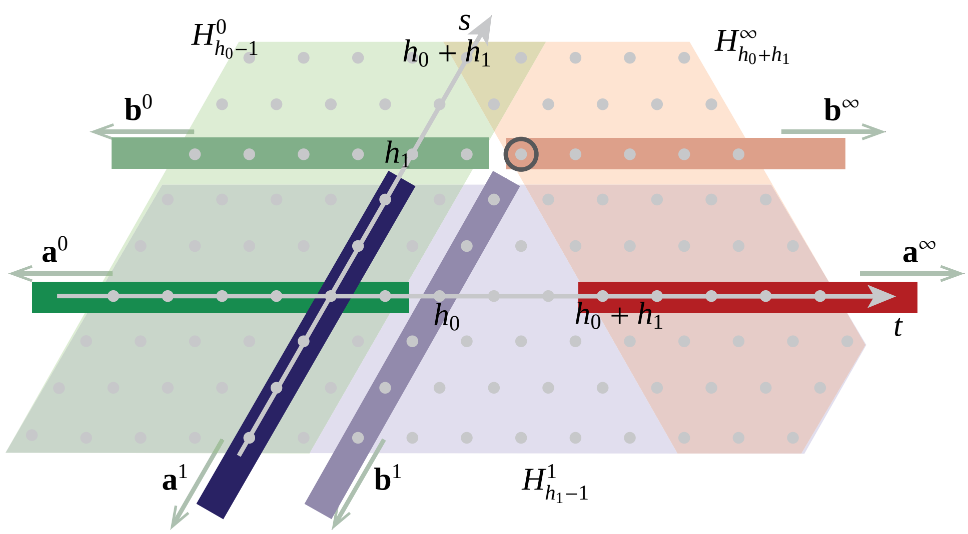

Figure 2.2: Propagation in the half-planes $H^\infty_{h_0+h_1}$, $H^0_{h_0-1}$ and $H^1_{h_1-1}$. Propagation to levels h_1, h_1 and h_0, respectively, is illustrated using lighter colors. Propagation to level 0 in each half-plane is illustrated using darker colors.

The following properties of propagation may at first seem surprising because it is not immediately apparent that the entries outside H^*_m are of no consequence for the sequence produced by propagation.

Lemma 2.3.2. *For $* = \infty, 0, 1$ and for all $\ell, \ell' \in \mathbb{Z}$ the following hold.*

(i) *Any two equivalent configurations which are both zero everywhere in* H^*_m *aside from level* ℓ*, are in fact equal on level* ℓ *in* H^*_m*. (So propagation of a configuration to level* ℓ *in* H^*_m *determines a unique sequence and propagating any two equivalent configurations to level* ℓ *in* H^*_m *produces the same sequence.)*

(ii) *If propagating a configuration* $\mathcal{K}$ *to level* ℓ *in* H^*_m *produces the sequence* $a_1, a_2, \ldots$*, then* a_p*, for* $p = 1, 2, \ldots$*, depends only on the restriction of* $\mathcal{K}$ *to*

$$\begin{cases} H^{\infty}_{m+p-1} & \textit{if } * = \infty \\ H^{0}_{m-p+1} & \textit{if } * = 0 \\ H^{1}_{m-p+1} & \textit{if } * = 1. \end{cases}$$

(iii) *The following defines a bijection on the set of finitely supported integer sequences. Given such a sequence, take the configuration which is everywhere-zero aside from level* ℓ *of* H^*_m *where one reads the sequence, and obtain a new sequence by propagating to level* ℓ' *in* H^*_m*. Indeed, this map is inverted by propagating back to level* ℓ*.*

Proof. We will explain only the case $* = \infty$. The cases $* = 0, 1$ are similar.

For (i), recall that the equivalence relation on configurations is generated by equivalences in which a triangle of only three adjacent entries is altered. Such alterations do not change the sequence obtained by propagating to level ℓ in H^{∞}_m by moving those above the level using ${}_b{}^a{}_c \sim {}_{a+b}{}^0{}_{a+c}$ and moving those below by using ${}_b{}^a{}_c \sim {}_{b-c}{}^{a+c}{}_0$. Consideration of the directions in which entries are moved by these two types of equivalences leads to (ii). For (iii) observe that the result is true when $|\ell - \ell'| = 1$. □

Corollary 2.3.3. *For all $k, l \in \mathbb{Z}$, each configuration is equivalent to a unique configuration supported on*

$$L_{k,l} \;:=\; \{\,(i,l) \mid i \in \mathbb{Z}\,\} \cup \{\,(k, l-1), (k, l-2), \ldots\,\},$$

specifically, that obtained by simultaneously propagating to level l in H^{∞}_{k+l} and H^{0}_{k-1} and to level k in H^{1}_{l-1}.

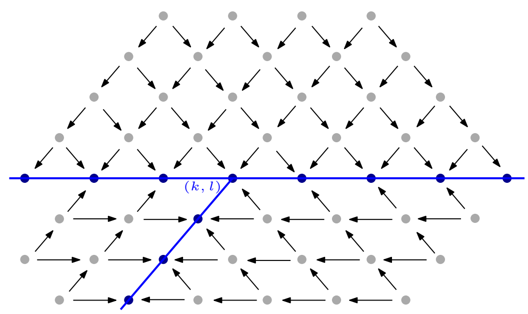

Figure 2.3: Propagation toward $L_{k,l}$ (in blue).

Figure 2.3 shows the strategy for pushing entries toward $L_{k,l}$.

Figure 2.1 shows an example of such a propagation with $k = \ell = 0$, and the transition from the central grid to the top grid in Figure 3.5 is an example with $k = 1$ and $\ell = 2$.

Remark 2.3.4. In the light of Lemma 2.2.1, Corollary 2.3.3 with $k = l = 0$ states that

$$\left\{\, 1, x^j, x^{-j}, (1+x)^{-j} \,\middle|\, j = 1, 2, \ldots \right\}$$

is a basis for $R[x, x^{-1}, (1+x)^{-1}]$ over R. (This is a special case of Lemma 3.6.1.)

CHAPTER 3

HOROCYCLIC PRODUCT OF TREES

This chapter is devoted to proving Theorem 1 and Theorem 2. First, we give a precise definition of horocyclic product of trees, then in Sections 3.3 and 3.4 we give a geometric proof of Theorem 1 in rank-1 and rank-2 cases. In Section 3.5 we give a proof of Theorem 2. We then proceed to proving Theorem 1 in the general case in Section 3.6.

As was mentioned in the introduction, for higher-rank torsion cases, $\Gamma_n(m)$, the theorem is due to Bartholdi, Neuhauser and Woess [3]. The aim here is to give a proof of their theorem and extend it to other rings including the general non-torsion case, Γ_n. We seek to give as elementary and transparent a treatment as possible. We work with lamplighter models in the rank-1 and rank-2 cases, and use these to illuminate a proof in the general case which uses polynomials and partial fractions.

3.1 R-branching trees

We let $\mathcal{T}_R$ denote the *R-branching tree*. This is the simplicial tree in which every vertex has $1 + |R|$ neighbors. Equip $\mathcal{T}_R$ with the natural path metric in which every edge has length one. Any choice of infinite directed geodesic ray $\rho : \mathbb{R} \to \mathcal{T}_R$ with $\mathbb{Z} \subseteq \mathbb{R}$ mapping to the vertices along the ray determines a height (or *Busemann*) function $h : \mathcal{T}_R \to \mathbb{R}$ by

$$h(p) \;=\; \rho^{-1}(q) + d(p, q)$$

where q is the point on the ray closest to p. Figure 3.1 gives some examples of calculations of heights.

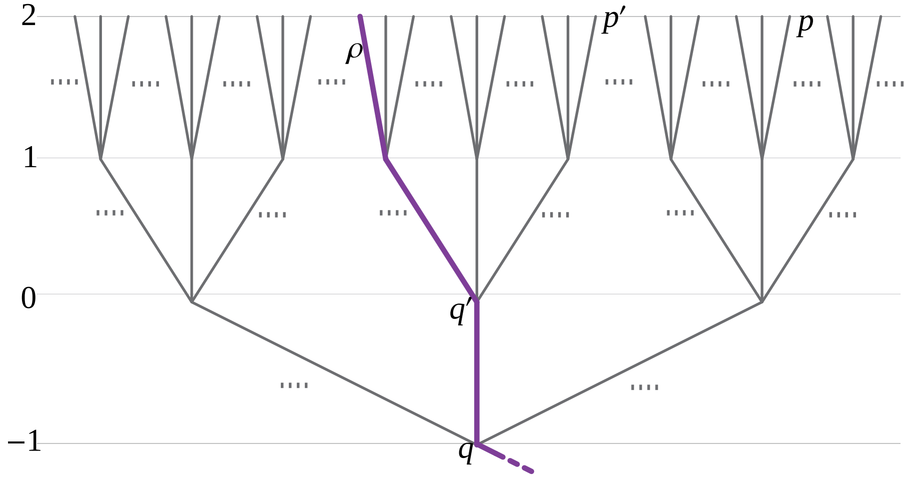

Figure 3.1: The tree $\mathcal{T}_R$ with an infinite geodesic ray ρ determining a height function h. For example, $h(p) = \rho^{-1}(q) + d(p, q) = -1 + 3 = 2$ and $h(p') = \rho^{-1}(q') + d(p', q') = 0 + 2 = 2$.

Label the edges emanating upwards from any given vertex in $\mathcal{T}_R$ by the elements of R in such a way that the edges traversed by ρ are all labeled 0. Then we can specify a unique *address* for each vertex in $\mathcal{T}_R$ as follows.

Lemma 3.1.1 (Addresses of vertices in $\mathcal{T}_R$)**.** *Vertices v in $\mathcal{T}_R$ are in bijective correspondence with pairs consisting of an integer (the height of v) and a finitely supported sequence of elements of R (the labels on the edges that a downwards path starting at v follows).*

Proof. Given a vertex v in $\mathcal{T}_R$, let the pair corresponding to it be given by the integer height of v and the sequence of elements of R obtained by reading off the edge-labels on the path that follows successive downwards edges starting at v. The sequences are finitely supported because the downwards path becomes confluent with ρ after the last non-zero entry in the sequence (if q is the point on the ray closest to v, then there are at most $d(v, q)$ non-zero entries in the sequence). $\square$

Another way to see the identification between the vertices in $\mathcal{T}_R$ and their addresses is by labeling the vertices by cosets of polynomials in the form $f + x^k R[x]$, where k is the height of the vertex and the coefficients of $x^{k-1}, x^{k-2}, \ldots$ in f give the finitely supported sequence of edge labels. Note that at each height, the coset labels at that height provide a partition of polynomials in $R[x]$. Figure 3.2 shows a branching at a vertex of $\mathcal{T}_R$ located at height k with the address given by the coefficients of $x^{k-1}, x^{k-2}, \ldots$ in $f = g + mx^{k-1}$.

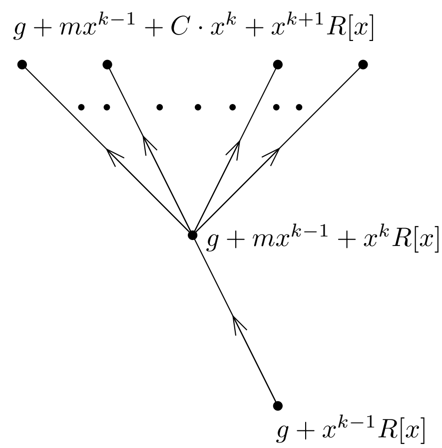

Figure 3.2: A branching at a vertex $(g + mx^{k-1}, k)$ of $\mathcal{T}_R$, for a given $k \in \mathbb{Z}$, $m \in R$, $g \in R[x, x^{-1}]$ such that the coefficients of x^i in g are zero for all $i \geq k - 1$. There is a single branch upwards for each $C \in R$ labeled $g + mx^{k-1} + C \cdot x^k + x^{k+1}R[x]$.

3.2 The horocyclic product of R-branching trees

The *horocyclic product* of $n+1$ copies of $\mathcal{T}_R$ is

$$\mathcal{H}_n(R) \;:=\; \left\{ (p_0, \ldots, p_n) \in \mathcal{T}_R^{n+1} \;\middle|\; \sum_{i=0}^{n} h(p_i) = 0 \right\}.$$

It is naturally an n-complex: $(p_0, \ldots, p_n)$ is in the k-skeleton if and only if

$$|\{i \mid h(p_i) \in \mathbb{Z}\}| \;\geq\; n-k.$$

Equivalently, if we view $\mathcal{T}_R^{n+1}$ as a cubical complex in the natural way, then the k-cells of $\mathcal{H}_n(R)$ are the intersections of the $(k+1)$-cells of $\mathcal{T}_R^{n+1}$ with $\mathcal{H}_n(R)$.

Figure 3.3 shows a horocyclic product of two 3-branching rooted trees of depth 2, and so a portion of $\mathcal{H}_1(\mathbb{Z}/3\mathbb{Z})$. Nine upwards- and nine downwards-3-branching trees are apparent in this graph.

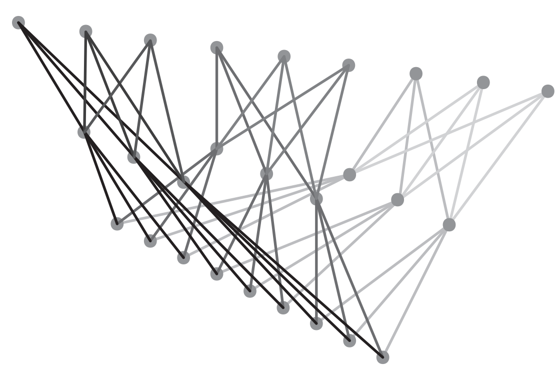

Figure 3.3: A portion of $\mathcal{H}_1(\mathbb{Z}/3\mathbb{Z})$, after a figure by Dymarz in [23].

In the article [1], we describe the cell-structure of $\mathcal{H}_n(R)$.

3.3 Proof of Theorem 1 in the case $n = 1$

Theorem 1 in the case $n = 1$ states that $\mathcal{H}_1(R)$ is the Cayley graph C of $\Gamma_1(R)$ with respect to the generating set $\{\lambda_r := (r, 1) \mid r \in R\}$. This generating set is, in fact, profligate—$\{\lambda_0, \lambda_1\}$ suffices to generate $\Gamma_1(R)$. This case includes $\Gamma_1 = \mathbb{Z} \wr \mathbb{Z}$ and lamplighters $\Gamma_1(m) = (\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z}$.

Proof of Theorem 1 for n = 1. (cf. [4, 15, 46]). An element of $\Gamma_1(R) = R[x, x^{-1}] \rtimes \mathbb{Z}$ is a pair (f, k) where $k \in \mathbb{Z}$ and $f = \sum f_j x^j$ with each $f_j \in R$ and only finitely many are non-zero. Recall from Lemma 3.1.1 that vertices in $\mathcal{T}_R$ are uniquely specified by their *addresses*—pairs consisting of a finitely supported sequence of elements of R (the edge-labels on the path proceeding downwards from the vertex) and an integer (the height).

Let Φ be the bijection between $\Gamma_1(R)$ and the vertices of $\mathcal{H}_1(R)$ that sends (f, k) to the pair of vertices (u, v) with addresses $((f_k, f_{k+1}, f_{k+2}, \ldots), -k)$ and $((f_{k-1}, f_{k-2}, f_{k-3}, \ldots), k)$, respectively. So, in effect, Φ splits the bi-infinite sequence of coefficients of f apart at k to give two infinite sequences as shown in the middle of Figure 3.4. The sequence at the locations shaded pink give the address of u and that shaded green gives the address of v.

In C, the edge labeled λ_r emanating from (f, k) leads to $(f, k)\lambda_r = (f + rx^k, k + 1)$, which is mapped by Φ to (u', v') where u' and v' have addresses $((f_{k+1}, f_{k+2}, \ldots), -k - 1)$ and $((f_k + r, f_{k-1}, f_{k-2}, \ldots), k + 1)$, respectively—see the top of Figure 3.4. So, as r varies over R, (u', v') varies over all the vertices adjacent to (u, v) that are reached by moving along the (unique) downwards edge in $\mathcal{T}_R$ emanating from u and moving along one of the R-indexed edges that emanate upwards from v.

The inverse of $\lambda_r = (r, 1)$ is $(-rx^{-1}, -1)$ since

$$(r, 1)(-rx^{-1}, -1) \;=\; (r + (-rx^{-1})x^1, 1 - 1) \;=\; (0, 0).$$

So, similarly, the family $(f, k)\lambda_r^{-1} = (f - rx^{k-1}, k - 1)$ with r ranging over R, is mapped by Φ to (u'', v'') where u'' and v'' have addresses $((f_{k-1}-r, f_k, f_{k+1}, \ldots), -k+1)$ and $((f_{k-2}, f_{k-3}, \ldots), k-1)$, respectively—see the bottom of Figure 3.4. These are the vertices obtained by moving along the one downwards edge in $\mathcal{T}_R$ from v and moving from u upwards along one of the R-indexed family of edges.

So, vertices that are joined by an edge in C are mapped by Φ to vertices that are joined by an edge in $\mathcal{H}_1(R)$. Moreover, every pair of vertices that are joined by an edge in $\mathcal{H}_1(R)$ can be reached in this way. So Φ extends to a graph-isomorphism $C \to \mathcal{H}_1(R)$, completing our proof. $\square$

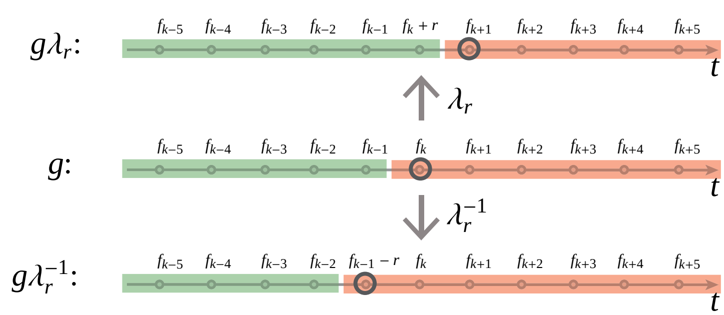

Figure 3.4: Here we use the lamplighter description of $\Gamma_1(R)$ to illustrate right-multiplication by the generators λ_r and their inverses. The middle line represents $g = (f, k)$ and the top and bottom represent $g\lambda_r$ and $g\lambda_r^{-1}$, respectively.

Remark 3.3.1. Perhaps the one subtlety in the above proof is that the edge in $\mathcal{T}_R$ from v to v' is labeled by $f_k + r$. The first guess one might make is that it would be the edge labeled r. But that would not work because (u', v') has to have some "memory" of f_k, else there would be no way for $\Phi^{-1}((u', v')\lambda_r^{-1})$ to equal $\Phi^{-1}(u, v)$.

