HEDGING IN LÉVY MARKETS

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This dissertation examines the hedging process in Lévy markets. It consists of three self-contained chapters.

In Chapter 1, we study some results from Malliavin calculus and their applications to the hedging problem. We set up a market with finite number of assets each driven by a Lévy process and focus on the hedging problem. A self-financing trading strategy is designed to perfectly hedge (if possible) a given contingent claim. If a perfect hedging is not possible, it is designed to approximately hedge a given contingent claim. In general, such a market is not complete and only a small subset of square integrable contingent claims can be hedged. For other claims, a minimum variance hedging strategy is given in literature in terms of Malliavin derivative of contingent claims and assets in the market. We review this result. Such a finite market can be completed by power jump assets. Malliavin derivative of each power jump asset is a polynomial and these polynomials are dense in square Lebesgue-integrable functions. In particular, Malliavin derivative of a contingent claim gives us a square integrable function and it can be approximated by Malliavin derivative of power jump assets. This result shows that enlarged market by infinitely many power jump assets gives us a market complete in the limit. In other words, a contingent claim in such a market (complete in the limit) can be either hedged perfectly or there exists a sequence of contingent claims, each can be hedged perfectly, such that mean squared hedging error goes to zero.

Chapter 2 focuses on hedging in Heath-Jarrow-Morton model (HJM). We assume that forward rates are driven by a Lévy process and summarize no–arbitrage conditions in HJM framework. We develop necessary and sufficient conditions for perfect hedging of a contingent claim by a self-financing trading strategy. These conditions give us two integral equations: one eliminates Brownian motion related risk and another eliminates jump risk. The latter one takes the form of a Fredholm integral equation of the first kind and it does not have a solution for all possible square integrable contingent claims. Both integral equations have kernels in terms of forward rates and both equations are solved for Malliavin derivatives in different directions. First, we derive the solutions for the hedging problem when the market is driven by a pure jump process. Our analysis shows that under certain conditions, all power jump assets can be perfectly hedged. This result gives us sufficient conditions for a market which is complete in the limit. Further, we extend our results for a general Lévy process. As an application of our approach, we develop a hedging strategy for a variance swap which can be approximately hedged. We also obtain the Malliavin derivative of a caplet.

Chapter 3 investigates hedging in exponential Lévy markets. First, we review the call option pricing model that is derived on the basis of Fourier transform. Using Malliavin calculus, we develop the necessary and sufficient conditions for perfect hedging of a contingent claim by a self-financing trading strategy of call options. These conditions give us two integral equations: the first one relates to hedge of Brownian motion related risk and the other deals with hedging jump risk. First, we assume that market is driven by a pure jump process and jump sizes are bounded. In this case, we have only one integral equation which is related to jump risk. The integral equation is transformed into a con-

volution type integral equation and a closed form solution for a dense subset of square integrable contingent claims is obtained in terms of Fourier transform and Malliavin calculus. We extend these results to general Lévy process with bounded jump sizes. Our approach can be used in practice to develop dynamic hedging strategies using call options. The above results have important implications for hedging in exponential Lévy market: one key implication is that for each possible jump size we need a call option with a strike value equal to jump size. We also show some results related to Asian options in exponential Lévy markets. Finally, we outline the major limitations of our approach.

BIOGRAPHICAL SKETCH

Yusuf Serkan Kıraç was born in Elazığ, a beautiful small city in east Turkey. Serkan spent much of his early childhood with his grandparents in Elazığ. He finished primary school and middle school in Elazığ, and high school in İstanbul. After graduating from Üsküdar Burhan Felek High School with unforgettable memories, he started at Boğaziçi University and graduated with an undergraduate degree in electrical and electronics engineering.

Before Serkan moved to the United States, he worked in finance industry as a software engineer. He holds a M.Sc. degree in operations research from Cornell University.

Dedicated to my grandmother Naime, my uncle Ökten and my mother Tuncay.

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CHAPTER 1

HEDGING IN A LÉVY MARKET WITH A FINITE NUMBER OF TRADED ASSETS

1.1 Introduction

In continuous time asset pricing, two important properties of a given market are very important: no–arbitrage and completeness. The former one has been studied by many researchers and from the first fundamental theorem of asset pricing, the existence of an equivalent martingale measure is a sufficient condition for a no–arbitrage market (Delbaen et al. [2006]). When the assets in the market are driven by a finite number of Brownian motions, finding such a measure by Girsanov's theorem is not a difficult task (Nunno et al. [2010], Medvegyev [2007]). It is also not difficult to check for uniqueness. If the measure is unique then the market is also complete by the second fundamental theorem of asset pricing (Delbaen et al. [2006]).

When the assets in the market are driven by a general Lévy process with jumps, $L(t)$, no–arbitrage condition is not difficult to check. On the other hand, under an equivalent martingale measure, $L(t)$ is not necessarily a Lévy process anymore. Also, the market is not complete in general and only a small subset of contingent claims can be hedged perfectly. In this study, we will focus on hedging contingent claims when the assets in a market are driven by a Lévy process.

There are two different groups of such markets driven by a Lévy process. In the first group, the market has a finite number of assets trading. In general, such a market doesn't contain a sufficient number of assets to hedge every contingent claim. In this case, if a given contingent claim cannot be hedged perfectly and there is no perfect hedge that leads to a zero mean squared hedging error. So, an investor can only use an approximate hedging strategy. In the second group, there are infinitely many assets trading in the market. One example belonging to this group is the class of default free term structure of interest models. Such a term structure of interest rates market can be complete in the limit under certain assumptions.

Another important aspect of hedging in a market is the set of admissible trading strategies. In general, there are two different types of hedging strategies. In the first group, we assume that the set of admissible trading strategies is self-financing. In this case, in an incomplete market, an investor chooses a self-financing strategy with minimal mean squared hedging error. The hedging strategy doesn't need to replicate the contingent claim perfectly.

When the market has a finite number of assets, a self-financing trading strategy with minimal mean squared hedging error is commonly used in the literature. This approach is called variance-optimal hedging and was studied by different researchers in the literature. Benth et al. [2003] developed a minimum variance hedging strategy by using Malliavin calculus. A self-financing trading strategy in a market with finite number of assets can be formulated as an inverse matrix equation in Malliavin calculus, and the solution exists if the matrix is invertible. On the other hand, this method requires computation of the Malliavin derivative of a contingent claim.

Another example of a market with infinite assets is a market that has all power jump assets. Corcuera et al. [2005] studied power jump assets to obtain a market complete in the limit. The authors initially assume a market that consists of a stock driven by a Lévy process and a money market account. Such a market is not complete. Further, the authors extend this market by including infinitely many power jump assets. We will define power jump assets formally later in this chapter. Briefly, for each $n \geq 2$, they define a power jump asset, which is defined in terms of the sum of n^{th} power of the $L(t)$'s jumps. These processes are assumed to be assets and traded in the extended market. They assume that there exists an equivalent martingale measure $\mathbb{Q}$ such that discounted asset price and discounted power jump assets are martingales under $\mathbb{Q}$. A square integrable contingent claim X can be either perfectly hedged by a self-financing trading strategy or there exists a sequence contingent claims, each can be hedged perfectly, that converges to X. Although the market is complete in the limit, it doesn't have any practical application since power jump assets are not real assets and not traded in financial markets.

In the second group, the set of admissible strategies is a bigger set, it also contains trading strategies that are not self-financing. In this group, investors aim to replicate a contingent claim perfectly at maturity with the exception of cash withdrawals or infusions. Investors can inject or withdraw money from the hedging portfolio. The total amount of discounted funds injected or withdrawn is called the discounted cost process, and the aim of the hedging portfolio is to replicate a given contingent claim perfectly using minimum discounted cost process. Föllmer et al. [1991] defined a locally risk minimizing strategy that is optimal and this optimal hedging strategy becomes a self-financing trading strategy when the market is complete.

In this chapter, we study Malliavin calculus for Lévy processes, power jumps processes, and their Malliavin derivatives followed by minimum hedging strategies. First, we define and review the necessary background in Malliavin calculus. The most important result of Malliavin calculus to be used is the Clark–Ocone theorem. It gives us a representation for square integrable variables. This representation has two components, each is a stochastic integral where the integrands are Malliavin derivatives in different directions. The difficulty of Malliavin calculus is finding these derivatives. Malliavin derivatives can be defined for square integrable random variables by a white noise analysis. However, in practice it is not a useful approach. In white noise analysis, the Hilbert space of square integrable random variables has an orthonormal basis in terms of Hermite polynomials and Hermite functions. A square integrable random variable has a decomposition with respect to this orthonormal basis. Malliavin derivatives are defined in terms of this decomposition. In practice, it is not possible to find this decomposition for a given square integrable random variable. In this thesis, we will use Malliavin derivatives in the white noise analysis only in the Clark–Ocone theorem.

Another method is to define it for a dense subspace of square integrable random variables by a Wiener-Itó chaos expansion. This dense subspace is computationally less complicated and sufficient for many applications. In this dissertation, we will obtain Malliavin derivatives for many contingent claims (for example, call option, caplets, variance swaps, Asian option etc.) and we will show that all of them are in this dense subspace.

Second, we move on to study power jump assets and orthonormalized power jump processes. Every power jump power jump asset is a sum of powers of jumps of the same Lévy process in a compact time interval. We know that if the jump size of a Lévy process is bounded below by a positive real number then in a finite time interval there can be at most finite jumps. In this case, a power jump asset is a finite sum of powers of these jumps. On the other hand, if the jumps sizes are not bounded below by a positive real number, every right–regular process (both left and right limits exists and process is right continuous) can have at most countably infinitely many jumps in a finite time integral (Medvegyev [2007]). In this case, a power jump asset is a countably infinite sum of powers of these jumps. The Malliavin derivative of each power jump is a power of the jump size and a linear combination of the power jump assets hedges a contingent claim with the number of assets held corresponding to a polynomial Malliavin derivative. In other words, if the Lévy process is a pure jump process, a dense subset of square integrable random variables can be hedged perfectly. Malliavin calculus will give us a better understanding of the use of power jump assets in hedging. We will develop these relations in the next sections.

Third, we will review a market with finite assets. In such a market, a small set of contingent claims can be hedged perfectly. We will review the necessary conditions for a contingent claim to be hedged perfectly by a self-financing trading strategy. This first step will show that the hedging problem using a self-financing trading strategy in a market driven by a Lévy process can be equivalently derived as the simultaneous solution of two linear equations. Finally, we will review minimum variance hedging using Malliavin calculus.

1.2 Malliavin Calculus

A Lévy process, L_t, is a sum of a Brownian motion, a linear drift and a pure jump process:

$$L_t = at + \sigma W_t + \int_0^t \int_{|x|<1} x\Big(N(du, dx) - v(dx)du\Big) + \int_0^t \int_{|x|\geq 1} xN(du, dx) \qquad (1.1)$$

where a is a constant, W_t is a Brownian motion, v is a Lévy measure, $N(dt, dx)$ is a Poisson random measure, and $N(du, dx) - v(dx)du$ is a compensated Poisson process (Medvegyev [2007], Protter [2005]). Therefore, it is defined on the product space of two probability spaces. In this part, we will first define the Malliavin derivative for the case where $L(t)$ is a Brownian motion. As mentioned in the introduction, it can be defined for two spaces : $\mathbb{D}_{1,2}$ and L^2. They will be defined formally in this chapter later. The former is based on a Wiener-Itó chaos expansion and the latter is based on white noise analysis. After we define the Malliavin derivative for the first space, we will define the Malliavin derivative for a pure jump process. Similarly, we will provide the different definitions for $\mathbb{D}_{1,2}$ and L^2. Further, we will define the Malliavin derivative for a general Lévy process. Most of the material in this section can be found in Benth et al. [2003], (Nunno et al. [2010], Nunno et al. [2004]).

1.2.1 In $\mathbb{D}_{1,2}$ When The Lévy Process Is A Brownian Motion

Malliavin calculus is based on the Wiener-Itô expansion of square integrable random variables. Every square integrable random variable has a unique decomposition, and Malliavin derivative of a random variable in a dense subset of square integrable random variables can be defined in terms of this decomposition. The first step is to define the dense subset $\mathbb{D}_{1,2}$ and then we will define the Malliavin derivative for this dense subspace. In this section we will assume that:

1-) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space.

2-) $W(t)$ $t \in [0, T]$ is a one dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$.

3-) $\mathcal{F}_t^W$ is the σ-algebra generated by $W(t)$, and $\mathcal{F}_t = \sigma\left(\mathcal{F}_t^W \cup \mathcal{N}\right)$ where $\mathcal{N}$ is the collection of $\mathbb{P}$-zero sets in $\mathcal{F}$. Therefore, $\mathcal{F}_t$ is a right continuous filtration and satisfies the usual conditions (Medvegyev [2007], Protter [2005]).

The Wiener-Itô expansion is based on iterated stochastic integrals with respect to the Brownian motion. Integrands are square integrable functions. A function f is in $L^2([0, T]^n)$ if

$$\int_0^T \int_0^T \cdots \int_0^T f(t_1, t_2, \cdots, t_{n-1}, t_n)^2 \lambda(dt_1, dt_2, \cdots, dt_{n-1}, dt_n) < \infty$$

In the expansion, integrands are also symmetric functions:

Definition 1.2.1 *$f : [0, T]^n \to \mathbb{R}$ is called symmetric if*

$$f(t_{\sigma_1}, t_{\sigma_2}, \cdots, t_{\sigma_{n-1}}, t_{\sigma_n}) = f(t_1, t_2, \cdots, t_{n-1}, t_n)$$

for any permutation $(\sigma_1, \sigma_2, \cdots, \sigma_{n-1}, \sigma_n)$ of $(1, 2, \cdots, n-1, n)$.

Proposition 1.2.1 *(The Wiener-Itô expansion by iterated integrals) Let X be a $\mathcal{F}_T$ measurable square integrable random variable. X has a unique expansion*

$$X = \sum_{n=0}^{\infty} I_n(f_n) \tag{1.2}$$

where f_n is a symmetric function in $L^2([0, T]^n)$ and

$$I_n(f_n) = n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f(t_1, t_2, \cdots, t_{n-1}, t_n) dW_{t_1} dW_{t_2} \cdots dW_{t_{n-1}} dW_{t_n} \tag{1.3}$$

is a stochastic integral with respect to the Brownian motion W_t.

Note that if $m \neq n$ then $I_m(f)$ and $I_n(g)$ are orthogonal, i.e. $\mathbb{E}^{\mathbb{P}}[I_m(f)I_n(g)] = 0$ when $m \neq n$. Although this decomposition exists for every square integrable random variable, the following summation 1.4 which is a norm is not finite for every square integrable random variable.

