FAMILIES OF DYNAMICAL SYSTEMS ASSOCIATED TO TRANSLATION SURFACES

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Kathryn Anne Lindsey

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Kathryn A. Lindsey, Ph.D.

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Dynamical systems associated to translation surfaces naturally arise in families equipped with a topological structure. How is the topological structure on the family related to the dynamical properties of the individual systems? We investigate this question for three types of families of dynamical systems associated to translation surfaces.

The collection of translation surfaces of finite type has a natural stratification based on the number and type of cone points. Each stratum has a topology with respect to which the stratum consists of between one and three connected components. We investigate how which connected component of a stratum a translation surface belongs to determines how many invariant components any translation flow on that surface may have. Specifically, we characterize the numbers of minimal and periodic components possible for translation flows on surfaces in the hyperelliptic connected components of strata.

The group $SL(2,\mathbb{R})$ acts on each stratum of translation surfaces. The horocycle flow is the action of the one-parameter subgroup of matrices $h_t = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. For any fixed translation surface M, we examine how the closures of the orbits under the horocycle flow of the surface $r_\theta \cdot M$ are related to each other, where $r_\theta = \left(\begin{smallmatrix} \cos\theta & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$. We show that for a residual set of angles θ, the closure of the orbit of $r_\theta \cdot M$ under the horocycle flow is equal to the closure of the orbit of M under $SL(2,\mathbb{R})$. We then apply this result to obtain a new characterization of lattice surfaces in terms of minimal sets for the horocycle flow.

A large class of translation surfaces of infinite type can be thought of as "limits" of a sequence of translation surfaces of finite type. Analogously to the way Rauzy diagrams describe the renormalization dynamics (and first return maps to a transversal of the translation flow) of finite type translation surfaces, Bratteli diagrams encode the renormalization dynamics (and first return maps to a transversal of the translation flow) for these surfaces of infinite type. Finite truncations of Bratteli diagrams correspond to surfaces of finite type, so Bratteli diagrams hint at a "moduli space" encompassing translation surfaces of both finite and infinite type. We investigate how the translation flow on the “limit” surface of infinite type is related to the translation flows on the finite type surfaces in the associated sequence. We prove that any finite entropy, measure-preserving flow on a standard Lebesgue space is measurably isomorphic to the translation flow on a translation surface of infinite type.

BIOGRAPHICAL SKETCH

Kathryn Lindsey grew up in Amherst, Massachusetts. As an elementary school student, she attended the Smith College Campus School, in Northampton, MA, and then the the Common School, in Amherst, MA. She completed grades 7 to 9 in the Amherst Regional school system, and grades 10-12 as a day student at Deerfield Academy, in Deerfield, MA, graduating in 2002. In fall 2002, she participated in Colby College's semester abroad program in Dijon, France. In spring 2003, Kathryn participated in Sea Education Association's Sea Semester program on board the S.S.V. Corwith Cramer and then worked as a deckhand on the schooner Ernestina until entering Williams College in Fall 2003. At Williams, she majored in Mathematics and Statistics and also concentrated in Chinese language. She participated in the ergodic theory group of the Williams College SMALL REU program in summer 2006 and went on complete a senior honors thesis in ergodic theory under the supervision of Professor Cesar Silva. In her free time, Kathryn continued to sail and teach on board Sea Education Association's tall ships, and she was awarded U.S.C.G. merchant mariner licenses as a 200 GRT Near Coastal Mate and 100 GRT Inland Master with AB Sail certification. In 2007, Kathryn graduated from Williams College and entered the mathematics Ph.D. program at Cornell University. Kathryn spent summer 2008 studying Mandarin Chinese at Heilongjian University in Harbin, China, on a U.S. State Department Critical Languages Scholarship. At Cornell, Kathryn conducted research on translation surfaces under the supervision of her advisor, Professor John Smillie, and investigated topics in complex dynamics under the direction of Professors Bill Thurston and John Hubbard. In Fall 2013, she was a Research Fellow at the Institute for Computational and Experimental Research in Mathematics in Providence, RI. Kathryn was supported during her graduate studies by a N.S.F. Graduate Research Fellowship and a Dept. of Defense National Defense Science and Engineering Graduate Fellowship.

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INTRODUCTION

A translation surface is a closed, orientable surface endowed with a metric such that the surface is flat except at a set of isolated conical singularities and the linear holonomy group of the surface is trivial. The collection of all translation surfaces of finite type forms a space that has a natural stratification based on the number and type of singularities of the flat metrics; a stratum $\mathcal{H}(k_1, \ldots, k_n)$ of the moduli space of translation surfaces consists of all translation surfaces with cone points of orders $k_1, \ldots, k_n$.

The flow in a fixed direction on a translation surface S determines a decomposition of S into closed invariant sets called components, each of which is either periodic or minimal. In chapter 2, we study this decomposition for translation surfaces in the hyperelliptic connected components $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$ of the corresponding strata of the moduli space of translation surfaces. Specifically, in Theorem 2.1.1 we characterize the pairs of nonnegative integers (p, m) for which there exists a translation surface in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$ with precisely p minimal components and m minimal components. This result extends work by Naveh ([Nav08]), who obtained tight upper bounds on number of invariant components for each stratum as a whole.

The group $SL(2, \mathbb{R})$ acts on strata via postcomposition with manifold charts of a translation surface. The stabilizer in $SL(2, \mathbb{R})$ of a translation surface S is called the Veech group of S. A translation surface whose Veech group is a lattice (has finite co-volume in $SL(2, \mathbb{R})$ is said to be a lattice surface. In Chapter 3, which is based on joint work with Jon Chaika, we prove a characterization of lattice surfaces in terms of the horocycle flow, which is the action of the one parameter subgroup $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, $t \in \mathbb{R}$. Specifically, Theorem 3.3.1 proves that a translation surface M is a lattice surface if and only if every subset of the closure of orbit of M under $SL(2, \mathbb{R})$ that is a minimal set for the horocycle flow is a periodic orbit of the horocycle flow. This result adds an additional entry to the list of equivalent characterizations of lattice surfaces presented in [SW10]. We also prove that for every translation surface M, for all but a meager set of angles θ, the closure of the horocycle orbit of $r_{-\theta} \cdot M$ (here $r_{-\theta} \cdot M$ denotes M "rotated" by the angle $-\theta$) equals the $SL(2, \mathbb{R})$-orbit closure of M.

Chapter 4, which is based on joint work with Rodrigo Treviño, explores dynamical systems associated to translation surfaces of infinite type – generalized versions of translation surfaces in which the requirements that the genus and cone points of the surface be finite are dropped. In this chapter, we present a general setting for constructing and studying infinite type surfaces. Our method connections between translation flows on infinite type translation surfaces, adic transformations defined on Bratteli diagrams, and cutting-and-stacking transformations. We do so by introducing a technique which takes an adic transformation and constructs a flat surface whose vertical translation flow admits a cross section for which the first return map is measurably isomorphic to the adic transformation. Theorem 4.6.1 proves that any finite entropy, measure-preserving flow on a standard Lebesgue space is measurably isomorphic to the translation flow on a flat surface obtained through our technique. We present specific examples of infinite type flat surfaces such that the vertical translation flows on these surfaces exhibit dynamical properties that cannot be realized by translation flows on finite type translation surfaces.

Chapter 1

Background

Translation surfaces arise in numerous contexts, and there are several equivalent ways of defining them. In §1.1, we describe translation surfaces as being surfaces constructed by "gluing" together polygons cut from sheets of "graph paper." In §1.2, we approach translation surfaces from the point of view of complex analysis, thinking of translation surfaces as coming from quadratic differentials on Riemann surfaces, and we connect these to elements of the cotangent space to Teichmüller space.

1.1 Euclidean geometry definition of translation surfaces

Definition 1.1.1. *A translation surface is a 2-real-dimensional manifold S with a subset $\Sigma \subset S$ such that the restriction to $S \setminus \Sigma$ of each transition map between charts of S is a translation.*

The statement that "the restriction to $S \setminus \Sigma$ of each transition map between charts of S is a translation" means that if $\phi : U \to V$ is a transition map between charts of the manifold S, with $U, V \subset \mathbb{R}^2$, then restriction $\phi|_{U \setminus \Sigma}$ can be written as $\phi|_{U \setminus \Sigma}(z) = z + c$ for and some $c \in \mathbb{R}^2$ and all z in $U \setminus \Sigma$. We will denote such a translation surface by the pair (S, Σ) or, in cases where the set Σ is clear from the context, simply by S.

Requiring a translation surface to have this particular type of manifold atlas immediately gives rise to two additional structures on the surface: a metric and a set of "directions" or foliations.

- **The canonical Euclidean metric** on (S, Σ). This condition on the manifold transition maps implies that the Euclidean metric on the restriction to $S \setminus \Sigma$ of each manifold chart is invariant under transition maps, resulting in a well-defined metric on $S \setminus \Sigma$. Identifying the metric completion of this metric on $S \setminus \Sigma$ with S yields the "canonical Euclidean metric" on S, when such an identification is possible. For any two points $x_1, x_2 \in S$, we define $d(x_1, x_2)$ to be the infimum of the lengths (with respect to the canonical Euclidean metric) of rectifiable paths in S connecting x_1 to x_2.

- **The foliation $\mathcal{F}_\theta$ of (S, Σ) in direction θ.** Since translations in $\mathbb{R}^2$ preserve "direction" (e.g. the "vertical direction," etc.), "directions" are well-defined on $S \setminus \Sigma$. Consequently, for any direction $\theta \in S^1 = \mathbb{R} \bmod 2\pi$, there is a well-defined foliation $\mathcal{F}_\theta$ of S in direction θ, defined by pulling back the straight-line foliation in direction θ on the images in $\mathbb{R}^2$ of the manifold charts.

One way of reformulating these two observations is to state that the tangent space $T(S \setminus \Sigma)$ has a canonical global trivialization, i.e. $T(S \setminus \Sigma)$ can be identified in a canonical way with $(S \setminus \Sigma) \times \mathbb{R}^2$. Consequently, a translation surface has trivial linear holonomy. That is, the group of linear maps on the tangent space $T_p(S \setminus \Sigma)$ induced by parallel transport of an element of $T_p(S \setminus \Sigma)$ along closed loops in $S \setminus \Sigma$ based at p is trivial.

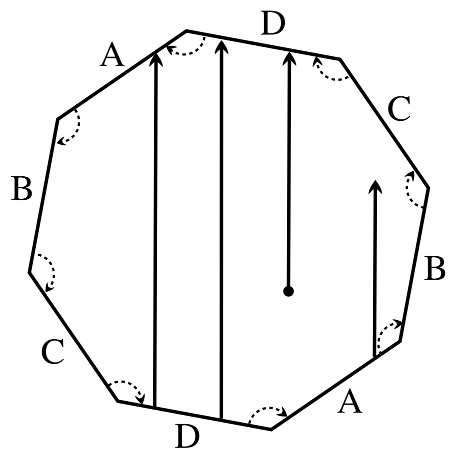


Figure 1.1: A translation surface formed by identifying opposite sides of an octagon embedded in the plane. Letters indicate the side identifications. Black vectors in the surface indicate a portion of the orbit of a point under the vertical translation flow. All vertices of the octagon are identified to form a single cone point. The cone angle at the cone point is 6π; circling around the cone point on the dotted lines, one passes through a total angle of 6π relative to the cone point.

Definition 1.1.2. *An orientation-preserving homeomorphism $\psi : (S_1, \Sigma_1) \to (S_2, \Sigma_2)$ such that $\psi(\Sigma_1) = \Sigma_2$ and such that the restriction $\psi|_{S_1 \setminus \Sigma_1}$ is affine in each chart is called an* **affine isomorphism**. *We denote by $D(\psi)$ the linear part (in $GL(2, \mathbb{R})$) of an affine isomorphism ψ. An affine isomorphism whose linear part is the identity is called a* **translation equivalence.**

We consider two translation surfaces to be **equivalent** if there exists a translation equivalence between them.

Definition 1.1.3. *A half-translation surface is a 2-real-dimensional manifold S with a subset $\Sigma \subset S$ such that the closure $\overline{S \setminus \Sigma} = S$ and the restriction to $S \setminus \Sigma$ of each transition map between charts of S has the form $z \mapsto \pm z + c$ for some $c \in \mathbb{R}^2$.*

A half-translation equivalence is an affine isomorphism whose linear part is $\pm Id$. Two half-translation surfaces are considered equivalent if there is a half-translation equivalence between them. Unlike in the case of translation surfaces, "directions" on half-translation surfaces are only defined up to $\pm Id$.

The definition of a translation surface (or half-translation surface), in its full generality, allows for some very complicated objects; in order to restrict our consideration to translation surfaces that are more "manageable," we shall wish to impose some finiteness conditions. Some possible restrictions for a translation surface (S, Σ) include: Σ is a finite or discrete set of points in S, (S, Σ) has finite area (with respect to the canonical Euclidean area form), S is compact, S has no boundary, or S is of finite genus. The class of translation surfaces which have been studied the most are translation surfaces of finite type, which, perhaps unsurprisingly, satisfy all of the conditions listed above.

Definition 1.1.4. *A translation surface or half-translation surface (S, Σ) is of* **finite type** *if S is a closed, connected surface of finite genus and Σ is a finite set of distinct points of S. A translation surface that is not of finite type is said to be of infinite type.*

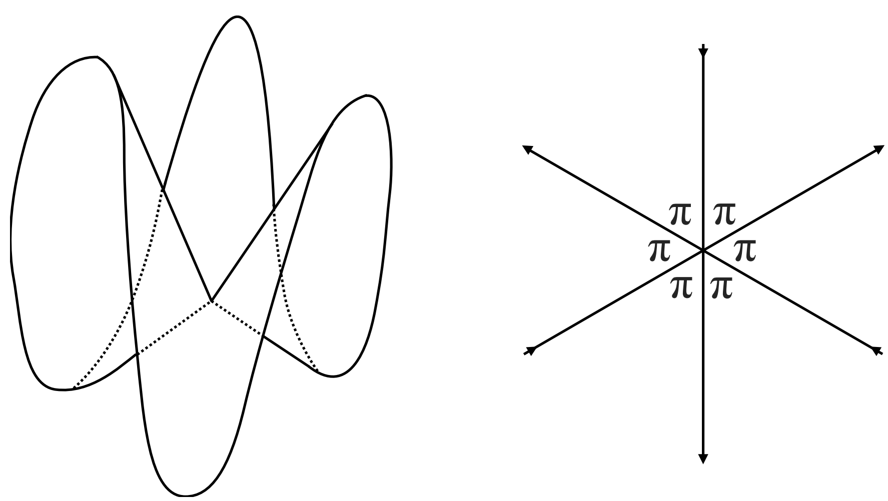


Figure 1.2: Two different depictions of a neighborhood of a cone point with cone angle 6π. The image at left depicts how this neighborhood might look if built from pieces of paper in $\mathbb{R}^3$. The cone angle of 6π is achieved by making the surface "ripple" around the cone point. The image a right is a more abstract representation of the neighborhood in $\mathbb{R}^2$; each of six sectors is assigned a cone angle of π. The arrows on the lines that divide the sectors indicate the positive vertical direction.

Points in Σ are called **singularities**. In general, the geometry of a translation surface near singularities can be quite wild (see, for example, [BV13]). A **cone point of a translation surface** is an isolated singular point which has a neighborhood that is isometric to a neighborhood of the ramification point of a finite-sheeted cover of the Euclidean plane ramified at a point. The **cone angle** of a cone point of a translation surface is the total angle around the cone point, which is equal to 2π times the number of sheets in the ramified cover, can be written as $2\pi(1+n)$ for some nonnegative integer n; n is said to be the **order** of the cone point.

A **cone point of a half-translation surface** is similar, except it may have a cone angle that is any nonnegative integer multiple of π. There is some debate in the literature over whether a cone point of a half-translation surface with cone angle $2\pi(1+n/2)$ should be said to have order n or order $n/2$. For the sake of consistency between the Euclidean geometry approach to translation surfaces and half-translation surfaces, we will adopt the convention that a half-translation surface with a cone point of cone angle $2\pi(1+n/2)$ is of order $n/2$.

A **saddle connection** is a finite-length leaf of the foliation in some direction which has a cone point a both ends. The Gauss-Bonnet Theorem implies that if (S,Σ) is a translation surface of finite type, every point in Σ is a cone point, and

$$\sum_{p\in\Sigma}(\text{cone angle of } p-2\pi)=2\pi(2g-2),$$

or, equivalently,

$$\sum_{p\in\Sigma}\text{order}(p)=2g-2,$$

where g is the genus of the underlying topological surface.

Delaunay triangulations are a tool for for decomposing translation surfaces into a collection of polygons. (Although we will define Delaunay triangulations only for translation surfaces of finite type, the notion of Delaunay triangulations can be generalized to some types of translation surfaces of infinite type.)

Definition 1.1.5. *Let (S,Σ) be a translation surface of finite type. For each cone point $\sigma\in\Sigma$, the*

Voronoi cell V_σ *is the set*

$$V_\sigma = \{x \in S \mid d(x,\sigma) = \inf_{p\in\Sigma} d(x,p)\}.$$

The **Voronoi diagram** *of* (S,Σ) *is the tuple of cells* $(V_\sigma)_{\sigma\in\Sigma}$

Since for finite type surfaces Σ is a finite set and the metric d is defined in terms of the lengths of geodesic paths, it follows that each cell of a Voronoi diagram for a translation surface is a closed, connected subset of S containing a unique cone point and that the boundary of each Voronoi cell is composed of a finite number of geodesic line segments.

Definition 1.1.6. *A* **Delaunay triangulation** *of a translation surface* (S,Σ) *of finite type is a triangulation of* S *determined by taking the graph* G *in* S *(with edges that are saddle connections) that is dual to the Voronoi diagram of* (S,Σ)*, and, if necessary, subdividing any tile with more than three edges into triangles by adding to* G *additional saddle connections contained entirely within this tile.*

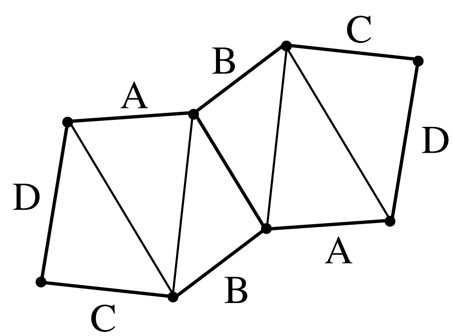


Figure 1.3: A Delaunay triangulation of a translation surface of genus 2 with one cone point. Letters indicate the edge identifications.

Because the triangulation of Definition 1.1.6 is constructed from the Voronoi diagram, it is clear that Delaunay triangulations exist for all finite type translation surfaces. By construction, each triangle of a Delaunay triangulation has a boundary consisting of three saddle connections and has no cone points in its interior. If each vertex of the Voronoi diagram has exactly three incident edges, the Delaunay triangulation is unique; if four or more edges of the Voronoi diagram meet at a vertex, a finite number of possible Delaunay triangulations exist for the surface, corresponding to the different possible ways to subdivide the corresponding tile into triangles. In any case, we see that any finite type translation surface can be triangulated in such a way that all vertices of the triangles are cone points, all edges of the triangles are saddle connections, and the interiors of the triangles contain no cone points.

A Delaunay triangulation of (S, Σ) may also be defined as a triangulation of S by geodesic paths with endpoints in Σ such that no point of Σ is in the interior of an isometrically immersed disk circumscribing the vertices of any triangle in the triangulation. This approach to Delaunay triangulations, which is equivalent to ours, is described in [MS91] and [Bow12].

The discussion of Delaunay triangulations shows that any translation surface of finite type can be thought of as a collection of polygons embedded in the Euclidean plane. This leads to another equivalent definition of a translation surface of finite type:

Definition 1.1.7. *Let $P_1, \dots, P_n$ be a collection of (not necessarily convex) Euclidean polygons embedded in the Euclidean plane, and assign the counter-clockwise orientation to the boundary of each polygon. Partition the set of edges of all polygons P_i into pairs so that each pair consists of parallel oppositely-oriented edges of the same length. Identify pairs of edges via translations, and identify vertices of the polygons only when forced to by a series of edge identifications. The resulting object is a translation surface of finite type.*

1.2 Complex analysis approach to translation surfaces

The analogues of translation surfaces and half-translation surfaces in complex analytic language are Abelian differentials and quadratic differentials on Riemann surfaces, respectively.

Definition 1.2.1. *An* **Abelian differential** *ω on a Riemann surface X is holomorphic 1-form on X.*

We may reformulate Definition 1.2.1 in terms of local coordinates on X: if X has charts $\phi_i : U_i \to V_i$, a holomorphic 1-form is a collection of holomorphic functions $w_i : V_i \to \mathbb{C}$ such that for any local coordinates z_j and z_k,

$$\omega_j(a_j)dz_j = \omega_k(a_k)dz_i$$

whenever $a_j = \phi_j(p)$ and $a_k = \phi_k(p)$ for $p \in U_j \cap U_k$.

Definition 1.2.2. *A* **quadratic differential** *on a Riemann surface X is a global section of the tensor square of the sheaf of holomorphic 1-forms on X.*

Definition 1.2.2 may also be stated in terms of local coordinates: denoting the the charts of X by the maps $\phi_i : U_i \to V_i$, a quadratic differential ω on X is a collection of holomorphic functions $\omega_i : V_i \to \mathbb{C}$ such that for any local coordinates z_j on V_j and z_k on V_k,

$$\omega_j(a_j)dz_j^2 = \omega_k(a_k)dz_k^2$$

whenever $a_j = \phi_j(p)$ and $a_k = \phi_k(p)$ for $p \in U_j \cap U_k$.

One type of "finiteness condition" commonly used in complex analysis is the condition that the Riemann surface on which a collection of Abelian or quadratic differentials is defined is of finite type. (A Riemann surface is of finite type if it is conformally equivalent to a compact surface from which at most finitely many points have been removed.) In this setting, Abelian or quadratic differentials may have poles at the surface's punctures. For a compact Riemann surface X of genus g and a meromorphic 1-form ω on X, the number of zeros of ω minus the number of poles of ω, counted with multiplicity, is $2g - 2$.

In order to obtain translation or half-translation surfaces of finite type, we will need to impose restrictions on the differentials themselves. There is a natural norm on the vector space of quadratic differentials on a Riemann surface X of finite type, given by $||q||_1 := \int_X |q|$. Similarly, for an Abelian differential α, we can define $||\alpha||_1 := \int_X |\alpha^2|$.

Definition 1.2.3. *Let X be a Riemann surface of finite type. An Abelian differential α on X is* **integrable** *if $||\alpha||_1 < \infty$. A quadratic differential q on X is* **integrable** *if $||q||_1 < \infty$.*

One can compute that a quadratic differential q on a hyperbolic Riemann surface of finite type is integrable if and only if q has at worst simple poles at the punctures of X.

We will describe how integrable Abelian and quadratic differentials on Riemann surfaces of finite type determine translation and half-translation surface structures, respectively.

Let X be a Riemann surface of finite type, together with an integrable Abelian differential ω on X; we will use ω to construct a translation surface structure on X. For a smooth path γ in X, the integral $\int_\gamma \omega(z_i)dz_i$ is well-defined. Thus, the map $\varphi_{x_0} : x \mapsto \int_{x_0}^x \omega(z)dz$ determines a bijection between a neighborhood U_{x_0} of a regular point x_0 in X to a neighborhood of 0 in $\mathbb{C}$. Using a different nearby basepoint, say x_1, determines a bijection φ_{x_1} of the form $x \mapsto \int_{x_1}^x \omega(z)dz$ from a neighborhood U_{x_1} to a neighborhood of 0 in $\mathbb{C}$. Notice that for $x \in U_{x_0} \cap U_{x_1}$, we have

$$\varphi_{x_0}(x) + \varphi_{x_0}(x_1) = \varphi_{x_1}(x).$$

In other words, φ_{x_0} and φ_{x_1} determine manifold charts such that the transition map between these charts is a translation. In this way, ω determines an atlas of charts on $X \setminus \{\text{zeros of } \omega\}$ such that transition maps between these charts are translations. The completion of this manifold with respect to the canonical Euclidean metric on these charts can then be identified with X, yielding a marked translation surface.

We now consider the case when ω is an integrable quadratic differential on X, and show that this defines a half-translation surface. Each regular point p has a neighborhood U on which a single-valued branch of $\sqrt{\omega}$ can be chosen. So for a path γ from p to a point x contained entirely within U, we can compute $\int_\gamma \sqrt{\omega(z)}dz$ using either of the branches. If $\Phi_1(x)$ and $\Phi_2(x)$ are the two possible integrals, we evidently have $\Phi_1(x) = \pm\Phi_2(x) + c$ for some constant $c \in \mathbb{C}$. Furthermore, since either branch of $\sqrt{\omega}$ is holomorphic in U, the integrals are path independent, i.e. if γ_1 and γ_2 are two smooth paths from p to x in U, then $\int_{\gamma_1} \sqrt{\omega(z)}dz = \int_{\gamma_2} \sqrt{\omega(z)}dz$. Thus, the integral Φ (using any branch of the square root) defines a local coordinate, say w, on U centered at p:

$$w(x) := \int_p^x \sqrt{\omega(z)}dz.$$

Then $dw = \sqrt{\omega(z)}$ and squaring we obtain $dw^2 = \omega(z)dz^2$. This means that in terms of the parameter w, ω has the representation given identically by 1. If $\widetilde{w}$ is another parameter near p with this property, then $dw^2 = d\widetilde{w}^2$ implies $w = \pm\widetilde{w} + const$. w is called the canonical parameter near p.

Note that the norm $|| \cdot ||_1$ corresponds to the Euclidean area of the associated translation or half-translation surface.

Integrable quadratic differentials also arise in complex analysis as elements of the cotangent (or tangent) space to Teichmüller space. Although remainder of this manuscript does not explicitly use this connection to Teichmüller theory, the idea that quadratic differentials describe a family of complex structures foreshadows the construction of spaces of half-translation surfaces in §1.3.

We will briefly present this approach to quadratic differentials in the specific context of hyperbolic Riemann surfaces of finite type; our presentation follows [Hub06].

For a Riemann surface X, we will denote by $L^\infty_*(TX, TX)$ the Banach space of measurable $\mathbb{C}$-antilinear bundle maps $\nu : TX \to TX$ with the norm

$$||\nu||_\infty = \text{ess sup}_{x \in X} |\nu_x| < \infty,$$

where $|\nu_x|$ is the operator norm of the restriction of ν to the fiber $T_x X$. (A map $f : V \to W$ between complex vector spaces is said to be $\mathbb{C}$-antilinear if $f(ax + by) = \overline{a} f(x) + \overline{b} f(x)$ for all $a, b \in \mathbb{C}$ and $x, y \in V$.)

Definition 1.2.4. *An L^∞* **Beltrami form** *on a Riemann surface X is an element of the open unit ball of $L^\infty_*(TX, TX)$. The* **space of Beltrami forms** *on X, which we denote by $\mathcal{M}(X)$, is the open unit ball of $L^\infty_*(TX, TX)$.*

Theorem 1.2.5 (The Mapping Theorem). *For any open subset $U \subset \mathbb{C}$ and $\mu \in L^\infty(U)$ with $||\mu||_\infty < 1$, there exists a quasiconformal map $f : U \to \mathbb{C}$ such that*

$$\frac{\partial f}{\partial \overline{z}} = \mu \frac{\partial f}{\partial z}, \tag{1.1}$$

and that any other quasiconformal solution to Equation (1.1) differs from f by composition with a conformal map.

Consequently, for a Riemann surface X and a Beltrami form μ on X, we can apply the mapping theorem to each chart $\varphi_i : U_i \to V_i$ of X to obtain a quasiconformal map $f_i : V_i \to \mathbb{C}$ so that $\frac{\partial f_i}{\partial \overline{z}} = \mu \frac{\partial f_i}{\partial z}$; the maps $f_i \circ \varphi_i : U_i \to \mathbb{C}$ thus determine an atlas for **the Riemann surface $\mathbf{X}_\mu$**. Hence, the space of Beltrami forms $\mathcal{M}(X)$ parametrizes a family of Riemann surfaces. The **universal curve** $\mathcal{M}(X) \times X$ is an analytic Banach manifold whose atlas of charts $\Psi_i : \mathcal{M}(X) \times U_i \to \mathcal{M}(X) \times \mathbb{C}$ are given by $(\mu, x) \mapsto (\mu, f_i \circ \varphi_i(x))$.

We may think of the space of quadratic differentials $Q(X)$ as being dual to the space of Beltrami forms $\mathcal{M}(X)$. In a local coordinate z, a Beltrami form $\mu \in L^\infty_*(TX, TX)$ is written $\mu(z)\frac{d\overline{z}}{dz}$, and a quadratic differential $q \in Q(X)$ is written $q(z)dz^2$, so the product is

$$q\mu = q(z)\mu(z)d\overline{z}dz = q(z)\mu(z)|dz|^2.$$

Thus, we can define $\langle \mu, q \rangle := \int_X q\mu$.