Remark 3.3.2. In this rank-1 case we could use any group G in place of the ring R, and identify a Cayley graph of the (restricted) wreath product $G \wr \mathbb{Z}$ as a horocyclic product. Specifically, view elements of $G \wr \mathbb{Z}$ as pairs (p, k) where $k \in \mathbb{Z}$ and p is a finitely supported function $\mathbb{Z} \to G$, and let p_g denote the map sending $1 \mapsto g$ and $i \mapsto 1_G$ for all $i \neq 1$. Then the Cayley graph of $G \wr \mathbb{Z}$ with respect to the generating set $\left\{\lambda_g := (p_g, 1) \mid g \in G\right\}$ is the horocyclic product of two G-branching trees. This appears to break down in higher rank where we would need G to be abelian (e.g. to define the lamplighter description in Section 2.2).

3.4 Proof of Theorem 1 in the case $n = 2$

In this section we will prove Theorem 1 when $n = 2$: the 1-skeleton of $\mathcal{H}_2(R)$ is the Cayley graph of $\Gamma_2(R)$ with respect to the generating set

$$\left\{\lambda_r := (r, \mathbf{e}_0),\ \mu_r := (r, \mathbf{e}_1),\ \nu_r := \lambda_r {\mu_r}^{-1} \,\middle|\, r \in R\right\}.$$

This case includes Baumslag and Remeslennikov's metabelian group, which is Γ_2. By presentation *(iii)* of Proposition 2.1.3 we know that $\{\lambda_r, \mu_r, \nu_r \mid r \in \mathbb{Z}\}$ form a generating set for Γ_2. A similar proof shows that $\{\lambda_r, \mu_r, \nu_r \mid r \in \mathbb{Z}/m\mathbb{Z}\}$ form a generating set for $\Gamma_2(m)$.

We will denote a vertex in $\mathcal{H}_2(R)$ by a triple of vertices in $\mathcal{T}_R$, each designated by their addresses in the sense of Lemma 3.1.1. First we will establish a bijection Φ from $\Gamma_2(R)$ to the vertices of $\mathcal{H}_2(R)$, defined by sending $g = (f, (h_0, h_1)) \in \Gamma_2(R)$

to the vertex $((\mathbf{a}^\infty, h_\infty), (\mathbf{a}^0, h_0), (\mathbf{a}^1, h_1))$ found as follows. Represent f using the lamplighter model as some configuration $\mathcal{K}$. Let $h_\infty = -h_0 - h_1$. Let $\mathbf{a}^\infty$, $\mathbf{a}^0$, and $\mathbf{a}^1$ be the sequences obtained by (independently) propagating $\mathcal{K}$ to level 0 in the half-planes $H^\infty_{h_0+h_1}$, $H^0_{h_0-1}$, and $H^1_{h_1-1}$, respectively—see Figure 2.2 for a general illustration and Figure 3.5 for a particular example.

Here is why Φ is a bijection. Let $\mathcal{K}'$ be the configuration of Corollary 2.3.3 that is equivalent to $\mathcal{K}$ and is supported on L_{h_0,h_1}. As that corollary points out, $\mathcal{K}'$ is determined by the sequences $\mathbf{b}^\infty$, $\mathbf{b}^0$, and $\mathbf{b}^1$ obtained from $\mathcal{K}$ by propagating $H^\infty_{h_0+h_1}$ and $H^0_{h_0-1}$ to level h_1, and $H^1_{h_1-1}$ to level h_0. But, given h_0 and h_1, the bijection of Lemma 2.3.2(*iii*) tells us that $\mathbf{b}^\infty$, $\mathbf{b}^0$, and $\mathbf{b}^1$ are determined by (and determine) $\mathbf{a}^\infty$, $\mathbf{a}^0$, and $\mathbf{a}^1$, respectively. So, given any vertex $\mathbf{v} = ((\mathbf{a}^\infty, h_\infty), (\mathbf{a}^0, h_0), (\mathbf{a}^1, h_1))$ in $\mathcal{H}_2(R)$, there is a unique $g = (f, (h_0, h_1))$ such that $\Phi(g) = \mathbf{v}$: specifically, take the f corresponding to $\mathcal{K}'$. (This is a special case of Proposition 3.6.9.)

Next we claim that for all $r \in R$,

$$\begin{aligned}
\Phi(g\lambda_r) &= \Big(\Big((a^\infty_2, a^\infty_3, \ldots), h_\infty - 1\Big),\ \Big(\big(r+\alpha, a^0_1, a^0_2, \ldots\big), h_0 + 1\Big),\ \Big(\mathbf{a}^1, h_1\Big)\Big),\\
\Phi(g\lambda_r^{-1}) &= \Big(\Big((-r+\alpha', a^\infty_1, a^\infty_2, \ldots), h_\infty + 1\Big),\ \Big(\big(a^0_2, a^0_3, \ldots\big), h_0 - 1\Big),\ \Big(\mathbf{a}^1, h_1\Big)\Big),\\
\Phi(g\mu_r) &= \Big(\Big((a^\infty_2, a^\infty_3, \ldots), h_\infty - 1\Big),\ \Big(\mathbf{a}^0, h_0\Big),\ \Big(\big((-1)^{h_0} r+\beta, a^1_1, a^1_2, \ldots\big), h_1 + 1\Big)\Big),\\
\Phi(g\mu_r^{-1}) &= \Big(\Big((-r+\beta', a^\infty_1, a^\infty_2, \ldots), h_\infty + 1\Big),\ \Big(\mathbf{a}^0, h_0\Big),\ \Big(\big(a^1_2, a^1_3, \ldots\big), h_1 - 1\Big)\Big),\\
\Phi(g\nu_r) &= \Big(\Big(\mathbf{a}^\infty, h_\infty\Big),\ \Big(\big(r+\gamma, a^0_1, a^0_2, \ldots\big), h_0 + 1\Big),\ \Big(\big(a^1_2, a^1_3, \ldots\big), h_1 - 1\Big)\Big),\\
\Phi(g\nu_r^{-1}) &= \Big(\Big(\mathbf{a}^\infty, h_\infty\Big),\ \Big(\big(a^0_2, a^0_3, \ldots\big), h_0 - 1\Big),\ \Big(\big((-1)^{h_0} r+\gamma', a^1_1, a^1_2, \ldots\big), h_1 + 1\Big)\Big)
\end{aligned}$$

where $\alpha, \alpha', \beta, \beta', \gamma$, and γ' depend only on g (and not on r).

As we will see, much of the explanation for these equations is contained in Figure 3.6. The central grid represents g: the lamplighter is at (h_0, h_1) and the se-

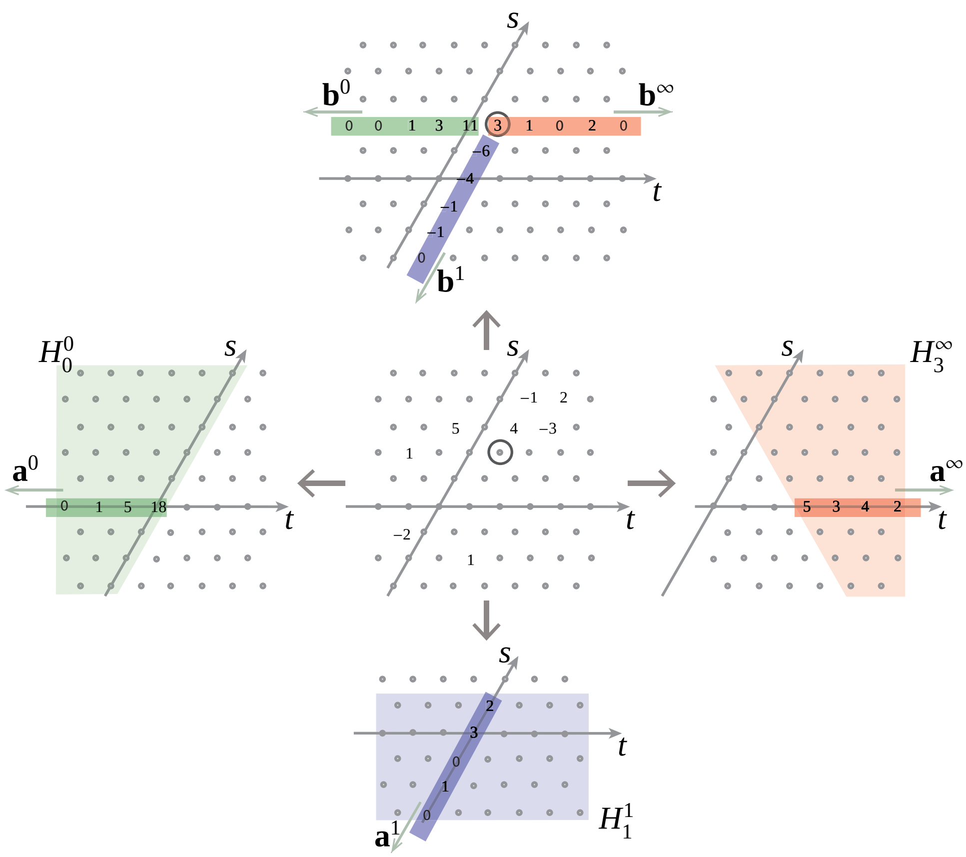

Figure 3.5: An example of a calculation of $\Phi(g)$, where g is the element of Γ_2 represented on the central grid. The lamplighter is at $(1, 2)$, so $h_\infty = -1 - 2 = -3$, $h_0 = 1$, and $h_1 = 2$. The right, left, and lower grid illustrate the calculation of $\mathbf{a}^\infty = (5, 3, 4, 2, 0, 0, \ldots)$, $\mathbf{a}^0 = (18, 5, 1, 0, 0, \ldots)$, and $\mathbf{a}^1 = (2, 3, 0, 1, 0, 0, \ldots)$, respectively, by propagation to level 0 in H_3^∞, H_0^0, and H_1^1. The upper grid illustrates a configuration which is supported on $L_{1,2}$, is equivalent to that of the central grid, and yields the sequences $\mathbf{b}^\infty = (3, 1, 0, 2, 0, 0, \ldots)$, $\mathbf{b}^0 = (11, 3, 1, 0, 0, \ldots)$, and $\mathbf{b}^1 = (-6, -4, -1, -1, 0, 0, \ldots)$, which feature in our proof of case $n = 2$ of Theorem 1.

quences $\mathbf{a}^\infty$, $\mathbf{a}^0$, $\mathbf{a}^1$, $\mathbf{b}^\infty$, $\mathbf{b}^0$, and $\mathbf{b}^1$ associated to f are obtained from the locations indicated (in the manner set out earlier). On right-multiplying g by λ_r, λ_r^{-1}, μ_r, μ_r^{-1}, ν_r, or ν_r^{-1}, the lamplighter moves as shown and r is added to or subtracted

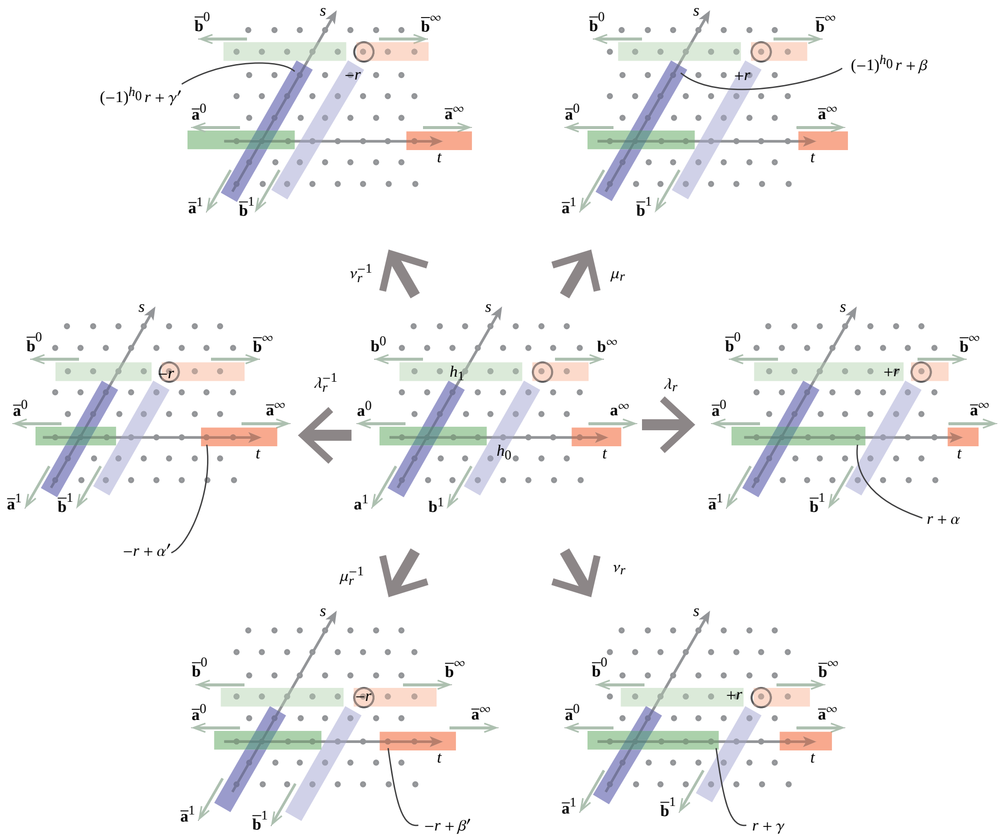

Figure 3.6: Obtaining $\Phi(g\lambda_r)$, $\Phi(g\lambda_r^{-1})$, $\Phi(g\mu_r)$, $\Phi(g\mu_r^{-1})$, $\Phi(g\nu_r)$, and $\Phi(g\nu_r^{-1})$ from $\Phi(g) = ((\mathbf{a}^\infty, h_\infty), (\mathbf{a}^0, h_0), (\mathbf{a}^1, h_1))$. The sequences associated to the former are denoted here by $\overline{\mathbf{a}}^\infty$, $\overline{\mathbf{a}}^0$, $\overline{\mathbf{a}}^1$, $\overline{\mathbf{b}}^\infty$, $\overline{\mathbf{b}}^0$, and $\overline{\mathbf{b}}^1$. The central grid represents g and the six outer grids represent $g\lambda_r$, $g\lambda_r^{-1}$, $g\mu_r$, $g\mu_r^{-1}$, $g\nu_r$, and $g\nu_r^{-1}$, as indicated.

from one entry in the configuration (also as shown). The locations from which the sequences $\overline{\mathbf{a}}^\infty$, $\overline{\mathbf{a}}^0$, $\overline{\mathbf{a}}^1$, $\overline{\mathbf{b}}^\infty$, $\overline{\mathbf{b}}^0$, and $\overline{\mathbf{b}}^1$ associated to the new configurations are obtained also shift as shown.

Here is the justification for the first coordinates on the righthand sides of the six equations above.

Here is why the first coordinate of $\Phi(g\lambda_r)$ is $\left(\left(a_2^\infty, a_3^\infty, \ldots\right), h_\infty - 1\right)$. Since

$$g\lambda_r \;=\; (f + r\cdot(h_0,h_1),\ (h_0,h_1)+\mathbf{e}_0) \;=\; (f + rx^{h_0}(1+x)^{h_1}, (h_0+1,h_1)),$$

the representation of $g\lambda_r$ in the lamplighter model is obtained from that of g by adding r to the entry in $\mathcal{K}$ at (h_0,h_1) and moving the lamplighter to (h_0+1,h_1). The second entry is $h_\infty - 1$ because $(h_\infty - 1)+(h_0+1)+h_1 = 0$, and $\overline{\mathbf{a}}^\infty$ is $\left(a_2^\infty, a_3^\infty, \ldots\right)$ by Lemma 2.3.2(*ii*), since the sequence obtained by propagating $H^\infty_{h_0+h_1+1}$ to level 0 is the same as that obtained by propagating $H^\infty_{h_0+h_1}$ to level 0 and discarding the first entry.

The first coordinate of $\Phi(g\mu_r)$ can be identified likewise.

Similarly, since

$$\begin{aligned} g\lambda_r^{-1} &= (f,(h_0,h_1))(-r\cdot(-\mathbf{e}_0), -\mathbf{e}_0) \\ &=\; (f - r\cdot(h_0,h_1)\cdot(-\mathbf{e}_0),\ (h_0,h_1)-\mathbf{e}_0) \\ &=\; (f - rx^{h_0-1}(1+x)^{h_1}, (h_0-1,h_1)), \end{aligned}$$

the representation of $g\lambda_r{}^{-1}$ is obtained by moving the lamplighter left to $(h_0 - 1, h_1)$ and subtracting r from the entry there. We claim that $\Phi(g\lambda_r{}^{-1})$ has first coordinate

$$\left(\left(-r+\alpha', a_1^\infty, a_2^\infty, \ldots\right), h_\infty + 1\right)$$

where α' depends only on g. The second entry is $h_\infty + 1$ because $(h_\infty + 1) + (h_0 - 1) + h_1 = 0$. All but the first entry of the sequence $\overline{\mathbf{a}}^\infty$ can again be identified by using Lemma 2.3.2(*ii*). In propagation in $H^\infty_{h_0+h_1-1}$, entries on the boundary line (that through $(h_0+h_1-1, 0)$ and $(0, h_0+h_1-1)$) advance only along that line: they are unchanged as they propagate and they do not affect any other entries in the resulting sequence. So the r subtracted from the entry at $(h_0 - 1, h_1)$ moves, undisturbed to $(h_0 + h_1 - 1, 0)$. The α' is the first entry in the sequence when the portion of $\mathcal{K}$ in $H^\infty_{h_0+h_1-1}$ is propagated to level 0. So it depends only on g.

The first coordinate of $\Phi(g{\mu_r}^{-1})$ can be identified likewise.

Since $\nu_r = \lambda_r {\mu_r}^{-1}$, the representation of $g\nu_r$ is obtained by adding r to the entry in $\mathcal{K}$ at (h_0, h_1), moving the lamplighter to $(h_0 + 1, h_1)$, then moving the lamplighter to $(h_0 + 1, h_1 - 1)$, and then subtracting r from the entry at $(h_0 + 1, h_1 - 1)$. Equivalently, it is obtained by moving the lamplighter to $(h_0 + 1, h_1 - 1)$ and adding r to the entry at $(h_0, h_1 - 1)$. So the first coordinate of $\Phi(g\nu_r)$ is $(\mathbf{a}^\infty, h_\infty)$: the second entry is h_∞ because $h_\infty + (h_0 + 1) + (h_0 - 1) = 0$ and $\overline{\mathbf{a}}^\infty = \mathbf{a}^\infty$ because $\overline{\mathbf{a}}^\infty$ and $\mathbf{a}^\infty$ are both obtained by propagating in $H^\infty_{h_0+h_1}$, and the altered entry in the configuration is outside $H^\infty_{h_0+h_1}$.