Definition 1.2.2 *($\mathbb{D}_{1,2}$) Let X be a $\mathcal{F}_T$ measurable square integrable random variable. From previous theorem, X has a unique expansion. $X \in \mathbb{D}_{1,2}$ if*

$$\| X \|^2_{\mathbb{D}_{1,2}} = \sum_{n=0}^{\infty} n n! \| f_n \|^2_{L^2([0,T]^n)} < \infty \tag{1.4}$$

It is obvious that, chaos expansion can be approximated by a finite sum and for a random variable with a finite sum decomposition, $\| \cdot \|^2_{\mathbb{D}_{1,2}}$ is finite. This shows that $\mathbb{D}_{1,2}$ is a dense subset of $L^2(\mathbb{P})$, square integrable random variables in $(\Omega, \mathcal{F}, \mathbb{P})$. Now, we can define a Malliavin derivative in $\mathbb{D}_{1,2}$.

Definition 1.2.3 *(Malliavin derivative in $\mathbb{D}_{1,2}$) Let $X \in \mathbb{D}_{1,2}$. The Malliavin derivative of X is defined as*

$$D_t X = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)),\ t \in [0,T] \tag{1.5}$$

where

$$I_{n-1}(f_n(\cdot, t)) = (n-1)! \int_0^t \cdots \int_0^{t_3} \int_0^{t_2} f(t_1, t_2, \cdots, t_{n-1}, t) dW_{t_1} dW_{t_2} \cdots dW_{t_{n-1}} \tag{1.6}$$

is a stochastic integral with respect to the Brownian motion W_t (Medvegyev [2007]).

Properties of Malliavin derivative in $\mathbb{D}_{1,2}$

In the next chapters, we will need following properties of Malliavin derivative in $\mathbb{D}_{1,2}$:

a-)$\mathbb{E}^{\mathbb{P}}\left[\int_0^T (D_t X)^2 dt\right] = \| X \|^2_{\mathbb{D}_{1,2}}$.

b-)$\mathbb{D}_{1,2}$ is a dense subset of $L^2(\mathbb{P})$.

c-)Assume that $X \in L^2(\mathbb{P})$, $X_n \in \mathbb{D}_{1,2}$, $X_n \to X$ in $L^2(\mathbb{P})$ and $D_t X_n$ converges to Y in $\|\cdot\|^2_{\mathbb{D}_{1,2}}$. Then $X \in \mathbb{D}_{1,2}$ and $D_t X = Y$.

d-)Assume that $X_1, X_2, ..., X_n \in \mathbb{D}_{1,2}$, $f(x_1, x_2, ..., x_n) \in C^1(\mathbb{R}^n)$ and $\left|\frac{\partial f}{\partial x_i}\right|$ is bounded for every i. Then $f(X_1, X_2, ..., X_n) \in \mathbb{D}_{1,2}$ and $D_t f(X_1, X_2, ..., X_n) = \sum_{i=1}^{n} \frac{\partial f(X_1, X_2, ..., X_n)}{\partial x_i} D_t X_i$.

(a) and (b) are trivial, proofs for (c) and (d) can be found in (Nunno et al. [2010]).

1.2.2 In L^2 When The Lévy Process Is A Brownian Motion

Although the Malliavin derivative for $\mathbb{D}_{1,2}$ can be defined in terms of a Wiener-Itô chaos expansion, we need a different expansion to extend it to $L^2(\mathbb{P})$. First we need to define Ω and $\mathcal{F}$. In this section we will assume that $\Omega = \mathcal{S}(\mathbb{R})'$ is the dual of the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$ (Stein et al. [2003], Stein et al. [2005]).

Definition 1.2.4 *$\mathcal{S}(\mathbb{R})$, Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$, is the set of all indefinitely differentiable functions f such that for every $k, l \geq 0$*

$$\sup_{x \in \mathbb{R}} |x|^k \mid f^{(l)}(x) < \infty$$

It is also called the space of tempered distributions. We define $\mathcal{F}$ as the set of Borel subsets of Ω equipped with the *weak** topology. Bochner-Minlos-Sazonov and Kolmogorov theorems ensure that there exists a probability measure $\mathbb{P}$ and a continuous process W_t such that it is a Wiener process in $(\Omega, \mathcal{F}, \mathbb{P})$ (Gelfand et al. [1964]). We will assume that $\mathcal{F}_t$ is the same as we defined in the previous section. First, we define Hermite polynomials and functions.

Definition 1.2.5 *(Hermite polynomials and functions) For every $n = 0, 1, 2, ...,$ we define h_n, the Hermite polynomial as*

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} \left(e^{-\frac{1}{2}x^2}\right) \tag{1.7}$$

We also define the k^{th} Hermite function as

$$e_k(x) = \pi^{-\frac{1}{4}}((k-1)!)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} h_{k-1}(\sqrt{2}x), \text{ k=1,2,...} \tag{1.8}$$

Using Hermite polynomials, we can decompose any square integrable random variable (in general this decomposition is valid for a larger space then L^2 space but in this study we focus on square integrable contingent claims).

Proposition 1.2.2 *(The Wiener-Itô expansion by Hermite polynomials) Let X be a $\mathcal{F}_T$ measurable square integrable random variable. X has a unique expansion*

$$X = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \tag{1.9}$$

where $\mathcal{J}$ is the set of all finite dimensional index vectors of non-negative integers $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$, $m = 1, 2, 3, ...$ and

$$H_\alpha = \prod_{j=1}^{m} h_{\alpha_j}\left(\int e_k(t) dW_t\right) \tag{1.10}$$

Note that, $\{H_\alpha\}_{\alpha \in \mathcal{J}}$ is an orthogonal basis of $L^2(\mathbb{P})$. Now, we can define the Malliavin derivative in $L^2(\mathbb{P})$.

Definition 1.2.6 *(Malliavin derivative in $L^2(\mathbb{P})$) Let $X \in L^2(\mathbb{P})$. The Malliavin derivative of X is defined as*

$$D_t X = \sum_{\alpha \in \mathcal{J}} \sum_{\alpha_k > 0} c_\alpha \alpha_k e_k(t) H_{\alpha - \epsilon_k} \tag{1.11}$$

where ϵ_k is a m dimensional vector of all zeros except 1 at position k.

We have two definitions of the Malliavin derivative: one in $L^2(\mathbb{P})$ and one in $\mathbb{D}_{1,2} \subset L^2(\mathbb{P})$. For a random variable $X \in \mathbb{D}_{1,2}$, both definitions of the Malliavin derivative give the same result (Nunno et al. [2010]).

1.2.3 In $\mathbb{D}_{1,2}$ When The Lévy Process Is A Pure Jump Process

Before we define the Malliavin derivative for a general Lévy process, we will define it for a pure jump process. From the Lévy-Itó decomposition theorem, a Lévy process can be decomposed into three parts: a Brownian motion term with drift, a compounded Poisson process and a limit of square integrable martingales. When the first component doesn't exist, a Lévy process is called a pure jump process. In this section we will assume that:

1-) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space.

2-) v is a Lévy measure on $\mathbb{R} \setminus \{0\}$.

3-) $N(t, x)$ $t \in [0, T]$ is a compensated Poisson random measure on $(\Omega, \mathcal{F}, \mathbb{P})$.

4-) λ is a Lebesque measure on $[0, T]$.

5-) $\mathcal{F}_t^N$ is the σ-algebra generated by $L(t) = \int_0^t \int_{\mathbb{R}\setminus\{0\}} xN(du, dx)$. $\mathcal{F}_t = \sigma\left(\mathcal{F}_t^N \cup \mathcal{N}\right)$ where $\mathcal{N}$ is the collection of $\mathbb{P}$-zero sets in $\mathcal{F}$. Therefore, $\mathcal{F}_t$ is a right continuous filtration and satisfies usual conditions. All of these definitions can be found in any of (Medvegyev [2007], Protter [2005], Sato [1999]).

Proposition 1.2.3 *(The Wiener-Itô expansion for Poisson random measures) Let X be a $\mathcal{F}_T$ measurable square integrable random variable. X has a unique expansion*

$$X = \sum_{n=0}^{\infty} I_n(f_n) \tag{1.12}$$

where f is a symmetric function in $L^2((\lambda \times v)^n)$ and

$$I_n(f_n) = n! \int_0^T \int_{\mathbb{R}\setminus\{0\}} \int_0^{t_n-} \int_{\mathbb{R}\setminus\{0\}} \cdots \int_0^{t_2-} \int_{\mathbb{R}\setminus\{0\}} f(t_1, x_1, \cdots, t_n, x_n) N(dt_1, dx_1) \cdots N(dt_n, dx_n) \tag{1.13}$$

A function f is in $L^2((\lambda \times v)^n)$ if

$$\int_0^T \int_{\mathbb{R}\setminus\{0\}} \cdots \int_0^T \int_{\mathbb{R}\setminus\{0\}} f(t_1, x_1, \cdots, t_n, x_n)^2 (\lambda \times v)(dt_1, dx_1, \cdots, dt_n, dx_n) < \infty$$

In the expansion, the integrands are also symmetric functions:

Definition 1.2.7 *Let $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. A function $f : ([0, T] \times \mathbb{R}_0)^n \to \mathbb{R}$ is called symmetric if*

$$f(t_{\sigma_1}, x_{\sigma_1}, t_{\sigma_2}, x_{\sigma_2} \cdots, t_{\sigma_{n-1}}, x_{\sigma_{n-1}}, t_{\sigma_n}, x_{\sigma_n}) = f(t_1, x_1, t_2, x_2, \cdots, t_{n-1}, x_{n-1}, t_n, x_n)$$

for any permutation $(\sigma_1, \sigma_2, \cdots, \sigma_{n-1}, \sigma_n)$ of $(1, 2, \cdots, n-1, n)$.

If $m \neq n$ then $I_m(f)$ and $I_n(g)$ are orthogonal. We will define the Malliavin derivative in $\mathbb{D}_{1,2}$ in terms of chaos expansion.

Definition 1.2.8 *($\mathbb{D}_{1,2}$) Let X be a $\mathcal{F}_T$ measurable square integrable random variable. From previous theorem, X has a unique expansion. $X \in \mathbb{D}_{1,2}$ if*

$$\| X \|^2_{\mathbb{D}_{1,2}} = \sum_{n=0}^{\infty} n n! \| f_n \|^2_{L^2((\lambda \times v)^n)} < \infty \tag{1.14}$$

Definition 1.2.9 *(Malliavin derivative in $\mathbb{D}_{1,2}$) Let $X \in \mathbb{D}_{1,2}$. The Malliavin derivative of X is defined as*

$$D_{t,x} X = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t, x)),\ t \in [0, T],\ x \in \mathbb{R} \setminus \{0\} \tag{1.15}$$

where

$$I_{n-1}(f_n(\cdot,t,x)) = (n-1)! \int_0^{t-} \int_{\mathbb{R}\setminus\{0\}} \cdots \int_0^{t_2-} \int_{\mathbb{R}\setminus\{0\}} f(t_1,x_1,\cdots,t_{n-1},x_{n-1},t,x) \, N(dt_1,dx_1)\cdots N(dt_{n-1},dx_{n-1})$$

For technical reasons, we need to define another dense subspace of $L^2(\mathbb{P})$.

Definition 1.2.10 *($\mathbb{D}^{\epsilon}_{1,2}$) Let $f \in L^2([0,T])$ and $F = e^{\int_0^T \int_{\mathbb{R}\setminus\{0\}} x f(t) N(dt,dx)}$. Define $\mathbb{D}^{\epsilon}_{1,2}$ as the set of all finite linear combinations of such random variables F.*

Note : $\mathbb{D}^{\epsilon}_{1,2}$ is a dense subset of $\mathbb{D}_{1,2}$ (Nunno et al. [2010]).

Properties of Malliavin derivative in $\mathbb{D}_{1,2}$ We will need some of the following properties of Malliavin derivative in $\mathbb{D}_{1,2}$.

a-)$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_{\mathbb{R}\setminus\{0\}} (D_{t,x}X)^2 \nu(dx)dt\right] = \| X \|^2_{\mathbb{D}_{1,2}}$.

b-)Both $\mathbb{D}_{1,2}$ and $\mathbb{D}^{\epsilon}_{1,2}$ are dense subspaces of $L^2(\mathbb{P})$.

c-)Assume that $X \in L^2(\mathbb{P})$, $X_n \in \mathbb{D}_{1,2}$, $X_n \to X$ in $L^2(\mathbb{P}\times\lambda\times\nu)$ and $D_{t,x}X_n$ converges to Y in $\|\cdot\|^2_{\mathbb{D}_{1,2}}$. Then $X \in \mathbb{D}_{1,2}$ and $D_{t,x}X = Y$.

d-)Assume that $X \in \mathbb{D}_{1,2}$, f is a continuous function. If $f(X) \in L^2(\mathbb{P})$ and $f(X + D_{t,x}X) \in L^2(\mathbb{P}\times\lambda\times\nu)$ then $f(X) \in \mathbb{D}_{1,2}$ and $D_{t,x}f(X) = f(X + D_{t,x}X) - f(X)$.

e-)Assume that $X, Y \in \mathbb{D}^{\epsilon}_{1,2}$. Then $XY \in \mathbb{D}^{\epsilon}_{1,2}$ and $D_{t,x}(XY) = XD_{t,x}Y + YD_{t,x}X + (D_{t,x}X)(D_{t,x}Y)$.

(a) and (b) are obvious, proofs for (c),(d) and (e) can be found in (Nunno et al. [2010]).

1.2.4 In L^2 When The Lévy Process Is A Pure Jump Process

Although the Malliavin derivative for $\mathbb{D}_{1,2}$ can be defined in terms of a Wiener-Itô chaos expansion, contingent claims are in $L^2(\mathbb{P})$. We need a different expansion to extend the Malliavin derivative to random variables in $L^2(\mathbb{P})$.
In this section, we will assume that for $\epsilon > 0$, there exists an $\gamma > 0$ such that

$$\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} e^{\gamma|x|}v(dx) < \infty \tag{1.16}$$

When v has compact support, the above condition is satisfied. In this thesis, we will always assume that v has a compact support, i.e. jump sizes are bounded.

