LAMINATIONS ON THE CIRCLE AND HYPERBOLIC GEOMETRY

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We develop a program studying group actions on the circle with dense invariant laminations. Such actions naturally and frequently arise in the study of hyperbolic manifolds. We use the convergence group theorem to prove that for a discrete torsion-free group acting on the circle, it is topologically conjugate to a Möbius group if and only if it admits three very-full invariant laminations with a certain transversality condition. We also discuss the case when a group acts on the circle with two very-full invariant laminations with disjoint endpoints. We show that such a group shares many interesting features with the fundamental group of a hyperbolic 3-manifold which fibers over the circle.

BIOGRAPHICAL SKETCH

Hyungryul Baik was born in Seoul, South Korea on December 17, 1985. He received his Bachelor of Science degree in Mathematical Science from KAIST (Korea Advanced Institute of Science and Technology) in 2009 and in the same year, he joined the doctoral program in Mathematics at Cornell University. In 2011, he passed his A-exam and received a Master of Science degree in Mathematics in 2012 from Cornell University.

This thesis is dedicated to Bill Thurston whose teaching and inspirations will live with me forever.

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CHAPTER 1

INTRODUCTION

Thurston-Perelman's geometrization theorem says that every closed orientable 3-manifold can be split along a finite collection of disjoint spheres and tori into pieces with one of eight very nice geometric structures after filling in the resulting boundary spheres with balls; see [17] for the eight geometric structures. Among the eight geometric structures, Thurston taught us that hyperbolic geometry is the most important one to study (most 3-manifolds admit hyperbolic structures in a suitable sense). Since Thurston revolutionized the field, many people have developed the various ways of understanding topology and geometry of 3-manifolds. However, we still do not have a coherent picture of how those different understandings fit together.

In [18], Thurston proposed a way of combining the theory of foliations in 3-manifolds and 3-manifold group actions on 1-manifolds (especially, the circle). In particular, he showed that if an atoroidal 3-manifold M admits a cooriented taut foliation $\mathcal{F}$, then $\pi_1(M)$ acts faithfully on the circle by orientation-preserving homeomorphisms (called a universal circle for $\mathcal{F}$) with a pair of transverse dense invariant laminations. Here, a lamination on the circle is a set of pairs of points which can be obtained as endpoints of leaves of a geodesic lamination on the hyperbolic plane. This suggests a potential connection between hyperbolic geometry in low-dimensional manifolds and group actions on the circle with invariant laminations. In this paper, we show that there is indeed a close connection between these two subjects. Most notably, we give an alternative characterization of Fuchsian groups. For definitions, see Chapter 2 and 3.

Theorem (Main Theorem (weak version)) *Let G be a torsion-free discrete subgroup of $Homeo_+(S^1)$. Then G is conjugate to a Fuchsian group H such that $\mathbb{H}^2/H$ has no cusps if and only if G admits three very-full laminations with disjoint endpoints.*

Theorem (Main Theorem (strong version)) *Let G be a torsion-free discrete subgroup of $Homeo_+(S^1)$. Then G is conjugate to a Fuchsian group H such that $\mathbb{H}^2/H$ is not the thrice-punctured sphere if and only if G admits a pants-like collection of three very-full laminations.*

In fact, Fuchsian groups have infinitely many very-full invariant laminations. Hence, our main theorem shows that once a group has three very-full laminations, then it has infinitely many. There is another interesting class of groups from low-dimensional topology which have two very-full invariant laminations but no third lamination. Let M be a hyperbolic 3-manifolds which fibers over the circle. Then M is a suspension of a pseudo-Anosov surface diffeomorphism. In this case, $\pi_1(M)$ acts faithfully on the circle with two invariant laminations - stable and unstable laminations of the pseudo-Anosov monodromy. In fact, this is a special case of Thurston's theorem for tautly foliated 3-manifold groups, and the invariant laminations coming with a universal circle are not very-full in general. There are some evidences that this special case can be completely characterized by a pair of very-full invariant laminations in a similar flavor with the Main Theorem.

Conjecture *Let G be a finitely generated torsion-free discrete subgroup of $Homeo_+(S^1)$. Then the followings are equivalent.*

1. *G is a closed hyperbolic 3-manifold group.*
2. *G virtually admits two very-full loose invariant laminations with disjoint endpoints (but cannot have three of such laminations at the same time).*

In the above conjecture, 1. implies 2. due to the recent resolution of virtual fibering conjecture of Ian Agol.

The rest of this paper is organized in the following way. In Chapter 2, we will review the theory of universal circles for taut foliations which motivates the study of group actions with invariant laminations. In Chapter 3, we see some examples of groups acting on the circle with invariant laminations, and make the notion of a laminar group precise. In Chapter 4, we study laminations invariant under Fuchsian group actions on the ideal boundary of the hyperbolic plane. In Chapter 5, we give a complete characterization of Fuchsian groups in terms of topological invariant laminations. In Chapter 6, we look for a characterization of fibered 3-manifold groups which is analogous to the main theorem of Chapter 5.

CHAPTER 2

UNIVERSAL CIRCLE AND LAMINATIONS ON THE CIRCLE

Thurston [18], Calegari-Dunfield [5], and Calegari [4] developed the theory of universal circles for 3-manifold with taut foliations (or essential laminations with additional assumptions). This provides a large family of 3-manifold groups which act on the circle with a pair of transverse dense invariant laminations and motives a study of invariant laminations on the circle for group actions in the context of low-dimensional topology. In this chapter, we give a quick review on the relevant part of the subject. A comprehensive treatment can be found in the book by Calegari [4].

2.1 Leafwise Hyperbolic Structure and a Master Circle

Suppose M is a closed orientable connected 3-manifold. Since $\chi(M)$ is zero, M admits a smooth codimentions-1 foliation.

Definition 2.1.1 *A smooth codimension-1 foliation $\mathcal{F}$ in the 3-manifold M is said to be* ***taut*** *if there exists a closed embedded loop γ in M transverse to $\mathcal{F}$ which meets every leaf of $\mathcal{F}$ at least once.*

Throughout the paper, we will only consider orientable, connected 3-manifolds and cooriented taut foliations. We have the following theorem by the work of Novikov [15] and improvements by Rosenberg [16].

Theorem 2.1.2 (Novikov, Resenberg) *Let $\mathcal{F}$ be a taut foliation of M. Suppose M is not finitely covered by $S^1 \times S^2$. Then the following properties are satisfied.*

1. *M is irreducible.*
2. *Leaves are incompressible; ie., the inclusion $\lambda \to M$ induces a monomorphism $\pi_1(\lambda) \to \pi_1(M)$.*
3. *Every loop γ transverse to $\mathcal{F}$ is essential in $\pi_1(M)$.*

In particular, the universal cover $\widetilde{M}$ of such a manifold M is homeomorphic to $\mathbb{R}^3$. For a foliation $\mathcal{F}$ in M, let $\widetilde{\mathcal{F}}$ be the foliation in $\widetilde{M}$ which covers $\mathcal{F}$.

Corollary 2.1.3 *Let M admit a taut foliation $\mathcal{F}$, and suppose M is not covered by $S^1 \times S^2$. Then $\pi_1(M)$ is infinite, and every leaf of $\widetilde{\mathcal{F}}$ is properly embedded plane in $\widetilde{M}$.*

On the other hand, one of the consequences of Candel's uniformization theorem [7] for Riemann surface laminations is that if M is atoroidal and admits a taut foliation $\mathcal{F}$, then there exists a Riemannian metric on M which is leafwise hyperbolic, ie., the restriction of the metric to each leaf of $\mathcal{F}$ is a metric of constant curvature -1. With respect to such a metric, each leaf of the covering foliation $\widetilde{\mathcal{F}}$ in the universal cover $\widetilde{M}$ becomes a copy of the hyperbolic plane. In particular, each leaf λ has an associated ideal circle at infinity $S^1_\infty(\lambda)$. Note that the deck transformation action of $\pi_1(M)$ on $\widetilde{M}$ maps leaves of $\widetilde{\mathcal{F}}$ to leaves of $\widetilde{\mathcal{F}}$, and it induces maps between corresponding circles at infinity. Thurston [18] constructed one master circle which "contains" all the circles at infinity from the leaves of $\widetilde{\mathcal{F}}$ and $\pi_1(M)$-action on that master circle which is compatible with the action on circles at infinity. Here, the "containment" can be made more precise by saying there exists a monotone map from the master circle to each circle at infinity.

Definition 2.1.4 *Let S^1_X, S^1_Y be homeomorphic to S^1. A continuous map $\phi : S^1_X \to S^1_Y$ is called* ***monotone*** *if the preimage of each point of S^1 under f is connected and* *contractible. For such a map ϕ, the* ***gaps*** *of ϕ are the maximal open connected intervals in S^1_X in the preimage of single points of S^1_Y. The* ***core*** *of ϕ is the complement of the union of the gaps.*

Definition 2.1.5 (Universal Circle) *Let $\mathcal{F}$ be a taut foliation of an atoroidal 3-manifold M. A* ***universal circle*** *for $\mathcal{F}$ is a circle S^1_{univ} together with the following data:*

1. *There is a faithful representation*

$$\rho_{univ} : \pi_1(M) \to \mathrm{Homeo}_+(S^1_{univ})$$

2. *For every leaf λ of $\widetilde{\mathcal{F}}$, there is a monotone map*

$$\phi_\lambda : S^1_{univ} \to S^1_\infty(\lambda)$$

3. *For every leaf λ of $\widetilde{\mathcal{F}}$ and every $\alpha \in \pi_1(M)$ the following diagram commutes:*

$$\begin{tikzcd} S^1_{univ} \arrow[r, "\rho_{univ}(\alpha)"] \arrow[d, "\phi_\lambda"'] & S^1_{univ} \arrow[d, "\phi_{\alpha(\lambda)}"'] \\ S^1_\infty(\lambda) \arrow[r, "\alpha"] & S^1_\infty(\alpha(\lambda)) \end{tikzcd}$$

4. *If λ and μ are incomparable leaves of $\widetilde{\mathcal{F}}$ then the core of ϕ_λ is contained in the closure of a single gap of ϕ_μ and vice versa.*

For now, let's ignore the condition 4. in the definition. Incomparability will be properly defined in Section 2.3.

Theorem 2.1.6 (Thurston [18], Calegari-Dunfield [5]) *Let $\mathcal{F}$ be a cooriented taut foliation of an atoroidal, oriented 3-manifold. Then there is a universal circle for $\mathcal{F}$.*

2.2 Laminations on the Circle

A *geodesic lamination* on a Riemannian manifold X is a closed union of disjoint geodesics of X. Recall the ideal boundary $\partial\mathbb{H}^2$ of the hyperbolic plane $\mathbb{H}^2$ is topologically a circle. Each geodesic in $\mathbb{H}^2$ determines two points on the ideal boundary. Hence, forgetting what is in the hyperbolic plane but remembering only what is in the ideal boundary, a geodesic lamination of $\mathbb{H}^2$ gives a set of pairs of points on the circle $\partial\mathbb{H}^2$ with a certain compatibility condition. A lamination on the circle is a set of pairs of points on the circle which can be obtained as an endpoint data of a geodesic lamination of $\mathbb{H}^2$ in this fashion.

To be more precise, let's start with the set of all unordered pairs of points of the circle S^1, denote by $\mathcal{M}$. Note that $\mathcal{M}$ is topologically an open Möbius band. The following compatibility condition mimics the disjointness of two geodesics in $\mathbb{H}^2$.

Definition 2.2.1 *Two pairs (a, b) and (c, d) of points of S^1 are called **linked** if each connected component of $S^1 \setminus \{a, b\}$ contains precisely one of the points c and d. Otherwise, the two pairs are called **unlinked**.*

There is a geometric interpretation of the linking condition. Identify S^1 with the boundary of a closed disk. Then two pairs (a, b) and (c, d) of points on S^1 are linked if and only if the chord connecting (a, b) intersects the chord connecting (c, d) in the interior of the disk. Now we give a precise definition of a lamination on the circle. By definition, if a, b, c are three distinct points on the circle, then the pairs (a, b) and (a, c) are unlinked.

Definition 2.2.2 *A **lamination** Λ on the circle is a closed subset of $\mathcal{M}$ whose elements are pairwise unlinked. Elements of Λ are called **leaves** of Λ, and for a leaf $l = (a, b)$ of*

*Λ, the points a, b are called **endpoints** of the leaf l. The set of endpoints of all leaves of Λ is denoted by* $\text{ends}(\Lambda)$.

Note that we can get a geodesic lamination of $\mathbb{H}^2$ from a lamination Λ on S^1. First, identify the circle S^1 with the ideal boundary $\partial\mathbb{H}^2$ of the hyperbolic plane, and then for each element of the lamination, add a geodesic connecting the endpoints. For a given lamination Λ on S^1, we call a geodesic lamination of $\mathbb{H}^2$ obtained this way *a geometric realization* of Λ. A geometric realization is unique up to isotopy of the closed disk relative to the boundary. Hence we may call it *the geometric realization* of Λ.

Sometimes we need to use the term lamination in a slightly more general sense. More precisely, sometimes a lamination Λ on the circle is a subset of $\overline{\mathcal{M}}$ such that $\Lambda \cap \mathcal{M}$ is a lamination in the usual sense. The elements in $\Lambda \cap \partial\overline{\mathcal{M}}$ are called **degenerate leaves** of Λ. Whenever we say a leaf without a modifier, it always means a non-degenerate leaf.

Definition 2.2.3 *A **complementary region** (or a **gap**) of a geodesic lamination L of the hyperbolic plane $\mathbb{H}^2$ is the metric completion of a connected component of $\mathbb{H}^2 \setminus L$ (with respect to the path metric on the component). A **complementary region** (or a **gap**) of a lamination Λ on the circle is a gap of the geometric realization of Λ.*

Definition 2.2.4 *A lamination Λ on the circle S^1 is called*

- ***dense** if the set* $\text{ends}(\Lambda)$ *is dense in S^1,*
- ***very-full** if each gap of Λ is a finite-sided ideal polygon (in other words, each connected component of the complement of the geometric realization of Λ in $\mathbb{H}^2$ has finite area),*

- ***boundary-full*** *if the closure of Λ in $\overline{\mathcal{M}}$ contains the entire $\partial\overline{\mathcal{M}}$,*

- ***loose*** *if whenever two leaves of Λ share a common endpoint, they must be boundary leaves of a single gap of Λ.*

2.3 Laminations for A Universal Circle

In Section 2.1, we saw that a tautly foliated atoroidal 3-manifold group admits a faithful action on the circle which comes from a leafwise hyperbolic metric. On a universal circle S^1_{univ} for a cooriented taut foliation in a 3-manifold, there exists a pair of laminations naturally constructed from the structure of the foliation.

