RAMSEY THEORY AND BANACH SPACE GEOMETRY

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We present two main results, one related to the original construction of Tsirelson's space and one that elaborates on a Ramsey type theorem formulated and proved by Gowers to obtain the stabilization of Lipschitz functions on the unit sphere of c_0.

We obtain an expression for the norm of the space constructed by Tsirelson. This study of the original construction is aimed at providing a description of the space that could lend itself to the use of tools coming from Ramsey theory. The expression can be modified to give the norm of the dual of any mixed Tsirelson space. In particular, our results can be adapted to give the norm for the dual of the space S constructed by Schlumprecht.

We give a constructive proof of the finite version of Gowers' FIN_k Theorem and analyse the corresponding upper bounds. We compare the finite FIN_k Theorem with the finite stabilization principle in the case of spaces of the form ℓ_∞^n, $n \in \mathbb{N}$ and establish a more slowly growing upper bound for the finite stabilization principle in this particular case.

BIOGRAPHICAL SKETCH

Diana was born in Bogotá, Colombia. She earned a double major in Mathematics and Mechanical Engineering from the Universidad de los Andes in Bogotá. Her sister Claudia studied Physics also at Los Andes. With Claudia and her parents' encouragement and support, Diana started her PhD studies at Cornell in the Fall of 2009. From an early age, Claudia and Diana showed interest in science. This interest grew in part out of the wide selection of books and encyclopaedias that Roberto, their dad, collected over the years. Roberto's book collection keeps growing to this day.

During the time Diana spent in Ithaca, she discovered how much she enjoys biking, cooking and gardening. During her PhD she lived a balanced life inspired by everything she learned from her mom, Rosita. There would not be many pictures to document Diana's life in Ithaca, if it wasn't for the hundreds of photos Rosita took each time the family came to visit.

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make me feel at home. I'm glad I got to meet her better over this last year. My friends Irina and Natalie were always up for any cooking project and in the kitchen we discovered we had more interests and perspectives in common. I enjoyed all the conversations about life with my friend Janos and I hope they will continue to happen.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

We present two main results, one related to the original construction of Tsirelson's space and one that elaborates on a Ramsey type theorem formulated and proved by Gowers [11] to obtain the stabilization of Lipschitz functions on the unit sphere of c_0. Starting with the construction of spreading models of Banach spaces by Brunel and Sucheston [2], Ramsey theory has served as an important tool in the study of the structure of Banach spaces. Gowers' result is another example of the connection between these two areas, and the study of the original construction of Tsirelson's space is aimed at providing a description of the space that could lend itself to the use of tools coming from Ramsey theory.

It was observed by Milman (see [19, p.6]) that, given a real-valued Lipschitz function defined on the unit sphere of an infinite dimensional Banach space, one can always find a finite dimensional subspace Y of any given dimension such that the oscillation of the function restricted to the unit sphere of Y is as small as we want. This motivated the question of whether in this setting one could also pass to an infinite dimensional subspace with the same property.

It was only in 1992 when W.T. Gowers proved in [11] that c_0, the classical Banach space of real sequences converging to 0 endowed with the supremum norm, has this property. For the special case of Lipschitz functions defined on the unit sphere of c_0 not depending on the sign of the canonical coordinates (i.e. such that $f(\sum a_i \mathbf{e}_i) = f(\sum |a_i| \mathbf{e}_i)$ for every (a_i) in the sphere of c_0) we can restrict our attention to the positive sphere of c_0, the set of elements of the sphere with non-negative canonical coordinates. Gowers associated a discrete structure to a net for the positive sphere of c_0, and proved a partition theorem for the structure that we shall refer to as the FIN_k *Theorem.*

The FIN_k Theorem is a generalization of Hindman's Theorem and its proof uses the Galvin-Glazer methods of ultrafilter dynamics. These methods are now widely used to give streamlined proofs of Ramsey type theorems such as Hindman's Theorem and different variations of the Hales-Jewett Theorem (see [29], [16]).

One of our main results is an inductive proof of the finite version of the FIN_k Theorem, together with the calculation of the upper bounds for the corresponding Ramsey numbers given by the proof. We also calculate upper bounds for the quantitative version of Milman's result about the stabilization of Lipschitz functions on finite dimensional Banach spaces for the special case of the spaces ℓ_∞^n, $n \in \mathbb{N}$. We find bounds for the finite stabilization principle that grow much more slowly than the bounds we find for the finite FIN_k Theorem.

The question about stabilization of Lipschitz functions defined on the unit sphere of an infinite dimensional Banach space is closely related to the notion of distortion. Informally, a space is distortable if it admits an equivalent norm which is far from being a scalar multiple of the original one. It was proved by James [17] that the classical spaces c_0 and ℓ_1 are not distortable.

In the 1970s Tsirelson [31] constructed a space which was the first example of a distortable space. The special properties of Tsirelson's space T derive from certain saturation properties of its unit ball. The original construction of the space is indirect: one defines a subset V of ℓ_∞ with certain properties and takes T to be the linear span of V with the norm that makes V be the unit ball. This provides no expression for the norm of T, which makes it difficult to study the space.

Later, Figiel and Johnson [10] gave an implicit expression for the norm of the dual of T. It is the dual of the space originally constructed by Tsirelson that came to be known as Tsirelson's space and it is denoted in the literature by T. Since we are interested in analysing the original construction due to Tsirelson, we call the space he constructed T, the space F will be the completion of c_{00} with respect to the norm given by Figiel and Johnson.

Many results about T^* followed and the properties of T^* were widely studied (see [5]). The space T^* is asymptotically ℓ_1 whereas the space T constructed by Tsirelson is asymptotically c_0 (see [25]). As witnessed by Gowers' argument, the space c_0 lends itself to the use of tools from Ramsey theory, more that the space ℓ_1 does. It is for this reason that we were interested in providing a new description of the original construction of Tsirelson.

The other main result of this thesis is an expression for the norm of the space T. The expression can be modified to give the norm of the dual of any mixed Tsirelson space. In particular, our results can be adapted to give the norm for the dual of the space S constructed by Schlumprecht in [28].

The thesis is organized as follows: Chapter 2 includes the notation and basic definitions from Banach space theory that we shall use, as well as the background that motivates the results presented in Chapter 4. We mainly talk about distortion and oscillation stability. We sketch the proof of the non-distortion of ℓ_1. We also define Tsirelson's space and the space S constructed by Schlumprecht, sketch the proof of the distortion of these spaces, and outline how the distortion of ℓ_2 was obtained by Odell and Schlumprecht [23] from the distortion of S.

In Chapter 4 we obtain an expression for the norm of the space originally constructed by Tsirelson, and specify how this expression can be adapted to obtain the norm of the dual of mixed Tsirelson spaces.

In Chapter 3 we mention some classical Ramsey type theorems such as the Hales-Jewett Theorem and Hindman's Theorem. We present the fundamentals of the Galvin-Glazer methods of ultrafilter dynamics and sketch the proofs of Hindman's Theorem and the FIN_k Theorem using these tools. We also sketch the proof of the oscillation stability of c_0 from the FIN_k Theorem.

In Chapter 5 we present the inductive proof of the finite FIN_k Theorem, calculate the resulting upper bounds as well as the upper bounds for the finite stabilization principle in the special case of the spaces ℓ_∞^n, $n \in \mathbb{N}$.

CHAPTER 2

BANACH SPACE GEOMETRY

2.1 Notation and basic definitions

In this section we present the basic results and notation from Banach space geometry that we shall use later on. We denote by $\mathbb{N}$ the set of natural numbers starting at zero, and take a natural number n to be the set of its predecessors $\{0, 1, \ldots, n-1\}$. We denote the cardinality of a finite set S by $\#S$. We will omit the range of sequences and summations when it is clear from the context. We shall use bold face for elements of a vector space. The *space generated* by a sequence of vectors $(\mathbf{x}_n)$ in a normed vector space is the closure of the linear span of the sequence and is denoted by $[\mathbf{x}_n]$.

A good reference for Banach space theory is [9]. We denote the unit ball and the unit sphere of a Banach space $(X, ||\cdot||)$ by B_X and S_X respectively. If we are considering different norms on X we shall write $B_{||\cdot||}$ and $S_{||\cdot||}$ for the unit ball and unit sphere with respect to the norm $||\cdot||$.

We will work with the classical real sequence spaces c_0 and ℓ_p $(1 \leq p \leq \infty)$. Recall that ℓ_∞ is the vector space of bounded real sequences with the supremum norm and that c_0 is the closed subspace of ℓ_∞ containing the sequences converging to zero. Additionally, for $1 \leq p < \infty$, ℓ_p is the space of real sequences $\mathbf{a} = (a_n)$ such that $\sum |a_n|^p < \infty$ with the norm $||\mathbf{a}|| = \left(\sum |a_n|^p\right)^{1/p}$. We shall also work with spaces which are the completion of c_{00}, the vector space of finitely supported real sequences, with respect to some norm.

Let X, Y be normed vector spaces. We say a linear transformation $T : X \to Y$ is *bounded* if there exists a constant C such that $||T(\mathbf{x})|| \le C\,||\mathbf{x}||$ for all $\mathbf{x} \in X$. If T is also invertible and the inverse is bounded, we say T is an *isomorphism*. The *norm of a bounded linear transformation* is given by

$$||T|| = \sup\{||T(\mathbf{x})|| : ||\mathbf{x}|| = 1\}.$$

We say X and Y are C*-isomorphic* if there exists an isomorphism $T : X \to Y$ such that $||T|| \cdot ||T^{-1}|| \le C$.

Two norms $||\cdot||$, $||\cdot||'$ on a vector space X are *equivalent* if there exist constants C_1, C_2 such that $\frac{1}{C_1}\,||\mathbf{x}|| \le ||\mathbf{x}||' \le C_2\,||\mathbf{x}||$ for every $\mathbf{x} \in X$ (i.e., if $(X, ||\cdot||)$ and $(X, ||\cdot||')$ are isomorphic).

Let X be an infinite dimensional Banach space. A sequence $(\mathbf{x}_i)$ in X is a *Schauder basis* if for every $\mathbf{x} \in X$ there is a unique sequence of scalars (a_i) such that $\mathbf{x} = \sum a_i\mathbf{x}_i$. If X is finite dimensional, the notion of Schauder basis coincides with the notion of algebraic basis. A standard reference on Schauder bases is [18]; we follow the notation therein. A sequence $(\mathbf{x}_i)$ in a Banach space X is called a *basic sequence* if it is a Schauder basis for the closure of the span of $(\mathbf{x}_i)$. The *canonical projections* $P_n : X \to X$ associated to a Schauder basis $(\mathbf{x}_i)$ are defined by

$$P_n\left(\sum_{i\in\mathbb{N}} a_i\mathbf{x}_i\right) = \sum_{i\in n} a_i\mathbf{x}_i$$

for $n \in \mathbb{N}$ and $\sum a_i\mathbf{x}_i \in X$. The canonical projections are uniformly bounded and the *basis constant* is the supremum of $\{||P_n|| : n \in \mathbb{N}\}$.

For $j \in \mathbb{N}$ we define the *coefficient functional* $\mathbf{x}_j^* : X \to \mathbb{R}$ by $\mathbf{x}_j^*(\mathbf{x}) = a_j$ for $\mathbf{x} = \sum a_i\mathbf{x}_i$. We say the basis $(\mathbf{x}_i)$ is *shrinking* if X^* is the closure of the span of the sequence of coefficient functionals $(\mathbf{x}_j^*)$ with respect to the norm of X^*.

For $C \geq 1$, we say two basic sequences $(\mathbf{x}_i), (\mathbf{y}_i)$ in Banach spaces X, Y respectively are *C-equivalent* if there exist constants C_1 and C_2 such that $C_1 \cdot C_2 \leq C$ and for every $n \in \mathbb{N}$ and scalars $(a_i)_{i \in n}$

$$\frac{1}{C_1} \left\| \sum_{i \in n} a_i \mathbf{x}_i \right\|_X \leq \left\| \sum_{i \in n} a_i y_i \right\|_Y \leq C_2 \left\| \sum_{i \in n} a_i \mathbf{x}_i \right\|_X .$$

Given a Banach space X with Schauder basis $(\mathbf{x}_i)_{i \in I}$, $I = \mathbb{N}$ or $I = n$ for some $n \in \mathbb{N}$, for $\mathbf{x} = \sum_{i \in I} a_i \mathbf{x_i} \in X$ we define the *support* $\mathrm{supp}(\mathbf{x})$ of $\mathbf{x}$ by $\mathrm{supp}(\mathbf{x}) = \{i \in I : a_i \neq 0\}$, we say $\mathbf{x}$ is *positive* with respect to $(\mathbf{x}_i)_{i \in I}$ if each a_i is non negative. A sequence of vectors $(\mathbf{y}_i)_{i \in J}$ with $J = \mathbb{N}$ or $J = n$ for some $n \in \mathbb{N}$ is a *block subsequence* of $(\mathbf{x}_i)_{i \in I}$ if each $\mathbf{y}_i$ has finite support and $\max \mathrm{supp}(\mathbf{y}_i) < \min \mathrm{supp}(\mathbf{y}_j)$ for every $i < j \in J$. A space generated by a block sequence is called a *block subspace*. The sequence $(\mathbf{y}_i)_{i \in I}$ is *positive* with respect to $(\mathbf{x}_i)_{i \in I}$ if each $\mathbf{y}_i$ is positive; it is *normalized* if each $\mathbf{y}_i$ has norm 1. A subspace generated by a positive normalized block sequence is called a *positive subspace*. The *positive unit sphere* of a positive subspace Y of X, denoted by PS_Y, is the set of positive vectors in the unit sphere of Y.

When talking about subspaces of a Banach space with a Schauder basis we shall mainly consider block subspaces. This is justified by the following result of Pełczyński:

Theorem 1. [9, Section 4.4] *Let X be an infinite dimensional Banach space with Schauder basis $(\mathbf{x}_i)$ then every infinite dimensional subspace of X contains for every $K > 1$ a sequence which is K-equivalent to a block subsequence of $(\mathbf{x}_i)$.*

In Section 4.1 we will use the Bipolar Theorem, for this we need some additional notation.

Let X be a Banach space, and for $A \subset X$, $B \subset X^*$ we define

$$\begin{aligned} A^\circ &= \{\mathbf{f} \in X^* : \text{for all } \mathbf{x} \in A, |\mathbf{f}(\mathbf{x})| \leq 1\} \\ B^\circ &= \{\mathbf{x} \in X : \text{for all } \mathbf{f} \in B, |\mathbf{f}(\mathbf{x})| \leq 1\}. \end{aligned}$$

We shall need the following instance of the Bipolar Theorem:

Theorem 2 (Alaoglu, Banach)**.** [9, Section 3.4] *Let X be a Banach space. For every $A \subset X^*$, $A^{\circ\circ}$ is the weak*- closure of the convex hull of $A \cup -A$.*

Given a Banach space X with a shrinking basis $(\mathbf{x}_n)$, we can identify $\mathbf{f} \in X^*$ with the sequence of scalars (a_n) such that $\mathbf{f} = \sum a_n \mathbf{x}_n^*$. It is clear that if $(\mathbf{f}_m)$ is a sequence in X^* with $\mathbf{f}_m = \sum a_n^m \mathbf{x}_n^*$ and such that $\mathbf{f}_m$ converges to $\mathbf{f}$ with respect to the weak*-topology, then $\mathbf{f} = \sum b_n \mathbf{x}_n^*$ where $\lim\limits_{m\to\infty} a_n^m = b_n$. The following proposition shows that in fact, the weak*- topology and the topology of pointwise convergence coincide in the unit ball of X^*.

Proposition 3. *Let X be a Banach space with a shrinking basis $(\mathbf{x}_n)$ and let $(\mathbf{f}_m) \subset B_{X^*}$ be such that $\mathbf{f}_m = \sum a_n^m \mathbf{e}_n^*$ and for each $n \in \mathbb{N}$, $\lim_{m\to\infty} a_n^m = b_n$. If $\mathbf{f} = \sum b_n \mathbf{e}_n^*$, then $\mathbf{f} \in X^*$ and $\mathbf{f}_m$ converges to $\mathbf{f}$ with respect to the weak*-topology.*

We shall also consider the following geometric property of Banach spaces. We say a Banach space X is *uniformly convex* if for every $0 < \epsilon \leq 2$ there exist $\delta(\epsilon) > 0$ such that for all $\mathbf{x}, \mathbf{y} \in X$ such that $||\mathbf{x}|| = ||\mathbf{y}|| = 1$ and $||\mathbf{x} - \mathbf{y}|| > \epsilon$, we have that

$$\left|\left|\frac{\mathbf{x}+\mathbf{y}}{2}\right|\right| \leq 1 - \delta(\epsilon).$$

If X is a uniformly convex space and $\mathbf{x}, \mathbf{y} \in X$ we define the *angle* between $\mathbf{x}$ and $\mathbf{y}$ by

$$\alpha(\mathbf{x}, \mathbf{y}) = \left|\left|\frac{\mathbf{x}}{||\mathbf{x}||} - \frac{\mathbf{y}}{||\mathbf{y}||}\right|\right|.$$

As the next proposition shows, a stronger triangle inequality holds in uniformly convex Banach spaces:

Proposition 4. [6] *Let* X *be a uniformly convex space and let* $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{k-1} \in X$ *and* $\mathbf{y} = \sum_{i<k} \mathbf{x}_i$*. If* $\alpha_i = \alpha(\mathbf{x}_i, \mathbf{y})$*, then*

$$||\mathbf{y}|| \leq \sum_{i<k} (1 - 2\delta(\alpha_i)) \, ||\mathbf{x}_i|| \, .$$

2.2 Distortion

If $(X, ||\cdot||)$ is a Banach space, we say $|\cdot|$ is an *equivalent norm* if there exist $K, L \in \mathbb{R}$ such that $L|\mathbf{x}| \leq ||\mathbf{x}|| \leq K|\mathbf{x}|$ for every $\mathbf{x} \in X$ (i.e. if the identity map $id : (X, ||\cdot||) \to (X, |\cdot|)$ is an isomorphism). Note that equivalent norms generate the same topology and any scalar multiple of $||\cdot||$ gives an equivalent norm. Given an equivalent norm $|\cdot|$ one can ask whether there is an infinite dimensional subspace of X where $|\cdot|$ is close to being a scalar multiple of the original norm. More precisely, for $\lambda > 1$ and a Banach space $(X, ||\cdot||)$ we say an equivalent norm $|\cdot|$ λ-*distorts* the space if for every infinite dimensional subspace Y of X, we have that

$$\sup \left\{ \frac{|\mathbf{y}_1|}{|\mathbf{y}_2|} : \mathbf{y}_1, \mathbf{y}_2 \in Y \text{ and } ||\mathbf{y}_1|| = ||\mathbf{y}_2|| = 1 \right\} \geq \lambda,$$

so that that the new norm is far form being a scalar multiple of the original norm on any infinite dimensional subspace of X. If a space admits such an equivalent norm, we say it is λ-*distortable*.

