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Fuzzy Revealed Preference Theory*

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Rationality has traditionally been defined as choice behaviour which can be explained in terms of some implicit binary preference. The point of departure of this paper lies in permitting the binary preference relation to be "fuzzy." Concepts from fuzzy set theory are used to formalise different notions of rationality, including degrees of rationality. The relation between these and traditional concepts is formally explored. In welfare economics, *quasi*-orderings have often been used to capture the inherent imprecisions of human value judgements. It is argued here that, in many situations, a more appropriate tool for this may be fuzzy orderings. *Journal of Economic Literature* Classification Numbers: 022, 024.

1. INTRODUCTION

That human preferences are typically characterised by different degrees of indeterminacy has been emphasised often enough. Attempts to incorporate such indeterminacy into our analyses have, however, been much more limited. It has also been argued persuasively that our rankings of societies in terms of social characteristics (e.g., equality, real national income, etc.) ought to reflect the inherent imprecisions of human perception instead of providing artificial "exact" rankings (Sen [14]). While I do comment on this (Section 5), the present paper is basically concerned not with evaluation but with individual rationality, in particular the theory of revealed preference. Here, as well as in normative analysis, the standard attempt to capture imprecision has been in terms of quasi-orderings (reflexive and transitive binary relations which may not be complete). While this does go some distance, a quasi-ordering has the difficulty that over each pair of alter-

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natives it does not permit *degrees* of imprecision.[1] A natural way of permitting this is by using the idea of "fuzzy" binary relation. And fortunately for this we do not have to begin from scratch. Recent advances in the theory of fuzzy sets have meant a ready stock of tools and concepts.

In standard theory, a *set* is a *well-defined* collection in the sense that each element of the universal set is either a full member of it (gets a mark of 1) or not a member (gets 0). A *fuzzy set* is a collection which allows partial membership (i.e., an element could get a mark anywhere between 0 and 1). Now, if we think of a binary relation as a subset of a Cartesian product, then a *fuzzy binary relation* is a fuzzy subset of the Cartesian product.

While fuzzy preference relations could be used in many diverse areas of economics, the present focus is on revealed preference analysis—the kind developed by Samuelson [11], Arrow [1], Richter [9], Sen [13], Suzumura [15] and others. An individual is characterised by a choice function (i.e., a specification of chosen elements for every feasible set of alternatives). One line of approach pursued by Richter, Sen and Suzumura is to check whether the individual's choice function could be thought of as the outcome of some preference maximisation. This implicit preference in this theory is required to be "exact." In reality, however, human preferences are often marked by a certain amount of fuzziness and though their actual choices are perforce "exact," they are nevertheless the outcome of fuzzy preferences. Consequently, this paper allows for the possibility that the choice function may be rationalizable by a *fuzzy* preference relation. The idea of "exact" or unfuzzy choices based on fuzzy preferences is formalised using two standard concepts from the theory of fuzzy sets.

A fuzzy revealed preference theory has the advantage that it allows us to think in terms of human beings as possessed of different degrees of rationality instead of the harsher notion of a partition of rational and irrational persons.

The principle aim of this paper is to initiate the use of fuzzy set theory in preference analysis. Hence, a large part of the present effort is directed towards developing useful concepts. And, in fact, even the theorems should not be viewed as theorems of "surprise" but rather as propositions which establish some of the important properties of these new concepts.

[1] In utility theory, in comparing inter- and intra-personal welfares there is a natural way of capturing a certain kind of indeterminacy. If we think of an individual as characterised by a collection of utility functions and his preferences as those which are in accord with each function in this collection, then an increasing amount of indeterminacy could be incorporated by making the collection increasingly broad [12, 6, 4]. Another approach, which has been used for revealed preference analysis, is to suppose that an individual's choice is determined by a preference ordering which is stochastically chosen from a collection of possible orderings [2].

2. Concepts in the Theory of Fuzzy Sets

Eubulides was a fourth century B. C. Greek philosopher who had devoted a substantial fraction of his life to discovering paradoxes. One of the most important paradoxes he discovered, which has many variants, is the Heap paradox. In essence it says: If from a heap of grain a person keeps removing one grain at a time, it will no doubt, after a while, cease to be a heap. Most of us would agree that there does not exist an integer n such that a collection of n grains would be described as a heap while a collection of $n-1$ grains would be described as a non-heap. How does one reconcile these two statements? This paradox has many ramifications which we cannot go into here, but one thing is clear—the paradox arises because we think of a heap as an "exact" concept, i.e., each collection of grains must be either a heap or a non-heap. One way out is to think of each collection of being a heap of a certain degree *between* 0 and 1, with 0 signifying a complete non-heap and 1 a complete heap with the numbers strictly between these extremes denoting the extent to which a collection is a heap. Not only would this help us out of Eubulides' paradox but this is probably closer to our conception of a heap than the $0-1$ concept which standard set theory commits us to. Given that a large number of definitions (e.g., the set of all bright red objects) are characterised by hazy boundaries, the motivation for a theory of fuzzy sets is obvious.