Let M be an atoroidal 3-manifold and $\mathcal{F}$ be a coorientated taut foliation in M. Let $\widetilde{\mathcal{F}}$ be the foliation in the universal cover $\widetilde{M}$ of M covering $\mathcal{F}$, and $L = L(\widetilde{\mathcal{F}}$ be the leaf space of $\widetilde{\mathcal{F}}$. L is a simply connected (non-Hausdorff) 1-manifold. A coorientation on $\mathcal{F}$ determines a coorientation on $\widetilde{\mathcal{F}}$, and therefore an orientation on embedded intervals contained in L. Now we define a partial order $<$ on L. For two leaves λ, μ in L, we say $\lambda < \mu$ if and only if there exists an oriented embedded interval in L from λ to μ. We say λ and μ are incomparable if we have neither $\lambda < \mu$ nor $\mu < \lambda$ (now recall the definition of a universal circle).

Definition 2.3.1 (Branching of a cooriented taut foliation) *Let $M, \mathcal{F}, \widetilde{\mathcal{F}}, L$ be as above. Then $\mathcal{F}$ is said to*

1. *be* ***$\mathbb{R}$-covered*** *if $L = \mathbb{R}$.*
2. *have* ***one-sided branching in the positive direction*** *if for any $\lambda_1, \lambda_2 \in L$, there exists $\mu \in L$ such that $\mu < \lambda_1$ and $\mu < \lambda_2$.*

3. *have **one-sided branching in the negative direction** if for any $\lambda_1, \lambda_2 \in L$, there exists $\mu \in L$ such that $\lambda_1 < \mu$ and $\lambda_2 < \mu$.*

4. *have **one-sided branching** if it has either one-sided branching in the positive direction or one-sided branching in the negative direction.*

5. *have **two-sided branching** if it is not $\mathbb{R}$-covered, and does not have one-sided branching.*

In Section 2.3.1, we show that if $\mathcal{F}$ has two-sided branching, then the structure of the leaf space $L = L(\widetilde{\mathcal{F}})$ determines a pair of dense invariant laminations. Constructing the laminations is a bit tricky when $\mathcal{F}$ has at most one-sided branching. In Section 2.3.2, we introduce another way of showing the existence of a pair of dense invariant laminations using pseudo-Anosov flow. We give only brief outlines for both cases. The details can be found from Chapter 8 and Chapter 9 of [4].

2.3.1 Laminations on a universal circle for a taut foliation with two-sided branching

Let $M, \mathcal{F}, \widetilde{\mathcal{F}}, L$ be as before, and assume $\mathcal{F}$ has two-sided branching. While a universal circle for $\mathcal{F}$ is not unique, we could consider minimal ones.

Definition 2.3.2 *A universal circle is **minimal** if for any distinct $p, q \in S^1_{univ}$, there exists a leaf λ of $\widetilde{\mathcal{F}}$ such that $\phi_\lambda(p) \neq \phi_\lambda(q)$ where ϕ_λ is the monotone map appearing in the definition of a universal circle.*

It is easy to see that if S^1_{univ} is a universal circle for $\mathcal{F}$, then there exists a minimal universal circle S^1_m with monotone maps $\phi^m_\lambda : S^1_m \to S^1_{univ}(\lambda)$ and a monotone map $m : S^1_{univ} \to S^1_m$ such that for all $\lambda \in L$, we have $\phi^m_\lambda \circ m = \phi_\lambda$ (see Definition 2.1.5). If S^1_{univ} is not minimal, one can define an equivalence relation on S^1_{univ} by saying $x \sim y$ if $\phi_\lambda(x) = \phi_\lambda(y)$ for all $\lambda \in L$. Then m is the quotient map collapsing each equivalence class to a point. Throughout the rest of the section, hence, we assume that S^1_{univ} is a minimal universal circle for $\mathcal{F}$.

For each λ in L, $L - \lambda$ has two connected components. Let $L^+(\lambda)$ denote the component containing at least one leaf μ with $\mu > \lambda$, and $L^-(\lambda)$ be the other component. For $X \subset L$, the set core(X) denotes the union $\bigcup_{\lambda \in X} \text{core}(\phi_\lambda)$. Let $\Lambda(\text{core}(X))$ be the boundary of the convex hull of the closure of core(X) in $\mathbb{H}^2$. Finally, define

$$\Lambda^+(\lambda) = \Lambda(\text{core}(L^+(\lambda)))$$

and

$$\Lambda^+_{univ} = \overline{\bigcup_{\lambda \in L} \Lambda^+(\lambda)}$$

where the closure is taken in $\mathcal{M}$. Define Λ^-_{univ} similarly.

Note that $\Lambda^\pm_{univ}$ are completely determined by the structure of L. To get a intuitive understanding, first consider a smooth surface embedded in $\mathbb{R}^3$ with an appropriate height function from the Morse theory. At a critical point, the level set changes from a single circle to a set of two circles. One can see this as if a single circle equipped with a lamination with only one leaf becomes a figure-eight shape by collapsing the leaf. Let's pretend for a moment that L is a simplicial tree. Let N be a tubular neighborhood of L, and consider the infinite surface ∂N. The ordering $>$ on L allows us to define a height function. For each $\lambda \in L$ at which L branches, the level set at λ can be understood as collapsing some lamination on S^1_{univ} which is precisely $\Lambda^+(\lambda)$ or $\Lambda^-(\lambda)$ depending on the direction of the branching of L at λ.

The following theorem of Calegari shows that $\Lambda^{\pm}_{univ}$ are the desired laminations.

Theorem 2.3.3 (Calegari [4]) *Let $\mathcal{F}$ be a taut foliation of an atoroidal 3-manifold M, and let S^1_{univ} be a minimal universal circle for $\mathcal{F}$. Then $\Lambda^{\pm}_{univ}$ are laminations of S^1_{univ} which are preserved by the natural action of $\pi_1(M)$. Furthermore, if L branches in the positive direction, then Λ^{+}_{univ} is nonempty, and if L branches in the negative direction, then Λ^{-}_{univ} is.*

Proof (sketch). The proof consists of two parts. First we need to show that no leaf of $\Lambda^+(\lambda)$ links any leaf of $\Lambda^+(\mu)$ for $\lambda, \mu \in L$. This is fairly straightforward to see except when $\lambda \in L^+(\mu)$ and $\mu \in L^+(\lambda)$.

In this case, we have $L = \Lambda^+(\mu) \cup \Lambda^+(\lambda)$. The minimality of S^1_{univ} implies that every point in S^1_{univ} is a limit of a sequence of points in $\mathrm{core}(\phi_{\lambda_i})$ for some sequence λ_i. As a result, $\mathrm{core}(L) = \mathrm{core}(\Lambda^+(\mu)) \cup \mathrm{core}(\Lambda^+(\lambda)) = S^1_{univ}$. Therefore, the boundaries of the convex hulls of $\mathrm{core}(\Lambda^+(\mu))$ and $\mathrm{core}(\Lambda^+(\lambda))$ (i.e., the leaves of $\Lambda^+(\mu)$ and $\Lambda^+(\lambda)$) do not cross in $\mathbb{H}^2$.

The second part is to show that Λ^{+}_{univ} is nonempty when L branches in the positive direction. Since the core of any monotone map is a perfect set, it suffices to show that $\mathrm{core}(\phi_\lambda)$ is not equal to S^1_{univ} for any $\lambda \in L$. It is not hard to show that there exists a leaf μ such that $\lambda \in L^-(\mu)$ and $\mu \in L^-(\lambda)$. In this case, one can show that $\mathrm{core}(L^+(\lambda))$ and $\mathrm{core}(L^+(\mu))$ are unlinked as subsets. This implies, in particular, that $\mathrm{core}(L^+(\lambda))$ is not dense in S^1_{univ}. $\square$

2.3.2 Laminations on a universal circle for a taut foliation which either is $\mathbb{R}$-covered or has one-sided branching

The following theorem of Calegari provides a useful tool in this case.

Theorem 2.3.4 (Calegari [4]) *Let M be a 3-manifold, and let $\mathcal{F}$ be a taut foliation of M which branches in at most one direction. Then either M is toroidal, or there is a pseudo-Anosov flow X transverse to $\mathcal{F}$.*

A *pseudo-Anosov flow* is defined as a generalization of an Anosov flow.

Definition 2.3.5 *A **pseudo-Anosov flow** ϕ_t on a 3-manifold M is a flow such that, away from finitely many closed orbits, the flow lines preserve a decomposition of TM into $TM = E^s \oplus E^u \oplus TX$, $TX \oplus E^s$ and $TX \oplus E^u$ are integrable, and the time t flow uniformly extends E^u and contracts E^s, ie., there exist constants $a \geq 1, b > 0$ such that*

$$\|d\phi_{t(v)}\| \leq ae^{-bt}\|v\| \text{ for any } v \in E^s, t \geq 0$$

$$\|d\phi_{-t(v)}\| \leq ae^{-bt}\|v\| \text{ for any } v \in E^u, t \geq 0$$

Since $TX \oplus E^s$ is integrable, it is tangent to a foliation in M which we call the weak stable foliation $\mathcal{F}^{ws}$. Similarly, $TX \oplus E^u$ is tangent to the weak unstable foliation $\mathcal{F}^{wu}$.

If X is a flow on a 3-manifold M, let $\widetilde{X}$ be the pulled-back flow on the universal cover, and P_X be the leaf space of $\widetilde{X}$ (flow lines are elements of this space). Due to Lemma 6.53 of [4], P_X is homeomorphic to $\mathbb{R}^2$ when X is pseudo-Anosov-flow. Moreover, Theorem 6.55 of [4] shows that P_X can be compactified by adding a circle, and $pi_1(M)$ acts on this circle faithfully.

Now consider the covering singular foliations $\widetilde{\mathcal{F}^{ws}}, \widetilde{\mathcal{F}^{wu}}$ of $\mathcal{F}^{ws}, \mathcal{F}^{wu}$. Observe that a nonsingular leaf of $\widetilde{\mathcal{F}^{ws}}$ or $\widetilde{\mathcal{F}^{wu}}$ projects down to a properly embedded line in P_X. On the other hand, a singular leaf projects down to a union of ≥ 3 properly embedded rays which are disjoint from a basepoint. Hence, the projections of $\widetilde{\mathcal{F}^{ws}}$ and $\widetilde{\mathcal{F}^{wu}}$ to P_X define a pair of singular foliations of P_X which we denote by Λ^s and Λ^u.

Choosing an arbitrary identification between $\overline{P_X}$ and $\overline{\mathbb{H}^2}$, Λ^s and Λ^u can be pulled tight to define a pair of geodesic laminations, hence a pair of laminations on the circle $\partial\overline{P_X}$.

Let M be an atoroidal 3-manifold with a cooriented taut foliation $\mathcal{F}$. Then X be a pseudo-Anosov flow transverse to $\mathcal{F}$ whose existence is guaranteed by Theorem 2.3.4. In this case, Calegari showed that there exists a unique minimal universal circle S^1_{univ} for $\mathcal{F}$, and $\partial\overline{P_X}$ is naturally identified with S^1_{univ} (see Chapter 9 in [4]). Therefore, Λ^s and Λ^u define a pair of laminations on the universal circle in this case.

CHAPTER 3

INVARIANT LAMINATIONS ON THE CIRCLE FOR GROUP ACTIONS

In this chapter, we consider groups acting faithfully on the circle by orientation-preserving homeomorphisms. In other words, we consider subgroups of the group, $\mathrm{Homeo}_+(S^1)$, of all orientation-preserving homeomorphisms of the circle. We will assume some basics about the groups acting on the circle. For a coherent exposition on the subject, we refer the readers to [12].

We propose a program to study those groups acting faithfully on S^1 in terms of number of pairwise transverse dense invariant laminations. In this Chapter, we give some examples of groups which admit a small number of invariant laminations as an introduction to such groups, and in the following chapters, we will see that studying these groups is closely connected to Hyperbolic geometry.

3.1 Laminar Groups

Let G be a subgroup of $\mathrm{Homeo}_+(S^1)$. Then the action of G on S^1 induces an action on the space $\mathcal{M}$ of unordered pairs of points of S^1. A lamination Λ is called G-*invariant* if Λ is an invariant subset under the G-action on $\mathcal{M}$, ie., $G(\Lambda) \subset \Lambda$. Note that G may have more than one invariant lamination. We also simply say Λ is an invariant lamination if the group in consideration is clear from the context.

We first give definitions to some types of elements of $\mathrm{Homeo}_+(S^1)$.

Definition 3.1.1 *Let f be an element of* $\mathrm{Homeo}_+(S^1)$*. Let Fix_f be the set of fixed points of f on S^1 and $|Fix_f|$ be the cardinality of the set. We say f is*

1. ***elliptic*** *if* $Fix_f = \emptyset$*,*

2. ***parabolic*** *if* $|Fix_f| = 1$*,*

3. ***hyperbolic*** *if* $|Fix_f| = 2$ *with one attracting fixed point and one repelling fixed point,*

4. ***pseudo-Anosov-like*** *(or simply* ***p-A-like****) if* $|Fix_f| = 2n$ *for some* $n > 1$*, and the fixed points alternate between attracting and repelling fixed points along the circle.*

Definition 3.1.2 *For a subgroup* G *of* $\mathrm{Homeo}_+(S^1)$*, the fixed point of some parabolic element of* G *is called a* ***cusp point*** *of* G*. Let* $C(G) \subset S^1$ *denote the set of all cusp points of* G*.*

Definition 3.1.3 *Let* Λ_1 *and* Λ_2 *be two laminations on the circle* S^1*. The laminations* Λ_1, Λ_2 *are said to be* ***transverse*** *if* $\Lambda_1 \cap \Lambda_2 = \emptyset$*, and* ***strongly transverse*** *if* $\mathrm{ends}(\Lambda_1) \cap \mathrm{ends}(\Lambda_2) = \emptyset$*. If* $\{\Lambda_i\}_{i \in I}$ *is a collection of* G*-invariant laminations for some subgroup* G *of* $\mathrm{Homeo}_+(S^1)$ *and an index set* I*, it is called* ***pants-like*** *if* $\Lambda_i \cap \Lambda_j = C(G)$ *for any* $i \neq j \in I$*.*

The name *pants-like* will be justified by the construction given in Section 4.2 and Section 4.3 for surface groups. In that construction, a cusp point of a surface group will be associated to a cusp of the surface.

Definition 3.1.4 (Laminar Groups) *Let* G *be a subgroup of* $\mathrm{Homeo}_+(S^1)$*.* G *is said to be* ***laminar*** *if it admits a dense invariant lamination* Λ *on* S^1*. A laminar group* G *is said to be*

- COL_n *if it admits pairwise transverse* n *dense invariant laminations,*

- ***strongly*** COL_n *if admits pairwise strongly transverse n dense invariant laminations,*
- ***strictly*** COL_n *if it is* COL_n *but not* COL_{n+1}*,*
- ***pants-like*** COL_n *if it admits a pants-like collection of n very-full laminations.*

The main subject of this paper is to show that there exists close connection between hyperbolic geometry and theory of pants-like COL_n groups, and there is a dramatic difference between pants-like COL_2 groups and pants-like COL_3 groups.