Definition 1.2.6. *A* **quasiconformal surface** *is a $\sim_{qc}$-equivalence class in the space of triples $(\varphi : S \to X)$ such that S is a topological surface, X is a Riemann surface, and $\varphi : S \to X$ is a homeomorphism, where $\sim_{qc}$ is the equivalence relation defined by*

$$(\varphi_1 : S_1 \to X_1) \sim_{qc} (\varphi_2 : S_2 \to X_2)$$

if and only if there exists a quasiconformal homeomorphism $h : X_1 \to X_2$.

Note that the equivalence relation $\sim_{qc}$ in Definition 1.2.6 does not explicitly depend on the topological model surface S or map φ, although the requirement that the maps $\varphi_i : S_i \to X_i$ be homeomorphisms implies that the model surfaces S_1 and S_2 be homeomorphic, and hence the same topological surface. Hence, the quasiconformal surface $qc(X)$ associated to a Riemann surface X is well-defined.

For a quasiconformal surface $S = qc(X)$, where X is a Riemann surface, we will define the space of Beltrami forms on S, $\mathcal{M}(S)$ to be $\mathcal{M}(X)$. This definition is well-defined in that if $\varphi_1 : S_1 \to X_1$ and $\varphi_2 : S_2 \to X_2$ are two representatives of S, the map

$$(\varphi_2 \circ \varphi_1^{-1})^* : \mathcal{M}(X_1) \to \mathcal{M}(X_2)$$

is an analytic isomorphism. Thus, we may represent an element of $\mathcal{S}$ by a pair $((\varphi : S \to X), \mu)$, where X is a Riemann surface representing the quasiconformal surface S and $\mu \in \mathcal{M}(X)$.

For a quasiconformal surface S, we will denote by $\mathbf{QC}(S)$ the group of quasiconformal homeomorphisms of S, and by $\mathbf{QC^0}(S)$ the subgroup of quasiconformal homeomorphisms of S that fix $I(S)$ and are isotopic to the identity rel $I(S)$. (Since any quasiconformal map between two Riemann surfaces extends to a homeomorphism between their closures, the ideal boundary $I(S)$ can be defined as $I(X)$ for any Riemann surface X with $qc(X) = S$. In the case that X is of finite type, $I(X)$

is empty.) The group $\mathbf{QC}(S)$ acts on S via precomposition with the "marking" map: for $f \in \mathbf{QC}(S)$ and $m \in \mathcal{M}(S)$ represented by $((\varphi : S \to X), \mu)$, then $f \cdot S$ is represented by $((\varphi \circ f : S \to X), \mu)$. If S is of finite type, $MCG(S)$ is just the set of homotopy classes of orientation-preserving homeomorphisms of S that fix the punctures, if any.

Definition 1.2.7. *Let S be a quasiconformal surface, and define a map Φ_S on $\mathcal{M}(S)$ by*

$$\Phi_S\left(((\varphi : S \to X), \mu)\right) := (\varphi : S \to X_\mu).$$

The **Teichmüller space** *$\mathcal{T}_S$ of S is the quotient space*

$$\{\Phi_S(m) \mid m \in \mathcal{M}(S)\} / \sim_{QC^0},$$

where $\sim_{QC^0}$ is the equivalence relation $\Phi_S(m_1) \sim_{QC^0} \Phi_S(m_2)$ if and only if there exists $f \in \mathbf{QC^0}(S)$ such that $m_1 = f \cdot m_2$.

We now comment on Definition 1.2.7. The Teichmüller space $\mathcal{T}_S$ consists of all Riemann surfaces of a given quasiconformal type. Representing an element of $\mathcal{T}_S$ by a triple $(\varphi : S \to X_\mu)$, the map φ serves to identify the Riemann surface structure X_μ with the topological model surface S. Since elements of $\mathbf{QC^0}$ fix the ideal boundary $I(S)$, quotienting out by the equivalence relation $\sim_{QC^0}$ means that the "marking" provided by φ is consistently defined on $I(S)$ and on the punctures of S (which are not in S) but is only defined up to isotopy rel $I(S)$ on the interior of S. Thus, $\mathcal{T}_S$ consists of all Riemann surfaces "marked" by punctures and elements of $I(S)$, of a given quasiconformal type.

The **Mapping Class Group** of a quasiconformal surface S is the quotient group $MCG(S) := \mathbf{QC}(S)/\mathbf{QC^0}(S)$. When S is of finite type, $MCG(S)$ is a discrete group.

Definition 1.2.8. *Let S be a quasiconformal surface. The* **moduli space** *of S, denoted Moduli(S), is the quotient space $\mathcal{T}_S/MCG(S)$.*

Quotienting out by $MCG(S)$ in the definition of Moduli(S) allows us to "forget" the "markings" on the surfaces. Thus, $Moduli(S)$ may be thought of as consisting of all "unmarked" Riemann surfaces of a given quasiconformal type.

Proposition 1.2.9. *Let $\tau \in \mathcal{T}_S$ be represented by $\varphi :\to X$. The pairing $L^\infty_*(TX, TX) \times Q^1(X) \to \mathbb{C}$ given by*

$$(v, q) \mapsto \int_X vq \tag{1.2}$$

induces a pairing $T_\tau \mathcal{T}_S \times Q^1(X) \to \mathbb{C}$. This pairing induces an isomorphism $T_\tau \mathcal{T}_S \to (Q^1(X))^\perp$.

1.3 Spaces of translation surfaces

Definition 1.3.1. *A* **marked translation surface of finite type** *is a triple $((Z, \Sigma'), f, (S, \Sigma))$ consisting of*

1. *a closed, connected topological surface Z of finite genus with a set Σ' of distinct points of Z,*
2. *a translation surface (S, Σ) of finite type, and*
3. *a homeomorphism $f : Z \to S$ with $f(\Sigma') = \Sigma$.*

Since closed, connected, orientable topological surfaces are classified their genus, we will consider any two marked translation surfaces of finite type with the same genus to have the same underlying topological model surface Z (the "marking" is the identification of Z with the translation surface). Two marked translation surfaces of finite type that a built on the same "model surface" (Z, Σ'), say

$$((Z, \Sigma'), f_1, (S_1, \Sigma_1)) \quad \text{and} \quad ((Z, \Sigma'), f_2, (S_2, \Sigma_2)),$$

are considered to be equivalent if there exists a translation equivalence $h : (S_1, \Sigma_1) \to (S_2, \Sigma_2)$ such that $h \circ f_1$ is isotopic rel Σ' to f_2.

Definition 1.3.2. *The set of marked translation surfaces of genus g and n marked points is the set $\widetilde{\mathcal{H}}(Z_g, \Sigma'_n)$ consisting of all marked translation surfaces $((Z_g, \Sigma'_n), f, (S, \Sigma))$ where Z_g is a closed, connected, orientable surface of genus g with a set Σ'_n of n distinct points.*

A **marked half-translation surface of finite type** is defined similarly, and the set of marked half-translation surfaces of genus g with n marked points is denoted $\widetilde{\mathcal{Q}}(Z_g, \Sigma_n')$.

The set of all marked (half-)translation surfaces of a given type admits a natural stratification based on the number and type of cone points.

Definition 1.3.3. *Let $k_1, ..., k_n$ be natural numbers such that $\sum_{i=1}^n k_i = 2g-2$ for some $g \in \mathbb{N}$, and let Z_g be a closed, connected, orientable topological surface of genus g with a set Σ_n' of n marked points. The* **stratum $\widetilde{\mathcal{H}}(k_1, ..., k_n)$ of the space of marked translation surfaces** *consists of the set of marked translation surfaces $((Z_g, \Sigma_n'), f, (S, \Sigma))$ whose cone points have orders $k_1, \ldots, k_n$.*

Definition 1.3.4. *Let $k_1, ..., k_n$ be elements of the set $\{\frac{-1}{2}\} \cup \{\frac{1}{2}, \frac{2}{2}, \ldots, \frac{4g-4}{2}\}$ such that $\sum_{i=1}^n k_i = 2g-2$ for some nonnegative integer g, and let Z_g be a closed, connected, orientable topological surface of genus g with a set Σ_n' of n marked points. The* **stratum $\widetilde{\mathcal{Q}}(k_1, ..., k_n)$ of the space of marked half-translation surfaces** *consists of the set of marked half-translation surfaces $((Z_g, \Sigma_n'), f, (S, \Sigma))$ whose cone points have orders $k_1, \ldots, k_n$.*

Given a path γ on a marked translation surface, define the holonomy coordinates of γ to be

$$hol(\gamma) = \left(\int_\gamma dx, \int_\gamma dy \right).$$

Holonomy coordinates determine a map from a stratum $\widetilde{\mathcal{H}}(k_1, \ldots, k_n)$ of marked translations surfaces to $H^1(Z, \Sigma_n'; \mathbb{R}^2)$ as follows. We may think of an element of $H^1(Z, \Sigma_n'; \mathbb{R}^2)$ as assigning a number in $\mathbb{R}^2$ to each homotopy (rel Σ_n') class of paths in Z. Given a marked translation surface $((Z_g, \Sigma_n'), f, (S, \Sigma))$, define the corresponding element of $H^1(Z_g, \Sigma_n'; \mathbb{R}^2)$ to be the element that assigns to each relative homotopy class $[\gamma]$ of paths in Z_g the holonomy coordinates of the path $f \circ \gamma$ in S. Thus, holonomy defines a map

$$\widetilde{\mathcal{H}}(k_1, \ldots, k_n) \to H^1(Z_g, \Sigma_n'; \mathbb{R}^2) \simeq \mathbb{R}^{2(2g+n-1)}.$$

This map defines a topology and local coordinates on the stratum of $\widetilde{\mathcal{H}}(k_1, \ldots, k_n)$ of marked translation surfaces. A slightly perturbed marked translation surface has the same combinatorial triangulation, but the sidelengths of the triangles are slightly different. However, the surface may be deformed enough that it no longer admits the same combinatorial triangulation; hence holonomy provides local, but not global, coordinates on $\widetilde{\mathcal{H}}(k_1, \ldots, k_n)$.

Definition 1.3.5. *The* **mapping class group** $MCG(Z, \Sigma')$ *of a closed topological surface* Z *with a finite set* Σ' *of (distinct) marked points in* Z *is the group of isotopy classes of homeomorphisms of* (S, Σ') *(homeomorphisms of* S *that preserve* Σ' *as a set).*

The group $MCG(Z, \Sigma')$ acts on each stratum of the set of marked surfaces $\widetilde{\mathcal{H}}(Z, \Sigma')$ by precomposition with the marking map – i.e. the image of a marked translation surface

$$((Z, \Sigma'), f, (S, \Sigma))$$

under the action an element $[\beta]$ of $MCG(Z, \Sigma')$ represented by a map β is the marked translation surface

$$((Z, \Sigma'), f \circ \beta, (S, \Sigma)).$$

The group $MCG(Z, \Sigma')$ acts properly discontinuously on each stratum of finite type marked (half)-translation surfaces. We can "forget" the markings on a marked translation surface by identifying all marked translation surfaces which differ by precomposition with an element of the mapping class group:

Definition 1.3.6. *Let* $k_1, ..., k_n$ *be natural numbers such that* $\sum_{i=1}^{n} k_i = 2g - 2$ *for some* $g \in \mathbb{N}$*, and let* Z *be a closed, connected, orientable topological surface of genus* g *with a set* Σ'_n *of* n *marked points. The* **stratum** $\mathcal{H}(k_1, ..., k_n)$ **of the moduli space of translation surfaces** *is the quotient* $\widetilde{\mathcal{H}}(k_1, \ldots, k_n)/MCG(Z_g, \Sigma'_n)$.

Definition 1.3.7. *Let* $k_1, ..., k_n$ *be elements of the set* $\{\frac{-1}{2}\} \cup \{\frac{1}{2}, \frac{2}{2}, \ldots, \frac{4g-4}{2}\}$ *such that* $\sum_{i=1}^{n} k_i = 2g - 2$ *for some nonnegative integer* g*, and let* Z *be a closed, connected, orientable topological surface of genus* g *with a set* Σ'_n *of* n *marked points. The* **stratum** $\mathcal{Q}(k_1, ..., k_n)$ **of the moduli space of half-translation surfaces** *is the quotient* $\widetilde{\mathcal{Q}}(k_1, \ldots, k_n)/MCG(Z_g, \Sigma'_n)$.

We will sometimes find it convenient to consider only (half)-translation surfaces of area 1. We will denote by $\widetilde{\mathcal{H}}^1(k_1, \ldots, k_n)$, $\mathcal{H}^1(k_1, \ldots, k_n)$, $\widetilde{\mathcal{Q}}^1(k_1, \ldots, k_n)$, and $\mathcal{Q}^1(k_1, \ldots, k_n)$, the subsets of

$\widetilde{\mathcal{H}}(k_1,\ldots,k_n)$, $\mathcal{H}(k_1,\ldots,k_n)$, $\widetilde{\mathcal{Q}}(k_1,\ldots,k_n)$, and $\mathcal{Q}(k_1,\ldots,k_n)$, respectively, consisting of surfaces of area 1.

The stratum $\mathcal{H}(k_1,\ldots,k_n)$ of the moduli space of translation surfaces is naturally endowed with the quotient topology coming from $\widetilde{\mathcal{H}}(k_1,\ldots,k_n)$. The stratum $\mathcal{H}(k_1,\ldots,k_n)$ is thus an orbifold of real dimension $2 \cdot \dim\left(H^1(Z,\Sigma_n';\mathbb{R}^2)\right) = 2(2g+|\Sigma|-1)$.

Although the topology of most strata (of unmarked translation surfaces) is still rather mysterious, the connected components of each stratum were classified in [KZ03]. Connected components are characterized by whether or not they are "hyperelliptic components" and, in the case all cone points are of even orders, by the parity of the spin structure associated to a surface.

For every $g \in \mathbb{N}$, the strata $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1,g-1)$ each have a connected component which is called the hyperelliptic component of that stratum and denoted $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1,g-1)$, respectively. Each half-translation surface $S \in \mathcal{Q}(l_1,...,l_n)$ that is not itself a translation surface is associated in a canonical way with a translation surface T. T is the "orientation double cover" of the half-translation surface, and is constructed as follows. At any regular point of a half-translation surface, "direction" is defined up to $\pm Id$, so there are two possible local choices for the "positive" vertical direction. We form an isometric double cover $\hat{S}$ of $S \setminus \Sigma$ so that the fiber over each point consists of two points, corresponding to the two choices of "positive" direction. This cover is trivial near a cone point if and only if the cone angle at that point is an integer multiple of 2π. Hence, we compactify $\hat{S}$ by adding two points to the fiber over a singular point x in S if the cone angle at x is an integer multiiple of 2π, and adding one point if the cone angle at x is of the form $(2n-1)\pi$ for some $n \in \mathbb{N}$; the resulting surface is T. We are interested in the following two special cases of this map:

$$\mathcal{Q}((-1/2)^{2g_T+1}, g_T-3/2) \to \mathcal{H}(2g_T-2)$$
$$\mathcal{Q}((-1/2)^{2g_T+2}, g_T-1) \to \mathcal{H}(g_T-1,g_T-1).$$

(The exponential notation $\mathcal{Q}((-1/2)^{2g'+1}, g'-3/2)$ refers to half-translation surfaces with $2g'+1$ cone points of order $-1/2$ and one cone point of order $g'-3/2$.) In these two cases, the map is an injective immersion, the dimension of the domain equals the dimension of the range, and the domain is a nonempty, connected stratum whose elements are topological spheres. The **hyperelliptic component** $\mathcal{H}^{hyp}(2g-2)$ consists of the image of the map $\mathcal{Q}(-1^{2g+1}, 2g-3) \to \mathcal{H}(2g-2)$. The **hyperelliptic component** $\mathcal{H}^{hyp}(g-1, g-1)$ consists of the image of the map $\mathcal{Q}(-1^{2g+2}, 2g-2) \to \mathcal{H}(g-1, g-1)$.

Each surface in any hyperelliptic component admits a unique **hyperelliptic involution** - an affine equivalence ϕ from the surface to itself with $D(\phi) = -Id$ and which fixes precisely $2g+2$ points, where g is the genus of the surface. We will refer to those points in a hyperelliptic surface which are fixed by the hyperelliptic involution as **Weierstrass points**; the Weierstrass points that are not cone points of the translation surface correspond to cone points of angle π (or, equivalently, simple poles of the quadratic differential) in the half-translation surface which is the quotient of the translation surface by the hyperelliptic involution.

Recall that a *saddle connection* in a translation surface is a geodesic path in the surface whose end points are (not necessarily distinct) cone points of the surface. We will call a saddle connection whose midpoint is a Weierstrass point a **Weierstrass edge**. Note that these are precisely the saddle connections which are invariant (as a set) under the hyperelliptic involution.

The **parity of the spin structure** associated to a translation surface is defined only for translation surfaces whose cone points all have even order. It takes values in $\mathbb{Z}/2\mathbb{Z}$ and is invariant under continuous deformation of a translation surface within a stratum. We describe here a topological approach to the computation of the parity of spin structure.

Let S be a translation surface whose cone points are all of even order. Choose smooth, oriented, simple closed curves $\{\alpha_i, \beta_i\}_{i=1,\ldots,g}$ representing a basis for $H_1(S, \mathbb{Z})$ that is symplectic with respect to geometric intersection number. Define $\text{ind}(\alpha_i)$ to be the winding number of that loop (i.e. the integer number of times the tangent vector to the path spins around as you travel once around the loop). Then the parity of the spin structure associated to S is defined as

$$\sum_{i=1}^{g} (\text{ind}(\alpha_i) + 1)(\text{ind}(\beta_i) + 1) \pmod 2.$$

Theorem 1.3.8. *([KZ03]) All connected components of any stratum of Abelian differentials on a curve of genus $g \geq 4$ are described by the following list:*

1. *The stratum $\mathcal{H}(2g-2)$ has three connected components: the hyperelliptic one, $\mathcal{H}^{hyp}(2g-2)$, and two other components: $\mathcal{H}^{even}(2g-2)$ and $\mathcal{H}^{odd}(2g-2)$ corresponding to even and odd spin structures.*

2. *If g is odd, the stratum $\mathcal{H}(g-1,g-1)$ has three connected components: the hyperelliptic one, $\mathcal{H}^{hyp}(2l,2l)$, and two other components: $\mathcal{H}^{even}(2l,2l)$ and $\mathcal{H}^{odd}(2l,2l)$.*

3. *If g is even, the stratum $\mathcal{H}(g-1,g-1)$ has two connected components: one of them, $\mathcal{H}^{hyp}(g-1,g-1)$, is hyperelliptic; the other, $\mathcal{H}^{nonhyp}(g-1,g-1)$, is not.*

4. *All other strata of the form $\mathcal{H}(2l_1,...,2l_j)$, where all $l_i \geq 1$, have two connected components: $\mathcal{H}^{even}(2l_1,...,2l_j)$ and $\mathcal{H}^{odd}(2l_1,...,2l_j)$, corresponding to even and odd spin structures.*

5. *All the other strata of Abelian differentials on the curves of genera $g \geq 4$ are nonempty and connected.*

Theorem 1.3.9. *([KZ03])*

1. *The moduli space of Abelian differentials on a curve of genus $g=2$ contains two strata: $\mathcal{H}(1,1)$ and $\mathcal{H}(2)$. Each of them is connected and coincides with its hyperelliptic component.*

2. *Each of the strata $\mathcal{H}(2,2)$, $\mathcal{H}(4)$ of the moduli space of Abelian differentials on a curve of genus $g=3$ has two connected components: the hyperelliptic one and the one having odd spin structure. The other strata are connected for genus $g=3$.*

For a finite type translation surface (S,Σ), let systole(S,Σ) denote the length of a shortest saddle connection in (S,Σ). For any $\epsilon > 0$, and any stratum $\mathcal{H}$ of (unmarked) finite type translation surfaces, the set

$$C_\epsilon := \{(S,\Sigma) \in \mathcal{H} \mid \text{systole}(S,\Sigma) \geq \epsilon\}$$

is compact.

1.4 Dynamics

There are two primary “levels” of dynamics associated to translation surfaces: translation flows on a fixed translation surface, and the action of $SL(2,\mathbb{R})$ on a stratum of translation surfaces. These two “levels” of dynamics are intimately related.

Definition 1.4.1. *The* **translation flow** *in direction θ on a translation surface (S,Σ) is the unit-speed (with respect to the canonical Euclidean metric) parametrization of the flow on $S \setminus \Sigma$ along the leaves of $\mathcal{F}_\theta$.*

A theorem of **Kerchoff, Masur, and Smillie** ([KMS86]) asserts that for any finite type translation surface (S,Σ) and for almost every direction $\theta \in S^1$, the translation flow in direction θ is uniquely ergodic. However, in some directions, the translation flow in a fixed direction determines a decomposition of S into more than 1 closed, invariant sets called **components**. There are two types of components: minimal components and periodic components. A **periodic component** is the closure of a maximal cylinder of periodic orbits. A **minimal component** is the closure of a non-periodic orbit. (See [Bos88] for a proof that any interval exchange map admits a partition of the interval into finitely many subintervals, each of nonzero width, such that every subinterval of this partition belongs to a unique minimal or periodic component of the interval exchange map. Since the first return map to a cross-section of a flow in a fixed direction on a translation surface is an interval exchange map, this proves the statement that the flow has finitely many invariant components, each of which are minimal or periodic.) The boundaries of these invariant components necessarily consist of saddle connections in the direction of the flow.

The group $SL(2,\mathbb{R})$ (or $GL(2,\mathbb{R})$ acts on the collection of translation surfaces as follows. Given $A \in SL(2,\mathbb{R})$ and a translation surface (S,Σ), the flat surface given by $A \cdot (S,\Sigma)$ is given by post-composing the charts of S with A. The actions of two subgroups of $SL(2,\mathbb{R})$ on strata are of particular importance. The **Teichmüller flow** or **geodesic flow** is the action of the one-parameter subgroup consisting of all matrices of the form

$$g_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \qquad t \in \mathbb{R}.$$

The **horocycle flow** is the action of the one-parameter subgroup consisting of all matrices of the form

$$h_t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad t \in \mathbb{R}.$$

The theorem of Kerckhoff, Masur and Smillie, like many other theorems about dynamics on translation surfaces, uses the tool of *renormalization*. A key idea is that the geodesic flow acts on translation surfaces by contracting the vertical direction and thus "speeding up" the vertical translation flow, so that the asymptotic behavior of the trajectory $g_t \cdot (S, \Sigma)$ provides information about the dynamics of the vertical translation flow.

Masur's criterion states that if the vertical flow on a translation surface (S, Σ) of finite type is not uniquely ergodic, then the geodesic flow $g_t \cdot (S, \Sigma)$ is divergent in the stratum of moduli space (i.e. it leaves every compact set of the stratum).

Definition 1.4.2. *Let (S, Σ) be a (unmarked) translation surface of finite type. The* **Veech group** *of (S, Σ), which we denote by $SL(S, \Sigma)$, is the stabilizer in $SL(2, \mathbb{R}) \subset GL(2, \mathbb{R})$ of (S, Σ).*

Equivalently, if $\mathrm{Aff}^+(S, \Sigma)$ is the group of affine automorphisms of (S, Σ), $SL(S, \Sigma)$ is the group of derivatives of elements of $\mathrm{Aff}^+(S, \Sigma)$.

Definition 1.4.3. *A translation surface (S, Σ) is a* **lattice surface** *if $SL(S, \Sigma)$ is a lattice, i.e. has finite co-volume in $SL(2, \mathbb{R})$.*

Theorem 1.4.4 (The Veech Dichotomy.). *If (S, Σ) is a lattice surface, then for every direction $\theta \in S^1$, precisely one of the following is true:*

1. *(S, Σ) admits a cylinder decomposition in direction θ, or*

2. *the translation flow in direction θ on (S, Σ) is uniquely ergodic.*

Definition 1.4.5. *A translation surface (S, Σ) is said to be*

- **periodic** *in direction θ if (S, Σ) admits a cylinder decomposition in direction θ,*

- **completely periodic** *if (S, Σ) is periodic in every direction in which (S, Σ) has at least one cylinder,*

- **uniformly completely periodic** *if* (S, Σ) *is completely periodic and there exists* $c > 0$ *such that for any direction* θ *for which* (S, Σ) *is periodic, the ratio of lengths of any two saddle connections in direction* θ *is at most* c*.*

- **parabolic** *in direction* θ *if* (S, Σ) *is periodic in direction* θ *and the moduli of all the cylinders in direction* θ *are commensurable.*

- **uniformly completely parabolic** *if* (S, Σ) *is uniformly completely periodic and* (S, Σ) *is parabolic in every periodic direction.*

Smillie and Weiss prove the equivalence of a long list of characterizations of lattice surfaces, which includes the following characterizations:

Theorem 1.4.6 ([SW10])**.** *The following are equivalent for a finite type (unmarked) translation surface* (S, Σ)*:*

- (S, Σ) *is a lattice surface.*

- (S, Σ) *is uniformly completely periodic.*

- (S, Σ) *is uniformly completely parabolic.*

- (S, Σ) *has "no small triangles" (see [SW10] for a precise definition).*

- *The* $SL(2, \mathbb{R})$ *orbit of* (S, Σ) *is closed.*

- *There is a compact subset* K *of the stratum* $\mathcal{H}$ *containing* (S, Σ) *such that for any* $\alpha \in SL(2, \mathbb{R})$*, the orbit of* $\alpha \cdot (S, \Sigma)$ *under the geodesic flow intersects* K*.*

A closed $SL(2, \mathbb{R})$ orbit (in a stratum of unmarked finite type translation surfaces) is called a **Teichmüller curve**. Attempting to classify those translation surfaces which generate Teichmüller curves has been a major area of inquiry in the past two decades. Although there has been some success for surfaces of low genus (see e.g. [McM07, Cal04, McM06]), a complete classification is still not known.

While we may not know which translation surfaces have closed $SL(2, \mathbb{R})$ orbits, a recent result of Eskin, Mirzakhani and Mohammedi yields insight into the structure of the closures of arbitrary

$SL(2,\mathbb{R})$ orbits. We will first state the theorem, and then define the term *affine invariant submanifold*.

Theorem 1.4.7 ([EMM]). *Let (S,Σ) be a finite type translation surface in a stratum $\mathcal{H}$. Then, the orbit closure $\overline{SL(2,\mathbb{R})\cdot(S,\Sigma)}$ is an affine invariant submanifold of $\mathcal{H}$.*

In §1.3, we defined the holonomy map

$$hol : \widetilde{\mathcal{H}}(k_1,\ldots,k_n) \to H^1(Z_g,\Sigma'_n;\mathbb{R}^2).$$

This map, when we identify $\mathbb{R}^2$ with $\mathbb{C}$ and interpret an element of $H^1(Z_g,\Sigma'_n;\mathbb{R}^2)$ as an element of $H^1(Z_g,\Sigma'_n;\mathbb{C})$, defines **period coordinates** on $\widetilde{\mathcal{H}}$. For any neighborhood $\widetilde{U}\subset\mathcal{H}$ over which the bundle $H^1(Z_g,\Sigma'_n;\mathbb{C})$ is trivializable, we represent the period coordinates on $\widetilde{U}$ by a map

$$\widetilde{\Phi} : \widetilde{U} \to H^1(Z_g,\Sigma'_n;\mathbb{C}) \simeq \mathbb{C}^n,$$

and this descends to a map

$$\Phi : U \to H^1(S,\Sigma;\mathbb{C}) \simeq \mathbb{C}^n$$

for a neighborhood U in $\mathcal{H} = \widetilde{\mathcal{H}}/MCG(Z_g,\Sigma'_n$ of (S,Σ), using the trivialization of the $H^1(Z_g,\Sigma'_n;\mathbb{C})$-bundle over $\widetilde{U}$ to identify it with $U\times H^1(S,\Sigma;\mathbb{C})$. Such a map Φ defines **local period coordinates** on $U\subset\mathcal{H}$.

Recall that $\mathcal{H}^1(k_1,\ldots,k_n)$ consists of the subset of surfaces in $\mathcal{H}(k_1,\ldots,k_n)$ that have area 1. For $(S,\Sigma)\in\mathcal{H}^1(k_1,\ldots,k_n)$, we denote by $\mathbb{R}(S,\Sigma)$ the set of surfaces in $\mathcal{H}(k_1,\ldots,k_n)$ that are (S,Σ) "scaled" uniformly by a factor of $r\in\mathbb{R}^+$.