James [17] proved that ℓ_1 and c_0 are not distortable, and it was not until 1974 that the first example of a distortable space was constructed by Tsirelson [31]. We shall first present the analysis of distortion for the classical spaces c_0 and ℓ_1 and then sketch the original construction of Tsirelson's space. This space has given rise to a new generation of Banach spaces, including the space S constructed by Schlumprecht [28], which played an important role in the proof of the distortion of ℓ_2. The question of whether ℓ_2 was distortable became known as the *Distortion Problem* which was settled by Odell and Schlumprecht [23]. We shall also present the main ideas behind their proof.

2.2.1 Classical spaces

The proofs of the results presented in this section can be found in [33]. The following propositions show that any Banach space containing an isomorphic copy of ℓ_1 (or c_0) contains a copy of ℓ_1 (resp. c_0) that is as close as we want to being a rescaled isometric copy of ℓ_1 (resp. c_0).

Proposition 5. [17] *Suppose a Banach space $(X, ||\cdot||)$ contains a subspace isomorphic to ℓ_1, then for every $\epsilon > 0$ there exists a sequence $(\mathbf{z}_n) \subset S_X$ such that for all $k \in \mathbb{N}$ and all $\alpha_0, \ldots, \alpha_{k-1} \in \mathbb{R}$,*

$$(1-\epsilon)\sum_{n<k}|\alpha_n| \leq \left|\left|\sum_{n<k}\alpha_n\mathbf{z}_n\right|\right| \leq \sum_{n<k}|\alpha_n|$$

Proof. Fix $\epsilon > 0$ and let $(\mathbf{x}_n) \subset X$ be a sequence equivalent to the standard basis for ℓ_1. Let M, N be such that for all $k \in \mathbb{N}$ and all $\alpha_0, \ldots, \alpha_{k-1} \in \mathbb{R}$,

$$M\sum_{n<k}|\alpha_n| \leq \left|\left|\sum_{n<k}\alpha_n\mathbf{x}_n\right|\right| \leq N\sum_{n<k}|\alpha_n|\,.$$

Set

$$K_m = \inf\left\{\left|\left|\sum_{n=m}^{k}\alpha_n\mathbf{x}_n\right|\right| : \sum_{n=m}^{k}|\alpha_n| = 1, k \in \mathbb{N}\right\}.$$

This is well defined because this set is bounded below by M. The sequence (K_m) is increasing and bounded above by N so let $K = \lim_{m\to\infty} K_m$. Let $\delta > 0$ be such that $\frac{1-\delta}{1+\delta} > 1-\epsilon$. Fix m_0 such that $K_{m_0} \geq (1-\delta)K$ and let $\alpha^0_{m_0+1}, \ldots, \alpha^0_{m_1}$ be scalars such that

$$\mathbf{y}_0 = \sum_{i=m_0+1}^{m_1} \alpha_i^0 \mathbf{x}_i$$

be such that $\sum_{i=m_0+1}^{m_1} |\alpha_i^0| = 1$ and $|\mathbf{y}_0| \leq (1+\delta)K$. Then let

$$\mathbf{y}_1 = \sum_{i=m_1+1}^{m_2} \alpha_i^1 \mathbf{x}_i$$

be such that $\sum_{i=m_1+1}^{m_2} |\alpha_i^1| = 1$ and $|\mathbf{y}_1| \leq (1+\delta)K$. Continue in this way to get a sequence $(\mathbf{y}_n)$ and let $\mathbf{z}_n = \frac{\mathbf{y}_n}{||\mathbf{y}_n||}$. Clearly for any $k \in \mathbb{N}$ and $\alpha_0, \ldots, \alpha_{k-1} \in \mathbb{R}$ we have that $\left|\left|\sum_{n\in k} \alpha_n \mathbf{z}_n\right|\right| \leq \sum_{n\in k} |\alpha_n|$. For the other inequality we have

$$\left|\left|\sum_{n=0}^{k} \alpha_n \mathbf{y}_n\right|\right| = \left|\left|\sum_{n=0}^{k} \alpha_n \sum_{i=m_n+1}^{m_{n+1}} \alpha_i^n \mathbf{x}_i\right|\right| \geq K_{m_0} \sum_{n=0}^{k} |\alpha_n| \sum_{i=m_n+1}^{m_{n+1}} |\alpha_i^n| = K_{m_0} \sum_{n=0}^{k} |\alpha_n|,$$

where the first inequality follows from the definition of K_{m_0}. Then

$$\begin{aligned}
\left|\left|\sum_{n=0}^{k} \alpha_n \mathbf{z}_n\right|\right| &\geq \frac{1}{(1+\delta)K} \left|\left|\sum_{n=0}^{k} \alpha_n \mathbf{y}_n\right|\right| \geq \frac{K_{m_0}}{(1+\delta)K} \sum_{n=0}^{k} |\alpha_n| \\
&\geq \frac{1-\delta}{1+\delta} \sum_{n=0}^{k} |\alpha_n| \geq (1-\epsilon) \sum_{n=0}^{k} |\alpha_n|.
\end{aligned}$$

□

Similarly for c_0 we have the following:

Proposition 6. [17] *Suppose a Banach space $(X, ||\cdot||)$ contains a subspace isomorphic to c_0, then for every $\epsilon > 0$ there exists a sequence $(\mathbf{z}_n) \subset S_X$ such that for all $k \in \mathbb{N}$ and all $\alpha_0, \ldots, \alpha_{k-1} \in \mathbb{R}$,*

$$\max |\alpha_n| \leq \left|\left|\sum_{n<k} \alpha_n \mathbf{z}_n\right|\right| \leq (1+\epsilon) \max |\alpha_n|$$

The proof is analogous to the proof in the case of ℓ_1 that we just presented.

2.2.2 Tsirelson-type spaces

Tsirelson's space was constructed in response to the question of whether or not Banach spaces must contain copies of c_0 or ℓ_p $(1 \leq p < \infty)$. Classical Banach spaces do contain copies of c_0 or some ℓ_p and there are some results, including Rosenthal's ℓ_1 Theorem and Dvoretsky's Theorem, that fueled the classical hope that this could be true for general Banach spaces.
In the 1970s Tsirelson [31] constructed a space with no isomorphic copies of c_0 or ℓ_p $(1 \leq p < \infty)$. Since any infinite dimensional Banach space which does not contain copies of c_0 or any ℓ_p, must contain a distortable subspace (see [24]), Tsirelson's space provided the first example of a distortable space.

The special properties of Tsirelson's space T derive from certain saturation properties of its unit ball. The original construction of the space is geometric: one defines a subset V of ℓ_∞ with certain properties and then takes T to be the linear span of V with the norm that makes V be the unit ball.

We shall outline the steps of the original construction of T; more details can be found in [31]. First set A to be the smallest subset of ℓ_∞ with the following properties:

(i) A is contained in the unit ball of ℓ_∞ and contains the vectors $(\mathbf{e}_n)_{n\in\mathbb{N}}$,

(ii) for every $\mathbf{x} \in A$ and $\mathbf{y} \in \ell_\infty$ such that $|\mathbf{y}| \leq |\mathbf{x}|$ pointwise (that is, $y_n \leq x_n$ for all $n \in \mathbb{N}$), we have that $\mathbf{y} \in A$,

(iii) if $N \in \mathbb{N}$ and $(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ is a block subsequence of $(\mathbf{e}_n)_{n\in\mathbb{N}}$ contained in A, then $\frac{1}{2}Q_N(\mathbf{x}_1 + \ldots + \mathbf{x}_N) \in A$, where Q_N is the projection onto $span\{\mathbf{e}_n :$

$n \geq N\}$.

Let K be the closure of A in the topology of pointwise covergence, then K is contained in a ball of c_0, properties (i)-(iii) are preserved and in addition one gets

(iv) For every $\mathbf{x} \in K$ there exists $n \in \mathbb{N}$ such that $2Q_n(\mathbf{x}) \in K$.

Note that in a ball on c_0 the weak topology coincides with the topology of pointwise convergence and in this topology K is compact. Finally one must argue that setting V to be the closed convex hull of K, one obtains a convex weakly compact set in c_0 with properties $(i)-(iv)$.

The space T is the closed linear span of V endowed with the norm $||\cdot||_T$ that makes V be the unit ball. This is how conditions $(i)-(iv)$ on V imply the desired properties of the space T:

Condition (i) makes the standard vectors $(\mathbf{e}_n)$ have norm 1 in T. Condition (iv) guarantees that $(\mathbf{e}_n)$ is a Schauder basis for T since it implies that for every $\mathbf{x} \in V$ and every $q \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $2^q Q_n(\mathbf{x}) \in V$, and hence the sequence $(Q_n(\mathbf{x}))_{n\in\mathbb{N}}$ converges to zero in norm. Condition (ii) makes the basis $(\mathbf{e}_n)$ *1-unconditional*, that is. if $k \in \mathbb{N}$, $a_n \leq b_n$ for $n < k$ and we let $\mathbf{y} = \sum_{n<k} a_n \mathbf{e}_n$, $\mathbf{x} = \sum_{n<k} b_n \mathbf{e}_n$ then $\mathbf{x}/||\mathbf{x}||_T \in V$ and so by condition (ii) $\mathbf{y}/||\mathbf{x}||_T \in V$, therefore $||\mathbf{y}||_T \leq ||\mathbf{x}||_T$.

We say a Banach space X is *finitely universal* if there exists $C > 1$ such that for each finite dimensional normed space E there exists a subspace $F \subseteq X$ of the same dimension which is isomorphic to E with constant smaller than C. If E is a finite dimensional normed space, then E can be embedded with constant ϵ in ℓ_∞^N for some N sufficiently large. Therefore to see that a space is finitely universal it suffices to verify the above condition for spaces of the form ℓ_∞^N, $N \in \mathbb{N}$. From condition (iii) we get that for any block subsequence $(\mathbf{x}_n)$ of $(\mathbf{e}_n)$, $N \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$

$$\max |\lambda_j| \leq ||\lambda_1 \mathbf{x}_{N+1} + \ldots + \lambda_N \mathbf{x}_{2N}||_T \leq 2 \max |\lambda_j|$$

Hence any infinite dimensional block subspace of T contains 2-isomorphic copies of ℓ_∞^N for every $N \in \mathbb{N}$ and is therefore finitely universal. The spaces ℓ_p $(1 < p < \infty)$ are uniformly convex and it can be shown (see [31]) that a uniformly convex space cannot be finitely universal. It follows that T contains no isomorphic copies of ℓ_p $(1 < p < \infty)$.

Finally one can show that T is reflexive and since it has an unconditional basis, it follows that T contains no copies of c_0 or ℓ_1.

Note that the construction of T provides no expression for its norm, which makes it difficult to study the space. Shortly after Tsirelson's construction, Figiel and Johnson [10] gave an expression for the norm of T^*, the dual of T.

We first need to introduce some notation. Recall that c_{00} is the vector space of all real valued sequences whose elements are eventually zero. If E, F are finite subsets of $\mathbb{N}$ we write $E < F$ if $\max E < \min F$. If $E \subseteq \mathbb{N}$ and $\mathbf{x} = (\mathbf{x}_i)$ a real sequence, not necessarily with finite support, then $E\mathbf{x}$ is the sequence such that

$(E\mathbf{x})_i = x_i$ if $i \in E$ and $(E\mathbf{x})_i = 0$ otherwise. In this notation the norm $||\cdot||_F$ found by Figiel and Johnson is given by

$$||\mathbf{x}||_F = \max\left\{||\mathbf{x}||_\infty, \frac{1}{2}\max\left\{\sum_{i=1}^{k}||E_i\mathbf{x}||_F : k \in \mathbb{N}, k \leq E_1 < \cdots < E_k\right\}\right\}, \quad (2.1)$$

and such that the standard basic vectors $\mathbf{e}_i$ have norm 1. This formula defines implicitly the norm of T^*. Note that this expression defines $||\cdot||_F$ on c_{00} inductively in the cardinality of the support, alternatively, one can define a sequence of norms on c_{00} whose pointwise limit is a norm that satisfies (2.1). Figiel and Johnson proved that T^* is isometrically isomorphic to the completion of c_{00} with respect to $||\cdot||_F$.

It is the dual of the space originally constructed by Tsirelson that came to be known as Tsirelson's space and it is denoted in the literature by T. Since we are interested in analysing the original construction due to Tsirelson, we call the space he constructed T, the space F will be the completion of c_{00} with respect to the norm given by Figiel and Johnson.

After the above expression for its norm was available, many results about F followed and its properties were widely studied (see [5]). We shall now sketch a direct proof of the distortion of F:

Proposition 7. [24] *F is $(2-\epsilon)$-distortable for every $\epsilon > 0$.*

Proof. We will consider vectors of the form $\mathbf{y} = 1/k\sum_{i=1}^{k} y_i$ where $(\mathbf{y}_i)_{1\leq i\leq k}$ is equivalent to the standard basis of ℓ_1^k, such a vector is called an ℓ_1^k-*average.* Let $\epsilon > 0$ and choose $n \in \mathbb{N}$ such that $1/n < \epsilon/3$. Define for $\mathbf{x} \in F$ the norm

$$|\mathbf{x}| = \sup\left\{\sum_{i=1}^{n}||E_i\mathbf{x}||_F : E_1 < \ldots < E_n\right\}.$$

This norm is equivalent to $||\cdot||_F$ since $||\mathbf{x}||_F \leq |\mathbf{x}| \leq n\,||\mathbf{x}||_F$ for $\mathbf{x} \in F$. It suffices to prove that for any normalized block subsequence $(\mathbf{x}_i)$ of $(\mathbf{e}_i)$ we have that

$$\begin{aligned}
\inf\{|\mathbf{x}| : ||\mathbf{x}||_F = 1, \mathbf{x} \in [\mathbf{x}_i]\} &= 1, \\
\sup\{|\mathbf{z}| : ||\mathbf{z}||_F = 1, \mathbf{z} \in [\mathbf{x}_i]\} &> 2/(1+\epsilon),
\end{aligned}$$

where $[\mathbf{x}_i]$ denotes the closed subspace generated by the sequence $(\mathbf{x}_i)$. We shall use the fact that in any block subsequence of $(\mathbf{e}_i)$ we can find for every $k, m \in \mathbb{N}$ l_1^k-averages whose supports start after m.

Let $(\mathbf{x}_i)$ be a normalized block sequence of $(\mathbf{e}_i)_{i>n}$. For any $k > n$ one can find a normalized block subsequence $(\mathbf{y}_i)_{1\leq i\leq k}$ of $(\mathbf{x}_i)_{i>k}$ which is equivalent to the standard basis of ℓ_1^k with the equivalence constant as close to 1 as we want. Let $\mathbf{y}$ be the ℓ_1^k average $\mathbf{y} = 1/k\sum_{i=1}^k y_i$ and note that $||\mathbf{y}||_F \approx 1$. Also if $E_1 < \ldots < E_n$ then setting $I = \{i : E_j \cap \operatorname{supp}(\mathbf{y}_i) \neq \emptyset \text{ for at most one } j\}$ and $J = \{1, \ldots, k\} \setminus I$ we have that $|J| \leq n$ and

$$\begin{aligned}
\sum_{i=1}^n ||E_i\mathbf{y}||_F &\leq \frac{1}{k}\left(\sum_{i\in I} ||\mathbf{y}_i||_F + \sum_{i\in J}\sum_{j=1}^n ||E_j y_i||_F\right) \\
&\leq \frac{1}{k}\left(\sum_{i\in I} ||\mathbf{y}_i||_F + \sum_{i\in J} 2\,||\mathbf{y}_i||_F\right) \\
&\leq \frac{1}{k}(k - |J| + 2|J|) \\
&\leq 1 + \frac{n}{k}
\end{aligned}$$

Thus $\inf\{|\mathbf{x}| : ||\mathbf{x}||_F = 1, \mathbf{x} \in [\mathbf{x}_i]\} = 1$. Now let

$$\mathbf{z} = \frac{2}{n}\sum_{i=1}^n \mathbf{z}_i \in [\mathbf{x}_i]_{i\geq n}$$

be such that each $\mathbf{z}_i$ is an $\ell_1^{k_i}$-average where k_{i+1} is chosen very large depending on $\max(supp\mathbf{z}_i)$ and ϵ. Since $||\mathbf{z}_i||_F \approx 1$, it follows that $|\mathbf{z}| \geq 2/n\sum ||\mathbf{z}_i||_F \approx 2$. Now if $m \leq E_1 < \ldots < E_m$ and i_0 is the smallest i such that $m < \max(\text{supp}(\mathbf{z}_i))$. By the choice of the k_i, $1 \leq i \leq n$ we have that $k_i > m$ for $i_0 < i \leq m$. Hence by the previous calculations we have that

$$\begin{aligned}
\frac{1}{2}\sum_{j=1}^{m} ||E_j z||_F &= \frac{1}{n}\sum_{j=1}^{m} \left|\left| E_j\left(\sum_{i=1}^{n} z_i\right)\right|\right|_F \\
&\leq \frac{1}{n}\left(\sum_{j=1}^{m} ||E_j z_{i_0}||_F + \sum_{i=i_0+1}^{n}\sum_{j=1}^{m} ||E_j z_i||_F\right) \\
&\leq \frac{1}{n}\left(2\,||\mathbf{z}_{i_0}||_F + \sum_{i=i_0+1}^{n}\left(1+\frac{m}{k_i}\right)\right) \\
&\leq \frac{2}{n} + 1 + \epsilon/3 \\
&< 1+\epsilon.
\end{aligned}$$

By the definition of the norm, we get $||\mathbf{z}||_F \leq 1+\epsilon$. Hence

$$\sup\{|\mathbf{z}| : ||\mathbf{z}||_F = 1, \mathbf{z} \in [\mathbf{x}_i]\} > 2/(1+\epsilon).$$

□

We shall also define Mixed Tsirelson spaces. These are spaces whose norm is given by a variation of the expression found by Figiel and Johnson.

Let $\mathcal{M}$ denote a compact family (in the topology of pointwise convergence) of finite subsets of $\mathbb{N}$ which includes all singletons. We say that a family $E_1 < \cdots < E_n$ of subsets of $\mathbb{N}$ is $\mathcal{M}$-admissible if there exists $M = \{m_i\}_{i=1}^n$ in $\mathcal{M}$ such that $m_1 \leq E_1 < m_2 \leq E_2 < \cdots < m_n \leq E_n$.

Definition 8. Let $(\mathcal{M}_n)_n, (\theta_n)_n$ be two sequences with each $\mathcal{M}_n$ a compact family of finite subsets of $\mathbb{N}$, $0 < \theta_n < 1$ and $\lim_n \theta_n = 0$. The *mixed Tsirelson space*

$\mathcal{T}[(\mathcal{M}_n, \theta_n)_n]$ is the completion of c_{00} with respect to the norm

$$\|\mathbf{x}\|_* = \max\{||\mathbf{x}||_\infty, \sup_n \sup \theta_n \sum_{i=1}^{k} \|E_i \mathbf{x}\|_*\},$$

where the inside sup is taken over all choices $E_1 < E_2 < \cdots < E_k$ of $\mathcal{M}_n$-admissible families.