3.2 Groups which are not laminar

While there is no known characterization of laminar groups, being laminar gives some information about the group.

Consider the unit circle in the plane $\mathbb{R}^2$. For an angle $\theta \in [0, 2\pi]$, let R_θ be the rigid rotation around the origin by the angle θ which defines an element of $\mathrm{Homeo}_+(S^1)$ (and denote it by R_θ again).

Lemma 3.2.1 *Suppose θ is an irrational multiple of π. Then there exists no lamination on the circle which is invariant under R_θ.*

Proof. Suppose Λ is a lamination on S^1 which is invariant under R_θ. Let l be a leaf of Λ. The complement of endpoints of l in S^1 consists of two open arcs and let I be the shorter one. Label the endpoints of l by a, b such that the triple (a, b, c) are positively oriented for an arbitrary point c in I. Since every orbit of R_θ is dense, there exists a positive integer n such that $R_\theta^n(a)$ lies in I and $R_\theta^n(b)$ lies in the other arc $(S^1 \setminus (\{a, b\} \cup I))$ where R_θ^n is the nth iterate of R_θ. But this means that the leaves l and $R_\theta^n(l)$ are linked, thus we get a contradiction. □

Definition 3.2.2 *For two actions $\rho_{1,2} : G \to \mathrm{Homeo}_+(S^1)$ on the circle by a group G, we say ρ_1 is* semi-conjugate *to ρ_2 if there exists a monotone map $f : S^1 \to S^1$ such that $f \circ \rho_1 = \rho_2 \circ f$.*

Lemma 3.2.3 *If Λ is a lamination on the circle and $f : S^1 \to S^1$ is a monotone map, then $f(\Lambda)$ is again a lamination except that it possibly has some degenerate leaves.*

Remark. There is a way to avoid talking about degenerate leaves of a lamination obtained as the image of another lamination under a monotone map. Note that a monotone map defines a map from $\mathcal{M}$ to $\overline{\mathcal{M}}$.

Proposition 3.2.4 *Suppose $\varphi \in \mathrm{Homeo}_+(S^1)$ has an irrational rotation number. Then there exists no very-full lamination on S^1 invariant under φ.*

Proof. If an element of $\mathrm{Homeo}_+(S^1)$ has an irrational rotation number, then it is semi-conjugate to an irrational rotation. Hence, there exist an irrational angle θ and a monotone map $f : S^1 \to S^1$ such that $f \circ \varphi = R_\theta \circ f$.

Suppose Λ is a φ-invariant lamination. Then each leaf l of Λ has both endpoints in $f^{-1}(p)$ for some $p \in S^1$. Otherwise, the image $f(l)$ of the leaf l under f is a non-degenerate leaf of $f(\Lambda)$, but Lemma 3.2.1 says that $f(\Lambda)$ cannot have any non-degenerate leaf. Also note that $f^{-1}(p)$ is a connected arc, since f is monotone.

Let $N = \{p \in S^1 : f^{-1}(p)$ is not a single point.$\}$. Note that if $p \in N$, then the entire orbit of p under R_θ is contained in N. Then $f^{-1}(N)$ is the union of infinitely many closed disjoint arcs of the form $f^{-1}(p)$ for some $p \in N$, and $S^1 \setminus f^{-1}(N)$ has empty interior (each orbit under R_θ is dense).

Let Π be a lamination on the circle defined by

$$\{(a, b) : a, b \text{ are endpoints of a connected component of } f^{-1}(N)\}.$$

There exists a unique gap P which contains no open arcs of S^1, since $S^1 \setminus f^{-1}(N)$ has empty interior. The fact that each leaf of Λ is supported by a connected component of $f^{-1}(N)$ implies that no leaf of Λ is linked to a leaf of Π. In other words, Λ has a gap which contains P. In particular, Λ cannot be very-full. □

Corollary 3.2.5 *Let $G \subset \mathrm{Homeo}_+(S^1)$. If G is laminar, then G contains no irrational rotations. If G is laminar and further it admits a very-full invariant lamination, then G contains no elements with irrational rotation numbers.*

There is another rather important example of a non-laminar subgroup of $\mathrm{Homeo}_+(S^1)$, namely Thompson's group F. For the definition, please see Section 3.2 in [12] (note that Ghys used G to denote the group instead of F). The following proposition shows that Thompson's group is too "big" to be laminar.

Proposition 3.2.6 *The induced action of Thompson's group F on $\mathcal{M}$ is minimal, ie., every orbit is dense.*

Proof. Regard S^1 as the quotient of the closed unit interval $[0, 1]$ by identifying the endpoints 0 and 1, and let d denote the corresponding standard metric on S^1. It suffices to show that for an arbitrary small $\epsilon > 0$ and any four distinct points p, q, r, s on the circle, there exists an element $f \in F$ such that $d(f(p), r), d(f(q), s) < \epsilon$.

Reparametrizing S^1 if necessary, we may assume that $0 < p, q, r, s < 1$. Let δ be a small positive number. If $x \in [p - \delta, p], y \in [q, q + \delta], z \in [r - \epsilon/2, r]$ and $w \in [s, s + \epsilon/2]$, then the affine map from the interval $[x, y]$ to the interval $[z, w]$ has a slope at most $\alpha := \frac{s-r+\epsilon}{q-r}$. Note that α depends on ϵ but not on δ. Now set $\delta = \epsilon/\alpha$ and pick x, y, z, w in those intervals as dyadic rational numbers (dyadic rationals form a dense subset of the real line, so such a choice is possible). Define a piecewise linear map $f : [0, 1] \to [0, 1]$ so that f is affine in the intervals $[0, x], [x, y], [y, 1]$ and $f(0) = 0, f(x) = z, f(y) = w, f(1) = 1$. Let $\beta = \frac{w-z}{y-x}$. We saw that $\beta \leq \alpha$.

Since $x \in [p - \delta, p]$, $f(p)$ lies in the interval $[z, z + \delta\beta] \subset [z, z + \delta\alpha] = [z, z + \epsilon]$. But then $z \in [r - \epsilon/2, r]$ implies that both r and $f(p)$ lie in $[z, z + \epsilon]$, in particular, $d(r, f(p)) \leq \epsilon$. Similarly, one can see that $d(s, f(q)) \leq \epsilon$. Now f defines a desired element of Thompson's group F in the quotient $S^1 = [0, 1]/0 \sim 1$. □.

Corollary 3.2.7 *Thompson's group F is not laminar.*

One should be a bit careful. We are talking about a specific representation of F into $\mathrm{Homeo}_+(S^1)$ and did not prove that there exist no laminar actions of F on S^1.

3.3 Strictly COL_1 Groups

In this section, we construct an example of a strictly COL_1 group. For a group $G \subset \mathrm{Homeo}_+(S^1)$, being strictly COL_1 means that any two G-invariant laminations have common leaves (not transverse). In fact, the example we will construct has a stronger property; it has a unique G-invariant lamination and any other laminations are not preserved by the G-action. In fact, the main ideas are already given in the proof of Proposition 3.2.4.

We will describe a special case of so-called **Denjoy blow-up**. For the general construction, see the Construction 2.45 in [4]. Let θ be an irrational angle and $R_\theta : S^1 \to S^1$ be the rotation by the angle θ. Pick a point $p \in S^1$ and let O_p denote the orbit of p under R_θ, ie., $O_p = \{R_\theta^n(p) : n \in \mathbb{Z}\}$. For each $n \in \mathbb{Z}$, replace the point $R_\theta^n(p)$ by a closed interval I_n of length $1/2^{|n|}$. The total length of the intervals added is finite, since the series $\sum_{n\in\mathbb{Z}} 1/2^{|n|}$ converges. For each $n \in \mathbb{Z}$, choose a homeomorphism $\varphi_n : I_n \to [0,1]$ where $[0,1]$ is the standard unit interval. Define an action of R_θ on the intervals I_n's as

$$R_\theta|_{I_n} : I_n \to I_{n+1} := \varphi_{n+1}^{-1} \circ \varphi_n, \text{ for each } n \in \mathbb{Z},$$

let it act on the complement of those intervals as before. Let $\overline{R_\theta}$ denote this new homeomorphism of the circle.

Note that $\overline{R_\theta}$ is semi-conjugate to R_θ by a monotone map which reserves the blow-up process. Namely, let $f : S^1 \to S^1$ be a map which is the identity on the complement of the intervals I_n's and collapses each I_n to a point. Then $f \circ \overline{R_\theta} = R_\theta \circ f$.

Let l denote the pair of endpoints of $I_0 (= f^{-1}(p))$. Define a lamination Λ_1 as

$$\Lambda_1 := \{\overline{R_\theta}^n(l) : n \in \mathbb{Z}\}.$$

Each leaf of Λ_1 is the set of endpoints of a closed arc of the form $f^{-1}(R_\theta^n(p))$, and since O_p is dense, there exists a unique gap, say P_1, of Λ_1 which contains no open arcs of the circle. Just as in the proof of Proposition 3.2.4, we have the following lemma.

Lemma 3.3.1 *No $\overline{R_\theta}$-invariant lamination contains a leaf intersecting the interior of the gap P_1 (in terms of its geometric realization).*

Proof. The negation of the conclusion gives us an $\overline{R_\theta}$-invariant lamination Λ such that $f(\Lambda)$ is an R_θ-invariant lamination with non-degenerate leaves. □.

Let L be an arbitrary (orientation-reversing) homeomorphism of the circle of order 2 whose fixed points are precisely the endpoints of I_0. One way to construct such L is first identifying S^1 with $\partial_\infty \mathbb{H}^2$ and then taking L as the reflection along the geodesic connecting the endpoints of I_0.

Define Λ_2 be the lamination

$$\Lambda_2 = \bigcup_{n\in\mathbb{Z}} \overline{R_\theta}^n \circ L \circ \overline{R_\theta}^{-n}(\Lambda_1).$$

Note that $\overline{R_\theta}^n \circ L \circ \overline{R_\theta}^{-n}$ is a homeomorphism of the circle of order 2 whose fixed points are the endpoints of the leaf $R_\theta^n(l)$. Hence one imagine Λ_2 obtained from Λ by adding the images of Λ_1 under the reflection (by some conjugates of L) along the leaves of Λ_1.

Note that any leaf of $\Lambda_2 \setminus \Lambda_1$ is the image of l under $\overline{R_\theta}^n \circ L \circ \overline{R_\theta}^{-n} \circ \overline{R_\theta}^m$ for some $n, m \in \mathbb{Z}$. Hence, $(\overline{R_\theta}^n \circ L \circ \overline{R_\theta}^{-n} \circ \overline{R_\theta}^m) \circ L \circ (\overline{R_\theta}^n \circ L \circ \overline{R_\theta}^{-n} \circ \overline{R_\theta}^m)^{-1}$ is a conjugate of L which fixes the given leaf. Use this to define the third lamination Λ_3 as

$$\Lambda_3 = \bigcup_{n,m\in\mathbb{Z}} (\overline{R_\theta}^n \circ L \circ \overline{R_\theta}^{-n} \circ \overline{R_\theta}^m) \circ L \circ (\overline{R_\theta}^n \circ L \circ \overline{R_\theta}^{-n} \circ \overline{R_\theta}^m)^{-1}(\Lambda_2).$$

We can continue this process and inductively define Λ_{n+1} from Λ_n by adding the images of Λ_n under the reflections (by some conjugates of L by a word in $\overline{R_\theta}$ and L) along the leaves in $\Lambda_n \setminus \Lambda_{n-1}$. Finally define Λ_θ as the union $\bigcup_{n\in\mathbb{Z}} \Lambda_n$. Let G'_θ be the group generated by $\overline{R_\theta}$ and L, and let G_θ be a subgroup consisting of orientation-preserving elements of G'_θ. By the construction, Λ_θ is G'_θ-invariant, and hence G_θ-invariant.

Proposition 3.3.2 *Λ_θ is the unique G_θ-invariant lamination.*

Proof. Suppose Λ is another G_θ-invariant lamination. Note that each gap of Λ_θ is an image of P under some element of G'_θ. Since Λ is different from Λ_θ, it must have leaf γ which intersects the interior of some gap P_2 of Λ_θ. The stabilizer of P_2 in G contains an element $g\overline{R_\theta}g^{-1}$ for some $g \in G'_\theta$. But then $P_2 = g(P_1)$. Hence, $g^{-1}(\gamma)$ is a leaf intersecting the interior of P_1, hence contradicting Lemma 3.3.1. □

Therefore, the pair $(G_\theta, \Lambda_\theta)$ is a desired example of strictly COL_1 group with its unique invariant lamination.

3.4 Strictly COL_2 Groups

A family of examples of strictly COL_2 group which we will discuss in this section provides us the first connection between the theory of laminar groups and hyperbolic geometry. In fact, we will show that the fundamental groups of hyperbolic 3-manifolds which fiber over the circle are examples of strictly COL_2 groups.

Let Σ be a hyperbolic surface of finite area, and $\varphi : \Sigma \to \Sigma$ be self-diffeomorphism of the surface Σ. A mapping torus M_φ with the monodromy φ is the 3-manifold defined as $\Sigma \times [0, 1]/(x, 1) \sim (\varphi(x), 0)$.

Since the 3-manifold M_φ is tautly foliated by copies of Σ, Thurston's theorem implies that there exists a universal circle for this foliation. In fact, we do not need to invoke Thurston's big hammer in this case. Note that the universal cover $\widetilde{\Sigma}$ of Σ can be identified with $\mathbb{H}^2$, and let $f : \mathbb{H}^2 \to \Sigma$ be the covering map. Let $\widetilde{\varphi} : \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of φ. To be more precise, let $p \in \Sigma$ be the base point of $\pi_1(\Sigma)$ (i.e., it means $\pi_1(\Sigma, p)$), and pick a lift $\widetilde{p}$ of p and $\widetilde{\varphi}(p)$ of $\varphi(p)$. Now, $\widetilde{\varphi}$ is determined by requiring to map $\widetilde{p}$ to $\widetilde{\varphi}(p)$.