The first coordinate of $\Phi(g{\nu_r}^{-1})$ is $(\mathbf{a}^\infty, h_\infty)$ likewise.

The entries in the second and third coordinates are explained analogously except for $\Phi(g\mu_r)$ and $\Phi(g\nu_r^{-1})$, where there is an added complication. When, in the case of $\Phi(g\mu_r)$, the r added at (h_0, h_1) is propagated to $(0, h_1)$ it changes sign with each step and so becomes $(-1)^{h_0} r$. Similarly, for $\Phi(g{\nu_r}^{-1})$, the r subtracted from $(h_0 - 1, h_1)$ changes sign with each step as it propagates to $(0, h_1)$, and so also becomes $(-1)^{h_0-1}(-r) = (-1)^{h_0} r$.

Finally, we explain why Φ extends to an isomorphism from the Cayley graph C to the 1-skeleton of $\mathcal{H}_2(R)$.

Suppose $g \in \Gamma_2(R)$. The set of vertices $\mathcal{V}$ in $\mathcal{H}_2(R)$ that are reached by traveling from $\Phi(g)$ along a single edge partitions into six subsets: travel along the unique downwards edge in one coordinate-tree, travel upwards along one of an R-indexed family of edges in another, and remain stationary in the last. Since α, α', β, β', γ, and γ' only depend on g, each of $g\lambda_r \mapsto \Phi(g\lambda_r)$, $g{\lambda_r}^{-1} \mapsto \Phi(g{\lambda_r}^{-1})$, $g\mu_r \mapsto \Phi(g\mu_r)$, $g{\mu_r}^{-1} \mapsto \Phi(g{\mu_r}^{-1})$, $g\nu_r \mapsto \Phi(g\nu_r)$, and $g{\nu_r}^{-1} \mapsto \Phi(g{\nu_r}^{-1})$ is a map onto one such subset, and together they give a bijection from the neighbors of g in C to $\mathcal{V}$.

There are no double-edges and no edge-loops in either graph: for the 1-skeleton of $\mathcal{H}_2(R)$ this is straightforward from the definition, and it therefore follows from the above for the Cayley graph. So Φ extends to an isomorphism between the two graphs, and this completes our proof.

Remark 3.4.1. It may be tempting to try to express directly the group multiplication in $\Gamma_2(R)$ in terms of the representations of elements as triples of addresses of vertices in $\mathcal{T}_R$. It is striking how spectacularly awkward this turns out to be, as the following special case of multiplication by a generator $\zeta \in \left\{\lambda_r^{\pm 1}, \mu_r^{\pm 1}, \nu_r^{\pm 1}\right\}$ illustrates.

We have $\Phi(g) = ((\mathbf{a}^\infty, h_\infty), (\mathbf{a}^0, h_0), (\mathbf{a}^1, h_1))$. To find $\Phi(g\zeta)$ we call on the sequences $\mathbf{b}^\infty$, $\mathbf{b}^0$ and $\mathbf{b}^1$. Since the propagation (of the bijection established in Lemma 2.3.2(*iii*)) in a half-plane proceeds in the manner of Pascal's triangle, we can explicitly express $\mathbf{a}^*$ in terms of $\mathbf{b}^*$ and $\mathbf{b}^*$ in terms of $\mathbf{a}^*$:

$$a_p^* = \begin{cases} \displaystyle\sum_{i=0}^{m} (-1)^{\delta} b_{p+i}^* \binom{m}{i} & \text{if } m \geq 0 \\ \displaystyle\sum_{i=0}^{\infty} (-1)^{\epsilon} b_{p+i}^* \binom{i-1-m}{i} & \text{if } m < 0, \end{cases}$$

$$b_p^* = \begin{cases} \displaystyle\sum_{i=0}^{-m} (-1)^{\delta} a_{p+i}^* \binom{-m}{i} & \text{if } m \leq 0 \\ \displaystyle\sum_{i=0}^{\infty} (-1)^{\epsilon} a_{p+i}^* \binom{i-1+m}{i} & \text{if } m > 0, \end{cases}$$

when $* = \infty, 0$ and $m = h_1$, $\epsilon = i$, and $\delta = 0$, and when $* = 1$, and $m = h_0$, $\epsilon = |h_0|$, and $\delta = i + |h_0|$. The infinite sums make sense since all but finitely many entries of the sequences $\mathbf{a}^*$ and $\mathbf{b}^*$ are zero.

These formulae could be used to express $\alpha, \alpha', \beta, \beta', \gamma$, and γ' in terms of $\mathbf{a}^\infty$, $\mathbf{a}^0$, $\mathbf{a}^1$, h_0 and h_1: obtain $\mathbf{b}^\infty$, $\mathbf{b}^0$, and $\mathbf{b}^1$ using the second formula, then shift them and add or subtract r appropriately to get the $\overline{\mathbf{b}}^\infty$, $\overline{\mathbf{b}}^0$ and $\overline{\mathbf{b}}^1$ associated to $\Phi(g\zeta)$, and finally obtain $\alpha, \alpha', \beta, \beta', \gamma$, and γ' using the first formula.

For example, to calculate α' first obtain $\mathbf{b}^\infty$ and b_1^0 from $\mathbf{a}^\infty$ and $\mathbf{a}^0$ using the second formula with $m = h_1$, then let $\overline{\mathbf{b}}^\infty = (b_1^0 - r, b_1^\infty, b_2^\infty, b_3^\infty, \ldots)$, then obtain $\overline{\mathbf{a}}^\infty$ from $\overline{\mathbf{b}}^\infty$ using the first formula with $m = h_1$, and then, as $-r + \alpha' = \overline{a}_1^\infty$, we have found α'.

The complexity of the formulae that would result stands in marked contrast to the "$f_k + r$" in our proof in Section 3.3 of Theorem 1 in the case where $n = 1$.

Remark 3.4.2. Given that equivalence classes of configurations correspond to elements of $R[x, x^{-1}, (1+x)^{-1}]$, the above analysis can all be rephrased in terms of polynomials—the point-of-view we will take in the next section. In the light of Lemma 2.2.1, Corollary 2.3.3 amounts to the statement that for each pair $(k, l) \in \mathbb{Z}^2$,

$$\left\{ x^{k+i}(1+x)^l \;\middle|\; i \in \mathbb{Z} \right\} \cup \left\{ x^k(1+x)^{j+l} \;\middle|\; j = -1, -2, \cdots \right\}$$

is a basis for $R\left[x, x^{-1}, (1+x)^{-1}\right]$ over R.

The sequence $\mathbf{a}^\infty$ lists the coefficients of $x^0, x^1, \ldots$ in $x^{h_\infty} f$, when expressed as a linear combination of the basis

$$\left\{ x^i \;\middle|\; i \in \mathbb{Z} \right\} \cup \left\{ (1+x)^j \;\middle|\; j = -1, -2, \cdots \right\}.$$

Likewise, $\mathbf{a}^0$ lists the coefficients of $x^{-1}, x^{-2}, \ldots$ in $x^{-h_0} f$, and $\mathbf{a}^1$ lists those of $(1+x)^{-1}, (1+x)^{-2}, \ldots$ in $(1+x)^{-h_1} f$.

If we multiply f by $x^{-h_0}(1+x)^{-h_1}$ to give $\hat{f}$ (in effect, shifting the origin from $(0,0)$ to (h_0, h_1)), then $\mathbf{b}^\infty$ lists the coefficients of $x^0, x^1, \ldots$ in $\hat{f}$, and $\mathbf{b}^0$ lists the coefficients of $x^{-1}, x^{-2}, \ldots$, and $\mathbf{b}^1$ lists those of $(1+x)^{-1}, (1+x)^{-2}, \ldots$.

3.5 $\mathcal{H}_2(\mathbb{Z})$ as a Cayley 2–complex

In this section we show that $\mathcal{H}_2(\mathbb{Z})$ is the Cayley 2-complex of

$$\Gamma_2 \;=\; \left\langle\, \lambda_i, \mu_i, \nu_i \ (i \in \mathbb{Z}) \;\middle|\; \lambda_i = \nu_i \mu_i, \ \lambda_{i+j} = \mu_i \nu_j \ (i, j \in \mathbb{Z}) \,\right\rangle,$$

proving Theorem 2.

Identify the Cayley graph (the 1-skeleton of the Cayley 2-complex) with the 1-skeleton of $\mathcal{H}_2(\mathbb{Z})$ as per the $n = 2$ case of Theorem 1 (proved in Section 3.4).

First we show that every 2-cell in $\mathcal{H}_2(\mathbb{Z})$ is bounded by an edge-loop which corresponds to a defining relation of Γ_2. Suppose a point $\mathbf{p} = (p_0, p_1, p_2) \in \mathcal{H}_2(\mathbb{Z})$ is in the interior of a 2-cell X. Then each p_j is in the interior of an edge I_j of the tree $\mathcal{T}_\mathbb{Z}$. Let $\ell_j = \min_{u \in I_j} h(u)$ and $x_j = h(p_j) - \ell_j$ for $j = 0, 1, 2$. It follows from $h(p_0) + h(p_1) + h(p_2) = 0$ and $0 < x_j < 1$ that $\ell_0 + \ell_1 + \ell_2$ is either -1 or -2. So $x_0 + x_1 + x_2$ is 1 or 2. Say X is of "type 1" or "2" accordingly. Examples are shown in Figure 3.7 (with the vertices of the triangles labeled by (x_0, x_1, x_2)-coordinates).

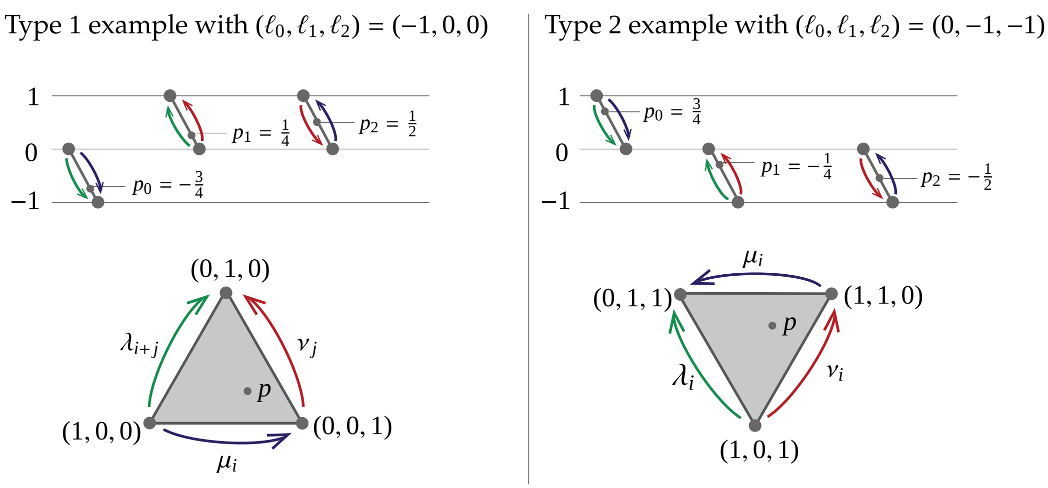

Figure 3.7: Examples of 2-cells of type 1 and 2 in $\mathcal{H}_2(\mathbb{Z})$.

Consider moving $\mathbf{p}$ within X as parametrized by (x_0, x_1, x_2). It is on an edge in ∂X when one of the p_i is at an end of I_i and is on a vertex when two (and hence all three) are at an end of I_i. So if X is of type 1, it has vertices, $(x_0, x_1, x_2) = (1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and ∂X is traversed by following the edges $(1 - r, r, 0)_{0 \le r \le 1}$, then $(0, 1 - r, r)_{0 \le r \le 1}$, and then $(r, 0, 1 - r)_{0 \le r \le 1}$. If X is of type 2, it has vertices, $(x_0, x_1, x_2) = (0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$, and ∂X is traversed by following $(r, 1 - r, 1)_{0 \le r \le 1}$, then $(1, r, 1 - r)_{0 \le r \le 1}$, and then $(1 - r, 1, r)_{0 \le r \le 1}$.

Now, ∂X corresponds to a length-3 relator in Γ_2, and matching the changes in heights as ∂X is traversed with the height-changes indicated in the family of six displayed equations in our proof of the $n = 2$ case of Theorem 1 in Section 3.4, that relator must be $\lambda_k {\nu_j}^{-1} {\mu_i}^{-1}$ for type 1, and $\lambda_i^{-1} \nu_j \mu_k$ for type 2, for some $i, j, k \in \mathbb{Z}$.

The workings of lamplighter model illustrated in Figure 3.6 allow us to see that $\lambda_k \nu_j^{-1} \mu_i^{-1} = 1$ in Γ_2 if and only if $k - j - i = 0$ since $\lambda_k \nu_j^{-1} \mu_i^{-1}$ does not move the lamplighter and increments the lamp at the lamplighter's location by $k - j - i$. That is, the relation is $\lambda_{i+j} = \mu_i \nu_j$ for some $i, j \in \mathbb{Z}$. Similarly, $\lambda_i^{-1} \nu_j \mu_k = 1$ in Γ_2 if and only if $i = j = k$ since $\lambda_i^{-1} \nu_j \mu_k$ does not move the lamplighter and transforms a triangle of numbers ${}_0{}^0{}_0 \mapsto {}_j{}^{-i}{}_k$ (with the lamplighter being located to the right of the $-i$). That is, the relation is $\lambda_i = \nu_i \mu_i$ for some $i \in \mathbb{Z}$. So around ∂X we read one of the defining relations in the presentation given in the theorem.

Finally, we show that every edge-loop in $\mathcal{H}_2(\mathbb{Z})$ which corresponds to a defining relation bounds a 2-cell. So suppose $\rho : S^1 \to \mathcal{H}_2(\mathbb{Z})$, given by $r \mapsto \rho(r) = (p_0(r), p_1(r), p_2(r))$, is a loop in the 1-skeleton of $\mathcal{H}_2(\mathbb{Z})$ and around ρ we read one of the defining relations. Then for each j, such are the defining relations, the image of the loop $r \mapsto p_j(r)$ is in a single edge I_j of $\mathcal{T}_\mathbb{Z}$ and, by a similar analysis to that above,

$$\left\{ (u_0, u_1, u_2) \in \mathcal{T}_\mathbb{Z}^3 \;\middle|\; u_j \in I_j \text{ and } h(u_0) + h(u_1) + h(u_2) = 0 \right\}$$

is a 2-cell of $\mathcal{H}_2(\mathbb{Z})$ with boundary circuit ρ.

So, as no edge-loop in either $\mathcal{H}_2(\mathbb{Z})$ or in the Cayley 2-complex is the boundary of two 2-cells, the result it proved.

3.6 The general case of Theorem 1

The standing assumptions in this section are that n is any fixed positive integer and R is any commutative ring with unity in which $2, 3, \ldots, n-1$ are invertible. We will prove Theorem 1 in full generality: the 1-skeleton of $\mathcal{H}_n(R)$ is the specified Cayley graph.

3.6.1 Preliminaries

Recall that

$$A_n(R) \;=\; R\left[x, x^{-1}, (1+x)^{-1}, \ldots, (n-1+x)^{-1}\right].$$

The following lemma generalizes Corollary 2.3.3 and is vital to the proof of Theorem 1. Baumslag and Stammbach [8] prove a very similar result as do Bartholdi, Neuhauser and Woess [3, Section 3]. We include a proof for completeness and because this and the lemmas that follow are where the hypothesis that $2, 3, \ldots, n-1$ are invertible is used.

Lemma 3.6.1 (*adapted from* Baumslag and Stammbach, Lemma 2.1 in [8])**.**

$$\left\{\, 1, x^j, x^{-j}, (1+x)^{-j}, \ldots, (n-1+x)^{-j} \,\middle|\, j = 1, 2, \ldots \right\}$$

is a basis for $A_n(R)$ over R.

Proof. First we show that the given set spans.

Suppose $S \subseteq \{0, 1, \ldots, n-1\}$.

For $l \in S$, let

$$\lambda_l \ := \ \prod_{i \in S \setminus \{l\}} (i-l)^{-1},$$

understanding this product to be 1 when $S \setminus \{l\} = \emptyset$. This is well defined because $2, 3, \ldots, n-1$ are invertible. Then, by induction on n,

$$\prod_{l \in S} (l+x)^{-1} \ = \ \sum_{l \in S} \lambda_l (l+x)^{-1}$$

in $A_n(R)$, the crucial calculation for the induction step being that

$$(l+x)^{-1}(m+x)^{-1} \ = \ (m-l)^{-1}(l+x)^{-1} + (l-m)^{-1}(m+x)^{-1}$$

for all $m \in \{1, 2, \ldots, n-1\}$ and $l \in \{0, 1, \ldots, m-1\}$. So $\prod_{l \in S}(l+x)^{-1}$ is in the span.

Next consider $x^{h_0}(1+x)^{h_1} \cdots (n-1+x)^{h_{n-1}}$ where each h_i is a non-positive integer. We show it too is in the span by inducting on $\sum_{i=0}^{n-1} |h_i|$. The base case is immediate and the previous paragraph gives the induction step: let $S = \{i \mid h_i < 0\}$ and

$$\varepsilon_i \ = \ \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$

for each i, then

$$\begin{aligned} & x^{h_0}(1+x)^{h_1} \cdots (n-1+x)^{h_{n-1}} \\ & \qquad = \ \left(x^{h_0+\varepsilon_0}(1+x)^{h_1+\varepsilon_1} \cdots (n-1+x)^{h_{n-1}+\varepsilon_{n-1}}\right) \prod_{l \in S}(l+x)^{-1} \\ & \qquad = \ \left(x^{h_0+\varepsilon_0}(1+x)^{h_1+\varepsilon_1} \cdots (n-1+x)^{h_{n-1}+\varepsilon_{n-1}}\right) \sum_{l \in S} \lambda_l (l+x)^{-1}. \end{aligned}$$

To complete the proof that the given set spans it is enough to show that $p(x)(m+x)^{-k}$ is in the span whenever $p(x) \in R[x]$, $m \in \{0, \ldots, n-1\}$, and $k > 0$. After all, any element of $A_n(R)$ is an R-linear combination of products of powers of

$x, (1+x), \ldots, (n-1+x)$ and so by the previous result is an R-linear combination of some such $p(x)(m+x)^{-k}$. Well, write $p(x) = (m+x)q(x) + s$ for some $q(x) \in R[x]$ and $s \in R$. Then

$$p(x)(m+x)^{-k} \;=\; \left(q(x) + s(m+x)^{-1}\right)(m+x)^{-k+1}$$

which by induction on $|k|$ is in the span.