The definitions above and further properties of mixed Tsirelson spaces can be found in [1, Part A, Chapter 1]. In this notation, the space S constructed by Schlumprecht in [28], is the mixed Tsirelson space $\mathcal{T}[(\mathcal{A}_n, \frac{1}{\log(n+1)})_n]$, where $\mathcal{A}_n = \{F \subset \mathbb{N} : \#F \leq n\}$. That is, the space S is the completion of c_{00} with respect to the norm

$$||\mathbf{x}||_S = \max\left\{||\mathbf{x}||_\infty, \max\left\{\frac{1}{\log(l+1)} \sum_{i=1}^{l} ||E_i \mathbf{x}||_S : l \in \mathbb{N}, E_1 < \cdots < E_l\right\}\right\}.$$

We say a space is *arbitrarily distortable* if it is λ-distortable for all $\lambda > 1$. The space S was the first known example of an arbitrarily distortable space. The proof of the arbitrary distortion of S follows the argument for the distortion of F we just presented. We shall outline the details below; the complete proof can be found in [28].

For each $l \in \mathbb{N}$, define the norm

$$||\mathbf{x}||_l = \sup\left\{\frac{1}{\log(l+1)} \sum_{i=1}^{l} ||E_i \mathbf{x}||_S\right\},$$

for $\mathbf{x} \in S$. These are equivalent norms since for $\mathbf{x} \in S$

$$\frac{1}{\log(l+1)} ||\mathbf{x}||_S \leq ||\mathbf{x}||_l \leq ||\mathbf{x}||_S.$$

It follows from the following theorem that $||\cdot||_l$ is an $\log(l+1)$-distortion for each $l \in \mathbb{N}$:

Theorem 9. [28] *For each $l \in \mathbb{N}$, $\epsilon > 0$, and each infinite dimensional subspace Z of S there are $\mathbf{z}_1, \mathbf{z}_2 \in Z$ with $||\mathbf{z}_1||_S = ||\mathbf{z}_2||_S = 1$ and*

$$||\mathbf{z}_1||_l \geq 1 - \epsilon, \ \text{and} \ ||\mathbf{z}_2||_l \leq \frac{1+\epsilon}{\log(l+1)}.$$

We shall sketch the proof of this theorem.

Proof. One must first show that ℓ_1 is finitely block representable in any block subspace of S. Let Z be a block subspace of S and let $\epsilon > 0$. Most of the work in the proof goes into finding a sequence $(\mathbf{y}_i)_{1 \leq i \leq l}$ of ℓ_1-averages such that $\mathbf{y}_1 < \ldots < \mathbf{y}_l$ and $||\mathbf{y}_i||_S \geq 1 - \epsilon$ for $1 \leq i \leq l$ and $\left|\left|\sum_{i=1}^{l} y_i\right|\right|_S \leq \frac{l}{\log(l+1)}$. If $(\mathbf{y}_i)_{1 \leq i \leq l}$ is such a sequence, one can set

$$\mathbf{z}_1 = \sum_{i=1}^{l} \mathbf{y}_i \Big/ \left|\left|\sum_{i=1}^{l} \mathbf{y}_i\right|\right|_S.$$

Hence

$$\begin{aligned} ||\mathbf{z}_1||_l &\geq \frac{1}{\log(l+1)} \sum_{i=1}^{l} ||\mathbf{y}_i||_S \Big/ \left|\left|\sum_{i=1}^{l} y_i\right|\right|_S \\ &\geq 1 - \epsilon. \end{aligned}$$

Now to find $\mathbf{z}_1$, let $n \in \mathbb{N}$ be such that $2l/n \leq \epsilon$ and choose $(\mathbf{y}_i)_{1 \leq i \leq n}$ to be a block sequence in Z equivalent to the standard basis of ℓ_1^n with the equivalence constant as close to 1 as we want. Let $\mathbf{z}_2$ be the ℓ_1^n-average $\mathbf{z}_2 = \frac{1}{n}\sum_{i=1}^{n} \mathbf{y}_i$ and note that $||\mathbf{y}||_S \approx 1$. Let $E_1 < \ldots < E_l$ be such that

$$||\mathbf{z}_2||_l = \frac{1}{\log(l+1)} \sum_{i=1}^{l} ||E_i z_2||_S.$$

We may assume each E_i is an interval. For $1 \leq i \leq l$ let

$$\tilde{E}_i = \bigcup \{\mathrm{supp}(\mathbf{y}_j) : 1 \leq j \leq n, \mathrm{supp}(\mathbf{y}_j) \subseteq E_i\}.$$

Let $\mathbf{z}_2' = \sum_{i=1}^{l} \left(E_i\mathbf{z}_2 - \tilde{E}_i\mathbf{z}_2 \right)$ Then

$$\begin{aligned} ||E_i z_2||_S &\leq ||E_i(\mathbf{z}_2 - \mathbf{z}_2')||_S + ||E_i z_2'||_S \\ &= \left|\left|\tilde{E}_i(\mathbf{z}_2)\right|\right|_S + ||E_i\mathbf{z}_2'||_S \\ &\lesssim \left|\left|\tilde{E}_i(\mathbf{z}_2)\right|\right|_S + \frac{2}{n}. \end{aligned}$$

Observe that $(\tilde{E}_i(\mathbf{z}_2))_{1\leq i\leq l}$ is a block subsequence of $(\mathbf{y}_i)_{1\leq i\leq n}$ so the corresponding normalized sequence is again equivalent to the standard basis of ℓ_1^l with equivalence constant as close to 1 as we want. Therefore

$$\left|\left|\sum_{i=1}^{l} \tilde{E}_i(\mathbf{z}_2)\right|\right|_S \approx \sum_{i=1}^{l} \left|\left|\tilde{E}_i(\mathbf{z}_2)\right|\right|_S .$$

So we have that

$$\begin{aligned} ||\mathbf{z}_2||_l &\leq \frac{1}{\log(l+1)} \left(\frac{2l}{n} + \sum_{i=1}^{l} \left|\left|\tilde{E}_i(\mathbf{z}_2)\right|\right|_S \right) \\ &\lesssim \frac{1}{\log(l+1)} \left(\frac{2l}{n} + 1 \right) \\ &\leq \frac{1+\epsilon}{\log(l+1)}. \end{aligned}$$

This finishes the proof of the theorem. □

2.2.3 The distortion problem

In 1993 Odell and Schulmprecht proved that ℓ_2 and moreover the spaces ℓ_p $(p > 1)$ are arbitrarily distortable. For uniformly convex spaces there is a condition equivalent to distortion that suggests how to transfer the distortion from one space to another. We shall need a few definitions to state this precisely.

For a Banach space X, $A \subseteq S_X$ is *asymptotic* if for every infinite dimensional subspace Y of X and every $\epsilon > 0$ we have that $S_Y \cap A_\epsilon \neq \emptyset$, where

$A_\epsilon = \{\mathbf{y} : ||\mathbf{x}-\mathbf{y}|| < \epsilon \text{ for some } \mathbf{x} \in A\}$. We say $A, B \subseteq S_X$ are separated if $\inf\{||\mathbf{x}-\mathbf{y}|| : \mathbf{x} \in A, \mathbf{y} \in B\} > 0$. The following proposition gives a condition under which a uniformly convex space is distortable:

Proposition 10. [23] *A Banach space X is distortable if there exist separated asymptotic sets $A, B \subseteq S_X$.*

Proof. Let X be a uniformly convex Banach space and suppose $A, B \subseteq S_X$ are separated and asymptotic. Let V be the closed convex hull of $A \cup -A \cup \gamma B_{||\cdot||}$, where $-A = \{-\mathbf{x} : \mathbf{x} \in A\}$ and $\gamma B_{||\cdot||}$ is the ball of radius γ around the origin for some $\gamma > 0$. Define $|\cdot|$ to be the norm whose unit ball is V. Note that $\gamma B_{||\cdot||} \subseteq B_{|\cdot|} \subseteq B_{||\cdot||}$ and so $|\cdot|$ is an equivalent norm. Let Y be an infinite dimensional subspace of X. Since A, B are asymptotic, there exist $\mathbf{x} \in Y \cap A_\epsilon$, $\mathbf{y} \in Y \cap B_\epsilon$. Let k be such that $ky \in V$. Suppose $ky = \sum_{i<n} a_i \mathbf{x}_i$ for some $\mathbf{x}_0, \ldots, \mathbf{x}_{n-1} \in A$ and $a_0, \ldots, a_{n-1}$ such that $\sum_{i<n} a_i = 1$. By Proposition 4 we have that

$$||\mathbf{y}|| = \left|\left|\sum_{i<n} \frac{a_i}{k}\mathbf{x}_i\right|\right| \leq \sum_{i<n} (1-2\delta(C))\,\frac{a_i}{k} = \frac{1-2\delta(C)}{k},$$

where $C = \inf\{||\mathbf{x}-\mathbf{y}|| : \mathbf{x} \in A, \mathbf{y} \in B\}$. If ky is a convex combination of a collection of elements of $A \cup -A \cup \gamma B_{||\cdot||}$ that includes some elements of $\gamma B_{||\cdot||}$, say $\mathbf{y} = \sum_{i<n} a_i \mathbf{x}_i$ with $\mathbf{x}_0 \in \gamma B_{||\cdot||}$ and $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1} \in A$, then

$$||\mathbf{y}|| = \left|\left|\sum_{i<n} \frac{a_i}{k}\mathbf{x}_i\right|\right| \leq \frac{||\mathbf{x}_0||}{k} + \frac{1-2\delta(C)}{k}.$$

Since $||\mathbf{y}|| = 1$, in any case it follows that $k \leq 1 - 2\delta(C) + \gamma$, hence

$$\frac{|\mathbf{x}|}{|\mathbf{y}|} \geq \frac{1}{1-2\delta(C)+\gamma} > 1$$

for small enough γ. □

The spaces ℓ_2 and more generally ℓ_p $(p > 1)$ are uniformly convex, therefore to obtain the distortion of these spaces it suffices to obtain a pair of separated asymptotic sets. Odell and Schlumprecht followed this strategy to obtain the distortion of the spaces ℓ_p, $(p > 1)$. We shall outline the steps of the proof as presented in [22], the details can be found in [23].

Their proof is based on the idea of transferring separated asymptotic sets from one space to the other using uniform homeomorphisms. A *uniform homeomorphism* is a uniformly continuous bijection with uniformly continuous inverse. An example of such a map is the *Mazur map* defined by

$$\begin{aligned} M_p : S_{\ell_1} &\rightarrow S_{\ell_p} \\ (a_i)_{i\in\mathbb{N}} &\mapsto (sign(a_i)|a_i|^{1/p})_{i\in\mathbb{N}}. \end{aligned}$$

The map M_p preserves asymptotic sets and more generally uniform homeomorphisms preserve separated sets, so it suffices to find separated asymptotic sets in ℓ_1. Note that since ℓ_1 is not uniformly convex, the existence of such pair of sets does not contradict the fact that ℓ_1 is not distortable.

The separated asymptotic sets come from the space S, which as the next proposition shows contains a sequence of separated asymptotic sets:

Proposition 11. [23] *There are sequences of sets $A_k \subseteq S_S$, $A_k^* \subseteq S_{S^*}$ and a monotone sequence (ϵ_k) decreasing to zero such that:*

(i) *A_k is asymptotic in S for all k,*

(ii) *$|\mathbf{x}_k^*(\mathbf{x}_l)| \leq \epsilon_{\min(k,l)}$ for $k \neq l$, $\mathbf{x}_k^* \in A_k^*$ and $\mathbf{x}_l \in A_l$,*

(iii) *for all k and $\mathbf{x} \in A_k$ there exists $\mathbf{x}^* \in A_k^*$ such that $\mathbf{x}^*(\mathbf{x}) > 1 - \epsilon_k$.*

It also follows from the existence of this sequence of asymptotic sets that S is arbitrarily distortable via the collection of norms $|\mathbf{x}|_k = \frac{1}{k}\,||\mathbf{x}||_S + \sup\{|\mathbf{x}^*(\mathbf{x})| : \mathbf{x}^* \in A_k^*\}$.

Using the sets given by Proposition 11 one can define the following subsets of ℓ_1

$$B_k = \left\{ \frac{\mathbf{x}_k^* \circ \mathbf{x}_k}{||\mathbf{x}_k^* \circ \mathbf{x}_k||_{\ell_1}} : \mathbf{x}_k \in A_k, \mathbf{x}_k^* \in A_k^*, \text{ and } ||\mathbf{x}_k^* \circ \mathbf{x}_k||_{\ell_1} \geq 1 - \epsilon_k \right\},$$

where $\mathbf{x}^* \circ \mathbf{x}$ denotes the sequence obtained by pointwise multiplication. Thus if $\mathbf{x}^* = \sum a_i \mathbf{e}_i^*$ and $\mathbf{x} = \sum b_i \mathbf{e}_i$, then $\mathbf{x}^* \circ \mathbf{x} = (a_i b_i)$, which is an element of ℓ_1. The sets B_k are asymptotic, the proof of this fact is technical and can be found in [23]. Hence the distortion of ℓ_2 follows by taking $C_k = M_2(B_k)$.

CHAPTER 3

RAMSEY THEORY

3.1 Overview

Ramsey theory takes its name from Frank P. Ramsey who stated and proved a higher dimensional version of the pigeon-hole principle [27]. *Ramsey's Theorem* states that for any natural numbers r, d and any r-coloring of $[\mathbb{N}]^d$, the set of d-element subsets of $\mathbb{N}$, there exists an infinite subset A of $\mathbb{N}$ such that $[A]^d$, the collection of all d-element subsets of A, is monochromatic. We see $[A]^d$ as a copy of $[\mathbb{N}]^d$. This reveals the general essence of Ramsey-type theorems, that is, given a finite coloring of a structure one looks for a copy of the original structure which realizes fewer colors, or in some cases finite but arbitrarily large substructures that realize fewer colors. The notion of copy varies depending on the context. An important example of a Ramsey-type theorem is van der Waerden's Theorem:

Theorem 12 (Van der Waerden)**.** [32] *For any natural number r and any r-coloring of $\mathbb{N}$ there exists $i < r$ such that there are arbitrarily long arithmetic progressions with color i.*

Note that one cannot hope to obtain for any coloring an infinite monochromatic arithmetic progression. This is easily seen by considering the 2-coloring that assigns red to 0, then blue to $1, 2$, then red to $3, 4, 5$ and so on, alternating between red and blue on blocks of increasing length.

One important generalization of van der Waerden's Theorem is the Hales-Jewett Theorem. We need to introduce some notation before stating the theorem. Let $L = \bigcup L_n$ be a given alphabet decomposed into an increasing chain of finite subsets

L_n and let v be a variable symbol different from all the symbols in L. We denote by W_L the collection of finite words in the language L. A word w in the language generated by $L \cup \{v\}$ is called a *variable word* if w has at least one occurrence of the symbol v. For a variable word w and $a \in L$ we denote by $w[a]$ the word obtained from w by replacing every occurrence of v by a. We are now ready to state the theorem:

Theorem 13 (Infinite Hales-Jewett)**.** [4],[3] *For any natural number r and any r-coloring of W_L there exists an infinite sequence of variable words (w_n) such that*

$$[w_n]_L = \left\{ w_{n_0}[\lambda_0]^\frown w_{n_1}[\lambda_1]^\frown \ldots {}^\frown w_{n_k}[\lambda_k] : n_0 < n_1 < \ldots < n_k, \lambda_i \in L_i \text{ for } i \leq k, k \in \mathbb{N} \right\},$$

the language generated by the sequence (w_n), is monochromatic.

One more result in Ramsey theory that is of particular interest for us is Hindman's Theorem:

Theorem 14 (Hindman)**.** [15] *For any natural number r and any r-coloring of $\mathbb{N}$ there exists an infinite sequence (x_n) of distinct natural numbers such that the collection of non-repeating finite sums of elements of the sequence is monochromatic.*

The theorems we have just presented have finite counterparts. The general form of the finite versions is, given a number of colors r and a number m there exists a number n big enough so that for any r-coloring of a structure of size n there exists a substructure of size m that realizes fewer colors. We state below the finite version of Ramsey's Theorem:

Theorem 15 (Finite Ramsey)**.** *For any $m, d, r \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for any r-coloring of $[n]^d$, the d-element subsets of n, there exists $A \subset n$ of size m such that $[A]^d$, the set of d-element subsets of A, is monochromatic.*

Let $R_d(m)$ be the minimum n satisfying the conditions in the theorem for $r = 2$.

We now state the finite versions of van der Waerden's Theorem, Hales Jewett Theorem and also state Folkman's Theorem which is finite version of Hindman's Theorem.

Theorem 16 (Finite Van der Waerden)**.** *For every $m, r \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for any r-coloring of n there exists a monochromatic arithmetic progression of length m.*

Let $W(m)$ be the the minimum n satisfying the conditions in the theorem for $r = 2$.

Theorem 17 (Finite Hales-Jewett)**.** [14] *For every $m, r \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for any alphabet L of size m and any r-coloring of $W_L(n)$, the words in the alphabet L of length n, there exists a variable word w such that the set $\{w[\lambda] : \lambda \in L\}$ is monochromatic.*

Let $HJ(m)$ be the minimum n satisfying the conditions in the theorem for $r = 2$.

Theorem 18 (Folkman)**.** [20] *For every $m, r \in \mathbb{N}$ there exists n such that for any r-coloring of n there exist pairwise disjoint subsets $x_0, x_1, \ldots, x_{m-1}$ of n such that the collection of unions of these sets is monochromatic.*

In Section 5.1, we shall present a proof of a variation of Folkman's theorem where the sets $x_0, x_1, \ldots, x_{m-1}$ are not only disjoint but they are also in block position. That is, $\max x_i < \min x_{i+1}$ for $i < m-1$. We will also analyse the resulting upper bounds.

The finite versions just presented can be obtained from the infinite versions using compactness, however such arguments give no information about the numbers $R_d(m), W(m)$ and $HJ(m)$.