Lemma 3.4.1 *For any $\alpha \in \pi_1(\Sigma)$, we have $\widetilde{\varphi}^{-1} \circ \alpha \circ \widetilde{\varphi} \in \pi_1(\Sigma)$.*

Proof. $\widetilde{\varphi}$ is a lift of φ means that $f \circ \widetilde{\varphi} = \varphi \circ f$. Let $x \in \mathbb{H}^2$ and let $\alpha \in \pi_1(\Sigma)$. Then, $f(\alpha(\widetilde{\varphi}(x))) = f(\widetilde{\varphi}(x)) = \varphi(f(x))$. On the other hand, $f((\widetilde{\varphi}^{-1} \circ \alpha \circ \widetilde{\varphi})(x)) = \varphi^{-1}(f((\alpha \circ \widetilde{\varphi})(x)))$. Therefore, $f((\widetilde{\varphi}^{-1} \circ \alpha \circ \widetilde{\varphi})(x)) = \varphi^{-1}(\varphi(f(x)) = f(x)$. But this implies that $(\widetilde{\varphi}^{-1} \circ \alpha \circ \widetilde{\varphi})(x)$ and x differ by an action of $\pi_1(\Sigma)$. This proves the lemma. □

What Lemma 3.4.1 really says is that we have an action of the group $\pi_1(\Sigma) \rtimes \mathbb{Z}$ on $\mathbb{H}^2$ where $\mathbb{Z}$ is generated by $\widetilde{\varphi}$. In fact, it is not hard to see that this is naturally isomorphic to $\pi_1(M_\varphi)$. Since Σ is of finite area, it is straightforward to check that $\widetilde{\varphi}$ acts on $\mathbb{H}^2$ as a quasi-isometry, and the action continuously extends as a homeomorphism on $\partial_\infty \mathbb{H}^2$. Hence, we obtain an action of $\pi_1(M_\varphi)$ on the circle $S^1 \cong \partial_\infty \mathbb{H}^2$.

Meanwhile, since φ is a pseudo-Anosov map, there exists a pair of laminations Λ^+ and Λ^- invariant under φ up to isotopy (called *stable* and *unstable* laminations). Once we lift them to $\mathbb{H}^2$ and take the endpoints of the geodesic leaves, we obtain a pair of laminations on $S^1 \cong \partial_\infty \mathbb{H}^2$, which we denote by Λ^+, Λ^- again. By the construction, these laminations are invariant under the action of $\pi_1(M_\varphi)$. We have proved that $\pi_1(M_\varphi)$ is a COL_2 group.

Theorem 3.4.2 *Let φ be a pseudo-Anosov diffeomorphism of a hyperbolic surface Σ of finite type. Then the fundamental group of the mapping torus of φ, $\pi_1(M_\varphi)$ is a strictly* COL_2 *group.*

Proof. Let Λ be a lamination on S^1 which is invariant under the action of $\pi_1(M_\varphi)$ on $\partial_\infty \mathbb{H}^2$. Abusing the notation, consider Λ as its geometric realization. Then Λ is invariant under the action of $\pi_1(M_\varphi)$ on $\mathbb{H}^2$ up to isotopy which fixes the boundary pointwise. Therefore, it projects down to a lamination on Σ which is invariant under φ up to isotopy. But we know that stable and unstable laminations are only such laminations. □

Corollary 3.4.3 *Let M be a closed hyperbolic 3-manifold. Then $\pi_1(M)$ is virtually a strictly* COL_2 *group.*

Proof. By Agol's Virtual Fibering Theorem [2], M has a finite-sheeted cover N which fibers over the circle. Then $\pi_1(N)$ can be regarded as a subgroup of $\pi_1(M)$ of finite index, and Theorem 3.4.2 tells us that $\pi_1(N)$ is strictly COL_2. □

3.5 COL_∞ Groups

In this paper, a *surface group* means the fundamental group of an orientable connected hyperbolic surface of finite type (equivalently, of finite area). Let Σ be a hyperbolic surface of finite type and γ be a simple closed curve on Σ. An elementary theory of Riemann surfaces shows that there exists a unique simple closed geodesic in the free homotopy class of γ, so we may assume that γ is a geodesic. Since Σ is a hyperbolic surface, we can identify its universal cover with the hyperbolic plane $\mathbb{H}^2$. Let $\pi : \mathbb{H}^2 \to \Sigma$ be the covering map.

Clearly, the union of pre-images of γ under π is a geodesic lamination on $\mathbb{H}^2$. Hence, it determines a lamination Λ_γ on the circle $\partial_\infty \mathbb{H}^2$. The deck transformation action of $\pi_1(\Sigma)$ on $\mathbb{H}^2$ continuously extends to an action on $\partial_\infty \mathbb{H}^2$ by homeomorphisms. Hence, one can consider $\pi_1(\Sigma)$ as a subgroup of $\mathrm{Homeo}_+(S^1)$ and Λ_γ as its invariant lamination.

Lemma 3.5.1 Λ_γ *is a dense lamination.*

Proof. Let I be an arbitrary open arc of $S^1 (= \partial_\infty \mathbb{H}^2)$. Let L be the geodesic in $\mathbb{H}^2$ connecting the endpoints of the arc I and H be the connected component of $\mathbb{H}^2 \setminus L$ which touches I. Since Σ is a hyperbolic surface of finite area, some fundamental domain of it, say F, must be contained in H. The pre-image of γ intersecting F is a geodesic such that at least one endpoint of it lies in the interior of I. This proves the claim. □

Proposition 3.5.2 $\pi_1(\Sigma)$ *is a* COL_∞ *group.*

Proof. Let $\mathcal{S}$ be the set of simple closed geodesics on Σ. For each γ in $\mathcal{S}$, the lift to the universal cover determines a lamination Λ_γ on $\partial_\infty \mathbb{H}^2$. Lemma 3.5.1 shows that those laminations are dense. Since two different laminations Λ_γ and Λ_δ are lifts of two non-homotopic simple closed curves, they are obviously transverse. Since there are infinitely many distinct homotopy classes in $\pi_1(\Sigma)$, the set $\mathcal{S}$ is infinite. □

We proved that all surface groups are COL_∞ groups. In fact, a much stronger statement is true. In the next chapter, we will show that any surface group is pants-like COL_∞.

CHAPTER 4

LAMINATIONS ON HYPERBOLIC SURFACES

At the end of the previous chapter, we saw that surface groups are examples of COL_∞-groups. In this section, we will take a closer look at the space of laminations on hyperbolic surfaces. Our goal is to find a nice collection of laminations on surfaces which distinguishes surface groups from other groups acting on the circle. We will discuss a possible candidate in this chapter, and then in next chapter, we will show that our candidate indeed completely characterizes surface groups. Here, we do not restrict our discussion to surfaces of finite type.

4.1 Pants-decomposable Surfaces

In this section, we study laminations on surfaces which could be decomposed into pairs of pants.

Definition 4.1.1 *A surface Σ with a complete hyperbolic metric is called* ***pants-decomposable*** *if there exists a non-empty multi-curve X on Σ consisting of simple closed geodesics so that the closure of each connected component of the complement of X is a pair of pants. The fundamental group of some pants-decomposable surface is called a pants-decomposable surface group. The multi-curve X used in the pants-decomposition of Σ will be called a* ***pants-curve****.*

Note that all hyperbolic surfaces of finite area except the thrice-puncture sphere are pants-decomposable. The thrice-puncture sphere is excluded by the definition, since we require the existence of a 'non-empty' pants-curve. If Σ is a surface of genus g with n punctures, then a pants-curve of Σ consists of $3g-3+n$

simple closed curves. For a hyperbolic surface with infinite area, we still have a similar decomposition but some component of complement of the closure of a multi-curve could be a half-annulus or a half-plane. For the precise statement, we refer to Theorem 3.6.2. of [13].

Let Σ be a pants-decomposable surface, and X be a pants-curve. Note that we can decompose each pair of pants into two ideal triangles. Let P be a pair of pants, and let $\{b_1, b_2, b_3\}$ be the set of all boundary components and punctures of P. Then for each $i \neq j$, we can put a bi-infinite geodesic in P such that one end converges to b_i and the other end converges to b_j. If b_i is a boundary component, then the convergence means that the geodesic spirals toward and accumulates onto b_i. If b_i is a puncture, then it just means that the geodesic escapes to that puncture. After adding three bi-infinite geodesics to each pair of pants in this way, one gets a lamination Λ_X on Σ which contains X and each complementary region is an ideal triangle. In particular, Λ_X is a very-full lamination. In this way, one can always construct a very-full lamination from a pants-curve on a pants-decomposable surface.

4.2 Pants-like Collection of Very-full Laminations Invariant Under Surface Groups

Let $\Sigma = \Sigma_{g,n}$ be a surface of genus g with n punctures, and let $\varphi : \Sigma \to \Sigma$ be a pseudo-Anosov map. Then ϕ has two invariant laminations, the stable lamination $\mathcal{L}^s$ and the unstable lamination $\mathcal{L}^u$. For two (simple) multi-curves C and D on Σ, we say C and D are *far* (or, C is *far from* D) if they are disjoint in the curve complex of Σ, i.e.., no component of C is homotopic to a component of D.

Lemma 4.2.1 *Let C be a (not necessarily simple) finite multi-curve and D be a simple multi-curve. Then for infinitely many positive integers m, $\varphi^m(D)$ is far from C.*

Proof. Suppose the negation of the claim. Then for all but finite positive integers m, $\varphi^m(D)$ has a component which is homotopic to a curve in C. Since C is finite, then some curve, say γ, in C is homotopic to a component of $\varphi^m(D)$ for infinitely many m.

On the other hand, the laminations $\mathcal{L}^{s,u}$ do not have any simple closed curve, so any simple closed curve (or even any simple multi-curve) is transverse to both $\mathcal{L}^s$ and $\mathcal{L}^u$ somewhere. Furthermore, since D is simple, $\varphi^m(D)$ converges to $\mathcal{L}^s$ (in Hausdorff topology) as m goes to $+\infty$. Therefore, there exists N such that for all $m \geq N$, $\varphi^m(D)$ is transverse to γ somewhere, which is a contradiction. □

Lemma 4.2.2 *There exists a countably infinite set of pants-curves on Σ which are pair-wise far.*

Proof. Let P_1 be an arbitrary pants-curve on Σ. By Lemma 4.2.1, we can take a positive integer n_1 such that $\varphi^{n_1}(P_1)$ is far from P_1. Since the image of a pants-curve under a homeomorphism is again a pants-curve, the multi-curve $\varphi^{n_1}(P_1)$ is a pants-curve, call it P_2. By using Lemma 4.2.1 again, we can find $n_2 > n_1$ such that $P_3 := \varphi^{n_2}(P_1)$ be a pants-curve which is far from $P_1 \cup P_2$. Repeating this argument, we can inductively define a pants-curve P_n for each n such that P_n is far from $P_1 \cup P_2 \cup \ldots \cup P_{n-1}$. The set $\{P_i\}_{i \in \mathbb{Z}_{>0}}$ is a desired set of pants-curves. □

For a pants-curve X on Σ, let $\Sigma(X)$ be a very-full lamination as constructed in Section 4.1. Let $\widetilde{\Lambda_X}$ denote the lift of Λ_X to the universal cover of Σ which is naturally identified with $\mathbb{H}^2$. Let E be the set $\mathrm{ends}(\Lambda_X) \setminus C(\pi_1(\Sigma))$. Recall that $C(\pi_1(\Sigma))$ is the set of cusp points of $\pi_1(\Sigma)$.

Lemma 4.2.3 *Each point in E is a fixed point of a hyperbolic element of $\pi_1(\Sigma)$.*

Proof. By the construction of Λ_X, each point in E is an endpoint of a closed leaf of Λ_X. □

Lemma 4.2.4 *Let G be a* COL *group with an invariant lamination Λ and $g \in G$ be a hyperbolic element. If Λ has leaf l one end of which is fixed by g, then Λ has a leaf joining two fixed points of g.*

Proof. Either $g^n(l)$ or $g^{-n}(l)$ converges to the axis of g as n goes to ∞. □

Lemma 4.2.5 *Let G be a* COL_n *group for some $n \geq 1$ and let $\{\Lambda_\alpha\}$ be a collection of n pairwise transverse dense invariant laminations of G. If $x \in S^1$ is a fixed point of a hyperbolic element g of G, then there exists at most one lamination Λ_α which has a leaf with x as an endpoint.*

Proof. This is a consequence of Lemma 4.2.4 and the transversality of the laminations. □

Theorem 4.2.6 *Any surface group is a pants-like* COL_∞ *group.*

Proof. Let Σ be a hyperbolic surface of finite type. By Lemma 4.2.2, there exists a collection $\{P_i\}_{i \in \mathbb{Z}_{>0}}$ of pants-curves such that P_i and P_j are far for any $i \neq j$. Now consider the collection $\{\Lambda(P_i)\}_{i \in \mathbb{Z}_{>0}}$ of very-full laminations each of which is constructed from a pants-curve P_i as in Section 4.1. By the construction, they are obviously pairwise transverse.

Suppose $p \in \mathrm{ends}(\Lambda(P_i)) \cap \mathrm{ends}(\Lambda(P_j))$ for some $i \neq j$. By Lemma 4.2.4 and Lemma 4.2.5, p cannot be a fixed point of a hyperbolic element of $\pi_1(\Sigma)$. Then Lemma 4.2.3 implies that p is a cusp point of $\pi_1(\Sigma)$. This shows that the collection of laminations is in fact pants-like. □

4.3 Pants-like Collection of Very-full Laminations Invariant Under General Fuchsian Groups

In this section, we slightly generalize the discussion in the last section. For a surface which is not necessarily of finite type, we again construct a collection of pants-curves which are pairwise far as before.

Proposition 4.3.1 *Let Σ be a pants-decomposable surface. Then there exists pants-curves X_0, X_1, X_2 which are pairwise far.*

Proof. We already proved this claim for a surface of finite area. Assume that Σ is a pants-decomposable surface of infinite area.

First we take an arbitrary pants-curve X_0. Seeing X_0 as some set of simple closed geodesics, choose a subset B of X_0 such that no two curves in B are boundary components of a single pair of pants, and each connected component of $\Sigma \setminus B$ is a finite union of pairs of pants, ie., of finite area. Let $(S_i)_{i \in \mathbb{N}}$ be the enumeration of the connected components of $\Sigma \setminus B$. Let X_1, X_2 be pants-curves such that for each $i = 1, 2$, X_i contains B, and $X_i \cap S_j$ is far from $X_0 \cap S_j$ as multi-curves on S_j for each $j \in \mathbb{N}$. Such pants-curves exist due to Lemma 4.2.2.