In this section we will also assume that Ω is the dual of the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$. It is also called the space of tempered distributions. We define $\mathcal{F}$ as the set of Borel subsets of Ω equipped with the *weak** topology. The Bochner-Minlos-Sazonov and Kolmogorov theorems ensure that there exists a probability measure $\mathbb{P}$ and a compensated Poisson random measure N such that $\int_0^t \int_{\mathbb{R}\setminus\{0\}} xN(ds,dx)$ is a pure jump Lévy process in $(\Omega, \mathcal{F}, \mathbb{P})$. (Gelfand et al. [1964])

Definition 1.2.11 *Let $\rho(dx) = x^2 v(dx)$ and $\{l_0 = 1, l_1, l_2, ...\}$ be the orthogonalization of $\{1, x, x^2, ...\}$ in $L^2(\rho)$. We define*

$$p_i(x) = x\frac{l_{i-1}(x)}{\| l_{i-1} \|_{L^2(\rho)}} \tag{1.17}$$

Let $\eta(i,j)$ be a bijection from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}$. Define

$$\delta_{\eta(i,j)}(t,x) = e_i(t)p_j(x) \tag{1.18}$$

Definition 1.2.12 *Let*

$$\delta_j^{\otimes k}(t_1, x_1, ..., t_k, x_k) = \delta_j(t_1, x_1)...\delta_j(t_k, x_k) \tag{1.19}$$

and $\mathcal{J}$ be the set of all finite dimensional index vectors of non-negative integers $\beta = (\beta_1, \beta_2, ..., \beta_m)$, $m = 1, 2, 3, ...$ For any $\beta = (\beta_1, ..., \beta_m) \in \mathcal{J}$ we define

$$\delta^{\hat{\otimes}\beta}(t_1, x_1, ..., t_{|\beta|}, x_{|\beta|}) = \delta^{\otimes\beta_1}\hat{\otimes}...\hat{\otimes}\delta^{\otimes\beta_m}(t_1, x_1, ..., t_{|\beta|}, x_{|\beta|}) \qquad (1.20)$$

where $|\beta| = \beta_1 + ... + \beta_m$ and $\hat{\otimes}$ is symmetrized tensor product.

After this setup, we can define the chaos expansion using a white noise analysis.

Proposition 1.2.4 *(The Wiener-Itô expansion by a white noise analysis) Let X be a $\mathcal{F}_T$ measurable square integrable random variable. X has a unique expansion*

$$X = \sum_{\beta \in \mathcal{J}} c_\beta K_\beta \qquad (1.21)$$

where $\mathcal{J}$ is the set of all finite dimensional index vectors of non-negative integers $\beta = (\beta_1, \beta_2, ..., \beta_m)$, $m = 1, 2, 3, ...$ and

$$K_\beta = I_{|\beta|}(\delta^{\hat{\otimes}\beta}) \qquad (1.22)$$

Note : $\{K_\beta\}_{\beta\in\mathcal{J}}$ is an orthogonal basis of $L^2(\mathbb{P})$.

Definition 1.2.13 *(Malliavin derivative in $L^2(\mathbb{P})$) Let $X \in L^2(\mathbb{P})$. The Malliavin derivative of X is defined as*

$$D_{t,x}X = \sum_{\beta\in\mathcal{J}} \sum_k c_\beta K_{\beta-\epsilon_k} \delta_k(t, x) \qquad (1.23)$$

where ϵ_k is a m dimensional vector of all zeros except 1 at position k.

Note : For a random variable $X \in \mathbb{D}_{1,2}$, both of these definitions of the Malliavin derivative give the same result (Nunno et al. [2010]).

1.2.5 Malliavin Calculus In $\mathbb{D}_{1,2}$

In this section we will assume that:

1-) $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i)$ $i = 1, 2, ..., d_1$ are independent complete probability spaces.

2-) W_t^i $t \in [0, T]$ are one dimensional independent Brownian motions in $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i)$.

3-) $\mathcal{F}_t^i$ is the filtration generated by W_s^i, $0 \leq s \leq t$. $\mathcal{F}_t^i$ that contains all the $\mathbb{P}^i$-zero sets in $\mathcal{F}^i$.

4-) $(\Omega^j, \mathcal{G}^j, \mathbb{P}^j)$ $j = d_1 + 1, d_1 + 2, ..., d$ are independent complete probability spaces.

5-) $N^j(t, x)$ $t \in [0, T]$ $j = d_1 + 1, d_1 + 2, ..., d$ are independent compensated Poisson random measures in $(\Omega^j, \mathcal{G}^j, \mathbb{P}^j)$.

6-) ν is a Lévy measure on $\mathbb{R} \setminus \{0\}$

7-) λ is the Lebesque measure on $[0, T]$.

8-) $\mathcal{G}_t^j$ is the filtration generated by $L(t) = \int_0^t \int_{\mathbb{R}\setminus\{0\}} x N^j(du, dx)$, $0 \leq s \leq t$. $\mathcal{G}_t^j$ that also contains all the $\mathbb{P}^j$-zero sets in $\mathcal{G}^j$.

9-) $\Omega = \Omega^1 \times \Omega^2 \times \cdots \times \Omega^d$, $\mathcal{F} = \mathcal{F}^1 \otimes \mathcal{F}^2 \times \cdots \times \mathcal{F}^d$, $\mathbb{P}(dw_1, dw_2, ..., dw_d) = \mathbb{P}^1(dw_1)\mathbb{P}^2(dw_2) \cdots \mathbb{P}^d(dw_d)$.

10-) $\mathcal{F}_t$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_t = \mathcal{F}_t^1 \otimes \mathcal{F}_t^2 \otimes \cdots \otimes \mathcal{F}_t^d$.

Definition 1.2.14 *($\mathbb{H}_\alpha$) Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$ be a d dimensional vector of positive natural numbers, $f_i, \in L^2(\lambda^{\alpha_i})$ $i \in \{1, 2, ..., d_1\}$ and $g_j \in L^2((\nu \times \lambda)^{\alpha_j})$ $j \in \{d_1 + 1, d_1 + 2, ..., d\}$ are symmetric functions (definition 1.2.1 and 1.2.7). We define $\mathbb{H}_\alpha(f_1, f_2, ..., f_{d_1}, g_{d_1+1}, g_{d_1+2}, ..., g_d)$ as*

$$\mathbb{H}_\alpha(f_1, ..., f_{d_1}, g_{d_1+1}, ..., g_d) = I_{\alpha_1}(f_1) \cdots I_{\alpha_{d_1}}(f_{d_1}) I_{\alpha_{d_1+1}}(g_{d_1+1}) \cdots I_{\alpha_d}(g_d) \tag{1.24}$$

where I_{α_i} $i \in \{1, 2, ..., d_1\}$ is an α_i dimensional iterated Itó integral with respect to W_t^i and I_{α_j} $j \in d_1 + 1, ..., d$ is an α_j dimensional iterated Itó integral with respect to $N^j(dt, dx)$.

Note : If $\alpha \neq \alpha'$ then $\mathbb{H}_\alpha(f_1, ..., g_d)$ and $\mathbb{H}_{\alpha'}(f'_1, ..., g'_{d'})$ are orthogonal.

Note : $\| \mathbb{H}_\alpha(f_1, ..., g_d) \|_{L^2(\mathbb{P})} = \alpha_1!...\alpha_d! \| f_1 \|_{L^2(\lambda^{\alpha_1})} ... \| g_d \|_{L^2((\nu\times\lambda)^{\alpha_d})}$

Proposition 1.2.5 *(The Wiener-Itô expansion by iterated integrals) Let X be a $\mathcal{F}_T$ measurable square integrable random variable. X has a unique expansion*

$$X = \sum_{\alpha\in\mathcal{J}^d} \mathbb{H}_\alpha(f_1, ..., g_d) \tag{1.25}$$

where $\alpha = (\alpha_1, ..., \alpha_d)$ be a d dimensional vector of positive natural numbers, $f_i \in L^2(\lambda^{\alpha_i})$ and $g_j \in L^2((\nu \times \lambda)^{\alpha_j})$ are symmetric functions.

Definition 1.2.15 *($\mathbb{D}_{1,2}$) Let X be a $\mathcal{F}_T$ measurable square integrable random variable. From the previous theorem, X has a unique expansion. $X \in \mathbb{D}_{1,2}$ if*

$$\| X \|^2_{\mathbb{D}_{1,2}} = \sum_\alpha (\alpha_1 + ... + \alpha_d) \|\mathbb{H}_\alpha\|^2_{L^2(\mathbb{P})} < \infty \tag{1.26}$$

Definition 1.2.16 *(Malliavin derivative in $\mathbb{D}_{1,2}$) Let $X \in \mathbb{D}_{1,2}$. The Malliavin derivative of X is defined as*

$$DX = (D_{1,t}X, ...D_{d_1,t}X, D_{d_1+1,t,x}X, ..., D_{d,t,x}X) \tag{1.27}$$

where

$$D_{i,t}X = \sum_{\alpha,\alpha_i\geq 1}^{\infty} \alpha_i I_{\alpha_1}(f_{\alpha_1}) \cdots I_{\alpha_{i-1}}(f_{\alpha_{i-1}}) I_{\alpha_i-1}(f_{\alpha_i-1}(\cdot, t)) I_{\alpha_{i+1}}(f_{\alpha_{i+1}}) \cdots I_{\alpha_d}(g_{\alpha_d}) \tag{1.28}$$

$$D_{i,t,x}X = \sum_{\alpha,\alpha_j\geq 1}^{\infty} \alpha_j I_{\alpha_1}(f_{\alpha_1}) \cdots I_{\alpha_{j-1}}(g_{\alpha_{j-1}}) I_{\alpha_j-1}(g_{\alpha_j-1}(\cdot, t, x)) I_{\alpha_{j+1}}(g_{\alpha_{j+1}}) \cdots I_{\alpha_d}(g_{\alpha_d}) \tag{1.29}$$

Note :

$$\|X\|^2_{\mathbb{D}_{1,2}} = \|D_{1,t}X\|^2_{L^2(P\times\lambda)} + ... + \|D_{d_1,t}X\|^2_{L^2(P\times\lambda)}$$

$$
\begin{aligned}
& +||D_{d_1+1,t,z}X||^2_{L^2(P\times\nu\times\lambda)} + ... + ||D_{d,t,z}X||^2_{L^2(P\times\nu\times\lambda)} \\
= & \sum_{\alpha\in\mathcal{J}^d} \alpha_1(\alpha)! \left(\prod_{i=1}^{d_1} ||f_i||^2_{L^2(\lambda^{\alpha_i})}\right)\left(\prod_{j=d_1+1}^{d} ||g_j||^2_{L^2((\nu\times\lambda)^{\alpha_j})}\right) \\
& +... \\
& \\
& +... \\
& + \sum_{\alpha\in\mathcal{J}^d} \alpha_{d_1}(\alpha)! \left(\prod_{i=1}^{d_1} ||f_i||^2_{L^2(\lambda^{\alpha_i})}\right)\left(\prod_{j=d_1+1}^{d} ||g_j||^2_{L^2((\nu\times\lambda)^{\alpha_j})}\right) \\
& + \sum_{\alpha\in\mathcal{J}^d} \alpha_{d_1+1}(\alpha)! \left(\prod_{i=1}^{d_1} ||f_i||^2_{L^2(\lambda^{\alpha_i})}\right)\left(\prod_{j=d_1+1}^{d} ||g_j||^2_{L^2((\nu\times\lambda)^{\alpha_j})}\right) \\
& +... \\
& \\
& +... \\
& + \sum_{\alpha\in\mathcal{J}^d} \alpha_{d}(\alpha)! \left(\prod_{i=1}^{d_1} ||f_i||^2_{L^2(\lambda^{\alpha_i})}\right)\left(\prod_{j=d_1+1}^{d} ||g_j||^2_{L^2((\nu\times\lambda)^{\alpha_j})}\right) \\
= & \sum_{\alpha\in\mathcal{J}^d} \left[(\alpha)! \left(\prod_{i=1}^{d_1} ||f_i||^2_{L^2(\lambda^{\alpha_i})}\right)\left(\prod_{j=d_1+1}^{d} ||g_j||^2_{L^2((\nu\times\lambda)^{\alpha_j})}\right)\left(\sum_{i=1}^{d}\alpha_i\right)\right] \\
= & \sum_{\alpha\in\mathcal{J}^d} (\alpha_1 + \cdots \alpha_d)||\mathbb{H}_\alpha||^2_{L^2(\mathbb{P})} < \infty
\end{aligned}
$$

1.2.6 Malliavin Calculus In L^2

In this section we will assume that Ω^i $i \in \{1, 2, ..., d_1\}$ is the dual of the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$. We define $\mathcal{F}^i$ as the set of Borel subsets of Ω^i equipped with the *weak** topology. The Bochner-Minlos-Sazonov and Kolmogorov theorems ensure that there exists a probability measure $\mathbb{P}^i$ and a continous process W^i_t such that it is a Wiener process in $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i)$ (Gelfand et al. [1964]).

We will also assume that for $\epsilon > 0$, there exists an $\gamma > 0$ such that

$$\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} e^{\gamma|x|}\nu(dx) < \infty$$

and Ω^j $j = d_1 + 1, ..., d$ is another copy of the dual of the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$. We define $\mathcal{F}^j$ as the set of Borel subsets of Ω^j equipped with the *weak** topology. The Bochner-Minlos-Sazonov and Kolmogorov theorems ensure that there exists a probability measure $\mathbb{P}^j$ and a compensated Poisson random measure N^j such that $\int_0^t \int_{\mathbb{R}\setminus\{0\}} xN^j(ds, dx)$ is a pure jump Lévy process in $(\Omega^j, \mathcal{F}^j, \mathbb{P}^j)$. We define the product space:

$$\Omega = \Omega^1 \times \Omega^2 \times \cdots \times \Omega^d$$

$$\mathcal{F} = \mathcal{F}^1 \otimes \mathcal{F}^2 \otimes \cdots \otimes \mathcal{F}^d$$

$$\mathbb{P}(dw_1, dw_2, \cdots, dw_d) = \mathbb{P}^1(dw_1)\mathbb{P}^2(dw_2) \cdots \mathbb{P}^d(dw_d)$$

Let $\mathcal{F}_t$ be the filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ defined as:

$$\mathcal{F}_t = \mathcal{F}_t^1 \otimes \mathcal{F}_t^2 \otimes \cdots \otimes \mathcal{F}_t^d$$

A square integrable random variable has the following expansion:

Proposition 1.2.6 *(The Wiener-Itô expansion) Let X be a $\mathcal{F}_T$ measurable square integrable random variable. X has a unique expansion*

$$X = \sum_{\alpha \in \mathcal{J}^d} c_\alpha H_{\alpha_1} \cdots H_{\alpha_{d_1}} K_{\alpha_{d_1+1}} \cdots K_{\alpha_d} \tag{1.30}$$

Note : $\{H_{\alpha_1} \cdots H_{\alpha_{d_1}} K_{\alpha_{d_1+1}} \cdots K_{\alpha_d}\}_{\alpha \in \mathcal{J}^d}$ is an orthogonal basis of $L^2(\mathbb{P})$.

Malliavin derivative of a square integrable random variable is defined in terms of its Wiener-Itô expansion.