It turns out that these numbers grow very rapidly and, in order to deal with such rapidly growing functions, we use the Ackermann Hierarchy. The Ackermann hierarchy is the sequence of functions $f_i : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$\begin{aligned} f_1(x) &= 2x \\ f_{i+1}(x) &= f_i^{(x)}(1) \end{aligned}$$

Already the function f_3 grows very fast with $f_3(5) = 2^{65536}$ a number with nearly 20,000 decimal digits (see [13, section 2.7]). The function f_3 is called TOWER and f_4 is called WOW. The Ackermann function is obtained by diagonalization and grows even faster than any f_i, $i \in \mathbb{N}$:

$$f_\omega(x) = f_x(x)$$

There is a slight variation of the function TOWER in the Ackermann Hierarchy which is useful to express upper bounds for the Ramsey numbers $R_d(m)$ and the van der Waerden numbers $W(m)$. The tower functions $t_i(x)$ are defined inductively by

$$\begin{aligned} t_1(x) &= x \\ t_{i+1}(x) &= 2^{t_i(x)} \end{aligned}$$

We shall use the following well known upper bounds for $R_d(m)$ and $W(m)$:

$$\begin{aligned} R_d(m) &\leq t_d(c_d m) \\ W(m) &\leq 2^{2^{2^{2^{2^{m+9}}}}} = t_6(m+9). \end{aligned}$$

(Where c_d is a constant that depends on d). See [13, Section 4.7] for a deduction of the bound for $R_d(m)$. The bound for $W(m)$ was found by Gowers in [12]. For a more detailed discussion of Ramsey numbers, van der Waerden numbers and Hales Jewett numbers, see [13, Ch. 4].

3.2 The FIN_k Theorem

The FIN_k Theorem is a generalization of Hindman's Theorem. For a fixed $k \in \mathbb{N}$, FIN_k is the set of all functions $f : \mathbb{N} \to \{0, 1, \ldots, k\}$ that attain the maximum value k and whose support $\mathrm{supp}(f) = \{n \in \mathbb{N} : f(n) \neq 0\}$ is finite. Given $f, g \in \mathrm{FIN}_k$ we say that $f < g$ if the maximum of the support of f is smaller than the minimum of the support of g. We consider two operations in FIN_k defined pointwise as follows:

(i) *Sum*: $(f + g)(n) = f(n) + g(n)$ for $f < g$,

(ii) *Tetris*: $T : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$. For $f \in \mathrm{FIN}_k$, $(Tf)(x) = \max\{0, f(x) - 1\}$.

Note that if $f_0 < \ldots < f_{n-1} \in \mathrm{FIN}_k$ then $T^{l_0}(f_0) + \ldots + T^{l_{n-1}}(f_{n-1}) \in \mathrm{FIN}_k$ as long as one of $l_0, \ldots, l_{n-1}$ is zero. A sequence $(f_i)_{i \in I}$ of elements of FIN_k with $I = \mathbb{N}$ or $I = n$ for some $n \in \mathbb{N}$ such that $f_i < f_j$ for all $i < j \in I$ is called a *block sequence.* The FIN_k Theorem states that for any finite coloring $c : \mathrm{FIN}_k \to \{0, 1, \ldots, r - 1\}$ there exists an infinite block sequence $(f_i)_{i \in \mathbb{N}}$ such that the combinatorial space

$\langle f_i \rangle_{i\in\mathbb{N}}$ generated by the sequence $(f_i)_{i\in\mathbb{N}}$,

$$\langle f_i \rangle_{i\in\mathbb{N}} = \{T^{l_1}(f_{i_1}) + \ldots + T^{l_n}(f_{i_n}) : i_1 < \ldots < i_n, \min\{l_1, \ldots, l_n\} = 0\},$$

is monochromatic.

The proof uses the Galvin-Glazer methods of ultrafilter dynamics. These methods are now widely used to give streamlined proofs of Ramsey type theorems such as Hindman's Theorem and different variations of the Hales-Jewett Theorem. Detailed proofs of these and other Ramsey type theorems can be found in ([29, Chapter 2]). We shall present here the fundamentals behind these methods, use this machinery to obtain Hindman's Theorem and sketch the proof of the infinite FIN_k Theorem.

We say $(S, *)$ is a *partial semigroup* if $*$ is a binary operation partially defined on S satisfying the associative law, $(x * y) * z = x * (y * z)$ whenever both sides of the equality are defined. We say $(S, *)$ is *directed* if for every finite sequence $x_0, \ldots, x_{n-1}$ of elements of S there exists $y \in S$ such that $y \neq x_i$ and $x_i * y$ is defined for all $i < n$.

Here we only consider countably infinite partial semigroups. Given a directed partial semigroup $(S, *)$, let βS be the Stone-Cech compactification of S. For what follows it is useful think of an ultrafilter on S in terms of the quantifier it defines as follows: $\mathcal{U}xP(x)$ means that the set $\{x \in S : P(x)\}$ is an element of $\mathcal{U}$.

We focus on the collection of ultrafilters on S such that for every $x \in S$ we have that $\mathcal{U}y(x * y$ is defined$)$, let γS denote this collection of ultrafilters. Note that γS is a non-empty closed subspace of βS, so it is a compact Hausdorff space with the topology generated by sets of the form $\mathcal{A} = \{\mathcal{U} \in \gamma S : A \in \mathcal{U}\}$, where $A \subseteq S$. We extend the operation $*$ defined on S to a total operation on γS by

$$A \in \mathcal{U} * \mathcal{V} \text{ if and only if } \mathcal{U}x\mathcal{V}y(x * y \in A)$$

A *compact semigroup* is a nonempty semigroup with a compact Hausdorff topology for which the map

$$x \mapsto xs$$

is continuous for all elements s of the semigroup. Note that γS with the operation we just defined is a compact semigroup. We shall use the following result by Ellis about compact semigroups:

Lemma 19 (Ellis)**.** [8], [29] *Every compact semigroup $(S, *)$ has an idempotent, that is, there exists $s \in S$ such that $s * s = s$.*

Given a directed partial semigroup S, the aim is to find a non-principal idempotent ultrafilter in γS. The following lemma gives conditions under which we can find such an idempotent:

Lemma 20. [29] *If $(S, *)$ is a semigroup without idempotents or with left cancellation then there is a non-principal idempotent in γS.*

Using these two lemmas we get the main result:

Theorem 21 (Galvin-Glazer)**.** [29] *Let $(S, *)$ be a directed partial semigroup without idempotents or with left cancellation. Then for every finite coloring of S there is a sequence $(x_i)_{i \in \mathbb{N}}$ of pairwise distinct elements of S, such that whenever $n_0 < \ldots < n_l$, the product $x_{n_0} * \ldots * x_{n_l}$ is defined and the space $\langle x_i \rangle_{i \in \mathbb{N}}$ generated by the sequence,*

$$\langle x_i \rangle_{i \in \mathbb{N}} = \{x_{n_0} * \ldots * x_{n_l} : l \in \mathbb{N}, n_0 < \ldots < n_l \in \mathbb{N}\}$$

is monochromatic.

Proof. Fix a finite coloring of S and let $\mathcal{U} \in \gamma S$ be a non-principal idempotent. We shall construct a sequence $(P_n)_{n\in\mathbb{N}}$ of elements of the ultrafilter and a sequence of $(x_n)_{n\in\mathbb{N}}$ of elements of S such that

(i) $P_{n+1} \subseteq P_n$,

(ii) $x_i \in P_i$ for $i \in \mathbb{N}$,

(iii) all products $x_{i_0} * \ldots * x_{i_l}$ with $i_0 < \ldots < i_l \in \mathbb{N}$ are defined,

(iv) $x_i * y \in P_i$ for all $y \in P_{i+1}$, $i \in \mathbb{N}$.

To construct such sequence let P_0 be a monochromatic subset of S such that $P_0 \in \mathcal{U}$. Since $\mathcal{U} * \mathcal{U} = \mathcal{U}$, we have that $\mathcal{U}x\mathcal{U}y(x * y \in P_0)$ so let $x_0 \in P_0$ be such that $P_1 = \{y \in P_0 : x_0 * y \in P_0\} \in \mathcal{U}$. Suppose $x_0, \ldots, x_{n-1}$ and $P_0 \supseteq P_1, \ldots \supseteq P_n$ satisfy properties (i)-(iv) above. Since $P_n \in \mathcal{U}$, we have that $\mathcal{U}x\mathcal{U}y(x * y \in P_n)$. Let $x_n \in P_n$ be such that $P_{n+1} = \{y \in P_n : x_n * y \in P_n\} \in \mathcal{U}$.

Claim 22. *[29] For any $l \in \mathbb{N}$ and $n_0 < n_1 < \ldots < n_l$, $x_{n_0} * x_{n_1} * \ldots * x_{n_l} \in P_{n_0}$.*

Proof of Claim. We proceed by induction on l. The base case is clear. Suppose the claim holds for l let $n_0 < n_1 < \ldots < n_l < n_{l+1}$. By induction hypothesis $x = x_{n_1} * \ldots * x_{n_l} \in P_{n_1}$. Since $n_1 \geq n_0 + 1$, we have that $P_{n_1} \subseteq P_{n_0+1}$ and by property (iv) we have that $x_{n_0} * x \in P_{n_0}$. □

It follows that $(x_n)_{n\in\mathbb{N}}$ is the sequence we were looking for. □

As a corollary we obtain the finite unions formulation of Hindman's Theorem:

Theorem 23 (Hindman). [7] *For any finite coloring of* FIN, *the collection of finite subsets of* $\mathbb{N}$, *there is an infinite sequence* $(x_i)_{i\in\mathbb{N}} \subseteq \mathrm{FIN}$ *such that* $\max x_i < \min x_{i+1}$ *for all* $i \in \mathbb{N}$ *and all finite unions of elements of the sequence have the same color.*

We now sketch the proof of the infinite FIN_k Theorem as presented in [29]:

Theorem 24. [11] *For every finite coloring of* FIN_k *there exists a block sequence* $(f_i)_{i\in\mathbb{N}}$ *such that the space* $\langle f_i\rangle_{i\in\mathbb{N}}$ *generated by the sequence* $(f_i)_{i\in\mathbb{N}}$,

$$\langle f_i\rangle_{i\in\mathbb{N}} = \{T^{l_1}(f_{i_1}) + \ldots + T^{l_n}(f_{i_n}) : i_1 < \ldots < i_n, \min\{l_1, \ldots, l_n\} = 0\},$$

is monochromatic.

The proof of the infinite FIN_k Theorem builds on the proof we just presented for Theorem 21. Let $k \in \mathbb{N}$, $k \geq 1$ be fixed. Note that for $1 < j \leq k$ $(\mathrm{FIN}_j, +)$ is a partial semigroup and we can extend the tetris operation defined on FIN_j to $T : \gamma\mathrm{FIN}_j \to \gamma\mathrm{FIN}_{j-1}$ as follows:
For $\mathcal{U} \in \gamma\mathrm{FIN}_j$ define $T(\mathcal{U}) = \{A \subseteq \mathrm{FIN}_{j-1} : \mathcal{U}x(T(x) \in A)\}$.
It is easy to check that T is a continuous onto homomorphism. We shall also work with the structure $\mathrm{FIN}_{[1,k]} = \bigcup_{i=1}^{k} \mathrm{FIN}_i$ with addition and tetris operation defined just as in FIN_k. Note that $(\mathrm{FIN}_k, +)$ and $(\mathrm{FIN}_{[1,k]}, +)$ are directed partial semigroups and $\gamma\mathrm{FIN}_k$ is a two-sided ideal of

$$\gamma\mathrm{FIN}_{[1,k]} = \bigcup_{i=1}^{k} \gamma\mathrm{FIN}_i$$

The following lemma provides a sequence of idempotent ultrafilters that we shall use in the proof of Theorem 24.

Lemma 25. [29] *For every $k \in \mathbb{N}$, there exists a sequence of ultrafilters $(\mathcal{U}_i)_{1 \le i \le k}$ such that for $1 \le i < j \le k$*

(i) $\mathcal{U}_i$ is a non-principal idempotent in $\gamma\mathrm{FIN}_i$,

(ii) $T^{(i-j)}(\mathcal{U}_i) = \mathcal{U}_j$,

(iii) $\mathcal{U}_j + \mathcal{U}_i = \mathcal{U}_i + \mathcal{U}_j = \mathcal{U}_j$.

Proof. Let $\mathcal{U}_1$ be and idempotent in $\gamma\mathrm{FIN}_1$. Suppose $\mathcal{U}_i$ is defined for $1 \le i < j$. Let $S_j = \{\mathcal{U} \in \gamma\mathrm{FIN}_j : T(\mathcal{U}) = \mathcal{U}_{j-1}\}$, this set is non-empty since T is onto and it is closed because T is continuous. Also $S_j + \mathcal{U}_{j-1}$ is a closed subsemigroup of $\gamma\mathrm{FIN}_j$ so there is an idempotent $\mathcal{W} \in S_j + \mathcal{U}_{j-1}$. Let $\mathcal{V} \in S_j$ be such that $\mathcal{W} = \mathcal{V} + \mathcal{U}_{j-1}$ and set $\mathcal{U}_j = \mathcal{U}_{j-1} + \mathcal{V} + \mathcal{U}_{j-1}$. It is easy to check that $\mathcal{U}_j$ is an idempotent, $T(\mathcal{U}_j) = \mathcal{U}_{j-1}$ and $\mathcal{U}_{j-1} + \mathcal{U}_j = \mathcal{U}_j + \mathcal{U}_{j-1} = \mathcal{U}_j$. □

We are now ready for the proof of the infinite FIN_k Theorem.

Proof of Theorem 24. Fix $k \in \mathbb{N}$ and a finite coloring of FIN_k. Let $(\mathcal{U}_j)_{1 \le j \le k}$ be the sequence of ultrafilters given by Lemma 25. We will construct inductively the sequence $(x_n)_{n\in\mathbb{N}}$ and for $1 \le l \le k$ a decreasing sequence $(A_n^j)_{n\in\mathbb{N}} \subseteq \mathcal{U}_j$ such that for $n \in \mathbb{N}$ and $i, j < k$

(i) $x_n \in A_n^k$,

(ii) $T^{k-j}(A_n^k) = A_n^j$,

(iii) $\mathcal{U}_k x(T^{k-i}(x_n) + T^{k-j}(x) \in A_n^{\max\{i,j\}})$ and for $n < m$ this is witnessed by A_m^k.

Let $A_0^k \in \mathcal{U}_k$ be a piece of the partition and for $1 \le j < k$ let $A_0^j = T^{k-j}(A_0^k)$. Since $A_0^{\max\{i,j\}} \in \mathcal{U}_{\max\{i,j\}} = T^{k-i}(\mathcal{U}_k) + T^{k-j}(\mathcal{U}_k)$ for any $i, j < k$, we have that $\mathcal{U}_k$

many $x_0 \in A_0^k$ satisfy condition (iii). Let x_0 be one such element of A_0^k.

Now suppose we have constructed $(x_n)_{n<m}$ and $(A_n^j)_{n<m}$, $1 \leq j \leq k$. For each $n < m$ and $1 \leq i, j \leq k$ let

$$C_n^{i,j} = \{x : T^{k-j}(x_n) + T^{k-i}(x) \in A_n^{\max\{i,j\}}\}.$$

By induction hypothesis $C_n^{i,j} \in \mathcal{U}_k$ so we can let

$$A_m^k = A_{m-1}^k \cap \bigcap_{\substack{n<m \\ 1\leq i,j\leq k}} C_n^{i,j}.$$

This way A_m^k witnesses (iii) for all $n < m$ as we wanted. We now check that the sequence $(x_n)_{n\in\mathbb{N}}$ works.

Claim 26. [29] *For all* $p \geq 1$, $T^{(k-l_0)}(x_{n_0}) + \ldots + T^{(k-l_{p-1})}(x_{n_{p-1}}) + y \in A_{n_0}^{\max\{l_0,\ldots,l_{p-1}\}}$ *whenever* $n_0 < \ldots < n_p$, $l_0, \ldots, l_p \in \{1, \ldots k\}$ *and* $y \in A_{n_p}^{l_p}$.

Proof of claim. We proceed by induction on p. To verify the base case let $n_0 < n_1$, $l_0, l_1 \in \{1, \ldots k\}$, $y \in A_{n_1}^{l_1} = T^{(k-l_1)}(A_{n_1}^k)$. By condition (iii) we have that $\mathcal{U}_k x(T^{(k-l_0)}(x_{n_0}) + T^{(k-l_1)}(x) \in A_{n_0}^{\max\{l_0,l_1\}})$ and this is witnessed by $A_{n_1}^k$. Let $x \in A_{n_1}^k$ be such that $T^{(k-l_1)}(x) = y$ then $T^{(k-l_0)}(x_{n_0}) + y = T^{(k-l_0)}(x_{n_0}) + T^{(k-l_1)}(x) \in A_{n_0}^{\max\{l_0,l_1\}}$.

Now suppose the statement holds for $p - 1$ and let $n_0 < \ldots < n_p$, $l_0, \ldots, l_p \in \{1, \ldots k\}$ and $y \in A_{n_p}^{l_p}$ be given. Let $y' = T^{(k-l_1)}(x_{n_1}) + \ldots + T^{(k-l_{p-1})}(x_{n_{p-1}}) + y$. By induction hypothesis $y' \in A_{n_1}^{\max\{l_1,\ldots,l_p\}}$. Let $l = \max\{l_1, \ldots, l_p\}$ and let $y^* \in A_{n_1}^k$ be such that $T^{(k-l)}(y^*) = y'$. We want to see that $T^{(k-l_0)}(x_{n_0}) + y' \in A_{n_0}^{\max\{l_0,l\}}$. By condition (iii) we have that $\mathcal{U}_k x(T^{(k-l_0)}(x_{n_0}) + T^{(k-l)}(x) \in A_{n_0}^{\max\{l_0,l\}})$ and this is witnessed by $A_{n_0+1}^k \supseteq A_{n_1}^k$. Since $y^* \in A_{n_1}^k$, it follows that $T^{(k-l_0)}(x_{n_0}) + y' = T^{(k-l_0)}(x_{n_0}) + T^{(k-l)}(y^*) \in A_{n_0}^{\max\{l_0,l\}}$. □

□

3.3 Oscillation stability of c_0

The FIN_k Theorem presented in the previous section was formulated and proved by W.T. Gowers in [11] to obtain the oscillation stability of the Banach space c_0. Let X be a Banach space with Schauder basis $(\mathbf{x}_i)$, we say that a Lipschitz function $f : S_X \to \mathbb{R}$ *stabilizes on the positive sphere* if for every $\epsilon > 0$ there exists an infinite dimensional positive subspace Y such that

$$\mathrm{osc}(f \upharpoonright PS_Y) = \sup\{|f(\mathbf{x}) - f(\mathbf{y})| : \mathbf{x}, \mathbf{y} \in PS_Y\} < \epsilon.$$

We shall prove next the stabilization of Lipschitz functions on the positive sphere of c_0:

Theorem 27. [11] *Every Lipschitz function defined on the sphere of c_0 stabilizes on the positive sphere of c_0.*

We will need some additional terminology. For a Banach space X, subsets $\Delta, Y \subseteq X$ and $\delta > 0$, we say Δ is a *δ-net for Y* if for every $y \in Y$ there exists $x \in \Delta$ such that $||x - y|| < \delta$. That is, we can approximate elements of Y using elements of Δ with an error smaller than δ.