Let U be the universal (unfuzzy) set. A is a *fuzzy subset* of U, or simply a *fuzzy set*, implies $A: U \to [0, 1]$. For all $x \in U$, $A(x)$ is interpreted as the extent to which x belongs to A. If a fuzzy set A is such that $A(U) \subset \{0, 1\}$ then A is described as an *ordinary* or an *exact* set.[2] If A is an exact set, we may write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$; and we may think of A as a collection, $\{y, z, \ldots, \}$, of exactly those elements x of U, for which $x \in A$. It is clear from this that a fuzzy set is a generalisation of our usual notion of a set since an exact set is just a special kind of a fuzzy set.

It is important to emphasise that given a fuzzy set A, $A(x)$ is not the probability that x belongs to A. Nowadays many clubs take in partial members (e.g., affiliate members, half-members, etc.) which means that the set of members of a club maybe a fuzzy set. Clearly in that case to say that i is a half-member of a club is in a fundamental sense different from the assertion that there is a probability of $\frac{1}{2}$ that i is a member of the club.

In our analysis, we shall need some concept of distances between fuzzy sets. The following is a generalisation of Hamming distance between sets.

[2] Some authors prefer to call it an *unfuzzy* set [8] or a *hard set* [5].

Given two fuzzy sets A and B in U (where $\#U < \infty$), the *generalised Hamming distance* (Kaufman [7]), $d(A, B)$, may be defined as

$$d(A, B) = \sum_{x \in U} |A(x) - B(x)|.$$

Let X ($\#X < \infty$) be a (exact) set of alternatives. R is a *fuzzy binary relation* (FBR) on X means $R: X \times X \to [0, 1]$, i.e., R is a fuzzy subset of $X \times X$. As before, an *exact* binary relation on X is a FBR, R, such that $R(X \times X) \subset \{0, 1\}$.

The FBR, R, on X is said to be[3]

(a) *reflexive* iff $\forall x \in X$, $R(x, x) = 1$;

(b) *complete* iff $\forall x, y \in X$, with $x \neq y$, $R(x, y) + R(y, x) \geqslant 1$;

(c) *transitive* iff $\forall x, y \in X$, $R(x, y) \geqslant \frac{1}{2}R(x, z) + \frac{1}{2}R(z, y)$, $\forall z \in X \backslash \{x, y\}$ such that $R(x, z) \neq 0$, $R(z, y) \neq 0$;

(d) a *fuzzy ordering* if it is reflexive, complete and transitive.

Given a FBR, R, on X and an exact non-empty set S which is a subset of X, the *greatest set* in S is denoted by $G(S, R)$ and is defined as $G(S, R): S \to [0, 1]$ such that $\forall x \in S$, $G(S, R)(x) = \min_{y \in S} R(x, y)$. Note that $G(S, R)$ is a fuzzy set[4] but if R is exact $G(S, R)$ coincides with what Sen [12] calls the choice set and Suzumura [15] calls the greatest set.

[3] My definition of transitivity is non-standard. By one definition (see [8, 7]) a FBR, R, on X, is transitive if $\forall x, y \in X$, $R(x, y) \geqslant \max_{z \in X} \min\{R(x, z), R(z, y)\}$. This definition has one difficulty. Suppose $X = \{x, y, z\}$ and $R(x, z) = 0.5$, $R(z, y) = 0.5$. Then the lower bound on $R(x, y)$ is 0.5. Consider, alternatively, that $R(x, z) = 1$, $R(z, y) = 0.5$. Even in this case the lower bound on $R(x, y)$ is 0.5. Clearly in the second example $R(x, y)$ may be expected to be above that in the first example. One general way of correcting this is to define transitivity in terms of the property $R(x, y) \geqslant \alpha \max\{R(x, z), R(z, y)\} + \beta \min\{R(x, z), R(z, y)\}$, $\forall z \in X \backslash \{x, y\}$, such that $R(x, z) \neq 0$, $R(z, y) \neq 0$; and with $\alpha + \beta = 1$, $\alpha, \beta > 0$. A simpler way (which is in fact a special case of this) is the definition I use. The important principle in defining fuzzy set theoretic concepts is that the concepts must coincide with the standard ones in the special case of the set being exact. It is easily checked that if $R(X \times X) \subset \{0, 1\}$, then my definitions of reflexivity, completeness, transitivity and ordering coincide with the traditional definitions (as, e.g., in Sen [12]).

[4] It is natural that if R is fuzzy, the greatest set will be fuzzy as well. However, given our practice of thinking of the greatest set as an exact concept, we may be tempted to treat the points of S which score the highest with the function $G(S, R)$ as the truly greatest elements and the rest as not greatest. That such an approach is contrived is obvious if we adopt a dual perspective: that is, first try to locate the not-greatest elements and then treat the residual as greatest. To retain the same spirit as above, we should define the lowest scorers with the function $G(S, R)$ as the not-greatest elements. These two approaches would give contradictory results. Also, my definition has the advantage of having parallels to Orlovsky's definition of "best" (see footnote 5).