We are not quite done yet, since all X_0, X_1, X_2 contain B. For each curve γ in B, we choose a simple closed curve $\delta(\gamma)$ as in Figure 4.1. They show three different possibilities for γ as red, blue, and green curves, and in each case, $\delta(\gamma)$ is drawn as the curve colored in magenta. By definition of B, $\delta(\gamma)$ is disjoint from $\delta(\gamma')$ for $\gamma \neq \gamma' \in B$. Let D be the positive multi-twist along the multi-curve $Y = \cup_{\gamma \in B} \delta(\gamma)$. Let $X_1' = D(X_1)$. Note that D does nothing to a curve in X_1 which has a zero geometric intersection number with Y, and clearly it has no homotopic curves

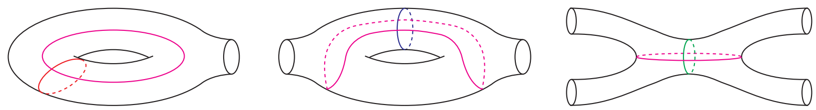

Figure 4.1: This shows how to choose the multi-curve along which we will perform the positive multi-twist to produce a new pants-curve. **(a):** The red curve is the boundary component of a single pair of pants. **(b):** The blue curve is one of two shared boundary components of two pairs of pants. **(c):** The green curve is the unique shared boundary components of two pairs of pants.

in X_0. On the other hand, if a curve γ in X_1 has a non-zero intersection number with Y, then $D(\gamma)$ has positive geometric intersection number with B. Since no curve in X_0 has positive geometric intersection number with B, we are done for this case too.

Changing X_2 is a bit trickier. Let $(b_i)_{i\in\mathbb{N}}$ be an enumeration of the curves in B. Let D_i be the positive Dehn twist along $\delta(b_i)$. One can take k_i for each i so that each curve in $D_i^{k_i}(X_2)$ either is the image of a curve in X_2 which has zero geometric intersection number with $\delta(b_i)$ (so remains unchanged) or has positive geometric number with b_i which is strictly larger than 2. Since $\delta(b_i)$ are disjoint, the infinite product $D' := \prod_{i\in\mathbb{N}} D_i^{k_i}$ is well-defined. Define X_2' as $X_2' = D'(X_2)$. Note that the geometric intersection number between a curve γ in X_1' and b_i for some i is at most 2. Now it is clear that X_0, X_1', X_2' are desired pants-curves. □

Theorem 4.3.2 *Any Fuchsian group G such that $\mathbb{H}^2/G$ is not the thrice-punctured sphere is a pants-like* COL_3 *group.*

Proof. We start with the case when G is the fundamental group of a pants-decomposable surface S. Let $(X_i)_{i=0,1,2}$ be the pants-curves as in Proposition 4.3.1.

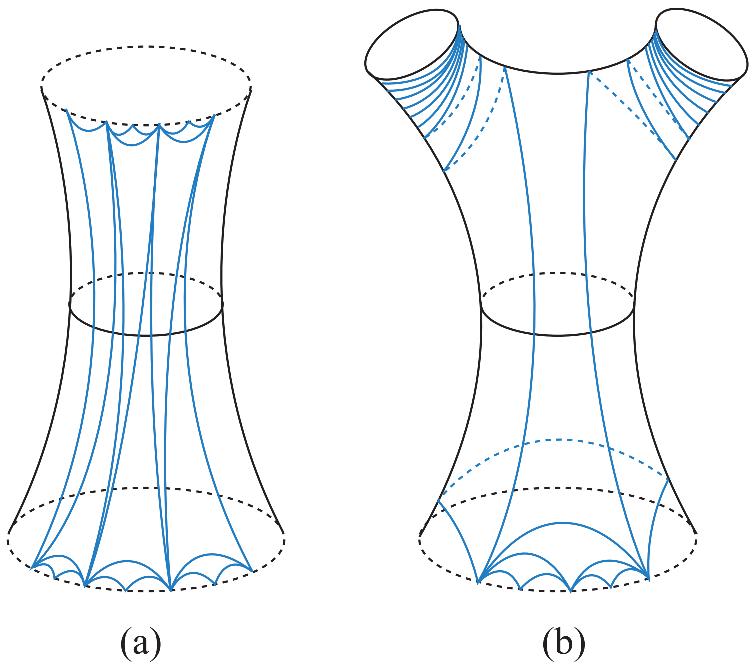

Figure 4.2: **(a):** This is the case when the quotient surface is an infinite annulus. **(b):** This is the case when there exists an annulus component glued to a pair of pants. In any case, one can put arbitrarily many pairwise transverse very full laminations on the annulus components in this way.

For each i, let L_i be the lamination on S obtained from X_i by decomposing the interior of each pair of pants into two ideal triangles. It is possible to put a hyperbolic metric on S so that L_i is a geodesic lamination. Identify the universal cover of S with $\mathbb{H}^2$. G acts on the circle at infinity. Let Λ_i be the lamination of the circle at infinity obtained by lifting L_i to $\mathbb{H}^2$ and taking the end-points data. Since all the complementary regions are ideal triangles, it is very full. Also any leaf of L_i is either a simple closed geodesic, or an infinite geodesic each of whose end either accumulates to a simple closed geodesic or escapes to a cusp. Hence each end is a fixed point of some parabolic or hyperbolic element of G. Now the pants-like property follows from Lemma 4.2.5 and the transversality of the

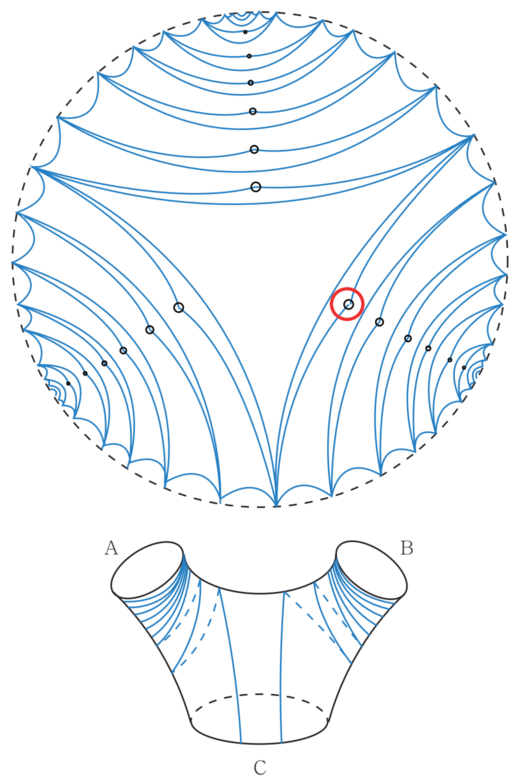

Figure 4.3: **Top:** It shows an example of the subsurface which we consider in the proof of Theorem 4.3.2. One can choose the endpoints of the leaves on the ideal boundary arbitrarily so that we can put as many pairwise transverse very-full laminations on such a subsurface as we want. **Bottom:** This is a pair of pants which could be glued to the part of the surface above indicated by the red circle along the boundary component labeled by C. The boundary component labeled by C is not included in X_0 but those labeled by A and B are. The boundary curves A and B could be cusps or glued along each other.

laminations.

We would like to get the same conclusion as in case that G is a Fuchsian group but its quotient surface $\mathbb{H}^2/G$ is neither a thrice-punctured sphere nor pants-decomposable.

Let's deal with the half-annulus components first. Suppose that X is a multi-curve on the quotient surface S such that $S \setminus X$ consists of pairs of pants and half-annuli. If two half-annuli are glued along a simple closed geodesic, our surface is actually an annulus and the lamination could be taken as in Figure 4.2 (a). Since we can take the ends of such a lamination arbitrarily, it is obvious that there are arbitrarily many such invariant laminations which are pairwise transverse. If the surface is not an annulus, a half-annulus component needs to be attached to a pair of pants. Let X_0 be the collection of simple closed geodesics obtained from X by removing those boundaries of half-annulus components. Let S' be the complement of the half-annuli. As in the proof of Proposition 4.3.1, we can find other pants-curves X_1, X_2 on S' so that X_0, X_1, X_2 are disjoint in the curve complex of S'. Now we decompose the interior of each pair of pants into two ideal triangles as before.

We need to put more leaves on each component of $S \setminus X_i$ for any i which is the union of one half-annulus and one pair of pants glued along a cuff. We construct a lamination inside such a component as in Figure 4.2 (b). Again, we can put an arbitrary lamination on the ideal boundary part of the half-annulus. Note that we construct each lamination so that all gaps are finite-sided, thus we are done.

Now we consider the case where S has even half-plane components. In the Theorem 3.6.2. of [13], it is also shown that if Z denotes the set of points of a pants-curve X, then components of $\overline{Z} \setminus Z$ are simple infinite geodesics bounding half-planes, i.e., we know exactly how the half-plane components arise in the decomposition of a complete hyperbolic surface. Let X be a multi-curve and let Z be the set of points of simple closed curves in X such that $S \setminus \overline{Z}$ consists of pairs of pants, half-annuli, and half-planes, and the boundaries of half-plane components form the set $\overline{Z} \setminus Z$. We will define X_0 by removing some geodesics from X. As before, we remove all the boundary curves of half-annulus components. Observe that there is a part of a surface which is homeomorphic to a half-plane with families of cusps and geodesic boundaries which converge to the ideal boundary (see Figure 3.6.3 on the page 86 of [13] for example). On this subsurface, there are infinitely many components of $\overline{Z}$ so that this subsurface is decomposed into pairs of pants and some half-planes. We remove all the components of X appearing on this type of subsurface. Again, X_0 is a pants-curve of a pants-decomposable subsurface S' of S with geodesic boundaries. On S', we construct X_1, X_2 as before. Among the connected components of $S \setminus S'$, the one containing a half-annulus can be laminated as we explained in the previous paragraph. In the connected component which is homeomorphic to an open disk with punctures, we can do this as in Figure 4.3. Once again, since the ideal boundary part is invariant, we can put an arbitrary lamination there. It is also obvious that the way we construct a lamination gives a very full lamination.

We have shown the theorem. □

Remark 4.3.3 *We constructed a pants-like collection of laminations for Fuchsian groups using pants-decompositions in the proof of Theorem 4.3.2. This is where the name 'pants-like' comes from.*

We would like to see if the converse of Theorem 4.3.2 is also true. In order to answer that question, one needs to analyze the properties of pants-like COL_3 groups.

CHAPTER 5

A CHARACTERIZATION OF MÖBIUS GROUPS

To study basic properties of pants-like COL groups, we will start by studying the structure of very-full invariant laminations. The ends of very-full laminations can be characterized by so-called rainbows. Using this tool, we will prove that all pants-like COL_3 groups are Möbius-like. Our final goal is to prove they are in fact Möbius groups, not just Möbius-like groups. In order to show that, we will use the Convergence Group Theorem. Once again, we will go back and forth between a lamination on the circle and its geometric realization freely even without mentioning it.

5.1 Very-full Laminations Have Enough Rainbows

We would like to take a little detour here, and study about a specific type of laminations.

Definition 5.1.1 *Let Λ be a lamination on S^1. A* ***rainbow of*** *Λ* ***at*** *$p \in S^1$ is an infinite sequence $\{(a_i, b_i)\}_{i\in\mathbb{N}}$ of leaves of Λ such that the sequences (a_i) and (b_i) both converge to p but from the opposite sides (See Figure 5.1).*

Definition 5.1.2 *A lamination Λ on S^1 is said to* ***have enough rainbows*** *if for each $p \in S^1$, either $p \in \mathrm{ends}(\Lambda)$ or there exists a rainbow of Λ at p.*

The following lemma is more or less an observation.

Lemma 5.1.3 *Let Λ be a very full lamination of S^1. Then Λ is dense. Further, for any gap P of Λ, if $x \in S^1 \cap P$, then x is an endpoint of some leaf of Λ.*

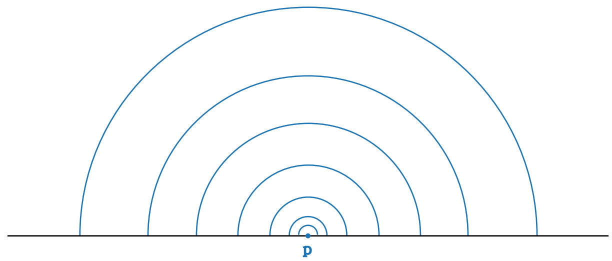

Figure 5.1: This is a schematic picture of a rainbow at p.

Proof. Suppose Λ is not dense. Then we can take an open connected arc I of S^1 where the leaves of Λ have no endpoints. Let l be a geodesic connecting the endpoints of I. Λ has no leaf intersecting l. Take a point p on l. Clearly the gap containing p cannot be a finite-sided ideal polygon. Suppose P is a gap of Λ and $x \in S^1 \cap P$. Since P is a finite-sided ideal polygon, it intersects S^1 only at points to which two sides of P converges. Hence x is an endpoint of some leaf. $\square$

The proof of the following lemma is easily provided from basic facts of hyperbolic geometry.

Lemma 5.1.4 *Consider a very full lamination Λ of S^1. Let $x \in \mathbb{H}^2$. For $p \in S^1$, a gap of Λ containing x contains p if and only if there is no leaf of Λ crossing the geodesic ray from x to p.*

Theorem 5.1.5 (Rainbow Lemma) *Very-full laminations have enough rainbows.*

Proof. It is clear that if $p \in E_\Lambda$, there is no rainbow. Suppose there is no rainbow for p. Then p has a neighborhood U so that if a leaf of Λ has both endpoints in U, then both endpoints are contained in the same connected component of $U \setminus \{p\}$. Replacing U by a smaller neighborhood, we may assume that no leaf connects the endpoints of U.

Identify S^1 with the boundary of the hyperbolic plane $\mathbb{H}^2$ and realize Λ as a geodesic lamination on $\mathbb{H}^2$. Let q_1 be a point on the geodesic connecting the endpoints of U. We may assume that there is no leaf passing through q_1. The only way not having such a point is that the entire $\mathbb{H}^2$ is foliated and p is an endpoint of one of the leaves. Let L be the geodesic passing through q_1 and ending at p. We denote the part of L between a point x on L and p by L_x. Note that any leaf of Λ crossing L_{q_1} has one end in U so that the other end must be outside U by the assumption on U.

If there is no leaf of Λ intersecting L_{q_1}, then the gap containing q_1 contains p by Lemma 5.1.4. Hence p must be an endpoint of a leaf by Lemma 5.1.3 and we are done. Suppose there is a leaf l_1 which crosses L_{q_1} at x. Then let q_2 be a point in L_x so that there is no leaf of Λ passing through it. If there was no such q_2, it means there is a leaf passing through each point of L_x so there must be a leaf ending at p whose other end is necessarily outside U. So, we may assume that such q_2 exists.

If there is no leaf of Λ crossing L_{q_2}, then the gap containing q_2 contains p, we are done. Otherwise, a leaf, say l_2, crosses L_{q_2} at x_2. See Figure 5.2. Repeat the process until we obtain an infinite sequence (q_i) on L which converges to p. This is possible, since otherwise we must have some x on L such that no leaf of Λ crosses L_x. Since (q_i) converges to p, the endpoints of the sequence (l_i) of leaves in U form a sequence converging to p, and the other endpoints are all outside U. By the compactness of S^1, we can take a convergent subsequence so that p is an endpoint of the limiting leaf. In any case, p must be an endpoint of a leaf.