For linear independence, suppose

$$0 \;=\; \sum_{j=0}^{d_\infty} \mu_j x^j + \sum_{i=0}^{n-1} \sum_{j=1}^{d_i} \lambda_{i,j}(i+x)^{-j}$$

in $A_n(R)$ for some $\mu_j, \lambda_{i,j} \in R$. Multiplying through by $x^{d_0}(1+x)^{d_1} \cdots (n-1+x)^{d_{n-1}}$ and comparing coefficients we see that $0 = \mu_0 = \mu_1 = \cdots = \mu_{d_\infty}$. The constant term on the right hand side is $\lambda_{0,d_0} \cdot 1^{d_1} \cdot 2^{d_2} \cdot \ldots \cdot (n-1)^{d_{n-1}}$. As $2, \ldots, (n-1)$ are invertible in R, we must have $\lambda_{0,d_0} = 0$. Repeatedly dividing through by x and analyzing the constant term gives $\lambda_{0,j} = 0$ for all j. Viewing the resulting polynomial as a polynomial in $x-1$ rather than x and applying the same technique yields $\lambda_{1,j} = 0$ for all j. Then viewing it as a polynomial in $x-2$, then $x-3$, and so on, gives $\lambda_{i,j} = 0$ for all i, j. □

In the light of this lemma we will, in the remainder of this section and the next talk about *the* $(*+x)^{-j}$ *or the* x^j *coefficient of a* $p \in A_n(R)$, meaning the coefficient of that term when p is expressed as a linear combination of the basis established in Lemma 3.6.1.

Lemma 3.6.2. *Suppose* $* \in \{0, 1, \ldots, n-1\}$ *and* $q_0, \ldots, q_{n-1} \in \mathbb{Z}$, *and* $q_* = 0$. *Given* $\lambda_{*,1}, \lambda_{*,2}, \ldots$ *in* R, *all but finitely many of which are zero, take* p *to be any element of* $A_n(R)$ *such that the coefficients of* $(*+x)^{-1}, (*+x)^{-2}, \ldots$ *are* $\lambda_{*,1}, \lambda_{*,2}, \ldots$. *Let* $\lambda'_{*,1}, \lambda'_{*,2}, \ldots$

be the coefficients of $(* + x)^{-1}, (* + x)^{-2}, \ldots$ *in*

$$p' \;:=\; x^{q_0}(1+x)^{q_1}\cdots(n-1+x)^{q_{n-1}}p.$$

Then $\lambda'_{*,1}, \lambda'_{*,2}, \ldots$ *depend only on* $\lambda_{*,1}, \lambda_{*,2}, \ldots$ *and*

$$(\lambda_{*,1},\ \lambda_{*,2},\ \ldots) \;\mapsto\; (\lambda'_{*,1},\ \lambda'_{*,2},\ \ldots)$$

is a bijection from the set of finitely supported sequences of elements of R *to itself. Moreover, if* $0 = \lambda_{*,2} = \lambda_{*,3} = \cdots$, *then*

$$(\lambda'_{*,1},\ \lambda'_{*,2},\ \lambda'_{*,3},\ \ldots) = \left(\lambda_{*,1} \prod_{i\in\{0,\ldots,n-1\}\setminus\{*\}} (i-*)^{q_i},\ 0,\ 0,\ \ldots\right).$$

Proof. It is enough to prove this in the special case $p' = (i+x)p$ where one of $q_0, \ldots, q_{n-1}$, denoted q_i, is 1 and all others are 0, for a general instance can be reached by composing a suitable sequences of instances of this special case (and its 'inverse'). Note that $i \neq *$, and so we will be able to invert $(i - *)$.

Express

$$p \;=\; \sum_{j=0}^{\infty}\mu_j x^j + \sum_{l=0}^{n-1}\sum_{j=1}^{\infty}\lambda_{l,j}(l+x)^{-j}, \tag{3.1}$$

$$p' \;=\; \sum_{j=0}^{\infty}\mu'_j x^j + \sum_{l=0}^{n-1}\sum_{j=1}^{\infty}\lambda'_{l,j}(l+x)^{-j} \tag{3.2}$$

where each $\mu_j, \mu'_j, \lambda_{l,j}, \lambda'_{l,j} \in R$ (and only finitely many are non-zero)—that is, as a linear combinations of the basis established in Lemma 3.6.1. We prove the special case by calculating $(\mu'_0, \mu'_1, \ldots)$ and $(\lambda'_{l,1}, \lambda'_{l,2}, \ldots)$.

For $i, l \in \{0, \ldots, n-1\}$,

$$(i+x)\sum_{j=0}^{\infty}\mu_j x^j \;=\; i\mu_0 + (\mu_0 + i\mu_1)x^1 + (\mu_1 + i\mu_2)x^2 + \cdots$$

and, as $(i+x)(l+x)^{-j} = (l+x)^{-j+1} + (i-l)(l+x)^{-j}$,

$$\begin{aligned}(i+x)\sum_{j=1}^{\infty}\lambda_{l,j}(l+x)^{-j} &= \sum_{j=1}^{\infty}\lambda_{l,j}(l+x)^{-j+1} + \sum_{j=1}^{\infty}\lambda_{l,j}(i-l)(l+x)^{-j} \\ &= \lambda_{l,1} + \sum_{j=1}^{\infty}\Big(\lambda_{l,j+1} + \lambda_{l,j}(i-l)\Big)(l+x)^{-j}. \qquad (3.3)\end{aligned}$$

So

$$(\lambda'_{*,1}, \lambda'_{*,2}, \ldots) = (\lambda_{*,2} + \lambda_{*,1}(i-*),\ \lambda_{*,3} + \lambda_{*,2}(i-*),\ \ldots),$$

and evidently the only coefficients from (3.1) this depends on are $\lambda_{*,1}, \lambda_{*,2}, \ldots$. Also we find that if $0 = \lambda_{*,2} = \lambda_{*,3} = \cdots$, then

$$(\lambda'_{*,1},\ \lambda'_{*,2},\ \lambda'_{*,3},\ \ldots) = (\lambda_{*,1}(i-*),\ 0,\ 0,\ \ldots),$$

which leads to the final claim. To see that

$$(\lambda_{*,1},\ \lambda_{*,2},\ \ldots) \mapsto (\lambda'_{*,1},\ \lambda'_{*,2},\ \ldots)$$

is invertible when $i \neq *$, consider any m such that $\lambda'_{*,q} = 0$ for all $q > m$. Then $0 = \lambda_{*,m+1} = \lambda_{*,m+2} = \cdots$ as otherwise the sequence $\lambda_{*,1}, \lambda_{*,2}, \ldots$ would not be finitely supported. And

$$\begin{aligned}\lambda_{*,m} &= (i-*)^{-1}\lambda'_{*,m} \\ \lambda_{*,m-1} &= (i-*)^{-1}(\lambda'_{*,m-1} - \lambda_{*,m}) \\ &\vdots \\ \lambda_{*,1} &= (i-*)^{-1}(\lambda'_{*,1} - \lambda_{*,2}).\end{aligned}$$

□

Lemma 3.6.3. *Suppose $q_0, \ldots, q_{n-1} \in \mathbb{Z}$ and $\sum_i q_i = 0$. Given $\mu_0, \mu_1, \ldots$ in R, all but finitely many of which are zero, take p to be any element of $A_n(R)$ such that the coefficients of $x^0, x^1, \ldots$ are $\mu_0, \mu_1, \ldots$. Let $\mu'_0, \mu'_1, \ldots$ be the coefficients of $x^0, x^1, \ldots$ in*

$$p' := x^{q_0}(1+x)^{q_1}\cdots(n-1+x)^{q_{n-1}}p.$$

Then $\mu'_0, \mu'_1, \ldots$ depend only on $\mu_0, \mu_1, \ldots$ and

$$(\mu_0,\ \mu_1,\ \ldots) \ \mapsto\ (\mu'_0,\ \mu'_1,\ \ldots)$$

is a bijection from the set of finitely supported sequences of elements of R to itself. Moreover, if $0 = \mu_1 = \mu_2 = \cdots$, then $(\mu'_0, \mu'_1, \mu'_2, \ldots) = (\mu_0, 0, 0, \ldots)$.

Proof. We follow a similar approach to our proof of Lemma 3.6.2. This time, as $\sum_i q_i = 0$, it is enough to prove the result in the special case $p' = x^{-1}(i+x)p$ where $q_0 = -1$, $q_i = 1$ and all $q_j = 0$ for all $j \neq 0, i$.

Again, consider p and p' expressed as in (3.1) and (3.2). The crucial calculations this time are that

$$x^{-1}(i+x)\sum_{j=0}^{\infty}\mu_j x^j \ =\ i\mu_0 x^{-1} + (\mu_0 + i\mu_1)x^0 + (\mu_1 + i\mu_2)x^1 + \cdots$$

and for $l \in \{0, 1, \ldots, n-1\}$, using (3.3),

$$x^{-1}(i+x)\sum_{j=1}^{\infty}\lambda_{l,j}(l+x)^{-j} \ =\ \lambda_{l,1}x^{-1} + \sum_{j=1}^{\infty}\left(\lambda_{l,j+1} + (i-l)\lambda_{l,j}\right)x^{-1}(l+x)^{-j}$$

which has no $x^0, x^1, \ldots$ terms when written as a linear combination of the basis elements since, by induction on j and when $l \neq 0$,

$$x^{-1}(l+x)^{-j} \ =\ l^{-j}x^{-1} - l^{-j}(l+x)^{-1} - l^{-j+1}(l+x)^{-2} - \cdots - l^{-1}(l+x)^{-j}.$$

So

$$(\mu'_0,\ \mu'_1,\ \ldots) \ =\ (\mu_0 + i\mu_1,\ \mu_1 + i\mu_2,\ \ldots),$$

and the final claim of the lemma is evident. To see that

$$(\mu_0,\ \mu_1,\ \ldots) \ \mapsto\ (\mu'_0,\ \mu'_1,\ \ldots)$$

is invertible, recall that $i \in \{1, 2, \dots, n-1\}$ (so i is invertible), and consider any m such that $\mu'_q = 0$ for all $q > m$. Then $0 = \mu_{m+1} = \mu_{m+2} = \cdots$ as otherwise we would have $\mu_{q+1} = -i^{-1}\mu_q$ for all $q > m$ and so the sequence $\mu_0, \mu_1, \mu_2, \dots$ would not be finitely supported. So

$$\begin{aligned} \mu_m &= \mu'_m \\ \mu_{m-1} &= \mu'_{m-1} - i\mu_m \\ &\vdots \\ \mu_0 &= \mu'_0 - i\mu_1. \end{aligned}$$

$\square$

Corollary 3.6.4. *If $k_* = 0$, then the coefficients of $(* + x)^{-1}, (* + x)^{-2}, \dots$ in $x^{k_0}(1 + x)^{k_1} \cdots (n-1+x)^{k_{n-1}}$ are all zero.*

Proof. This is the final statement of Lemma 3.6.2 in the special case $p = 1$ (and hence $\lambda_{*,j} = 0$ for all j), and $q_l = k_l$ for all l. $\square$

Corollary 3.6.5. *If $k_* = -1$, then the coefficient of $(*+x)^{-1}$ in $x^{k_0}(1+x)^{k_1} \cdots (n-1+x)^{k_{n-1}}$ is*

$$\prod_{i \in \{0,\dots,n-1\} \setminus \{*\}} (i - *)^{k_i}.$$

Proof. This is the final statement of Lemma 3.6.2 in the special case $p = (* + x)^{-1}$ (so $\lambda_{*,1} = 1$ and $\lambda_{*,j} = 0$ for all $j \neq 1$), $q_* = k_* + 1 = 0$ and $q_l = k_l$ for all $l \neq *$. $\square$

Corollary 3.6.6. *If $k_\infty := -\sum_{i=0}^{n-1} k_i > 0$, then the coefficients of $x^0, x^1, \dots$ in $x^{k_0}(1 + x)^{k_1} \cdots (n-1+x)^{k_{n-1}}$ are all zero.*

Proof. This is the final statement of Lemma 3.6.3 with $q_0 = k_0 + k_\infty$ and $q_i = k_i$ for all other i (so $\sum_{i=0}^{n-1} q_i = 0$ as required) in the special case $p = x^{-k_\infty}$ (and since $k_\infty > 0$, we have $\mu_j = 0$ for all j). □

Corollary 3.6.7. *If $\sum_{i=0}^{n-1} k_i = 0$, then in $x^{-k_0}(1+x)^{-k_1}\cdots(n-1+x)^{-k_{n-1}}$ the coefficient of x^0 is 1 and the coefficients of $x^1, x^2, \ldots$ are all zero.*

Proof. This is the final statement of Lemma 3.6.3 in the special case $p = 1$ (so $\mu_0 = 1$ and $\mu_j = 0$ for all $j \neq 0$) and $q_i = -k_i$ for all i. □

Lemma 3.6.8. *For $p \in A_n(R)$,*

(*i*) *the coefficients of $x^0, x^1, \ldots$ in p equal those of $x^1, x^2, \ldots$ in xp,*

(*ii*) *the coefficients of $(*+x)^{-1}, (*+x)^{-2}, \ldots$ in $(*+x)p$ equal those of $(*+x)^{-2}, (*+x)^{-3}, \ldots$ in p.*

Proof. Calculate in the manner of our proof of Lemma 3.6.2. The crucial point for (*i*) is that $x(l+x)^{-j} = (l+x)^{-j+1} - l(l+x)^{-j}$ has no $x^1, x^2, \ldots$ terms when $j \geq 1$. The crucial points for (*ii*) are that $(*+x)(l+x)^{-i} = (l+x)^{-i+1} + (*-l)(l+x)^{-i}$ and $(*+x)x^j = *x^j + x^{j+1}$ have no $(*+x)^{-1}, (*+x)^{-2}, \ldots$ terms when $l \in \{0, 1, \ldots, n-1\} \smallsetminus \{*\}$ and $i \geq 1$ and when $j \geq 0$. □

3.6.2 The bijection Φ between $\Gamma_n(R)$ and the vertices of $\mathcal{H}_n(R)$

Define a map Φ from $\Gamma_n(R) = A_n(R) \rtimes \mathbb{Z}^n$ to the vertices of $\mathcal{H}_n(R)$ by

$$(f, (h_0, \ldots, h_{n-1})) \mapsto ((\mathbf{a}^\infty, h_\infty), (\mathbf{a}^0, h_0), \ldots, (\mathbf{a}^{n-1}, h_{n-1}))$$

where $h_\infty := -h_0 - \cdots - h_{n-1}$ and the sequences $\mathbf{a}^\infty, \mathbf{a}^0, \ldots, \mathbf{a}^{n-1}$ will be defined as follows (guided by Remark 3.4.2). They list the coefficients of elements of $A_n(R)$, expressed as linear combinations of the basis from Lemma 3.6.1, specifically, for $* = 0, \ldots, n-1$,

- $\mathbf{a}^\infty$ lists the coefficients of $x^0, x^1, \ldots$ in $x^{h_\infty} f$, and
- $\mathbf{a}^*$ lists the coefficients of $(* + x)^{-1}, (* + x)^{-2}, \ldots$ in $(* + x)^{-h_*} f$.

Our proof that Φ is a bijection will involve

$$\hat{f} \; := \; x^{-h_0}(1 + x)^{-h_1} \cdots (n - 1 + x)^{-h_{n-1}} f$$

and further sequences $\mathbf{b}^\infty, \mathbf{b}^0, \ldots, \mathbf{b}^{n-1}$ defined by:

- $\mathbf{b}^\infty$ lists the coefficients of $x^0, x^1, \ldots$ in $\hat{f}$, and
- $\mathbf{b}^*$ lists the coefficients of $(* + x)^{-1}, (* + x)^{-2}, \ldots$ in $\hat{f}$.

Proposition 3.6.9. *Φ is a bijection.*

Proof. Suppose $\mathbf{v} = ((\mathbf{a}^\infty, h_\infty), (\mathbf{a}^0, h_0), \ldots, (\mathbf{a}^{n-1}, h_{n-1}))$ is a vertex of $\mathcal{H}_n(R)$ and so $h_\infty = -h_0 - \cdots - h_{n-1}$. We will explain that there is a unique $g = (f, (h_0, \ldots, h_{n-1}))$ with $\Phi(g) = \mathbf{v}$.

The idea is to find the sequences $\mathbf{b}^\infty$, $\mathbf{b}^0$, …, $\mathbf{b}^{n-1}$, for then we can recover $\hat{f}$ (and therefore f) from them since they list all its coefficients when expressed as a linear combination of the basis from Lemma 3.6.1.

For $* = \infty$, this is possible (and unique) by Lemma 3.6.3 applied with $p = x^{h_\infty} f$ and $p' = \hat{f}$ (and so $q_0 = -(h_\infty + h_0)$, and $q_i = -h_i$ for $i = 1, \ldots, n-1$). It establishes a bijection taking $(\mu_0, \mu_1, \ldots) = \mathbf{a}^\infty$, which lists the coefficients of $x^0, x^1, \ldots$ in $x^{h_\infty} f$, to $\mathbf{b}^\infty = (\mu'_0, \mu'_1, \ldots)$, which lists the coefficients of $x^0, x^1, \ldots$ in $\hat{f}$. Likewise, for $* = 0, 1, \ldots, n-1$, apply Lemma 3.6.2 with $p = (* + x)^{-h_*} f$ and $p' = \hat{f}$ (and so $q_i = -h_i$ for $i = 0, 1, \ldots, n-1$ except that $q_* = 0$). It establishes a bijection taking $(\lambda_{*,1}, \lambda_{*,2}, \ldots) = \mathbf{a}^*$, which lists the coefficients of $(* + x)^{-1}, (* + x)^{-2}, \ldots$ in $(* + x)^{-h_*} f$, to $\mathbf{b}^* = (\lambda'_{*,1}, \lambda'_{*,2}, \ldots)$, which lists the coefficients of $(* + x)^{-1}, (* + x)^{-2}, \ldots$ in $\hat{f}$. □

3.6.3 Extending Φ

Next we show that Φ extends to a graph-isomorphism from the Cayley graph C of $\Gamma_n(R)$ with respect to the generating set

$$\left\{ (r, \mathbf{e}_i),\ (r, \mathbf{e}_j)(r, \mathbf{e}_k)^{-1} \;\middle|\; r \in R,\ 0 \le i, j, k \le n-1 \text{ and } j < k \right\}$$

to the 1-skeleton of $\mathcal{H}_n(R)$.