Clearly, given a binary relation there are different ways of defining the "best" elements in a set. The above definition of a greatest set is *one* formalisation of the general idea of a set of "best" elements. There can indeed be others. The definition used in Orlovsky [8][5] can be shown to be the fuzzy counterpart of the concept of a maximal set in traditional choice theory (see [12, 15]). It is possible to construct alternative approaches to revealed preference theory by using different definitions as illuminated by Suzumura [15] but here we restrict attention to the greatest set.

3. Choice Functions and Rationality

Let X ($3 \leqslant \#X < \infty$) be the basic (exact) set of alternatives. Let K be the set of all exact subsets of X containing two or more elements. An individual is characterised by a *choice function*, $C(\cdot)$, which is defined as

$$C: K \to K \qquad \text{and} \qquad \forall S \in K, \qquad C(S) \subset S.$$

In this paper—as also in Arrow [1], Sen [13], Suzumura [15, 16] and others[6]–the choice function is a primitive. It is implicitly treated as an observable.[7] Having observed an individual's choice function, we want to decide whether it may be described as "rational" or not. There is a vast literature around essentially this problem. The attention here is on a particularly ingenious idea of Richter [9, 10] that a choice function is rational if it can be thought of as having been "generated" by a (exact) binary relation, i.e., if $C(\cdot)$ can be thought of as the outcome of preference maximisation. Thus $C(\cdot)$ is *Richter rational* iff there exists an exact binary relation R, on C such that for all $S \in K$, $C(S) = G(S, R)$. In this case the R which *rationalises* $C(\cdot)$ need not satisfy any properties. A more demanding and probably a more satisfactory definition of rationality used by Richter [10] and widely discussed since, is regular rationality.

[5] Given a FBR, R, Orlovsky defines strict preference, $P(R)$, as follows: $P(R)(x, y) = \max\{R(x, y) - R(y, x), 0\}$ and then he defines the set of "nondominated elements", $M(S, R)$ as follows: $M(S, R)(x) = 1 - \max_{y \in S} P(R)(y, x)$. If R is exact, clearly this definition coincides with the standard definition of a maximal set.

[6] In Arrow and Sen, the choice function is defined on the entire domain K. Suzumura's framework is more general in that he allows for the possibility that the domain of the choice function may be a subset of K. This difference is critical. Actually what is crucial in the Arrow–Sen framework is that the domain includes all those subsets of X which contain 2 or 3 elements.

[7] Though the problem is ignored here, it is important to note that *observing* a choice function has more than mere practical difficulties, as I have discussed elsewhere [3].

Regular Rationality

$C(\cdot)$ is *regular rational* iff $\exists$ an exact binary ordering, R, on X such that $\forall S \in K$, $C(S) = G(S, R)$.

Regular rationality is of particular interest because of its equivalence to other standard axioms of rationality, like Samuelson's [11] weak axiom—as interpreted by Arrow [1].

The spirit of these definitions is that a person is rational if (i) he possesses an exact binary relation (with or without some properties) and (ii) he adheres to it when choosing. In reality, most human preferences are inexact; and hence the objective of this paper is to allow for fuzziness in preference, i.e., to develop concepts of rationality without using (i). That is, a person will be described as rational if his choice function, $C(\cdot)$, could be thought of, in some sense, as the overt expression of an underlying fuzzy preference. This may be formalised in different ways depending on how we associate an *exact* set of best elements to a *fuzzy* binary relation. I explore two routes here using two concepts from the existing work on fuzzy set theory.

Consider first Orlovsky's [8] concept of "unfuzzy dominance" based on a FBR and its corresponding definition of rationality.

Unfuzzy Dominance

Given a FBR, R, on X the *unfuzzy dominant set* in $S \in K$ is denoted by $D(S, R)$ and defined as $D(S, R) = \{x \in S \mid G(S, R)(x) = 1\}$.

D-Rationality

$C(\cdot)$ is *D-rational* iff $\exists$ a fuzzy ordering, R, on X such that $\forall S \in K$, $C(S) = D(S, R)$.

D-rationality permits individuals to have fuzzy preferences and therefore has a motivation which is distinct from the numerous existing rationality concepts. But, unfortunately, it gives us no extra mileage because it turns out that D-rationality is behaviourally indistinguishable from regular rationality.

THEOREM 1. *An individual is D-rational iff he is regular rational.*

Proof. Let $C(\cdot)$ be regular rational. Hence $\exists$ an exact ordering, $\hat{R}$, on X such that $\forall S \in K$, $C(S) = G(S, \hat{R})$. Hence, $\forall S \in K$, $G(S, \hat{R})$ is exact which implies that $\forall x \in S$, $G(S, \hat{R})(x) \in \{0, 1\}$. It follows from the definition of $D(S, \hat{R})$ that $G(S, \hat{R})(x) = 0 \rightarrow x \notin D(S, \hat{R})$ and $G(S, \hat{R})(x) = 1 \rightarrow x \in D(S, \hat{R})$. Hence $D(S, \hat{R}) = G(S, \hat{R})$. Therefore, $\forall S \in K$, $C(S) = D(S, \hat{R})$, i.e., $C(\cdot)$ is D-rational.