Proof. Let $f : S_{c_0} \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant 1 and let $\epsilon > 0$. The main idea of the proof is to identify FIN_k for some $k \in \mathbb{N}$ with a δ-net for PS_{c_0} for some $\delta > 0$. Let $\delta = \epsilon/3$ and choose $k \in \mathbb{N}$ such that $(1+\delta)^{-(k-1)} < \delta$. Define $\Delta_k \subseteq PS_{c_0}$ to be the collection of all finitely supported functions

$$\xi : \mathbb{N} \to \left\{0, \frac{1}{(1+\delta)^{k-1}}, \ldots, \frac{1}{1+\delta}, 1\right\}$$

that take the value 1. Note that Δ_k is a δ-net for PS_{c_0} and there is a natural correspondence between Δ_k and FIN_k, namely, let $\Phi : \Delta_k \to \mathrm{FIN}_k$ be defined by

$$\Phi(x)(n) = \max\left\{k + \log_{1+\delta} x(n), 0\right\}.$$

Note that $\Phi((1+\delta)^{-1}x) = T(\Phi(x))$ and $\Phi(x+y) = \Phi(x) + \Phi(y)$.
Suppose the range of f is contained in an interval $[c,d]$. We can partition this interval into subintervals of length δ and color $\xi \in \Delta_k$ depending on the interval containing $f(\xi)$. This induces a coloring of FIN_k and by the FIN_k theorem there is a block sequence $(g_i)_{i\in\mathbb{N}} \subseteq \mathrm{FIN}_k$ such that $\langle g_i\rangle_{i\in\mathbb{N}}$ is monochromatic. For each $i \in \mathbb{N}$ let $x_i = \Phi^{-1}(g_i)$ and let Y be the space generated by the positive sequence $(x_i)_{i\in\mathbb{N}}$. Since the tetris operation T corresponds to scalar multiplication by $(1-\delta)^{-1}$, it follows that the preimage of $\langle g_i\rangle_{i\in\mathbb{N}}$ under Φ is a δ-net for PS_Y. Therefore $osc(f \upharpoonright PS_Y) < \epsilon$. $\square$

We say that a space X is *oscillation stable* if every Lipschitz function stabilizes on the unit sphere, that is, for every $\epsilon > 0$ there exists an infinite dimensional subspace Y such that

$$\operatorname{osc}(f \upharpoonright S_Y) = \sup\{|f(\mathbf{x}) - f(\mathbf{y})| : \mathbf{x}, \mathbf{y} \in S_Y\} < \epsilon.$$

Note that a space X is not distortable if and only if every equivalent norm stabilizes, and since an equivalent norm is an example of a Lipschitz function, it follows that oscillation stable spaces are not distortable.

To obtain the stabilization of Lipschitz functions on the whole unit sphere of c_0, Gowers used a modification of the combinatorial structure FIN_k to account for the change of signs, and in this case proved an approximate Ramsey type theorem for it and obtained:

Theorem 28. *[11] The space c_0 is* oscillation stable*.*

It turns out that only "c_0-like" spaces are oscillation stable (see [24, p. 1349]). However, given a Lipschitz function defined on the unit sphere of an infinite dimensional Banach space X, we can always pass to a finite dimensional subspace Y of any given dimension such that the oscillation of the function restricted to the unit sphere of Y is as small as we want. This was first observed by Milman (see [19, p.6]). The following theorem gives the quantitative version of this fact:

Theorem 29. [19],[21] *For every $C, \epsilon > 0$ and $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that if F is a Banach space of dimension n and $(f_i)_{i \in n}$ is a basis for F with constant at most C and $f : S_F \to \mathbb{R}$ is C-Lipschitz, then there is an m-dimensional subspace G of F such that $osc(f \restriction S_G) < \epsilon$.*

In Section 5.3 we present a proof of this theorem for the special case of spaces of the form ℓ_∞^n. We shall also give an upper bound on how big n should be for given values of C, ϵ and m.

CHAPTER 4

TSIRELSON'S SPACE

In this chapter we will give an expression for the norm of Tsirelson's space T defined in Section 2.2.2. We will prove that there is a norm $||\cdot||$ on c_{00} that satisfies the following implicit equation

$$\begin{aligned} ||\mathbf{x}|| &= \min\{2\min\{\max_{1\le i\le k}||E_i\mathbf{x}|| : k \le E_1 < \cdots < E_k, \mathbf{x} = \sum_{i=1}^{k} E_i\mathbf{x}\}, \\ &\quad \inf\{||\mathbf{y}|| + ||\mathbf{z}|| : \mathbf{x} = \mathbf{y} + \mathbf{z}, \mathrm{supp}(\mathbf{y}) \subseteq \mathrm{supp}(\mathbf{x})\}\}. \end{aligned}$$

Or equivalently,

$$||\mathbf{x}|| \le 2\min\{\max_{1\le i\le k}||E_i\mathbf{x}|| : k \le E_1 < \cdots < E_k, \mathbf{x} = \sum_{i=1}^{k} E_i\mathbf{x}\}.$$

In fact, we shall prove that the norm of the space T is maximal (in the point-wise sense) among the norms satisfying the implicit equation above and such that $||\mathbf{e}_i|| = 1$ for every vector $\mathbf{e}_i$ in the standard basis of c_{00}. Note that, as opposed to the implicit equation given by Figiel and Johnson, this expression doesn't allow us to calculate the norm of finitely supported vectors inductively in the cardinality of the support.

The expression for this norm can be adapted for the dual of any mixed Tsirelson space. In particular one can get an expression for the norm of the dual of the space S defined in Section 2.2.2.

4.1 Definition of the norm

The norm $||\cdot||_F$ of the dual of T can be obtained as a limit of norms in the following way. For $\mathbf{x} \in c_{00}$ define

$$\begin{aligned} ||\mathbf{x}||_0 &= ||\mathbf{x}||_\infty \\ ||\mathbf{x}||_{n+1} &= \max\left\{ ||\mathbf{x}||_n, \frac{1}{2}\max\left\{ \sum_{i=1}^{k} ||E_i\mathbf{x}||_n : k \in \mathbb{N}, k \leq E_1 < \cdots < E_k \right\}\right\} \end{aligned}$$

It is clear that if we let $||\mathbf{x}||_F = \lim_{n\to\infty} ||\mathbf{x}||_n$, then $||\cdot||_F$ satisfies the implicit equation (2.1). To get an expression for the norm of T we shall take the limit of a sequence of positive scalar functions on c_{00}. It is not the case that each scalar function is a norm but the sequence is defined in such a way that the pointwise limit is a norm on c_{00}.

Definition 30. For $\mathbf{x} \in c_{00}$ let

$$\begin{aligned} \rho_0(\mathbf{x}) &= ||\mathbf{x}||_{\ell_1} \\ \rho_{n+1}(\mathbf{x}) &= \min\{2\min\{\max_{1\leq i\leq k} \rho_n(E_i\mathbf{x}) : k \leq E_1 < \cdots < E_k, \mathbf{x} = \sum_{i=1}^{k} E_i\mathbf{x}\}, \\ &\qquad \inf\{\rho_n(\mathbf{w}^1) + \rho_n(\mathbf{w}^2) : \mathbf{x} = \mathbf{w}^1 + \mathbf{w}^2\}\} \\ ||\mathbf{x}|| &= \lim_{n\to\infty} \rho_n(\mathbf{x}) \end{aligned}$$

Lemma 31. *The function $||\cdot||$ defines a norm on c_{00} and whenever $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n) \in c_{00}$ are such that $|y_n| \leq |x_n|$, we have $||\mathbf{y}|| \leq ||\mathbf{x}||$.*

Proof. Let $\mathbf{x} = (x_n), \mathbf{y} = (y_n) \in c_{00}$ be such that $|y_n| \leq |x_n|$. We prove by induction on n that $\rho_n(\mathbf{y}) \leq \rho_n(\mathbf{x})$. The base case is clear so suppose the inequality holds for n. Let $k \leq E_1 < \cdots < E_k$ be such that $\mathbf{x} = \sum E_i\mathbf{x}$. Note that since $\text{supp}(\mathbf{y}) \subseteq \text{supp}(\mathbf{x})$, we have that $\mathbf{y} = \sum E_i\mathbf{y}$.

By induction hypothesis $\max_{1\leq i\leq n} \rho_n(E_i\mathbf{y}) \leq \max_{1\leq i\leq n} \rho_n(E_i\mathbf{x})$ so

$$\min\left\{ \max_{1\leq i\leq n} \rho_n(E_i\mathbf{y}) : k \leq E_1 < \cdots < E_k, \mathbf{y} = \sum E_i\mathbf{y} \right\} \leq \max_{1\leq i\leq n} \rho_n(E_i\mathbf{x}).$$

Since $k \leq E_1 < \cdots < E_k$ was arbitrary, we have that

$$\rho_{n+1}(\mathbf{y}) \leq 2 \min \left\{ \max_{1 \leq i \leq n} \rho_n(E_i \mathbf{x}) : k \leq E_1 < \cdots < E_k, \mathbf{x} = \sum E_i \mathbf{x} \right\}.$$

Let $\mathbf{x}^1, \mathbf{x}^2$ be such that $\mathbf{x} = \mathbf{x}^1 + \mathbf{x}^2$, then we can find $\mathbf{y}^1, \mathbf{y}^2$ such that $\mathbf{y} = \mathbf{y}^1 + \mathbf{y}^2$ and $|y_n^i| \leq |x_n^i|$, hence

$$\rho_{n+1}(\mathbf{y}) \leq \inf\{\rho_n(\mathbf{w}^1) + \rho_n(\mathbf{w}^2) : \mathbf{x} = \mathbf{w}^1 + \mathbf{w}^2\}.$$

It follows that $\rho_{n+1}(\mathbf{y}) \leq \rho_{n+1}(\mathbf{x})$. Therefore for all $n \in \mathbb{N}$, $\rho_n(\mathbf{y}) \leq \rho_n(\mathbf{x})$ and it follows that $||\mathbf{y}|| \leq ||\mathbf{x}||$.

This monotonicity implies that $|| \cdot ||$ is bounded below by the sup norm. Now we prove that $|| \cdot ||$ defines a norm.

We summarize the properties of the sequence $(\rho_n)_n$ that will be used in the proof:

(i) For all $\mathbf{x} \in c_{00}$, $(\rho_n(\mathbf{x}))_n$ is monotone decreasing,

(ii) For $\mathbf{x}, \mathbf{y} \in c_{00}$, $\rho_{n+1}(\mathbf{x} + \mathbf{y}) \leq \rho_n(\mathbf{x}) + \rho_n(\mathbf{y})$.

(iii) For any $\lambda \in \mathbb{R}$ and $\mathbf{x} \in c_{00}$, $\rho_n(\lambda \mathbf{x}) = |\lambda| \rho_n(\mathbf{x})$.

The only non trivial property we must verify is the triangle inequality. Let $\mathbf{x}, \mathbf{y} \in c_{00}$; first we prove that for all $m \in \mathbb{N}$,

$$||\mathbf{x} + \mathbf{y}|| - ||\mathbf{x}|| \leq \rho_m(\mathbf{y}). \tag{4.1}$$

For $m \in \mathbb{N}$ and $n > m$ we have

$$||\mathbf{x} + \mathbf{y}|| \leq \rho_{n+1}(\mathbf{x} + \mathbf{y}) \leq \rho_n(\mathbf{x}) + \rho_n(\mathbf{y}) \leq \rho_n(\mathbf{x}) + \rho_m(\mathbf{y}),$$

therefore $||\mathbf{x} + \mathbf{y}|| - \rho_m(\mathbf{y}) \leq \rho_n(\mathbf{x})$ and this holds for all $n > m$ so

$$||\mathbf{x} + \mathbf{y}|| - \rho_m(\mathbf{y}) \leq \inf_{n > m} \rho_n(\mathbf{x}) = ||\mathbf{x}||,$$

hence $||\mathbf{x} + \mathbf{y}|| - ||\mathbf{x}|| \leq \rho_m(\mathbf{y})$.

Since (4.1) holds for all $m \in \mathbb{N}$, we have that $||\mathbf{x} + \mathbf{y}|| - ||\mathbf{x}|| \leq \inf_m \rho_m(\mathbf{y}) = ||\mathbf{y}||$. Hence $|| \cdot ||$ defines a norm on c_{00}. $\square$

We will now see that the norm $||\cdot||$ satisfies an implicit expression, dual to that found by Figiel and Johnson.

Proposition 32. *For any* $\mathbf{x} \in c_{00}$*, we have that*

$$\begin{aligned} ||\mathbf{x}|| &= \min\{2\min\{\max_{1\le i\le k}||E_i\mathbf{x}|| : k \le E_1 < \cdots < E_k, x = \sum E_i\mathbf{x}\}, \\ &\quad \inf\{||\mathbf{y}|| + ||\mathbf{z}|| : \mathbf{x} = \mathbf{y} + \mathbf{z}, \operatorname{supp}(\mathbf{y}) \subseteq \operatorname{supp}(\mathbf{x})\}\}. \end{aligned}$$

Furthermore, if $||\cdot||'$ *is a norm satisfying the implicit equation above and such that* $||\mathbf{e}_i||' = 1$ *for every vector* $\mathbf{e}_i$ *in the standard basis of* c_{00}*, then for all* $\mathbf{x} \in c_{00}$*,* $||\mathbf{x}||' \le ||\mathbf{x}||$*.*

Proof. Since $||\cdot||$ satisfies the triangle inequality, $||\mathbf{x}|| = \inf\{||\mathbf{y}|| + ||\mathbf{z}|| : \mathbf{x} = \mathbf{y} + \mathbf{z}, \operatorname{supp}(\mathbf{y}) \subseteq \operatorname{supp}(\mathbf{x})\}$. So we have to check that

$$||\mathbf{x}|| \le 2\min\{\max_{1\le i\le k}||E_i\mathbf{x}|| : k \le E_1 < \cdots < E_k, \mathbf{x} = \sum E_i\mathbf{x}\}.$$

Let $k \le E_1 < \cdots < E_k$ be such that $\mathbf{x} = \sum E_i\mathbf{x}$, and define

$$J = \{j \le k : \max_{1\le i\le k}||E_i\mathbf{x}|| = ||E_jx||\}.$$

Let $j_0 \in J$ be such that for some cofinal $C \subseteq \mathbb{N}$ we have that for $n \in C$, $\max\limits_{1\le i\le k}\rho_n(E_i\mathbf{x}) = \rho_n(E_{j_0}\mathbf{x})$.

Then for $n \in C$

$$||\mathbf{x}|| \le \rho_{n+1}(\mathbf{x}) \le 2\max_{1\le i\le k}\rho_n(E_i\mathbf{x}) = 2\rho_n(E_{j_0}\mathbf{x}).$$

So $||\mathbf{x}|| \le 2\,||E_{j_0}\mathbf{x}|| = 2\max\limits_{1\le i\le k}||E_i\mathbf{x}||$.

Now suppose $||\cdot||'$ is a norm on c_{00} that satisfies the implicit equation and such that $||\mathbf{e}_i||' = 1$ for every vector $\mathbf{e}_i$ in the standard basis of c_{00}. It follows easily by induction that for all $n \in \mathbb{N}$ and all $\mathbf{x} \in c_{00}$, $||\mathbf{x}||' \le \rho_n(\mathbf{x})$. Hence for all $\mathbf{x} \in c_{00}$, $||\mathbf{x}||' \le ||\mathbf{x}||$. $\square$

Unlike the expression given by Figiel and Johnson, this implicit expression does not allow us to calculate the norm of a vector recursively in the cardinality of its support. This is because of the infimum term in each ρ_n; this term is necessary in order to have the triangle inequality hold in the limit.

Since T is reflexive and $T^* = F$, the space T is isometrically isomorphic to the dual of F. We will prove that the norm $||\cdot||$ we defined and the norm of F^* coincide in c_{00}.

We observed before that the sequence $(\mathbf{e}_n)_n$ is a Schauder basis for F. Let $(\mathbf{e}_n^*)_n$ be the corresponding coefficient functionals. We shall define a subset of F^* that contains all the information needed to calculate the norm of a given vector $\mathbf{x} \in F$. Let

$$\begin{aligned} V_0 &= \{\pm \mathbf{e}_k^* : k \in \mathbb{N}\} \\ V_{n+1} &= V_n \cup \left\{ \frac{1}{2}(f_1 + \cdots + f_k) : k \in \mathbb{N}, k \leq f_1 < \cdots < f_k, f_i \in V_n \right\} \\ V &= \bigcup_n V_n. \end{aligned}$$

Proposition 33. *For every* $\mathbf{x} \in F$*,* $||\mathbf{x}||_F = sup\{f(\mathbf{x}) : f \in V\}$*.*

Proof. Let $(||\cdot||_{F,n})_n$ be the sequence of norms defined by Figiel and Johnson. For each $n \in \mathbb{N}$ and $x \in c_{00}$ define $\tau_n(\mathbf{x}) = \sup\{f(\mathbf{x}) : f \in V_n\}$. It is easy to prove by induction on n that $\tau_n(\mathbf{x}) = ||\mathbf{x}||_{F,n}$ for every $\mathbf{x} \in c_{00}$. Therefore

$$||\mathbf{x}||_F = \lim_{n\to\infty} ||\mathbf{x}||_{F,n} = \lim_{n\to\infty} \tau_n(\mathbf{x}) = \sup\{f(\mathbf{x}) : \mathbf{x} \in V\}.$$

□

We denote the convex hull of a set A by $\text{conv}(A)$. We are now ready to use the instance of the Bipolar Theorem presented in Section 2.1 to prove the following

Proposition 34. *The unit ball of the dual of F is the weak*-closure of the convex hull of V. Also,* $B_{F^*} \cap c_{00} = \text{conv}(V)$*.*

Proof. By Proposition 33, $V^\circ = B_F$ and $V^{\circ\circ} = (B_F)^\circ = B_{F^*}$. Hence by the Bipolar theorem, we have that B_{F^*} is the weak*- closure of the convex hull of V.

For the second part, let $\mathbf{x} \in B_{F^*} \cap c_{00}$, $G = \mathrm{supp}(\mathbf{x})$. Then $V_G = \{f \in V : \mathrm{supp}(f) \subseteq G\}$ is a compact subset of a finite dimensional subspace of F^*, therefore $\mathrm{conv}(V_G)$ is compact. Let $P_G : F^* \to [\{\mathbf{e}_i^* : i \in G\}]$ denote the projection map with $P_G y = Gy$. Note that P_G is w^*-continuous. Then

$$\mathbf{x} \in P_G[B_{F^*}] = P_G[\overline{\mathrm{conv}(V)}^{w^*}] \subseteq \overline{\mathrm{conv}(P_G[V])}^{w^*} = \overline{\mathrm{conv}(V_G)}^{w^*} = \mathrm{conv}(V_G).$$

□

In order to prove that X is the dual of F, we need the following lemma:

Lemma 35. *For $\mathbf{x} \in c_{00}$, if $\rho_n(\mathbf{x}) \leq 1$ then $\mathbf{x} \in B_{F^*}$.*

Proof. Note that B_{F^*} has the following properties:

(i) The sequence $(\mathbf{e}_n^*)$ is contained in B_{F^*},

(ii) if $f \in B_{F^*}$ and $|\alpha| \leq 1$, then $\alpha f \in B_{F^*}$,

(iii) if $f_1, \cdots, f_n \in B_{F^*} \cap c_{00}$ are such that $n \leq f_1 < \cdots < f_n$, then $1/2(f_1 + \cdots + f_n) \in B_{F^*} \cap c_{00}$.