Recall that we denote the standard basis for $\mathbb{Z}^n$ by $\mathbf{e}_0, \ldots, \mathbf{e}_{n-1}$. So, if $\mathbf{h} = (h_0, \ldots, h_{n-1}) \in \mathbb{Z}^n$, then $\mathbf{h} + \mathbf{e}_i = (h_0, \ldots, h_{i-1}, h_i + 1, h_{i+1}, \ldots, h_{n-1})$. Recall that for such $\mathbf{h}$ and for $f \in A_n(R)$,

$$f \cdot \mathbf{h} \;=\; f x^{h_0} (1 + x)^{h_1} \cdots (n - 1 + x)^{h_{n-1}}.$$

(Warning: $f \cdot \mathbf{0} = f$ and $f \cdot (\mathbf{h} + \mathbf{h}')$ equals $(f \cdot \mathbf{h}) \cdot \mathbf{h}'$, and not in general $f \cdot \mathbf{h} + f \cdot \mathbf{h}'$.) Also recall that the group operation on $\Gamma_n(R)$ is $(f, \mathbf{h})(\hat{f}, \hat{\mathbf{h}}) = (f + \hat{f} \cdot \mathbf{h}, \mathbf{h} + \hat{\mathbf{h}})$.

Suppose $g = (f, \mathbf{h}) \in \Gamma_n(R)$ where $f \in A_n(R)$ and $\mathbf{h} \in \mathbb{Z}^n$. We show below that post-multiplying g by the elements of the generating set and their inverses gives

$$g(r,\mathbf{e}_j) \;=\; \Big(f + r\cdot\mathbf{h},\ \mathbf{h}+\mathbf{e}_j\Big), \tag{3.4}$$

$$g(r,\mathbf{e}_j)^{-1} \;=\; \Big(f - r\cdot(\mathbf{h}-\mathbf{e}_j),\ \mathbf{h}-\mathbf{e}_j\Big), \tag{3.5}$$

$$g(r,\mathbf{e}_j)(r,\mathbf{e}_k)^{-1} \;=\; \Big(f + (k-j)r\cdot(\mathbf{h}-\mathbf{e}_k),\ \mathbf{h}+\mathbf{e}_j-\mathbf{e}_k\Big), \tag{3.6}$$

$$g(r,\mathbf{e}_k)(r,\mathbf{e}_j)^{-1} \;=\; \Big(f + (j-k)r\cdot(\mathbf{h}-\mathbf{e}_j),\ \mathbf{h}+\mathbf{e}_k-\mathbf{e}_j\Big) \tag{3.7}$$

for all $r \in R$ and all $j, k \in \{0, \ldots, n-1\}$. The explanation is that (3.4) is immediate from how group multiplication is defined, (3.5) uses that

$$(r,\mathbf{e}_j)^{-1} \;=\; (-r\cdot(-\mathbf{e}_j), -\mathbf{e}_j),$$

the key calculation for (3.6) is that

$$r\cdot\mathbf{h} - r\cdot(\mathbf{h}+\mathbf{e}_j-\mathbf{e}_k) \;=\; r\left(1-\frac{j+x}{k+x}\right)\cdot\mathbf{h} \;=\; r\frac{k-j}{k+x}\cdot\mathbf{h} \;=\; (k-j)r\cdot(\mathbf{h}-\mathbf{e}_k),$$

and (3.7) is immediate from (3.6).

Suppose

$$\Phi(g) \;=\; ((\mathbf{a}^\infty, h_\infty), (\mathbf{a}^0, h_0), \ldots, (\mathbf{a}^{n-1}, h_{n-1})).$$

We claim next that Φ maps

$$\begin{aligned}
g(r,\mathbf{e}_j) &\;\mapsto\; \Big(\Big((a_2^\infty, a_3^\infty, \ldots), h_\infty - 1\Big),\ \ldots, \Big(\Big(\alpha_j + r\beta_j, a_1^j, a_2^j, \ldots\Big), h_j+1\Big),\ \ldots\ \Big),\\
g(r,\mathbf{e}_j)^{-1} &\;\mapsto\; \Big(\Big(\Big(\alpha'_j - r\beta'_j, a_1^\infty, a_2^\infty, \ldots\Big), h_\infty + 1\Big),\ \ldots, \Big(\Big(a_2^j, a_3^j, \ldots\Big), h_j - 1\Big),\ \ldots\Big),\\
g(r,\mathbf{e}_j)(r,\mathbf{e}_k)^{-1} &\;\mapsto\; \Big(\ldots, \Big(\Big(\alpha_{jk} + r\beta_{jk}, a_1^j, a_2^j, \ldots\Big), h_j+1\Big),\ \ldots, \Big(\Big(a_2^k, a_3^k, \ldots\Big), h_k - 1\Big), \ldots\Big),\\
g(r,\mathbf{e}_k)(r,\mathbf{e}_j)^{-1} &\;\mapsto\; \Big(\ldots, \Big(\Big(a_2^j, a_3^j, \ldots\Big), h_j - 1\Big),\ \ldots, \Big(\Big(\alpha'_{jk} + r\beta'_{jk}, a_1^k, a_2^k, \ldots\Big), h_k+1\Big), \ldots\Big),
\end{aligned}$$

where the pairs indicated by ellipses are unchanged from the corresponding $(\mathbf{a}^i, h_i)$ in $\Phi(g)$, and in terms of linear combinations of the basis established in

Lemma 3.6.1,

$$\alpha_j \text{ is the coefficient of } (j+x)^{-1} \text{ in } (j+x)^{-h_j-1} f,$$

$$\alpha'_j \text{ is the coefficient of } x^0 \text{ in } x^{h_\infty+1} f,$$

$$\alpha_{jk} \text{ is the coefficient of } (j+x)^{-1} \text{ in } (j+x)^{-h_j-1} f,$$

$$\alpha'_{jk} \text{ is the coefficient of } (k+x)^{-1} \text{ in } (k+x)^{-h_k-1} f,$$

$$\beta_j = \prod_{i\in\{0,\ldots,n-1\}\smallsetminus\{j\}} (i-j)^{h_i}, \text{ the coefficient of } (j+x)^{-1} \text{ in } (j+x)^{-h_j-1}\cdot \mathbf{h},$$

$$\beta'_j = 1, \text{ the coefficient of } x^0 \text{ in } x^{h_\infty+1}\cdot(\mathbf{h}-\mathbf{e}_j),$$

$$\beta_{jk} = \prod_{i\in\{0,\ldots,n-1\}\smallsetminus\{j\}} (i-j)^{h_i}, \text{ the coefficient of } (j+x)^{-1} \text{ in } (k-j)(j+x)^{-h_j-1}\cdot(\mathbf{h}-\mathbf{e}_k),$$

$$\beta'_{jk} = \prod_{i\in\{0,\ldots,n-1\}\smallsetminus\{k\}} (i-k)^{h_i}, \text{ the coefficient of } (k+x)^{-1} \text{ in } (j-k)(k+x)^{-h_k-1}\cdot(\mathbf{h}-\mathbf{e}_j).$$

(The values of the coefficients β_j, β_{jk} and β'_{jk} are as stated as a consequence of Corollary 3.6.5 and β'_j as a consequence of Corollary 3.6.7.)

Here is why. First note that the second entries (those involving $h_\infty, h_1, \ldots, h_{n-1}$) of all the coordinates are correct: they can be read off the vectors in the second coordinates of the righthand sides of (3.4)–(3.7). Secondly, note that the case of $\Phi(g(r,\mathbf{e}_j)(r,\mathbf{e}_k)^{-1})$ is identical to that of $\Phi(g(r,\mathbf{e}_k)(r,\mathbf{e}_j)^{-1})$, save that j and k are interchanged. So we will only address the former.

Here is why the $(\mathbf{a}^i, h_i)$ indicated by ellipses in the above four equations are indeed the same as the corresponding $(\mathbf{a}^i, h_i)$ in $\Phi(g)$. We compare the $(*+x)^{-1}, (*+x)^{-2}, \ldots$ coefficients of the appropriate polynomials.

Case $\Phi(g(r,\mathbf{e}_j))$. The polynomials in question are $(*+x)^{-h_*}(f+r\cdot\mathbf{h})$ and $(*+x)^{-h_*} f$.

The relevant coefficients agree when $* \notin \{\infty, j\}$ since those of

$$(* + x)^{-h_*} r \cdot \mathbf{h} \;=\; r \prod_{l \in \{0,1,\ldots,n-1\} \setminus \{*\}} (l + x)^{h_l}$$

are all zero by Corollary 3.6.4.

Case $\Phi(g(r, \mathbf{e}_j)^{-1})$. Similarly, the relevant coefficients of

$$(* + x)^{-h_*}(-r \cdot (\mathbf{h} - \mathbf{e}_j)) \;=\; -r(j + x)^{-1} \prod_{l \in \{0,1,\ldots,n-1\} \setminus \{*\}} (l + x)^{h_l}$$

are all zero when $* \notin \{\infty, j\}$ by the same corollary.

Case $\Phi(g(r, \mathbf{e}_j)(r, \mathbf{e}_k)^{-1})$. Similarly, when $* \notin \{\infty, j, k\}$ the relevant coefficients of $(* + x)^{-h_*}(k - j)r \cdot (\mathbf{h} - \mathbf{e}_k)$ are all zero. And, for the $* = \infty$ case, the coefficients of $x^0, x^1, \ldots$ in $x^{h_\infty}(k - j)r \cdot (\mathbf{h} - \mathbf{e}_k)$ are all zero by Corollary 3.6.6 (with $k_0 = h_\infty + h_0$, $k_k = h_k - 1$ and $k_l = h_l$ for all other l) since $h_\infty + h_0 + \cdots + h_{n-1} - 1 = -1 < 0$.

Now we turn to the coordinates which differ after multiplication by a generator.

Why the ∞-coordinate of $\Phi(g(r, \mathbf{e}_j))$ *is* $\left(\left(a_2^\infty, a_3^\infty, \ldots\right), h_\infty - 1\right)$. We need to determine the coefficients of $x^0, x^1, \ldots$ in $x^{h_\infty - 1}(f + r \cdot \mathbf{h})$. Those of $x^{h_\infty - 1} r \cdot \mathbf{h}$ are all zero by Corollary 3.6.6. Lemma 3.6.8(i) tells us that the coefficients of $x^0, x^1, \ldots$ in $x^{h_\infty - 1} f$ equal those of $x^1, x^2, \ldots$ in $x^{h_\infty} f$, and so are $a_2^\infty, a_3^\infty, \ldots$ by definition.

Why the j-coordinate of $\Phi(g(r, \mathbf{e}_j)^{-1})$ *is* $\left(\left(a_2^j, a_3^j, \ldots\right), h_j - 1\right)$. The $(j + x)^{-1}, (j + x)^{-2}, \ldots$ coefficients of $(j + x)^{-h_j + 1}(f - r \cdot (\mathbf{h} - \mathbf{e}_j))$ are $a_2^j, a_3^j, \ldots$ since those of $(j + x)^{-h_j + 1} r \cdot (\mathbf{h} - \mathbf{e}_j) = (j + x)^{-h_j} r \cdot \mathbf{h}$ are all zero by Corollary 3.6.4 and those of $(j + x)^{-h_j + 1} f$ equal the $(j + x)^{-2}, (j + x)^{-3}, \ldots$ coefficients of $(j + x)^{-h_j} f$ by Lemma 3.6.8(ii).

Why the k-coordinate of $\Phi(g(r, \mathbf{e}_j)(r, \mathbf{e}_k)^{-1})$ *is* $\left(\left(a_2^k, a_3^k, \ldots\right), h_k - 1\right)$. The $(k + x)^{-1}, (k +$

$x)^{-2}, \ldots$ coefficients of $(k+x)^{-h_k+1}(f+(k-j)r\cdot(\mathbf{h}-\mathbf{e}_k))$ are $a_2^k, a_3^k, \ldots$ similarly to the previous case.

Why the j-coordinate of $\Phi(g(r,\mathbf{e}_j))$ is $\left(\left(\alpha_j + r\beta_j, a_1^j, a_2^j, \ldots\right), h_j+1\right)$. We need to check that the $(j+x)^{-1}, (j+x)^{-2}, \ldots$ coefficients of $(j+x)^{-h_j-1}(f+r\cdot\mathbf{h})$ are $\alpha_j + r\beta_j, a_1^j, a_2^j, \ldots$. The $(j+x)^{-2}, (j+x)^{-3}, \ldots$ coefficients are $a_1^j, a_2^j, \ldots$ since those of $(j+x)^{-h_j-1}r\cdot\mathbf{h} = (j+x)^{-1}((j+x)^{-h_j}r\cdot\mathbf{h})$ are all zero by Corollary 3.6.4 and those of $(j+x)^{-h_j-1}f$ equal the $(j+x)^{-1}, (j+x)^{-2}, \ldots$ coefficients of $(j+x)^{-h_j}f$ by Lemma 3.6.8(*ii*) for the same reasons as in earlier cases. Its $(j+x)^{-1}$-coefficient is $\alpha_j + r\beta_j$ by definition.

Why the ∞-coordinate of $\Phi(g(r,\mathbf{e}_j)^{-1})$ is $\left(\left(\alpha'_j - r\beta'_j, a_1^\infty, a_2^\infty, \ldots\right), h_\infty+1\right)$. The $x^0, x^1, \ldots$ coordinates of $x^{h_\infty+1}(f - r\cdot(\mathbf{h}-\mathbf{e}_j))$ are $\alpha'_j - r\beta'_j, a_1^\infty, a_2^\infty, \ldots$ for similar reasons.

Why the j-coordinate of $\Phi(g(r,\mathbf{e}_j)(r,\mathbf{e}_k)^{-1})$ is $\left(\left(\alpha_{jk} + r\beta_{jk}, a_1^j, a_2^j, \ldots\right), h_j+1\right)$. The $(j+x)^{-1}, (j+x)^{-2}, \ldots$ coefficients of $(j+x)^{-h_j-1}(f+(k-j)r\cdot(\mathbf{h}-\mathbf{e}_k))$ are $\alpha_{jk} + r\beta_{jk}, a_1^j, a_2^j, \ldots$ likewise.

The set of vertices $\mathcal{V}$ in $\mathcal{H}_n(R)$ that are reached by traveling from $\Phi(g)$ along a single edge partitions into $(n+1)n$ subsets: travel along the unique downwards edge in one of the $n+1$ coordinate-trees, travel upwards along one of an R-indexed family of edges in another, and remain stationary in the rest.

As we have seen, for each element x of the generating set

$$\left\{ (r,\mathbf{e}_i),\ (r,\mathbf{e}_j)(r,\mathbf{e}_k)^{-1} \;\middle|\; r\in R,\ 0\le i,j,k\le n-1 \text{ and } j<k \right\}$$

the location of $\Phi(gx)$ and $\Phi(gx^{-1})$ falls in one of these subsets. Thereby the union of this generating set together with the set of the inverses of its elements has $(n+1)n$ subsets which correspond to the $(n+1)n$ subsets of $\mathcal{V}$. Indeed, each subset contains one R-indexed family of generators or inverse-generators.

Since α_j and β_j do not depend on r and β_j is invertible (since $2, 3, \ldots, n-1$ are invertible), for fixed j, the map $r \mapsto \alpha_j + r\beta_j$ is a bijection $R \to R$. So $g(r, e_j) \mapsto \Phi(g(r, e_j))$ is a bijection between a subset of the neighbors of g in the Cayley graph C and one of these subsets of $\mathcal{V}$.

Likewise, because $\beta'_j, \beta_{jk}, \beta'_{jk}$ are invertible (since $2, 3, \ldots, n-1$ are invertible),

$$r \mapsto \alpha'_j - r\beta'_j,$$

$$r \mapsto \alpha_{jk} + r\beta_{jk},$$

$$r \mapsto \alpha'_{jk} + r\beta'_{jk}$$

are all bijections $R \to R$. So as α'_j, α_{jk}, α'_{jk}, β'_j, β_{jk}, and β'_{jk} do not depend on r, there are similar bijections between subsets of neighbors of g and subsets of $\mathcal{V}$. Combined, these bijections give a bijection from the neighbors of g in C to the neighbors of $\Phi(g)$ in $\mathcal{V}$.

There are no double-edges and no edge-loops in either graph: for the 1-skeleton of $\mathcal{H}_n(R)$ this is straightforward from the definition, and it therefore follows from the above for C. So Φ extends to an isomorphism from C to the 1-skeleton of $\mathcal{H}_n(R)$, completing our proof.

CHAPTER 4

DISTORTION IN Γ_2

Throughout this chapter we will use the presentation

$$\left\langle\, a, s, t \;\middle|\; [a, a^t] = 1,\ [s, t] = 1,\ a^s = aa^t \,\right\rangle$$

of Γ_2.

In [18], Cleary showed that the subgroup $\Gamma_1 = \langle a, t \rangle$ is exponentially distorted in Γ_2. For completeness, we include a short proof of this result in Section 4.2. Then in Section 4.3 we show that, in contrast, the subgroup $\langle a \rangle$ is undistorted in Γ_2.

4.1 Motivation

Our original motivation for studying distortion came from trying to show a quadratic upper bound for the filling length function of Γ_2. If the distortion were to occur, it would lead to 'shortcut diagrams' (see Section 5 of [30] for details) which could be useful tools in the construction of small van Kampen diagrams. If the subgroup $\langle a \rangle$ were to be exponentially distorted, the shortcut diagrams could have been helpful to show that the filling length of Γ_2 grows at most quadratically.

We were able to show that, in fact, the subgroup $\langle a \rangle$ is undistorted in Γ_2, which is a nice result on its own, but it fails to lead to an upper bound on filling length using shortcut diagrams.

In Chapter 5, we work towards showing that there is a quadratic upper bound for the filling length function of Γ_2 modulo a combinatorial open question about propagating configurations.

4.2 The subgroup $\Gamma_1 = \langle a, t\rangle$ is exponentially distorted

It is well-known and not hard to see that the subgroup Γ_1 is exponentially distorted in Γ_2. Recall the lamplighter model discussed in Section 2.2 and consider a word $w_n = a^{s^n} \in \Gamma_2$ whose configuration $\mathcal{K}_{w_n}$ has zeros at all the lattice points, except at $(0, n)$ where the entry is 1. Using the relation $a^s = aa^t$, we can rewrite it as an element in $\langle a, t\rangle$ by propaging this configuration to the t-axis. This gives a configuration $\mathcal{K}_{w'_n}$ that is zero everywhere except at points $(i, 0)$ where the entries are $\binom{n}{i}$ for $0 \le i \le n$. Thus, $w'_n = a^{\binom{n}{0}} t a^{\binom{n}{1}} t a^{\binom{n}{2}} t \cdots t a^{\binom{n}{n}} t^{-n} \in \Gamma_1$ and so $|w'_n| = 2n + \sum_{i=0}^{n} \binom{n}{i} = 2n + 2^n$. From Corollary 2.3.3 it can be seen that, in fact, this is a shortest word in the subgroup Γ_1 representing w_n since the lamplighter must flip the switches at least 2^n times and must travel at least a distance of $2n$ to get to the lamp at location $(n, 0)$ and back to the origin. But as an element of Γ_2, w_n has length $|w_n| = |a^{s^n}| = 2n + 1$. So, the subgroup Γ_1 is exponentially distorted in Γ_2.