Now consider a $C(\cdot)$ that is D-rational. Therefore, $\exists$ a fuzzy ordering, say R^*, such that $\forall S \in K$, $C(S) = D(S, R^*)$. Define an exact binary relation $\tilde{R}$ as

$$\forall x, y \in X, \tilde{R}(x, y) = \begin{cases} 1, & \text{if } R^*(x, y) = 1 \\ 0, & \text{if } R^*(x, y) < 1. \end{cases}$$

Let $S \in K$. $x \in G(S, \tilde{R}) \leftrightarrow \forall y \in S,\ \tilde{R}(x, y) = 1$

$$\begin{aligned}
&\leftrightarrow \forall y \in S, R^*(x, y) = 1\\
&\leftrightarrow G(S, R^*)(x) = 1\\
&\leftrightarrow x \in D(S, R^*).
\end{aligned}$$

Hence $\forall S \in K$, $C(S) = D(S, R^*) = G(S, \tilde{R})$. The proof is completed by showing that $\tilde{R}$ is an ordering. The reflexivity of $\tilde{R}$ is an immediately consequence of the reflexivity of R^*. Since by the definition of a choice function $C(S) \neq \varnothing$, $\forall S \in K$, it follows that $\forall x, y \in X$, $D(\{x, y\}, R^*) \neq \varnothing$. Hence $R^*(x, y) = 1$ or $R^*(y, x) = 1$, which implies $\tilde{R}(x, y) = 1$ or $\tilde{R}(y, x) = 1$, i.e., $\tilde{R}$ is complete. Finally, to prove transitivity consider x, y, $z \in X$.

$$\begin{aligned}
\tilde{R}(x, y) = 1, \tilde{R}(y, z) = 1 &\rightarrow R^*(x, y) = 1, R^*(y, z) = 1\\
&\rightarrow R^*(x, z) = 1, \text{ by transitivity of } R^*\\
&\rightarrow \tilde{R}(x, z) = 1. \quad \blacksquare
\end{aligned}$$

Theorem 1 implies that whenever a person's behaviour is such that it can be rationalised using a *fuzzy* ordering, there must exist an *exact* ordering which rationalizes it in the sense of Richter's regular rationality. This renders the use of a *fuzzy* ordering inconsequential in practice. A more interesting approach is suggested by an alternative method of constructing a dominant exact set from a FBR. This makes use of the concept of the "nearest unfuzzy subset" (see Kaufman [7]) though my definition is slightly unconventional. In what ensues, the assumption of finiteness of X is important. (Note that this rules out a *direct* application to traditional consumption sets.)

Nearest Exact Set

Given a fuzzy set A in X, a *nearest exact set* of A, denoted $N[A]$, is any exact set which is nearest to A in terms of the generalised Hamming distance (see Section 2). It follows that $N[A]$ is any set with the property

$$\{x \in X \mid A(x) > 0.5\} \subset N[A] \subset \{x \in X \mid A(x) \geqslant 0.5\}.$$

N-Rationality

$C(\cdot)$ is *N-rational* iff $\exists$ a fuzzy ordering, R, on X such that $\forall S \in K$, $C(S) = N[G(S, R)]$. A fuzzy ordering R which has this property is said to *N-rationalise* $C(\cdot)$.

D-rationality, we saw above, makes no extra allowances than regular rationality. N-rationality goes to the other extreme.

THEOREM 2. *All individuals are N-rational.*

Proof. Define a *FBR*, $\hat{R}$, on X as follows:

$$\forall\, x, y \in X, \hat{R}(x, y) = \begin{cases} 0.5, & \text{if } \; x \neq y; \\ 1, & \text{if } \; x = y. \end{cases}$$

Hence $\forall S \in K, G(S, \hat{R})$ has the property that $\forall x \in S$, $G(S, \hat{R})(x) = 0.5$. Hence, $\forall$ exact set $T \subset S$, $T = N[G(S, \hat{R})]$, i.e., T is a nearest exact set of $G(S, \hat{R})$. Hence $\forall C(\cdot)$, $\forall S \in K$, $C(S) = N[G(S, \hat{R})]$. Clearly $\hat{R}$ is an ordering. Hence $\hat{R}$, defined as above, can N-rationalise any choice function. ∎

At first sight this is a disappointing result. But on reflection it becomes clear that while N-rationality is indeed a vacuous concept, Theorem 2 is precisely what one expects from a fuzzy revealed preference theory. In fuzzy preference analysis we ask the question: Can a person's choice behaviour be explained in terms of a fuzzy ordering? And Theorem 2 answers: Yes, as long as we are willing to allow fuzziness of any degree. As mentioned earlier, following Richter [9], in a large number of papers including the present one, the concept of raionality has two suppositions (i) a person has a preference ordering and (ii) he chooses in adherence to it. What Theorem 2 suggests is the following: If an individual has a (fuzzy or exact) preference ordering but he fails to adhere to it, then we could always conceive of another person who has a fuzzier preference ordering and adheres to it and whose behaviour is identical to that of the first.