Definition 1.4.8. *An ergodic $SL(2,\mathbb{R})$-invariant Borel probability measure ν_1 on $\mathcal{H}^1(k_1,\ldots,k_n)$ is* **affine** *if the following hold:*

1. *The support $\mathcal{M}_1$ of ν_1 is an immersed submanifold of $\mathcal{H}^1(k_1,\ldots,k_n)$, meaning there exists a manifold $\mathcal{N}$ and a proper continuous map $f:\mathcal{N}\to\mathcal{H}^1(k_1,\ldots,k_n)$ so that $\mathcal{M}_1 = f(\mathcal{N})$.*

2. *The self-intersection points of $\mathcal{M}_1$ (i.e. the set of points of $\mathcal{M}$ which do not have a unique preimage under f) is a closed subset of $\mathcal{M}$ of ν-measure 0.*

3. *Each point in $\mathcal{N}$ has a neighborhood U such that locally $\mathbb{R}f(U)$ is given by a complex linear subspace defined over $\mathbb{R}$ in the period coordinates.*

4. *Denote by ν the measure supported on $\mathbb{R}\mathcal{M}$ such that $d\nu = d\nu_1 da$, where a is the area of a translation surface. Each point in $\mathcal{N}$ has a neighborhood U such that the restriction to $\mathbb{R}f(U)$ is a linear measure in the period coordinates on $\mathbb{R}f(U)$, i.e. it is, up to normalization, the restriction of Lebesgue measure to the subspace $\mathbb{R}f(U)$.*

Definition 1.4.9. *An* **affine invariant submanifold** *of $\mathcal{H}^1(k_1, \dots, k_n)$ is a suborbifold $\mathcal{M}_1$ of $\mathcal{H}^1(k_1, \dots, k_n)$ that is the support of an affine ergodic $SL(2, \mathbb{R})$-invariant Borel probability measure on $\mathcal{H}^1(k_1, \dots, k_n)$.*

In particular, an affine invariant submanifold is a closed subset of $\mathcal{H}^1(k_1, \dots, k_n)$ which is invariant under $SL(2, \mathbb{R})$ and which, in local period coordinates, is an affine subspace.

Chapter 2

Counting invariant components

2.1 Chapter Overview

As described in §1.4, the translation flow in a fixed direction on a translation surface S determines a decomposition of S into closed invariant sets, each of which is either periodic or minimal. (See Figure 2.1.) In this chapter, we study this decomposition for translation surfaces in the hyperelliptic connected components $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1,g-1)$ of the corresponding strata of the moduli space of translation surfaces. Specifically, Theorem 2.1.1 characterizes the pairs of nonnegative integers (p,m) for which there exists a translation surface in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$ with precisely p periodic components and m minimal components. The content of this chapter was previously published in [Lin13].

Theorem 2.1.1. *Let g, m, and p be nonnegative integers such that $g \geq 2$ and m and p do not vanish at the same time.*

1. *There exists a translation surface in the hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$ with precisely p periodic components and m minimal components if and only if*

$$3m + 2p - 1 \leq 2g.$$

2. *There exists a translation surface in the hyperelliptic component $\mathcal{H}^{hyp}(g-1,g-1)$ with precisely p periodic components and m minimal components if and only if*

$$3m + 2p - 2 \leq 2g.$$

An interesting feature of Theorem 2.1.1 is that when trying to maximize the number of invariant components of surfaces in the hyperelliptic components, minimal components "count" one and a half times as much as periodic components do against an upper bound. Theorem 2.1.1 extends results (Theorems 2.1.2 and 2.1.3 below) by Naveh ([Nav08]), who obtained tight upper bounds on m and, for each fixed value of m, on $m+p$, with the bounds taken over each stratum as a whole. Naveh's bounds in the cases of the strata we are considering are as follows:

1. Over the stratum $\mathcal{H}(2g-2)$, the tight upper bound on m is $g-1$, and for each fixed $0 \leq m \leq$

$g-1$ the tight upper bound p is $g-m$.

2. Over the stratum $\mathcal{H}(g-1,g-1)$, the tight upper bound on m is g. When $m=g$, $p=0$. For each fixed value of m with $0 \le m \le g-1$, the tight upper bound on p depends on the parity of g. If g is odd, the tight upper bound on p is $g+1-m$. If g is even and $m=g-1$, the tight upper bound on p is 1. If g is even and $m<g-1$, the tight upper bound on p is $g+1-m$.

In contrast, Theorem 2.1.1 gives $p \le g+\frac{1}{2}-\frac{3m}{2}$ for the component $\mathcal{H}^{hyp}(2g-2)$ and $p \le g+1-\frac{3m}{2}$ for the component $\mathcal{H}^{hyp}(g-1,g-1)$. This shows that, except in the cases for which Naveh's bounds coincide with those of Theorem 2.1.1, the surfaces constructed in [Nav08] which realize Naveh's bounds are not hyperelliptic.

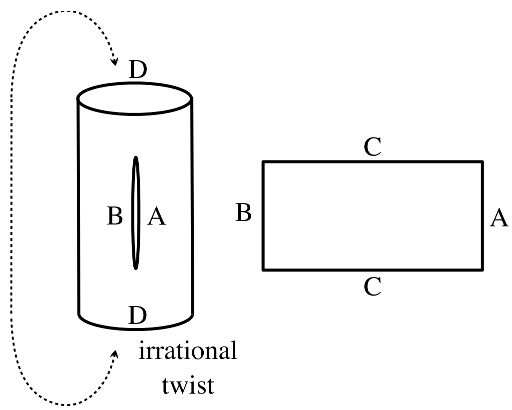


Figure 2.1: A surface in $\mathcal{H}(2)$ composed of one periodic component and one minimal component f(or the vertical translation flow). The minimal component is formed by taking a vertical "tube" and gluing the top of the tube to the bottom of the tube via an irrational twist. A vertical "slit" is cut in the tube, and we glue the ends of a horizontal cylinder - the periodic component - to the sides of this slit.

The idea underlying the proof of Theorem 2.1.1 is to decompose translation surfaces into subsurfaces, each of which represents an invariant component, and analyze how these subsurfaces are "glued" together. In Section 2.2 we introduce *invariant component diagrams*, graph-like objects which describe how a surface in a hyperelliptic component of moduli space decomposes into invariant components. By relating the Euler characteristic of a surface with the number of edges in the associated invariant component diagram, we obtain restrictions describing which invariant component diagrams are possible for surfaces in a given hyperelliptic component of moduli space. Proposition 2.2.10 uses this relationship to show that every surface in a hyperelliptic component must satisfy the inequalities in Theorem 2.1.1. In Section 2.3, we use invariant component diagrams as "blueprints" for constructing surfaces with specified numbers of periodic and minimal components. We create a toolbox of "pieces," i.e. closed hyperelliptic subsurfaces P_n and M_n, which can be linked together to build the surfaces described certain invariant component diagrams. Proposition 2.3.3 proves that for any m, p, and g which satisfy the requirements of Theorem 2.1.1, we can construct a surface in the target hyperelliptic component with that number of periodic and minimal components. The proof of Theorem 2.1.1, located at the end of Section 2.3, combines these two propositions.

Related Results

The Poincaré-Bendixson Theorem implies that every minimal component of a translation surface has genus at least one. A classical result states that a continuous flow on a closed, orientable surface (not necessarily a translation surface) of genus g has at most g distinct sets which are orbit closures of non-periodic, recurrent points. Moreover, any such surface admits a continuous flow which achieves this bound [Mar70]. Naveh discovered the following two theorems.

Theorem 2.1.2. *([Nav08]) Let $\mathcal{H} = \mathcal{H}(k_1, ..., k_n)$ be a stratum in the moduli space of translation surfaces of genus g.*

1. *If $k_i \leq g-1$, $i = 1, \ldots, n$, then for every flat surface in $\mathcal{H}$ an upper bound on the number of minimal components is g, and this bound is tight.*

2. *Otherwise, for every flat surface in $\mathcal{H}$ an upper bound on the number of minimal components is $g-1$, and this bound is tight.*

Theorem 2.1.3. *([Nav08]) Let $\mathcal{H} = \mathcal{H}(k_1, ..., k_n)$ be a stratum in the moduli space of translation surfaces of genus $g \geq 2$. Denote $B = \{i : k_i \text{ is odd}\}$. Fix $0 \leq M \leq g-1$ and denote $m = \max\{0, M - [g-1-|B|/2]\}$. Then, if there exists a surface in $\mathcal{H}$ with M minimal components and P periodic components, it satisfies*

$$M + P \leq g - 1 + n - m.$$

Furthermore, for each M such that $0 \leq M \leq g-1$, this bound is tight (meaning there exists a surface in $\mathcal{H}$ with M minimal components and $g-1+n-m-M$ periodic components). If $M = g$, then $P = 0$ and $M + P = g$.

2.2 Invariant component diagrams and upper bounds

In this section, we develop a theory of invariant component diagrams for surfaces in the hyperelliptic connected components. These graph-like objects describe how the subsurfaces corresponding to the various invariant components of a translation surface "sit next to each other" in the surface. A key observation is that every invariant component of such a surface is invariant (as a set) under the hyperelliptic involution (Lemma 2.2.1). In fact, if we cut along the boundaries of an invariant component C of a surface S and then "heal" the cuts by gluing together edges of C that are paired by the hyperelliptic involution of S, the resulting "piece" is itself a hyperelliptic translation surface consisting of a single invariant component (Lemma 2.2.3). Since the quotient of surface in a hyperelliptic connected component by the hyperelliptic involution has genus 0, it follows that these "pieces" are arranged in a tree (Corollary 2.2.5). There is an easy formula relating the Euler characteristic (or total cone angle) of a surface associated to an invariant component diagram to the numbers of various types of edges in the diagram (Lemma 2.2.9). Proposition 2.2.10 uses this formula to establish a lower bound on the genus a surface in a hyperelliptic connected component with specified numbers of minimal and periodic components may have.

We will adopt the convention that the translation flow on a surface is in the vertical direction unless otherwise specified.

Lemma 2.2.1. *For any $g > 0$, let S be a translation surface in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$, and denote by γ the hyperelliptic involution of S. Then $\gamma(C) = C$ for every invariant component C of S.*

Proof. If $g = 1$, the translation surface S has no saddle connections and so consists of a single invariant component. If S consists of a single invariant component the conclusion is immediate, regardless of g.

So assume $g \geq 2$ and assume S has more than one invariant component (in the vertical direction). Let A be the maximum of the areas (with respect to the flat metric) of the invariant components of S. For $t > 0$, let $M_t \in GL_2(\mathbb{R})$ be the matrix $\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$. For $t > 0$, denote by S_t the surface made by gluing together $S - C$ and $M_t(C)$ in the same way as in the original surface S. (The boundary of C in S consists of vertical saddle connections, and the matrices M_t preserve the direction and lengths of vertical saddle connections, so we can still glue along the vertical saddle connections to form S_t.)

Pick $p > 1$ such that the area of $M_p(C)$ is greater than A. The surfaces S_p and S are in the same connected component of moduli space, the hyperelliptic component, since $\{S_t : 1 \leq t \leq p\}$ forms a path in moduli space from S to S_p. Let γ_p be the hyperelliptic involution of S_p. Since γ_p is an isometry which maps vertical geodesics to vertical geodesics, γ_p maps minimal components to minimal components and periodic components to periodic components (of the same area). Since $area(M_p(C)) > A$, we must have $\gamma_p(M_p(C)) = M_p(C)$. Hence $\gamma_p(S_p - M_p(C)) = S_p - M_p(C)$.

Now we will use γ_p to construct a hyperelliptic involution γ_1 on S. On $S - C$, let $\gamma_1 = \gamma_p$. On C, let $\gamma_1 = M_p^{-1} \circ \gamma_p \circ M_p$. Since γ_p is an involution and has $2g-2$ fixed points, so does γ_1. Thus γ_1 is a hyperelliptic involution on S. Since hyperelliptic involutions are unique for surfaces of genus $g \geq 2$, $\gamma = \gamma_1$. Thus $\gamma(C) = \gamma_1(C) = C$. $\square$

For any invariant component C, we will refer to a vertical saddle connection (recall we are assuming the flow to be in the vertical direction) in the boundary of C as a *boundary edge* of C.

Lemma 2.2.2. *Let C and D be invariant components of a surface S in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$. Then the number of boundary edges of C that are glued to D is even, and the hyperelliptic involution interchanges pairs of these boundary edges.*

Proof. Assume there exists a boundary edge e_1 of C which is glued to D. Let p be the midpoint of e_1 and let γ be the hyperelliptic involution of S. Suppose $\gamma(e_1) = e_1$. Then since γ is an isometry, γ restricted to e_1 is either the identity or rotation about the midpoint of e_1. Since γ has only finitely many fixed points in S, γ cannot be the identity. Hence p is a Weierstrass point, and in a small neighborhood of p in S, γ acts as a rotation around p by π radians. Since $\gamma(C) = C$ by Lemma 2.2.1, this means a small disk about p is contained in C, contradicting the fact that $p \in e_1$ is in the boundary of C.

Therefore $\gamma(e_1) \neq e_1$. Let $e_2 = \gamma(e_1)$. Since both C and D are fixed by γ, e_2 is also a boundary edge of C that is glued to D. Since γ is an involution $\gamma(e_2) = e_1$. $\square$

We will now define the dissection $\hat{S}$ of a hyperelliptic surface S in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$. The first step in constructing $\hat{S}$ is to cut along all boundary edges of all the invariant components of S. This results in a number of "pieces" (surfaces with boundary) – one for each invariant component of S. Each boundary edge of an invariant piece C is paired with another boundary edge of C by the hyperelliptic involution by Lemma 2.2.2. The second step in constructing $\hat{S}$ is to glue the paired edges together via translations. This yields a union of closed connected surfaces, say $S_1, ..., S_k$, where k is the number of invariant components of S. We will call each surface S_i a *piece* of the dissection $\hat{S}$. (Lemma 2.2.3 will show that each S_i is a translation surface, as opposed to a half-translation surface). Define the set of *augmented cone points* to be the set consisting of all preimages in $S_1 \cup ... \cup S_k$ of the cone points of S. (Every cone point in any surface S_i is the preimage of a cone point of S, but not every preimage in a S_i of a cone points of S is a cone point (i.e. has cone angle $> 2\pi$) of S_i.) The **dissection** $\hat{S}$ of S is the collection of pieces $\{S_1, ..., S_k\}$ together with the set of augmented cone points.

For example, suppose S is a surface in $\mathcal{H}(1,1)$ which consists of two minimal tori glued along a vertical slit. To construct the dissection $\hat{S}$, the first step would cut along the slit. This yields two tori, each of which has a slit cut in it. Second, since the hyperelliptic involution interchanges the sides of each slit, we would "heal" (glue together the sides of) the slit in each torus. This yields two minimal tori S_1 and S_2 (without slits); these are the two pieces of $\hat{S}$. The set of augmented cone points consists of four points: the points in each torus that were the endpoints of the slit. The dissection $\hat{S}$ consists of the two pieces S_1 and S_2, along with the set of the four augmented cone points.

Define an *augmented saddle connection* of $\hat{S}$ to be a geodesic path in one of the S_i whose endpoints are both augmented cone points. If C is an invariant component of S, each pair of boundary edges of C becomes a vertical augmented saddle connection in the piece S_C in $\hat{S}$ corresponding to C. Furthermore, the restriction of the hyperelliptic involution to S_C defines an isometric involution on S_C for which the midpoint of this augmented saddle connection is a fixed point.

Lemma 2.2.3. *Each piece S_i of the dissection $\hat{S}$ is a translation surface.*

Proof. To show that S_i is a translation surface, as opposed to a half-translation surface, it suffices to show that when we describe S_i as a finite collection of polygons embedded in $\mathbb{R}^2$ whose boundaries are given a counter-clockwise orientation, edges which are identified are parallel and *have opposite orientations*. (Edge identifications for polygons comprising a half-translation surface are not required to identify oppositely-oriented edges.)

Represent S by a finite collection P of polygons embedded in $\mathbb{R}^2$ whose boundaries are oriented counter-clockwise, along with "gluing rules" which identify pairs of parallel, oppositely-oriented edges. Without loss of generality, we may assume the hyperelliptic involution interchanges pairs of congruent polygons. We may further assume that each polygon is contained in a unique invariant component of S. Let $P_i \subset P$ be the subset consisting of polygons which make up the invariant component of S which corresponds to piece S_i.

Every edge identification of polygons in P satisfies the requirement that edges have opposite orientation. The only edges of polygons in P_i which are not glued (in S) to other edges of polygons in P_i are those that belong to the boundary of the invariant component corresponding to the piece S_i. To form the piece S_i, we glue (via translations) pairs of these edges interchanged by the hyperelliptic involution. Thus, the restriction of the hyperelliptic involution to S_i fixes the midpoint of such a pair of identified edges, and in a neighborhood of this point, acts as a rotation by π radians about this point. Since the hyperelliptic involution interchanges pairs of congruent polygons, this implies that the embeddings in $\mathbb{R}^2$ of paired congruent polygons differ by a rotation by π radians. Consequently, if e_1 and e_2 are edges of polygons in P_i such that e_1 and e_2 are contained the boundary edges of the invariant component $S_i \subset S$ and e_1 and e_2 are paired by the hyperelliptic involution, then e_1 and e_2 have opposite orientations. Therefore S_i is a translation surface. $\square$

Lemma 2.2.4. *Let S be an element of $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$.*

1. *The quotient of each piece S_i of the dissection $\hat{S}$ by the isometric involution that is the restriction of the hyperelliptic involution of S to the piece S_i has genus 0.*

2. *The pieces of $\hat{S}$ are arranged in a tree.*

Proof. Let S_1,...,S_n be the pieces of $\hat{S}$. Let S^* denote the quotient surface S/γ, and let $S_1^*, ..., S_n^*$ denote the quotient of the pieces by the isometric involutions which are the restrictions of the hyperelliptic involution γ of S to each piece. The surface S is formed from the pieces $S_1, ..., S_n$ by cutting along and gluing pairs of augmented saddle connections in the pieces. Let $p_1, ..., p_m$ be a list of the pairs of augmented saddle connections which are joined together to form S. Let $p_1^*, ..., p_m^*$ be a list of the pairs of geodesic segments in $S_1^*, ..., S_n^*$ which are the images of $p_1, ..., p_m$. Then the quotient surface S^* is formed from $S_1^*, ..., S_n^*$ but cutting along and gluing the pairs $p_1^*, ..., p_m^*$.

We will express the Euler characteristic $\chi(S^*)$ in terms of the Euler characteristics $\chi(S_1^*), ..., \chi(S_n^*)$. We have that $\chi(S^*) = 2$ by the definition of a hyperelliptic surface. The surface obtained by cutting along and gluing a pair of segments p_i^* is homeomorphic to the connected sum of the two quotient pieces. The Euler characteristic of the connected sum of any two surfaces X_1 and X_2 equals $\chi(X_1) + \chi(X_2) - 2$. Therefore,

$$2 = \chi(S^*) = -2m + \sum_{i=1}^{n} \chi(S_i^*). \tag{2.1}$$

Denote the genus of S_i^* by g_i^*. Then $\chi(S_i^*) = 2 - 2g_i^*$. In order to connect all the pieces $S_1, ..., S_n$, we must have that $m \geq n - 1$. Thus,

$$-2m + \sum_{i=1}^{n} \chi(S_i^*) \leq -2(n-1) + \sum_{i=1}^{n} \chi(S_i^*) = 2 - \sum_{i=1}^{n} 2g_i^*. \tag{2.2}$$

Combining equations (2.1) and (2.2) yields

$$2 \leq 2 - \sum_{i=1}^{n} 2g_i^*,$$

implying $g_i^* = 0$ for all i. This proves part 1 of the proposition.

Using the fact that $g_i^* = 0$ for all i, we have

$$2 = \chi(S^*) = -2m + \sum_{i=1}^{n} \chi(S_i^*) = -2m + 2n - \sum_{i=1}^{n} 2g_i^* = 2(n-m).$$

Hence, $m = n - 1$. Since we have n pieces connected along $m = n - 1$ slits, the pieces must be arranged as a tree, proving part 2 of the proposition. □

Combining Lemmas 2.2.3 and 2.2.4 yields the following classification of surfaces in the hyperelliptic components $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1,g-1)$:

Corollary 2.2.5. *Each translation surface in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$ with n invariant components consists of n hyperelliptic translation surfaces which have slits cut along augmented Weierstrass edges, and these hyperelliptic translation surfaces are glued together in a tree configuration by identifying pairs of slits via translations.*

We now define the **invariant component diagram** associated to a surface S in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$, a graph-like object that describes how the pieces of $\hat{S}$ can be connected to form S. (Figures 2.2 and 2.3 illustrate examples of an invariant component diagram.) The graph has n vertices, where n is the number of pieces of $\hat{S}$, and a bijection relates the vertices with the pieces of $\hat{S}$. The graph has a number of *half-edges*, line segments which have one end at a vertex and the other end free-floating. There are two different types of half-edges, which we will call solid half-edges and dotted half-edges. Two solid half-edges incident to two distinct vertices may joint to form a full solid edge.

Each solid half-edge incident to a vertex represents a vertical augmented saddle connection on the corresponding piece of $\hat{S}$. Each dotted half-edge incident to a vertex represents a pair of augmented critical leaves of the vertical foliation of the corresponding piece of $\hat{S}$ which do not form an augmented saddle connection (i.e. a dotted half-edge represents a "broken" augmented saddle connection). Define a bijection relating the half-edges of the graph and the augmented saddle connections (intact or "broken") of the pieces of $\hat{S}$.

To form S from the pieces of $\hat{S}$, we cut along some of the vertical augmented saddle connections in the pieces, turning these augmented saddle connections into slits, and then we glue together pairs of slits. For each pair of these slits which are glued together to form S, connect the corresponding two solid half-edges in the graph so that they form a full solid edge.

Algorithmic definition of the invariant component diagram associated to S:

1. Draw n labeled vertices, where n is the number of pieces of the dissection $\hat{S}$. Say that vertex v_i represents piece C_i of the dissection $\hat{S}$.

2. For each i, draw k_i solid half-edges incident to vertex v_i, where k_i is the number of augmented vertical saddle connections in the surface C_i.

3. For each i, draw j_i dotted half-edges incident to vertex v_i, where j_i is the number of augmented "broken" vertical saddle connections in the surface C_i. (A "broken" saddle connection means a pair of critical leaves of the vertical foliation which are of infinite length and do not form a saddle connection.)

4. For each pair of vertical saddle connections in the original surface S which were boundary edges of components C_m and C_n (and which we cut when forming the dissection $\hat{S}$), connect a pair of solid half-edges incident to vertices v_m and v_n to form a full solid edge between these two vertices.

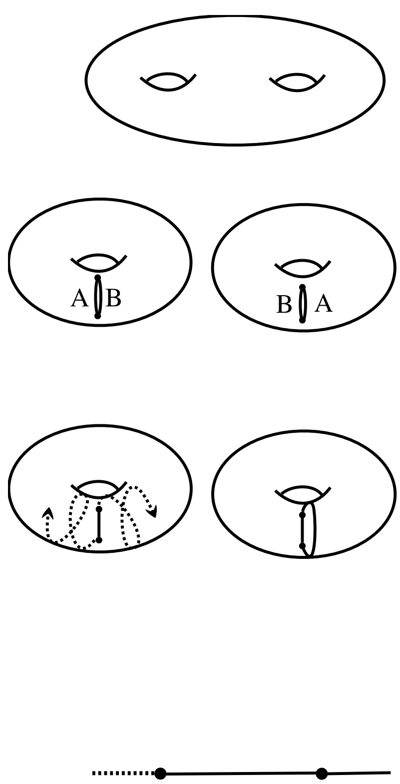
Lemma 2.2.6. *Let S be in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$. Denote by n_i the number of half-edges incident to the i^{th} vertex of the invariant component diagram associated to S. Then the total cone angle of S equals $2\pi\sum_i n_i$.*

Proof. A cone point with cone angle $k\pi$ has k vertical rays (going either "up" or "down") which emanate from the cone point. A vertical (possibly augmented) saddle connection requires two such rays (one going up, one going down, meeting in the middle). A "broken" saddle connection (i.e. a pair of non-closed augmented critical leaves of the vertical foliation) also requires two such rays. Each half-edge incident to a vertex in the invariant component diagram represents a vertical augmented (possibly broken) saddle connection in the corresponding piece of the dissection $\hat{S}$. Hence, 2π multiplied by the number of half-edges incident to a fixed vertex equals the sum over the augmented cone points in that piece of the cone angle at those points. Hence, the total cone angle of S equals the sum over all the augmented cone points in all the pieces in $\hat{S}$ of the cone angle at those points. □

Lemma 2.2.7. *Let S be a translation surface such that every critical leaf of the vertical foliation is closed. Then S admits a cylinder decomposition in the vertical direction.*

Proof. Assume every critical leaf of the vertical foliation of S is closed. Let x be a point in S that is located some small distance ϵ in the horizontal direction from a vertical saddle connection. Then every point of the leaf passing through x of the vertical foliation is also ϵ away from a vertical saddle connection. Because S has finitely many vertical saddle connections, each of which is of finite length, the leaf passing through x must be periodic. As x was arbitrary, this implies every vertical saddle connection has a neighborhood of periodic points. Since the boundaries between invariant components of S are vertical saddle connections, this implies every regular point of S is periodic. □

Dotted half-edges in the invariant component diagram represent "broken" saddle connections, so Lemma 2.2.7 immediately implies:



A surface S in $\mathcal{H}(1,1)$. Assume the vertical flow on S determines one minimal component and one periodic component.

Intermediate step in forming $\widehat{S}$. Cut along the edges of the invariant components. This forms two slit tori. One torus is minimal, the other is periodic.

The dissection $\widehat{S}$, formed by gluing together the A and B edge of each torus (these edges are paired by the hyperelliptic involution.) The torus on the left is a minimal component; the vertical trajectories indicated by dotted lines do not meet. The torus on the right is periodic.

The invariant component diagram associated to S. The four half-edges of the graph represent the four saddle connections in $\widehat{S}$ (one of which is the "broken" saddle connection consisting of the pair of critical leaves which do not meet). The central two half-edges form a full edge joining the two vertices (these represent the saddle connections which were slits in the intermediate step).

Figure 2.2: A simple example of an invariant component diagram.

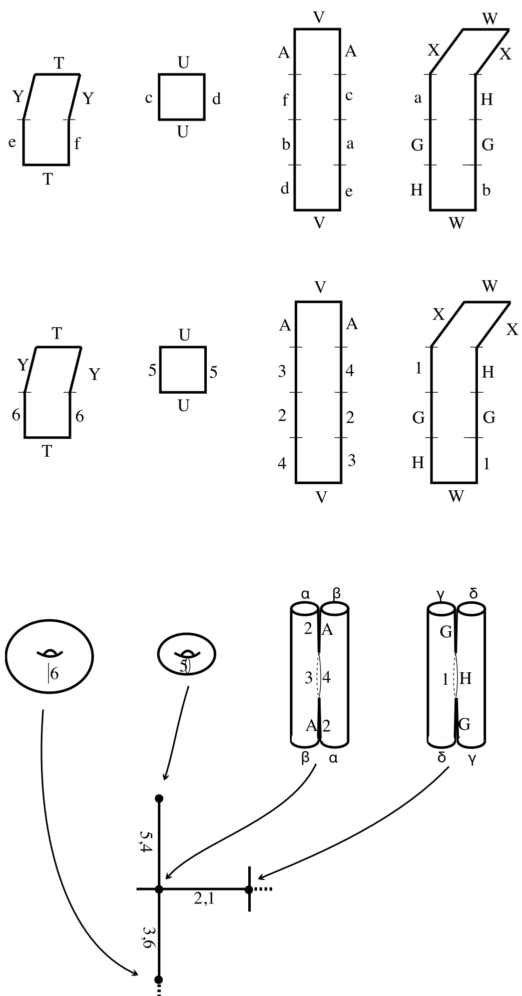
Corollary 2.2.8. *Each vertex in the invariant component diagram which corresponds to a minimal component in S has at least one incident dotted half-edge.*

Lemma 2.2.9. *Let S be a translation surface in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$ which has m minimal components and p periodic components. Then the total cone angle of S is at least $2\pi(3m+2p-2)$.*

Proof. The invariant component diagram associated to S must have

1. $m+p$ vertices (of which m represent minimal components and p represent periodic components),

2. at least one dotted half-edge incident to each of the "minimal" vertices, and



The original surface, which consists of four invariant components. Capital letters denote identifications of polygon edges which are NOT edges of the components, and lower case letters denote identifications of polygon edges which ARE edges of the component. The two pairs of slanted edges labeled by *X* and *Y* are each meant to indicate an "irrational twist" forming a minimal component.

We cut along the edges of the components, destroying the identifications labeled in the previous step by lower case letters. The hyperelliptic involution on the original surface determines an identification of pairs of edges of components. We use roman numeral to denote these pairings. The dissection consists of the four closed surfaces ("pieces") formed by gluing all pairs of edges marked at left.