4.3 The subgroup $\langle a\rangle$ is undistorted

We first prove the following result, which is used in the proof of Theorem 4, which is in turn used to prove Theorem 3.

Proposition 4.3.1. *Suppose $\mathcal{K}$ is a configuration equivalent to $\mathcal{K}_\epsilon$. If M is the entry*

0
0 0
0 -1 0
1 0 0 1
0 0 -2 0 0
0 2 0 0 2 0
-3 0 0 -1 0 0 -3
0 0 2 0 0 2 0 0
0 -2 0 0 -2 0 0 -2 0
3 0 0 3 0 0 3 0 0 3
0 0 -2 0 0 80 0 0 -2 0 0
0 2 0 0 3 0 0 3 0 0 2 0
-1 0 0 -2 0 0 -1 0 0 -2 0 0 -1
0 0 2 0 0 1 0 0 1 0 0 2 0 0
0 -1 0 0 0 0 0 -1 0 0 0 0 0 -1 0
0 0 0 2 0 0 2 0 0 2 0 0 2 0 0 0
0 0 0 0 0 -1 0 0 -2 0 0 -1 0 0 0 0 0

Figure 4.1: Trivial configuration with norm 160 and maximum value of 80.

with maximum absolute value in $\mathcal{K}$, then $|M| \leq |\mathcal{K}| - |M|$. (In other words, the maximum of the absolute values of the entries in $\mathcal{K}$ is less than or equal to the sum of the absolute values of all other entries in the configuration.)

Figure 4.1 shows a trivial configuration $\mathcal{K}$ with the maximal value $M = 80$ and $80 = |M| = |\mathcal{K}| - |M| = 160 - 80$. This shows that the bound in Proposition 4.3.1 cannot be improved. (Other simpler examples exist.)

Using Proposition 4.3.1, we will prove:

Theorem 4. *If a configuration $\mathcal{K}$ is equivalent to a configuration where all entries are zeros except for one entry which is M, then $\mathcal{K}$ has norm $|\mathcal{K}| \geq |M|$.*

Proof of Proposition 4.3.1. Without loss of generality, assume that the maximal element appearing in the configuration $\mathcal{K}$ is positive and is located at $(0, 0)$.

Let $r_0 = 1$, $r_1 = \frac{-1+i\sqrt{3}}{2}$ and $r_{-1} = \frac{-1-i\sqrt{3}}{2}$ be the three cube roots of unity. Note that $|r_0| = |r_1| = |r_{-1}| = 1$ and $r_0 + r_1 + r_{-1} = 0$.

The idea behind the proof is to realize the equivalence between $\mathcal{K}$ and $\mathcal{K}_\epsilon$ as a sequence of basic moves consisting of adding (or subtracting) a translate of the triangle ${}_{-1}{}^{1}{}_{-1}$, and to construct an invariant that remains zero after each basic move.

From the definition of the equivalence of configurations and the fact that $\mathcal{K}$ is equivalent to $\mathcal{K}_\epsilon$, we know that there is a finite sequence of configurations starting with $\mathcal{K}_\epsilon$ and ending with $\mathcal{K}$ in which each configuration differs from the next only in one triangle of adjacent integers which is ${}_{b}{}^{a}{}_{c}$ in one and is ${}_{b+z}{}^{a-z}{}_{c+z}$ for some $z \in \mathbb{Z}$ in the other. Hence, we can obtain $\mathcal{K}$ from the all-zero configuration by adding $z_{i,j}$ times ${}_{-1}{}^{1}{}_{-1}$ for some $z_{i,j} \in \mathbb{Z}$ to the triangle located at ${}_{(i,j-1)}{}^{(i,j)}{}_{(i+1,j-1)}$ for each $(i, j) \in \mathbb{Z}^2$ where only finitely many of $z_{i,j}$ are non-zero.

Thus, the entry at each location (i, j) in $\mathcal{K}$ is $d_{i,j} = z_{i,j} - z_{i-1,j+1} - z_{i,j+1}$.

Consider a new configuration $\mathcal{K}''$ obtained from $\mathcal{K}$ in two steps: first multiply those integers at every even row by -1 (so we can associate this configuration to the one obtained from all-zero configuration by adding $\pm z_{i,j}$ times ${}_{1}{}^{1}{}_{1}$ to the triangle located at ${}_{(i,j-1)}{}^{(i,j)}{}_{(i+1,j-1)}$ for each $(i, j) \in \mathbb{Z}^2$ where only finitely many of $z_{i,j}$ are non-zero), and then multiply the whole grid pointwise by

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & \vdots & & & & & & & & \\
 & \cdots & & r_0 & & r_1 & & r_{-1} & & r_0 & & r_1 & & r_{-1} & & r_0 & & \cdots \\
\cdots & & r_1 & & r_{-1} & & r_0 & & r_1 & & r_{-1} & & r_0 & & r_1 & & r_{-1} & & \cdots \\
 & \cdots & & r_0 & & r_1 & & r_{-1} & & r_0 & & r_1 & & r_{-1} & & r_0 & & \cdots \\
\cdots & & r_1 & & r_{-1} & & r_0 & & r_1 & & r_{-1} & & r_0 & & r_1 & & r_{-1} & & \cdots \\
 & \cdots & & r_0 & & r_1 & & r_{-1} & & r_0 & & r_1 & & r_{-1} & & r_0 & & \cdots \\
\cdots & & r_1 & & r_{-1} & & r_0 & & r_1 & & r_{-1} & & r_0 & & r_1 & & r_{-1} & & \cdots \\
 & & & & & & & & \vdots & & & & & & & &
\end{array}$$

where any r_0 is at $(0,0)$. Denote the resulting entries at points (i,j) by $c_{i,j}$. Note that these operations did not change the complex norm of the numbers that we started with. Each addition of ${}_1{}^1{}_1$ corresponds to the addition of either

$$ {}_{r_1}{}^{r_0}{}_{r_{-1}} \qquad \text{or} \qquad {}_{r_{-1}}{}^{r_1}{}_{r_0} \qquad \text{or} \qquad {}_{r_0}{}^{r_{-1}}{}_{r_1} $$

Since $r_0 + r_1 + r_{-1} = 0$, adding ${}_1{}^1{}_1$ at any lattice point keeps the sum (regular sum, not in an absolute value) of all the entries at zero. Hence, $\sum_{i,j\in\mathbb{Z}} c_{i,j} = 0$ and

$$|M| = |c_{0,0}| = \left| \sum_{i,j\in\mathbb{Z}\setminus(0,0)} c_{i,j} \right| \leq \sum_{i,j\in\mathbb{Z}\setminus(0,0)} |c_{i,j}|,$$

which is exactly the sum of the norms of all other elements in the original configuration as claimed. □

Remark 4.3.2. Note that in the last inequality, equality occurs if and only if the non-zero $c_{i,j}$'s are complex numbers with the same argument and sign. Equivalently, the entries in $\mathcal{K}$ are non-zero only at locations corresponding to r_0's and even rows have negative entries, while odd rows have positive entries.

Note that the non-zero entries in Figure 4.1 are in columns as predicted by Remark 4.3.2 and that the entries in the rows (if we disregard M) alternate signs.

Proof of Theorem 4. Without the loss of generality, assume that M is located at $(0,0)$ in the second configuration.

First, let us assume that the entry at location $(0,0)$ in $\mathcal{K}$ is zero. Then consider configuration $\mathcal{K}'$ which is obtained from configuration $\mathcal{K}$ by subtracting M at location $(0,0)$ (obtaining $-M$ at that location). Clearly, the resulting configuration $\mathcal{K}'$ is equivalent to $\mathcal{K}_\epsilon$. If M is not the largest entry in absolute value, then $|\mathcal{K}| \geq |M|$ and we are done. If M is the maximal value appearing in $\mathcal{K}'$ then by Proposition 4.3.1, $|M| \leq |\mathcal{K}'| - |M| = |\mathcal{K}|$ and we are done.

Now if the entry at $(0,0)$, call it $c_{0,0}$, in $\mathcal{K}$ were non-zero, we can use the above proof where we replace $\mathcal{K}$ by the same configuration but with a zero at location $(0,0)$, call it $\overline{\mathcal{K}}$, and replace M by $\overline{M} = M - c_{0,0}$. The above proof gives $|\overline{\mathcal{K}}| \geq |\overline{M}| = |M - c_{0,0}|$, so $|\mathcal{K}| = |\overline{\mathcal{K}}| + |c_{0,0}| \geq |M - c_{0,0}| + |c_{0,0}| \geq |M|$. $\square$

We are now ready to prove Theorem 3, which says that the subgroup $\langle a \rangle$ of Γ_2 is undistorted in Γ_2.

Proof of Theorem 3. Suppose we have an element $a^M \in \langle a \rangle$ which can be represented in Γ_2 by some word w with the configuration $\mathcal{K}_w$. We know that for any word, $|w| \geq |\mathcal{K}_w|$ and by Theorem 4, $|K_w| \geq |M|$. Hence, for all $M \in \mathbb{Z}$, a^M cannot be expressed as a word of length less than $|M|$ in Γ_2. $\square$

CHAPTER 5

FILLING LENGTH

Recall from Section 1.9 that the filling length of a trivial word w in a finitely presented group, is the minimal integer L such that w can be converted to the empty word through words of length at most L by applying relators and freely reducing/expanding. The filling length function $\mathrm{FL}\colon \mathbb{N} \to \mathbb{N}$, for a finitely presented group Γ, is defined by

$$\mathrm{FL}_\Gamma(n) \;=\; \max\left\{ \mathrm{FL}(w) \;\middle|\; w = 1 \text{ in } \Gamma \text{ and } |w| \leq n \right\}.$$

Recall that Γ_2 has the presentation

$$\Gamma_2 = \left\langle a, s, t \;\middle|\; [a, a^t] = 1, a^s = aa^t, [s, t] = 1 \right\rangle.$$

and can be viewed using the two-dimensional model involving the combinatorics of Pascal's triangle.

Our conjecture is that the filling length of Γ_2 is quadratic. In this chapter, we work toward finding an upper bound for this function. We are able to show the upper bound modulo a combinatorial open question on the configurations. One would also need to find a quadratic lower bound on the filling length function of Γ_2 to prove the conjecture.

5.1 Open questions on pushing configurations

Recall the rhombic grid model for Γ_2 introduced in Section 2.2. The job of finding an upper bound on the filling length of Γ_2 can be broken into two components:

bounding the number of times the lamplighter flips switches and bounding the length of the lamplighter's path. In Section 5.2, we show that we can bound the lamplighter's path. Here we pose some open questions, which if answered would give us a bound on the switch-flips, and hence a bound on the filling length.

Definition 5.1.1. *Let* $\mathbf{B_N} = \left\{ (i, j) \,\middle|\, |i| + |j| \leq N \right\}$ *denote the ball of size N around the origin in the lamplighter model.*

Remark 5.1.2. In the model, the ball B_N looks like a tilted rectangle. See Figure 5.1 for switches located in B_4. For any N, the ball B_N contains $N^2 + (N + 1)^2$ lattice points (in particlar, there are $\leq 5N^2$ lattice points in B_N).

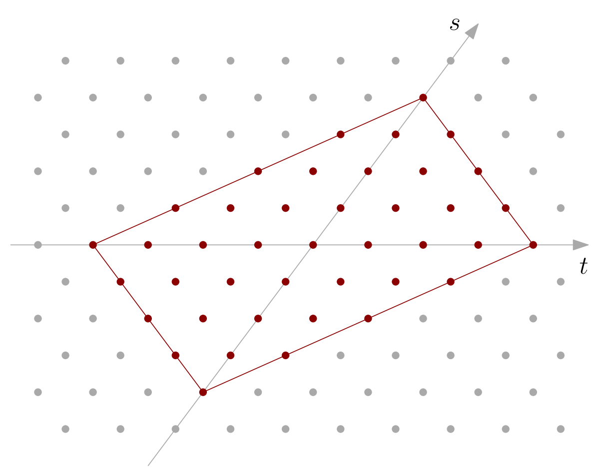

Figure 5.1: The dark points show elements of B_4.

Next, we consider a special (systematic) way of altering the switches (in a configuration) that the lamplighter presses, while not changing the word that the configuration represents, called *propagating row-by-row*.

Given a configuration for a word w lying entirely above the t-axis, we can propagate it row-by-row toward the t-axis via the following:

- consider the top row which has non-zero entries
- work from left to right replacing each integer in that row by a zero by using the equivalence

$$\begin{smallmatrix} & z & \\ x & & y \end{smallmatrix} \mapsto \begin{smallmatrix} & 0 & \\ x+z & & y+z \end{smallmatrix}$$

- repeat

See Figure 5.2 for an example of propagating row-by-row.

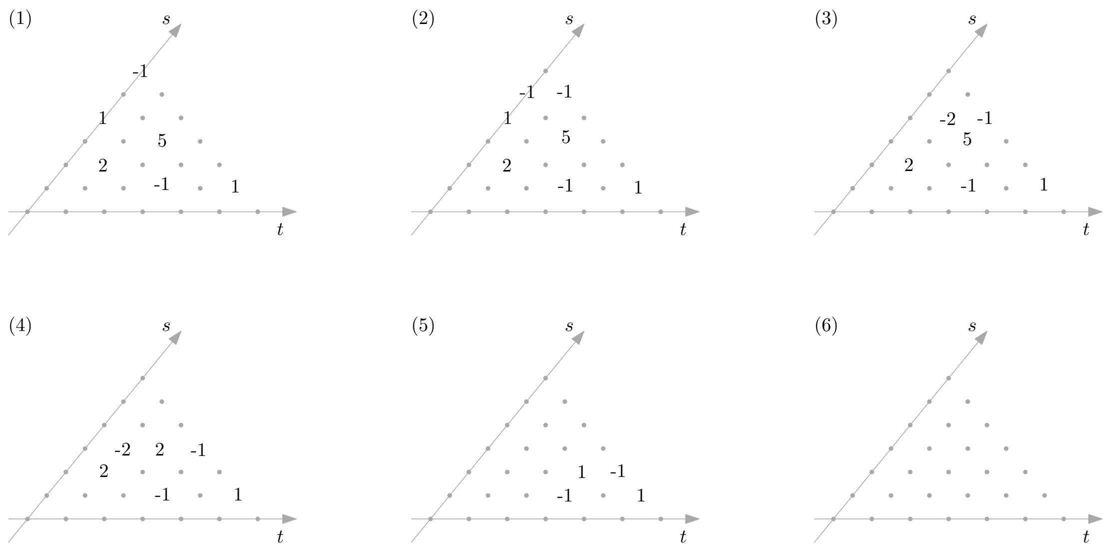

Figure 5.2: Example of propagating row-by-row.

Open Question 5.1.3. *Does there exist a constant C such that for any trivial word w in Γ_2 with configuration $\mathcal{K}_w$ whose support lies above the t-axis, when we propagate the configuration $\mathcal{K}_w$ row-by-row to the t-axis the norm of each of the intermediate configurations is at most $C \cdot |\mathcal{K}_w|$?*

Or a less restrictive question,

Open Question 5.1.4. *Does there exist a constant C such that given any trivial word $w \in \Gamma_2$ whose configuration is supported on B_N for some $N \in \mathbb{Z}$, there exists a sequence* *of configurations* $\mathcal{K}^1, \mathcal{K}^2, \ldots, \mathcal{K}^p$ *with* $\mathcal{K}^1 = \mathcal{K}_w$ *and* $\mathcal{K}^p = \mathcal{K}_\epsilon$*, each* $\mathcal{K}^i$ *is supported within* $B_{C \cdot N}$*, and* $|K^i| \leq C \cdot |K_w|$ *for all i.*

Note that in the Open Question 5.1.3, if $|w| = N$ then $w \in B_N$ and all the configurations that we encounter while propagating w row-by-row stay inside B_N. Since w represents a trivial word supported above the t-axis, when we reach the t-axis, we must have all zeros in the configuration.

Clearly, a positive answer to the Open Question 5.1.3 implies a positive answer to the Open Question 5.1.4 since we can force the sequence of the configurations to be the ones we encounter by propagating row-by-row.

5.2 How our open questions relate to the filling length of Γ_2

In this section, we will show how a positive answer to either one of the above questions leads to an upper bound on the filling length of Γ_2. A positive answer to the Open Question 5.1.3 leads to a quadratic upper bound on the filling length, while a positive answer to the more general question leads to a cubic upper bound. In fact, any "systematic-enough" way of propagating a configuration (such as propagating row-by-row) would lead to a quadratic upper bound on the filling length of Γ_2.

Theorem 5. *Assume there exists a constant C such that given any trivial word w in* Γ_2 *whose support lies above the t-axis, when we propagate its configuration* $\mathcal{K}_w$ *row-by-row to the t-axis the norm of each of the intermediate configurations stays at most* $C \cdot |\mathcal{K}_w|$*. Then* $\mathrm{FL}_{\Gamma_2}(n) \preccurlyeq n^2$.

To prove the theorem, we need to consider any trivial word w written in $\langle a, t, s\rangle$, and exibit an algorithm that would transform this word to an empty word ϵ through the words of length at most quadratic in $|w|$. We will prove Theorem 5 in two steps:

(i) We show that we can restrict our attention to the trivial words w, where the lamplighter only visits the first quadrant.

(ii) We show that after each application of ${}_{x}{}^{z}{}_{y} \sim {}_{x+z}{}^{0}{}_{y+z}$, we can re-write w to be in the form $w' = \prod a^{c_{n,m} s^n t^m}$ for some integers $c_{n,m}$, where the product is in lexicographic order of the pairs (n, m): for each consecutive pair $a^{s^{n_i} t^{m_i}} a^{s^{n_j} t^{m_j}}$, we have $n_i \leq n_j$ and if $n_i = n_j$, then $m_i < m_j$, while increasing the length of the word at most quadratically.

5.2.1 Proof of Theorem 5

The first part can be seen from the following: Given a trivial word w with $|w| = N$. Let $w' = t^N s^N w s^{-N} t^{-N}$. Then clearly, w' is a trivial word that lies entirely in the first quadrant. The length of w' is linear in the length of w, since $|w'| \leq 5N = 5|w|$. Thus, if we show that the filling length of each trivial word lying entirely in the first quadrant is some function $F(N)$, it will imply that the filling length of all trivial words is $\simeq F(N)$ as well.

The second part follows from a few lemmas below.