If we consider regular rationality as a kind of "full" rationality then Theorem 2 suggests the possibility of thinking of individuals as possessing different degrees of rationality: If $C(\cdot)$ and $\hat{C}(\cdot)$ are such that R and $\hat{R}$ are the "least"-fuzzy orderings (respectively) which N-rationalise them and R is "less" fuzzy than $\hat{R}$, then $C(\cdot)$ could be thought of as more rational than $\hat{C}(\cdot)$, particularly since $C(\cdot)$ is, in some sense, closer to a regular rational choice function. It is this idea which is developed in the next section.

4. The Extent of Rationality

In traditional theory, one person can never be *more* rational than another without him being completely rational and the other being completely irrational. As discussed above, fuzzy revealed preference theory leads, quite naturally, up to the idea of the *extent* of rationality. To formalise this we need some measure of the "fuzziness" of binary relations. Since a FBR is a fuzzy subset of a Cartesian product, we may consider the following standard definition.

For any FBR, R, on X, its index of fuzziness, $v(R)$, is defined as

$$v(R) = \frac{2d(R, N(R))}{\#(X \times X)},$$

where $N(R)$ is a nearest exact set of R, and $d(R, N(R))$ is the generalized Hamming distance (defined in Section 2) between R and $N(R)$.[8]

It is easy to check that if R is a completely fuzzy binary relation ($\forall x, y \in X$, $R(x, y) = 0.5$) then $v(R) = 1$ and if R is exact, $v(R) = 0$. In this paper our attention is on orderings rather than arbitrary binary relations. Note that a completely fuzzy ordering is defined as follows

$$\forall x, y \in X, R(x, y) = \begin{cases} 0.5, & \text{if } x \neq y; \\ 1, & \text{if } x = y. \end{cases}$$

Hence for a completely fuzzy *ordering* $v(R) < 1$, and the extent by which $v(R)$ falls short of one clearly depends on $\#X$. Hence for fuzzy *orderings* it is more apt to define an index of fuzziness (naming it *degree of fuzziness* to distinguish it from the standard measure $v(R)$) as follows:

For any fuzzy ordering, R, on X, its *degree of fuzziness*, $\delta(R)$, is defined as

$$\delta(R) = \frac{2d(R, N(R))}{\#(X \times X) - \#X}.$$

This measure has the advantage of taking values from 0 to 1, with the two ends signifying respectively complete fuzziness and exactitude.

Now we can formally define the extent of ones rationality based on the informal idea discussed at the end of Section 3.

For brevity, let $\mathscr{R}[C(\cdot)]$ be the set of all fuzzy orderings which N-rationalise $C(\cdot)$.

Degree of Fuzzy Rationality

$C(\cdot)$ is *fuzzy rational of degree* $\Omega[C(\cdot)]$, which is defined as follows:

$$\Omega[C(\cdot)] = 1 - \min_{R \in \mathscr{R}[C(\cdot)]} \delta(R).$$

By Theorem 2 we know that $\mathscr{R}[C(\cdot)]$ is non-empty for all $C(\cdot)$. Hence $\Omega[C(\cdot)]$ is well-defined and every individual is fuzzy rational of some degree.

[8] It is true that $N(R)$ is non-unique since one fuzzy set can have more than one nearest exact set. It is however easy to see that $d(R, N(R))$ is unique for all $N(R)$. Hence $v(R)$ is well-defined.

What is interesting is that the concept of degrees of fuzzy rationality has links with concepts in the standard theory of revealed preference, in particular with Samuelson's [11], weak axiom. *Following Arrow* [1], the *weak axiom of revealed preference* (WARP) may be defined as below.

WARP. $C(\cdot)$ satisfies WARP iff there does not exist $S_1, S_2 \in K$ such that for some $x, y \in X$, $x \in C(S_1)$, $y \in S_1$ and $y \in C(S_2)$, $x \in S_2 \backslash C(S_2)$.

We shall say that *WARP is violated at* (x, y) iff $\exists S_1, S_2 \in K$ such that $x \in C(S_1)$, $y \in S_1$ and $y \in C(S_2)$, $x \in S_2 \backslash C(S_2)$. If WARP is violated at (a, b), for all $a, b \in X$, $a \neq b$, we say that $C(\cdot)$ *everywhere violates WARP.* "$C(\cdot)$ *nowhere violates WARP*" is defined similarly. It is now clear that a person is rational in the sense of Samuelson, i.e., his choice function satisfies WARP, iff WARP is nowhere violated. A single violation is equated with irrationality.

The next theorem shows the connection between fuzzy rationality and the Arrovian version of Samuelson's weak axiom.