This follows from the original construction of T. We include the proof for completeness, since we want to generalize our arguments to the dual of mixed Tsirelson spaces.
Property (i) is clear by the definition of V. Note that the set V has the closure property described in property (iii). Let V' be the convex hull of V then V' has property (ii). Let $f_1, \cdots, f_n \in V'$ be such that $n \leq f_1 < \cdots < f_n$ and let $f = 1/2(f_1 + \cdots + f_n)$. For each $i = 1, \ldots, n$, f_i can be written as

$$f_i = \alpha_i^1 g_i^1 + \cdots + \alpha_i^m g_i^m$$

for some $m \in \mathbb{N}$, some $g_i^j \in V \cup \{0\}$ and some non negative scalars α_i^j such that $\sum_s \alpha_i^s = 1$ for all $i \leq n$. We may assume that $\mathrm{supp}(g_i^j) \subset \mathrm{supp}(f_i)$ for each i, j. Therefore for each choice of $s_1 < \cdots < s_n \leq m$ we have that $n \leq g_1^{s_1} < g_2^{s_2} < \cdots < g_n^{s_n}$, so

$$\frac{1}{2}(g_1^{s_1} + \cdots + g_n^{s_n}) \in V.$$

Since f can be written as a convex combination of vectors of this form, it follows that $f \in V'$. Hence V' has the closure property described in (iii).

We now prove by induction on n that for $\mathbf{x} \in c_{00}$, if $\rho_n(\mathbf{x}) \leq 1$ then $\mathbf{x} \in B_{F^*}$. For the base case, assume that $||\mathbf{x}||_{\ell_1} \leq 1$ and $\mathbf{x} = (x_i)_i^k$ for some k, then $\sum_1^k |x_i| \leq 1$ so $\mathbf{x} \in B_{F^*}$ by convexity of B_{F^*}.

Now suppose that $\rho_{n+1}(\mathbf{x}) \leq 1$. If $\rho_{n+1}(\mathbf{x}) = 2 \max\limits_{1 \leq i \leq k} \rho_n(E_i\mathbf{x})$ for some $k \leq E_1 < \cdots < E_k$ such that $\mathbf{x} = \sum E_i\mathbf{x}$, then $\rho_n(2E_i\mathbf{x}) \leq 1$ for all $1 \leq i \leq k$. By induction hypothesis this implies that $2E_i\mathbf{x} \in B_{F^*}$ for all $1 \leq i \leq k$. Hence $\mathbf{x} = \frac{1}{2}\sum 2E_i\mathbf{x} \in B_{F^*}$ by property (iii).

Now suppose that $\rho_{n+1}(\mathbf{x}) = \inf\{\rho_n(\mathbf{y}) + \rho_n(\mathbf{z}) : \mathbf{x} = \mathbf{y} + \mathbf{z}\}$. For each $k \in \mathbb{N}$ let $\mathbf{y}_k, \mathbf{z}_k \in c_{00}$ be such that $\mathbf{x} = \mathbf{y}_k + \mathbf{z}_k$ and $\rho_n(\mathbf{y}_k) + \rho_n(\mathbf{z}_k) \leq 1 + 1/k$. Then $\mathbf{u}_k := \mathbf{y}_k/\rho_n(\mathbf{y}_k), \mathbf{v}_k := \mathbf{z}_k/\rho_n(\mathbf{z}_k) \in B_{F^*}$, by induction hypothesis. Since

$$\frac{\mathbf{x}}{\rho_n(\mathbf{y}_k) + \rho_n(\mathbf{z}_k)} = \frac{\rho_n(\mathbf{y}_k)}{\rho_n(\mathbf{y}_k) + \rho_n(\mathbf{z}_k)}\mathbf{u}_k + \frac{\rho_n(\mathbf{z}_k)}{\rho_n(\mathbf{y}_k) + \rho_n(\mathbf{z}_k)}\mathbf{v}_k,$$

and B_{F^*} is convex, it follows that $/(\rho_n(\mathbf{y}_k) + \rho_n(\mathbf{z}_k)) \in B_{F^*}$, or in other words,

$$\mathbf{x} \in (\rho_n(\mathbf{y}_k) + \rho_n(\mathbf{z}_k))B_{F^*} \subset (1 + 1/k)B_{F^*}$$

for every k. Hence, $\mathbf{x} \in B_{F^*}$.

$\square$

Let X be the completion of c_{00} with respect to the norm $||\cdot||$ from Definition 30.

Theorem 36. *$B_{F^*} \cap c_{00} = B_X \cap c_{00}$. Hence $X = F^*$.*

Proof. It can be proved by induction using the implicit equation in Proposition 32 that $V_n \subset B_X$ for all $n \in \mathbb{N}$. Therefore the convex hull of V is contained in B_X and by

Proposition 34, this proves the first inclusion.

For the reverse inclusion, let $\mathbf{x} \in B_X \cap c_{00}$, we may assume that $||\mathbf{x}|| = 1$. Let $(n_k)_k$ be a sequence of natural numbers such that $\rho_{n_k}(\mathbf{x}) < 1 + 1/k$. By the previous lemma $\frac{\mathbf{x}}{1+1/k} \in B_{F^*}$, hence $\mathbf{x} \in B_{F^*}$.

Thus we have proved that the norm $||\cdot||$ coincides with $\|\cdot\|_{F^*}$ on c_{00}. Since c_{00} is dense in X and $[\mathbf{e}_n^*]$ is dense in F^*, it follows that $X = F^*$, that is F^* is the completion of c_{00} with respect to the norm $||\cdot||$.

□

4.2 A norm for the dual spaces of mixed Tsirelson spaces

We shall show how our results can be adapted to give an expression for the norm of the dual of any mixed Tsirelson space $\mathcal{T}[(\mathcal{M}_n, \theta_n)_n]$. As in Section 4.1, we have the following definition.

Definition 37. For $\mathbf{x} \in c_{00}$ let $\rho_0(\mathbf{x}) = ||\mathbf{x}||_{\ell_1}$ and let $\rho_{n+1}(x)$ be the minimum of the quantities

$$\min \left\{ \frac{1}{\theta_k} \max_{1 \le i \le k} \rho_n(E_i \mathbf{x}) : (E_i)_1^k \ \mathcal{M}_k\text{-admissible}, \mathbf{x} = \sum_{i=1}^{k} E_i \mathbf{x}, k \in \mathbb{N} \right\}$$

and

$$\inf \left\{ \rho_n(\mathbf{w}^1) + \rho_n(\mathbf{w}^2) : \mathbf{x} = \mathbf{w}^1 + \mathbf{w}^2 \right\}.$$

This defines a norm $||\cdot||$ on c_{00}, since the proof of Lemma 31 can be carried out for this modified expression. Specifically, the argument for monotonicity is independent of the Schreier condition and of the coefficient 2. Clearly the properties (i)-(iii) used in the proof of the lemma are also satisfied.

The proof of Proposition 32 goes through as well, so the norm defined above satisifies the following implicit equation

$$\begin{aligned} ||\mathbf{x}|| &= \min\{\min\{\frac{1}{\theta_l}\max_{1\leq i\leq k}||E_i\mathbf{x}|| : (E_i)_1^k\ \mathcal{M}_l\text{-admissible}, \mathbf{x} = \sum_{i=1}^{k} E_i\mathbf{x}, l\in\mathbb{N}\}, \\ &\quad \inf\{||\mathbf{y}||+||\mathbf{z}|| : \mathbf{x}=\mathbf{y}+\mathbf{z}, \mathrm{supp}(\mathbf{y})\subseteq \mathrm{supp}(\mathbf{x})\}\}, \end{aligned}$$

the standard basis of c_{00} is normalized with respect to $||\cdot||$, and is the maximum such norm.

To describe the unit ball of $\mathcal{T}[(\mathcal{M}_n,\theta_n)_n]^*$, we use the following sequence of sets:

$$\begin{aligned} V_0 &= \{\pm\mathbf{e}_k^* : k\in\mathbb{N}\} \\ V_{n+1} &= V_n\cup\{\theta_l(f_1+\cdots+f_k) : f_1<\cdots<f_k, f_i\in V_n, \\ &\quad (\mathrm{supp}(f_i))_i^k,\ \mathcal{M}_l\text{-admissible}, l\in\mathbb{N}\} \\ V &= \bigcup V_n. \end{aligned}$$

It is easy to see, just as for F, that for any $\mathbf{x}\in\mathcal{T}[(\mathcal{M}_n,\theta_n)_n]$,

$$\|\mathbf{x}\|_* = sup\{f(\mathbf{x}) : f\in V\}.$$

So by the Bipolar Theorem, the unit ball of the dual of $\mathcal{T}[(\mathcal{M}_n,\theta_n)_n]$ is the weak*-closure of the convex hull of V.

Let X be the completion of c_{00} with respect to the norm $||\cdot||$ defined in 37. We restate Theorem 36 for $\mathcal{T}[(\mathcal{M}_n,\theta_n)_n]^*$:

Theorem 38. $B_{\mathcal{T}[(\mathcal{M}_n,\theta_n)_n]^*}\cap c_{00} = B_X\cap c_{00}$. *Hence* $X=\mathcal{T}[(\mathcal{M}_n,\theta_n)_n]^*$.

The inclusion $B_{\mathcal{T}[(\mathcal{M}_n,\theta_n)_n]^*}\cap c_{00}\subseteq B_X\cap c_{00}$ can be proved just as we did for F^*. For the reverse inclusion, one can prove that the unit ball of $\mathcal{T}[(\mathcal{M}_n,\theta_n)_n]^*$ has the following properties:

(i) The sequence $(\mathbf{e}_n^*)$ is contained in $B_{\mathcal{T}[(\mathcal{M}_n,\theta_n)_n]^*}$,

(ii) if $f \in B_{\mathcal{T}[(\mathcal{M}_n,\theta_n)_n]^*}$ and $|\alpha| \leq 1$, then $\alpha f \in B_{\mathcal{T}[(\mathcal{M}_n,\theta_n)_n]^*}$,

(iii) for every $k \in \mathbb{N}$, if $f_1, \cdots, f_n \in B_{\mathcal{T}[(\mathcal{M}_n,\theta_n)_n]^*} \cap c_{00}$ are such that $f_1 < \cdots < f_n$, and $(\mathrm{supp}(f_i))_i^n$ is $\mathcal{M}_k$-admissible, then $\theta_k(f_1 + \cdots + f_n) \in B_{\mathcal{T}[(\mathcal{M}_n,\theta_n)_n]^*} \cap c_{00}$.

So the norm defined in 37 is the norm of the dual of the mixed Tsirelson space $\mathcal{T}[(\mathcal{M}_n, \theta_n)_n]$.

4.3 Additional remarks

At this moment it is not clear whether the norm of T satisfies an implicit equation that allows us to calculate the norm of finitely supported vectors inductively in the cardinality of the support. A natural attempt is to replace the infimum term in Proposition 32 by the ℓ_1 norm and let

$$\rho(\mathbf{x}) = \min\{||\mathbf{x}||_1, 2\min\{\max_{1\leq i\leq k} \rho(E_i\mathbf{x}) : k \leq E_1 < \cdots < E_k, \mathbf{x} = \sum_{i=1}^{k} E_i\mathbf{x}\}\}.$$

A few calculations show that ρ does not satisfy the triangle inequality. Nevertheless, the vectors $f \in B_{F^*}$ with the property that $\rho(f) = ||f||$, actually produce the unit ball of F^* in the sense stated below:

Proposition 39. $B_{F^*} = \overline{\mathrm{conv}\{f \in B_{F^*} : \rho(f) = ||f||\}}$.

Proof. By Proposition 34, it follows that $B_{F^*} \cap c_{00} = \mathrm{conv}\{f \in V : ||f|| = 1\}$. Note that for $f \in c_{00}$, we have that $||f|| \leq \rho(f)$. Also one can prove by induction on n that for every $f \in V_n$, $\rho(f) \leq 1$. It follows that if $f \in V$ is such that $||f|| = 1$, then we have that $||f|| = \rho(f)$. This finishes the proof. □

CHAPTER 5

FINITE STABILIZATION AND THE FINITE FIN_K THEOREM

As we saw in Section 3.3, the FIN_k Theorem is closely related to the oscillation stability of the Banach space c_0. In this chapter we give a constructive proof of the finite version of Gowers' FIN_k Theorem presented in Section 3.2. Namely we prove the following:

Theorem 40. *For all natural numbers m, k, r there exists a natural number n such that for every r-coloring of $\mathrm{FIN}_k(n)$, the functions in FIN_k supported below n, there exists a block sequence in $\mathrm{FIN}_k(n)$ of length m that generates a monochromatic combinatorial subspace.*

Let $g_k(m, r)$ be the minimum n satisfying the conditions in the theorem. This result follows easily from the infinite version by a compactness argument (For more details and examples of compactness arguments used to obtain finite Ramsey type theorems, see [13]). This compactness argument could be formalized within second order arithmetic and would yield a second order proof of the finite FIN_k Theorem, provided there was a proof of the infinite version within second order arithmetic. However, the existing proof of the infinite FIN_k Theorem uses ultrafilter dynamics and it has not been formalized in second order arithmetic. Towsner [30] showed how to carry out an analog of Glazer's proof of Hindman's Theorem in second order arithmetic. It is not known if a similar argument could yield a proof in second order arithmetic of the infinite FIN_k Theorem. J. Paris and L. Harrington [26] found the following example of a Ramsey type theorem which is unprovable in PA:

(PH) For all natural numbers n, k, r there exists a natural number m such that for any r-coloring of the k-element subsets of the interval of natural numbers $[n, m]$, there exists a monochromatic set S such that $\#S > \min S$.

The situation is different for the finite FIN_k Theorem since our proof uses only induction and can be proved using only the axioms of PA. In the notation of Theorem 40, the bounds we find for $k > 1$ and $m > 0$ are

$$g_k(m, 2) \leq f_{2k+2} \circ f_4(6m - 2),$$

where for $i \in \mathbb{N}$, f_i denotes the i-th function in the Ackermann Hierarchy.

Since the FIN_k Theorem was a crucial step in the proof of the oscillation stability of c_0, one might expect that the bounds for the quantitative version of Milman's result about the stabilization of Lipschitz functions on finite dimensional Banach spaces mentioned in Section 3.3, for the special case of the spaces ℓ_∞^n, $n \in \mathbb{N}$, would be comparable to those we find for the finite version of the FIN_k Theorem. However, this is not the case: we will establish much smaller bounds for the Finite Stabilization Principle in the special case of ℓ_∞^n- spaces and functions defined on their positive spheres. We repeat the statement of the Finite Stabilization Principle now in terms of block sequences:

Theorem 41. [21] *For every pair of real numbers $C, \epsilon > 0$ and every natural number m there is a natural number n such that for every n-dimensional normed space F with a Schauder basis $(\mathbf{x}_i)_{i=0}^{n-1}$, whose basis constant does not exceed C, and for every C-Lipschitz $f : F \to \mathbb{R}$, there is a block subsequence $(\mathbf{y}_i)_{i=0}^{m-1}$ of $(\mathbf{x}_i)_{i=0}^{n-1}$ so that*

$$osc(f \restriction S_{[\mathbf{y}_i]_{i=0}^{m-1}}) < \epsilon,$$

where S_Y is the unit sphere of $[\mathbf{y}_i]_{i=0}^{m-1}$ is the vector space generated by the sequence $(\mathbf{y}_i)_{i=0}^{m-1}$.

Let $N(C, \epsilon, m)$ be the minimum n such that for every C-Lipschitz function $f : PS_{\ell_\infty^n} \to \mathbb{R}$, there is a block sequence $(\mathbf{y}_i)_{i=0}^{m-1}$ of positive vectors such that $\mathrm{osc}(f \restriction PS_{[\mathbf{y}_i]_{i=0}^{m-1}}) < \epsilon$. By analysing the proof of Theorem 41 in [21] for the special case of the spaces ℓ_∞^n, $n \in \mathbb{N}$, we obtain the following upper bound for $N(C, \epsilon, m)$:

$$N(C, \epsilon, m) \lesssim f_3 \left(m^s \cdot \lceil \frac{C}{\epsilon} \rceil^{m^s} \right),$$

where $s = \log(\epsilon/12C)/\log(1-\epsilon/12C)+2$. For fixed ϵ and C, this upper bound is more slowly growing than the bound we found for $g_k(m,2)$ for a fixed $k \geq 2$.

5.1 The finite FIN_k Theorem

We start by fixing some notation based on the terminology introduced in Section 3.2. Let $k \in \mathbb{N}$ be given. For $N, d \in \mathbb{N}$ we define the *finite version of* FIN_k and its *d-dimensional version* by:

$$\begin{aligned}
\mathrm{FIN}_k(N) &= \{f \in \mathrm{FIN}_k : \max(\mathrm{supp}(f)) < N\} \\
\mathrm{FIN}_k(N)^{[d]} &= \{(f_i)_{i<d} |\ f_i \in \mathrm{FIN}_k(N) \text{ and } f_i < f_j \text{ for } i < j < d\}.
\end{aligned}$$

The elements of $\mathrm{FIN}_k(N)^{[d]}$ are called *block sequences*. The combinatorial space $\langle f_i \rangle_{i<d}$ generated by a sequence $(f_i)_{i<d} \in \mathrm{FIN}_k(N)^{[d]}$ is the set of elements of $\mathrm{FIN}_k(N)$ of the form $T^{l_0}(f_{i_0}) + \ldots + T^{l_{n-1}}(f_{i_{n-1}})$ where $n \in \mathbb{N}$, $i_0 < \ldots < i_{n-1} < d$ and $\min\{l_0, \ldots, l_{n-1}\} = 0$. A block subsequence of $(f_i)_{i<d}$ is a block sequence contained in $\langle f_i \rangle_{i<d}$. Just as we defined the d-dimensional version of $\mathrm{FIN}_k(N)$, if $(f_i)_{i<l}$ is a block sequence, we define $(\langle f_i \rangle_{i<l})^{[d]}$ to be the collection of block subsequences of $(f_i)_{i<l}$ of length d.