We have redrawn each component so as to emphasize our mental picture of each component as a topoogical surface. The setments labeled G, A, and 2 are vertical saddle connections, and the horizontal edges labeled by greek letters are glued via a horizontal twist so that these vertical saddle connections are not broken.

The invariant component diagram, with additional labels on the full edges. The label *(2,1)* on the horizontal edge indicates that in order to reconstruct the original surface from these pieces of the dissection, we would cut a slit along saddle connection *2* of the piece corresponding to the central vertex, cut a slit along saddle connection *1* of the piece corresponding to the rightmost vertex, and then glue these two pieces to each other along these slits.

Figure 2.3: An example of an invariant component diagram.

3. enough solid full edges to make the graph connected.

The diagram may have additional half-edges, which may be either solid or dotted. At a minimum, then, the diagram has m dotted half-edges, and $m + p - 1$ solid full edges. Each of the solid full edges consists of two half-edges. Thus the total number of half-edges in the invariant component diagram is at least $m + 2(m + p - 1) = 3m + 2p - 2$. By Lemma 2.2.6, the total cone angle of S is therefore at least $2\pi(3m + 2p - 2)$. $\square$

Proposition 2.2.10 follows immediately from Lemma 2.2.9 and the fact that the total cone angle (the sum of the cone angles at the singularities) of a surface in $\mathcal{H}(2g - 2)$ is $2\pi(2g - 1)$ while the total cone angle of a surface in $\mathcal{H}(g - 1, g - 1)$ is $2\pi(2g)$.

Proposition 2.2.10. *Fix $g \in \mathbb{N}$. Let (p, m) be a pair of nonnegative integers, at least one of which is nonzero.*

1. *If there exists a translation surface in the hyperelliptic component $\mathcal{H}^{hyp}(2g - 2)$ with precisely p periodic components and m minimal components then*

$$3m + 2p - 1 \leq 2g.$$

2. *If there exists a translation surface in the hyperelliptic component $\mathcal{H}^{hyp}(g-1, g-1)$ with precisely p periodic components and m minimal components then*

$$3m + 2p - 2 \leq 2g.$$

2.3 Constructing surfaces with specific numbers of periodic and minimal components

We will use invariant component diagrams as “blueprints” for constructing surfaces in $\mathcal{H}^{hyp}(2g - 2)$ and $\mathcal{H}^{hyp}(g - 1, g - 1)$ with specific numbers of periodic and minimal components. First, we will construct the “building blocks” we will be using – hyperelliptic surfaces with given numbers of Weierstrass edges and “broken” saddle connections.

The proof of the following lemma is left to the reader.

Lemma 2.3.1. *Let $X = \mathbb{R}/\mathbb{Z}$ and $Y = \mathbb{R}/\mathbb{Z}$ with the standard ordering on $[0, 1)$. Fix n distinct points $x_0 < x_1 < ...x_{n-1}$ in X. For each $i \in \{0, \ldots, n-1\}$, define $y_i \in Y$ by $y_i = -x_i \pmod 1$, and define φ_i be the translation sending the closed interval $[x_i, x_{i+1}] \subset X$ to the closed interval $[y_{i+1}, y_i] \subset Y$. Let $\sim$ be the equivalence relation on $X \cup Y$ generated by the relations $x \in X \sim y \in Y$ if there exists i such that $\varphi_i(x) = y$. Then*

1. *if n is odd, $\{x_0, ..., x_{n-1}, y_0, ..., y_{n-1}\}/\sim$ consists of a single equivalence class,*
2. *if n is even, $\{x_0, ..., x_{n-1}, y_0, ..., y_{n-1}\}/\sim$ consists of two equivalence classes, each of which has n preimages.*

For each natural number n, we will define hyperelliptic surfaces P_n and M_n. P_n will consist of a single periodic component and M_n will consist of a single minimal component. The surface P_n will have n vertical Weierstrass edges, and M_n will have $n-1$ vertical Weierstrass edges. (In the cases $n = 1$ and $n = 2$, P_n and M_n have no true cone points, but we will think of them as having n augmented cone points. P_1 has one vertical augmented Weierstrass edge, M_1 has no vertical augmented Weierstrass edges (and will not be used), P_2 has two vertical augmented Weierstrass edges, and M_2 has one vertical augmented Weierstrass edge.) If n is even, P_n and M_n will be genus $\frac{n}{2}$ surfaces in $\mathcal{H}^{hyp}(\frac{n}{2}-1, \frac{n}{2}-1)$. If n is odd, P_n and M_n will be genus $\frac{n+1}{2}$ surfaces in $\mathcal{H}^{hyp}(n-1)$.

The surfaces P_n and M_n do not have boundary. To connect two such surfaces together, we will cut along a vertical Weierstrass edge in each, and glue along the resulting slits.

The surfaces P_n: For $n \in \mathbb{N}$, define P_n as follows. Begin with a flat rectangle which measures n units in the vertical direction and 1 unit in the horizontal direction. Partition each of the vertical sides into n disjoint segments of length 1. On the left side of the rectangle, label the segments $s_1, ..., s_n$, in order, with s_1 at the top and s_n at the bottom. On the right side of the rectangle, again label the segments $s_1, ..., s_n$, but use the opposite order: label the top segment s_n, and the bottom segment s_1. For each i, identify the two vertical segments labeled s_i via a translation. Identify the two horizontal sides of the rectangle via a translation in the vertical direction (so that the vertical sides of the rectangle each become a closed curve). This surface is P_n. (See Figure 2.4.)

The only cone point(s) in P_n are formed by identifying the endpoints of the segments s_i. By Lemma 2.3.1, if n is odd all the endpoints of the s_i's are identified to form a single cone point. If n is even, Lemma 2.3.1 shows that the endpoints of the s_i's form two distinct cone points with equal cone angle. The total cone angle of P_n is $2\pi n$, so P_n has genus $\frac{n+1}{2}$ if n is odd and genus $\frac{n}{2}$ if n is even. It is easy to see that P_n admits an isometric involution with $2g+2$ fixed points: rotating the entire rectangle by a half-turn fixes the midpoint of each s_i, the midpoint of the rectangle, the midpoint of the horizontal sides of the rectangle, and the unique cone point if n is odd. Thus, P_n admits a hyperelliptic involution.

The surfaces M_n**:** For $n \in M$, define M_n as follows. Begin with the rectangular representation of the surface P_n defined above. Perform a vertical shear (apply a matrix $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$) so that the two segments labeled s_1 on the vertical sides of the rectangle are at the same vertical height. Then the horizontal path from the top of the left s_1 segment to the top of the right s_1 segment is a closed horizontal curve, as is the horizontal path from the bottom of the left s_1 segment to the bottom of the right s_1 segment. Apply an irrational horizontal shear (a matrix $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$ where α is irrational) to the rectangle whose vertical sides are the two s_1 segments and whose horizontal sides are the closed horizontal curves connecting the endpoints of the s_1 segments. The resulting surface is M_n. (See Figure 2.4.)

Since M_n is obtained by continuously deforming P_n and P_n is in the hyperelliptic connected component, M_n is also in the hyperelliptic connected component. M_n has the same number and type of cone points as P_n. The irrational twist on the rectangle whose vertical edges are s_1 destroys one vertical Weierstrass edge and makes the vertical foliation of M_n minimal.

We now describe how to construct surfaces associated to certain invariant component diagrams (those in which every vertex has at most one incident dotted half-edge). A vertex with $n \in \mathbb{N}$ incident (half-)edges corresponds to a surface P_n if all (half-)edges are solid and corresponds to a surface M_n if precisely one of the (half-)edges is dotted. For each solid full edge which connects two vertices, cut along a vertical Weierstrass edge in each to form slits, and glue the slits together via a translation. (Recall that for $n = 1$ and $n = 2$, we interpret the surfaces P_n and M_n to have n augmented cone points, and use the augmented Weierstrass edges instead of true saddle

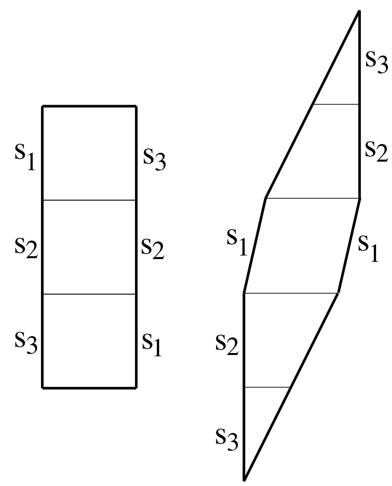


Figure 2.4: Polygonal representations of the surfaces P_3 (left) and M_3 (right).

connections.)

We will now define two families of invariant component diagrams. For each $k \in \mathbb{N}$ and each pair of nonnegative integers (p, m), at least one of which is nonzero and such that $3m + 2p - 2 \leq k$, we will define two invariant component diagrams: the *p-central diagram* $\mathcal{D}^{per}_{(k,p,m)}$ and the *m-central diagram* $\mathcal{D}^{min}_{(k,p,m)}$.

Construct the p-central diagram $\mathcal{D}^{per}_{(k,p,m)}$ as follows: Draw one central vertex. Now draw $m+p-1$ other vertices and connect each of these vertices to the central vertex with a full solid edge. Add a dotted half-edge to m of the non-central vertices. Add $y = k - (3m + 2p - 2)$ solid half-edges to the central vertex.

Construct the m-central diagram $\mathcal{D}^{min}_{(k,p,m)}$ as follows: Draw one central vertex. Now draw $m + p - 1$ other vertices and connect each of these vertices to the central vertex with a full solid edge. Add a dotted half-edge to the central vertex and to $m - 1$ of the non-central vertices. Add $y = k - (3m + 2p - 2)$ solid half-edges to the central vertex.

Lemma 2.3.2. *Fix an integer $k \geq 3$. The translation surface associated to an invariant component diagram $\mathcal{D}^{per}_{(k,p,m)}$ or $\mathcal{D}^{min}_{(k,p,m)}$ has one cone point if k is odd and two cone points having the same multiplicity if k is even.*

Proof. Fix any integer $k \geq 3$ and any such diagram $\mathcal{D}$. Let p' be the number of non-central periodic vertices of $\mathcal{D}$, let m' be the number of non-central minimal vertices of $\mathcal{D}$, and let y' be the number of half-edges (either dotted of solid) incident to the central vertex which are not part of a full edge. Regardless of whether $\mathcal{D}$ is m-central or p-central, we have $k = 2p' + 3m' + y'$. The central component is either a P_n or a M_n building block for some $n \in \mathbb{N}$. Each of the p' periodic non-central vertices represents a periodic cylinder glued to a Weierstrass edge in the central component. Gluing a cylinder to a Weierstrass edge in the central component forces the marked points at the top and bottom of the Weierstrass edge to coalesce. Thus, for each of the p' cylinders attached to the central component, the number of marked points on the two vertical boundaries of the rectangle representing the central component decreases by one. Gluing a minimal slit tori along a Weierstrass edge does not cause any marked points to coalesce.

The total number of half-edges (counting a full edge as two half-edges) in $\mathcal{D}$ is k. Of these k half-edges, $2p'$ are used to connect the central vertex to the non-central periodic vertices, and another $2m'$ of the half-edges are incident to non-central minimal vertices (each of the m' non-central minimal vertices has a dotted half-edge and a half-edge contained in the full edge connecting the vertex to the central vertex.) Thus, the number of half-edges incident to the central vertex which do not represent Weierstrass edges whose top and bottom points are forced by cylinders to coalesce is $y' + m' = k - 2p' - 2m'$. This is the number of distinct marked points on each of the two vertical boundaries of the rectangle representing the central component. By Lemma 2.3.1, these marked points are identified to form one marked point if $k - 2p' - 2m'$ is odd and two marked points with equal numbers of preimages if $k - 2p' - 2m'$ is even. (The condition that $k \geq 3$ excludes the diagram consisting of two vertices connected by a solid full edge; in this case the total surface is a torus and has no cone points.) Since $2p' + 2m'$ is even, the total surface therefore has one cone point if k is odd and two equal cone points if k is even. □

Proposition 2.3.3. *Fix an integer $k \geq 3$ and a pair of nonnegative integers (p, m), at least one of which is nonzero and such that $3m + 2p - 1 \leq k$. A translation surface surface associated to the invariant component diagram $\mathcal{D}^{per}_{(k,p,m)}$ or $\mathcal{D}^{min}_{(k,p,m)}$ is in $\mathcal{H}^{hyp}(\frac{k}{2} - 1, \frac{k}{2} - 1)$ if k is even and is in $\mathcal{H}^{hyp}(k - 1)$ if k is odd. Furthermore, such a surface has precisely p periodic components and m minimal components.*

Proof. Let S be the translation surface associated to one of these diagrams. Every building block B used to construct S is a hyperelliptic surface with hyperelliptic involution γ_B which fixes $2g_B + 2$

points, where g_B is the genus of B. When we glue two building blocks B_1 and B_2 together, we cut a slit along a Weierstrass edge in each; this destroys one fixed point in each building block. Since γ_{B_1} and γ_{B_2} agree along the slits, we can define an involution $\gamma_{B_1 \sqcup B_2}$ on the total surface by defining $\gamma_{B_1 \sqcup B_2}$ piecewise as whichever of γ_{B_1} and γ_{B_2} is defined. Thus $\gamma_{B_1 \sqcup B_2}$ fixes $2(g_{B_1} + g_{B_2}) + 2$ points and is the hyperelliptic involution on the total surface.

Similarly, when gluing together n building blocks $B_1, ..., B_n$ along Weierstrass edges, we can define a hyperelliptic involution $\gamma_{\sqcup_i B_i}$ on the total surface by defining $\gamma_{\sqcup_i B_i}$ piecewise to be whichever γ_{B_i} is defined. The involution $\gamma_{\sqcup_i B_i}$ fixes $2(g_{B_1} + ... + g_{B_n}) + 2$ points. Since every invariant component diagram $\mathcal{D}^{per}_{(k,p,m)}$ or $\mathcal{D}^{min}_{(k,p,m)}$ is a tree, the genus of any surface associated to such a diagram is the sum of the genera of the building blocks. Consequently $\gamma_{\sqcup_i B_i}$ fixes $2g + 2$ points, where g is the genus of the total surface.

The integer k is the total number of half-edges (counting a full edge as two half-edges) in the invariant component diagram. By Lemma 2.2.6, the total cone angle (the sum over the cone points of S of the cone angle at that point) of S is $2\pi k$. A translation surface in the stratum $\mathcal{H}(2g-2)$ has total cone angle $2\pi(2g-1)$ and a translation surface in the stratum $\mathcal{H}(g-1, g-1)$ has total cone angle $2\pi(2g)$. Then by Lemma 2.3.2, if k is odd, the surface S has genus $\frac{k+1}{2}$ and is in $\mathcal{H}^{hyp}(k-1)$; if k is even, the surface S has genus $\frac{k}{2}$ and is in $\mathcal{H}^{hyp}(\frac{k}{2}-1, \frac{k}{2}-1)$. $\square$

Proof of Theorem 2.1.1. The upper bounds in Theorem 2.1.1 are given in Proposition 2.2.10. Proposition 2.3.3 proves that given any hyperelliptic connected component ($\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$) for the moduli space of translation surfaces of genus $g \geq 2$, and any pair (p, m) satisfying the upper bounds, there exists a translation surface in that hyperelliptic connected component with precisely p periodic components and m minimal components. $\square$

Chapter 3

Lattice surfaces and the horocycle flow

3.1 Chapter Overview

This content of this chapter is based on joint work with Jon Chaika. In this chapter, we address the question: how are horocycle orbit closures related to $SL(2,\mathbb{R})$ orbit closures? Theorem 3.2.1 asserts that for any translation surface, the $SL(2,\mathbb{R})$ orbit closure equals the "horocycle orbit in direction θ" orbit closure for a residual set of directions $\theta \in S^1$. Theorem 3.3.1 characterizes lattice surfaces in terms of their minimal sets for the horocycle flow in every direction. Throughout this section, we denote by G the group $GL(2,\mathbb{R})$ and we denote by H the subgroup of $SL(2,\mathbb{R})$ consisting of the upper triangular matrices associated to the horocycle flow.

Although strata are not homogeneous spaces for the action of $SL(2,\mathbb{R})$, one might wonder which results from homogeneous dynamics also hold for the $SL(2,\mathbb{R})$ action on strata. (Only the stratum $\mathcal{H}(0)$ is a homogeneous space for $SL(2,\mathbb{R})$. One important result in the theory of homogeneous dynamics is the Mautner phenomenon, which, in the case of $SL(2,\mathbb{R})$, may be stated as follows:

Proposition 3.1.1 (The Mautner phenomenon). *Let $\mathfrak{H}$ be a Hilbert space and let $\phi : SL(2,\mathbb{R}) \to U(\mathfrak{H})$ be a continuous unitary representation on $\mathfrak{H}$. Then any element $v \in \mathfrak{H}$ that is invariant under the subgroup $H = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$ is also invariant under $SL(2,\mathbb{R})$.*

(The statement "$\phi : SL(2,\mathbb{R}) \to U(\mathfrak{H})$ is a continuous unitary representation on $\mathfrak{H}$" means ϕ is a homomorphism into the group of unitary automorphisms $\mathbb{U}(\mathfrak{H})$ of $\mathfrak{H}$ such that for every $v \in \mathfrak{H}$, the element $\phi(g)(v) \in \mathfrak{H}$ depends continuously on $g \in SL(2,\mathbb{R})$. References for the Mautner phenomenon include, e.g. [Ein06, Moo80, Wit03].)

The action of $SL(2,\mathbb{R})$ on a stratum $\mathcal{H}^1$ with an $SL(2,\mathbb{R})$-invariant probability measure μ_1 determines a continuous unitary representation on $\mathfrak{H} = L^2(\mathcal{H}^1, \mu_1)$ defined by

$$\alpha \in SL(2,\mathbb{R}) \mapsto (f \in \mathfrak{H} \mapsto f \circ \alpha \in \mathfrak{H}) \in U(\mathfrak{H}).$$

For flows, ergodicity can be characterized as the condition that "the only invariant elements of $L^2(\mathcal{H}^1, \mu_1)$ are constant functions." Consequently, a $SL(2,\mathbb{R})$-invariant measure on $\mathcal{H}$ is ergodic for the horocycle flow if and only if it is ergodic for the $SL(2,\mathbb{R})$ action. Thus, from the viewpoint of analogy with homogeneous dynamics, the assertion in Theorem 3.2.1 that the horocycle flow in "most" directions has the same orbit closure as the $SL(2,\mathbb{R})$ action is not surprising.

An immediate observation relating the horocycle flow with the dynamics of $SL(2,\mathbb{R})$ is that the closure in a stratum $\mathcal{H}$ of any $SL(2,\mathbb{R})$ orbit contains a horocycle flow orbit closure, and hence by [SW04] in fact contains a minimal set for the horocycle flow. (A *minimal set* for the action of a group A on a space X is a nonempty, closed, A-invariant subset of X that is minimal with respect to inclusion.)

Smillie and Weiss classified the minimal sets in a stratum $\mathcal{H}$ or $\mathcal{Q}$ for the horocycle flow in [SW04].

Proposition 3.1.2 ([SW04])**.** *Any closed invariant set for the horocycle flow contains a minimal set, and a minimal set is compact.*

Proposition 3.1.3 ([SW04])**.** *Let S be a half-translation surface that is periodic in the horizontal direction. Let $\mathcal{O} = \overline{H \cdot S}$. Then*

1. *S admits a cylinder decomposition $S = C_1 \cup \cdots \cup C_r$, where each C_i is a cylinder whose interior is a union of horizontal core curves.*

2. *There is an isomorphism between $\mathcal{O}$ and a d-dimensional torus, where d is the dimension of the $\mathbb{Q}$-linear subspace of $\mathbb{R}$ spanned by the moduli of $C_1, \ldots, C_r$. This isomorphism conjugates the H-action on $\mathcal{O}$ with a one-parameter translation flow.*

3. *The restriction of the H-action to $\mathcal{O}$ is minimal.*

Theorem 3.1.4 ([SW04])**.** *If S is a half-translation surface such that $\overline{H \cdot S}$ is contained in a compact subset of a single stratum, then the flow along any leaf of the horizontal foliation is periodic. In particular, any minimal set for the horocycle flow is as described in Proposition 3.1.3.*

This characterization of minimal sets for the horocycle flow iss an ingredient in Wright's proof of the "cylinder deformation theorem" ([Wri]).

Given a collection of horizontal cylinders $\mathcal{C}$ of a surface M, we define $\eta_{\mathcal{C}} \in T_M(\mathcal{M}) \subset H^1(S, \Sigma; \mathbb{C})$ to be the derivative (with respect to t) of $h_t^{\mathcal{C}}$ at M in local period coordinates, where $h_t^{\mathcal{C}}$ is the "cylinder shear" which applies the matrix h_t to the cylinders of $\mathcal{C}$ and leaves the rest of the surface unchanged. Denote by $a_t^{\mathcal{C}}$ the "cylinder stretch" which applies the matrix $a_t = \left(\begin{smallmatrix} 1 & 0 \\ 0 & e^t \end{smallmatrix}\right) \in GL(2,\mathbb{R})$ to the cylinders of $\mathcal{C}$ and leaves the rest of the surface unchanged.

Definition 3.1.5. *Let M be a flat surface and $\mathcal{M} = \overline{G \cdot M}$. Two cylinders of M are*

1. *$\mathcal{M}$-parallel if they are parallel at M and at every nearby surface $M' \in \mathcal{M}$.*

2. *$\mathcal{M}$-collinear if their core curves $\alpha, \beta \in H_1(M, \Sigma; \mathbb{Z})$ have collinear images in $T^M(\mathcal{M})$.*

Wright observes in [Wri] that 1.4.7 implies that two cylinders are $\mathcal{M}$-parallel if and only if they are $\mathcal{M}$-collinear.

Lemma 3.1.6 (Lemma 4.11, [Wri])**.** *For any horizontally periodic surface $M \in \mathcal{M}$ and equivalence class $\mathcal{C}$ of $\mathcal{M}$-parallel horizontal cylinders, $\eta_{\mathcal{C}} \in T^M(\mathcal{M})$.*

3.2 Relating $SL(2,\mathbb{R})$ and horocycle orbit closures

For any subgroup Y of $SL(2,\mathbb{R})$ acting on a stratum $\mathcal{H}$, denote the closure in $\mathcal{H}$ of the y-orbit of a surface $M \in \mathcal{H}$ by $\overline{Y \cdot M}$. Denote by $H \subset SL(2,\mathbb{R})$ the horocycle flow.

A subset A of a topological space X is **meager** if it can be expressed as the union of countably many nowhere dense subsets of X. A **residual set** is the complement of a meager set.

Theorem 3.2.1. *For any translation surface M, there exists a residual set $A \subset S^1$ such that*

$$\overline{Hr_\theta \cdot M} = \overline{SL(2,\mathbb{R}) \cdot M}$$

for all $\theta \in A$.

We state Theorem 3.2.1 here without proof. The proof will appear in a forthcoming joint research paper with Jon Chaikia.

3.3 A characterization of lattice surfaces in terms of the horocycle flow

In this section, we use Theorem 3.2.1 to prove Theorem 3.3.1.

Theorem 3.3.1. *The following are equivalent:*

1. M *is a lattice surface.*

2. *For every angle* θ*, every* H*-minimal subset of* $\overline{Hr_{-\theta} \cdot M}$ *is a periodic* H*-orbit.*

3. *Any* H*-minimal subset of* $\overline{G \cdot M}$ *is a periodic* H*-orbit.*

Proof. We prove Theorem 3.3.1 in a series of Lemmas. Lemma 3.3.2 is the implication (1) implies (2), Lemma 3.3.3 is (2) implies (3), and Lemma 3.3.12 is (3) implies (1). □

Lemma 3.3.2. *(1) implies (2)*

Proof. Fix a lattice surface M and angle θ. Let L be any surface in any H-minimal subset A of $\overline{Hr_{-\theta} \cdot M}$. Since a lattice surface has, by definition, a closed $SL(2,\mathbb{R})$ orbit, L is the image of M under some element of $SL(2,\mathbb{R})$, and hence L is also a lattice surface. By Theorem 3.1.4, L is periodic in the horizontal direction, and by Theorem 1.4.6 L is uniformly completely parabolic, so the moduli of all the horizontal cylinders of L are commensurable. Hence, by Proposition 3.1.3, A is the periodic orbit of L under H. □

Lemma 3.3.3. *(2) implies (3)*

Proof. Assume (2) holds for the flat surface M. Let $\mathcal{A}$ be an H-minimal subset of $\overline{G \cdot M}$. Then $\mathcal{A}$ has the form $\mathcal{A} = \overline{H \cdot N}$ for some flat surface $N \in \overline{G \cdot M}$. We may assume without loss of generality that N is in $\overline{SL(2,\mathbb{R}) \cdot M}$ (all the surfaces in $\mathcal{A}$ necessarily have the same area, so we may rescale them to have the same area as that of M, and this subset of rescaled surfaces will still be an H-minimal set). By Theorem 3.2.1, there exists an angle θ such that $\overline{Hr_{-\theta} \cdot M} = \overline{SL(2,\mathbb{R}) \cdot M}$. Thus

$$\mathcal{A} \subset \overline{SL(2,\mathbb{R}) \cdot N} \subset \overline{SL(2,\mathbb{R}) \cdot M} = \overline{Hr_{-\theta} \cdot M}.$$

By (2), $\mathcal{A}$ is a periodic H-orbit. □

Lemma 3.3.4. *(3) implies that for any surface $N \in \overline{G \cdot M}$, N is parabolic in every periodic direction.*

Proof. Suppose N is periodic in direction θ and the moduli of the cylinders (in direction θ) are not rationally related. Then $Hr_{-\theta} \cdot N$ is a H-minimal set that is not periodic. □

Lemma 3.3.5. *(3) implies that for any surface $N \in \overline{G \cdot M}$, N is completely periodic.*

Proof. Suppose there is a direction θ in which N has at least one cylinder and at least one minimal component. Without loss of generality, we assume this direction is the horizontal direction. By Theorem 3.1.4, fix a surface $N' \in \overline{H \cdot N} \subset \overline{G \cdot M}$ with a horizontal cylinder decomposition, and let $\mathcal{C}$ be the cylinders in N' which "came from" the horizontal cylinders of N. Clearly the cylinders of $\mathcal{N}'$ which are not in $\mathcal{C}$ are not in the same $\mathcal{M}$-equivalence class as any of the cylinders of $\mathcal{C}$. Thus, by Lemma 3.3, $\eta_{\mathcal{C}} \in T_{N'}(\mathcal{M})$. Following the approach in [Wri], for some small $\epsilon > 0$, there exists a surface $N'' \in \mathcal{M}$ corresponding to $[\omega] + \epsilon i \eta_{\mathcal{C}}$, where $[\omega]$ is element of $H^1(S, \Sigma; \mathbb{C})$ corresponding to N', that is horizontally periodic and such that the moduli of the cylinders of N'' that "came from" $\mathcal{C}$ are not rationally related to the moduli of the cylinders of N'' that did not "come from" $\mathcal{C}$. Hence N'' is horizontally periodic but not parabolic, and by Lemma 3.3.4 this contradicts (3). □

Corollary 3.3.6. *(3) implies that for any surface $N \in \overline{G \cdot M}$ and any direction θ that is periodic for N, N has a unique $\mathcal{M}$-equivalence class of cylinders.*

Proof. Suppose N has at least two $\mathcal{M}$-equivalence classes of cylinders in a periodic direction θ. Without loss of generality, assume assume θ is the horizontal direction. By Lemma , we can stretch the cylinders in one of these equivalence classes vertically while keeping the rest of the surface unchanged to obtain a horizontally periodic surface $N' \in \mathcal{M}$ which has at least two horizontal cylinders whose moduli are not rationally related, violating the conclusion of Lemma 3.3.4. □

Corollary 3.3.7. *(3) implies $dim_{\mathbb{C}} p(T_N(\mathcal{M})) = 2$, where N is any point in $\mathcal{M} = \overline{G \cdot M}$.*

Proof. This follows immediately from Lemma 3.3.5 and Wright's Theorem 1.7, which says that if $dim_{\mathbb{C}} p(T(\mathcal{M})) > 2$ then there exist translation surfaces in $\mathcal{M}$ which are not completely periodic. □

Corollary 3.3.8. *(3) implies that for any surface $N \in \overline{G \cdot M} = \mathcal{M}$ and for any direction θ that is periodic for N, N has a unique $\mathcal{M}$-equivalence class of cylinders in direction θ.*

Definition 3.3.9. *Let $M \in \mathcal{M}$ be horizontally periodic. The* twist space *of $\mathcal{M}$ at M is the subspace Twist$(M, \mathcal{M})$ of $T^M(\mathcal{M})$ of cohomology classes in $T^M(\mathcal{M})$ which are zero on all horizontal saddle connections.*

Definition 3.3.10. *Let $M \in \mathcal{M}$ be horizontally periodic. The* cylinder preserving space *of $\mathcal{M}$ at M is the subspace of Pres$(M, \mathcal{M})$ of $T^M(\mathcal{M})$ of cohomology classes which are zero on the core curves of all horizontal cylinders.*

Lemma 3.3.11. *(3) implies $dim(T^S(\mathcal{M})) = 2$, for any point $S \in \mathcal{M}$.*

Proof. Pick a horizontally periodic surface $N \in \mathcal{M} = \overline{G \cdot M}$ with the maximal number of cylinders. By Wright's Lemma 8.6, Twist$(N, \mathcal{M}) =$ Pres$(N, \mathcal{M})$, and by Wright's Corollary 8.3, Twist$(N, \mathcal{M}) = span(\eta_{C_i})_{i=1}^n$, where $C_1, \ldots, C_n$ are the horizontal cylinders.