THEOREM 3. *If* $C(\cdot)$ *satisfies WARP then* $\Omega[C(\cdot)] = 1$ *and if* $C(\cdot)$ *everywhere violates WARP then* $\Omega[C(\cdot)] = 0$. *Further,* $\Omega[C(\cdot)] = 1$ *implies* $C(\cdot)$ *satisfies WARP.*

Proof. First note that $[C(\cdot)$ is regular rational$] \leftrightarrow [\Omega[C(\cdot)] = 1]$. This is an immediate consequence of the fact that if R is an exact ordering then $d(R, N(R)) = 0$. Now, given the known theorem (see [13]; see also [15]), that WARP is equivalent to regular rationality when the domain of the choice function, $C(\cdot)$, includes all subsets of X containing 2 or 3 elements, it follows that $[C(\cdot)$ satisfies WARP$] \leftrightarrow [\Omega[C(\cdot)] = 1]$.

What remains to prove is that if $C(\cdot)$ everywhere violates WARP, then $\Omega[C(\cdot)] = 0$. Suppose $C(\cdot)$ everywhere violates WARP. Let R^* be a solution of $\min_{R \in \mathscr{R}[C(\cdot)]} \delta(R)$. Hence R^* N-rationalises $C(\cdot)$. Let x, $y \in X$, $x \neq y$. Since $C(\cdot)$ everywhere violates WARP, $\exists S_1, S_2 \in K$ such that

$$x \in C(S_1), \qquad y \in S_1,$$
$$y \in C(S_2), \qquad x \in S_2 \backslash C(S_2).$$

Since R^* N-rationalises $C(\cdot)$, it follows that

$$x \in N[G(S_1, R^*)], \qquad y \in S_1,$$
$$y \in N[G(S_2, R^*)], \qquad x \in S_2 \backslash N[G(S_2, R^*)].$$

It follows from the definitions of a greatest set and a nearest exact set that $R^*(x, y) \geqslant 0.5$, $R^*(y, x) \geqslant 0.5$; and that $\exists z \in S_2$ such that $R^*(x, z) \leqslant 0.5$.

There are two possibilities: $z=y$ or $z\neq y$. If $z=y$, then it immediately follows that $R^*(x,y)=0.5$. If $z\neq y$, then by the transitivity of R^*

$$R^*(x,z)\geqslant\tfrac{1}{2}R^*(x,y)+\tfrac{1}{2}R^*(y,z).$$

Since $z\in S_2$ and $y\in N[G(S_2,R^*)]$, hence $R^*(y,z)\geqslant 0.5$. Since, also, $R^*(x,y)\geqslant 0.5$ and $R^*(x,z)\leqslant 0.5$, for the above inequality to be valid it is necessary that $R^*(x,y)=0.5$. Given that (x,y) was arbitrarily chosen, it follows that $\forall x,y\in X$, with $x\neq y$, $R^*(x,y)=0.5$. Since R^* is an ordering and therefore reflexive, $\forall x\in X$, $R^*(x,x)=1$. Hence,

$$d(R^*,N(R^*))=\tfrac{1}{2}\{\#(X\times X)-\#X\}$$

which implies $\delta(R^*)=1$, i.e., $\Omega[C(\cdot)]=0$. ∎

In the standard theory of revealed preference, irrationality is equated with a single act of irrationality. Thus a person is rational in the sense of Samuelson if and only if he *nowhere* violates WARP. This theory cannot distinguish between a person in whose choice behaviour *there exists* a case of irrationality to one who is consistently irrational. But as Theorem 3 shows in fuzzy revealed preference analysis a person who satisfies WARP everywhere and one who nowhere satisfies it occupy two ends of a range of possible degrees of irrationality.

It may be useful to show with an example a partially rational person.

EXAMPLE 1. Let $X=\{x,y,z\}$ and $C(\cdot)$ be as follows:

$$C(X)=\{x,y\},\ C(\{x,y\})=\{y\},\ C(\{y,z\})=\{y\},\ C(\{x,z\})=\{x\}.$$

Clearly the person chooses "oddly" over $\{x,y\}$ but is quite "sane" in his choice over all other pairs. It is easy to check that he is irrational in terms of WARP and is therefore also *not* regular rational. Fuzzy revealed preference theory, however, distinguishes him from a totally irrational person. Note that $R\in\mathscr{R}[C(\cdot)]\leftrightarrow R$ is a fuzzy ordering and

$$N[G(X,R)]=\{x,y\},\qquad N[G(\{x,y\},R]=\{y\},$$
$$N[G(\{y,z\}),R]=\{y\},\qquad N[G(\{x,z\},R]=\{x\}.$$

Hence $R\in\mathscr{R}[C(\cdot)]\leftrightarrow R$ is a fuzzy ordering and

$$\begin{array}{llll} R(x,y)\geqslant 0.5, & R(y,x)\geqslant 0.5, & R(x,z)\geqslant 0.5, & R(x,y)\leqslant 0.5 \\ R(y,z)\geqslant 0.5, & R(z,x)\leqslant 0.5, & R(z,y)\leqslant 0.5 & \end{array} \tag{1}$$

Now consider all fuzzy *orderings* on X which satisfy property (1). It is easy to check that the least fuzzy ordering (in the sense of $\delta(R)$) in this class is an ordering R^* defined as follows:

$$R^*(x, y) = 0.5,$$
$$R^*(y, x) = R^*(y, z) = R^*(x, z) = R^*(x, x) = R^*(y, y) = R^*(z, z) = 1,$$
$$R^*(z, y) = R^*(z, x) = 0.$$