Therefore, $p \in E_\Lambda$ if and only if p has no rainbow. □

Corollary 5.1.6 *Let G be a group acting on S^1 and Λ be a G-invariant very-full lam-*

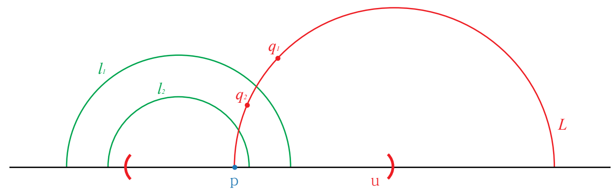

Figure 5.2: This shows a situation when we have no rainbow for p.

ination. For $x \in S^1$ which is the fixed point of a parabolic element g of G, there exist infinitely many leaves which have x as an endpoint.

Proof. Let $x \in S^1$ be the fixed point of a parabolic element g of G and pick Λ_α. By Theorem 5.1.5, if x is not an endpoint of a leaf of Λ_α, x has a rainbow. But any leaf none of whose ends is x must be contained in a single fundamental domain of g to stay unlinked under the iterates of g (here, a fundamental domain is the arc connecting y and $g(y)$ in S^1 for some y different from x). Then the existence of a rainbow would imply that g is a constant map whose image is x but it is impossible since g is a homeomorphism. Hence x is an endpoint of some leaf l of Λ_α. Then $(g^n(l))_{n\in\mathbb{Z}}$ are infinitely many distinct leaves of Λ_α all of which have x as an endpoint. □

Corollary 5.1.7 (Boundary-full laminations) *Suppose a group G acts on S^1 faithfully and minimally. Let Λ be a lamination of S^1 invariant under the G-action. If Λ is very full and totally disconnected, then Λ is boundary-full.*

Proof. The minimality of the action implies that once the closure of the lamination in $\overline{\mathcal{M}}$ contains at least one point in $\partial\mathcal{M}$, then it contains $\partial\mathcal{M}$ and thus the lamination is very-full (this is a simple diagonalization argument).

Let l_1 be any leaf of Λ. Due to the minimality, some element of G maps one of the ends of l_1 somewhere in the middle of the shortest two arcs joining the endpoints of l. Let l_2 be the image of l_1 under the action of this element. Again due to the minimality, one can find an element of G which maps one of the ends of l_2 somewhere in the middle of the shortest arcs in the complement of the endpoints of l_1 and l_2 in S^1. Let l_3 be the image of l_2 under that element. Repeating this procedure, one gets a sequence (l_n) of leaves for which the distance between their endpoints tends to zero, hence giving a desired point in ∂M. □

In fact, the laminations we constructed for pants-decomposable surface groups satisfy the hypotheses of Corollary 5.1.7. Hence all of them are boundary-full laminations.

5.2 Pants-like COL_3 Groups are Möbius-like Groups

As usual, identifying $\partial_\infty \mathbb{H}^2$ with S^1, we regard $\mathrm{PSL}_2(\mathbb{R})$ as a subgroup of $\mathrm{Homeo}_+(S^1)$.

Definition 5.2.1 *A subgroup G of $\mathrm{Homeo}_+(S^1)$ is called a **Möbius-like** group if each element of G is topologically conjugate to an element of $\mathrm{PSL}_2(\mathbb{R})$, i.e., for each $g \in G$, there exists a $h \in \mathrm{Homeo}_+(S^1)$ such that $hgh^{-1} \in \mathrm{PSL}_2(\mathbb{R})$.*

In particular, each element of a Möbius-like group has at most two fixed points in S^1. Let's check how many fixed points an element of a pants-like COL_3 group can have.

Lemma 5.2.2 *Let f be a non-identity orientation-preserving homeomorphism of S^1 with $3 \leq |Fix_f|$. Then any very full lamination Λ invariant under f has a leaf connect-*

ing two fixed point of f. Moreover, for any connected component I of $S^1 \setminus Fix_f$ with endpoints a and b, at least one of a and b is an endpoint of a leaf of Λ.

Proof. Let I be a connected component of $S^1 \setminus Fix_f$ with endpoints a and b. Since Fix_f has at least three points, one can take $c \in Fix_f \setminus \{a, b\}$. Relabeling a and b if necessary, we may assume that the triple a, b, c are counterclockwise oriented.

Suppose a is not an endpoint of a leaf of Λ. Then there exists a rainbow in Λ at a by Theorem 5.1.5. In particular, there exists a leaf l such that one end of l lies in I and the other end lies outside I, call the second one d. If d is a fixed point of f, then replace c by d. Otherwise, we may assume that a, c, d are counterclockwise oriented and there is no fixed point of f between c and d after replacing c by another fixed point if necessary. Clearly, either $f^n(l)$ or $f^{-n}(l)$ converges to the leaf connecting b and c (may not be the same c as the c at the beginning). This proves the lemma. □

Corollary 5.2.3 *Let G be a pants-like* COL_3 *group. Then for any $g \in G$, one must have $|Fix_g| \leq 2$.*

Proof. Let $\{\Lambda_1, \Lambda_2, \Lambda_3\}$ be a pants-like collection of G-invariant laminations. Suppose that there exists an element g of G which has at least three fixed points on S^1. Let I be a connected component of $S^1 \setminus Fix_g$ with endpoints a and b. Then by Lemma 5.2.2, each of a and b is an endpoint of a leaf of some Λ_i. Hence, if none of a, b is the fixed point of a parabolic element of G, we get a contradiction to the pants-like property.

Suppose a is the fixed point of a parabolic element $h \in G$. By Corollary 5.1.6 (or rather the proof of it), there must be a leaf l of Λ_i for any choice of i such that one end of l is a and the other end lies in I. Then either $g^n(l)$ or $g^{-n}(l)$ converges to the leaf connecting a and b as n increases. Hence each Λ_i must have the leaf connecting a and b, contradicting to the transversality. Similarly b cannot be a cusp point either. This completes the proof. $\square$

Lemma 5.2.4 *Let G be a group acting on S^1 and Λ be a very-full G-invariant lamination. For each hyperbolic element $g \in G$ with fixed points a and b, if Λ does not have (a, b) as a leaf, there must be leaf l of Λ_α which separates a, b (ie., not both endpoints of l lies in the same connected component of $S^1 \setminus \{a, b\}$).*

Proof. This is just an observation using the existences of a rainbow. $\square$

Lemma 5.2.5 *Let G be a pants-like COL_3 group. If $f \in G$ is parabolic, then its fixed point is a parabolic fixed point. If f is hyperbolic, it has one attracting and one repelling fixed points, ie., it has North pole-South pole dynamics.*

Proof. Let f be parabolic. It is obvious if it is attracting or repelling, it necessarily has another fixed point. Hence, it must be attracting on one side and repelling on the other side, ie., a parabolic fixed point. Now let f be hyperbolic. The only way of not having North pole-South pole dynamics is that both fixed points are parabolic fixed points. But we have G-invariant laminations with no leaves connecting the fixed points of g (by the transversality, all but at most one lamination are like that. See Lemma 4.2.5). For each of those laminations, there must be a leaf connecting two components of the complement of the fixed points by Lemma 5.2.4. They cannot stay unlinked under f if both fixed points are parabolic fixed points. $\square$

Lemma 5.2.6 *Let G be a pants-like COL_3 group. Any elliptic element of G is of finite order.*

Proof. Let $f \in G$ be an elliptic element. If its rotation number is rational, then some power f^n of f must have fixed points. By Corollary 5.2.3, Fix_{f^n} has either one or two points unless f^n is identity. Suppose f^n has only one fixed point first. Then f must have one fixed point too, contradicting to the assumption. Thus f^n has two fixed points. But since f has no fixed points, both fixed points must be parabolic fixed points, which contradicts to Lemma 5.2.5 Hence the only possibility is $f^n = Id$, so f is of finite order.

Suppose f has irrational rotation number. It cannot be conjugate to a rigid rotation with irrational angle, since any irrational rotation has no invariant lamination at all as we observed before. So f must be semiconjugate to a irrational rotation, say R. We may assume that the action of f on S^1 is obtained by Denjoy blow-up for a orbit of a point under R. Any invariant lamination should be supported by the blown-up orbit. But such a lamination cannot be very full. Hence f cannot have an irrational rotation number. □

Theorem 5.2.7 *Suppose $G \subset \mathrm{Homeo}_+(S^1)$ is a pants-like* COL_3 *group.*

Then each element of G is either a torsion, parabolic, or hyperbolic element.

Proof. This follows from Corollary 5.2.3, Lemma 5.2.5 and Lemma 5.2.6. □

Corollary 5.2.8 *Pants-like* COL_3 *groups are Möbius-like.*

Theorem 5.2.7 provides a classification of elements of pants-like COL_3 groups just like the one for Fuchsian groups (or Möbius groups). But it is known that a Möbius-like group is not necessarily a Möbius group. In the following sections, we will show that pants-like COL_3 groups are convergence groups.

5.3 Convergence Group Theorem

For general reference, we state the following well-known lemma without proof.

Lemma 5.3.1 *Let G be a group acting on a space X. Let K be a compact subset of X such that $g(K) \cap K \neq \emptyset$ for infinitely many g_i. Then there exist a sequence (x_i) in K converging to x and a sequence of the set $\{(g_i)\}$, also call it (g_i) for abusing the notation, such that $g_i(x_i)$ converges to a point x' in K.*

Let G be a discrete subgroup of $\mathrm{Homeo}_+(S^1)$. Then G is said to have the *convergence property* if for an arbitrary infinite sequence of distinct elements (g_i) of G, there exist two points $a, b \in S^1$ (not necessarily distinct) and a subsequence (g_{i_j}) of (g_i) so that g_{i_j} converges to a uniformly on compact subsets of $S^1 \setminus \{b\}$. If G has the convergence property, then we say G is a *convergence group*.

Let T be the space of ordered triples of three distinct elements of S^1. By Theorem 4.A. in [19], a group $G \subset \mathrm{Homeo}_+(S^1)$ is a convergence group if and only if it acts on T properly discontinuously. If one looks at the proof of this theorem, we do not really use the group operation. Hence we get the following statement from the exactly same proof.

Proposition 5.3.2 *Let C be a set of homeomorphisms of S^1. C has the convergence property if and only if C acts on T properly discontinuously.*

Convergence group theorem says that a group acting faithfully on the circle is a convergence group if and only if it is a Fuchsian group (this theorem was proved for a large class of groups in [19], and in full generality in [11] and [9]). For the general background for convergence groups, see [19].

5.4 Pants-like COL_3 Groups are Convergence Groups

For pants-like COL_3 group G, we can define its limit set in a similar way as for the case of Fuchsian groups. Let $\Omega(G)$ be the set of points of S^1 where G acts discontinuously, ie., $\Omega(G) = \{x \in S^1 :$ there exists a neighborhood U of x such that $g(U) \cap U = \emptyset$ for all but finitely many $g \in G\}$, and call it *domain of discontinuity* of G. Let $L(G) = S^1 \setminus \Omega(G)$ and call it *limit set* of G. For our conjecture to have a chance to be true, $\Omega(G)$ and $L(G)$ have the same properties as those for Fuchsian groups.

For the rest of this section, we fix a torsion-free pants-like COL_3 group G with a pants-like collection $\{\Lambda_1, \Lambda_2, \Lambda_3\}$ of G-invariant laminations. Let $F(G)$ be the set of all fixed points of elements of G, ie., $F(G) = \cup_{g \in G} Fix_g$. We do not need the following lemma but it shows another similarity of pants-like COL_3 groups with Fuchsian groups.

Lemma 5.4.1 *$\overline{F(G)}$ is either a finite subset of at most 2 points or an infinite set. When $\overline{F(G)}$ is infinite, it is either the entire S^1 or a perfect nowhere dense subset of S^1.*

Proof. When all the elements of G share fixed points, then $|\overline{F(G)}| \leq 2$ as $|Fix_g| \leq 2$ for all $g \in G$. If that is not the case, say we have $g, h \in G$ whose fixed point sets are distinct. Then for $x \in Fix_h \setminus Fix_g$, $g^n(x)$ are all distinct for $n \in \mathbb{Z}$ and $g^n(x)$ is a fixed point of $g^n h g^{-n}$, hence $\overline{F(G)}$ is infinite. Note that $\overline{F(G)}$ is a closed minimal G-invariant subset of S^1. Hence, it is either the entire S^1 or a nowhere dense subset of S^1. Let x be a fixed point of $g \in G$ and y be a point of $S^1 \setminus Fix_g$. Then either $g^n(y)$ or $g^{-n}(y)$ converges to x (both can happen when g is parabolic). Thus $\overline{F(G)}$ is a perfect set. □

Next lemma itself will not be used to prove our main theorem but the proof is important.

Lemma 5.4.2 $L(G) = \overline{F(G)}$.