The subspace Pres$(N, \mathcal{N})$ is defined by the equations $\alpha_i^* = 0$, $i = 1, \ldots, n$, where $\alpha, \ldots, \alpha_n$ are the elements of $H_1(N, \Sigma; \mathbb{C})$ corresponding to the core curves of the cylinders. Thus,

$$dim\left(Pres(N, \mathcal{M})\right) + dim\left(T^N(\mathcal{M}) \cap span(\alpha_i^*)\right) = dim\left(T^N(\mathcal{M})\right).$$

Hence,

$$dim\left(T^N(\mathcal{M}) \cap span(\eta_{C_i})_{i=1}^n)\right) + dim\left(T^N(\mathcal{M}) \cap span(\alpha_i^*)_{i=1}^n\right) = dim\left(T^N(\mathcal{M})\right).$$

The horizontal cylinders of N are belong to a unique $\mathcal{M}$-equivalence class by Lemma 3.3.8. Thus, all horizontal cylinders of N are $\mathcal{M}$-collinear. Hence, $dim\left(span(\alpha_i^*) \cap T^N(\mathcal{M})\right) = 1$.

Now suppose $dim\left(T^N(\mathcal{M}) \cap span(\eta_{C_i})\right) \geq 2$. Then we could vertically stretch some of the horizontal cylinders of N while leaving the other horizontal cylinders unchanged to obtain a surface $N' \in \mathcal{M}$ with at least two horizontal cylinders whose moduli are not rationally related, contradicting Lemma 3.3.4. Hence $dim\left(T^N(\mathcal{M}) \cap span(\eta_{C_i})\right) = 1$. □

Lemma 3.3.12. *(3) implies (1).*

Proof. The statement $dim(T^S(\mathcal{M})) = 2$, for any point $S \in \mathcal{M}$, asserts that $\mathcal{M}$ is a closed $SL(2,\mathbb{R})$ orbit. Thus, Lemma 3.3.11, in conjunction with Theorem 3.1.4, implies that each point of $\mathcal{M}$ is a lattice surface. $\square$

Chapter 4

Translation surface models of ergodic systems

4.1 Chapter Overview

The content of this chapter is based on joint work with Rodrigo Treviño. In this chapter, we introduce a new method for constructing and describing infinite type translation surfaces of finite area, and then use this technique to investigate the dynamics of the vertical translation flow on such surfaces.

The existing body of scientific literature contains isolated examples of infinite type finite area translation surfaces, with each example or class of examples apparently coming from a different construction. In this chapter, we propose a general setting and technique for describing and constructing infinite type finite area translation surfaces.

Finite type translation surfaces can be described using finite interval exchange maps and the "zippered rectangle" construction (see e.g. Viana). The zippered rectangle approach involves a finite collection of rectangles with vertical and horizontal sides of specified widths and heights, and a finite interval exchange map which describes how to glue the "tops" of the rectangles to the "bottoms" of the rectangles (the interval exchange map is the first return map to a transversal of the vertical translation flow). Additional data specifies how to glue other edges. Rauzy induction allows us to describe how the surface evolves under the geodesic flow by describing the associated sequence of interval exchange maps.

Building finite type translation surfaces using the the zippered rectangle construction may be viewed as a special case of building a flow under a function, along with some additional data describing the surface from the point of view of the horizontal flow, where the base transformation is the interval exchange map, and the "function" is the heights of the rectangles. Thus, a natural setting for a general constructive theory of infinite type translation surfaces is the setting of flows built under functions where the base transformation is an interval exchange with infinitely many intervals, and the height function is piecewise constant. But how do we "glue" the vertical sides? A simple way is to define a second infinite interval exchange on the union of the vertical sides of the rectangles. The construction we propose in this chapter uses a finite collection of rectangles with specified heights and widths and a pair of infinite interval exchange transformations. One of these interval exchange transformations is defined on the union of the horizontal sides of the rectangles, and specifies how to glue the horizontal sides together, and the other is defined on the union of the vertical sides, and specifies how to glue the vertical sides together.

Infinite interval exchange maps may be described by a *cutting and stacking* process, which itself may be described by a *Bratteli diagram*. We define a generalization of a Bratteli diagram – we call it simply a *diagram* – each of which determines a pair of infinite interval exchange maps and a collection of rectangles. These, together, define a translation surface. Consequently, each diagram determines a translation surface. We propose diagrams as a general setting for constructing and defining a large class of infinite type translation surfaces.

A nice feature of diagrams is that they give rise to a natural analogue of Rauzy-Veech induction. The analogous renormalization map on the level of diagrams is defined simply by "shifting" the indices on the levels of the diagram by 1. (This process is described in detail in §4.5.) Consequently, the evolution of the associated translation surface under the geodesic flow (and the vertical translation flow on the surface) has a nice description in terms of the geometric structure of the diagram.

By developing a dictionary between the languages of diagrams and translation surfaces, we can translate various theorems and techniques from ergodic theory into the translation surface world. In particular, we prove the existence of translation flows which are mixing (§4.6.1), construct concrete specific examples of translation surfaces for which the vertical translation flow has positive topological entropy (§4.6.2.3), surfaces whose translation flow is minimal and has uncountably many ergodic invariant probability measures (§4.6.2.2), and surfaces for which the first return map to a transversal of the translation flow is measure-theoretically mixing (§4.6.2.1). These properties are not possible for surfaces of finite type. We prove in Theorem 4.6.1 that any finite entropy, finite measure-preserving flow on a standard Lebesgue space is realizable (measure-theoretically isomorphic to) as the translation flow on a translation surface surface.

4.2 Bratteli diagrams

Bratteli diagrams were introduced in [Bra72] to study C^*-algebras; Vershik associated dynamical systems to these diagrams in [Ver89]. These maps, which are called Bratteli-Vershik or adic transformations, are defined on the space of infinite paths starting at a root vertex in a Bratteli diagram; the transformation maps a path to its successor (when possible) under a given ordering. Vershik showed that every measure-preserving transformation on a Lebesgue space is measure-theoretically isomorphic to an adic transformation ([Ver89]). In §4.2.1, we review some of the theory of Bratteli diagrams. In §4.2.2, we introduce bi-infinite generalizations of Bratteli diagrams. Definition 4.2.29 defines a *diagram*, a bi-infinite Bratteli diagram together with some additional data; diagrams are the basic combinatorial objects we will associate to infinite type translations surfaces.

4.2.1 Bratteli diagrams

In this section we present some background and definitions in the study of Bratteli diagrams. For more information on the theory of Bratteli diagrams and associated dynamical systems, see, for example, [HPS92, BKM09, DHS99].

Definition 4.2.1. *A* Bratteli diagram $B = (V, E)$ *is a connected infinite directed graph together with partitions of the vertex set V and edge set E of the graph into countable unions of pairwise disjoint nonempty finite sets*

$$V = \bigsqcup_{i \geq 0} V_i \text{ and } E = \bigsqcup_{i > 0} E_i$$

such that $s(E_i) = V_{i-1}$ and $r(E_i) = V_i$ for all $i > 0$, where s and r are the associated source *and* range *maps ($s, r : E \to V$), respectively.*

The set V_i of vertices is called the i^{th} level of the Bratteli diagram. We will denote $|V_i|$ by c_i. Note that the conditions $s(E_i) = V_{i-1}$ and $r(E_i) = V_i$ for all $i > 0$ imply that every vertex in V_0 is the source of an edge in E_1 and every vertex $v \in V_i$ for $i > 0$ is both the source vertex of an edge in E_{i+1} and the range vertex of an edge in E_i.

Given a Bratteli diagram B, for $i > 0$, the incidence matrix $F_i = [f^i_{v,w}]$ is a $c_i \times c_{i-1}$ matrix whose entries $f^i_{v,w}$ are the number of edges between the vertices $v \in V_i$ and $w \in V_{i-1}$:

$$f^i_{v,w} = |\{e \in E_i \mid r(e) = v \text{ and } s(e) = w\}|.$$

It follows from the conditions $s(E_i) = V_{i-1}$ and $r(E_i) = V_i$ for all $i > 0$ that none of the matrices F_k have a row or column which consists of all zero entries. Given an initial vector $h^0 = (h_1^0, \ldots, h_{|V_0|}^0) \in \mathbb{R}_+^{|V_0|}$ with all positive entries, for each $i \geq 0$, we define (recursively) a *height* vector $h^i = (h_1^i, \ldots, h_{c_i}^i) \in \mathbb{R}^{c_i}$. The height vectors are then given by the recursive formula

$$h^{i+1} = F_i h^i. \tag{4.1}$$

For nonnegative integers $k < l$, a *finite path* from a vertex in V_k to a vertex in V_l is a set of edges $e_{k+1}, \ldots, e_l$, such that $e_i \in E_i$ and $r(e_i) = s(e_{i+1})$ for all i. We will denote such a path by $(e_k, \ldots, e_l)$. For a path $p = (e_i, \ldots, e_j)$, we define $s(p) = s(e_i)$ and $r(p) = r(e_j)$.

For a Bratteli diagram B, we denote by X_B the set of all infinite paths in B which start at a vertex in V_0. For a point $x \in X_B$, denote by x_i the i^{th} edge of the path x. We topologize X_B by specifying a clopen basis of all cylinder sets

$$U(e_1, \ldots, e_n) := \{x \in X_B \mid x_i = e_i \text{ for all } i \in \{1, \ldots, n\}\},$$

where $(e_1, \ldots, e_n)$ is a finite path starting at a vertex in V_0. As such, X_B is a compact, Haudorff, zero-dimensional space with a countable basis of clopen sets.

Definition 4.2.2. *An* ordered *Bratteli diagram $(B, \leq_r)$ is a Bratteli diagram $B = (V, E)$ together with a partial order $\leq_r$ on E so that edges e and e' are comparable under $\leq_r$ if and only if $r(e) = r(e')$.*

To pass from Bratteli diagrams to cutting and stacking maps (and flat surfaces) in a canonical way, we will want an additional partial order that compares edges with the same source vertex. Thus, we define fully ordered Bratteli diagrams:

Definition 4.2.3. *A* fully ordered *Bratteli diagram $(B, \leq_{r,s})$ is an ordered Bratteli diagram $(B, \leq_r)$ together with a partial order $\leq_s$ on $E \cup V_0$ so that any two edges e, e' are comparable under $\leq_s$ if and only if $s(e) = s(e')$, $\leq_s$ is a total order on V_0, and edges are not comparable with vertices.*

The partial order $\leq_r$ in an ordered Bratteli diagram $(B, \leq_r)$ induces a lexicographic partial order on the set of all finite paths from V_i to V_j for any $j > i$. Namely, we write

$$(e_{i+1}, e_{i+2}, \ldots, e_j) <_r (f_{i+1}, f_{i+2}, \ldots, f_j)$$

if and only if the exists $k \in \{i+1, \ldots, j\}$ such that $e_l = f_l$ for $k < l \leq j$ and $e_k <_r f_k$. Two infinite paths x and y in X_B are comparable under $\leq_r$ if the agree after some some level n ($x_k = y_k$ for all $k > n$) and $x_n \neq y_n$; then we define $x <_r y$ if and only if $x_n <_r y_n$.

An infinite path $x \in X_B$ is *maximal* under $\leq_r$ if x_i is a maximal edge according to $\leq_r$ for all $i \in \mathbb{N}$. Denote by X_{max} the set of maximal paths in X_B; X_{min} is defined similarly. Given any path $x \in X_B \setminus X_{max}$, there exists a smallest integer i such that x_i is not maximal. Since there exist only finitely many (finite) paths from a vertex in V_0 to the vertex $r(x_i)$, the infimum $\inf\{y \in X_B \mid y >_r x\}$ is achieved by a path in X_B.

Definition 4.2.4. *Let $(B, \leq_r)$ be an ordered Bratteli diagram. For a point $x \in X_B \setminus X_{max}$, define the* successor *of x to be*

$$\alpha = \inf\{y \in X_B \mid y >_r x\}$$

Definition 4.2.5. *Let $(B, \leq_r)$ be an ordered Bratteli diagram. The* Bratteli-Vershik or adic *transformation $T : X_B \backslash X_{max} \to X_B \backslash X_{min}$ is the map which sends a point $x \in X_B \backslash X_{max}$ to its successor.*

Definition 4.2.6. *Let $B = (V, E)$ be a Bratteli diagram. The* tail equivalence relation *is a relation $\sim$ on X_B defined by*

$$x \sim y \quad \text{if and only if} \quad \exists N \geq 0 \text{ such that } x_k = y_k \quad \text{for all} \quad k > N.$$

Note that the tail equivalence relation is independent of the many possible choices of orders $\leq_{r,s}$ on a Bratteli diagram.

Definition 4.2.7. *A Bratteli diagram is* aperiodic *if every tail equivalence class of X_B is infinite. In this case any adic transformation defined on X_B is also called* aperiodic.

Definition 4.2.8. *A Bratteli diagram is* completely periodic *if every tail equivalence class of X_B is finite. In this case any adic transformation defined on X_B is also called* completely periodic.

The notion of complete periodicity will be made more clear in the decomposition (4.2) below.

Remark 4.2.9. *Whenever $|X_{min}| = |X_{max}| < \infty$, the adic transformation can be extended to all of X_B and defines a homeomorphism. In particular, any finite tail equivalence class has a unique maximal* *path and a unique minimal path in X_B; in this case, it is natural to extend the adic transformation so that it maps this maximal path to this minimal path. Thus, the (natural extension of the) adic map on a completely periodic Bratteli diagram (defined in Definition 4.2.8) is periodic.*

Definition 4.2.10. *A minimal subset X' of X_B, for a Bratteli diagram $B = (V, E)$, is a set that is closed under the tail equivalence relation $\sim$ and is minimal among such sets with respect to inclusion. A Bratteli diagram B is* minimal *if X_B has no proper minimal subsets.*

Remark 4.2.11. *Definition 4.2.10 is equivalent to the following condition: X_B is minimal if for any $x = (x_1, x_2, \dots) \in X_B$, $k > 0$, and $v \in V_k$, there exists an integer $j > k$ and a path $(e_{k+1}, \dots, e_j)$, with $e_i \in E_i$ for all i, such that $s(e_{k+1}) = v$ and $r(e_j) = s(x_{j+1})$.*

Definition 4.2.12. *A Borel probability measure μ on X_B is an* invariant measure for the tail equivalence relation *if for any two infinite paths $p_1 = (e_1, e_2, \dots)$ and $p_2 = (f_1, f_2, \dots)$ in X_B with $p_1 \sim p_2$ and for any $l \in \mathbb{N}$ such that $e_k = f_k$ for all $k > l$, we have $\mu(U(e_1, \dots, e_l)) = \mu(U(f_1, \dots, f_l))$.*

Remark 4.2.13. *For an ordered Bratteli diagram B, a Borel probability measure on X_B that is invariant with respect to the adic transformation is also an invariant measure for the tail equivalence relation. The converse is not true: the support of a Borel probability measure which is invariant for the tail equivalence relation could be contained in X_{max} for some order $\leq_r$; this set has empty intersection with the domain of the adic transformation. In fact, it is possible that every invariant Borel probability measure for the tail equivalence relation have a support contained in X_{max}. See §4.6.2.4 for an example of an adic transformation which admits no invariant Borel probability measure but does admit an invariant infinite Borel measure.*

Using the Compact Representation Lemma [AS13] with the Krylov-Bogolyubov theorem we obtain the following basic result (see also [PS97]).

Proposition 4.2.14. *Let $(B, \leq_r)$ be an ordered Bratteli diagram. Then there is at least one Borel probability measure on X_B which is invariant for the adic transformation defined by the partial order $\leq_r$.*

For any Bratteli diagram B, there is a decomposition of X_B as

$$X_B = X_P \bigsqcup X_M, \tag{4.2}$$

where

$$X_P = \bigsqcup_i \bigcup_{x \in X_P^i} x$$

where each X_P^i is a finite tail-equivalence class, called a *periodic component*. The set X_M consists of the *minimal components*

$$X_M = \bigsqcup_i X_M^i,$$

where each X_M^i is a minimal subset. This decomposition will be analogous to the decomposition of a flat surface into minimal and periodic components. If a Bratteli diagram is not made up only of a single minimal component, there is a clear obstruction to ergodicity of any adic transformation defined from it.

Definition 4.2.15. *A* weight function *for a Bratteli diagram $B = (V, E)$ is a map $w : V_0 \cup E \to (0, \infty)$ such that*

1. *for any vertex $v \in V$ and any two positively oriented finite paths $(e_1, \ldots, e_j)$ and $(f_1, \ldots, f_j)$ from vertices in V_0 to v,*

$$w(s(e_1)) \cdot \prod_{i=1}^{j} w(e_i) = w(s(f_1)) \cdot \prod_{i=1}^{j} w(f_i).$$

2. *for any $v \in V$,*

$$\sum_{e \in s^{-1}(v)} w(e) = 1,$$

3. *for any infinite path $x = (x_1, x_2, \ldots) \in X_B$ that does not belong to a finite tail equivalence class (i.e. is an element of a minimal component),*

$$\lim_{n \to \infty} w(s(x_1)) \cdot \prod_{i=1}^{n} w(x_i) = 0.$$

For a weight function w and $v \in V_k$ with $k > 0$, we can define the the quantity $w(v)$ by

$$w(v) = w(s(e_1)) \cdot \prod_{i=1}^{j} w(e_i) \tag{4.3}$$

for any path $(e_1, \ldots, e_k)$ with $r(e_k) = v$ from V_0 to v. By (i) in Definition 4.2.15, it is independent of the path $(e_1, \ldots, e_k)$ taken.

Definition 4.2.16. *A weight function w on a Bratteli diagram $B = (V, E)$ is said to be a* probability weight function *if*

$$\sum_{v \in V_0} w(v) = 1$$

and is said to be a finite weight function *if*

$$\sum_{v \in V_0} w(v) < \infty.$$

The following lemma, whose proof is straightforward and is left to the reader, records the fact that weight functions on Bratteli diagrams correspond to invariant measures for the tail equivalence relation. This correspondence between a measure μ and weight w to which we refer is obtained by setting $w(v) = \mu(v)$ for $v \in V_0$ and $w(e) = \frac{\mu(r(e))}{\mu(s(e))}$ for $e \in E$.

Lemma 4.2.17. *A probability weight function w on a Bratteli diagram $B = (V, E)$ determines a unique invariant Borel probability measure for the tail equivalence relation. Conversely, an invariant Borel probability measure for the tail equivalence relation determines a unique probability weight function on B.*

Remark 4.2.18. *In section §4.4, we will develop a correspondence between weighted, fully ordered Bratteli diagrams and cutting and stacking maps (§4.3). Each vertex $v \in V_i$ in Bratteli diagram $B = (V, E)$ will correspond to a tower in the stack S_i, and the value assigned to a vertex by the Borel measure associated to a weight function (as in Lemma 4.2.17) will be the width of the levels of that tower. We will see that condition 1 means that two subtowers of S_i which are stacked on top of each other to form a tower of stack S_{i+1} have the same width. Condition 2 reflects the fact that the sum of the widths of the subtowers into which a given tower is cut must equal the width of that tower. Condition 3 says that the widths of the stacks which limit to a minimal set for the limit map must go to zero.*

Definition 4.2.19. *Let $B = (V, E)$ be a Bratteli diagram. Let m, n be distinct non-negative integers with $m < n$, and for each i, $m \leq i \leq n$, let e_i be an edge in E_i such that $r(e_j) = s(e_{j+1})$ for* *all* $m \leq j < n$*. The ordered sequence* $e_m, e_{m+1}, \ldots, e_n$ *is a* positively oriented path *in* B*, and the sequence* $e_n, e_{n-1}, \ldots, e_m$ *is a* negatively oriented path *in* B*.*

Denote by $E_{m,n}$ the set of positively oriented finite paths connecting vertices in V_m with vertices in V_n, and denote by $E_{n,m}$ the set of negatively oriented finite paths connecting vertices in V_n with vertices in V_m.

Definition 4.2.20. *Let* $B = (V, E)$ *be a Bratteli diagram and let*

$$0 = m_0 < m_1 < m_2 < \cdots$$

be an increasing sequence in $\mathbb{N}$ *. For* $l \in \mathbb{N}$ *and* $k \in \{0, \ldots, l-1\}$*, we define another Bratteli diagram* $B' = (V', E')$ *by setting* $V'_0 = V_0$*,* $V'_n = V_{m_n}$ *for all* $n \in \mathbb{N}$*, and* E'_n *is identified with* E_{m_{n-1}, m_n}*. Then* $B' = (V', E')$ *is called the* telescoping *of* B *to* $\{m_n\}_{n \geq 0}$*.*

With the notation used in the definition of telescoping, the incidence matrices F'_n for $B' = (V', E')$ are given by

$$F'_n = F_{m_n} F_{m_n - 1} \ldots F_{m_{n-1}+1}.$$

4.2.2 Bi-infinite Bratteli diagrams

Definition 4.2.21. *A* bi-infinite Bratteli diagram $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ *is an infinite directed graph together with partitions of the vertex set* $\mathcal{V}$ *and edge set* $\mathcal{E}$ *of the graph into countable unions of pairwise disjoint nonempty countable sets*

$$\mathcal{V} = \bigsqcup_{i \in \mathbb{Z}} \mathcal{V}_i \text{ and } \mathcal{E} = \bigsqcup_{i \in \mathbb{Z} \setminus \{0\}} \mathcal{E}_i$$

with associated range and source maps $r, s : \mathcal{E} \to \mathcal{V}$ *such that* $s(\mathcal{E}_i) = \mathcal{V}_{i-1}$ *and* $r(\mathcal{E}_i) = \mathcal{V}_i$ *for all* $i \in \mathbb{N}$ *and* $s(\mathcal{E}_i) = \mathcal{V}_i$ *and* $r(\mathcal{E}_i) = \mathcal{V}_{i+1}$ *for all* $i < 0$*.*

Remark 4.2.22. *We will henceforth use uppercase letters in calligraphy font* $\mathcal{B}, \mathcal{V}, \mathcal{E}, \mathcal{F}$ *to refer to bi-infinite Bratteli diagrams, while we will use regular uppercase letters* B, V, E, F *to refer to "singly-infinite" Bratteli diagrams. If an adjective "bi-infinite" or "singly-infinite" is not explicitly stated, we will rely on font to make it clear which type of diagram we are referring to.*

Definition 4.2.23. *For a bi-infinite Bratteli diagram $\mathcal{B} = (\mathcal{V}, \mathcal{E})$, the* positive half *of $\mathcal{B}$, denoted $\mathcal{B}^+ = (\mathcal{V}^+, \mathcal{E}^+)$, is the restriction of $\mathcal{B}$ to the vertices in $\mathcal{V}_i$ for $i \geq 0$ and the edges in $\mathcal{E}_i$ for $i > 0$. The* negative half *of $\mathcal{B}$, denoted $\mathcal{B}^- = (\mathcal{V}^-, \mathcal{E}^-)$, is the restriction of $\mathcal{B}$ to the vertices in $\mathcal{V}_i$ for $i \leq 0$ and the edges in $\mathcal{E}_i$ for $i < 0$.*

For $m < n$, denote by $\mathcal{E}_{m,n}$ the set of positively oriented finite paths connecting vertices in $\mathcal{V}_m$ with vertices in $\mathcal{V}_n$, and denote by $\mathcal{E}_{n,m}$ the set of negatively oriented finite paths connecting vertices in $\mathcal{V}_n$ with vertices in $\mathcal{V}_m$. An (unoriented) *infinite path* x in $\mathcal{B}$ consists of a map $x : \mathbb{Z}\backslash\{0\} \to \mathcal{E}$ such that $x(i) \in \mathcal{E}_i$ and $r(x(i)) = s(x(i+1))$ for all $i \in \mathbb{Z}$. Denote the set of (unoriented) infinite paths in $\mathcal{B}$ by $X_\mathcal{B}$. For $x \in X_\mathcal{B}$, we will use x_i to denote the edge $x(i)$.

The set $X_\mathcal{B}$ has a natural product structure: let $B^+ = (V^+, E^+)$ be the Bratteli diagram defined by the positive part $\mathcal{B}^+$ of $\mathcal{B}$ and $B^- = (V^-, E^-)$ be the Bratteli diagram defined by the negative part $\mathcal{B}^-$ (interchanging the role of the source and range maps when we switch between $\mathcal{B}^-$ to B^- since we must switch between the indices taking values in $-\mathbb{N}$ and $\mathbb{N}$). Since $|V_0^+| = |V_0^-|$, we can identify each vertex in V_0^+ with one in V_0^- and make the identification

$$X_\mathcal{B} = \{(x, y) \in X_{B^+} \times X_{B^-} : s(x(0)) = s(y(0))\} \tag{4.4}$$

since $V_0^+ = V_0^-$.

A bi-infinite Bratteli diagram $\mathcal{B}$ formed from two Bratteli diagrams B^+ and B^- in this way, for some choice of a bijection between V_0^+ and V_0^-, is called *a joining* of B^+ and B^-. If $(B^+, \leq_{r,s}^+)$ and $(B^-, \leq_{r,s}^-)$ are both fully ordered Bratteli diagrams with $|V_0^+| = |V_0^-|$, there is a canonically chosen joining of B^+ and B^- determined by the partial orders $\leq_s^\pm$: since the vertices of $V_0^\pm$ are totally ordered by $\leq_s^\pm$, we identify each vertex in V_0^+ with the vertex in V_0^- that has the same relative place in the orders (i.e. the vertex in V_0^+ that is the greatest with respect to $\leq_s^+$ is identified with the vertex in V_0^- that is the greatest with respect to $\leq_s^-$, etc.). In this case, we call the joining of B^+ and B^- determined by $\leq_s^\pm$ *the* joining:

Definition 4.2.24. *The bi-infinite Bratteli diagram $\mathcal{B}$ that is the joining of two fully ordered Bratteli diagrams $(B^+, \leq_{r,s}^+)$ and $(B^-, \leq_{r,s}^-)$ according to the identifications determined by $\leq_s^\pm$ is* the joining *of B^+ and B^- and we will denote it by $\mathcal{B} = \mathcal{B}(B^+, B^-)$.*

When joining two diagrams B^+ and B^-, the 0^{th} level vertices of both B^+ and B^- "fuse" to into the vertex set $\mathcal{V}_0$ of $\mathcal{B} = \mathcal{B}(B^+, B^-)$, the i^{th} level vertices of B^+, for $i \in \mathbb{N}$, are identified with the i^{th} level vertices of $\mathcal{B}$, and the i^{th} level vertices of B^-, for $i \in \mathbb{N}$, are identified with the $(-i)^{th}$ level vertices of $\mathcal{B}$. The edges in $\mathcal{E}_i^{\pm}$ are identified with those in $E_{\pm i}^{\pm}$ for $i \in \mathbb{N}$ while the range and source maps of $\mathcal{B}^-$ are reversed for the negative part: for $e \in \mathcal{E}_i^-, v \in \mathcal{V}_i^-, v' \in \mathcal{V}_{i+1}^-$, we have $r(e) = v'$ and $s(e) = v$ if and only if $r(e) \in V_{-i}$ and $s(e) \in V_{i-1}$.