Definition 1.2.17 *(Malliavin derivative in L^2) Let $X \in L^2(\mathbb{P})$. The Malliavin derivative of X is defined as*

$$DX = (D_{1,t}X, D_{2,t}X, \cdots, D_{d_1,t}X, D_{d_1+1,t,x}X, \cdots, D_{d,t,x}X) \tag{1.31}$$

where

$$D_{i,t}X = \sum_{\alpha \in \mathcal{J}^d} \left(c_\alpha H_{\alpha_1} \cdots H_{\alpha_{i-1}} H_{\alpha_{i+1}} \cdots H_{\alpha_{d_1}} K_{\alpha_{d_1+1}} \cdots K_{\alpha_d} \sum_{(\alpha_i)_k \geq 1} (\alpha_i)_k \, H_{\alpha_i - \epsilon_k} \;\; e_k(t) \right) \quad (1.32)$$

$$D_{j,t,x}X = \sum_{\alpha \in \mathcal{J}^d} \left(c_\alpha H_{\alpha_1} \cdots H_{\alpha_{d_1}} K_{\alpha_{d_1+1}} \cdots K_{\alpha_{j-1}} K_{\alpha_{j+1}} \cdots K_{\alpha_d} \sum_{(\alpha_j)_k \geq 1} (\alpha_j)_k K_{\alpha_j - \epsilon_k} \delta_k(t,x) \right) \quad (1.33)$$

Note : For a random variable $X \in \mathbb{D}_{1,2}$, both definitions of Malliavin derivative give the same result (Nunno et al. [2010]).

1.3 Power Jump Assets

1.3.1 Model And Completeness

Corcura et al. [2005] used trading in power jump assets as an extension of the stock and money market account to hedge any contingent claim. We will define what is meant by power jump assets. In their model they assume that the stock price follows a geometric Lévy process driven by the SDE

$$dS_t = bS_t dt + S_t dZ_t \quad (1.34)$$

$$Z_t = \alpha t + cW_t + \int_0^t \int_{|x|<1} x \, (N(ds, dx) - \nu(dx) ds) + \int_0^t \int_{|x| \geq 1} x \, N(ds, dx) \quad (1.35)$$

where Z_t is a Lévy process with parameters $(\alpha, c^2, \nu(dx))$ under the probability measure $\mathbb{P}$ and N is a Poisson random measure under $\mathbb{P}$.
They assume that all moments of Z_t exist. In the first step, they define a compensated i–th–power jump process $Y_t^{(i)}$.

Definition 1.3.1 *(Compensated i–th–Power Jump Process) Assume that Z_t is a Lévy process under $\mathbb{P}$ with Lévy measure ν and $\int_{\mathbb{R}\backslash\{0\}} x^i \nu(dx) < \infty$ for all $i \in \{1,2,3,...\}$. Define the compensated i–th–power jump process $Y_t^{(i)}$ for $i \in \{2,3,4,...\}$ as*

$$Y_t^{(i)} = \sum_{0\leq s\leq t} (\Delta Z_s)^i - \mathbb{E}^{\mathbb{P}}\left[\sum_{0\leq s\leq t} (\Delta Z_s)^i\right] = \sum_{0\leq s\leq t} (\Delta Z_s)^i - t\int_{\mathbb{R}\backslash\{0\}} x^i \nu(dx) \tag{1.36}$$

where $\Delta Z_s = Z_s - Z_{s-}$ and for $i = 1$

$$Y_t^{(1)} = Z_t - t\mathbb{E}^{\mathbb{P}}[Z_1] \tag{1.37}$$

In the literature, $Y_t^{(i)}$ is also called a Teugels martingale of order i (Nualart et al. [2000]). From the definition, it is obvious that $Y_t^{(i)}$ is also a Lévy process.
Authors enlarge the market by assuming trading in power jump assets $H^{(i)}(t)$ $i \geq 2$ where

$$H_t^{(i)} = e^{rt} Y_t^{(i)} \tag{1.38}$$

The authors assume that there exists an equivalent martingale measure $\mathbb{Q}$ such that the discounted stock and discounted power jump assets are martingales under $\mathbb{Q}$. In this section, we assume that:

1-) $(\Omega, \mathcal{F}, \mathbb{Q})$ is a complete probability space and $\mathbb{Q}$ is an equivalent martingale measure.
2-) Z_t is a Lévy process under both $\mathbb{P}$ and $\mathbb{Q}$.
3-) $\mathcal{F}_t$, $t \in [0,T]$ is a filtration generated by Z_t and augmented by null sets in $\mathcal{F}$. It is right continuous (Medvegyev [2007]).
4-) Risk free rate r is deterministic and constant.

Under the equivalent martingale measure, the discounted $H_t^{(i)}$ process which is equal to $Y_t^{(i)}$ is a martingale. The discounted value of a self-financing trading strategy is equal to the sum of discounted $H_t^{(i)}$. Let us assume that $r = 0$, V_t is the value of our discounted self-financing trading strategy and $\phi_t^{(i)}$ is the number of units of $H_t^{(i)}$ at time t in our portfolio.

$$V_t = \sum_{i=1}^{\infty} \int_0^t \phi_s^{(i)} dY_s^{(i)} \tag{1.39}$$

Unfortunately, $Y_t^{(i)}$ are not strongly orthogonal martingales [1]. Therefore, V_t does not converge in general. In their initial work, they extended the market by orthonormalized *i*–th–power jump process instead of power jump assets.

Definition 1.3.2 *(Orthonormalized i–th–Power Jump Process with respect to a Lévy measure v) Let* $\rho(dx) = x^2 v(dx) + c\delta_0(dx)$ *and* $\{l_1 = 1, l_2, l_3, l_4, ...\}$ *be the orthogonalization of* $\{1, x, x^2, x^3, ...\}$ *in* $L^2(\rho)$*. Then*

$$l_i(x) = l_{i,1} + l_{i,2}x + l_{i,3}x^2 + \cdots + l_{i,i}x^{i-1} \tag{1.40}$$

Let

$$T_t^{(i)} = l_{i,1}Y_t^{(1)} + l_{i,2}Y_t^{(2)} + l_{i,3}Y_t^{(3)} + \cdots + l_{i,i}Y_t^{(i)} \tag{1.41}$$

$e^{rt}T^{(i)}(t)$ *is called orthonormalized i–th–power jump process.*

But the problem remaining is which Lévy measure should be used in the orthogonalization of the power jump assets. There are two Lévy measures, one under $\mathbb{P}$ and another under $\mathbb{Q}$. In their final work, they extended the market by including trading in power jump assets and they used the martingale measure $\mathbb{Q}$.

Definition 1.3.3 *(Extended Market) Extended market consists of* S_t*, money market account with constant rate r and* $\{H_t^{(i)}\}_{i=2,3,...}$*.*

[1]The martingales M and N are strongly orthogonal if $[M, N] = 0$.

They assume that, there exists an equivalent martingale measure $\mathbb{Q}$ such that $Z(t)$ remains a Lévy process under $\mathbb{Q}$ with Lévy measure ν', and the discounted stock price and the discounted power jump assets are martingales under $\mathbb{Q}$. They use the orthonormalized i–th–power jump process $T^{(i)}(t)$ with respect to ν'. The orthonormalized i–th–power jump processes with respect to ν' are strongly orthogonal martingales under $\mathbb{Q}$.

In Malliavin calculus, every square integrable random variable is decomposed into orthogonal components by a chaos expansion. We have a similar decomposition in terms of iterated stochastic integrals with respect to orthonormalized power jump processes.

Proposition 1.3.1 *(Chaos expansion) Let X be a square integrable random variable. X can be represented as follows:*

$$X = \mathbb{E}^{\mathbb{Q}}[X] + \sum_{\alpha \in \mathcal{J}} \int_0^T \int_0^{t_m-} \int_0^{t_{m-1}-} \cdots \int_0^{t_2-} f_\alpha(t_1, t_2, \cdots, t_m) dT_{t_1}^{(1)} dT_{t_2}^{(2)} \cdots dT_{t_m}^{(m)} \quad (1.42)$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ *is a vector of non–negative integers of size* $m = 1, 2, ...$

In the classical Black-Scholes option pricing theory, market completeness is based on using a martingale representation theorem. In the Black–Scholes setting, a market is complete if any given square integrable contingent claim can be perfectly hedged by a self–financing trading strategy of the assets in the market and a money market account. Existence of such a trading strategy is based on the martingale representation theorem.

As in the Black–Scholes setting, completeness in our setting is also based on a martingale representation theorem.

Definition 1.3.4 *(Complete market) Let* $\mathbb{Q}$ *be an equivalent martingale measure. A* *square integrable contingent claim X can be hedged perfectly if there exists a self-financing trading strategy that replicates X.*

A market is called complete if any square integrable contingent claim X can be perfectly hedged.

A market is called complete in the limit if a square integrable contingent claim X can be either perfectly hedged or there exists a sequence of square integrable contingent claims X_n such that X_n can be hedged perfectly and $X_n \to X$ in $L^2(\mathbb{Q})$.

In this regard, Nualart et al. [2000] developed a predictable representation property for square integrable random variables.

Proposition 1.3.2 *(Predictable Representation Property) Let X be a square integrable random variable. X can be represented in terms of $\{T_t^{(i)}\}_{i=1,2,3,...}$ as follows:*

$$\mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[X] + \sum_{i=1}^{\infty} \int_0^t h_s^{(i)} dT_s^{(i)} \tag{1.43}$$

where $h_s^{(i)}$ is predictable and

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \mid h_s \mid^2 ds\right] < \infty$$

Corcuera et al. [2005] showed that extended market is complete in the limit.

Proposition 1.3.3 *(Completeness) Any square integrable contingent claim X in the extended market can be obtained (either perfectly or in the limit) via a self-financing trading strategy by power jump assets and a money market account.*

In their initial work, they claimed that every contingent claim can be perfectly hedged by orthonormalized power jump assets which is not true. Such a market is not possible since orthonormalized power jump assets are defined in terms of an equivalent martingale measure $\mathbb{Q}$. We first need to define an equivalent martingale measure $\mathbb{Q}$ and then we can find orthonormalized power jump assets with respect to the Lévy measure ν' (ν' is the Lévy measure under $\mathbb{Q}$). In their final work, they show that a contingent claim can be hedged either perfectly or in the limit.

The idea behind the proof of this proposition is the martingale representation property of the model. Let X be a contingent claim measurable with respect to $\mathcal{F}_T$. Let M_t be the following martingale

$$M_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-rt}X \mid \mathcal{F}_t\right]$$

From the martingale representation property of the model, there exists a sequence of portfolios

$$M_t^n = M_0 + \sum_{i=1}^{n} \int_0^t h_s^{(i)} dT_s^{(i)}$$

such that

$$\mathbb{E}^{\mathbb{Q}}\left[(M_t^n - M_t)^2\right] \to 0$$

Since $T_t^{(i)}$ is a linear combination of $Y_t^{(j)}$ $j = 1, 2, ..., i$, M_t^n is a sum of stochastic integrals with respect to $Y_t^{(j)}$. Therefore, X can be approximately hedged by finite number of power jump assets.

The extended market is complete in the limit and every contingent claim can be hedged approximately by a self-financing strategy using the stock, a money market account and the power jump assets. On the other hand, such a market is not useful in practice. In OTC markets, only variance swaps are traded and higher order power jump assets are not traded.

1.3.2 Malliavin Derivative Of Power Jump Assets

In this section, we will explain the relationship between the Malliavin derivative of power jump assets and hedging. In the next chapters, the Clark–Ocone theorem will be discussed in details. We assume that Z_t is a Lévy process under both $\mathbb{P}$ and $\mathbb{Q}$, and its jump sizes are bounded. For $i \geq 2$, $Y_t^{(i)}$ can be written as an integral with respect to $N(dt, dx)$, a compensated Poisson random measure under $\mathbb{P}$.

$$Y_t^{(i)} = \int_0^t \int_{\mathbb{R}\setminus\{0\}} x^i N(dx, ds) \quad \text{i=2,3,4,...} \tag{1.44}$$

and

$$D_t Y_t^{(i)} = 0 \tag{1.45}$$

$$D_{t,x} Y_t^{(i)} = x^i \tag{1.46}$$

For $i = 1$

$$D_t Y_t^{(i)} = c \tag{1.47}$$

$$D_{t,x} Y_t^{(i)} = x \tag{1.48}$$

Under certain assumptions (in Girsanov's theorem, assume that Θ_t and $Y(t, x)$ are deterministic and Y is a function of x only), Z_t is a Lévy process under $\mathbb{Q}$, and every square integrable random variable X can be written in the following form (Clark-Ocone theorem) (Nunnoet al. [2010], Lokka [2004]):

$$X = \mathbb{E}^{\mathbb{Q}}[X] + \int_0^T \mathbb{E}^{\mathbb{Q}} \left[D_t X \mid \mathcal{F}_t\right] dW_t' + \int_0^T \int_{\mathbb{R}\setminus\{0\}} \mathbb{E}^{\mathbb{Q}} \left[D_{t,x} X \mid \mathcal{F}_t\right] N'(dx, dt) \tag{1.49}$$

where W_t' is a Brownian motion and $N'(dx, dt)$ is a compensated Poisson measure under $\mathbb{Q}$.

Hedging has two parts. One part is the approximation of $\mathbb{E}^{\mathbb{Q}} \left[D_{t,x} X \mid \mathcal{F}_t\right]$ by a linear combination of $\mathbb{E}^{\mathbb{Q}} \left[D_{t,x} Y_t^{(i)} \mid \mathcal{F}_t\right] = x^i$. In other words, it is the approximation of $\mathbb{E}^{\mathbb{Q}} \left[D_{t,x} X \mid \mathcal{F}_t\right]$ by polynomials. The second part is the approximation of

$\mathbb{E}^{\mathbb{Q}}[D_t X \mid \mathcal{F}_t]$ by the same linear combination of power jump assets at the same time.

Let us assume that $c = 0$, i.e. the Lévy process Z_t is a pure jump process. In this case, Clark–Ocone theorem gives us following representation:

$$X = \mathbb{E}^{\mathbb{Q}}[X] + \int_0^T \int_{\mathbb{R}\setminus\{0\}} \mathbb{E}^{\mathbb{Q}}[D_{t,x} X \mid \mathcal{F}_t] \, N'(dx, ds) \tag{1.50}$$

and the hedging problem is reduced to a representation/approximation of $\mathbb{E}^{\mathbb{Q}}[D_{t,x} X \mid \mathcal{F}_t]$ by a polynomial of x. For some contingent claims, it can be exactly represented by a polynomial. In general, any L^2 function can be approximated by a polynomial (Stein et al. [2003]). Intuitively, when the Lévy process is a pure jump process, the extended market is complete in the limit. In other words, a dense subset of square integrable contingent claims can be hedged perfectly. The main result of this section the following result:

Proposition 1.3.4 *Assume that, we have a market with infinitely many assets, jump sizes are bounded, and there exists an equivalent martingale measure $\mathbb{Q}$ such that all assets are driven by a Lévy process Z'_t under $\mathbb{Q}$. For every $i = 1, 2, ...$ there exists a portfolio that perfectly replicates the power jump asset $H_t^{(i)}$. Let X be a contingent claim in $L(\mathbb{Q})^2)$ and measurable with respect to σ-algebra generated by Z'_t $t \in [0, T]$. Then, for a given $\epsilon > 0$, there exists a self-financing trading strategy that replicates a contingent claim X' such that the hedging error*

$$\mathbb{E}^{\mathbb{Q}}\left[\left(e^{\int_0^T -r_t dt} X - e^{\int_0^T -r_t dt} X'\right)^2\right]$$

is less than ϵ.