Lemma 5.2.1. *The filling length of* $[a, a^{t^n}]$ *is at most quadratic with respect to n. Namely,* $\mathrm{FL}([a, a^{t^n}]) \leq 46n^2$.

Proof. Let us first only consider $n > 0$. We will prove the lemma by inducting on n. As the base case, we know that $\mathrm{FL}([a, a^t]) = 8$. For $n > 1$, we can transform $[a, a^{t^{n+1}}]$ to a trivial word using the following moves (see Figure 5.3).

$$\begin{aligned}
[a, a^{t^{n+1}}] &= a^{-1}a^{-t^{n+1}}aa^{t^{n+1}} && \\
&= a^{-1}a^{-t^{n+1}}a a^{-1}a^{-t^{n}}aa^{t^{n}} a^{t^{n+1}} && \mathrm{FL}([a, a^{t^n}]) + 4n + 8 \\
&= a^{-1}a^{-t^{n+1}}a^{-t^{n}}aa^{t^{n}}tt^{-1}a^{t^{n+1}} && 8n + 12 \\
&= a^{-1}a^{-t^{n+1}}a^{-t^{n}}aa^{t^{n}}taa^{t^{n}}a^{-1}a^{-t^{n}}t^{-1}a^{t^{n+1}} && \mathrm{FL}([a, a^{t^n}]) + 8n + 12 \\
&= a^{-1}a^{-t^{n+1}}a^{-t^{n}}aa^{t^{n}}a^{t}a^{t^{n+1}}a^{-t}a^{-t^{n+1}}a^{t^{n+1}} && 12n + 22 \\
&= a^{-1}a^{-t^{n+1}}a^{-t^{n}}aa^{t^{n}}a^{t}tt^{-1}a^{t^{n+1}}a^{-t} && 12n + 22 \\
&= a^{-1}a^{-t^{n+1}}a^{-t^{n}}aa^{t^{n}}a^{t}ta^{-1}a^{-t^{n-1}}aa^{t^{n-1}}t^{-1}a^{t^{n+1}}a^{-t} && \mathrm{FL}([a, a^{t^{n-1}}]) + 8n + 18 \\
&= a^{-1}a^{-t^{n+1}}a^{-t^{n}}aa^{t^{n}}a^{t}a^{-t}a^{-t^{n}}a^{t}a^{t^{n}}a^{t^{n+1}}a^{-t} && 12n + 24 \\
&= a^{-1}a^{-t^{n+1}}a^{-t^{n}}aa^{t}a^{t^{n}}a^{t^{n+1}}a^{-t} && 12n + 24 \\
&= a^{-1}a^{-t^{n+1}}a^{-t^{n}}aa^{t}a^{t^{n}}a^{t^{n+1}}a^{-t} && 8n + 16 \\
&= a^{-1}sa^{-t^{n}}s^{-1}sas^{-1}sa^{t^{n}}s^{-1}a^{-t} && 8n + 16 \\
&= a^{-1}sa^{-t^{n}}aa^{t^{n}}s^{-1}a^{-t} && 4n + 13 \\
&= a^{-1}sa^{-t^{n}}aa^{t^{n}}a^{-t^{n}}a^{-1}a^{t^{n}}as^{-1}a^{-t} && \mathrm{FL}([a, a^{t^n}]) + 4n + 9 \\
&= a^{-1}sas^{-1}a^{-t} && 8n + 13 \\
&= a^{-1}aa^{t}a^{-t} && 8 \\
&= 1. &&
\end{aligned}$$

For each line, the expression on the right hand side gives an upper bound on the length of the intermediate words we encounter while transforming the word from the line just above to the word on that line.

We see that at each step, the length of the intermediate words never exceeds

$\mathrm{FL}([a, a^{t^n}]) + 12n + 24$. So

$$\begin{aligned}
\mathrm{FL}([a, a^{t^{n+1}}]) &\leq \mathrm{FL}([a, a^{t^n}]) + 12n + 24 \\
&\leq \mathrm{FL}([a, a^{t^{n-1}}]) + 12(n-1) + 24 + 12n + 24 \\
&\leq \ldots \leq \\
&\leq \mathrm{FL}([a, a^{t}]) + 12 + 24 + 12 \cdot 2 + 24 + \ldots + 12n + 24 \\
&\leq 8 + \frac{12n(n+1)}{2} + 24n = 6n^2 + 30n + 8 \leq 44n^2.
\end{aligned}$$

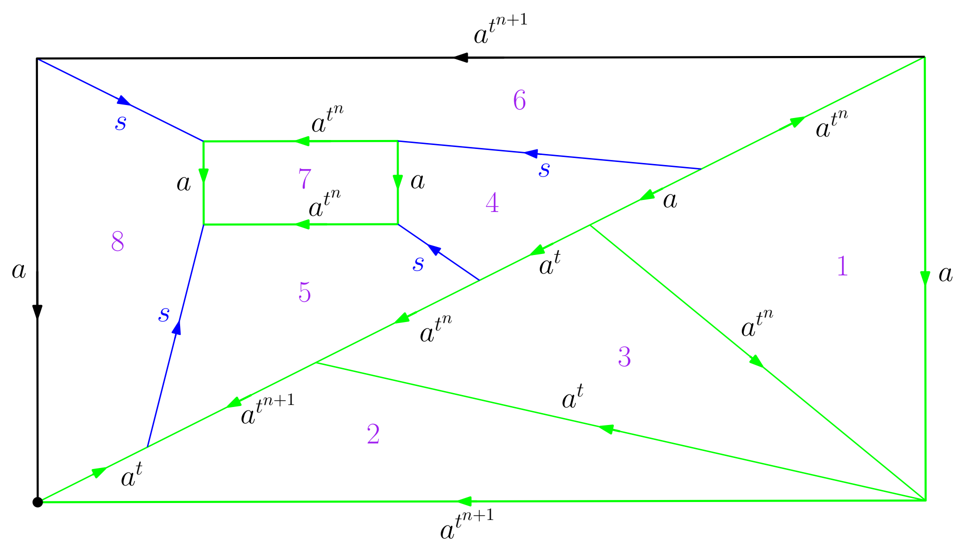

Figure 5.3: A sketch of a van Kampen diagram for $[a, a^{t^{n+1}}]$.

Figure 5.3 shows a sketch of a van Kampen diagram for the word $[a, a^{t^{n+1}}]$. The purple numbers inside the cells indicate the order in which we collapse these cells. We are able to collapse the green cells by induction — cells 1, 2 and 7 cost $\mathrm{FL}([a, a^{t^n}])$ to collapse, while cell 3 costs $\mathrm{FL}([a, a^{t^{n-1}}])$.

Now for $n < 0$, we get a similar bound by

$$
\begin{aligned}
[a, a^{t^{-|n|}}] &= a^{-1}a^{-t^{-|n|}}aa^{t^{-|n|}} \\
&= a^{-1}t^{-|n|}a^{-1}t^{|n|}at^{-|n|}at^{|n|} \\
&= t^{-|n|}t^{|n|}a^{-1}t^{-|n|}a^{-1}t^{|n|}at^{-|n|}at^{|n|} \\
&= t^{-|n|}[a^{t^{|n|}}, a]t^{|n|} \\
&= t^{-|n|}t^{|n|} \\
&= 1.
\end{aligned}
$$

Thus, $\mathrm{FL}([a, a^{t^n}]) \leq 46|n|^2$ for all $n \in \mathbb{Z}$. □

Lemma 5.2.2. *The filling length of* $[a, a^{s^n}]$ *is at most quadratic with respect to* n*. Namely,* $\mathrm{FL}([a, a^{s^n}]) \leq 46n^2$*.*

Proof. The proof is similar to that of Lemma 5.2.1 (see Figure 5.4). Using the relation $a^t = a^{-1}a^s$, we see that $(a^{s^n})^t = (a^t)^{s^n} = (a^{-1}a^s)^{s^n} = a^{-s^n}a^{s^{n+1}}$ (so some of the arrows in the figure are flipped). □

Lemma 5.2.3. *The filling length of* $[a, a^{s^m t^n}]$ *is at most quadratic with respect to* $|n|+|m|$*. Namely,* $\mathrm{FL}([a, a^{s^m t^n}]) \leq 48(|n| + |m|)^2$*.*

Proof. First, let us consider only $n \geq 0$. For a given m, we will induct on n. The previous lemma gives the base case since we showed that $\mathrm{FL}([a, a^{s^m}]) \leq 46|m|^2$. See Figure 5.5 and calculations below for details of the induction on n. We use the fact that $(a^{s^m t^n})^s = (a^s)^{s^m t^n} = (aa^t)^{s^m t^n} = a^{s^m t^n}a^{s^m t^{n+1}}$.

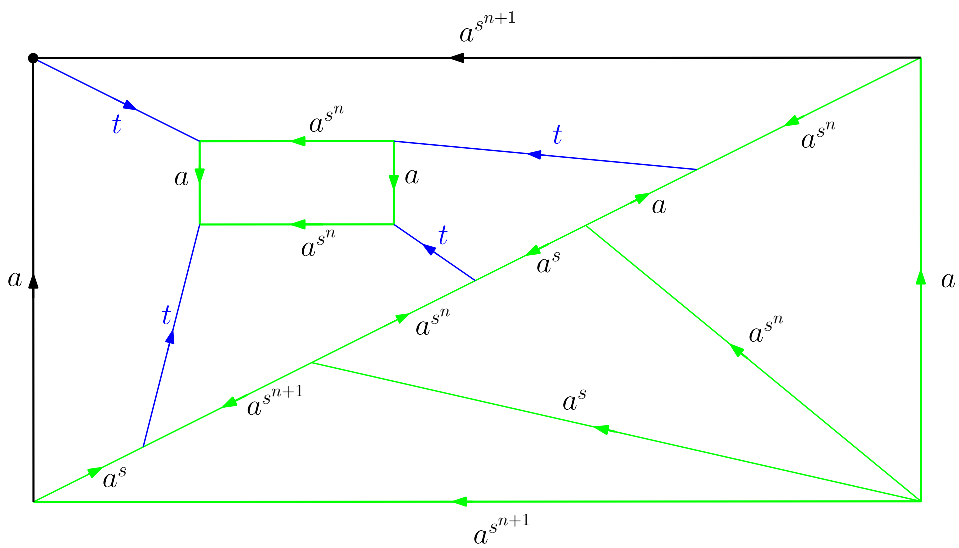

Figure 5.4: A sketch of a van Kampen diagram for $[a, a^{s^{n+1}}]$.

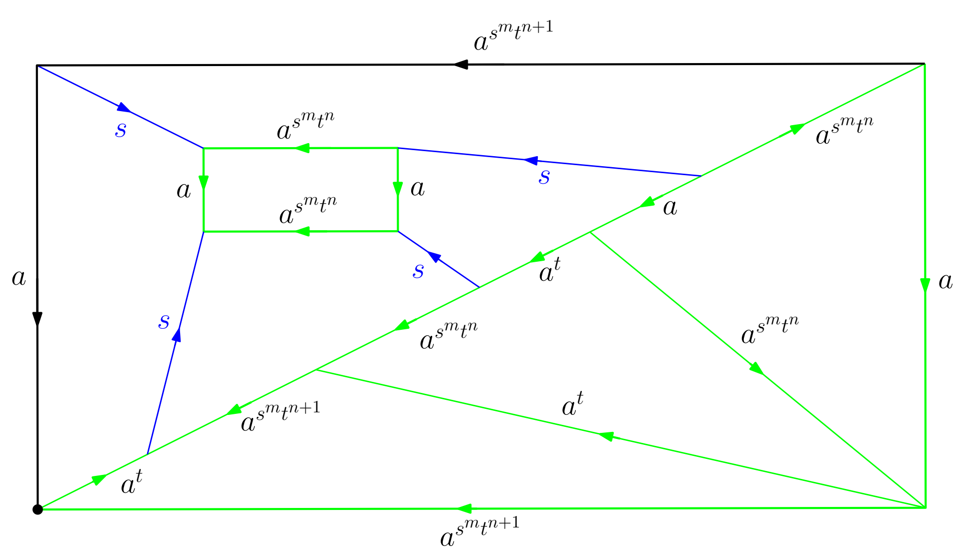

Figure 5.5: A sketch of a van Kampen diagram for $[a, a^{s^m t^{n+1}}]$.

More explicitly,

$$
\begin{aligned}
[a, a^{s^m t^{n+1}}] &= a^{-1} a^{-s^m t^{n+1}} a a^{s^m t^{n+1}} \\
&= a^{-1} a^{-s^m t^{n+1}} a a^{-1} a^{-s^m t^n} a a^{s^m t^n} a^{s^m t^{n+1}} \\
&= a^{-1} a^{-s^m t^{n+1}} a^{-s^m t^n} a a^{s^m t^n} a^{s^m t^{n+1}} \\
&= a^{-1} a^{-s^m t^{n+1}} a^{-s^m t^n} a a^{s^m t^n} a^t a^{s^m t^{n+1}} a^{-t} a^{-s^m t^{n+1}} a^{s^m t^{n+1}} \\
&= a^{-1} a^{-s^m t^{n+1}} a^{-s^m t^n} a a^{s^m t^n} a^t a^{s^m t^{n+1}} a^{-t} \\
&= a^{-1} a^{-s^m t^{n+1}} a^{-s^m t^n} a a^{s^m t^n} a^t a^{-t} a^{-s^m t^n} a^t a^{s^m t^n} a^{s^m t^{n+1}} a^{-t} \\
&= a^{-1} a^{-s^m t^{n+1}} a^{-s^m t^n} a a^t a^{s^m t^n} a^{s^m t^{n+1}} a^{-t} \\
&= a^{-1} a^{-s^m t^{n+1}} a^{-s^m t^n} a a^t a^{s^m t^n} a^{s^m t^{n+1}} a^{-t} \\
&= a^{-1} s a^{-s^m t^n} s^{-1} s a s^{-1} s a^{s^m t^n} s^{-1} a^{-t} \\
&= a^{-1} s a^{-s^m t^n} a a^{s^m t^n} s^{-1} a^{-t} \\
&= a^{-1} s a^{-s^m t^n} a a^{s^m t^n} a^{-s^m t^n} a^{-1} a^{s^m t^n} a s^{-1} a^{-t} \\
&= a^{-1} s a s^{-1} a^{-t} \\
&= a^{-1} a a^t a^{-t} \\
&= 1.
\end{aligned}
$$

As in the proof of Lemma 5.2.1, we see that at each step, the length of the current word is $\leq \mathrm{FL}([a, a^{s^m t^n}]) + 12(|m| + n) + 24$. So

$$\begin{aligned}
\mathrm{FL}([a, a^{s^m t^{n+1}}]) &\leq \mathrm{FL}([a, a^{s^m t^n}]) + 12(|m| + n) + 24 \\
&\leq \mathrm{FL}([a, a^{s^m t^{n-1}}]) + 12(|m| + n - 1) + 24 + 12(|m| + n) + 24 \\
&\leq \ldots \leq \\
&\leq \mathrm{FL}([a, a^{s^m t^0}]) + 12 \cdot |m| + 24 + \ldots + 12(|m| + n) + 24 \\
&\leq 46m^2 + 12(|m|)(n+1) + \frac{12n(n+1)}{2} + 24(n+1) \leq \\
&\leq 46m^2 + 12|m|n + 12|m| + 24 + 6n^2 + 30n \leq \\
&\leq 46m^2 + 48|m|n + 36n^2 \leq 46(|m| + n)^2.
\end{aligned}$$

Now for $n < 0$, we get a similar bound by

$$\begin{aligned}
[a, a^{s^m t^{-|n|}}] &= a^{-1} a^{-s^m t^{-|n|}} a a^{s^m t^{-|n|}} \\
&= a^{-1} s^m t^{-|n|} a^{-1} t^{|n|} s^{-m} a s^m t^{-|n|} a t^{|n|} s^{-m} \\
&= s^m t^{-|n|} t^{|n|} s^{-m} a^{-1} s^m t^{-|n|} a^{-1} t^{|n|} s^{-m} a s^m t^{-|n|} a t^{|n|} s^{-m} \\
&= s^m t^{-|n|} [a^{s^{-m} t^{|n|}}, a] t^{|n|} s^{-m} \\
&= s^m t^{-|n|} t^{|n|} s^{-m} \\
&= 1.
\end{aligned}$$

Thus, $\mathrm{FL}([a, a^{s^m t^n}]) \leq 46(|m| + |n|)^2 + 2(|m| + |n|) \leq 48(|m| + |n|)^2$ for all $n, m \in \mathbb{Z}$. □

Lemma 5.2.4. *For $n, m, n', m' \in \mathbb{Z}$,* $\mathrm{FL}([a^{s^m t^n}, a^{s^{m'} t^{n'}}]) \leq 50(|n| + |m| + |n'| + |m'|)^2$.