Hence $d(R^*, N(R^*)) = \frac{1}{2}$, $\delta(R^*) = \frac{1}{6}$ and $\Omega[C(\cdot)] = \frac{5}{6}$. Therefore $C(\cdot)$ is fuzzy rational of degree $\frac{5}{6}$. ∎

Note that while Theorem 3 provides a complete characterisation of rationality of degree 1, it provides only a partial characterisation of the other extreme of the rationality spectrum, namely fuzzy rationality of degree 0. This is because while it is true that everywhere violation of WARP does imply $\Omega[C(\cdot)] = 0$, the reverse implication is not valid. This is reasonable: Suppose $\# X = 3$. Then it is easy to check that it is impossible to violate the weak axiom everywhere, i.e., over all ordered pairs (x, y). If the reverse implication was valid, it would imply that in this case no choice function could be completely irrational (i.e., $\Omega[C(\cdot)] = 0$), by definition. Fortunately, as the next example shows, that is not so.

EXAMPLE 2. Let $X = \{x, y, z\}$ and $C(\cdot)$ be as follows:

$$C(X) = \{x\}, \quad C(\{x, y\}) = \{x\}, \quad C(\{y, z\}) = \{y\}, \quad C(\{x, z\}) = \{z\}.$$

While this person's choices appear thoroughly inconsistent, note that he does not everywhere violate WARP; in particular WARP is not violated over (x, y). Nevertheless, this person is fuzzy rational of degree 0.

Suppose $R \in \mathscr{R}[C(\cdot)]$. Then, by a process of reasoning similar to the one used in Example 1, we get

(i) $R(x, y) \geqslant 0.5$, (ii) $R(x, z) \geqslant 0.5$, (iii) $R(y, z) \geqslant 0.5$, (iv) $R(z, x) \geqslant 0.5$,

(v) $R(y, x) \leqslant 0.5$, (vi) $R(z, y) \leqslant 0.5$, (vii) $R(x, z) \leqslant 0.5$.

By transitivity, (iii), (iv) and (v): $R(y, z) = R(z, x) = R(y, x) = 0.5$.

By transitivity, (i), (iv) and (vi): $R(x, y) = R(z, y) = 0.5$

By (ii) and (vii): $R(x, z) = .5$.

Hence, $\forall a, b \in X$ with $a \neq b$, $R(a, b) = 0.5$. Since R is an ordering, $\forall a \in X$, $R(a, a) = 1$. Hence $\delta(R) = 1$. Since R was arbitrarily chosen from $\mathscr{R}[C(\cdot)]$, hence $\forall R \in \mathscr{R}[C(\cdot)]$, $\delta(R) = 1$. Hence $\Omega[C(\cdot)] = 0$. ∎

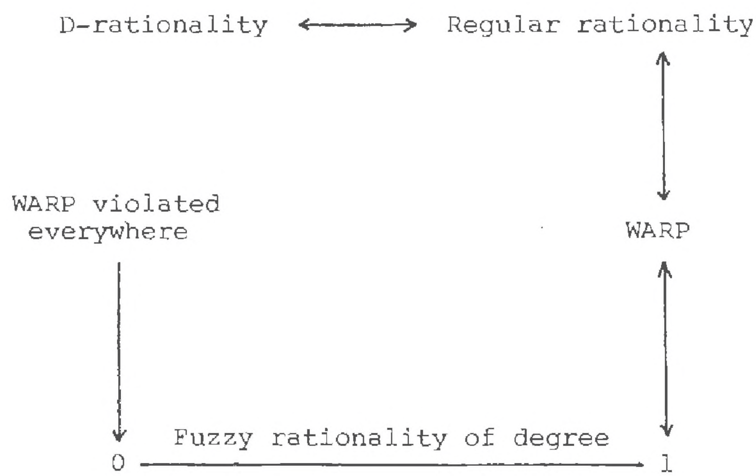

FIGURE 1

The results derived in this paper and also the earlier result that WARP is equivalent to regular rationality [15] are summed up with an implication diagram (Fig. 1). A one way arrow represents implication and a two way arrow equivalence. It may be worthwhile emphasizing that the results are based on the assumption that all 2 or 3 element subsets of the universal set of alternatives are members of the domain of the choice function, unlike in classical consumer theory in which the domain consists of a class of convex polyhedra.

5. EXACT QUASI-ORDERINGS AND FUZZY ORDERINGS

The relevance of fuzzy preference analysis in positive economics stems from the fact that human preferences are often marked by different degrees of indeterminateness and a theory based on the assumption of exact preferences may lead not only to wrong answers but, more importantly, to wrong questions. Thus questions like "Is an agent rational?" or "Does he consider x preferable to y?" may be fundamentally misleading, and should instead be replaced by (respectively) "How rational is an agent?" or "To what extent does he consider x preferable to y?"