Proof: Let X be a contingent claim. For a given $\epsilon > 0$, there exists a portfolio of

n power jump assets such that

$$\mathbb{E}^{\mathbb{Q}}\left[\left(e^{\int_0^T -r_t dt}X - \sum_{i=1}^{n}\int_0^T \varphi_t^{(i)} dY_t^{(i)}\right)^2\right] < \epsilon$$

If $h^{(i)}(t,\alpha)$ replicates $H_t^{(i)}$ (α is the index for assets in the market) then $\varphi(t,\alpha) = \sum_{i=1}^{n}\varphi_t^{(i)}h^{(i)}(t,\alpha)$ approximately hedges X where $\varphi(t,\alpha)$ is the number of assets with index α at time t in the self-financing trading strategy. □

1.4 Hedging in a Lévy Market With a Finite Number of Assets

In this section, we assume that:

1-) $(\Omega, \mathcal{F}, \mathbb{Q})$ is a complete probability space and $\mathbb{Q}$ is an equivalent martingale measure.

2-) $\mathcal{F}_t$, $t \in [0,T]$ is a filtration generated by Z_t and augmented by the null sets in $\mathcal{F}$. It is right continuous (Medvegyev [2007]).

3-) ν' is a Lévy measure on $\mathbb{R} \setminus \{0\}$.

4-) Risk free rate r is deterministic and constant.

We assume that we have a market with $K < \infty$ number of assets $S^{(i)}(t)$ $i = 1, 2, ..., K$. Each asset price is driven by an SDE and its discounted asset price is a martingale under $\mathbb{Q}$.

$$d\left(e^{-rt}S^{(i)}(t)\right) = S^{(i)}(t-)\sigma^{(i)}(t)dW'_t + S^{(i)}(t-)\int_{\mathbb{R}\setminus\{0\}}\gamma^{(i)}(t,x)N'(dt,dx) \tag{1.51}$$

where W'_t is a Brownian motion and $N'(dt,dx)$ is a compensated Poisson random measure under $\mathbb{Q}$, $\sigma^{(i)}(t)$ and $\gamma^{(i)}(t,x)$ are predictable processes satisfying

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \left(\sigma^{(i)}(t)\right)^2 dt + \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(\gamma^{(i)}(t,x)\right)^2 \nu'(dx)dt\right] < \infty \text{ for } i\text{=1,2,...,K} \tag{1.52}$$

Let V_t be value of a self-financing trading strategy. Without loss of generality, we can assume that $r = 0$.

$$\begin{aligned} dV_t &= \sum_{i=1}^{K} \phi^{(i)}(t) dS^{(i)}(t) \\ &= \int_0^T \Big(\sum_{i=1}^{K} \phi^{(i)}(t) S^{(i)}(t-) \sigma^{(i)}(t) \Big) dW'_t \\ &\quad + \int_0^T \int_{\mathbb{R}\setminus\{0\}} \Big(\sum_{i=1}^{K} \phi^{(i)}(t) S^{(i)}(t-) \gamma^{(i)}(t,x) \Big) N'(dt,dx) \end{aligned}$$

where $\phi^{(i)}(t)$ is the number of i–th asset in the portfolio at time t, it is a predictable process and satisfies the following condition

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T & \Big(\sum_{i=1}^{K} \phi^{(i)}(t) S^{(i)}(t-) \sigma^{(i)}(t) \Big)^2 dt \\ & + \int_0^T \int_{\mathbb{R}\setminus\{0\}} \Big(\sum_{i=1}^{K} \phi^{(i)}(t) S^{(i)}(t-) \gamma^{(i)}(t,x) \Big)^2 \nu'(dx)dt \Big] < \infty \end{aligned}$$

The Clark–Ocone theorem gives a representation for every square integrable payoff X:

$$X = \mathbb{E}^{\mathbb{Q}}[X] + \int_0^T \alpha(t) dW'_t + \int_0^T \int_{\mathbb{R}\setminus\{0\}} \beta(t,x) N'(dt,dx) \tag{1.53}$$

X can be perfectly hedged if and only if

$$\begin{aligned} V_0 &= \mathbb{E}^{\mathbb{Q}}[X] \\ \sum_{i=1}^{K} \phi^{(i)}(t) S^{(i)}(t-) \sigma^{(i)}(t) &= \alpha(t) \\ \sum_{i=1}^{K} \phi^{(i)}(t) S^{(i)}(t-) \gamma^{(i)}(t,x) &= \beta(t,x) \end{aligned}$$

Note that, the second equation is a functional equation and it has a smaller solution set. In general, the first equation is very easy to solve without solving the second equation. The second equation requires that $\beta(t,\cdot)$ is a linear combination of $\gamma^{(i)}(t,\cdot)$ for a fixed t.

We conclude that a very small subset of square integrable payoffs can be hedged if the market has a finite number of assets.

1.4.1 Minimal Variance Hedging Strategy

When a square integrable payoff cannot be hedged perfectly, the next question is the minimum variance hedging strategy. Benth et al. [2003] developed a solution by using the Malliavin derivative.

We will assume that we have a market with $K < \infty$ number of assets $S^{(i)}(t)$ $i = 1, 2, ..., K$. Discounted asset prices are driven by stochastic differential equations

$$d\Big(e^{-rt}S^{(i)}(t)\Big) = \sum_{j=1}^{K} \sigma_{i,j}(t)dW'_j(t) + \sum_{j=1}^{K} \int_{\mathbb{R}\setminus\{0\}} \gamma_{i,j}(t,x)N'_j(dt,dx) \tag{1.54}$$

where W'_j is a Brownian motion and N'_j is a compensated Poisson random measure under $\mathbb{Q}$, ν'_j is the corresponding Lévy measure, $\sigma_{i,j}(t)$ and $\gamma_{i,j}(t,x)$ are predictable processes. Jump sizes are bounded and the probability space is complete. The filtration is generated by $W'_j(t)$ and $\int_{\mathbb{R}\setminus\{0\}} xN'_j(dt,dx)$ $j = 1, 2, ..., K$ and satisfies usual conditions. We define the set of admissible trading strategies as:

Definition 1.4.1 *A predictable adapted process $\phi(t) = (\phi_1(t), \phi_2(t), \cdots, \phi_K(t))$ $t \in [0,T]$ is called an admissible trading strategy if*

$$\mathbb{E}^{\mathbb{Q}}\left[\sum_{j=1}^{K}\int_0^T \phi_j(t)^2\Big(\sum_{i=1}^{K}\sigma_{i,j}^2(t) + \int_{\mathbb{R}\setminus\{0\}}\gamma_{i,j}^2(t,x)\nu'_j(dx)\Big)dt\right] < \infty$$

The set of all $\mathcal{G}$-admissible trading strategies is denoted by $\mathcal{A}_{\mathcal{G}}$.

The minimal variance portfoilo is chosen as follows:

Proposition 1.4.1 *(Benth et al. [2003]) Let $X \in L^2(\mathbb{Q})$. The minimal variance portfolio $\phi^*(t) \in \mathcal{A}_{\mathcal{G}}$ that minimizes*

$$\mathbb{E}^{\mathbb{Q}}\left[\Big(X - \mathbb{E}^{\mathbb{Q}}[X] - \sum_{i=1}^{K}\int_0^T \phi_i(t)d(e^{-rt}S^{(i)}(t))\Big)^2\right]$$

is given by

$$\phi^*(t) = M^{-1}(t)R(t),\, t \in [0, T]$$

if the inverse exists. $M(t)$ is a $K \times K$ matrix

$$M_{i,j}(t) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=1}^{K}\left(\sigma_{i,k}(t)\sigma_{k,j}(t) + \int_{\mathbb{R}\setminus\{0\}} \gamma_{i,k}(t,x)\gamma_{k,j}(t,x)\nu_j(dx)\right) \mid \mathcal{F}_t\right]$$

and $R(t)$ is K dimensional vector process

$$R_i(t) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=1}^{K}\left(\sigma_{i,j}(t)\mathbb{E}^{\mathbb{Q}}\left[D_{j,t}X \mid \mathcal{F}_t\right] + \int_{\mathbb{R}\setminus\{0\}} \gamma_{i,j}(t,x)\mathbb{E}^{\mathbb{Q}}\left[D_{j,t,x}X \mid \mathcal{F}_t\right]\nu_j(dx)\right) \mid \mathcal{F}_t\right]$$

1.5 Conclusion

In this chapter, we reviewed Malliavin calculus, power jump assets, and hedging in finite markets. The second fundamental theorem of pricing gives us a connection between hedging and uniqueness of an equivalent martingale measure when the market has a finite number of assets. When there are finitely many assets in the market and assets have jumps, a perfect hedging is not possible for every square integrable contingent claim and the market is not complete in the limit. In other words, even for a dense subset of square integrable contingent claims, a perfect hedging is not possible.

When there are infinitely many assets in the market and assets have jumps, a perfect hedging is not possible for every square integrable. However, under certain conditions, perfect hedging can be possible for a dense subset of square integrable contingent claims. Björk et al. [1997] developed this result for an Heat–Jarrow–Morton market (Heath et al. [1992]). Later, Jarrow et al. [1999] generalized this result for other markets with infinitely many assets. In both studies, researchers obtained similar results, i.e. they showed that perfect hedging can be possible for only a dense subset of square integrable contingent claims. Björk et al. [1997] call this property as ″approximate hedging″, whereas Jarrow et al. [1999] use the term ″quasicompleteness″. We use ″complete in the limit″ to explain that there is a dense subset of contingent claims that can be hedged perfectly. In both of the studies, the market is complete in the limit if and only if the equivalent martingale measure is unique.

We use Malliavin calculus and it gives us a very useful technique to check whether a given market is complete in the limit. The idea is based on using a polynomial approximation of the Mallivaian derivatives. If the power jump assets can be hedged perfectly then the market is complete in the limit.

CHAPTER 2

HEDGING IN HJM FRAMEWORK DRIVEN BY A LÉVY PROCESS

2.1 Introduction

Interest rate models are used to explain bond prices and to price derivative products in fixed income markets. Early interest rate models assumed that the spot rate of interest followed a mean reverting process. Existence of arbitrage and pricing derivatives in the market can be solved in the Black-Scholes Merton framework, but requires identification of interest rate risk premium.

Heath et al. [1992] introduced the HJM model in 1992, a major departure from the earlier models. The HJM framework takes a continuum of instantaneous forward rates as the variables driving the model. It considers bond prices for all the maturities simultaneously. All of these infinitely many processes are driven by a finite dimensional Brownian motion. The main result of the HJM model is the no-arbitrage condition: in this context a particular relationship between the drift and volatility of forward rates. Under this condition, the market is also complete.

Later, Björk et al. [1997] extended the HJM model to forward rates driven by semi-martingales. They obtained a similar no-arbitrage condition as specified in the HJM framework. These authors showed that the HJM model under jumps can be approximately complete at most, which means that a contingent claim can be hedged in the limit. A necessary and sufficient condition for approximate completeness is the uniqueness of the equivalent martingale measure.

Eberlein et al. [2005] showed that under certain assumptions, a Lévy term structure model has a unique equivalent martingale measure. In this chapter, we will investigate hedging in a Lévy term structure model. Our main conclusion is that when the market is driven by a Lévy process, a dense subspace of square integrable contingent claims can be hedged perfectly under an equivalent martingale measure. We will develop a hedging strategy for a variance swap by using Malliavin calculus. Our method does not use the results from Eberlein et al. [2005] and all our results are obtained by using Malliavin calculus and power jump assets.

The structure of the remainder of this chapter is as follows. First, we provide a review the HJM framework. Thereafter, we present the uniqueness condition for the equivalent martingale measure. Subsequently, we show that the HJM model is complete in the limit when forward rates are driven by a pure jump process. In the last part of this chapter, we will generalize these results for non-homogenous Lévy processes.

2.2 HJM Model

In the HJM model, we assume that the market has a finite time horizon $[0, T^*]$. For every $T \in [0, T^*]$, there is a zero-coupon bond with maturity T. Let $P(t, T)$ be the discounted price of the zero-coupon bond with maturity T. We assume that face value of the bond is 1\$, ie. $P(T, T) = 1\$$. Initially, we will assume that only bonds with maturity $T \in J$ where J is a dense subset of $I = [0, T^*]$ are traded.

In the HJM framework, the fundamental building blocks of the framework are the forward rates. We assume that

$$f(t,T) = \int_0^t \alpha(s,T)ds + \int_0^t \gamma(s,T)dL(s) \tag{2.1}$$

where $L(t)$ is a non–homogeneous Lévy process under $\mathbb{P}$, $\alpha(s,T)$ and $\gamma(s,T)$ are adapted stochastic processes respectively. For every $T \in [0,T^*]$, $f(0,T)$ is deterministic, and

$$\sup_{T\in J} f(0,T) < \infty$$

α and γ are $\mathcal{P} \otimes \mathcal{B}(I)$ measurable where $\mathcal{P}$ is the predictable σ-field [1] on $\Omega \times I$.

$$\sup_{s,T\leq T^*} (|\alpha(w,s,t)| + |\gamma(w,s,t)|) < \infty$$

Later, we will define an equivalent martingale measure $\mathbb{Q}$. A Lévy process under $\mathbb{P}$ is not a Lévy process under $\mathbb{Q}$ necessarily but it is a non–homogeneous Lévy process. Similarly, a non–homogeneous Lévy process under $\mathbb{P}$ is a non–homogeneous Lévy process under $\mathbb{Q}$. This is why, we can assume that $L(t)$ is a non–homogeneous Lévy process under $\mathbb{P}$ to obtain more general results. A non–homogeneous Lévy process $L(t)$ is an additive processes (Sato [1999]). Additive processes are generalizations of Lévy processes where the process has independent increments and the increments are not necessarily stationary. We will assume that we have a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathcal{F}_t$ $t \in [0,T^*]$ generated by bond prices in the framework which include all $\mathbb{P}$-null sets. Such a filtration is right continuous and $L(t)$ has a right-regular modification (Medvegyev [2007]). We will assume this right-regular modification in the development of results. Every additive process satisfies the Lévy-Itô decomposition (Medvegyev [2007]).