Definition 4.2.25. *A* fully ordered *bi-infinite Bratteli diagram* $(\mathcal{B}, \leq_{r,s})$ *is a Bratteli diagram* $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ *together with partial orders* $\leq_r$ *and* $\leq_s$ *on* $\mathcal{E}$ *so that edges* e, e' *are comparable under* $\leq_r$ *if and only if* $r(e) = r(e')$ *and are comparable under* $\leq_s$ *if and only if* $s(e) = s(e')$*.*

Remark 4.2.26. *The joining* $\mathcal{B} = \mathcal{B}(B^+, B^-)$ *of two fully ordered Bratteli diagrams* $(B^+, \leq_{r,s}^+)$ *and* $(B^-, \leq_{r,s}^-)$ *is itself a fully ordered bi-infinite Bratteli diagram. Since the range and source maps are reversed for the negative part* $\mathcal{B}^-$ *when joining two diagrams* B^+, B^-*, the orders on the negative part of* $\mathcal{B}$ *are also reversed: the orders* $\leq_r^-$ *and* $\leq_s^-$ *at* $v \in V_i^-$ *become the orders* $\leq_s$ *and* $\leq_r$*, respectively, at* $v \in \mathcal{V}_{-i}$*.*

The definition of the incidence matrices $\mathcal{F}_i$ for Bratteli diagrams generalizes to the case of bi-infinite Bratteli diagram. In particular, when joining two Bratteli diagrams B^+, B^- with matrices F_i^+, F_i^- to obtain $\mathcal{B}(B^+, B^-)$, the matrices $\mathcal{F}_i$ are $\mathcal{F}_i = F_i^+$ for $i > 0$ and $\mathcal{F}_i = (F_{-i}^-)^T$ for $i < 0$. The notion of telescoping also extends to bi-infinite Bratteli diagrams: for any sequence $\{m_n\}_{n \in \mathbb{Z}}$ with $m_0 = 0$ and $m_i < m_j$ if and only if $i < j$, the telescoping of $\mathcal{B}$ to $\{m_n\}$ is obtained by telescoping the positive and negative parts of $\mathcal{B}$, respectively, to the positive and negative parts of $\{m_n\}$.

Definition 4.2.27. *A* probability weighted *bi-infinite Bratteli diagram is a bi-infinite Bratteli diagram* $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ *together with a pair of weight functions* $w^+ : \mathcal{V}_0 \cup \mathcal{E}^+ \to (0, \infty)$*,* $w^- : \mathcal{V}_0 \cup \mathcal{E}^- \to (0, \infty)$ *such that*

1. w^+ *is a probability weight function for* $\mathcal{B}^+ = (\mathcal{V}^+, \mathcal{E}^+)$*,*

2. w^- *is a finite weight function for* $\mathcal{B}^- = (\mathcal{V}^-, \mathcal{E}^-)$*,*

3.
$$\sum_{v \in \mathcal{V}_0} w^+(v) \cdot w^-(v) = 1$$

Remark 4.2.28. *Definition 4.2.27 involves a choice of normalization; we chose to make the w^+ weight a probability weight, while only requiring that the w^- weight be finite and satisfy condition 3. We will see in Section §4.4 that condition 3 means that the associated flat surface has area* 1.

Definition 4.2.29. *A* diagram *is a bi-infinite, fully-ordered, probability weighted Bratteli diagram* $(\mathcal{B}, w^{\pm}, \leq_{r,s})$.

4.3 Cutting and Stacking

Cutting and stacking is a basic tool in ergodic theory used to construct infinite I.E.T.s. This technique is described in, for example, [AOW85] and [Sil08]. We will review the cutting and stacking technique here.

A cutting and stacking transformation T is defined by constructing a sequence of maps $T_0, T_1, T_2, \ldots$ on subsets of the real line such that for all i

$$\text{domain}(T_i) \subseteq \text{domain}(T_{i+t})$$

and

$$T_{i+1}|_{\text{domain}(T_i)} = T_i.$$

We set

$$\text{domain}(T) = \bigcup_i \text{domain}(T_i)$$

and then define T to be the pointwise limit of the maps T_i.

We will always require that the range and domain of a cutting and stacking map be equal except for countably many points, i.e. there exist countable sets P and P' such that

$$\text{domain}(T) \setminus P = \text{range}(T) \setminus P'.$$

Associated with each map T_i is a **stack** S_i consisting of a finite number $c_i \in \mathbb{N}$ of columns $C_{i,1}, \ldots, C_{i,c_i}$. A **column** $C_{i,j}$, consists of a finite number $h_{i,j}$ of open subintervals $I_{i,j,1}, \ldots, I_{i,j,h_{i,j}}$ of the real line, all of equal, finite measure. The intervals $I_{i,j,k}$, for $k \in \{1, \ldots, h_{i,j}\}$, are called the **levels** of column $C_{i,j}$. We require that for every n, all levels of all columns of the stack S_n are pairwise disjoint. We think of the levels of a column $C_{i,j}$ as being "stacked" with level $I_{i,j,1}$ at the bottom of the column and level $I_{i,j,(h_{i,j})}$ at the top of the column.

The domain of the map T_i is the union of all levels of all columns of the stack S_i except for the top levels of the columns of S_i. That is,

$$\text{domain}(T_i) = \bigsqcup_{j=1}^{c_i} \bigsqcup_{k=1}^{(h_{i,j})-1} I_{i,j,k}.$$

For any point $x \in \text{domain}(T_i)$, we define $T_i(x)$ to be the point directly "above" x in the stack. In other words, if x is a point in the level $I_{i,j,k}$, then $T_i(x)$ is the point y in the level $I_{i,j,k+1}$ such that $\lambda([a_{i,j,k}, x]) = \lambda([a_{i,j,k+1}, y])$, where λ denotes the Lebesgue (or other) measure and $a_{i,j,k}$ is the left endpoint of the interval $I_{i,j,k}$. (The reason domain(T_i) does not include the top levels of S_i is because there are no levels in the stack S_i above the top levels for T_i to map points into.)

The stacks S_i (and thus the transformations T_i) are defined inductively. The initial data that defines a cutting and stacking map is the stack S_0 along with "rules" for the inductive steps, specifying how to obtain each stack S_{i+1} from stack S_i for each i. The "rules" consist of three types of moves.

The first type of move is "cutting" columns into finitely many subcolumns of specified positive widths. We take these intervals to be open, and specify that the endpoints of these open intervals are not in the domain of subsequent maps T_j for $j > i$. For example, a column may be cut into two subcolumns of equal width (measure). To do this, divide each level of the column into two open subintervals of equal width – a "left half" and a "right half." Now, all the "left halves" form a subcolumn (keeping the stacking order of the levels) and all the "right halves" for a subcolumn (with the same order).

The second type of move is adding **spacers**. The i^{th} step spacers are open intervals in $\mathbb{R}$ which are disjoint from the union of all levels in S_i (and disjoint from each other), and which are added above a subcolumn of S_i with the same width as that of the spacer. Finitely many spacers may be added (in a specified order) to the top of any subcolumn. The "rules" would specify which subinterval(s) in $\mathbb{R}$ is (are) "stacked" above which subcolumn.

The third type of move is stacking (sub)columns (possibly containing i^{th}-step spacers) of S_i to form the columns of S_{i+1}. (For example, if a column is cut into two subcolumns of equal width, we could stack the left subcolumn under the right subcolumn. The resulting column, which is a column of S_{i+1}, is half as wide and twice as tall as the original column. The left half of the top level of the original column is no longer a top level of the new column, and so is in domain(T_{i+1})). We require that (sub)columns which are stacked on top of each other have equal width.

4.3.0.1 Truncating cutting and stacking processes

The cutting and stacking process may be viewed as taking the limit of a sequence of periodic maps. The domain of each map T_i is the union of all levels except the top level of the corresponding stack S_i. For a fixed $i \in \mathbb{N}$, it is possible to extend T_i to a homeomorphism $\widetilde{T_i}$ of the union of *all* the levels of the stack S_i by specifying that $\widetilde{T_i}$ maps the top level of S_i to the bottom level of S_i in an isometric, orientation-preserving way. The map $\widetilde{T_i}$ is periodic; the period under $\widetilde{T_i}$ of a point in S_i is the height of S_i.

For a typical cutting and stacking construction, the width of a level in the stack S_i converges to 0 as i goes to ∞, so the limit map T is defined up to a set of measure 0. However, we will want to consider cutting and stacking processes in which only finitely many stacks $S_1, \ldots, S_N$ are defined, and the width of a level in S_N is nonzero. We consider this case to be the same as the case in which infinitely many stacks S_i are defined but $S_m = S_n$ for all $m, n > N$ for some $N \in \mathbb{N}$. Thus, throughout the paper, we will adopt the convention that in this case the "limit" map determined by the cutting and stacking process is the periodic map $\widetilde{T_N}$.

Remark 4.3.1. *In Section §4.4, we will make use of correspondence between the adic map on Bratteli diagrams and cutting and stacking maps. Beyond some finite level of the Bratteli diagram B, all infinite paths in a periodic component of X_B merge since the tail equivalence class is finite. The restriction of the adic map to this periodic component of X_B is not a priori defined on the maximal path in this component, but admits a natural extension that sends the maximal path in the periodic component to the minimal path in the periodic component. (Compare with Remark 4.2.9.) Therefore, we want the analogous cutting and stacking map - a map associated to a finite tower - to send the top level of* *the tower to the bottom level of the tower. Thus, a periodic component of X_B will be associated to a periodic cutting and stacking map of the form $\widetilde{T_N}$ for some N.*

4.4 The Dictionary

In this section we present a construction that associates a flat surface to a diagram, and develop a dictionary between diagrams and flat surfaces constructed from them. The dictionary is summarized in Table 4.1 at the end of the section.

4.4.1 Interpreting a diagram as a flat surface

The core idea of the technique is to interpret each "half" of a diagram $(\mathcal{B}, w^{\pm}, \leq_{r,s})$ as determining an interval exchange map (likely with infinitely many intervals), i.e., a piecewise isometry of a finite interval. One of these interval exchange maps will determine the dynamics of a "first return map" to a transversal of the vertical flow, and the other interval exchange map will determine the dynamics of the first return map to a transversal of a horizontal flow. We will divide our description of how to define this map into two steps: first we will describe how to construct a flat surface from a pair of interval exchange maps and a collection of rectangles, and second we will describe how to interpret a diagram as a pair of interval exchange maps together with a collection of rectangles.

4.4.1.1 Obtaining a flat surface from a pair of interval exchange maps and a collection of rectangles

By *rectangle* R, we mean a subset of $\mathbb{R}^2$ of the form $I_1 \times I_2$, where I_1 and I_2 are closed intervals. Using this notation, we will refer to the Euclidean length of I_1 as the *width* of R and to the Euclidean length of I_2 as the *height* of R.

Fix $n \in \mathbb{N}$, and real numbers $b_1, b_2 > 0$. Write the interval $[0, b_1]$ as a union of n intervals $X_1, \ldots, X_n$ which overlap only at endpoints, i.e., $X_i \cap X_{i+1}$ consists of one point, and write the interval $[0, b_2]$ as a union of n intervals $Y_1, \ldots, Y_n$ which also overlap only at endpoints. Now for each $i = 1, \ldots, n$, define the rectangle $R_i = X_i \times Y_i$. Thus, $R_1, \ldots, R_n$ is a collection of n rectangles, arranged diagonally in $\mathbb{R}^2$, whose widths sum to b_1 and whose heights sum to b_2. (See Figure 4.1.)

Let T be an interval exchange transformation defined on the interval $[0, b_1]$ and let S be an interval exchange transformation defined on the interval $[0, b_2]$. Each point $\tilde{x} \in [0, b_1]$ is the x-coordinate of a unique point on the "top" edge of one of the rectangles $R_1, \ldots, R_n$ (unless $\tilde{x}$ belongs to an edge of an interval X_i), and we denote this point $top(x)$. Denote by $bottom(\tilde{y})$ the unique point on the "bottom" edge of one of the rectangles $R_1, \ldots, R_n$ whose y-coordinate is $\tilde{y}$ (unless $\tilde{y}$ is on an edge of some interval Y_i). The functions $bottom$ and $left$ are defined analogously. More precisely, if we define the functions τ, ρ defined in the interior of the intervals $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, respectively, to be the functions such that $\tau(x) = i$ if and only if $x \in X_i$ and $\rho(y) = i$ if and only if $y \in Y_i$, then

$$top(x) = \left(x, \sum_{i=1}^{\tau(x)} |Y_i| \right) \qquad \text{and} \qquad bottom(x) = \left(x, \sum_{i=1}^{\tau(x)-1} |Y_i| \right),$$
$$right(y) = \left(\sum_{i=1}^{\rho(y)} |X_i|, y \right) \qquad \text{and} \qquad left(y) = \left(\sum_{i=1}^{\rho(y)-1} |X_i|, y \right).$$

Define

$$\Sigma_t = \{x \in [0, b_1] \mid T \text{ is not continuous at } x\},$$
$$\Sigma_r = \{y \in [0, b_2] \mid S \text{ is not continuous at } y\}.$$

For each $i \in \{1, \ldots, n\}$, let

$$G_i^+ = \bigcup_{x \in X_i - \Sigma_t} top(x) \cup bottom(T(x))$$
$$G_i^- = \bigcup_{y \in Y_i - \Sigma_r} right(y) \cup left(S(y))$$

and define

$$\Sigma = C_i \cup \partial \left(\bigcup_{i=1}^{n} R_i \right) \setminus \bigcup_{i=1}^{n} (G_i^+ \cup G_i^-)$$

where C_i are all the corners of the rectangle $X_i \times Y_i$.

The flat surface associated to the pair (T, S) and the collection of rectangles $R_1, \ldots, R_n$ is the

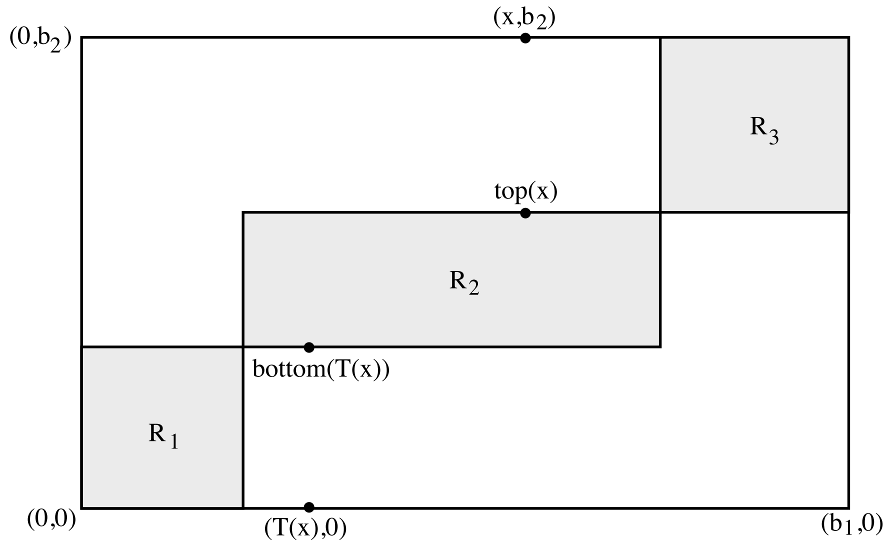


Figure 4.1: Constructing a flat surface from an interval exchange transformation T and a set of three rectangles R_1, R_2, R_3.

surface obtained by

$$\left(\bigcup_i R_i \setminus \Sigma\right) / \sim, \tag{4.5}$$

where $\sim$ is the equivalence relation defined, for $x \in (X_i - (\Sigma^t \cup \partial X_i))$,

$$top(x) \sim bottom(T(x)) \qquad \text{and} \qquad right(y) \sim left(S(y))$$

for $y \in (Y_i - (\Sigma^r \cup \partial Y_i))$. See Figure 4.1.

The associated holomorphic 1-form α on this surface is defined such that its vertical foliation coincides with lines locally of the form $x = const.$ and horizontal foliation locally of the form $y = const.$ when representing the surface as in (4.5).

4.4.1.2 Interpreting a diagram as a pair of interval exchange maps and a collection of rectangles

Let $(\mathcal{B}, w^{\pm}, \leq_{r,s})$ be a diagram. We will use it to define a collection of rectangles and two interval exchange maps; we will then use the construction introduced in §4.4.1.1 to construct a flat surface $S(\mathcal{B}, w^{\pm}, \leq_{r,s})$ from the obtained interval exchange transformatins.

We will define a collection of $c_0 = |\mathcal{V}_0|$ rectangles. Let $v_1, \ldots, v_{c_0}$ be the vertices in $\mathcal{V}_0$. For each integer i, $1 \leq i \leq c_0$, we will define a rectangle R_i of width $w^+(v_i)$ and height $w^-(v_i)$, and we arrange these rectangles "diagonally." That is,

$$R_i = \left[\sum_{j<i} w^+(v_j), \sum_{j \leq i} w^+(v_j)] \right] \times \left[\sum_{j<i} w^-(v_j), \sum_{j \leq i} w^-(v_j) \right]. \tag{4.6}$$

We will describe how to obtain a cutting and stacking map from an weighted, fully ordered Bratteli diagram $(B, w, \leq_{r,s})$. We will then apply this construction to the positive half of $(\mathcal{B}, w^{\pm}, \leq_{r,s}$) to determine the map which we will associate to the union of the horizontal sides of the rectangles, as well as apply it to the negative half of $(\mathcal{B}, w^{\pm}, \leq_{r,s})$ to determine the map which we will associate to the union of the vertical sides of the rectangles. Recall that when considering the negative part $(\mathcal{B}, w^{\pm}, \leq_{r,s})$ of a diagram as a Bratteli diagram $(B^-, w, \leq_{r,s}^-)$, the orders $\leq_{r,s}$ are switched to obtain $\leq_{r,s}^-$ (see Remark 4.2.26).

We aim to interpret a weighted ordered Bratteli diagram $(B, w, \leq_{r,s})$ as combinatorial description of "cutting and stacking" instructions. We wish to construct a measure-preserving map ϕ from X_B to a real interval minus a countable set of points so that ϕ conjugates the adic map on X_B to a cutting-and-stacking map on the interval. A point in X_B and its successor (which is determined by the partial order $\leq_r$) must be mapped by ϕ to a point and the point directly "above" it in the cutting and stacking process. However, this requirement does not determine a unique cutting and stacking process, since it does not, for example, specify whether a given subcolumn is to the left or right of the other subcolumns which comprise a column. Thus, the order $\leq_r$ on the Bratteli diagram determines a family of measurably isomorphic cutting-and-stacking maps. The partial order $\leq_s$ will be used to pick out a unique such map. Namely, we will use $\leq_s$ to give the relative orders of subcolumns of a column, as well as the order of the 0^{th} level intervals.

By the decomposition (4.2), it will suffice to describe the construction for minimal components and for periodic components of X_B.

Assume B is minimal and let $I = [0, \sum_{v \in V_0} w(v)]$. We will define a family of injective maps $f_i : D_i \to I$, $D_i \subset I$, indexed by $i \in \mathbb{N}$, such that

1. $D_i \subset D_{i+1}$ for all i,

2. $\bigcup_i D_i = I \setminus \Sigma$ for some set $\Sigma \subset I$ of Lebesgue measure 0,

3. the restrictions $f_i|_{D_j} = f_j$ for all $j < i$.

We will then define the map $f : I \setminus \Sigma \to I$ to be the pointwise limit $f = \lim_{i \to n} f_i|_{I \setminus \Sigma}$. This will be an interval exchange transformation, i.e., a piecewise isometry of an interval.

By our order $\leq_s$ we have an order on the level V_0. Therefore, for $v_i \in V_0$, $1 \leq i \leq |V_0|$, we define the "(stage 0) column over v_i" to be the interval

$$J_i^0 = \left(\sum_{j=1}^{i-1} w(v_j), \sum_{j=1}^{i} w(v_j) \right). \tag{4.7}$$

Up to finitely many points, these intervals cover I. For each fixed $v_i \in V_0$, denote by $e_1^i, \ldots, e_n^i$ the edges coming out of v_i in increasing order with respect to $\leq_s$. Partition the level 0 tower over v_i into open subintervals

$$J_1(v_i) = (j_0, j_1), \quad J_2(v_i) = (j_1, j_2), \quad \ldots, \quad J_n(v_i) = (j_{n-1}, j_n)$$

with $\sum_{j<i} w(v_j) = j_0 < j_1 < \ldots < j_n = \sum_{j \leq i} w(v_j)$ and such that the Lebesgue measures of the intervals $J_1(v_i), \ldots, J_n(v_i)$ are, respectively,

$$w(e_1^i), w(e_2^i), \ldots, w(e_n^i).$$

Thus, each edge e_j^i is associated to one subinterval of the column over v_i.

For each vertex $v \in V_1$, we will form the "(stage 1) column over v" as follows. Let $e'_1, \ldots, e'_m$ denote the edges that terminate at v in increasing order with respect to $\leq_r$. Stack the subintervals from the level 0 columns associated with the edges $e'_1, .., e'_m$ in order, so that the subinterval associated with e'_1 is the bottom of the stack, and the subinterval associated with e'_m is at the top of the stack.

The domain D_1 of the map f_1 will be the union over all $v \in V_1$ of all but the top level of the stage 1 column over v. Because the weight function w is compatible with X_B, all subintervals in a stack will have the same width. The map f_1 is defined by mapping a point x in a subinterval to the corresponding point in the subinterval directly above it.

We define D_k and f_k by induction on k. Assume the (stage k) columns over the vertices in V_k have been defined. For each vertex $v_i \in V_k$, denote by $e_1^i, \ldots, e_n^i$ the edges coming out of v_i in increasing order with respect to $\leq_s$. Cut each level of the (stage k) column over v_i into open subintervals such that the relative lengths of the subintervals (in increasing order from left to right) are, respectively,

$$w(e_1^i), w(e_2^i), \ldots, w(e_n^v).$$

In this way, for each j the edge e_j^i is associated to the subset (a "subcolumn") of the column over v_i consisting of the j^{th} subinterval of each level of the column over v_i.

For each vertex v' in V_{k+1} we will form the "(stage $k+1$) column over v'" as follows. Let $e'_1, \ldots, e'_m$ denote the edges coming into v' in increasing order with respect to $\leq_r$. Each edge e'_i is associated to a subset of a stage n column. Stack the subcolumns associated with the edges $e'_1, \ldots, e'_m$ in order, so that the subcolumn associated with e'_1 is on the bottom of the column and the subcolumn associated to e'_m is on the top of the column.

The domain D_{k+1} of the map f_{k+1} is the union over all vertices $v' \in V_{k+1}$ of all but the top level of the (stage $k+1$) column over v'. Because the Bratteli diagram is compatibly weighted, all levels of each stage $k+1$ column will have the same width. The map f_k is defined by mapping any point x in a non-maximal level of any stage $k+1$ column to the corresponding point in the level immediately above it.

Define

$$\Sigma = \bigcap_{k=1}^{\infty} \text{TopLevels}(k) \tag{4.8}$$

where TopLevels(k) denotes the union over $v \in V_k$ of the top level of the (stage k) tower over v. Since B is minimal, by (iii) in Definition 4.2.15, we have that Σ has Lebesgue measure 0. Note that the set Σ is in bijection with X_{max}. The sets TopLevels(k) are nested; for any $x \in I \setminus \Sigma$, there exists $n \in \mathbb{N}$ such that $N > n$ implies x is in some non-top level of a stage N tower. Thus, $\lim_{n\to\infty} f_n(x)$ is well-defined for all $x \in I \setminus \Sigma$. Thus, the pointwise limit function $f = \lim_{n\to\infty} f_n$ is well-defined on $I \setminus \Sigma$. Furthermore, f is injective and Lebesgue measure-preserving.

Let us now assume that B consists of a single periodic component according to (4.2), i.e.,

$|X_B| < \infty$. The finite set of paths of X_B is ordered by the ordering $\leq_r$ in $r^{-1}(v^*)$, where $v^* \in V_k$ is the first vertex after which all paths in X_B coincide. As such, there are $|X_B|$ open intervals of length $|X_B|^{-1}$, bijectively identified to paths starting at V_0 and ending at v^* which are permuted by the map according to the order $\leq_r$. The interval corresponding to the maximal path in X_B is mapped to the one corresponding to the minimal path. Therefore we have defined a periodic interval exchange transformation $f : I\backslash\Sigma \to I\backslash\Sigma$ of period $|X_B|$, where $\Sigma = \{\frac{i}{|X_B|} : i \in \{0, \ldots, |X_B|\}\}$. Note that mapping the maximal path in a periodic component to the minimal path in that periodic component agrees with the convention established in subsection §4.3.0.1 of interpreting finitely many steps (or an infinite process with only finitely many nontrivial steps) of a cutting and stacking process as determining a periodic map.

Let $(B, w, \leq_{r,s})$ be any fully ordered, weighted Bratteli diagram and assume that w is a probability weight function. We can define an injective map $f : I\backslash\Sigma \to I$ by definining a map on each minimal component X^i_M and periodic component X^i_P as above. Since (4.2) is a decomposition into invariant subsets of the tail equivalence relation, the union of the maps for each component gives the map on $I\backslash\Sigma$ which corresponds to the cutting and stacking transformation defined on I by weigthed, fully ordered Bratteli diagram $(B, w, \leq_{r,s})$.

Let $(\mathcal{B}, w^{\pm}, \leq_{r,s})$ be a probability weighted, fully ordered Bratteli diagram. By the construction above we have two interval exchange maps $T^{\pm}$ defined on full measure subsets of $I^+ = [0, 1]$ and I^- constructed as cutting and stacking transformations. Using the construction from §4.4.1.1, using these maps along with the rectangles (4.6), we can build a unique flat surface $S(\mathcal{B}, w^{\pm}, \leq_{r,s})$ associated to $(\mathcal{B}, w^{\pm}, \leq_{r,s})$.

Remark 4.4.1. *It follows from condition 3 of Definition 4.2.27 that a surface constructed from a diagram has surface area* 1*. The adjective "probability" in condition 1 of the same definition implies that the sum of the areas of the rectangles is* 1*.*

4.4.1.3 Conventions for drawing diagrams

We now establish conventions for representing a diagram $(\mathcal{B}, w^{\pm}, \leq_{r,s})$ as a picture, and we will adhere to these conventions throughout the paper. Dots representing vertices in the same level of $\mathcal{B}$ will be drawn in a horizontal row. Vertices in negative levels will be towards the "top" of the picture, and vertices in positive levels will be towards the "bottom" of the picture. Vertices in $\mathcal{V}_0$ will be arranged from left to right in ascending order according to $\leq_s$. Edges whose source is a vertex v will be drawn coming out of v in order from left to right according to $\leq_s$ (i.e., in a neighborhood immediately "below" v in the picture, edge e_1 will be to the left of e_2 if $e_1 \leq_s e_2$). Edges whose range is a vertex v will be drawn entering v in order from left to right according to $\leq_r$ (i.e. in a neighborhood immediately "above" v in the picture, edge e_1 will be to the left of e_2 if $e_1 \leq_r e_2$). Weights assigned to each vertex v in $\mathcal{V}_0$ by w^+ (resp. w^-) will be written just below (resp. above) the dot in the picture corresponding to v. Weights assigned to edges in $\mathcal{E}^+$ by w^+ and to edges in $\mathcal{E}^-$ by w^- will be written next to those edges.

4.4.2 First examples: constructing surfaces from diagrams

We present two basic examples illustrating how to interpret diagrams as flat surfaces. The Bratteli diagram for Chamanara's surface (4.4.2.1) is one of the simplest diagrams, and is thus a good "warm up" before considering more complicated constructions. The Chacón middle third example (4.4.2.2) is included here because it illustrates how to represent "spacers" in a cutting and stacking process using a diagram. This will illustrate our unified point of view through the dictionary developed.

4.4.2.1 Chamanara's Surface

In this section we present five different descriptions of the dyadic odometer and its connection to one of the first and best-known examples of a flat surface of infinite genus and finite area which was given by Chamanara [Cha04]; we obtain this surface via a suspension of the dyadic odometer.

Description 1. The *dyadic odometer* is the map $\Phi : X \to X$, where $X = \{0,1\}^{\mathbb{N}}$, defined as addition by 1 in base two of $.1000\ldots$ with infinite carry to the right. The dyadic odometer is a minimal, non weak-mixing, and uniquely ergodic transformation of the Cantor set X.