While thus far our concern was exclusively with positive issues, some of the fuzzy relational concepts used here could have important applications in normative economics. This is because concepts like inequality or social welfare may be inherently imprecise. Of course we are, in principle, free to define away the imprecision. But as Sen [14, p. 47–48] argues in the context of "inequality," "In a trivial sense ⋯ one can define "inequality" precisely as one likes, and as long as one is explicit and consistent one may think that one is above criticism. But the force of the expression "inequality," and indeed our interest in the concept, derive from the meaning that is associated with the term, and we are not really free to define it purely arbitrarily. And—as it happens—the concept of inequality has different facets which may point in different directions..." From this persuasive axiom what Sen goes on to attack in conventional evaluative rankings is the property of completeness (see Sen [14, pp. 47–8, 72–6]). Thus he argues for the use of quasi-orderings for comparing inequality and real income in different economies. While within the confines of unfuzzy binary relations this is an obvious conclusion, it is not so in a broader framework. In the latter, it is not clear that an argument for the eschewal of orderings on the ground that most evaluative concepts have "different facets which may point in different directions," is a case for the adoption of quasi-orderings rather than fuzzy orderings.

In particular quasi-orderings have one important difficulty which is in some sense similar to the very one which motivates its use in preference to orderings. With a quasi-ordering, for any pair of alternatives, $\{x, y\}$, the preference relation is either not defined at all or completely defined. There can be no indeterminacy there! This is precisely where the main advantage of fuzzy orderings lie. Not only in this area but even elsewhere, it is possible that the motivation which has prompted the use of quasi-orderings may well be captured with greater aptness by fuzzy relations or orderings.

This may be illustrated with a suggestion made by Sen [14, pp. 72–74] for inequality rankings which allow for the inherent multi-facetedness of the concept of equality:

Suppose here are k basic criteria, $C^1, \ldots, C^k$, each representing one facet of the concept and each yielding a complete ordering. Then one way of ranking economies is in terms of the intersection, Q, of these k orderings,—the intersection being defined as follows:

$$\forall x, y, xQy \text{ iff } xC^iy, \forall i \in \{1, \ldots, k\}.$$

Q, it may be checked, is a quasi-ordering. Sen illustrates the use of this *intersection method* by actually computing the quasi-ordering of income distributions in five countries based on three criteria, the Gini coefficient, the coefficient of variation and the standard deviation of logarithms.

The difficulty with the intersection method lies in the fact that it errs on the side of caution. Suppose that all but one criterion judges x superior to y and only one criterion considers y to be superior. By the intersection method we can pass no judgement on x and y. This is a particularly serious problem if k happens to be large. Moreover, this method has a kind of discontinuity: Beginning from a case of full consensus if one criterion changes its ranking between x and y, the quasi-ordering Q switches from full judgement to no judgement at all between x and y.

One way of overcoming this while retaining the spirit of Sen's suggestion is to define a FBR as an aggregation of the k criteria in the following sense: Given k basic (exact) orderings $C^1, \ldots, C^k$, let the FBR, $\tilde{Q}$, be defined as follows:

$$\forall x, y, \tilde{Q}(x, y) = \sum_{i=1}^{k} \alpha_i C^i(x, y), \qquad \sum \alpha_i = 1 \text{ and } \alpha_i > 0, \forall i.$$

For each i, α_i measures the relative importance of the ith criterion. Remember that since C^i is exact, $C^i(x, y) = 1$ or 0. It can be checked that $\tilde{Q}$ is reflexive and complete. This follows from the reflexivity and completeness of C^i, for all i. $\tilde{Q}$, however, does not satisfy transitivity (as defined above). This need not be too disturbing since over domains of alternatives where $\tilde{Q}$ happens to be exact, transitivity is ensured, i.e., $[\tilde{Q}(x, y) = 1, \tilde{Q}(y, z) = 1] \rightarrow \tilde{Q}(x, z) = 1$. Hence $\tilde{Q}$ does satisfy a kind of "weak" transitivity property which coincides with the standard definition of transitivity where the binary relation in question is exact.

$\tilde{Q}$ has the advantage that, unlike Q, it does not go completely silent as soon as there is the smallest conflict of opinion. It thereby avoids the "discontinuity" problem which characterises Q: If in comparing x with y, a small subset of the collection of basic criteria differs from the rest, a small amount of fuzziness enters the overall ranking between x and y.

In closing, it is important to point out that there is one aspect in which a fuzzy *ordering* is more restrictive than a quasi-ordering. Given a fuzzy ordering, R, $R(x, y) + R(y, x) \geqslant 1$; for x, y. This means that a fuzziness over (y, x) must have a complementary unfuzziness over (x, y). Unlike a quasi-ordering, a fuzzy ordering does not permit a two-way ignorance between x and y, i.e., we can never have $R(x, y) = R(y, x) = 0$. This may be an advantage or a disadvantage depending on the problem at hand.

In all these exercises of evaluative ranking, it is important however to appreciate that while there is much to criticise in attempts to artificially rank societies as orderings, the other extreme of complete fuzziness is barren. The art of constructing appealing measures of social characteristics, like inequality and real rational income, depends critically on how one strikes a balance.

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