Proof. It is very easy to show $\overline{F(G)} \subset L(G)$. Let x be a fixed point of $g \in G$. Note that g cannot be a torsion element. Now it is clear that for any neighborhood U of x, $g^n(U)$ intersects U nontrivially, for instance x lies in the intersection. Since $L(G)$ is a closed set, we get $\overline{F(G)} \subset L(G)$

For the converse, we use laminations. Let $x \in L(G)$. If x is the fixed point of a parabolic element, then we are done. Hence we may assume that it is not the case and take Λ_α such that $x \notin \text{ends}(\Lambda_\alpha)$. Let (l_i) be a sequence of leaves which forms a rainbow for x (such a sequence exists by Theorem 5.1.5) and let (I_i) be a sequence of open arcs in S^1 such that each I_i is the component of the complement of the endpoints of l_i containing x. Since G acts not discontinuously at x, we can choose $g_i \in G$ such that $g_i(I_i)$ intersects I_i nontrivially. This cannot happen arbitrarily, but one must have either $g_i(\overline{I_i}) \subset \overline{I_i}$, $\overline{I_i} \subset g_i(\overline{I_i})$, or $I_i \cup g_i(I_i) = S^1$, since the endpoints of I_i form a leaf. In either case, it is easy to see that g_i must have a fixed point in $\overline{I_i}$ (see Figure 5.3 for possible configurations, and the reason why our situation is restricted to these cases is described in Figure 5.4). Since I_i shrinks to x, this implies that x is a limit point of fixed points of (g_i). □

Lemma 5.4.3 *Suppose (g_i) is a sequence of elements of G and $x \in S^1$ such that for any neighborhood U of x, $g_i(U)$ intersects U nontrivially for all large enough i. Then x is a limit point of the fixed points of g_i.*

Proof. If we can find Λ_α such that $x \notin \text{ends}(\Lambda_\alpha)$, then we are done by the proof of Lemma 5.4.2. Suppose it is not the case, ie., x is the fixed point of a parabolic element h. The key point here is to figure out how to construct I_i for x in order to

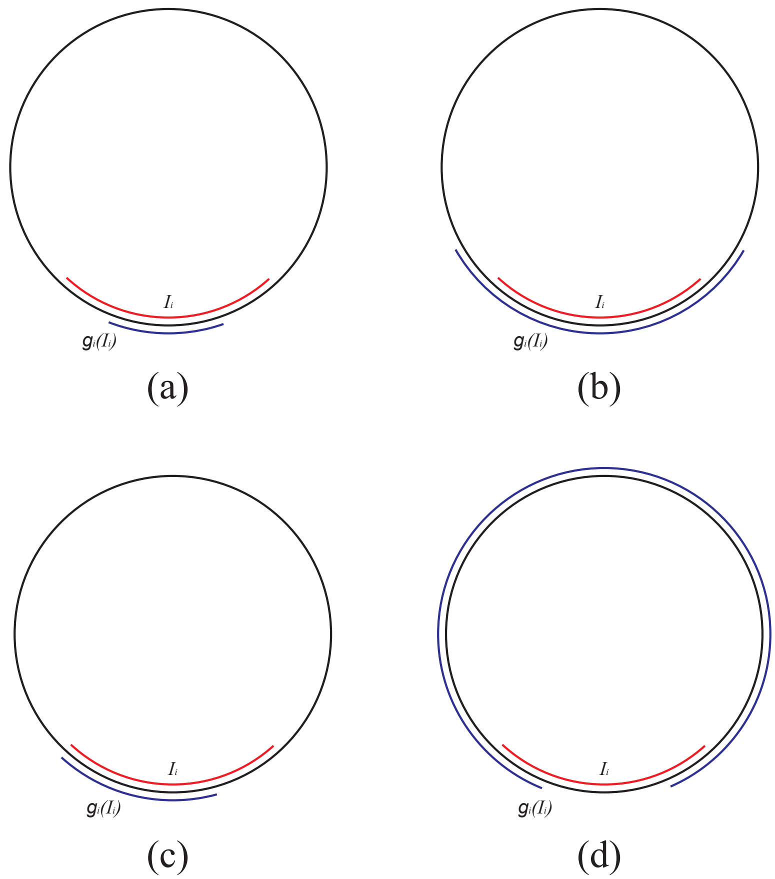

Figure 5.3: This shows the possibilities of the image of I_i under g_i. I_i is the red arc (drawn inside the disc) and $g_i(I_i)$ is the blue arc (drawn outside the disc) in each figure. **(a):** $g_i(I_i)$ is completely contained in I_i. **(b):** $g_i(I_i)$ completely contains I_i. **(c):** $g_i(I_i)$ is completely contained in I_i but one endpoint of I_i is fixed. **(d):** The union of I_i and $g_i(I_i)$ is the whole circle.

mimic the proof of Lemma 5.4.2. Take arbitrary invariant lamination Λ. There exists a leaf l which has x as an endpoint. Then $h^n(l)$'s form an infinite family of such leaves. For all $i \in \mathbb{N}$, let a_i be the other endpoint of $h^i(l)$ and b_i be the other endpoint of $h^{-i}(l)$. Let I_n be an open interval containing x with endpoints a_n, b_n for each $n \in \mathbb{N}$. Now we have a sequence of intervals shrinking to x. Passing to a subsequence if necessary, we may assume that $g_i(I_{i+1}) \cap I_{i+1} \neq \emptyset$ for all i.

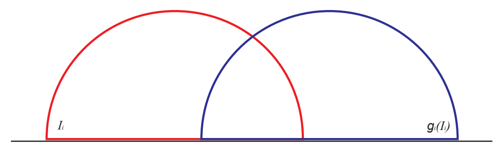

Figure 5.4: This case is excluded, since the leaf connecting the endpoints of I_i is linked with its image under g_i.

Note that it is possible that neither $g_i(I_{i+1}) \subset I_{i+1}$ nor $I_{i+1} \subset g_i(I_{i+1})$ holds (see Figure 5.5 (a)), but one must have either $g_i(I_i) \subset I_i$ or $I_i \subset g_i(I_i)$ to avoid having any linked leaves (see Figure 5.5 (b)). Now the same argument shows that x is a limit point of fixed points of g_i. □

Proposition 5.4.4 *Suppose we have a sequence (x_i) of points in S^1 which converges to $x \in S^1$ and a sequence (g_i) of elements of G such that $g_i(x_i)$ converges to $x' \in S^1$. Then either x or x' lies in the limit set $L(G)$. Moreover, either x is an accumulation point of the fixed points of the sequence $(g_{i+1}^{-1} \circ g_i)$ or x' is an accumulation point of the fixed points of the sequence $(g_i \circ g_{i+1}^{-1})$.*

Proof. Take U any neighborhood of x'. Then for large enough N, we have $g_i(x_i) \in U$ for all $i \geq N$. Suppose $g_i^{-1}(U)$ does not contain any x_j with $j \neq i$ for each $i \geq N$. But $g_i^{-1}(U)$ contains x_i and the sequence (x_i) converges to x. Hence $g_i^{-1}(U)$ for $i \geq N$ form a sequence of disjoint open intervals shrinking to x, implying that $g_i^{-1}(x')$ converges to x. Now let V be a neighborhood of x. Replacing N by a larger number if necessary, $g_i^{-1}(x')$ lies in V for all $i \geq N$. Then $(g_{i+1}^{-1} \circ g_i)(V)$ intersects nontrivially V for all $i \geq N$.

Now suppose such U does not exist. Take an arbitrary neighborhood U of x'. Passing to a subsequence, we may assume that $g_i(x_i) \in U$ for all i. By the

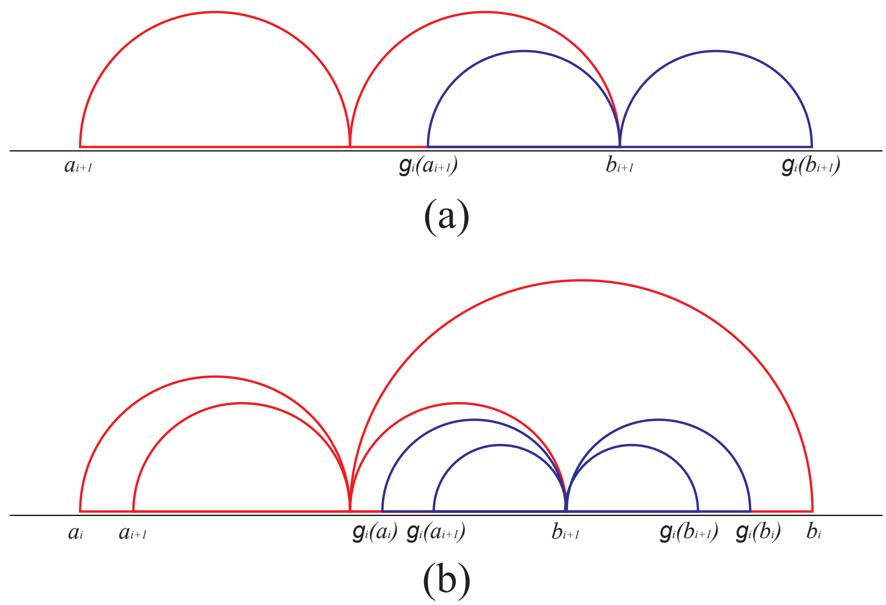

Figure 5.5: The nested intervals for a cusp point need more care. (a) shows that it might have a problematic intersection, but (b) shows we can take a one-step bigger interval to avoid that. The endpoints of I_i are marked as a_i, b_i and the endpoints of I_{i+1} are marked as a_{i+1}, b_{i+1}.

assumption, for each i, there exists $n_i \neq i$ such that $x_{n_i} \in g_i^{-1}(U)$. Taking a subsequence of x_i, we may assume that (x_i) converges to x monotonically. Thus we are allowed to assume that either $n_i = i + 1$ or $n_i = i - 1$ for all i. In the former case, $g_{i+1} \circ g_i^{-1}(U)$ intersects U nontrivially for each i, and in the latter case, $g_i \circ g_{i+1}^{-1}$ does the same thing (note that $g_i \circ g_{i+1}^{-1}$ and $g_{i+1} \circ g_i^{-1}$ have the same fixed points).

Now the result follows by applying Lemma 5.4.3. □

We are ready to prove the main theorem of the paper.

Theorem 5.4.5 *Let G be a torsion-free discrete subgroup* $\mathrm{Homeo}_+(S^1)$*. If G admits a pants-like collection of very-full invariant laminations $\{\Lambda_1, \Lambda_2, \Lambda_3\}$, then G is a Fuchsian group.*

Proof. By the Convergence Group Theorem, it suffices to prove that G is a convergence group. Suppose not. Then there exists a sequence (g_i) of distinct elements of G such that for any pair of (not necessarily distinct) points α, β of S^1, no subsequence converges to α uniformly on compact subsets of $S^1 \setminus \{\beta\}$. This implies that this sequence, as a set, acts on T not properly discontinuously. Then we have three sequences $(x_i), (y_i), (z_i)$ converging to x, y, z and a sequence (h_i) of the set $\{g_i\}$ such that $h_i(x_i) \to x', h_i(y_i) \to y', h_i(z_i) \to z'$ where x, y, z are all distinct and x', y', z' are all distinct. Note that the sequence (h_i) could have been taken as a subsequence of (g_i), so let's assume that.

From Proposition 5.4.4, we can take a subsequence of (h_i) (call it again (h_i) by abusing the notation) such that either two of x, y, z are accumulation points of the fixed points $h_{i+1}^{-1} \circ h_i$ or two of x', y', z' are accumulation points of fixed points of $h_i \circ h_{i+1}^{-1}$. Without loss of generality, suppose x', y' are accumulation points of fixed points of $h_i \circ h_{i+1}^{-1}$.

We would like to pass to subsequences so that the fixed points of the sequence $h_{i+1}^{-1} \circ h_i$ (or $h_i \circ h_{i+1}^{-1}$) have at most two accumulation points. But this cannot be done directly, since a subsequence of $(h_{i+1}^{-1} \circ h_i)$ is not from a subsequence of (h_i) in general. Instead, we proceed as follows.

Take a subsequence $(h_{i_k+1}^{-1} \circ h_{i_k})$ of $(h_{i+1}^{-1} \circ h_i)$ such that there are at most two points where the fixed points of $(h_{i_k+1}^{-1} \circ h_{i_k})$ accumulate (Such a subsequence exists due to Corollary 5.2.3 and the compactness of S^1). Similarly, let $(h_{i_{k_j}} \circ h_{i_{k_j}+1}^{-1})$ be a subsequence of $(h_i \circ h_{i+1}^{-1})$ such that there are at most two points where the fixed points of $(h_{i_{k_j}} \circ h_{i_{k_j}+1}^{-1})$ accumulate.

Since x', y' are accumulation points of fixed points of $h_i \circ h_{i+1}^{-1}$, they are accumulated points of fixed points of $(h_{i_{k_j}} \circ h^{-1}_{i_{k_j}+1})$. But the fixed points of $(h_{i_{k_j}} \circ h^{-1}_{i_{k_j}+1})$ have at most two accumulation points, and x', y', z' are three distinct points, so that z' cannot be an accumulation point of fixed points of $(h_{i_{k_j}} \circ h^{-1}_{i_{k_j}+1})$. This also implies that z' is not an accumulation point of fixed points of $(h_i \circ h^{-1}_{i+1})$. By our choice of (h_i), this implies that z must be an accumulation fixed points of $(h^{-1}_{i+1} \circ h_i)$ (so it is an accumulation point of fixed points of $(h^{-1}_{i_k+1} \circ h_{i_k})$).

Let (a_i) be the sequence of fixed points of $(h^{-1}_{i+1} \circ h_i)$ which converges to z. Now we consider a subsequence such that $(h_{i_{k_{jw}}}(a_{i_{k_{jw}}}))$ converges to a point, say a. Note that for each i, we have $(h_i \circ h^{-1}_{i+1})(h_i(a_i)) = h_i(a_i)$. Hence, $h_{i_{k_{jw}}}(a_{i_{k_{jw}}})$ are fixed points of a subsequence of $(h_{i_{k_j}} \circ h^{-1}_{i_{k_j}+1})$, so that a must be x' or y'. Without loss of generality, let's assume that $a = x'$.

Then we have the followings;

(1) $(a_{i_{k_{jw}}})$ converges to z, since a_i converges to z.

(2) $(h_{i_{k_{jw}}}(a_{i_{k_{jw}}}))$ converges to x'.

(3) $(h_{i_{k_{jw}}}(z_{i_{k_{jw}}}))$ converges to z', since $h_i(z_i)$ converges to z'.

(4) $(h_{i_{k_{jw}}}(x_{i_{k_{jw}}}))$ converges to x', since $h_i(x_i)$ converges to x'.

Now for notational simplicity, we drop the subscripts $i_{k_{jw}}$ and simply denote them as i (we are passing to subsequences here).

We want to have a nested strictly decreasing sequence of intervals for each of x', z. Suppose for now that none of them is a cusp point. By the pants-like property, we can take Λ_α so that no leaf ends in $\{x', z\}$. In this case, we have a rainbow for each of x', z and take intervals as in the proof of Lemma 5.4.2. For $p \in \{x', z\}$, let (I^p_i) be the sequence of nested decreasing intervals containing

p. By taking subsequences, we may assume that for each i, we have $a_i, z_i \in I_i^z$ but $x_i \notin I_i^z$, and $h_i(x_i), h_i(a_i) \in I_i^{x'}$. Then, in particular, $h_i(I_i^z)$ intersects $I_i^{x'}$ non-trivially. But since h_i is a homeomorphism and $h_i(x_i) \in I_i^{x'}$, it is impossible to have $h_i(I_i^z) \supset I_i^{x'}$. Hence there are two possibilities: either $h_i(I_i^z) \subset I_i^{x'}$ or I_i^z is expanded by h_i so that $h_i(I_i^z) \cup I_i^{x'} = S^1$. If the latter happens infinitely often, we can take a subsequence which has only the latter case for all i. Then $S^1 \setminus (I_i^z \cup I_n^{x'})$ is mapped completely into $I_n^{x'}$ by h_i. This shows that the sequence h_i has the convergence property with the two points z, x', a contradiction to our assumption. Hence we may assume that this does not happen, ie., $h_i(I_i^z)$ are completely contained in $I_n^{x'}$. But we know that $z_i \in I_i^z$ and $h_i(z_i) \to z' \neq x'$. This is a contradiction.

If some of them are cusp points, we take intervals as in the proof of Lemma 5.4.3. As we saw, one needs to be slightly more careful to choose $(I_i^{x'}), (I_i^z)$ so that the case $h_i(I_i^{x'}) \not\subset I_i^z, h_i(I_i^{x'}) \not\supset I_i^z$ and $h_i(I_i^{x'}) \cup I_i^z \neq S^1$ does not happen; one can avoid this as we did in the proof of Lemma 5.4.3 (recall the Figure 5.5). Then the same argument goes through.