In order to define stochastic integrals driven by additive processes, the process must be a semimartingale. An additive process is not necessarily a semi-

[1] $\mathcal{P}$ is the σ-algebra of the subsets of $\Omega \times I$ generated by the adapted, continuous processes. (Medvegyev [2007])

martingale. A continuous non-random process with an unbounded variation trajectory is an additive processes but not a semimartingale. Eberlein et al. [2005] consider the subclass of additive processes which are semimartingales.

A non–homogeneous Lévy process can have different jump size distributions at different time points. Its characteristic function is defined in terms of a triple (b_t, c_t, F_t) where b_t is an adapted process, c_t is a non–negative adapted process, F_t is a Lévy measure for each t. Its characteristic function is equal to

$$\mathbb{E}^{\mathbb{P}}[e^{iuL(t)}] = \exp\left\{\int_0^t \left(iub_s - \frac{1}{2}c_s u^2\right) ds + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left(e^{iux} - 1 - iux1_{|x|\leq 1}\right) F_s(dx)ds\right\} \tag{2.2}$$

A non–homogeneous Lévy process can be decomposed into three parts:

$$L(t) = \int_0^t b_s ds + L(t)^c + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} x1_{|x|\leq 1}(\mu - \upsilon)(ds, dx) + \sum_{s\leq t} \Delta L(s) 1_{|\Delta L_s|>1} \tag{2.3}$$

and $L_t^c = c_t^{1/2} W(t)$ $(c_t \geq 0)$ is the continuous martingale part of L, $W(t)$ is a standard d-dimensional Brownian motion, μ is a random measure, $\upsilon(dt, dx) = dtF_t(dx)$ is the compensator, and

$$\int_0^{T^*} (|b_t| + c_t + \int_{\mathbb{R}^d\setminus\{0\}} (|x|^2 \wedge 1)F_t(dx))dt < \infty \tag{2.4}$$

and it is a semimartingale.

The discounted price at time $t \leq T$ of a default free zero coupon bond with maturity T is given by

$$P(t, T) = e^{-\int_0^t r(s)ds - \int_t^T f(t,s)ds} \tag{2.5}$$

where $r(s) := f(s, s)$ is the risk free spot rate.

After maturity, the bond price is assumed to be a constant, $P(t, T) = P(T, T) = 1$ for all $t \in (T, T^*]$. For $s > T$, $\alpha(w, s, T) = 0$, $\gamma(w, s, T) = 0$.

By changing the order of integrations and using a stochastic Fubini theorem (Medvegyev [2007]), we obtain

$$P(T)_0 := P(0,T) = e^{-\int_0^T f(0,s)ds}$$

$$P(T)_t := P(t,T) = P(T)_0 e^{-\int_0^T A(T)_s ds + \int_0^t \Gamma(T)_s^T dL_s} = P(T)_0 \mathcal{E}(H(T)_t)$$

where $A(T)_t = -\int_{t\wedge T}^T \alpha(t,s)ds$, $\Gamma(T)_t = -\int_{t\wedge T}^T \gamma(t,s)ds$. $\mathcal{E}(H)$ is the Doléans exponential of

$$\begin{aligned} H(T)_t &= \frac{1}{2}\int_0^t \Gamma(T)_s^T c_s \Gamma(T)_s ds + \int_0^t A(T)_s ds + \int_0^t \Gamma(T)_s^T dL_s \\ &\quad + \sum_{s\le t}\left(e^{\Gamma(T)_s^T \Delta L_s} - 1 - \Gamma(T)_s \Delta L_s\right) \\ &= \int_0^t a(T)_s ds + \int_0^t \Gamma(T)_s^T dL_s^c ds + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} (\Gamma(T)_s^T x 1_{|x|\le 1})(\mu - \nu)(ds,dx) \\ &\quad + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{\Gamma(T)_s^T x} - 1 - \Gamma(T)_s^T x 1_{|x|\le 1}\right)\mu(ds,dx) \end{aligned}$$

and $a(T)_t = A(T)_t + \Gamma(T)_t^T b_t + \frac{1}{2}\Gamma(T)_t^T c_t \Gamma(T)_t$.

One of the important concepts in stochastic calculus is the information set. An investor observes bond prices in the market and they generate a natural filtration. We will call $\mathcal{G}_t$ as the filtration generated by the prices $P(T)_t$, $T \in J$. Another filtration is generated by the original process $L(t)$. $\mathcal{F}_t$ is called the filtration generated by the non-homogeneous Lévy process $L(t)$. In general, these two filtrations are not equal and $\mathcal{G}_t \subseteq \mathcal{F}_t$.

Finally, we have two different probability spaces. Each of $\mathcal{F} = \mathcal{F}_{T^*}$ and $\mathcal{G} = \mathcal{G}_{T^*}$ defines a different σ–field and $\mathcal{G} \subseteq \mathcal{F}$. In hedging, we try to develop a self-financing trading strategy that replicates a given square integrable contingent claim. This contingent claim must be measurable with respect to the σ–field ($\mathcal{G}$ or $\mathcal{F}$). It means that hedging problem depends on the probability space. In

finance theory, we assume that a market is complete if every contingent claim measurable with respect to $\mathcal{G}$ can be hedged.
Note that, if α and γ are non-random then $\mathcal{G}_t$ is equal to a filtration generated by a non-homogeneous Lévy process $\bar{L}_t$.

Proposition 2.2.1 *(Eberlein et al. [2005]) Assume that α and γ are non-random. The filtration $\mathcal{G}_t$ is equal to the filtration generated by $\bar{L}_t = \int_0^t \Pi_s dL_s$ where Π_t is the orthogonal projection over $E_t = span\{\Gamma(T)_t : T \in J\}$.*

Change of Measure:

Before we apply Girsanov's theorem, we need to set up a framework that explains all different possible cases for equivalent martingale measures. In general, we have two different filtrations $\mathcal{G}_t$ and $\mathcal{F}_t$.
We will call $Q_{\mathcal{F}}$ as the set of equivalent probability measures on $(\Omega, \mathcal{F})$ under which the processes $P(T)_t$ are martingales for all $T \in J$ relative to $\mathcal{F}_t$.
Similarly, $Q_{\mathcal{G}}$ is the set of equivalent probability measures on $(\Omega, \mathcal{G})$ under which the processes $P(T)_t$ are martingales for all $T \in J$ relative to $\mathcal{G}_t$.
If the processes $P(T)_t$ are local martingales for all $T \in J$ relative to $\mathcal{F}_t$ under an equivalent measure, we will assume that it is a member of $Q_{\mathcal{F},loc}$.
Similarly, $Q_{\mathcal{G},loc}$ is the set of equivalent probability measures on $(\Omega, \mathcal{G})$ under which the processes $P(T)_t$ are local martingales for all $T \in J$ relative to $\mathcal{G}_t$.

The class of local martingales is too large and the interesting stochastic processes for our model are non–homogeneous Lévy process. We define $Q'_{\mathcal{F}}$ (respectively $Q'_{\mathcal{F},loc}$) as the set of equivalent probability measures on $(\Omega, \mathcal{F})$ under which the processes $P(T)_t$ are martingales (respectively local martingales) for all $T \in J$ relative to $\mathcal{F}_t$, and $L(t)$ is still a non–homogeneous Lévy process.

Our last set of equivalent probability measures is $\mathcal{Q}'_{\mathcal{G}}$ (respectively $\mathcal{Q}'_{\mathcal{G},loc}$) : The set of equivalent probability measures on $(\Omega, \mathcal{G})$ under which the processes $P(T)_t$ are martingales (respectively local martingales) for all $T \in J$ relative to $\mathcal{G}_t$, and $L(t)$ is still a non-homogeneous Lévy process.

In general, the relationships between these sets are :

$$\mathcal{Q}'_{\mathcal{F}} \subseteq \mathcal{Q}_{\mathcal{F}} \subseteq \mathcal{Q}_{\mathcal{F},loc}$$

$$\mathcal{Q}'_{\mathcal{F},loc} \subseteq \mathcal{Q}_{\mathcal{F},loc}$$

$$\mathcal{Q}'_{\mathcal{G}} \subseteq \mathcal{Q}_{\mathcal{G}} \subseteq \mathcal{Q}_{\mathcal{G},loc}$$

$$\mathcal{Q}'_{\mathcal{G},loc} \subseteq \mathcal{Q}_{\mathcal{G},loc}$$

The market is arbitrage free when $\mathcal{Q}_{\mathcal{G}}$ is not empty.

If $\mathbb{Q} \in \mathcal{Q}_{\mathcal{F}}$ is a martingale measure (with respect to $\mathcal{F}_t$), its restriction to $\mathcal{G}_t$ gives us a martingale measure with respect to $\mathcal{G}_t$.

$$\mathcal{Q}_{\mathcal{F}} \subseteq \mathcal{Q}_{\mathcal{G}}$$

In general its converse is not correct.

A change of measure is defined in terms of two process u_t and $Y(t,x)$. Let u_t be a predictable $\mathbb{R}^d$-valued stochastic process and Y be a $\mathcal{P} \otimes \mathcal{R}$ measurable $(0, \infty)$-valued function such that

$$\int_0^{T^*} u_t^T c_t u_t dt < \infty \;\; a.s.$$

$$\int_0^{T^*} dt \int_{\mathbb{R}^d \setminus \{0\}} \left(|Y(t,x) - 1| \wedge (Y(t,x) - 1)^2 \right) F_t(dx) < \infty \;\; a.s.$$

Two pairs of (u_1, Y_1) and (u_2, Y_2) are equivalent if

$$(u_1 - u_2)^T c (u_1 - u_2) = 0 \;\; m - a.e.$$

where $m(dw, dt) = \mathbb{P}(dw)dt$ is the measure defined on $\Omega \times I$ and $\bar{m}(dw, dt, dx) = \mathbb{P}(dw)dtF_t(dx)$ is the measure defined on $\Omega \times I \times \mathbb{R}^d$.

Two equivalent pairs define the same equivalent measure. Let $\mathcal{Y}_m(J)$ be the set of all equivalence classes of pairs (u, Y) such that for every $T \in J$

$$\int_0^{T^*} dt \int_{\mathbb{R}^d \setminus \{0\}} \left| Y(t,x) \left(e^{\Gamma(T)_t^T x} - 1 \right) - \Gamma(T)_t^T x 1_{|x|\leq 1} \right| F_t(dx) < \infty \ \ a.s. \tag{2.6}$$

and

$$a(T)_t + \Gamma(T)_t^T c_t u_t + \int_{\mathbb{R}^d \setminus \{0\}} \left(\left(e^{\Gamma(T)_t^T x} - 1 \right) Y(t,x) - \Gamma(T)_t^T x 1_{|x|\leq 1} \right) F_t(dx) < \infty \ \ m{-}a.e. \tag{2.7}$$

Björk [1997] showed that every equivalent measure can be defined in terms of u_t and $Y(t, x)$. The proof is an application of Girsanov's theorem.

Proposition 2.2.2 *(Björk) There is a one-to-one correspondence between the probability measures in $\mathcal{Q}_{\mathcal{F},loc}$ and the set $\mathcal{Y}_m(J)$. Moreover, the density process (Radon-Nikodym derivative) for a measure $\mathbb{Q}$ in $\mathcal{Q}_{\mathcal{F},loc}$ defined by (u, Y) is the Doléans exponential $\mathcal{E}(M)$ of the $\mathbb{P}$-local martingale*

$$M_t = \int_0^t u_s^T dL_s^c + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} (Y(s,x) - 1)(\mu - \nu)(ds, dx). \tag{2.8}$$

L_t becomes

$$L_t = \int_0^t b'_s ds + L_t'^c + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} x 1_{|x|\leq 1} (\mu - \nu')(ds, dx) + \sum_{s \leq t} \Delta L_s 1_{|\Delta L_s| > 1} \tag{2.9}$$

where $L_t'^c$ is the continuous martingale part of L under $\mathbb{Q}$, $b'_t = b_t + c_t u_t + \int_{\mathbb{R}^d \setminus \{0\}} (Y(t,x) - 1) x 1_{|x|\leq 1} F_t(dx)$ and $\nu'(w, dt, dx) = Y(w, t, x) \nu(w, dt, dx)$ is the compensator of μ with respect to $\mathbb{Q}$.

Note that, if $L(t)$ is a Lévy process under $\mathbb{P}$, it is not necessarily a Lévy process under $\mathbb{Q}$. On the other hand, if $Y(t, x)$ is a non-random function of x only, it is a Lévy process under $\mathbb{Q}$.

Eberlein et al. [2005] proved the following propositions for non–random α and γ :

Proposition 2.2.3 *(Eberlein et al. [2005]) If α and γ are deterministic then $Q'_{\mathcal{F},loc} = Q'_{\mathcal{F}}$*

This result shows that if there exists an equivalent measure $\mathbb{Q}$ and $L(t)$ is a non-homogenous Lévy process under $\mathbb{Q}$ then discounted bond prices are martingales under $\mathbb{Q}$. The following proposition show that if there exists a unique equivalent measure $\mathbb{Q}$ such that discounted bond prices are local martingales under $\mathbb{Q}$ then $L(t)$ is a non-homogenous Lévy process under $\mathbb{Q}$.

Proposition 2.2.4 *(Eberlein et al. [2005]) If α and γ are deterministic and either L has bounded jumps or its dimension d is equal to one then*
either $Q_{\mathcal{F},loc}, Q'_{\mathcal{F},loc}$ are both empty,
or $Q_{\mathcal{F},loc}, Q'_{\mathcal{F},loc}$ are both singleton,
or each of $Q_{\mathcal{F},loc}, Q'_{\mathcal{F},loc}$ has more than one element.

By using the previous two propositions, they show that

Proposition 2.2.5 *If α and γ are deterministic and either L has bounded jumps or its dimension d is equal to one then*
either $Q_{\mathcal{F},loc}, Q'_{\mathcal{F},loc}, Q'_{\mathcal{F}}$ are all empty,
or $Q_{\mathcal{F},loc}, Q'_{\mathcal{F},loc}, Q'_{\mathcal{F}}$ are all singleton,
or each of $Q_{\mathcal{F},loc}, Q'_{\mathcal{F},loc}, Q'_{\mathcal{F}}$ has more than one element.

When α and γ are non-random, the filtration $\mathcal{G}$ is equal to the filtration generated by the non-homogeneous Lévy process $\bar{L}_t$. If J is a dense subset of I, then these results are also true for $Q_{\mathcal{G},loc}, Q'_{\mathcal{G},loc}$ and $Q'_{\mathcal{G}}$.