Description 2. The directed graph B in Figure 4.2 is the Bratteli diagram for the dyadic odometer. The space X can be identified with the space of all infinite paths starting at the vertex and moving uniformly downwards along the diagram. Labeling the left and rights edges at every level

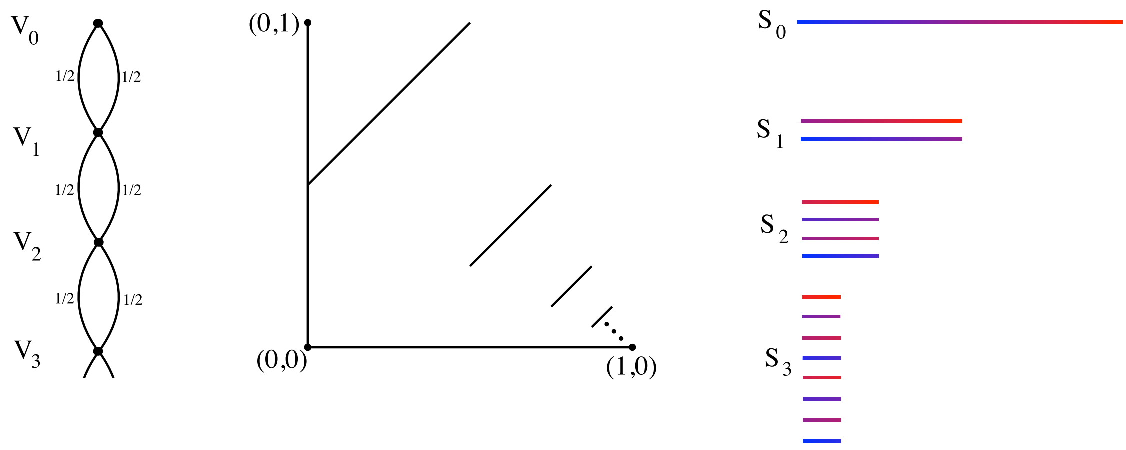


Figure 4.2: A Bratteli diagram and corresponding cutting-and-stacking representation for the dyadic odometer

in B with 0 and 1, respectvely, it can be easily seen that X and X_B are in fact the same sets. The dyadic odometer can be defined as a homeomorphisms of the space of all infinite paths on this diagram. This transformation is in fact the adic transformation corresponding to the orders coming from Figure 4.2.

Description 3. Define a map $B : X \to [0,1]$ by $B(a) = \sum_{i=1}^{\infty} a_i 2^{-i}$. Outside a countable set of X, this mapping is a bijection onto $I = [0,1]\backslash P$, where P is some countable subset of $[0,1]$. Consider the map $R : I \to I$ defined by $R(x) = B \circ \Phi \circ B^{-1}(x)$. It is a restriction to I of the map $\bar{R} : [0,1] \to [0,1]$ defined by

$$\bar{R}(1 - 2^{-n} + x) = 2^{-(n+1)} + x \qquad \text{for} \qquad 0 \leq x < 2^{-(n+1)}, \;\; n \in \mathbb{N} \tag{4.9}$$

and $\bar{R}(1) = 0$. The map $\bar{R}$ is also known as the *Van der Corput map*. It is a piecewise isometry of the unit interval which can also be described as an interval exchange transformation on infinitely many intervals.

Description 4. The map $\bar{R}$ can also be constructed via the process of cutting and stacking as follows (see Figure 4.2). Consider the interval $[0,1]$ and cut it into the two disjoint intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$. Consider the map $T_1 : [0, \frac{1}{2}) \to [0,1]$ defined as the unique isometry sending 0 to $\frac{1}{2}$. This

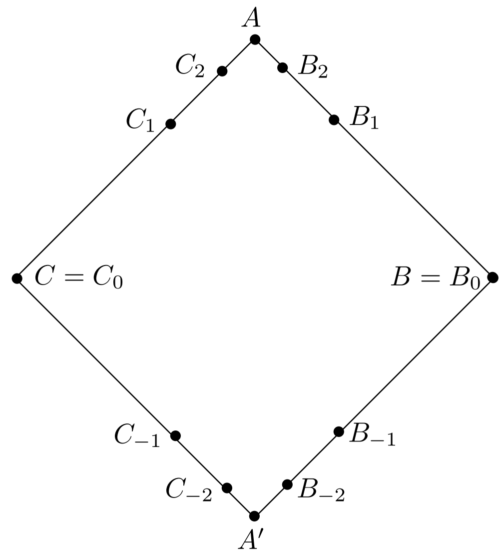


Figure 4.3: Construction of the surface S_p.

map can also be seen as the map defined by "stacking" the interval $[\frac{1}{2}, 1)$ over the interval $[0, \frac{1}{2})$, thereby creating a "tower" made up of two intervals, and mapping a point $x \in [0, \frac{1}{2})$ to the point directly above it in the upper level of the stack.

We now define a map $T_2 : [0, \frac{3}{4}) \to [0, 1]$ with the property that $T_2|_{[0,\frac{1}{2})} = T_1$. Considering the tower consisting of the interval $[\frac{1}{2}, 1)$ over the interval $[0, \frac{1}{2})$, we cut this tower into 4 intervals of equal length: $[0, \frac{1}{4})$ and $[\frac{1}{4}, \frac{1}{2})$ on the bottom and $[\frac{1}{2}, \frac{3}{4})$ and $[\frac{3}{4}, 1)$ on top. We now stack the two rightmost intervals ($[\frac{1}{4}, \frac{1}{2})$ and $\frac{3}{4}, 1]$) on top of the tower created by the leftmost intervals, thereby creating a tower consisting of 4 intervals of length $\frac{1}{4}$. The map T_2 is defined, for a point x on the three bottom intervals, as its image by moving up one level on the tower. As such it is a piecewise isometry and it satisfies $T_2|_{[0,\frac{1}{2})} = T_1$.

We can continue this process indefinitely and create a sequence of maps $T_k : [0, \frac{2^k-1}{2^k}) \to [0, 1]$ with the property that $T_{k+1}|_{[0,\frac{2^k-1}{2^k}]} = T_k$. Let $T : [0, 1] \to [0, 1]$ be the pointwise limit of this sequence of maps which maps 1 to 0. The limiting map T coincides with the Van der Corput map (4.9).

Description 5a. Let us consider the suspension flow ϕ_t for the map $\bar{R}$: it is the vertical flow generated by the vector field ∂_y on the surface S' obtained by gluing edges of unit square $[0, 1]^2$ through the identifications $(x, 1) \sim (R(x), 0)$. Identifying the vertical edges $\{0, 1\} \times [0, 1]$ of $S'_{2^{-1}}$

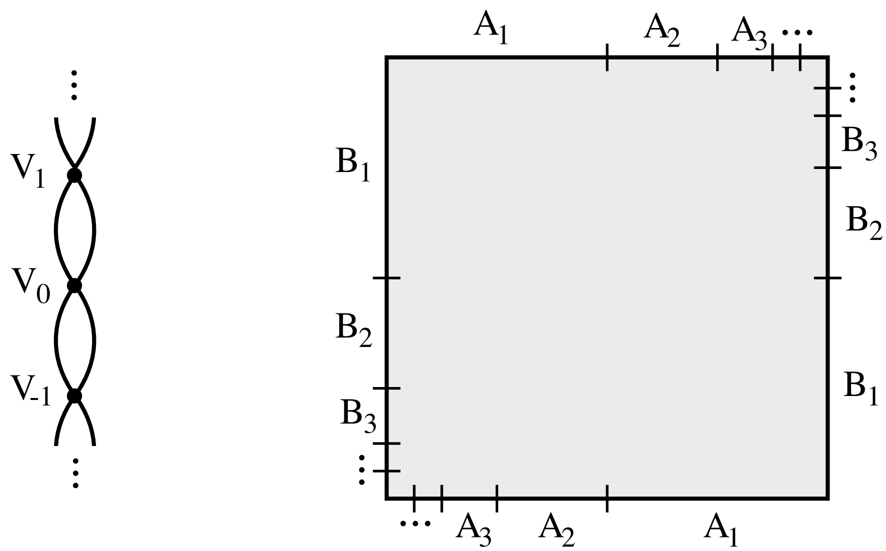


Figure 4.4: A diagram (left) that gives rise to one of Chamanara's surfaces (right).

through $(1, y) \sim (0, R(y))$ gives us a surface $S_{2^{-1}}$ (the 2 denotes the fact that we used the dyadic odometer to construct it) on which the horizontal flow generated by the vector field ∂_x is conjugated to the vertical flow ϕ_t through the involution $i : (x, y) \mapsto (y, x)$. The surface $S_{2^{-1}}$ is a non-compact surface of infinite topological type, has finite area, and has a flat metric almost everywhere.

Description 5b. Let us now consider the construction due to Chamanara of an infinite family of flat surfaces of infinite genus parametrized by $p \in (0, 1)$ [Cha04] (see Figure 4.3). Let $\mathcal{S} = ABA'C$ be a square centered at the origin in $\mathbb{C}$ such that its sides have length one and the diagonal BC is on the real line. Set $B_0 = B$ and $C_0 = C$. For $i \geq 1$ define B_i (respectively B_{-i}, C_i and C_{-i}) to be the point on the interval BA (respectively BA', CA and CA') such that the length of AB_i (respectively $A'B_{-i}$, AC_i, and $A'C_{-i}$) is p^i for some $0 < p < 1$. The sides B_iB_{i+1} and $C_{-(i+1)}C_{-i}$ are identified by a translation. This identifies all the points of the form B_{2k+1} and C_{2k} and the points of the form B_{2k} and C_{2k+1}. We denote the identification map by Q_p. The resulting surface obtained from the above is denoted by $S_p = Q_p(\mathcal{S})$ and it is clear that it is a flat surface of finite area. It is shown in [Cha04, Proposition 9] that it is an infinite genus surface with one end. It is also easy to see that it is the geometric limit of finite genus surfaces: let $\mathcal{S}^n$ be the subset of $\mathcal{S}$ bounded from above by C_nB_n and below by $C_{-n}B_{-n}$. Then for each n, $S_p^n = Q_p(\mathcal{S})$ is a translation surface of genus n with two singularities of order $n - 1$. Then limiting surface $S_p^n \longrightarrow S_p$ is our infinite genus surface with singularities of infinite order.

Now let $\mathcal{B}$ be the diagram whose positive and negative parts are the same and are the Bratteli diagram B corresponding to the dyadic odometer. The surface S associated to this diagram through the construction described in §4.4.1 can be seen to be the same as Chamanara's surface $S_{2^{-1}}$ in Figure 4.4 by rotating the former by $\pi/4$. In fact, for any prime p, any $S_{p^{-1}}$ can be constructed in a similar way by suspending the p-adic odometer (defined as addition by 1 in base p) as we did for the dyadic odometer. In other words, the adic transformation defined on a half-diagram $B^{\pm}$ with the 1×1 transition matrix $[p]$ at every level is isomorphic to the p-adic odometer and to the interval exchange transformation defined as the first return map to a transversal of a translation flow on $S_{p^{-1}}$. Through this identification we are therefore able to go from statements of the dynamics of translation flows on flat surfaces of infinite topological type to statements about the dynamics of the p-adic odometers.

4.4.2.2 The Chacón middle third transformation

A well-known example of a transformation which is mild mixing but not light mixing is the Chacón middle third transformation. Figure 4.5 gives the first few steps of the Bratteli diagram and cutting-and-stacking construction for this map. The union of all spacers used in the construction has measure $1/3$; at each stage, the unused spacers correspond to the rightmost vertex of each level in the Bratteli diagram. Start with a single interval of with $2/3$; at each stage in the cutting-and-stacking process, cut the column in three equal subcolumns, put a spacer over the middle subcolumn, and stack the subcolumns from left to right (with left on the bottom).

It should be noted that the tail equivalence relation is *not* minimal. Indeed, the path x_∞ in Figure 4.5 consisting of the right-most edge at every level is in its own tail equivalence class. This means that the decomposition (4.2) for this diagram has one minimal component and one periodic component consisting of x_∞. As such the tail equivalence is *not* uniquely ergodic: there is one invariant probability measured supported at x_∞ and another one supported on the minimal component which is the unique invariant measure on $X_B \backslash \{x_\infty\}$. The extra point x_∞ in Figure 4.5 is somewhat artificial byproduct of encoding cutting and stacking transformations with spacers using Bratteli diagrams. See §4.6.2.4 for another phenomenon which occurs when encoding the

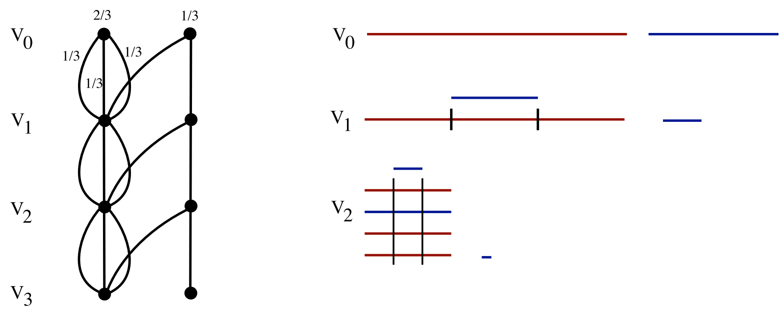


Figure 4.5: The first parts of the Bratteli diagram and cutting-and-stacking steps for the Chacón middle third transformation.

use of spacers in a Bratteli diagram.

4.4.2.3 The Arnoux-Yoccoz-Bowman surface

In the early 80's, Arnoux and Yoccoz [AY81] constructed a family of flat surfaces, one of every genus $g \geq 3$. These served as examples of surfaces carrying pseudo Anosov maps, which where not well-understood as the theory was still in its infancy. It was eventually shown that the Veech groups of these surfaces are quite peculiar: they do not contain parabolic elements [HL06]. One usually expects that if the Veech group of a flat surface has an infinite subgroup of hyperbolic automorphisms, then it is generated by parabolic elements. For the Arnoux-Yoccoz family of surfaces, this was shown not to be the case.

Bowman [Bow12] has taken the geometric limit of this family of surfaces as the genus goes to infinity. The limiting surface will be referred to as the *Arnoux-Bowman-Yoccoz surface*, and it is depicted in Figure 4.6. Bowman proved that the vertical and horizotal flows on this surface are uniquely ergodic. This surface has finite area and, much like its finite-genus "subsurfaces", the Veech group of this surface contains no parabolic elements. In fact Bowman showed that the Veech group of this surface is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$, where the infinite subgroup is generated by the map which expands the horizontal direction by a factor of 2 while contracting the vertical by a factor of $\frac{1}{2}$ (as shown in figure 4.6).

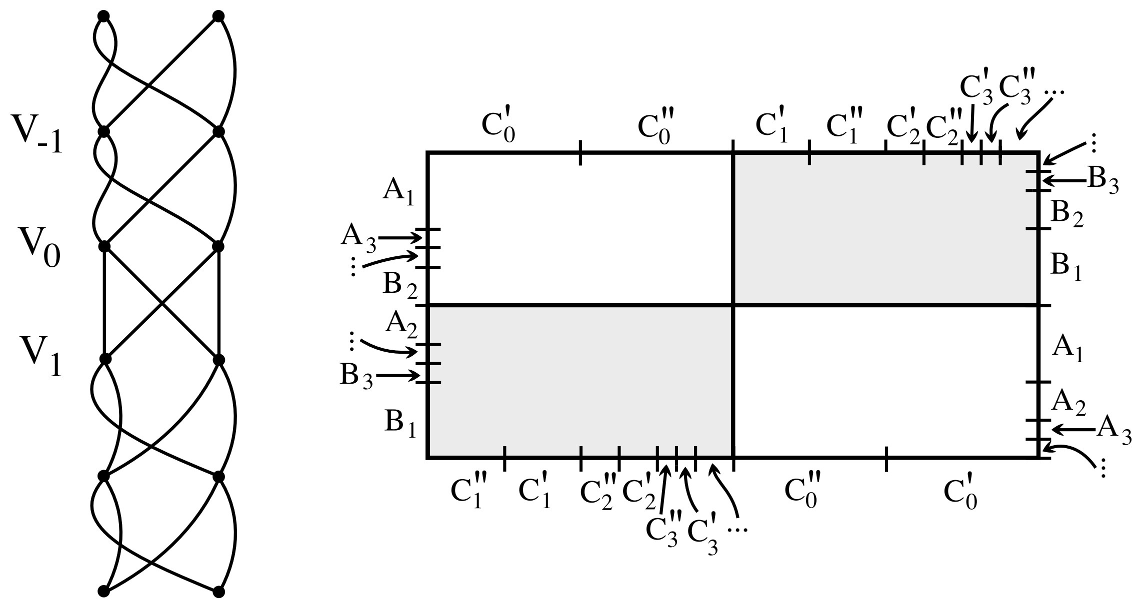


Figure 4.6: The Arnoux-Yoccoz-Bowman surface: The Bratteli diagram defining it and its rectangle representation as in (4.6).

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4.5 Renormalization

In this section we develop a mechanism which will serve as a renormalization tool for the vertical flow on flat surfaces $S(\mathcal{B}, w^{\pm}, \leq_{r,s})$ constructed from diagrams $(\mathcal{B}, w^{\pm}, \leq_{r,s})$. The spirit of the procedure is that as we deform the surface $S(\mathcal{B}, w^{\pm}, \leq_{r,s})$ by the Teichmüller deformation, we can perform a step of cutting and stacking on our surface and arrive at another surface which corresponds to the surface constructed from the shift of the diagram $(\mathcal{B}, w^{\pm}, \leq_{r,s})$. This is summarized in Propositiong 4.5.2. The 1-parameter deformation along with the renormalization maps can be seen as a generalization of Rauzy-Veech induction (see [Via06]).

Let $(\mathcal{B}, w^{\pm}, \leq_{r,s})$ be a diagram whose positive part is not completely periodic. Throughout this section, we will assume that, if the positive part of $\mathcal{B}$ is not aperiodic, then the invariant measure determined by w^+ assigns zero value to periodic components. Recall that we can define $h_v^k = w^-(v)$ for $v \in \mathcal{V}_k$ with $k > 0$ using (4.1) and $w^+(v)$ using (4.3). Define

$$\ell_v^k = w^+(v) \qquad \text{and} \qquad h_v^k = w^-(v) \tag{4.10}$$

for $v \in \mathcal{V}_k$, $i \in \mathbb{N}$. Notice also that $\sum_{v\in V_0} \ell_v^0 = 1$ by assumption.

We will define a sequence of maps

$$\mathcal{R}_k : S(\mathcal{B}, w^\pm, \leq_{r,s}) \to S(\mathcal{B}'_k, w_k^\pm, \leq_{r,s}^k)$$

taking surfaces constructed from diagrams to other such surfaces. The data defining the surfaces will be related as follows. For $\mathcal{B} = (\mathcal{V}, \mathcal{E})$, $\mathcal{B}'_k = (\mathcal{V}', \mathcal{E}')$ is obtained by shifting $\mathcal{B}$: $\mathcal{V}'_i = \mathcal{V}_{i+k}$ and $\mathcal{E}'_i = \mathcal{E}_{i+k}$ along with their orders $\leq_{r,s}$ and $w_k^\pm = e^{\pm t_k} w^\pm$, where the t_k belong to the sequence of renormalization times

$$t_k \equiv -\log\left(\sum_{v\in\mathcal{V}_k} \ell_v^k\right) = -\log\left(\sum_{v\in\mathcal{V}_k} w^+(v)\right) \tag{4.11}$$

for $k > 0$. By (iii) in Definition 4.2.15, we have that $t_k \to \infty$ if $\mathcal{B}^+$ is aperiodic. Moreover, up to telescoping, we can assume that $\inf_k(t_k - t_{k-1}) > 0$. The renormalized heights and widths are obtained from (4.10) and (4.11) by

$$\bar{h}_v^k = e^{-t_k} h_v^k \qquad \text{and} \qquad \bar{\ell}_v^k = e^{t_k} \ell_v^k \tag{4.12}$$

for any $v \in \mathcal{V}_k$.

Let $(\mathcal{B}, w^\pm, \leq_{r,s})$ be a diagram and $S(\mathcal{B}, w^\pm, \leq_{r,s})$ be the flat surface constructed from it through the construction in §4.4.1.2. Let

$$S_t(\mathcal{B}, w^\pm, \leq_{r,s}) = g_t S(\mathcal{B}, w^\pm, \leq_{r,s})$$

be the surface obtained by deforming $S(\mathcal{B}, w^\pm, \leq_{r,s})$ using the Teichmüller flow. Consider the surface $S_{t_1}(\mathcal{B}, w^\pm, \leq_{r,s})$, for t_1 defined in (4.11).

Choose some vertex $v_i \in \mathcal{V}_0$. By our deformation of the surface, the interior of every deformed rectangle $g_{t_1} R_i$ in (4.6) corresponding to the vertex v_i in $\mathcal{V}_0$ is isometric to $(0, \bar{\ell}_i^0) \times (0, \bar{h}_i^0)$. We cut the rectangle $g_{t_1} R_i$ (associated to the tower over v_i) into sub-rectangles of width $\bar{\ell}_{v_i}^0 w(e)$ and height $\bar{h}_{v_i}^0$ using the order $\leq_s$ on v_i for every e is an edge with $s(e) = v_i$. Doing this for every vertex $v_i \in \mathcal{V}_0$ we have $|E_1|$ subrectangles corresponding to edges in E_1 which were obtained as subrectangles of the R_j.

Now we stack our sub-rectangles into $|\mathcal{V}_1|$ new towers using the orders given by the order in $r^{-1}(v)$ for each $v \in \mathcal{V}_1$. For some $v \in \mathcal{V}_1$, let $(e_1, \ldots, e_n)$ be the ordered set of edges in $r^{-1}(v)$. For all $i \in \{1, \ldots, |r^{-1}(v)|-1\}$, we identify the interior of the top of the sub-rectangle corresponding to the edge e_i to the interior of the bottom edge of the sub-rectangle corresponding to the edge e_{i+1}. Denote the surface obtained by the process of deforming and cutting and stacking described above as $DS(\mathcal{B}, w^{\pm}, \leq_{r,s})$ and define

$$\mathcal{R} : S(\mathcal{B}, w^{\pm}, \leq_{r,s}) \longrightarrow DS(\mathcal{B}, w^{\pm}, \leq_{r,s}) \tag{4.13}$$

to be the map taking one surface to the other by this process. We point out that the map between $S_{t_1}(\mathcal{B}, w^{\pm}, \leq_{r,s})$ and $S(\mathcal{B}', w_1^{\pm}, \leq^1_{r,s})$ is an isometry: the cutting and stacking does not change the flat metric in any way.

Definition 4.5.1 (Shifting). *The diagram $(\mathcal{B}', w_1^{\pm}, \leq'_{r,s})$ with $\mathcal{B}' = (\mathcal{V}', \mathcal{E}')$ is the* shift *of $(\mathcal{B}, w^{\pm}, \leq_{r,s})$ with $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ if it can be constructed as follows. $\mathcal{V}'_i = \mathcal{V}_{i+1}$ for all $i \in \mathbb{Z}$ and $\mathcal{E}'_i = \mathcal{E}_{i+1}$ for all $i \neq -1$. For $i = -1$, $\mathcal{E}'_{-1} = \mathcal{E}_1$. As such, there is a bijection $\sigma : \mathcal{B}' \to \mathcal{B}$ corresponding to this shift.*

Let $w_1^{\pm} : \mathcal{V}'_0 \cup \mathcal{E}' \to (0,1)$ be the weight function obtained from $(\mathcal{B}, w^{\pm})$ as follows: for $v \in \mathcal{V}'_0$, $w_1^{\pm}(v) = e^{\pm t_1} w^{\pm}(\sigma(v))$, $w_1^{+}(e) = w^{+}(\sigma(e))$ for any $e \in \sigma(\mathcal{E}^{+}) \backslash \mathcal{E}'_{-1}$, and $w_1^{-}(e) = w^{-}(\sigma(e))$ for any $e \in \sigma(\mathcal{E}^{-})$. Let $\leq'_{r,s}$ be defined on $\mathcal{B}'$ by $e \leq'_{r,s} f$ if and only if $\sigma(e) \leq_{r,s} \sigma(f)$ on $\mathcal{B}$.

We will denote by $\sigma(\mathcal{B}, w^{\pm}, \leq_{r,s})$ the shift of $(\mathcal{B}, w^{\pm}, \leq_{r,s})$, and by σ^k the process of shifting k times. It is straightforward to check from the definition that if $(\mathcal{B}, w^{\pm}, \leq_{r,s})$ is a diagram, then so is $\sigma(\mathcal{B}, w^{\pm}, \leq_{r,s})$.

Proposition 4.5.2. *Let $(\mathcal{B}, w^{\pm}, \leq_{r,s})$ be a diagram. Then*

$$S(\sigma(\mathcal{B}, w^{\pm}, \leq_{r,s})) = \mathcal{R}(S(\mathcal{B}, w^{\pm}, \leq_{r,s})),$$

where $\mathcal{R}$ is the map defined in (4.13).

Proof. The construction of the map $\mathcal{R}$ was done through uniform deformation in addition to cutting and stacking. It is straightforward then by the definition of the shift σ that $S(\sigma(\mathcal{B}, w^{\pm}, \leq_{r,s}))$, through the construction described in §4.4.1.2, has the same number of rectangles of the same widths as those in $\mathcal{R}(S(\mathcal{B}, w^{\pm}, \leq_{r,s}))$. Moreover, the identifications on the top and bottom edges of $S(\sigma(\mathcal{B}, w^{\pm}, \leq_{r,s}))$ coincide with those of $\mathcal{R}(S(\mathcal{B}, w^{\pm}, \leq_{r,s}))$. It remains to show that the heights and left/right identifications of $S(\sigma(\mathcal{B}, w^{\pm}, \leq_{r,s}))$ coincide with those of $\mathcal{R}(S(\mathcal{B}, w^{\pm}, \leq_{r,s}))$.

Let $(B^{-}, \leq_{r,s})$ be the negative part of $(\mathcal{B}, \leq_{r,s})$, indexed now by $\mathbb{N} \cup \{0\}$ so that the orders $\leq_{r,s}$ are reversed. Let B' be the negative part of $\sigma(\mathcal{B}, \leq'_{r,s})$, also indexed now by $\mathbb{N} \cup \{0\}$. The shifting operation σ has the following effect: The $\mathcal{V}_1$ vertices, along with their orders, go to $\mathcal{V}_0$ vertices while the ones in $\mathcal{V}_0$ go to $\mathcal{V}_{-1}$ vertices. This means that when we consider the negative part B' of the shifted diagram, the $\leq_r$ orders at $\mathcal{V}_1$ become $\leq'_s$ orders at $V_0(B')$ and the $\leq_s$ orders at $\mathcal{V}_0$ become $\leq'_r$ orders at $V_1(B')$.

Let Y be the (ordered) path space consisting of all oriented infinite paths starting from $V_1(B')$, the first level of vertices in B'. There is an order-preserving bijection between Y and X_{B^-}. Consider the adic transformation $T : X_{B'} \to X_{B'}$. For $x = (x_1, x_2, \dots) \in X_{B'}$, suppose that x_1 is maximal (with respect to the order $\leq'_r$ used to define the adic transformation). Then the map at the point x depends only on the tail starting at x_2, that is, on the path $(x_2, x_3, \dots)$. Therefore, the map here coincides with the adic transformation on X_{B^-} through the order-preserving conjugacy. In other words, the cutting and stacking operations dictated by B' (after the first stage) coincide with those of B^-.

Suppose for $x = (x_1, x_2, \dots)$, x_1 is not maximal. Then the adic transformation sends $x \mapsto (x_1 + 1, x_2, \dots)$. But the order $\leq'_r$ at $V_0(B')$ came from the $\leq_s$ order at $\mathcal{V}_0$, meaning that the order in which the columns are stacked in the first step of cutting and stacking for B' comes from the order in which we cut the rectangles (4.6) for $(\mathcal{B}, w^{\pm}, \leq_{r,s})$. Therefore, the geometry is compatible with the combinatorics of the first step of cutting and stacking. Since the cutting and stacking steps of B' (after the first stage) agree with those of B^-, the left/right edge identifications for $S(\sigma(\mathcal{B}, w^{\pm}, \leq_{r,s}))$ given by the limit map f' (obtained from the cutting-and-stacking operations) used to define them agree with with those obtained by deforming the surface and cutting and stacking, i.e., the ones for the surface $\mathcal{R}(S(\mathcal{B}, w^{\pm}, \leq_{r,s}))$. $\square$

Table 4.1: A diagram-translation surface dictionary.

Diagram $(\mathcal{B}, w^{\pm}, \leq_{r,s})$	Translation Surface $S = S(\mathcal{B}, w^{\pm}, \leq_{r,s})$
$\lvert\mathcal{V}_0\rvert$	Number of rectangles used to draw S
Positive part of $(\mathcal{B}, w^{\pm}, \leq_{r,s})$	geometry & dynamics of vertical translation flow on S
Negative part of $(\mathcal{B}, w^{\pm}, \leq_{r,s})$	geometry & dynamics of horizontal translation flow on S
A vertex $v \in \mathcal{V}_k$, $k \in \mathbb{N} \cup \{0\}$	A rectangular subset of S of width $w^+(v)$ and height $w^-(v)$ obtained from k steps of cutting and stacking.
Weight functions $w^{\pm}$	Transverse measures to vertical/horizontal foliations
Minimal/periodic components in the decomposition (4.2) of the positive (resp. negative) part of $(\mathcal{B}, w^{\pm}, \leq_{r,s})$	Minimal/periodic components of the vertical (resp. horizontal) translation flow on S.
Shift operator σ	Teichmüller deformation

The shift σ on a Bratteli diagram yields a sequence of surfaces

$$S_k(\mathcal{B}, w^{\pm}, \leq_{r,s}) := S(\sigma^k(\mathcal{B}, w^{\pm}, \leq_{r,s})) = (\mathcal{B}_k, w_k^{\pm}, \leq_{r,s}^k)$$

which are obtained as a shift on the starting Bratteli diagram B and by rescaling the weights in the shifted diagram by the appropriate quantities. We will denote by $\mathcal{R}_k$ the map satisfying

$$\mathcal{R}_k(S(\mathcal{B}, w^{\pm}, \leq_{r,s})) = S(\sigma^k(\mathcal{B}, w^{\pm}, \leq_{r,s})), \tag{4.14}$$

which is obtained through composition of maps of the type defined in (4.13). By Proposition 4.5.2, for each k the surface $S_k(\mathcal{B}, w^{\pm}, \leq_{r,s})$ is obtained by deforming $S(\mathcal{B}, w^{\pm}, \leq_{r,s})$ for time t_k and then cutting and stacking.