Hence the set $\{g_i\}$ must act properly discontinuously on T, contradicting our assumption. Now the result follows. □

Remark 5.4.6 *In the proof of Theorem 5.4.5, the consequence of the pants-like property which we needed is that for arbitrary pair of points $p, q \in S^1$ which are not fixed by some parabolic elements, there exists an invariant lamination so that none of p, q is an endpoint of the leaf of that lamination.*

Corollary 5.4.7 (Main Theorem) *Let G be a torsion-free discrete subgroup of* $\mathrm{Homeo}_+(S^1)$. *Then G is a pants-like* COL_3 *group if and only if G is a Fuchsian group whose quotient is not the thrice-punctured sphere.*

Proof. This is a direct consequence of Theorem 4.3.2 and Theorem 5.4.5. □

Corollary 5.4.8 *Let G be a torsion-free discrete subgroup of* $\mathrm{Homeo}_+(S^1)$*. Then G admits pairwise strongly transverse three very-full invariant laminations if and only if G is a Fuchsian group whose quotient has no cusps.*

Proof. Replacing the pants-like property by the pairwise strong transversality is equivalent to saying that there are no parabolic elements. Hence, this is an immediate corollary of the Main Theorem. □

Corollary 5.4.9 *Let G be a torsion-free discrete pants-like* COL_3 *group. Then G-action on S^1 is minimal if and only if G is a pants-decomposable surface group.*

Proof. One direction is clear from the observation that the fundamental group of a pants-decomposable surface acts minimally on $\partial_\infty \mathbb{H}^2$. Suppose G is a pants-like COL_3 group. By Theorem 5.4.5, G is a Fuchsian group. Let S be the quotient surface $\mathbb{H}^2/G$. Note that S is not the thrice-punctured sphere, since it has infinitely many transverse laminations. If S is not pants-decomposable, then there still exists a multi-curve which decomposes S into pairs of pants, half-annuli, and half-planes (Theorem 3.6.2. of [13]). Thus any fundamental domain of G-action on $\overline{\mathbb{H}^2}$ contains some open arcs in $S^1 = \partial_\infty \mathbb{H}^2$. Let I be a proper closed sub-arc of such an open arc. Since it is taken as a subset of a fundamental domain, the orbit closure of I is a closed invariant subset of S^1 which has non-empty interior and is not the whole S^1. This contradicts to the minimality of G-action. □

Corollary 5.4.10 *Let M be a oriented hyperbolic 3-manifold whose fundamental group is finitely generated. If $\pi_1(M)$ admits a pants-like* COL_3*-representation into* $\mathrm{Homeo}_+(S^1)$*, then M is a homeomorphic to $S \times \mathbb{R}$ for some surface S. If we further* *assume that M has no cusps and is geometrically finite, then M is either quasi-Fuchsian or Schottky.*

Proof. The existence of a pants-like COL_3-representation into $\mathrm{Homeo}_+(S^1)$ implies that $\pi_1(M)$ is isomorphic to $\pi_1(S)$ for a hyperbolic surface S. Now it is a consequence of the Tameness theorem (independently proved by Agol [1] and Calegari-Gabai [6]). □

Remark 5.4.11 *In Theorem 4.3.2, one can work harder to show that Fuchsian groups are in fact pants-like* COL_∞ *groups. The result of Section 3 says there are pants-like* COL_2 *groups which are not pants-like* COL_3 *groups (we will see the distinction in more detail in the next section). Hence, any torsion-free discrete subgroup of* $\mathrm{Homeo}_+(S^1)$ *is a pants-like* COL_n *group where n is either* $0, 1, 2,$ *or infinity. This is like the cardinality of limit sets of Kleinian groups (or more generally, the boundaries of Gromov-hyperbolic groups).*

Remark 5.4.12 *In Theorem 5.4.5, it is easy to see that the torsion-free assumption is not necessary. We conjecture that the main theorem (Corollary 5.4.7) could be stated without the torsion-free assumption. To show that, one needs to construct pants-like collection of three very-full laminations on hyperbolic orbifolds. It is not too clear how to do so with simple geodesics.*

CHAPTER 6

BEYOND MÖBIUS GROUPS: MORE CONNECTIONS TO HYPERBOLIC GEOMETRY

In Chapter 5, we gave a complete characterization of Fuchsian Groups (or torsion-free discrete Möbius groups) in terms of invariant laminations. More precisely, we saw that having three *different* very-full laminations is the defining property of Fuchsian groups. In fact, there are two obvious directions to generalize this result.

Question 6.0.13 *What can one say about the torsion-free discrete subgroups of* $\mathrm{Homeo}_+(S^1)$ *which have only two* different *very-full laminations?*

Question 6.0.14 *One meaningful extension of Fuchsian groups is the fundamental groups of 3-manifolds which fiber over the circle. Is there a characterization of those groups which is analogous to Corollary 5.4.7?*

Surprisingly enough, there are many evidences that above two questions are essentially the same.

6.1 Pants-like COL_2 Groups

Let G be a subgroup of $\mathrm{Homeo}_+(S^1)$ which has two very-full G-invariant laminations Λ_1, Λ_2. For the sake of simplicity, we also assume that $|Fix_g| < \infty$ for each non-trivial element $g \in G$. We will see how serious this assumption is later.

Theorem 6.1.1 (Classification of Elements of Pants-like $\mathbf{COL_2}$ Groups) *The elements of G are either torsion, parabolic, hyperbolic or pseudo-Anosov-like.*

Proof. By the proof of Lemma 5.2.6, it suffices to analyze the case that some power g^n of a non-torsion element $g \in G$ has more than two fixed points. For simplicity, we replace g^n by g.

Let I be a connected component of $S^1 \setminus Fix_g$ with endpoints a and b. Lemma 5.2.2 implies that for each i, either $a \in \text{ends}(\Lambda_i)$ or $b \in \text{ends}(\Lambda_i)$. We also know that none of a and b can be the fixed point of a parabolic element (see the second half of the proof of Corollary 5.2.3). Hence the pants-like property implies that there is no i such that both a and b are in $\text{ends}(\Lambda_i)$. In particular, this implies that for each $p \in Fix_g$, there exists $i \in \{1, 2\}$ so that p is not in $\text{ends}(\Lambda_i)$. But this implies that there is a rainbow in Λ_i at p.

But a parabolic fixed point cannot have a rainbow. This implies that g has no parabolic fixed points, i.e., every fixed points of g is either a source or a sink. Hence g must have even number of fixed points which alternate between attracting and repelling fixed points, i.e., g is pseudo-Anosov-like. □

Conjecture 6.1.2 *Suppose G is torsion-free discrete and $|Fix_g| \leq 2$ for each $g \in G$. Then G is Fuchsian.*

For each pseudo-Anosov-like element g of G, the boundary leaves of the convex hull of the attracting fixed points form a ideal polygon, we call it the *attracting polygon* of g. The *repelling polygon* of g is defined similarly.

Theorem 6.1.3 *Let G, Λ_1, Λ_2 be as defined at the beginning of the section. Suppose that there exists $g \in G$ which has more than two fixed points (so there are at least 4 fixed points). Then each Λ_i contains either the attracting polygon of g or the repelling polygon of g.*

Proof. Say $Fix_g = \{p_1, \ldots, p_n\}$ such that if we walk from p_i along S^1 counter-clockwise, then the first element of Fix_g we meet is p_{i+1} (indexes are modulo n). Suppose $p_1 \in E_{\Lambda_1}$. Then by the argument in the proof of Theorem 6.1.1, both p_2 and p_n are not in E_{Λ_1}. If we apply this consecutively, one can easily see that $p_i \in E_{\Lambda_1}$ if and only if i is odd.

Let j be any even number. Since p_j is not in E_{Λ_1}, there exists a rainbow at j. In particular, there exists a leaf l in Λ_1 so that one end of l lies between p_j and p_{j+1}, and the other end lies between p_j and p_{j-1}. Hence either $g^n(l)$ or $g^{-n}(l)$ converges to the leaf (p_{j-1}, p_{j+1}) as n increases. So, the leaf (p_{j-1}, p_{j+1}) should be contained in Λ_1. Since j is an arbitrary even number, this shows that Λ_1 contains the boundary leaves of the convex hull of the fixed points of g with odd indices. Similarly, one can see that Λ_2 must contain the boundary leaves of the convex hull of the fixed points of g with even indices. Since the fixed points of g alternate between attracting and repelling fixed points along S^1, the results follows. □

This shows that not only the pseudo-Anosov-like elements resemble the dynamics of pseudo-Anosov homeomorphisms but also their invariant laminations are like stable and unstable laminations of pseudo-Anosov homeomorphisms.

We introduce a following useful theorem of Moore [14] and an application in our context.

Theorem 6.1.4 (Moore) *Let $\mathcal{G}$ be an upper semicontinuous decomposition of S^2 such that each element of $\mathcal{G}$ is compact and nonseparating. Then $S^2/\mathcal{G}$ is homeomorphic to S^2.*

A decomposition of a Hausdorff space X is *upper semicontinuous* if and only if the set of pairs (x, y) for which x and y belong to the same decomposition element is closed in $X \times X$. A lamination Λ of S^1 is called *loose* if no point on S^1 is an endpoint of two leaves of Λ which are not edges of a single gap of Λ.

Theorem 6.1.5 *Let G, Λ_1, Λ_2 be as defined at the beginning of the section. We further assume that G is torsion-free, each Λ_i is loose, and Λ_1 and Λ have distinct endpoints. Then G acts on S^2 by homeomorphisms such that $|Fix_g(S^2) := \{p \in S^2 : g(p) = p\}| \leq 2$ for each $g \in G$.*

Proof. Let D_1 and D_2 be disks glued along their boundaries, and we consider this boundary as the circle where G acts. So we get a 2-sphere, call it S_1, such that G acts on its equatorial circle. Put Λ_i on D_i for each $i = 1, 2$. One can first define a relation on S_1 so that two points are related if they are on the same leaf or the same complementary region of Λ_i for some i. Let $\sim$ be the closed equivalence relation generated by the relation we just defined.

It is fairly straightforward to see that $\sim$ satisfies the condition of Moore's theorem from the looseness. Looseness, in particular, implies that each equivalence class of $\sim$ has at most finitely many points in S^1.

This concludes that $S_2 := S_1 / \sim$ is homeomorphic to a 2-sphere, and let $p : S_1 \to S_2$ be the corresponding quotient map. Clearly, p is surjective even after restricted to the equatorial circle, call it p again. Now we have a quotient map $p : S^1 \to S_2 = S^2$, hence G has an induced action on S^2 by homeomorphisms.

If g is either parabolic or hyperbolic, the preimage of each point in $Fix_g(S^2)$ must be in $Fix_g(S^1)$. If g is pseudo-Anosov-like, then Theorem 6.1.3 implies that the preimage of each point in $Fix_g(S^2)$ is either the attracting polygon or the repelling polygons of g. In any case, $Fix_g(S^2)$ can have at most two points. □

In other words, G in Theorem 6.1.5 acts on S^2 as a Möbius-like group. It is not at all clear when this action is a convergence group action.

We will end this section by showing that the assumption that each element of the group has at most finitely many fixed points is not a strong assumption for those groups in Theorem 6.1.5.

Lemma 6.1.6 *Let G, Λ_1, Λ_2 be as in Theorem 6.1.5. If an element g of G has countably many fixed points, then it has at most finitely many fixed points.*

Proof. Suppose g has infinitely many fixed points. Then one can find a sequence of isolated fixed points x_i which monotonely converges to a fixed point x of g. Exchanging the roles of Λ_1 and Λ_2 if necessary, we may assume that x_1 is not in ends(Λ_1). Then, by the argument in the proof of Theorem 6.1.3, Λ_1 has leaves connecting x_i and x_{i+2} for each even number i. Since Λ_1 is loose, these leaves should be boundary leaves of a single gap. But it contradicts the fact that Λ_1 is very-full. □

6.2 Conjectures and Open Problems

From what we have seen in the previous section, it is conceivable that G contains a subgroup of the form $H \rtimes \mathbb{Z}$ where H is a pants-like COL_3 group and $\mathbb{Z}$ is generated by a pseudo-Anosov-like element (unless G itself is a pants-like COL_3 group). Maybe one can hope the following conjecture to be true.

Conjecture 6.2.1 *Let G be a finitely generated torsion-free discrete subgroup* $\mathrm{Homeo}_+(S^1)$*. Then G is virtually a pants-like* COL_2 *group with loose laminations (with* *no third such invariant lamination) if and only if G is virtually a hyperbolic 3-manifold group.*

If G is a hyperbolic 3-manifold group, then Agol's Virtual Fibering theorem in [2] says that G has a subgroup of finite index which fibers over the circle. Hence the result of Section 3 implies that such a subgroup is COL_2. The laminations we have are stable and unstable laminations of a pseudo-Anosov map of a hyperbolic surface, hence they form a pants-like collection of two very-full laminations. This proves one direction of the conjecture.

Cannon's conjecture [8] says that if a word-hyperbolic group with ideal boundary homeomorphic to S^2 acts on its boundary faithfully, then the group is a Kleinian group. One may try to prove the converse of Conjecture 6.2.1 modulo Cannon's conjecture by proving the following conjecture.

Conjecture 6.2.2 *The action on the 2-sphere obtained in Theorem 6.1.5 is a uniform convergence group action.*

A theorem of Bowditch [3] says if a group G acts on a compactum X as a uniform convergence group, then G is word-hyperbolic and X is G-equivariantly homeomorphic to the ideal boundary of G. Hence, if Conjecture 6.2.2 holds, then the Cannon's conjecture implies Conjecture 6.2.1.

In fact, there is a possible counterexample of Conjecture 6.2.1. Let M be a closed hyperbolic 3-manifold which admits a pseudo-Anosov flow $\mathcal{F}$. Let $\widetilde{\mathcal{F}}$ be the lift of the flow $\mathcal{F}$ to the universal cover of M. Then the work of Fenley shows that the space of flow-lines of $\widetilde{\mathcal{F}}$ is homeomorphic to an open disk. One can paraphrase his result as follows.

Theorem 6.2.3 (Fenley [10]) *Let M be a closed hyperbolic 3-manifold which admits a pseudo-Anosov flow $\mathcal{F}$ without a perfect fit. Then $pi_1(M)$ acts faithfully on S^1 by homeomorphisms with two very-full laminations with distinct endpoints.*

Question 6.2.4 *Can those laminations in Theorem 6.2.3 be loose when $\mathcal{F}$ is not a suspension flow?*

If Question 6.2.4 has an affirmative answer, then hyperbolic 3-manifolds with pseudo-Anosov flows without perfect fits would give counterexamples to Conjecture 6.2.1. In any case, one can ask the following question.

Question 6.2.5 *Can one characterize the actions in Theorem 6.2.3 in terms of the invariant laminations (in a similar fashion to Conjecture 6.2.1)?*

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