Proposition 2.2.6 *(Eberlein et al. [2005]) If α and γ are deterministic and either L has* *bounded jumps or its dimension d is equal to one, and J is dense in I then*

either $Q_{\mathcal{G},loc}, Q'_{\mathcal{G},loc}, Q'_{\mathcal{G}}$ *are all empty,*

or $Q_{\mathcal{G},loc}, Q'_{\mathcal{G},loc}, Q'_{\mathcal{G}}$ *are both singleton,*

or each of $Q_{\mathcal{G},loc}, Q'_{\mathcal{G},loc}, Q'_{\mathcal{G}}$ *has more than one element.*

Their main result is the following proposition. The last condition is the version of HJM no-arbitrage condition for jump–diffusion processes .

Proposition 2.2.7 *(Eberlein et al. [2005]) Assume that* α *and* γ *are deterministic and either L has bounded jumps or its dimension d is equal to one, and J is dense in I.*
1-) $Q_{\mathcal{G},loc}, Q_{\mathcal{G}}, Q'_{\mathcal{G}}$ *are equal, and they are either empty, or a singleton.*
2-) For a non-empty (singleton) $Q_{\mathcal{G},loc}$*, it is necessary and sufficient that there exists a non-random pair* $(u, Y) \in \mathcal{Y}_m(J)$ *that satisfies*

$$\int_0^{T^*} u_t^T c_t u_t dt < \infty \ \ a.s. \tag{2.10}$$

$$\int_0^{T^*} dt \int_{\mathbb{R}^d \setminus \{0\}} \left(|Y(t,x) - 1| \wedge (Y(t,x) - 1)^2 \right) F_t(dx) < \infty \ \ a.s. \tag{2.11}$$

$$\int_0^{T^*} dt \int_{\mathbb{R}^d \setminus \{0\}} \left| Y(t,x) \left(e^{\Gamma(T)_t^T x} - 1 \right) - \Gamma(T)_t^T x 1_{|x| \leq 1} \right| F_t(dx) < \infty \ \ a.s. \tag{2.12}$$

$$a(T)_t + \Gamma(T)_t^T c_t u_t + \int_{\mathbb{R}^d \setminus \{0\}} \left(\left(e^{\Gamma(T)_t^T x} - 1 \right) Y(t,x) - \Gamma(T)_t^T x 1_{|x| \leq 1} \right) F_t(dx) = 0 \ \ m{-}a.e. \tag{2.13}$$

In this case, the measure $\mathbb{Q} \in Q_{\mathcal{G}}$ *can be extended (not necessarily in a unique way) to a measure in* $Q'_{\mathcal{F}}$*, and the prices are exponentials of the processes with independent increments under* $\mathbb{Q}$ *as well as* $\mathbb{P}$*.*

One the most important conclusion of these results is the uniqueness of the equivalent martingale measure. If $Q_{\mathcal{G}}$ is not empty than it is a singleton. It means that if the market is arbitrage free than there is only one equivalent martingale measure.

The second important result is the change of measure. This unique measure $\mathbb{Q}$ is obtained by a non–random pair $(u, Y) \in \mathcal{Y}_m(J)$. If $L(t)$ is a non–homogeneous Lévy process under $\mathbb{P}$ then it is also a non–homogeneous Lévy process under $\mathbb{Q}$

One final remark is about the function $Y(t, x)$. If $Y(t, x)$ is a non–random process, $L(t)$ is a Lévy process under $\mathbb{P}$, and $Y(t, x)$ is a function of x only then $L(t)$ is a Lévy process under $\mathbb{Q}$ also.

If $L(t)$ is a Lévy process under $\mathbb{P}$, it can be written as:

$$L_t = bt + L_t^c + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} x1_{|x|\leq 1}(\mu - \nu)(ds, dx) + \sum_{s\leq t} \Delta L_s 1_{|\Delta L_s|>1} \tag{2.14}$$

and $L_t^c = \sigma W(t)$ $(\sigma \geq 0)$ is the continuous martingale part of L, W_t is a standard)d-dimensional Brownian motion, μ is the random measure, $\nu(dt, dx) = dtF(dx)$ is the compensator. Under the equivalent martingale measure it becomes

$$L_t = \int_0^t b_s' ds + L_t'^c + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} x1_{|x|\leq 1}(\mu - \nu')(ds, dx) + \sum_{s\leq t} \Delta L_s 1_{|\Delta L_s|>1} \tag{2.15}$$

where $L_t'^c = \sigma W_t'$ is the continous martingale part of L under $\mathbb{Q}$, $b_t' = b + \sigma u_t + \int_{\mathbb{R}\setminus\{0\}} (Y(t, x) - 1)x1_{|x|\leq 1}F(dx)$ and $\nu'(w, dt, dx) = Y(w, t, x)dtF(dx)$ is the compensator of μ with respect to $\mathbb{Q}$.

We conclude that if J is a finite subset of $I = [0, T^*]$, i.e. there are finite number of bonds in the market, the model is arbitrage free and the equivalent martingale measure is unique when $L(t)$ is continuous and its dimension is less then the number of bonds in the market. In this case, $L(t)$ is a Brownian motion with a drift and any contingent claim can be hedged by a self-financing strategy.

If J is a finite subset of $I = [0, T^*]$, i.e. there are finite number of bonds in the market, the model is arbitrage free when $L(t)$ is a Lévy processes with jumps. In general, an equivalent martingale measure is not unique in this case.

If J is dense in I, α and γ are deterministic and either L has bounded jumps or its dimension d is equal to one, then either the HJM market is arbitrage free and the equivalent martingale measure is unique or there is no equivalent martingale measure and the market is not arbitrage free.

2.2.1 Hedging In An HJM Market

In this section, we will develop hedging strategies in terms of Malliavin calculus. Although the following results are correct for higher dimensions, we assume that $d = 1$ for simplicity.

By Girsanov's Theorem, an equivalent martingale measure $\mathbb{Q}$ is obtained in the previous part and it is unique. L_t is a semimartingale and it is the integral of a non-random process with respect to a Brownian motion W'_t and a random measure with compensator $\nu'(dt, dx)$ under the new measure $\mathbb{Q}$.

Under the martingale measure, the prices are exponentials of processes with independent increments and they are martingales. From Itó's formula for noncontinous semimartingales, $P(T)_t$ becomes

$$P(T)_t = \int_0^t P(T)_{s-}\Gamma(T)_s \sqrt{c_s} dW'_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} P(T)_{s-}\left(e^{\Gamma(T)_s x} - 1\right)(\mu - \nu')(ds, dx) \quad (2.16)$$

In the model described in the previous section, a change of measure is done by a non-random pair (u, Y). If we additionally assume that

(a) F_t is a Lévy measure but not a function of time (i.e. $F_t = F$, $\nu(dt, dx) = F(dx)dt$), and

(b) $Y(t, x)$ is not a function of time (i.e. $Y(t, x) = Y(x)$, $\nu'(dt, dx) = Y(x)F(dx)dt$) then the bond prices under $\mathbb{Q}$ become

$$P(T)_t = \int_0^t P(T)_{s-}\Gamma(T)_s \sqrt{c_s} dW'_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} P(T)_{s-} \left(e^{\Gamma(T)_s x} - 1\right) N'(ds, dx) \quad (2.17)$$

where N' is a compensated Poisson measure under $\mathbb{Q}$ and its Lévy measure is $Y(x)F(dx)dt$.

In this case $\mathcal{F}_t$, the filtration generated by L_t, is a filtration generated by a Brownian motion $W(t)$ and a Poisson measure $N'(dt, dx)$ and in general it is not equal to the filtration generated by W'_t and $N'(dt, dx)$ where W'_t is a Brownian motion and $N'(dt, dx)$ is a compensated Poisson random measure under $\mathbb{Q}$. The latter is a subset of the former one. In financial applications, payoffs are functions of the assets in the market and they are measurable with respect to $\mathcal{G}_{T^*}$ and $\mathcal{G}_t$ is a subset $\mathcal{F}_t$. In this case, the generalized Clark-Ocone theorem can be used to obtain the hedging portfolio.

On the other hand, if u_t and $Y(t, x)$ are deterministic then the Clark-Ocone theorem gives us a representation with respect to W' and N' in a computationally tractable way. Therefore, the assumptions above not only guarantee that $L(t)$ is a Lévy process under $\mathbb{Q}$, but they also imply that the most simple form of the Clark-Ocone theorem can be used.

Our next step is the definition admissible trading strategies:

Definition 2.2.1 *A stochastic process $\phi : [0, T^*] \times [0, T^*] \times \Omega \to \mathbb{R}$ is called an admissible trading strategy if*

- $\phi(t, T) = 0$ *when* $t > T$.
- ϕ *is* $\mathcal{B}([0, T^*]) \otimes \mathcal{B}[0, T^*]) \otimes \mathcal{G}_{T^*}$ *measurable.*
- $\phi(\cdot, T)$ *is a predictable and adapted process (with respect to $\mathcal{G}$) for every* $T \in [0, T^*]$.
- $\mathbb{E}\left[\int_0^{T^*} \phi(t, T)^2 \left(P(T)_{t-}^2 \Gamma(T)_t^2 c_t + \int_{\mathbb{R}\setminus\{0\}} P(T)_{t-}^2 \left(e^{\Gamma(T)_t x} - 1\right)^2 Y(x)F(dx)\right) dt\right] < \infty$ *for all* $T \in [0, T^*]$.

- $\int_0^{T*}|\int_0^{T^*}\phi(t,T)P(T)_{t-}\Gamma(T)_t\sqrt{c_t}dW'_t|dT<\infty$ *almost surely.*
- $\int_0^{T*}|\int_0^{T^*}\phi(t,T)\int_0^t\int_{\mathbb{R}\backslash\{0\}}P(T)_{t-}\left(e^{\Gamma(T)_t x}-1\right)N'(dt,dx)|dT<\infty$ *almost surely.*

The set of all $\mathcal{G}$-admissible self-financing trading strategies is denoted by $\mathcal{A}_{\mathcal{G}}$.

The portfolio at time t can be approximated by a sum of n portfolios $X_{t,1},X_{t,2},X_{t,3},\ldots,X_{t,n}$. Each portfolio $X_{t,i}$ has only one bond with maturity $t+i\Delta T$ ($\Delta T=\frac{T^*-t}{n}$), and its amount in the portfolio is $\phi(t,t+i\Delta T)\Delta T$.

Our first result is the following proposition:

Proposition 2.2.8 *Suppose F is a square integrable (under both $\mathbb{P}$ and $\mathbb{Q}$), $\mathcal{G}_{T^*}$ measurable payoff at T^*. The self-financing admissible trading strategy $\phi(t,T)\in\mathcal{A}_{\mathcal{G}}$ perfectly hedges the final payoff F if and only if*

$$\int_0^{T^*}\phi(t,T)P(T)_{t-}\Gamma(T)_t\sqrt{c_t}dT=\mathbb{E}^{\mathbb{Q}}\left[D_t\left(e^{-\int_0^{T^*}r(s,s,)ds}F\right)|\mathcal{F}_t\right] \tag{2.18}$$

$$\int_0^{T^*}\phi(t,T)P(T)_{t-}\left(e^{\Gamma(T)_t x}-1\right)dT=\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}\left(e^{-\int_0^{T^*}r(s,s,)ds}F\right)|\mathcal{F}_t\right] \tag{2.19}$$

for all $t\in[0,T^*]$. D_tF and $D_{t,x}$ denote the Malliavian derivatives.

Proof: From the Clark-Ocone theorem, the discounted payoff can be written as:

$$\begin{aligned}e^{-\int_0^{T^*}r(s,s)ds}F= &\ \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{T^*}r(s,s)ds}F\right]+\int_0^{T^*}\mathbb{E}^{\mathbb{Q}}\left[D_t\left(e^{-\int_0^{T*}r(s,s)ds}F\right)|\mathcal{F}_t\right]dW'_t\\ &+\int_0^{T^*}\int_{\mathbb{R}\backslash\{0\}}\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}\left(e^{-\int_0^{T*}r(s,s)ds}F\right)|\mathcal{F}_t\right]N'(dt,dx)\end{aligned}$$

The discounted portfolio value at maturity is equal to

$$\begin{aligned}e^{-\int_0^{T^*}r(s,s)ds}X(T^*) &= X(0)+\int_0^{T^*}\left(\int_0^{T^*}\phi(t,T)d(P(T)_t)\right)dT\\ &= X(0)+\int_0^{T^*}\left(\int_0^{T^*}\phi(t,T)P(T)_{t-}\Gamma(T)_t\sqrt{c_t}dW'_t\right)dT\\ &\quad+\int_0^{T^*}\left(\int_0^{T^*}\int_{\mathbb{R}\backslash\{0\}}\phi(t,T)P(T)_{t-}\left(e^{\Gamma(T)_t x}-1\right)N'(dt,dx)\right)dT\end{aligned}$$

Since $\int_0^{T^*}\left|\int_0^{T^*}\phi(t,T)P(T)_{t-}\Gamma(T)_t\sqrt{c_t}dW'_t\right|dT < \infty$ almost surely, the order of integrals can be changed by a version of the Fubuni theorem for real valued semi-martingale integrators

$$\int_0^{T^*}\left(\int_0^{T^*}\phi(t,T)P(T)_{t-}\Gamma(T)_t\sqrt{c_t}dW'_t\right)dT = \int_0^{T^*}\left(\int_0^{T^*}\phi(t,T)P(T)_{t-}\Gamma(T)_t\sqrt{c_t}dT\right)dW'_t$$

Similarly $\int_0^{T^*}\left|\int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}}\phi(t,T)P(T)_{t-}\left(e^{\Gamma(T)_t x}-1\right)N'(dt,dx)\right|dT < \infty$ almost surely, and the order of integrals can be changed

$$\begin{aligned}\int_0^{T^*} & \left(\int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}}\phi(t,T)P(T)_{t-}\left(e^{\Gamma(T)_t x}-1\right)N'(dt,dx)\right)dT \\ &= \int_0^{T^*}\int_0^{T^*}\left(\int_{\mathbb{R}\setminus\{0\}}\phi(t,T)P(T)_{t-}\left(e^{\Gamma(T)_t x}-1\right)dT\right)N'(dt,dx)\end{aligned}$$

The discounted portfolio value becomes

$$\begin{aligned}e^{-\int_0^{T^*}r(s,s)ds}X(T^*) &= X(0)+\int_0^{T^*}\left(\int_0^{T^*}\phi(t,T)P(T)_{t-}\Gamma(T)_t\sqrt{c_t}dT\right)dW'_t \\ &\quad +\int_0^{T^*}\int_0^{T^*}\left(\int_{\mathbb{R}\setminus\{0\}}\phi(t,T)P(T)_{t-}\left(e^{\Gamma(T)_t x}-1\right)dT\right)N'(dt,dx)\end{aligned}$$

The self-financing admissible trading strategy $\phi(t,T)\in\mathcal{A}_{\mathcal{G}}$ should satisfy the following integral equations:

$$\int_0^{T^*}\phi(t,T)P(T)_{t-}\Gamma(T)_t\sqrt{c_t}dT = \mathbb{E}^{\mathbb{Q}}\left[D_t\left(e^{-\int_0^{T^*}r(s,s,)ds}F\right)|\mathcal{F}_t\right]$$

$$\int_0^{T^*}\phi(t,T)P(T)_{t-}\left(e^{\Gamma(T)_t x}-1\right)dT = \mathbb{E}^{\mathbb{Q}}\left[D_{t,x}\left(e^{-\int_0^{T^*}r(s,s,)ds}F\right)|\mathcal{F}_t\right]$$

□

The above proposition is the first step in hedging. It gives us a necessary and sufficient condition for a perfect hedging strategy. The expression on the right hand side of the first equation is an $\mathcal{F}_t$ measurable random variable and this equation has infinitely many solutions in general. The right hand side of the second equation is a function of jump size x and it is a Fredholm integral equation of first kind. Before we study solutions of these two equations, we will first find the Malliavin derivative of a caplet. These computations will indicate the forthcoming difficulties in the application of the previous result.