4.6 Dynamical properties of the translation flow

In this section we will exhibit flat surfaces whose translation flows exhibit a variety of phenomena which cannot occur for translation flows on flat surfaces of finite type. It is known that translation flows for compact flat surfaces are not mixing [Kat80], have zero topological entropy, and admit finitely many ergodic invariant measures [Vee78]. We show that these limitations do not apply to flat surfaces of infinite type and finite area. We show the existence of flat surfaces of infinite type and finite area whose translation flow is mixing (Corollary 4.6.2 of §4.6.1), flat surfaces whose translation flows have positive topological entropy (§4.6.2.3), and translation flows which are minimal and admit uncountably many ergodic invariant measures (§4.6.2.2). In fact, Theorem 4.6.1 shows that any finite entropy, ergodic aperiodic flow on a finite measure Lebesgue space can be realized by the translation flow of a flat surface.

4.6.1 The range of dynamical behaviors of translation flows

The main goal of this subsection is to establish the following theorem:

Theorem 4.6.1. *Let φ_t be a measurable ergodic aperiodic flow on a finite measure Lebesgue space (X, μ) with finite entropy $h(\varphi_1)$. Fix $p, q > 0$ such that p/q is irrational and $h(S_1) < \frac{2}{p+q}$. Then there exists an ordered, weighted Bratteli diagram $(\mathcal{B}, w^{\pm}, \leq_{r,s})$ with $|\mathcal{V}_0| = 2$ such that the vertical flow on $S(\mathcal{B}, w^{\pm}, \leq_{r,s})$ is isomorphic to φ_t.*

Theorem 4.6.1 shows that translation flows on flat surfaces of infinite type exhibit a wide range of measure-theoretic dynamical properties (in marked contrast to the much more restricted range of behaviors possible for finite type flat surfaces). In particular, for example, if we take φ_t to be the horocycle flow on the unit tangent bundle of a compact Riemann surface of constant negative curvature, we obtain the following result.

Corollary 4.6.2. *There exist translation flows on flat surfaces of infinite genus that are mixing.*

We note that although Corollary 4.6.2 proves that there exist mixing translation flows on surfaces on infinite type, at this time we do not know of any concrete example of a mixing transformation flow on a translation surface.

A *flow built under a function* is given by a quadruple (B, T, m, f), where B, the base, is a non-atomic Lebesgue space with measure m (either finite or σ-finite), T is a measure-preserving automorphism of B, and $f : B \to \mathbb{R}^+$ is an m-measurable map from B to $\mathbb{R}^+$ with $\sum_{i=0}^{\infty} f(T^i(b)) =$

∞ for all $b \in B$ and $\int_B f dm = 1$. On the set

$$\Omega = \{(b, x) : b \in B, 0 \leq x < f(b)\}$$

a measure is given by the restriction of the completed product measure $m \cdot \lambda$ to Ω, where λ denotes Lebesgue measure. The measure-preserving flow φ_t is defined by

$$\varphi_t(b, x) = \left(T^i(b), x + t - \sum_{j=0}^{i-1} f(T^j(b)) \right),$$

where i is the unique integer such that

$$\sum_{j=0}^{i-1} f(T^j(b)) \leq x + t < \sum_{j=0}^{i} f(T^j(b)).$$

Consider the simplest case, that of a flow built under a constant function on the unit interval with Lebesgue measure, say $([0, 1], T_1, \lambda, c)$. Clearly, if T_1 is measurably isomorphic to some other transformation T_2 of the base, then the flows $([0, 1], T_1, \lambda, c)$ and $([0, 1], T_2, \lambda, c)$ are measurably isomorphic.

Ambrose characterized the flows isomorphic to those built under a constant function: those whose corresponding unitary group has eigenfunction with nonzero eigenvalue ([Amb41]).

Theorem 4.6.3. *([AOW85]) Any measure-preserving automorphism of a unit measure Lebesgue space is measurably isomorphic to a cutting and stacking map on the unit interval with Lebesgue measure.*

Combining these two results we immediately yields a characterization of flows on flat surfaces built from a single rectangle according to our construction.

Corollary 4.6.4. *Let φ_t be a measure-preserving flow on a unit measure Lebesgue space. Then the flow φ_t is measurably isomorphic to the vertical flow on some surface $S(\mathcal{B}, w^{\pm}, \leq_{r,s})$ corresponding to a fully-ordered finite-weighted diagram $(\mathcal{B}, w^{\pm}, \leq_{r,s})$ with $|\mathcal{V}_0| = 1$ if and only if the unitary group corresponding to φ_t has an eigenfunction with nonzero eigenvalue.*

Ambrose ([Amb41]) proved that if φ_t is a measurable, measure-preserving ergodic flow on a Lebesgue space of finite measure, then there is a flow built under a function (B, T, m, f) such that

$m(B) < \infty$ and f is bounded strictly away from 0 and ∞ that is isomorphic to φ_t. A stronger version of this theorem proved by Rudolph ([Rud76]) shows that the function f can be chosen so that it takes on only two values, and the associated partition is generating:

Theorem 4.6.5. *([Rud76]) Let φ_t be a measurable ergodic aperiodic flow on a finite measure Lebesgue space (X, μ) with finite entropy $h(\varphi_1)$. Fix $p, q > 0$ such that p/q is irrational and $h(\varphi_1) < \frac{2}{p+q}$. Then there is a finite measure-preserving flow built under a function $(B, T, m, p\chi_P + q\chi_{P^c})$ with $m(B) = m(P \cup P^c) = 1$, that is isomorphic to φ_t, and (P, P^c) is a generating partition for T on B.*

We recall some definitions. For a measurable dynamical system $(X, \mathcal{A}, T, \mu)$ and a partition $Q = \{Q_1, \ldots, Q_k\}$ of X, the partition Q is said to be *generating* for T if $\bigvee_{j=0}^{\infty} T^{-j}(Q) = \mathcal{A}$. Given a Rohlin tower with height n and base B, and a finite partition $\mathcal{P}$, *purifying* the tower with respect to $\mathcal{P}$ means partitioning the base B into sets B_m, $1 \leq m \leq M$ such that for for all $0 \leq j < n$, the set $T^j(B_m)$ is contained within one set of the partition $\mathcal{P}$. The configuration $B_m, T(B_m), \ldots, T^{n-1}(B_m)$ is called a *pure* column with respect to $\mathcal{P}$.

We will use Theorem 4.6.5 to prove Theorem 4.6.1. Theorem 4.6.3 gives us that any flow satisfying the conditions of Theorem 4.6.5 is isomorphic to a flow built under a function where the base transformation is a cutting and stacking map and the function takes only the values p and q. However, we need the height function to be constant on each interval of the infinite interval exchange map on the base determined by the cutting and stacking map. The property in Theorem 4.6.5 that the partition (P, P^c) is generating for T enables us to make this assertion:

Proposition 4.6.6. *Let φ_t be a measurable ergodic aperiodic flow on a finite measure Lebesgue space with finite entropy $h(\varphi_1)$. Fix $p, q > 0$ such that p/q is irrational and $h(\varphi_1) < \frac{2}{p+q}$. Then there is a finite measure-preserving flow built under a function $(B, T, m, p\chi_P + q\chi_{P^c})$ such that $m(B) = m(P \cup P^c) = 1$, T is a cutting and stacking transformation on $[0, 1]$, and $p\chi_P + q\chi_{P^c}$ is constant on each interval of the infinite interval exchange map T on B associated to the cutting and stacking transformation.*

Proof. By Theorem 4.6.5, we may assume that φ_t is a flow built under a function

$$(B, T, m, p\chi_P + q\chi_{P^c})$$

such that $m(B) = m(P \cup P^c) = 1$ and (P, P^c) is generating for T. We need to show that T can be taken to be an infinite interval exchange transformation and that the height function can be taken to be constant on each interval of this interval exchange transformation. To do this, we now mimic the proof of Theorem 4.6.3 in [AOW85], with these goals in mind. Since φ_t is aperiodic, $(B, \mathcal{A}, m, T)$ is also an aperiodic measure-preserving dynamical system (here $\mathcal{A}$ denotes the σ-algebra m-measurable subsets of B).

Following [AOW85], we denote by $\mathcal{T}(L, n, \epsilon)$ a Rohlin tower of height n and base L with residual set of measure ϵ, so that the sets

$$L, T(L), \ldots, T^{n-1}(L)$$

are disjoint and $\mu(|\mathcal{T}|) = 1 - \epsilon$, where we define $|\mathcal{T}(L, n, \epsilon)| := \bigcup_{j=0}^{n-1} T^j(L)$. We fix a sequence of Rohlin towers $\{\mathcal{T}(L_j, n_j, \epsilon_j)\}_{n\in\mathbb{N}}$ such that $n_j \nearrow \infty$, $\epsilon \searrow 0$, and for each $j \in \mathbb{N}$,

$$\left(|\mathcal{T}_{j+1}| \setminus (L_j \cup T^{n_j-1}(L_j))\right) \supset |\mathcal{T}_j|,$$

where L_j denotes the base level of $\mathcal{T}_j$. That such a sequence of Rohlin towers exists is proven in [AOW85].

Define a sequence of partitions $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ of B by setting $\mathcal{P}_1 = (P, P^c)$ and for $i > 1$ setting

$$\mathcal{P}_i = \left(\bigvee_{j=1}^{i-1} (|\mathcal{T}_j|, |\mathcal{T}_j|^c)\right) \vee \left(\bigvee_{j=0}^{i-1} \left(T^{-j}(P), T^{-j}(P^c)\right)\right).$$

Each $\mathcal{P}_i$ is a finite partition of B, $\mathcal{P}_{i+1}$ is a refinement of $\mathcal{P}_i$, and $\bigvee_{i=1}^{\infty} \mathcal{P}_i = \mathcal{A}$ because (P, P^c) is generating for T.

We will now use the Rohlin towers $\mathcal{T}_i$ to define a cutting and stacking process on a real interval $I = [0, 1]$. We will use S_i to denote the i^{th} stack in the process.

To determine the stack S_1, we first purify the Rohlin tower $\mathcal{T}_1$ with respect to $\mathcal{P}_1$. In other words, partition L_1 into maximal sets $L_{1,1}, \ldots, L_{1,M_1}$ so that for all $0 \le n \le n_1$ and all $1 \le k \le M_1$, the set $T^n(L_{1,k})$ lies entirely within a single set of $\mathcal{P}_1$. For each $1 \le k \le M_1$, form a column $C_{1,k}$

in S_1 of height n_1 and width $m(L_{1,k})$. Thus S_1 consists of M_1 columns, each with n_1 levels, and there is a measure-preserving bijection (call the bijection ϕ_1) between the set of levels of S_1 and the set of levels of the $\mathcal{P}_1$-pure columns comprising $\mathcal{T}_1$. Furthermore, ϕ_1 preserves the property of one level being immediately above another level, i.e. a level A_1 is immediately above a level A_2 in a $\mathcal{P}_1$-pure column of $\mathcal{T}_1$ if and only if $\phi_1(A_1)$ is immediately above $\phi(A_2)$ in a column of S_1.

To determine the stack S_2, we first purify the Rohlin tower $\mathcal{T}_2$ with respect to $\mathcal{P}_2$. This determines a partition of L_2 into maximal sets $L_{2,1}, \ldots, L_{1,M_2}$ so that for all $0 \leq n \leq n_2$ and all $1 \leq k \leq M_2$, the set $T^n(L_{2,k})$ lies entirely within a single set of $\mathcal{P}_2$. Each $\mathcal{P}_2$-pure column in $\mathcal{T}_2$ contains "blocks" from $|\mathcal{T}_1|$ (i.e. n_1 successive levels consisting entirely of points from $\mathcal{T}_1$), as well as levels contained in $|\mathcal{T}_1|^c$. Since $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$, each of these"blocks" is a subset of a single $\mathcal{P}_1$-column in $\mathcal{T}_1$. The condition that

$$|\mathcal{T}_2| \setminus (L_2 \cup T^{n_2-1}(L_2)) \supset |\mathcal{T}_1|$$

ensures that each block is a whole n_1 levels – it does not run off the top or bottom of the $\mathcal{P}_2$-pure column. For each $1 \leq k \leq M_2$, we will form a column $C_{2,k}$ in S_2 of height n_2 and width $m(L_{1,k})$ as follows. For each level of the $\mathcal{P}_2$-pure column over $L_{2,k}$ that is contained in $\mathcal{T}_1^c$, add a "spacer" level of width $m(L_{2,k})$, and for each "block" of levels which is a subset of a $\mathcal{P}_1$-pure column of $\mathcal{T}_1$, cut and use a width-$m(L_{2,k})$ subcolumn of the column in S_1 corresponding via ϕ_1 to that $\mathcal{P}_1$-pure column of $\mathcal{T}_1$. Thus S_2 consists of M_2 columns, each of height n_2, and there is a measure-preserving bijection between the set of levels of S_2 and the set of levels of the $\mathcal{P}_2$-pure columns comprising $\mathcal{T}_2$. Furthermore, the bijection ϕ_2 preserves the property of one level being immediately over another level. We note also that S_2 is obtained from S_1 via cutting and stacking.

This procedure is repeated inductively, yielding a sequence of stacks S_i obtained via cutting and stacking such that S_i consists of M_i columns, each of height n_i, and there is a measure-preserving bijection ϕ_i between the set of levels of S_i and the set of levels of the $\mathcal{P}_i$-pure columns comprising $\mathcal{T}_i$, and ϕ_i preserves the property of one level being immediately above another level. Define R to be the limiting map on I defined by the cutting and stacking process.

The fact that $\bigvee_i \mathcal{P}_i = \mathcal{A}$, together with the maps ϕ_i, determines an measure-preserving embedding of $\mathcal{A}$ into the σ-algebra of Borel subsets of the interval I with Lebesgue measure. The measure space $(B, \mathcal{A}, m)$ is a standard Lebesgue space. As such, each point $x \in B$ is the intersection of a decreasing sequence of sets $\{C_i\}_{i\in\mathbb{N}}$ where C_i is a level of a pure column comprising $\mathcal{T}_i$. The sequence $\{\phi_i(C_i)\}_{i\in\mathbb{N}}$ is a decreasing sequence of nested intervals, say $I_1, I_2, \ldots$, whose intersection $\cap_{i=1}^{\infty}$ has measure 0, and is thus a point $a \in I$. We can thus define $\phi(x) = a$. The map $\phi : B \to I$ is thus a measurable, measure-preserving isomorphism between the systems $(B, \mathcal{A}, m, T)$ and $(I, \mathcal{B}, \lambda, R)$, where $\mathcal{B}$ denotes the Borel σ-algebra on I.

The stacks S_i have been constructed so that each level of each stack is in bijection with some level of a $\mathcal{P}_j$-pure column, for some j. Define a function f whose domain is the collection of levels of the columns S_i for all i, by $f(A) = (p\chi_P + q\chi_{P^c})(\phi_j^{-1}(A))$ for a level A in stack S_j. (The function f is well-defined because $p\chi_P + q\chi_{P^c}$ is constant on the set $\phi_j^{-1}(A)$, since it is a level of a $\mathcal{P}_j$-pure column, and hence contained in either P or P^c.) It follows immediately that the flows built under functions $(T, B, m, p\chi_P + q\chi_{P^c})$ and (R, I, λ, f) are isomorphic, and (R, I, λ, f) is the desired system. $\square$

Lemma 4.6.7. *The sets P and P^c in the statement of Proposition 4.6.6 may be assumed to each consist of a single subinterval of B.*

Proof. It did not matter which specific subintervals of I were used to form the stacks S_i in the proof of Proposition 4.6.6. The inductive nature of the construction of the stacks S_i provides an order on the intervals – we simply order them according to the order in which we first use an interval when performing the cutting and stacking process. Thus, we might, for example, take the intervals on which f takes the value p starting from the left side of I, and take the intervals on which f takes the value q starting from the right side of I, and work toward the middle. $\square$

The proof of Proposition 4.6.6 yields the construction of a cutting and stacking map from an measure preserving transformation. Each one of the steps of this construction can be recorded as the positive part of an ordered Bratteli diagram $\mathcal{B}$. Combining Proposition 4.6.6 and Lemma 4.6.7 immediately proves Theorem 4.6.1.

Remark 4.6.8. *Various theorems strengthening Ambrose's initial result exist. Ambrose and Kakutani ([AK42]) weakened the requirement of ergodicity; they called a flow φ* proper *if there is no measurable* *set A with $\mu(A) > 0$ such that for all measurable $A' \subset A$ and all t the symmetric difference $A' \triangle \varphi_t A'$ has μ-measure 0. They proved that any proper flow can be realized by (i.e. is measurably isomorphic to) a flow built under a function (not bounded below) if one allows m to be σ-finite.*

A sequence of two papers by first Rudolph ([Rud76]) and then Krengel ([Kre76]) strengthened this theorem so that the height function f can be chosen to take on only two prescribed values and with prescribed frequency:

Theorem 4.6.9. *([Kre76]) Let φ_t be a measurable, measure-preserving aperiodic flow on a finite measure Lebesgue space (Σ, μ), and fix $p, q > 0$ with p/q irrational and $0 < \rho < \infty$. Then there exists a quadruple (B, T, m, f) as above such that m is finite, f takes only the values p and q, $m(\{b \in B : f(b) = p\}) = \rho \cdot m(\{b \in B : f(b) = q\})$ and such that φ_t is isomorphic to the flow built under (B, T, m, f).*

As observed in [Kre76], this result implies that all aperiodic measure-preserving flows on Lebesgue spaces of measure 1 admit a representation on the same Ω and differ only by the automorphism T of the base. However, Krengel's theorem does not touch on the question of whether the two-set partition determined by p and q generates the σ-algebra of measurable sets; our proof of Proposition 4.6.6 requirement that the partition be generating.

4.6.2 Examples of surfaces which exhibit certain dynamical properties

4.6.2.1 Suspensions of staircase transformations

We mention a specific class of examples of cutting and stacking transformations which are known to be mixing. Informally, a staircase transformation is a cutting and stacking transformation with a sequence $\{r_n\}$ of natural numbers with $r_n \to \infty$ as $n \to \infty$ so that at the n^{th} stage the n^{th} stack, which consists of a single column, is cut into r_n subcolumns $\{c_{n,1}, \ldots, c_{n,r_n}\}$ of equal widths and $s_{n,j}$ spacers are added over the subcolumn $c_{n,j}$ before stacking the subcolumns in order from left to right. In this setup it is always assumed that the sequences r_n, $s_{n,j}$ are such that the limiting transformation is defined over a set of finite measure. The first staircase transformation explicitly shown to be mixing [Ada98] was Smorodinsky's staircase, where $r_n = n + 1$ and $s_{n,j} = j$, although Ornstein had proved that for a family of staircases defined over some parameter space, the typ-

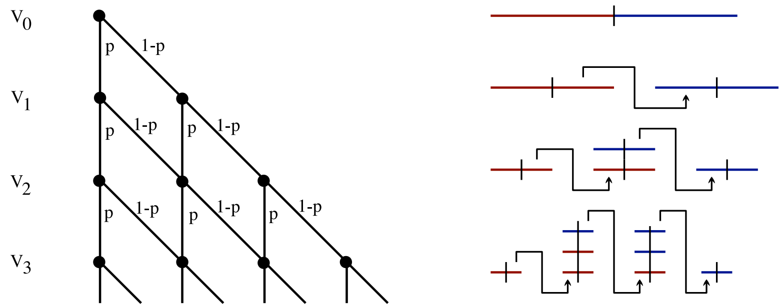

Figure 4.7: The first parts of the Bratteli diagram and cutting-and-stacking steps for the Pascal adic transformation.

ical staircase transformation is mixing [Orn72]. In [CS10] there is a characterization of mixing staircase transformations in terms of uniform convergence of certain averages of partial sums of the spacer sequence $\{s_{n,j}\}$. In particular, the authors show that all polynomial staircase transformations (roughly, $s_{n,j}$ is a polynomial $p_n(j)$, where the degree and coefficients of p_n are bounded uniformly for all n with some additional conditions) are mixing.

Since staircase transformations use spacers, they can be encoded in Bratteli diagrams $(B, w^+, \leq_{r,s}$ $)$ where each level has two vertices, one corresponding to the spacers, one for the rest (see §4.4.2.2 for a concrete example).

4.6.2.2 The Pascal adic transformation

Let $c_i = i + 1$ for each integer $i \geq 0$. (Recall $c_i := |V_i|$.) Denote the elements of V_i by $v_1^i, \ldots, v_{c_i}^i$ and order them accordingly. For each $i \in \mathbb{N}$, define the incidence matrix $F_i = [f_{v,w}^i]$ to be the $c_i \times c_{(i-1)}$ matrix with entries

$$f^i_{v_j^{i-1}, v_k^i} = \begin{cases} 1 & \text{if } k = j \text{ or } k = j+1 \\ 0 & \text{otherwise} \end{cases}$$

Define the partial orders $\leq_s$ and $\leq_r$ as indicated in Figure 4.7, using the left-right ordering convention described in §4.4.1.3. The Vershik map $T : X \setminus X_{max} \to X \setminus X_{min}$ is called the Pascal adic transformation.

For any $p \in (0, 1)$, define a weight function ω_p on the set of edges by

$$\omega_p(e) = \begin{cases} p & \text{if } k = j \\ (1-p) & \text{if } k = j+1 \end{cases}$$

where j and k are defined by $S(e) = v_j^{i-1}$ and $R(e) = v_k^i$.

It is well known that the invariant ergodic Borel probability measures for the Pascal adic transformation are precisely the measures ω_p (which are called Bernoulli measures) (see [MP05]). In fact, the Pascal adic transformation is totally ergodic (every power T^n is ergodic) for each ω_p. (However, whether or not the Pascal adic transformation is weak mixing is an open question.)

Consider the flat surface $S(\mathcal{B}, w^{\pm}, \leq_{r,s})$ where the positive part of $(\mathcal{B}, \leq_{r,s})$ coinsides with the Pascal Bratteli diagram in Figure 4.7. Then by the remark above, the translation flow on this surface is minimal and has uncountably many ergodic, finite invariant measures. This phenomenon (having infinitely many ergodic invariant probability measures) does not occur for translation flows on surfaces of finite type.

4.6.2.3 Chaotic translation flows

We will review the concept of *independent cutting and stacking*, considered in [Shi73] to construct cutting and stacking transformations with chaotic properties. Consider a collection of columns of intervals $C_0 = \{C^1, \dots, C^q\}$, where each C^i is a ordered collection of $h(C^i)$ intervals of the same width $w(C^i)$, each one stacked on top of the previous with the condition that $\sum_i h(C^i)w(C^i) = 1$. Denote by $C^i * C^j$ the stacking of column C^j on top of column C^i (for which it is necessary that $w(C^i) = w(C^j)$). We will denote $w(C_0) := \sum_i w(C^i)$.

Independent stacking is done as follows. Starting with the q intervals (columns of height 1) $C_0 = \{C^1, \dots, C^q\}$ of the same width, cut each column C^i into $2q$ subtowers C^i_j, $j = 1, \dots, 2q$, and stack them into q^2 towers $C_1 = \{\hat{C}^{i,j}\}_{i,j\leq q}$ by

$$\hat{C}^{i,j} = C^i_j * C^j_{q+i} \qquad 1 \leq i \leq q \qquad 1 \leq j \leq q.$$

Note that $\sum_l w(C^l) = 2\sum_{i,j} w(\hat{C}^{i,j})$. Iterating the independent cutting and stacking procedure, we obtain a sequence of collections of towers C_k where a map is defined on all but the top levels.

This map limits to a piecewise isometry of the unit interval $T(C_0)$, since the iterative construction depends on the starting tower C_0.

Let $\mathcal{P}^n$ be the partition of the unit interval making up C_n. The following summarizes the properties of transformations obtained through independent cutting and stacking.

Theorem 4.6.10 ([Shi73]). *The following hold for independent cutting and stacking.*

1. *$\mathcal{P}^n$ is a Markov partition for $T(C_0)$.*

2. *If two columns of some C_k have height which differ by 1 or, more generally, if for some k the greatest common divisor of the heights of the columns of k is 1, then $T(C_0)$ is mixing, and therefore Bernoulli.*

3. *The topological entropy of $T(C_0)$ is $w(C_0)\log(q_0)$, where q_0 is the number of towers in C_0.*

Any independent cutting and stacking procedure can be written as a weighted, ordered Bratteli diagram $(B^+, w^-, \leq^+_{r,s})$.

Corollary 4.6.11. *Let $(\mathcal{B}, w^\pm, \leq_{r,s})$ be a Bratteli diagram whose positive part is realized by an independent cutting and stacking transformation and negative part given by the $|C_0| \times |C_0|$ identity matrix. Then the vertical flow on $S(\mathcal{B}, w^\pm, \leq_{r,s})$ has topological entropy $w(C_0)\log(q_0)$, where $q_0 = |C_0|$ is the number of towers in C_0.*

4.6.2.4 The Hajian-Kakutani skyscraper

Cutting and stacking can also be used to define ergodic transformations on infinite intervals. The Hajian-Kakutani Skyscraper is an example of an infinite measure-preserving, invertible, rank-one, ergodic transformation. We briefly review the cutting and stacking procedure to define this transformation.

Starting with the interval $[0,1)$ (our zeroth tower), we cut it into two intervals of equal length, place 2 spacers over the second interval, and define the first map as the "moving up one level" linear map on the first tower obtained by stacking $[\frac{1}{2}, 1)$ and the two spacers above it above $[0, \frac{1}{2})$. The map is now defined on $[0, \frac{3}{2})$. Proceding inductively, we may define the $(k+1)^{st}$ map by cutting the k^{th} tower into two subtowers of equal width, adding 2^k spacers above the second subtower, and

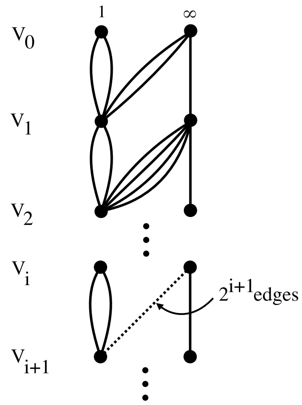

Figure 4.8: The Bratteli diagram corresponding to the Hajian-Kakutani skyscraper.

stacking that over the first subcolumn. It can be easily checked that the k^{th} map is defined on $4^k - 1$ intervals of length 2^{-k}. Therefore, the limiting map is defined on $[0, \infty)$.

Let us now describe this transformation using an ordered Bratteli diagram. See Figure 4.8, where the weight function on each edge emanating from each left-most vertex is $1/2$. By Proposition 4.2.14, there is a *probability* measure which is invariant for the tail equivalence relation for the corresponding diagram. In fact, there is a unique probability invariant measure – the atomic measure supported entirely on the point x_∞, by which we denote the point in the diagram corresponding to the path passing through the rightmost vertex at every level. From the point of view of cutting and stacking, the point x_∞ is artificial – it represents the "fake" column (interval) from which we take the spacers used in the cutting and stacking process.

Although the Hajian-Kakutani skyscraper is thus naturally an infinite measure preserving transformation, it is closely linked to a previously discussed finite measure preserving transformation: the dyadic odometer. On the interval $[0, 1)$, the first return map to $[0, 1)$ is precisely the dyadic odometer – thus, the Hajian-Kakutani skyscraper is just the dyadic odometer with extra spacers stuck in. Consequently, since the dyadic odometer is ergodic, the Hajian-Kakutani skyscraper is also ergodic. (See [HK70] for the original construction, and for example, the appendix in [AP00] for background on skyscrapers in general.)

Although an investigation of arbitrary infinite type translation surfaces of infinite area is beyond the scope of this work, this example suggests how some such surfaces could be constructed from infinite σ-finite measure-preserving transformations. By controlling the dynamics of the finite measure base transformation of a skyscraper, we may control the dynamical properties of the vertical flow on a infinite measure, infinite type translation surface.

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