Proof. Notice that

$$\begin{aligned}
[a^{s^m t^n}, a^{s^{m'} t^{n'}}] &= s^m t^n a^{-1} s^{m'-m} t^{n'-n} a^{-1} t^{n-n'} s^{m-m'} a s^{m'-m} t^{n'-n} a t^{n-n'} s^{m-m'} s^{-m} t^{-n} \\
&= s^m t^n [a, a^{s^{m'-m} t^{n'-n}}] s^{-m} t^{-n} = 1.
\end{aligned}$$

By Lemma 5.2.3, $\mathrm{FL}([a, a^{s^{m'-m}t^{n'-n}}]) \leq 48(|m'-m|+|n'-n|)^2$. So $\mathrm{FL}([a^{s^m t^n}, a^{s^{m'}t^{n'}}]) \leq 48(|m'-m|+|n'-n|)^2 + 2(|m|+|n|) \leq 48(|m'|+|m|+|n'|+|n|)^2 + 2(|m|+|n|) \leq 50(|m'|+|m|+|n'|+|n|)^2$. ◻

Lemma 5.2.5. *For $n, m, n', m', c_{nm}, c_{n'm'} \in \mathbb{Z}$,*

$$\mathrm{FL}([a^{c_{nm}s^m t^n}, a^{c_{n'm'}s^{m'}t^{n'}}]) \leq 51 \left|[a^{c_{nm}s^m t^n}, a^{c_{n'm'}s^{m'}t^{n'}}]\right|^2.$$

Proof. This can be done by induction on the powers c_{nm} and $c_{n'm'}$ of a. We explain the case where c_{nm} and $c_{n'm'}$ are positive, the other cases are done similarly. The idea is that we pass one of the $a^{-s^m t^n}$ through all but one of the $a^{-s^{m'}t^{n'}}$ and we pass one of the $a^{s^{m'}t^{n'}}$ through all but one of the $a^{s^m t^n}$. By Lemma 5.2.4, we know that each exchange increases the length of the word to at most $\left|[a^{c_{nm}s^m t^n}, a^{c_{n'm'}s^{m'}t^{n'}}]\right| + 50(|n|+|m|+|n'|+|m'|)^2$, but once the exchange is done the word returns to its original length of $\left|[a^{c_{nm}s^m t^n}, a^{c_{n'm'}s^{m'}t^{n'}}]\right|$. Finally, once we have $[a^{s^m t^n}, a^{s^{m'}t^{n'}}]$ inside our word, we can collapse it (by Lemma 5.2.4) using words of length at most $\left|[a^{c_{nm}s^m t^n}, a^{c_{n'm'}s^{m'}t^{n'}}]\right| + 50(|n|+|m|+|n'|+|m'|)^2 \leq 51 \left|[a^{c_{nm}s^m t^n}, a^{c_{n'm'}s^{m'}t^{n'}}]\right|^2$ to get a word $[a^{(c_{nm}-1)s^m t^n}, a^{(c_{n'm'}-1)s^{m'}t^{n'}}]$ of length less than $\left|[a^{c_{nm}s^m t^n}, a^{c_{n'm'}s^{m'}t^{n'}}]\right|$. Repeating this process gives the result. Here is how the word changes through one step of the process:

$$\begin{aligned}
[a^{c_{nm}s^m t^n}, a^{c_{n'm'}s^{m'}t^{n'}}] &= a^{-c_{nm}s^m t^n} a^{-c_{n'm'}s^{m'}t^{n'}} a^{c_{nm}s^m t^n} a^{c_{n'm'}s^{m'}t^{n'}} \\
&= a^{-(c_{nm}-1)s^m t^n} a^{-s^m t^n} a^{-(c_{n'm'}-1)s^{m'}t^{n'}} a^{-s^{m'}t^{n'}} a^{s^m t^n} a^{(c_{nm}-1)s^m t^n} a^{s^{m'}t^{n'}} a^{(c_{n'm'}-1)s^{m'}t^{n'}} \\
&= a^{-(c_{nm}-1)s^m t^n} a^{-(c_{n'm'}-1)s^{m'}t^{n'}} a^{-s^m t^n} a^{-s^{m'}t^{n'}} a^{s^m t^n} a^{s^{m'}t^{n'}} a^{(c_{nm}-1)s^m t^n} a^{(c_{n'm'}-1)s^{m'}t^{n'}} \\
&= a^{-(c_{nm}-1)s^m t^n} a^{-(c_{n'm'}-1)s^{m'}t^{n'}} [a^{s^m t^n}, a^{s^{m'}t^{n'}}] a^{(c_{nm}-1)s^m t^n} a^{(c_{n'm'}-1)s^{m'}t^{n'}} \\
&= a^{-(c_{nm}-1)s^m t^n} a^{-(c_{n'm'}-1)s^{m'}t^{n'}} a^{(c_{nm}-1)s^m t^n} a^{(c_{n'm'}-1)s^{m'}t^{n'}} \\
&= [a^{(c_{nm}-1)s^m t^n}, a^{(c_{n'm'}-1)s^{m'}t^{n'}}].
\end{aligned}$$

◻

We are now ready to complete the proof of Theorem 5.

Proof of Theorem 5. Consider a trivial word $w \in \Gamma_2$ with $|w| = N$ supported above the t-axis. Consider the path traveled by the lamplighter on the lamplighter model while reading off the letters of w and the configuration $\mathcal{K}_w$ associated to w. Use the relation $[s,t] = 1$ to re-write w (without increasing its length) in the form $w_0 = s^{l'_0}t^{k'_0}\prod_i a^{c_i}s^{l'_i}t^{k'_i}$ with $l'_i, k'_i \in \mathbb{Z}$ for all i. Now inserting suitable words on $\{t^{\pm 1}, s^{\pm 1}\}$ (namely, $t^{-\sum_{j=0}^{i} k'_j}s^{-\sum_{j=0}^{i} l'_j}s^{\sum_{j=0}^{i} l'_j}t^{\sum_{j=0}^{i} k'_j}$) after each a^{c_i}, we can rewrite w_0 as $w_1 = \prod_i a^{c_i s^{l_i} t^{k_i}}$. Note that we added less than $2N$ many letters from the set $\{t^{\pm 1}, s^{\pm 1}\}$ after each a^{c_i}. Hence $|w_1| < 2N^2$ and the configuration corresponding to w_1 is the same as that for w. Finally permute consecutive $a^{c_i s^{l_i} t^{k_i}}$ as needed so that (l_i, k_i) are in lexicographic order ($l_i \leq l_{i+1}$ and if $l_i = l_{i+1}$ then $k_i < k_{i+1}$) to get $w_2 = \prod_j a^{c_j s^{l_j} t^{k_j}}$. Note that at each step we re-rewrite a word $z = ua^{c_n s^{l_n} t^{k_n}} a^{c_m s^{l_m} t^{k_m}} v$ with $|z| < 2N^2$ by the word $z' = ua^{c_m s^{l_m} t^{k_m}} a^{c_n s^{l_n} t^{k_n}} v$ (of the same length), where $u, v \in \Gamma_2$. But for each pair (n, m), we know that $|a^{c_n s^{l_n} t^{k_n}} a^{c_m s^{l_m} t^{k_m}}| \leq 2N$ and so Lemma 5.2.5 says that throughout this process the words we encounter are of the length $\leq 2N^2 + 51(2N)^2 = 206N^2$. Hence, this permutation can be achieved encountering words of length at most $\leq 206N^2$ and the resulting word is of the same length as w_1, so $|w_2| < 2N^2$.

So we re-wrote w as a word $w_2 = \prod_j a^{c_j s^{l_j} t^{k_j}}$ with (l_j, k_j) in lexicographic order. Note that since we did not alter the configuration associated to w throughout the re-writing process, we have $\mathcal{K}_{w_2} = \mathcal{K}_w$, so in particular, w_2 is supported above t-axis and $|\mathcal{K}_{w_2}| = |\mathcal{K}_w|$. Recall that we assumed that there exists a universal constant C such that when we propagate the configuration for w_2 row-by-row to the t-axis the norm of the intermediate configurations stays at most $C \cdot \mathcal{K}_{w_2}$. Suppose that the configurations we encounter after propagating one row at a time are $\mathcal{K}_{w_2}, \mathcal{K}_{w_3}, \ldots, \mathcal{K}_{w_n} = \mathcal{K}_\epsilon$. Our assumption guarantees that $|\mathcal{K}_{w_p}| \leq C \cdot |\mathcal{K}_{w_2}| = C \cdot |\mathcal{K}_w|$ for all p. We show below that if we re-write the word w_p to be in the form $w_p = \prod_j a^{c_{p,j} s^{l_{p,j}} t^{k_{p,j}}}$ with $(l_{p,j}, k_{p,j})$ in lexicographic order after propagating each row, we reduce w_2 to the trivial word, while keeping the length of the words we encounter less than $464C^2 \cdot N^2$. Here is why.

Throughout the propagation, the sum of the absolute values of powers of a stay less than $C \cdot N$ by assumption of the open question. The lamplighter only needs to walk within the triangle bounded by the initial entries, so he will visit at most N entries in a given row. Note that since propagation row-by-row alters values one row at a time, the path that the lamplighter takes will always be shorter than $2N^2 + 2N^2$, where the first $2N^2$ accounts for his walk below the row being altered (the beginning piece of the word w_2) and second $2N \cdot N$ counts the walk from the origin to each element in the row being altered and back. So the total length of the word we get after propagating a row is less than $(C + 4)N^2$. Now to get the word in the desired form, notice that since each subword $a^{c_{nm} s^m t^n} a^{c_{n'm'} s^{m'} t^{n'}}$ has length at most $(C + 2)N$, Lemma 5.2.5 guarantees that exchanging these powers of a keeps the length of the intermediate words we encounter less than $(C + 4)N^2 + 51 \cdot (C + 2)^2 \cdot N^2 \leq 464C^2 \cdot N^2$. So, after propagation of each entry, we can re-write the words so that the powers of a are in lexicographic order, while keeping the length of the words we encounter smaller than $464C^2 \cdot N^2$. Since the initial configuration propagates to all-zero configuration on the t-axis, the final word that we encounter is the empty word, so the filling length of w is at most quadratic. □

Remark 5.2.6. If we were able to solve Open Question 5.1.4, then we would get a cubic upper bound on the filling length of Γ_2. The proof is similar to that above, except if we allow any propagation within $B_{C\cdot N}$, we can only guarantee that the path of the lamplighter is less than $2N^2 + 10C^2N^3$, where the first $2N^2$ accounts for unchanged part of the word as before and $2N\cdot(5C^2N^2)$ accounts for the walk from the origin to and from all the points inside the ball $B_{C\cdot N}$ (see Remark 5.1.2).

5.3 Reformulations and related questions

One approach we took while trying to prove a polynomial upper bound on the filling length of Γ_2 was to reduce the types of configurations that we need to consider. We already saw that we can restrict our attention to those configurations that are supported on the first quadrant. Proposition 5.3.2 shows that we can further restrict our attention to the configurations in the first quadrant that have just one positive entry above all the negative entries.

Definition 5.3.1. *For an integer $M \geq 0$, let $\mathbf{\Delta_M}$ be the equilateral triangle whose vertices are $(0,0)$, $(M,0)$ and $(0,M)$. In other words, the triangle bounded by the t-axis, s-axis and the line where the s− plus t−coordinate equals M.*

Proposition 5.3.2. *Given any configuration $\mathcal{K}$ supported in the first quadrant, we can use the equivalence relations ${}_1{}^0{}_0 \rightarrow {}_0{}^1{}_{-1}$, and ${}_0{}^0{}_1 \rightarrow {}_{-1}{}^1{}_0$ to iteratively rewrite $\mathcal{K} = \mathcal{K}_0 \rightarrow \mathcal{K}_1 \rightarrow \ldots \rightarrow \mathcal{K}_r$ such that each consecutive pair of $\mathcal{K}_i$'s differs in one of the relations above, $|\mathcal{K}_i| \leq |w|^2$ for all i and such that $\mathcal{K}_r$ either has an empty support or there exists an integer $M \geq 0$ such that $\mathcal{K}_r$ is supported on Δ_M and the only positive entry is at $(0,M)$.*

Proof. Consider a configuration $\mathcal{K}_w$ for a word w supported in the first quadrant.

Let $N = |w|$ and let M be the maximal distance of a non-zero element of $\mathcal{K}_w$ from the origin. Then $\mathcal{K}_w$ is supported on Δ_M. We can raise the positive entries in the configuration line-by-line while keeping its support inside Δ_M via the following:

- Consider the line with t-coordinate zero. Work from the bottom up in order of increasing s-coordinate from $(0,0)$ to $(0, M-1)$ replacing each positive integer along the way by two integers of the same magnitude using the equivalence

  $$ {}_{1}{}^{0}{}_{0} \mapsto {}_{0}{}^{1}{}_{-1}. $$

  Stop when there are no more positive integers on the line $t = 0$, except possibly at $(0, M)$.

- For $0 < i < M$ repeat this process: for the line with t-coordinate equal to i, work along the line from $(i, 0)$ to $(i, M-i-1)$ and use the equivalence

  $$ {}_{1}{}^{0}{}_{0} \mapsto {}_{0}{}^{1}{}_{-1} $$

  to push the positive entries up. Stop when there are no more positive integers on the line $t = i$, except possibly at $(i, M-i)$.

- Finally, consider the line where the $s-$ plus $t-$coordinate equals M (which at this point should contain all the positive entries in the configuration). Work along the line from $(M, 0)$ to $(1, M-1)$ and replace each positive integer along the way by two integers of same magnitude using the equivalence

  $$ {}_{0}{}^{0}{}_{1} \mapsto {}_{-1}{}^{1}{}_{0}. $$

  Stop when there are no more positive integers on this line, except possibly at $(0, M)$, so at the top of Δ_M.

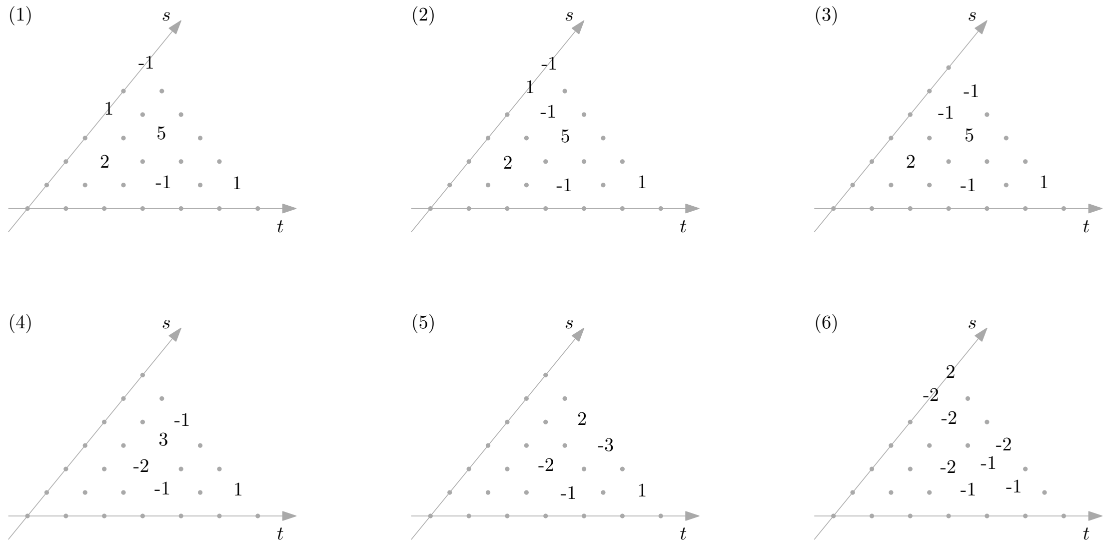

Figure 5.6: Example of raising a configuration row-by-row.

See Figure 5.6 for an example of raising a configuration line-by-line, where (1) has the initial configuration $\mathcal{K}$, (2) and (3) show the movement along the line $t = 0$, (4) shows the result of moving along the line $t = 1$, (5) shows the result of moving along the line $t = 2$ and $t = 3$ (since there is no positive entries on the line $t = 3$), and finally (6) shows the result of moving along the line whose $t-$ and $s-$coordinates add up to $M = 6$.

Note that each of the initial positive integers, z, moves at most M many times up during this procedure and so it deposits at most M many $-z$'s along the way. So the norm of all the intermediate (and the resulting) configurations is at most $M \cdot |\mathcal{K}_w| \leq N^2 = |w|^2$, since $M \leq N$ by definition. $\square$

Remark 5.3.3. Note that we can also restrict our attention to vertically symmetric configurations since given a configuration in the form of Proposition 5.3.2, we can add it to its reflection about the vertical line passing through the point $(0, M)$ to obtain a vertically symmetric configuration (while at most doubling the length of the word).

5.4 Consequences of the Pascal's triangle relation

In trying to solve the Open Question 5.1.3, we considered the combinatorics of propagation. Specifically, how introducing the integers on the lattice points effects the overall picture. Proposition 5.4.1 explores the propagation when there are no new integers introduced. It shows that if the numbers occurring during the propagation are bounded, then the maximum will persist through the propagation unless we introduce new integers to cancel it out.

Proposition 5.4.1. *Consider a configuration consisting of integers $a_{i,j}$ at locations (i, j) of the rhombic grid A_2 (not necessarily finitely supported). Suppose the configuration satisfies the Pascal's triange relation at each location (i.e. $a_{i,j} = a_{i-1,j+1} + a_{i,j+1}$). Suppose $m := \max |a_{i,j}|$ is finite, then the configuration must be*

$$
\begin{array}{ccccccccccccccccccc}
 & & & & & & & & & \vdots & & & & & & & & & \\
 & \dots & & -m & & m\text{-}x & & x & & & -m & & m\text{-}x & & x & & -m & & \dots \\
\dots & & x\text{-}m & & -x & & m & & x\text{-}m & & & -x & & m & & x\text{-}m & & -x & \dots \\
 & \dots & & -m & & m\text{-}x & & x & & & -m & & m\text{-}x & & x & & -m & & \dots \\
\dots & & x\text{-}m & & -x & & m & & x\text{-}m & & & -x & & m & & x\text{-}m & & -x & \dots \\
 & & & & & & & & & \vdots & & & & & & & & &
\end{array}
$$

for some $0 \le x \le m$.

Proof. For now, assume that the maximum m is realized by a non-negative entry. Around m, the configuration must be:

$$\begin{array}{ccccccc} -x-m+z & & m-z & & z & & -y-z \\ & -x & & m & & -y & \\ & & m-x & & m-y & & \\ & & & 2m-x-y & & & \end{array}$$

for some $x, y, z \in \mathbb{Z}$ and all other entries shown are forced by the Pascal's triangle relation.

Notice that

- In order to have $m - x \leq m$ and $m - y \leq m$, we must have $x, y \geq 0$. Since m is the maximum of the absolute values of the entries, we must have $0 \leq x, y \leq m$.
- Since $m - z \leq m$, we have $z \geq 0$.
- Since $2m - (x + y) \leq m$, we have $x + y \geq m$.
- Since $|-(y + z)| \leq m$ and $y, z \geq 0$, we have $y + z \leq m$.
- Combining the last two inequalities, gives $x + y \geq m \geq y + z$, so $x \geq z$.
- Since $x \geq z$, we have $m + (x - z) = |-m-(x-z)| = |-x-m+z| \leq m$ which implies $-x - m + z \geq -m$ which in turn implies that $x \leq z$. So $x = z$ and we can re-write our configuration as:

$$\begin{array}{ccccccc} -m & & m-x & & x & & -y-x \\ & -x & & m & & -y & \\ & & m-x & & m-y & & \\ & & & 2m-x-y & & & \end{array}$$

- Looking at the upper right position, we notice that $|-(x+y)| \leq m$ with $x, y \geq 0$, implies $0 \leq x+y \leq m$, but we had above that $x+y \geq m$, so $x+y=m$.

So using $y = m - x$, we can re-write the configuration around m as

$$\begin{array}{ccccccccc}
-m & & m-x & & x & & -m \\
& -x & & m & & x-m & \\
& & m-x & & x & & \\
& & & m & & &
\end{array}$$

Repeating this argument with signs reversed, shows that if the maximum value is attained at a negative entry, say $-m$ for some $m > 0$, then the configuration around $-m$ must look like

$$\begin{array}{ccccccccc}
m & & x-m & & -x & & m \\
& x & & -m & & m-x & \\
& & x-m & & -x & & \\
& & & -m & & &
\end{array}$$

for some $0 \leq x \leq m$. So we can cover the whole grid A_2 by covering the areas around each m and $-m$ in turn, leading to the claimed configuration. $\square$

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