The following definition is important when coding an element of FIN_k in a sequence of elements of FIN_{k-1}. Given $\mathbf{f} = (f_i)_{i<m} \in \mathrm{FIN}_k^{[m]}$, for

$$g = \sum_{i<m} T^{k-n_i} f_i,$$

we define $\mathit{supp}_k^{\mathbf{f}}(g)$ to be the set of all $i < m$ such that $n_i = k$. The cardinality of this set determines the length of the sequence we need in order to code g, as we shall describe in detail later on. The proof is by induction on k. The starting point is Folkman's Theorem. In the inductive step, the idea is to code an element of FIN_k in a finite sequence of elements of FIN_{k-1} and apply the result for FIN_{k-1} and its higher dimensional versions. We canonically identify FIN_1 with FIN, the collection of finite subsets of $\mathbb{N}$. The case

$k = 1$ of the finite FIN_k Theorem, phrased in terms of finite sets and finite unions, is a variation of Folkman's Theorem. We include a proof of Folkman's Theorem for the sake of completeness and more importantly because we are interested in analysing the corresponding upper bounds. The proof presented here is extracted from [13, Section 3.4].

Theorem 42 (Folkman)**.** [13] *For every pair of natural numbers m, r there exists $N \in \mathbb{N}$ such that for every $c : \mathrm{FIN}(N) \to r$ there exists $(x_i)_{i<m} \in \mathrm{FIN}(N)^{[m]}$ such that $c \restriction \langle x_i \rangle_{i<m}$ is constant.*

It easily follows from the Pigeon-Hole principle that the theorem reduces to the following:

Lemma 43. [13] *For every pair of natural numbers m, r there exists $N \in \mathbb{N}$ such that for all $c : \mathrm{FIN}(N) \to r$ there exists $(x_i)_{i<m} \in \mathrm{FIN}(N)^{[m]}$ such that $c \restriction \langle x_i \rangle_{i<m}$ is* min-*determined. That is, if $x = \bigcup_{i \in s} x_i, y = \bigcup_{i \in t} x_i$ with $s, t \subseteq m$ such that* $\min s = \min t$ *then $c(x) = c(y)$.*

We denote by $N(m, r)$ the minimal N satisfying the conditions of Lemma 43. We shall use van der Waerden's Theorem. For $n, r \in \mathbb{N}$, let $W(n, r)$ be the minimal m such that for any r-coloring of m there is a monochromatic arithmetic progression of length n.

Proof. We fix the number of colors $r \in \mathbb{N}$ and proceed by induction on m, the length of the desired sequence. The base case $m = 1$ is clear, so we suppose the statement holds for m and prove it for $m + 1$. By a repeated application of Ramsey's theorem, we fix $N \in \mathbb{N}$ such that given any r-coloring of $\mathrm{FIN}(N)$, there exists $A \subseteq N$ of cardinality $W(N(m, r), r)$ such that for all $i < W(N(m, r), r)$, the coloring c is constant on $[A]^i$, the color possibly depending on i.

Now let $c : \mathrm{FIN}(N) \to r$ be given and let $A \subset N$ be as above with $c \restriction [A]^i$ constant with value $c_i < r$. Define $d : W(N(m,r), r) \to r$ by

$$d(i) = c_i.$$

We use van der Waerden's Theorem to find $\alpha, \lambda < W(N(m,r), r)$ and $i_0 < r$ such that $d \restriction \{\alpha + \lambda j : j < N(m,r)\}$ is constant with value c_{i_0}. Let x_0 be the set consisting of the first α elements of A and let $y_1 < \ldots < y_{N(m,r)}$ be a block sequence of subsets of $A \setminus x_0$ each one of which has cardinality λ. Note that the combinatorial space generated by $(y_i)_{i<N(m,r)}$ is canonically isomorphic to $\mathrm{FIN}(N(m,r))$, therefore by induction hypothesis there exists a block subsequence $x_1 < \ldots < x_m$ of $(y_i)_{0<i\leq N(m,r)}$ such that $c \restriction \langle x_i \rangle_{i=1}^m$ is min-determined.
We shall see that $(x_i)_{i<m+1}$ is the sequence we are looking for. Fix $x, y \in \langle x_i \rangle_{0\leq i\leq m}$ with the same minimum. Suppose first that $x_0 \subseteq x$ then also $x_0 \subseteq y$ and $\#x = m + \lambda i, \#y = m + \lambda j$ for some $i, j < N(m,r)$. Hence $c(x) = c(y) = c_{i_0}$. Now suppose $x_0 \nsubseteq x$ then the same holds for y and consequently $x, y \in \langle x_i \rangle_{i=1}^m$. By the choice of $(x_i)_{i=1}^m$ it follows that $c(x) = c(y)$. $\square$

We now prove Theorem 40 in its multidimensional form.

Theorem 44. *For every $k, m, r, d \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every coloring $c : \mathrm{FIN}_k(n)^{[d]} \to r$ there exists $(f_i)_{i<m} \in \mathrm{FIN}_k(n)^{[m]}$ such that $c \restriction (\langle f_i \rangle_{i<m})^{[d]}$ is constant.*

Let $g_{k,d}(m,r)$ be the minimum n satisfying the conditions in Theorem 44. We prove the theorem by induction on k. If we have the theorem for some k and $d = 1$, we can deduce the theorem for k and dimensions $d > 1$ using a diagonalization argument that we shall describe in detail in Section 5.2 when calculating the upper bounds. We include the dimensions in the statement of the theorem because they play an important role in the proof and because we are interested in calculating upper bounds for $g_{k,d}(m,r)$.

Proof. The base case $k = 1$ in dimension 1 is Folkman's Theorem. Suppose the theorem holds for k and all $m, r, d \in \mathbb{N}$. We work to get the result for $k+1$. We need the following preliminary result:

Claim 45. *For every $N, r \in \mathbb{N}$ there exists $\bar{N}$ such that for every $c : \mathrm{FIN}_{k+1}(\bar{N}) \to r$ there exists $\boldsymbol{h} = (h_i)_{i<N} \in FIN_{k+1}(\bar{N})^{[N]}$ such that for*

$$\begin{aligned} f &= \sum_{i<N} T^{k+1-s_i}(h_i) \\ g &= \sum_{i<N} T^{k+1-t_i}(h_i), \end{aligned}$$

$c(f) = c(g)$ whenever $supp^{\boldsymbol{h}}_{k+1}(f) = supp^{\boldsymbol{h}}_{k+1}(g)$, that is, whenever for all $i < N$, $s_i = k+1$ if and only if $t_i = k+1$.

Let $\bar{N}_{k+1}(N, r)$ be the minimum $\bar{N}$ satisfying the conditions in Claim 45.

Proof of Claim 45. Let $N, r \in \mathbb{N}$ be given. By iteratively applying the induction hypothesis, fix $\bar{N}$ be such that for any sequence of r-colorings $(e_i)_{i<N}$ with $e_i : \mathrm{FIN}_k(\bar{N})^{[2i+3]} \to r$, there exists a block sequence $(f_j)_{j<3N}$ such that for each $i < N$, e_i is constant on $(\langle f_j \rangle_{j<3N})^{[2i+3]}$, its value possibly depending on i.
Let $c : \mathrm{FIN}_{k+1}(\bar{N}) \to r$ be given. Define $U : \mathrm{FIN}_k \to \mathrm{FIN}_{k+1}$ by

$$(Uf)(i) = \begin{cases} f(i) + 1 & \text{if } f(i) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

For each $i < N$ define the coloring $e_i : \mathrm{FIN}_k(\bar{N})^{[2i+3]} \to r$ by

$$e_i((h_j)_{j<2i+3}) = c\left(\sum_{j<2i+3} U^{j \ mod 2} h_j\right),$$

By the choice of $\bar{N}$, we can find a block sequence $(f_j)_{j<3N}$ such that for each $i < N$, e_i is constant on $(\langle f_j \rangle_{j<3N})^{[2i+3]}$. We shall see that the sequence $(h_i)_{i<N}$ defined by $h_i = f_{3i} + Uf_{3i+1} + f_{3i+2}$ for $i < N$ is the sequence we are looking for. Let $g_1, g_2 \in \langle h_i \rangle_{i<N}$ be such that $supp^{\mathbf{h}}_{k+1}(g_1) = supp^{\mathbf{h}}_{k+1}(g_2)$, and let l be the cardinality of $supp^{\mathbf{h}}_{k+1}(g_1)$.

Then we can write g_1, g_2 as

$$g_1 = \sum_{j<2(l-1)+3} U^{j \ mod2} w_j$$
$$g_2 = \sum_{j<2(l-1)+3} U^{j \ mod2} w'_j$$

for some $(w_j)_{j<2(l-1)+3}, (w'_j)_{j<2(l-1)+3} \in (\langle f_j \rangle_{j<3N})^{[2(l-1)+3]}$. Since e_{l-1} is constant on $(\langle f_j \rangle_{j<3N})^{[2(l-1)+3]}$, it follows that $c(g_1) = c(g_2)$. □

We now verify that for the case $k+1$, $d = 1$ in Theorem 44, we may take $n = \bar{N}_{k+1}(H, r)$ where $H = g_{1,1}(m, r)$. Let $c : \mathrm{FIN}_k(n) \to r$ be given. By the choice of n we can find $\mathbf{h} = (h_i)_{i<H}$ such that $c \restriction \langle h_i \rangle_{i<H}$ depends only on $supp^{\mathbf{h}}_{k+1}$. Define $d : \mathcal{P}(H) \to r$ by

$$d(x) = c\left(\sum_{i \in x} h_i\right).$$

By the choice of H we can find $x_0 < \ldots < x_{m-1}$ subsets of H such that $d \restriction \langle x_i \rangle_{i<m}$ is constant. For $i < m$ let $f_i = \sum_{j \in x_i} h_j$. Note that for $f \in \langle f_i \rangle_{i<m}$, $supp^{\mathbf{h}}_{k+1}(f)$ is a finite union of $x_0, \ldots x_{m-1}$. Therefore $c \restriction \langle f_i \rangle_{i<m}$ is constant.
The result for $d > 1$ is obtained by a standard diagonalization procedure that we shall describe in more detail in the next section. □

5.2 Bounds for the finite FIN_k Theorem

In this section we calculate upper bounds for the numbers $g_{k,d}(m, r)$ given by the proof in Section 5.1. Since we used Ramsey's Theorem and van der Waerden's Theorem in our arguments, we will need upper bounds for the numbers corresponding to these two theorems. We shall use the bounds presented in Section 3.1, which we repeat here for easy reference:

$$R_d(m) \leq t_d(c_d m) \tag{5.1}$$
$$W(m) \leq 2^{2^{2^{2^{2^{m+9}}}}} = t_6(m+9) \tag{5.2}$$

We now analyse the proof of Theorem 44 presented in Section 5.1 to get some information about the corresponding upper bounds. We consider only 2-colorings and so we will omit the number of colors in the arguments of our functions and write $g_{k,d}(m)$ for $g_{k,d}(m,2)$. That is, $g_{k,d}(m)$ is the minimal n such that for every coloring $c : \mathrm{FIN}_k(n)^{[d]} \to 2$ there exists $(f_i)_{i<m} \in \mathrm{FIN}_k(n)^{[m]}$ such that $c \restriction (\langle f_i \rangle_{i<m})^{[d]}$ is constant. We adopt the same convention for any other numbers defined in the course of the proof of Theorem 44 that have the number of colors as a parameter.

In what follows we will analyse each step in the proof presented in Section 5.1 and refer to the sequence of lemmas and claims presented therein. Recall that in the proof we proceeded by induction on k and in the inductive step from k to $k+1$ we used the inductive hypothesis in several dimensions $d > 1$. Therefore we will first find the upper bounds corresponding to the base case $k = 1$ in dimension 1, and then describe the diagonalization argument to obtain the result for $k = 1$ and dimension 2. The arguments are similar for higher dimensions and so we get upper bounds for $g_{1,d}(m)$, $d > 1$. To illustrate how the bounds behave when the value of k increases, we analyse the inductive step in the proof and obtain an upper bound for $g_{2,1}(m)$. The arguments to pass from k to $k+1$ and to increase the dimension are similar for bigger values of k and so we get upper bounds for $g_{k,d}(m)$, $k \geq 2$, $d \geq 1$.

We start by finding an upper bound for $N(m)$, the minimal $N \in \mathbb{N}$ such that for all $c : \mathrm{FIN}(N) \to r$ there exists $(x_i)_{i<m} \in \mathrm{FIN}(N)^{[m]}$ such that $c \restriction \langle x_i \rangle_{i<m}$ is min-determined. In the proof of Lemma 43 we had to apply Ramsey's Theorem in dimensions $1, 2, \ldots, W(N(m))$ in order to obtain a suitable set of cardinality $W(N(m))$. To iterate easily the upper bound (5.1), note that for any $i \in \mathbb{N}$ and any given constant c, if x is big enough then we have that $t_i(cx) \leq t_{i+1}(x)$. Using these estimates we get the following

recursive inequality:

$$\begin{aligned}
N(m+1) &\leq R_{W(N(m))} \circ \ldots R_2 \circ R_1(W(N(m))) \\
&\leq t_{l+1}(W(N(m))) \\
&= t_{l+6}(N(m)+9) \\
&\leq t_{t_6(N(m)+10)}(N(m)+9) \\
&\leq f_3(t_6(N(m)+10)+N(m)+9) \\
&\leq f_3(t_6(N(m)+11)) \\
&\leq f_3^3(N(m))
\end{aligned}$$

Where l counts the index of the tower function resulting from the iteration of the bound for $R_d(m)$, that is

$$l = \sum_{i=1}^{t_6(m+9)} i.$$

From the recursive inequality for $N(m)$, we get that

$$N(m) \leq (t^3)^m(1) = f_4(3m).$$

In order to obtain Folkman's Theorem from Lemma 43, we applied the Pigeon-Hole principle, and so we have that

$$g_{1,1}(m) \leq N(2(m-1)+1) \tag{5.3}$$

$$\leq f_4(6m-3). \tag{5.4}$$

We now work to obtain bounds for $g_{1,d}(m)$ for $d > 1$. To increase the dimension by 1, we use a standard diagonalization argument which results in upper bounds of the form $g_{1,d}(m) \leq f_5^d(7m + 2(d-1))$.

We describe the diagonalization argument we used to obtain Theorem 44 for $k = 1$ and $d = 2$ and calculate the resulting upper bound for $g_{1,2}(m)$. Let $c : \mathrm{FIN}(N)^{[2]} \to 2$ be given and let us calculate how large N should be in order to ensure the existence of a block sequence of length m generating a monochromatic combinatorial subspace. We define block sequences $S_0, \ldots, S_{p-1}$ and $a_0 < \ldots < a_{p-1}$ where $p = g_{1,1}(m)$ with the following properties:

(i) $S_0 = \{\{0\}, \ldots, \{N-1\}\}$,

(ii) a_j is the first element of S_j,

(iii) For $j > 0$, S_j is a block subsequence of S_{j-1} such that for each $x \in \langle a_i \rangle_{i<j}$, the coloring $c_x : \mathrm{FIN}(N \setminus x) \to 2$ of the finite subsets of $N \setminus x$ defined by $c_x(y) = c(x, y)$ is constant with value i_x when restricted to $\langle S_j \rangle$,

(iv) the sequence S_{p-1} has length 2.

Each S_j, $0 < j < p$ can be obtained by a repeated application of Theorem 44 for $k = 1$ in dimension 1. Let $S = \{a_j : j < p\}$ and consider the coloring $d : \langle S \rangle \to 2$ defined by $d(x) = i_x$.

By the choice of p, we can find a block subsequence of S of length m that generates a d-monochromatic combinatorial subspace, and by construction this sequence will also generate a c-monochromatic combinatorial subspace. Since the total number of refinements to obtain the sequences $(S_j)_{j<p}$ is $2^p - 1$, it suffices to start with $N \geq g_{1,1}^{2^p-1}(2)$ and so

$$g_{1,2}(m) \leq g_{1,1}^{2^p-1}(2) \tag{5.5}$$

One can prove by induction on l that $g_{1,1}^l(m) \leq f_4^l(6m + l - 4)$, so we have that

$$\begin{aligned} g_{1,2}(m) &\leq f_4^{2^p-1}(2^p + 7) && (5.6) \\ &\leq f_4^{f_4(6m-2)}(f_4(6m-2)) && (5.7) \\ &\leq f_4^{f_4(6m-2)+1}(6m-2) && (5.8) \\ &\leq f_5(f_4(7m)) && (5.9) \\ &\leq f_5^2(7m). && (5.10) \end{aligned}$$

In general the recursive inequality resulting from the diagonalization argument is

$$g_{1,d+1}(m) \leq g_{1,1}^{h_d(g_{1,d}(m))}(2), \tag{5.11}$$

where $h_d(l)$ for $l \in \mathbb{N}$, is the cardinality of $\mathrm{FIN}_1^{[d]}(l)$. Note that $h_d(l) \leq 2^{ld}$. From calculations like the ones in (5.6)-(5.10), one gets that for $d \geq 2$

$$g_{1,d}(m) \leq f_5^d(7m + 2(d-1)). \tag{5.12}$$

Now we find an upper bound for $g_{k,1}(m)$, $k > 1$. Recall that in the proof of Theorem 44 we proceeded by induction on k. In the inductive step from k to $k+1$ we used the higher dimensional versions of the result for k. We first found a subsequence where the coloring depends only on $supp_k$, which is the content of Claim 45. We then applied Folkman's Theorem to obtain the desired sequence in FIN_{k+1}.

We consider the case $k = 2$, the calculations for bigger values of k are similar. Let $N \in \mathbb{N}$ be given, in order to establish Claim 45 for N and $k = 2$ we used Theorem 44 for $k = 1$ in dimensions $2i + 3 (i < N)$. Using the notation in the proof, we see that

$$\begin{aligned}
\bar{N}_2(N) &\leq g_{1,2(N-1)+3} \circ \ldots \circ g_{1,5} \circ g_{1,3}(N) && (5.13)\\
&\leq g_{1,2(N-1)+3}^N(N) && (5.14)\\
&\leq f_5^{2N^2+N}(2N^2 + 10N - 1) && (5.15)\\
&\leq f_6(4N^2 + 11N - 1). && (5.16)
\end{aligned}$$

Where in (5.15) we have used the inequality $g_{1,d}^l(m) \leq (7m + 2(d-1) + (l-1)d)$, for $l, d \in \mathbb{N}$, $d > 1$. From the final application of Folkman's Theorem we get

$$\begin{aligned}
g_{2,1}(m) &\leq \bar{N}_2(g_{1,1}(m)) && (5.17)\\
&\leq g_{1,2(g_{1,1}(m)-1)+3} \circ \ldots \circ g_{1,5} \circ g_{1,3}(g_{1,1}(m)) && (5.18)\\
&\leq f_6(f_4(6m - 2)). && (5.19)
\end{aligned}$$

For the case $k = 2$, the bounds for the higher dimensional numbers we obtain are:

$$g_{2,d}(m) \leq f_7^d(f_4(6m - 2) + 2(d-1)), \tag{5.20}$$

In general from the inductive step we get that for any $k \in \mathbb{N}$,

$$\bar{N}_{k+1}(N) \leq g_{k,2(N-1)+3} \circ \ldots \circ g_{k,5} \circ g_{k,3}(N), \text{ and} \tag{5.21}$$

$$g_{k+1,1}(m) \leq g_{k,2(g_{1,1}(m)-1)+3} \circ \ldots \circ g_{k,5} \circ g_{k,3}(g_{1,1}(m)). \tag{5.22}$$

The diagonalization argument used to increase the dimension from d to $d+1$ in the case $k = 1$ is similar for bigger values of k so we get that

$$g_{k,d+1}(m) \leq g_{k,1}^{h_{k,d}(g_{k,d}(m))}(2), \tag{5.23}$$

where $h_{k,d}(l)$ for $l \in \mathbb{N}$, is the cardinality of $\mathrm{FIN}_k^{[d]}(l)$. Note that $h_{k,d}(l) \leq dl^k$. Using (5.21), (5.22) and (5.23), we can carry out similar calculations as the ones presented for the case $k = 2$ to obtain:

$$g_{k,1}(m) \leq f_{4+2(k-1)} \circ f_4(6m-2), \tag{5.24}$$

$$g_{k,d}(m) \leq f_{5+2(k-1)}^{d}(f_4(6m-2) + 2(d-1)), \tag{5.25}$$

where $d > 1$. We summarize in the following table the upper bounds we obtain.

k	*Dimension*	*Upper bound*
2	1	$f_6 \circ f_4(6m-2)$
	2	$f_7^2(f_4(6m-2)+2)$
	$\vdots$	
	d	$f_7^d(f_4(6m-2)+2(d-1))$
3	1	$f_8 \circ f_4(6m-2)$
	2	$f_9^2(f_4(6m-2)+2)$
	$\vdots$	
	d	$f_9^d(f_4(6m-2)+2(d-1))$
$\vdots$		
k	1	$f_{4+2(k-1)} \circ f_4(6m-2)$
	2	$f_{5+2(k-1)}^2(f_4(6m-2)+2)$
	$\vdots$	
	d	$f_{5+2(k-1)}^d(f_4(6m-2)+2(d-1))$

5.3 Bounds for the finite stabilization theorem

In this section we shall prove the Finite Stabilization Principle mentioned in Section 3.3 for the special case of spaces of the form ℓ_∞^n, namely we shall prove the following:

Theorem 46. [21] *For every $C, \epsilon > 0$ and $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that for every C-Lipschitz function $f : PS_{\ell_\infty^n} \to \mathbb{R}$ there is a positive block sequence $(\mathbf{y}_i)_{i<m}$ such that*

$$osc(f \restriction PS_{[\mathbf{y}_i]_{i<m}}) < \epsilon.$$

Let $n(C, \epsilon, m) \in \mathbb{N}$ be the minimum n satisfying the conditions in Theorem 46. This quantitative version is stated and proved in [21] for the sphere of arbitrary finite dimensional Banach spaces. At first, it seemed plausible that Theorem 46 would suffice to prove the finite FIN_k Theorem. Given a coloring of $\mathrm{FIN}_k(n)$ for some $n \in \mathbb{N}$, one would have to define a function on a subset of the positive sphere of ℓ_∞^n and extend this coloring to the positive sphere of ℓ_∞^n. Problems arise because we have no control over the Lipschitz constant of the resulting function on the positive sphere of ℓ_∞^n.

It is interesting to see how the bounds found in the previous section for the finite FIN_k Theorem compare to the bounds for Theorem 46. In what follows we shall outline the proof of Theorem 46 as presented in [21] and calculate the resulting upper bound for the function $n(C, \epsilon, m)$. The argument is organized in two claims. The statement of the first one as we present it here, is slightly different from [21]; in its proof we use Ramsey's Theorem explicitly. We reproduce the argument for the second claim and provide the details that allow us to calculate the upper bounds.

Let $(\mathbf{e}_i)_{i\in\mathbb{N}}$ be the standard basis of c_0.

Claim 47. [21] *For any $l \in \mathbb{N}$, $C, \epsilon > 0$, there exists $\bar{m} \in \mathbb{N}$ such that for any C-Lipschitz function $f : PS_{[\mathbf{e}_i]_{i<\bar{m}}} \to \mathbb{R}$, there exists $A \subset \bar{m}$ of cardinality m such that $f \restriction PS_{[\mathbf{e}_i]_{i\in A}}$ is ϵ-* almost spreading*, that is, for any $n_0 < \ldots < n_{l-1}, m_0 < \ldots < m_{l-1} \in A$, $l < m$ and any sequence of scalars $(a_i)_{i<l}$ such that $0 < a_i \leq 1$ and $\max_i a_i = 1$, we have that $|f(\sum_{i<l} a_i\mathbf{e}_{n_i}) - f(\sum_{i<l} a_i\mathbf{e}_{m_i})| < \epsilon$.*

Let $\bar{m}(C, \epsilon, m)$ be the minimum $\bar{m}$ satisfying the conditions in Claim 47.

Proof. Let $\epsilon > 0$. Given a C-Lipschitz function $f : PS_{[\mathbf{e}_i]_{i<d}} \to \mathbb{R}$, $d \in \mathbb{N}$ and a sequence of scalars $\mathbf{a} = (a_i)_{i<l}$ such that $0 < a_i \leq 1$ and $\max_i a_i = 1$, we define a coloring $c_{f,\mathbf{a}}$ of $[d]^l$ as follows: Let $(I_j)_{j<r}$ be a partition of the range of f into intervals of length at most $\epsilon/3$, where $r = \lceil 3C/\epsilon \rceil$. Define $c_{f,\mathbf{a}} : [d]^l \to r$ by $c_{f,\mathbf{a}}(\{n_0, \ldots, n_{l-1}\}_<) = j$ if and only if $f(\sum_{i<l} a_i\mathbf{e}_{n_i}) \in I_j$. Note that if $A \subset d$ is homogeneous for $c_{f,\mathbf{a}}$ then for any $n_0 < \ldots < n_{l-1}, m_0 < \ldots < m_{l-1} \in A$, we have that $|f(\sum_{i<l} a_i\mathbf{e}_{n_i}) - f(\sum_{i<l} a_i\mathbf{e}_{m_i})| < \epsilon/3$. We see that $\bar{m}$ should be big enough so that given a C-Lipschitz function $f : PS_{[\mathbf{e}_i]_{i<\bar{m}}} \to \mathbb{R}$, we can find $A \subset \bar{m}$ of cardinality m that is homogeneous for colorings $c_{f,\mathbf{a}}$ with $\mathbf{a}$ ranging over some $\epsilon/3C$-nets of $PS_{[\mathbf{e}_i]_{i<l}}$ for $l < m$. For each $l < m$ we use an $\epsilon/3C$-net for $PS_{[\mathbf{e}_i]_{i<l}}$ of cardinality $\lceil 3C/\epsilon \rceil^{l-1}$. Hence for each $l < m$ it suffices to use Ramsey's Theorem for r-colorings of l-tuples, consequently:

$$\bar{m}(C, \epsilon, m) \leq f_3\left(\left(m\left\lceil\frac{3C}{\epsilon}\right\rceil^m\right) \cdot \left\lceil \log_2 \frac{3}{\epsilon}\right\rceil\right)$$

□

For the next step we need to introduce some notation. We say that $\mathbf{x}, \mathbf{y} \in c_{00}$ have the *same distribution*, denoted by $\mathbf{x} \overset{dis}{=} \mathbf{y}$ if $\mathbf{x} = \sum_{i<k} a_i \mathbf{e}_{n_i}$ and $\mathbf{y} = \sum_{i<k} a_i \mathbf{e}_{m_i}$ for some $k \in \mathbb{N}$, $(a_i)_{i<k} \subset \mathbb{R}$, and $n_0 < \ldots < n_{k-1}$, $m_0 < \ldots < m_{k-1} \in \mathbb{N}$. For $\mathbf{x}, \mathbf{y} \in c_{00}$ let

$$dis(\mathbf{x}, \mathbf{y}) = \inf\{||\bar{\mathbf{x}} - \bar{\mathbf{y}}||_\infty : \bar{\mathbf{x}} \overset{dis}{=} \mathbf{x}, \text{ and } \bar{\mathbf{y}} \overset{dis}{=} \mathbf{y}\}.$$

For $0 < r < 1$ define recursively a sequence of finitely supported vectors $(\mathbf{y}_r^{(n)})_{n\in\mathbb{N}}$ as follows. Let $\mathbf{y}_r^{(0)} = \mathbf{e}_0$ and assuming $\mathbf{y}_r^{(n)} = \sum_{i<l_n} y_r^{(n)}(i)\mathbf{e}_i$ is already defined we let

$$\mathbf{y}_r^{(n+1)} = \sum_{i<l_n} \left(r^{n+1}\mathbf{e}_{3i} + y_r^{(n)}(i)\mathbf{e}_{3i+1} + r^{n+1}\mathbf{e}_{3i+2}\right)$$

(thus $\mathbf{y}_r^{(1)} = (r, 1, r, 0, \ldots), \mathbf{y}_r^{(2)} = (r^2, r, r^2, r^2, 1, r^2, r^2, r, r^2, 0, \ldots)$, etc.). Note that $\#supp(\mathbf{y}_r^{(n)}) = 3^n$ for $n \in \mathbb{N}$. We shall need the following observation:

Claim 48. [21] *Let $t, s \in \mathbb{N}$, $0 < r < 1$, and let $(\mathbf{z}_l)_{l<3^t}$ be a block sequence of vectors with the same distribution as $\mathbf{y}_r^{(st)}$. Then for every linear combination $\mathbf{z} = \sum_{l<3^t} r^{\alpha_l}\mathbf{z}_l$ with at least one $\alpha_l = 0$, there exists $\bar{\mathbf{z}} \overset{dis}{=} \mathbf{y}_r^{(st)}$ such that for every $j \leq (s-1)t$ and $i \in \mathbb{N}$ such that $\mathbf{z}(i) = r^j$, we have that $\bar{\mathbf{z}}(i) = r^{j+l}$ for some $0 \leq l \leq t$.*

Proof. Let $s \in \mathbb{N}$ and $0 < r < 1$. We prove the claim by induction on t. For the base case $t = 1$, let $(\mathbf{z}_l)_{l<3}$ be a block sequence of vectors distributed as $\mathbf{y}_r^{(s)}$. It is useful to note that

$$\begin{aligned}\mathbf{y}_r^{(s)} \quad &\overset{dis}{=} \quad r\overline{\mathbf{y}^{(s-1)}} + r^2\overline{\mathbf{y}^{(s-2)}} + \ldots r^{(s)}\overline{\mathbf{y}^{(0)}} + \overline{\overline{\overline{\mathbf{y}^{(0)}}}} \\ &\qquad + r^{(s)}\overline{\overline{\mathbf{y}^{(0)}}} + \ldots + r^2\overline{\overline{\mathbf{y}^{(s-2)}}} + r\overline{\overline{\mathbf{y}^{(s-1)}}},\end{aligned} \tag{5.26}$$

where for $i < s$, $\overline{\mathbf{y}^{(i)}} \overset{dis}{=} \overline{\overline{\mathbf{y}^{(i)}}} \overset{dis}{=} \mathbf{y}_r^{(i)}$, $\overline{\overline{\overline{\mathbf{y}^{(0)}}}} \overset{dis}{=} \mathbf{e}_0$ and are such that there is no overlap or gaps between the supports of the terms of the sum. Let $\mathbf{z} = \sum_{l<3} r^{\alpha_l}\mathbf{z}_l$ with at least one $\alpha_l = 0$. Since each $\mathbf{y}^{(i)}$, $i < s$ also has an structure as in (5.26), it is easy to find a vector distributed as $\mathbf{y}_r^{(s)}$ that satisfies the conclusion of the claim.

For the inductive step, let $(\mathbf{z}_l)_{l<3^{t+1}}$ be a block sequence of vectors with the same distribution as $\mathbf{y}_r^{(s(t+1))}$ and let $\mathbf{z} = \sum_{l<3^{t+1}} r^{\alpha_l}\mathbf{z}_l$ with at least one $\alpha_l = 0$. It is easy to see that we can write $\mathbf{z}$ as

$$\mathbf{z} = \sum_{m<3} r^{\beta_m}\mathbf{w}_m,$$

where each $\mathbf{w}_m$ is a linear combination of 3^t many vectors distributed as $\mathbf{y}_r^{(s(t+1))}$ and at least one $\beta_m = 0$. For each $m < 3$ let $\bar{\mathbf{z}}_m \overset{dis}{=} \mathbf{y}_r^{(st)}$ be the vector given by the induction hypothesis when applied to the vector obtained from $\mathbf{w}_m$ by restricting to the coordinates with values greater than or equal to r^{st}. Let $\bar{\mathbf{z}} = \sum_{m<3} r^{\beta_m}\bar{\mathbf{z}}_m$. We may now apply the claim in the case $t = 1$ to the vector obtained by restricting $\bar{\mathbf{z}}$ to the coordinates with value greater than or equal to r^s. Let $\bar{\bar{\mathbf{z}}}$ be the vector obtained in this way. It is easy to extend the vector $\bar{\bar{\mathbf{z}}}$ to a vector distributed as $\mathbf{y}_r^{(s(t+1))}$ with the desired property. □

Claim 49. [21] *For every $\epsilon > 0$ and $m \in \mathbb{N}$ there exists a normalized block subsequence $(\mathbf{z}_i)_{i<m}$ of $(\mathbf{e}_i)$ such that the positive sphere of $[z_i]_{i<m}$ has diameter less than ϵ with respect to $dis(\cdot,\cdot)$.*

Let $D(\epsilon, m)$ be the minimal n such that we can find a block subsequence $(\mathbf{z}_i)_{i<m}$ of $(\mathbf{e}_i)_{i<n}$ as in Claim 49.

Proof. Let $\epsilon > 0$, $m \in \mathbb{N}$ be given. To simplify the notation suppose $m = 3^t$ for some $t \in \mathbb{N}$. Let $0 < r < 1$ be such that $1 - r^t < \epsilon/4$ and let $s \in \mathbb{N}$ be such that $r^{(s-1)t} < \epsilon/4$. Take $n = s \cdot t$. Let $\mathbf{z}_0 < \ldots < \mathbf{z}_{m-1}$ be distributed as $\mathbf{y}_r^{(n)}$. Let $\mathbf{w}_0, \mathbf{w}_1$ be in the positive sphere of $[z_i]_{i<m}$. Let $\bar{\mathbf{w}}_i$, $i < 2$ be linear combinations of the vectors $\mathbf{z}_0, \ldots, \mathbf{z}_{m-1}$ of norm 1, whose coefficients are positive powers of r and such that $||\mathbf{w}_i - \bar{\mathbf{w}}_i|| < \epsilon/4$, $i < 2$. By Claim 48 we have that $dis(\bar{\mathbf{w}}_i, \mathbf{z}_0) < \epsilon/4$, and therefore $dis(\mathbf{w}_0, \mathbf{w}_1) < \epsilon$.

We now calculate the upper bound for $D(\epsilon, m)$ given by the proof. From the conditions $1 - r^t < \epsilon/4$ and $r^{(s-1)t} < \epsilon/4$ we get

$$\begin{aligned} s &> \frac{|\log(\epsilon/4)|}{|t \log r|} + 1 \\ |t \log r| &< |\log(1 - \epsilon/4)|, \end{aligned}$$

so $s > \frac{|\log(\epsilon/4)|}{|\log(1-\epsilon/4)|} + 1$ and we have that

$$D(\epsilon, 3^t) \leq 3^{(s+1)\cdot t} \leq (3^t)^{\frac{|\log(\epsilon/4)|}{|\log(1-\epsilon/4)|}+2}.$$

□

To see how Theorem 46 follows from Claims 47 and 49, and obtain the resulting upper bound for $n(C, \epsilon, m)$, let $C, \epsilon > 0$, $m \in \mathbb{N}$ be given. Suppose $m = 3^t$ for some $t \in \mathbb{N}$. Let

$$n = \bar{m}(C, \epsilon/3, D(\epsilon/3C, m)).$$

Let $f : \ell_\infty^n \to \mathbb{R}$ be C-Lipschitz. By Claim 47, we can find $A \subset n$ of cardinality $D(\epsilon/3C, m)$ such that f is $\epsilon/3$-almost spreading on $PS_{[\mathbf{e}_i]_{i \in A}}$. By Claim 49 we can find a block subsequence $(\mathbf{z}_i)_{i<m}$ of $(\mathbf{e}_i)_{i \in A}$ such that $PS_{[\mathbf{z}_0, \ldots, \mathbf{z}_{m-1}]}$ has diameter less than $\epsilon/3C$ with respect to $dis(\cdot, \cdot)$.

Let $\mathbf{y}_i \in PS_{[\mathbf{z}_0, \ldots, \mathbf{z}_{m-1}]}$, $i = 0, 1$. We can find $\bar{\mathbf{y}}_i \in PS_{[\mathbf{z}_0, \ldots, \mathbf{z}_{m-1}]}$ such that $\bar{\mathbf{y}}_i \stackrel{dis}{=} \mathbf{y}_i$, $i = 0, 1$ and $||\bar{\mathbf{y}}_0 - \bar{\mathbf{y}}_1||_\infty < \epsilon/3C$. Hence we have that

$$\begin{aligned} |f(\mathbf{y}_0) - f(\mathbf{y}_1)| &\leq |f(\mathbf{y}_0) - f(\bar{\mathbf{y}}_0)| + |f(\bar{\mathbf{y}}_0) - f(\bar{\mathbf{y}}_1)| + |f(\mathbf{y}_1) - f(\bar{\mathbf{y}}_1)| \\ &< \epsilon. \end{aligned}$$

Therefore

$$n(C, \epsilon, 3^t) \leq \bar{m}(C, \epsilon/3, D(\epsilon/3C, 3^t)) < f_3\left(3^{t \cdot s} \lceil \frac{9C}{\epsilon} \rceil^{3^{t \cdot s}} \lceil \log_2 \frac{9}{\epsilon} \rceil\right),$$

where $s = \lceil \frac{\log(\epsilon/12C)}{\log(1-\epsilon/12C)} \rceil + 2$.

5.4 Additional remarks

As far as we know there is no proof of the infinite FIN_k Theorem that avoids the use of idempotent ultrafilters. The proof we present of the finite version cannot be adapted to the infinite case. This is because when proving the result for $k+1$, we have to know how many dimensions of the inductive hypothesis we need, and this number depends on the desired length of the homogeneous sequence.

We found upper bounds for the Finite Stabilization Theorem in the special case of the spaces ℓ_∞^n that grow much more slowly than the upper bounds we have for the finite FIN_k Theorem. This suggests that the FIN_k Theorem is stronger than this special case of the Finite Stabilization Theorem. To make this comparison precise and also because it is interesting in its own right, we still have to find lower bounds for the functions $g_k(n), k \in \mathbb{N}$. This would amount to finding for any given $l \in \mathbb{N}$, a bad coloring of $\mathrm{FIN}_k(N)$ for some N, for which there is no sequence of length l generating a monochromatic combinatorial subspace. In this direction it would also be interesting to find a way for stepping up lower bounds for a given $k \in \mathbb{N}$ to larger values of k.

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