2.2.2 The Malliavin Derivative Of Discounted Caplet

Our first step is to find the Wiener-Itó Chaos Expansion of the discounted value process.

Wiener-Itó Chaos Expansion of the discount process:

The discount process is defined as $e^{-\int_0^t r(s)ds}$ where $r(s)$ is the risk free spot rate.

$$r(s) = f(s,s) = f(0,s) + \int_0^s \alpha(u,s)du + \int_0^s \gamma(u,s)dL_u$$

$$e^{-\int_0^t r(s)ds} = e^{-\int_0^t f(0,s)ds - \int_0^t \int_0^s \alpha(u,s)duds - \int_0^t \int_0^s \gamma(u,s)dL_u ds} = C' e^{-\int_0^t \int_0^s \gamma(u,s)dL_u ds}$$

where $C' = e^{\int_0^t \Gamma(t)_u b'_u du + \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^{\Gamma(t)_u x} - 1 - \Gamma(t)_u x 1_{|x|\le 1})\nu'(du,dx)}$ is a constant. By changing the order of integrations and using the Fubini theorem, it becomes

$$C' e^{\int_0^t \Gamma(t)_u dL_u} = C e^{\int_0^t \Gamma(t)_u \sqrt{c_u} dW'_u} e^{\int_0^t \int_{\mathbb{R}\setminus\{0\}} \Gamma(t)_u x N'(du,dx) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^{\Gamma(t)_u x} - 1 - \Gamma(t)_u x)\nu'(du,dx)}$$

where $C = C' e^{\int_0^t \Gamma(t)_u b'_u du + \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^{\Gamma(t)_u x} - 1 - \Gamma(t)_u x 1_{|x|\le 1})\nu'(du,dx)}$ is a constant.

Wiener-Itó Chaos expansion of the continous part :

The Hermite polynomials are defined as

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}\left(e^{-\frac{1}{2}x^2}\right), \quad n = 0, 1, 2, 3, ...$$

and for every $\beta > 0$,

$$e^{ux - \frac{u^2\beta}{2}} = \sum_{n=0}^{\infty} \frac{u^n \beta^{n/2}}{n!} h_n\left(\frac{x}{\sqrt{\beta}}\right)$$

Let $g \in L^2([0,T])$. By choosing $x = \int_0^T g(s)dW'_s$, $u = 1$, and $\beta = ||g||_{L^2([0,T])}$

$$e^{\int_0^T g(s)dW'_s} = e^{\frac{1}{2}||g||^2} \sum_0^{\infty} \frac{||g||^n}{n!} h_n\left(\frac{\int_0^T g(s)dW'_s}{||g||}\right)$$

Since $\frac{g}{||g||}$ has a $L^2([0,T])$ norm equal to 1,

$$I_n(g^{\otimes n}) := n! \int_o^T \int_0^{t_n} \cdots \int_o^T \int_0^{t_2} g(t_1)\cdots g(t_n) dW'(t_1)\cdots dW'(t_n) = ||g||^n h_n\left(\frac{\int_0^T g(s)dW'_s}{||g||}\right)$$

Therefore,

$$e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u} = \sum_{n=0}^{\infty} I(f_n)$$

where $f_n = \frac{1}{n!} e^{\frac{1}{2}||\Gamma(T)\sqrt{c}||^2_{L^2([0,T])}} (\Gamma(T)\sqrt{c})^{\otimes n}$.

The $\mathbb{D}_{1,2}$ norm of the continuous part is defined as

$$\begin{aligned}
|| e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u} ||^2_{\mathbb{D}_{1,2}} &:= \sum_{n=1}^{\infty} nn! ||f_n||^2_{L^2([0,T])^n)} \\
&= \sum_{n=1}^{\infty} nn! \frac{1}{n!n!} e^{||\Gamma(T)\sqrt{c}||^2_{L^2([0,T])}} ||\Gamma(T)\sqrt{c}||^{2n}_{L^2([0,T])} \\
&= e^{||\Gamma(T)\sqrt{c}||^2_{L^2([0,T])}} ||\Gamma(T)\sqrt{c}||^2_{L^2([0,T])} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} ||\Gamma(T)\sqrt{c}||^{2(n-1)}_{L^2([0,T])} \\
&= e^{2||\Gamma(T)\sqrt{c}||^2_{L^2([0,T])}} ||\Gamma(T)\sqrt{c}||^2_{L^2([0,T])}
\end{aligned}$$

and it is finite. Therefore $e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u} \in \mathbb{D}_{1,2}$.
Let $F = \int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u$. F is Malliavin differentiable and $D_t F = \Gamma(T)_t \sqrt{c_t}$.

$$||\Gamma(T)_t \sqrt{c_t} e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}||^2_{L^2(\mathbb{Q}\times\lambda)} = ||\Gamma(T)\sqrt{c}||^2_{L^2([0,T])} \mathbb{E}^{\mathbb{Q}}\left[e^{2\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}\right] < \infty$$

where λ is the Lebesque measure, and $\mathbb{E}^{\mathbb{Q}}\left[e^{2\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}\right]$ is finite where $\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u$ is a normal random variable with variance $\int_0^T \Gamma(T)^2_u c_u du$. By the chain rule, the Malliavin derivative of the continuous part is equal to

$$D_t\left(e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}\right) = \Gamma(T)_t \sqrt{c_t} e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}$$

Wiener-Itó Chaos expansion of the non-continous part :

The non-continous part $Z_t = e^{\int_0^t \int_{\mathbb{R}\backslash\{0\}} \Gamma(T)_u x N'(du,dx) - \int_0^t \int_{\mathbb{R}\backslash\{0\}} \left(e^{\Gamma(T)_u x} - 1 - \Gamma(T)_u x\right) \nu'(du,dx)}$ is the solution of the following SDE

$$dZ_t = Z_{t-} \int_{\mathbb{R}\backslash\{0\}} \left(e^{\Gamma(T)_t x} - 1\right) N'(dt, dx)$$

Its Wiener-Itó Chaos expansion is given by

$$Z_T = \sum_0^\infty I_n(f_n)$$

where $f_n(u_1, x_1, ..., u_n, x_n) = \frac{1}{n!} \prod_{i=1}^n \left(e^{\Gamma(T)_{u_i} x_i} - 1\right) = \frac{1}{n!} \left(e^{\Gamma(T)_u x} - 1\right)^{\otimes n} (u_1, x_1, ..., u_n, x_n)$, and

$$I_n(f_n) := n! \int_0^T \int_{\mathbb{R}\backslash\{0\}} \int_0^{t_n-} \int_{\mathbb{R}\backslash\{0\}} \cdots \int_0^{t_2-} \int_{\mathbb{R}\backslash\{0\}} f_n(t_1, x_1, ..., t_n, x_n) N'(t_1, x_1) ... N'(t_n, x_n)$$

The $\mathbb{D}_{1,2}$ norm of the non-continous part is defined as

$$\begin{aligned} \| Z_T \|^2_{\mathbb{D}_{1,2}} &:= \sum_{n=1}^\infty n n! \|f_n\|^2_{L^2((\lambda\times\nu')^n)} \\ &= \sum_{n=1}^\infty n n! \frac{1}{n! n!} \|e^{\Gamma(T)_u x} - 1\|^{2n}_{L^2(\lambda\times\nu')} < \infty \end{aligned}$$

and it is finite. Therefore $Z_T \in \mathbb{D}_{1,2}$. Its Malliavin derivative is equal to

$$D_{t,x} Z_T = \sum_0^\infty n I_{n-1} f_n(., t, x) = Z_T \left(e^{\Gamma(T)_t x} - 1\right)$$

Hedging a caplet :

A caplet is a derivative product with payoff $(r(T) - K)^+$. Let $F = e^{-\int_0^T r(s)ds}(r(T) - K)^+$ be the discounted payoff.

$$\begin{aligned} F = \; & C \cdot e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u} \cdot e^{\int_0^T \int_{\mathbb{R}\backslash\{0\}} \Gamma(T)_u x N'(du,dx) - \int_0^T \int_{\mathbb{R}\backslash\{0\}} (e^{\Gamma(T)_u x} - 1 - \Gamma(T)_u x) \nu'(du,dx)} \\ & \cdot \left(\int_0^T \gamma(u,T) \sqrt{c_u} dW'_u + \int_0^T \int_{\mathbb{R}\backslash\{0\}} \gamma(u,T) x N'(du,dx) - K'\right)^+ \end{aligned}$$

where C, and $K' - K - f(0,T) - \int_0^T \alpha(u,T) du - \int_0^T \gamma(u,T) b'_u du - \int_0^T \int_{\mathbb{R}\backslash\{0\}} \gamma(u,T) x 1_{x>1} \nu'(du,dx)$ are constants.

Malliavin derivative of F when $\mathcal{F}_t$ is generated by only the Brownian Motion:

When $\mathcal{F}_t$ is generated by only the Brownian Motion, the discounted payoff becomes

$$\begin{aligned} F &= e^{-\int_0^T r(s)ds}(r(T)-K)^+ \\ &= Ce^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}\left(\int_0^T \gamma(u,T)\sqrt{c_u}dW'_u - K'\right)^+ \\ &= C\left(e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}\int_0^T \gamma(u,T)\sqrt{c_u}dW'_u - K'e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}\right)^+ \\ &= C(G)^+ \end{aligned}$$

where C, and K' are constants, and

$$G = e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}\int_0^T \gamma(u,T)\sqrt{c_u}dW'_u - K'e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}$$

Lemma 2.2.1 $G \in \mathbb{D}_{1,2}$, *and*

$$D_t G = e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}\left(\gamma(t,T)\sqrt{c_t} + \left(\int_0^T \gamma(u,T)\sqrt{c_u}dW'_u - K'\right)\left(\Gamma(T)_t\sqrt{c_t}\right)\right)$$

Proof: From the previous part, $e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u} \in \mathbb{D}_{1,2}$, and

$$D_t e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u} = \Gamma(T)_t\sqrt{c_t}e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}$$

Let $\varphi(x_1,x_2) = x_1 e^{x_2}$, $X_1 = \int_0^T \gamma(u,T)\sqrt{c_u}dW'_u$, $X_2 = \int_0^T \Gamma(T)_u\sqrt{c_u}dW'_u$, and $Y = \left(\int_0^T \gamma(u,T)\sqrt{c_u}dW'_u\right)e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}$.

$$Y = \varphi(X_1, X_2)$$

Since $D_t X_1 = \gamma(t,T)\sqrt{c_t} \in L^2(\mathbb{Q})$, and $D_t X_2 = \Gamma(T)_t\sqrt{c_t} \in L^2(\mathbb{Q})$,

$$\frac{\partial\varphi}{\partial x_1}(X_1,X_2)D_t X_1 = e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}\left(\gamma(t,T)\sqrt{c_t}\right) \in L^2(\mathbb{Q}\times\lambda)$$

$$\frac{\partial\varphi}{\partial x_2}(X_1,X_2)D_t X_2 = e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u}\left(\int_0^T \gamma(u,T)\sqrt{c_u}dW'_u\right)\left(\Gamma(T)_t\sqrt{c_t}\right) \in L^2(\mathbb{Q}\times\lambda)$$

($\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u$ and $\int_0^T \gamma(u,T)\sqrt{c_u} dW'_u$ are normal random variables). By the chain rule,

$$D_t Y = \frac{\partial \varphi}{\partial x_1}(X_1, X_2) D_t X_1 + \frac{\partial \varphi}{\partial x_2}(X_1, X_2) D_t X_2$$

$$D_t G = e^{\int_0^T \Gamma(T)_u \sqrt{c_u} dW'_u} \left(\gamma(t,T)\sqrt{c_t} + \left(\int_0^T \gamma(u,T)\sqrt{c_u} dW'_u - K' \right) \left(\Gamma(T)_t \sqrt{c_t} \right) \right)$$

Also, each term in $D_t G$ is in $L^2(\mathbb{Q} \times \lambda)$, so $G \in \mathbb{D}_{1,2}$. □

Now, we can show that F is in the $\mathbb{D}_{1,2}$ subspace.

Lemma 2.2.2 *$F \in \mathbb{D}_{1,2}$, and $D_t F = 1_{[0,\infty]}(G) D_t G$.*

Proof: From the previous part, $D_t G \in \mathbb{D}_{1,2} \subset L^2(\mathbb{Q})$.

Let $f(x) = (x)^+$ and f_n be a sequence of functions such that $|f_n - f| \leq \frac{1}{n}$, and $f_n(x) = f(x)$ when $|x| \geq \frac{1}{n}$,

$$\|F - f_n(G)\|_{L^2(\mathbb{Q})} = \|f(G) - f_n(G)\|_{L^2(\mathbb{Q})} \leq \frac{1}{n} \to 0$$

$D_t f_n(G) = f'_n(G) D_t G$ and from the dominated convergence theorem

$$\|D_t f_n(G) - 1_{[0,\infty]}(G) D_t G\|_{L^2(\mathbb{Q}\times\lambda)} = \|\left(f'_n(G) - 1_{[0,\infty)}(G)\right) D_t G\|_{L^2(\mathbb{Q}\times\lambda)} \to 0$$

Therefore, $f_n(G) \to f(G) = F$ in $L^2(\mathbb{Q})$, and $D_t f_n(G) \to 1_{[0,\infty]}(G) D_t G$ in $L^2(\mathbb{Q} \times \lambda)$. Since the Malliavin derivative is a closed operator (Nunno et al. [2010]), $F \in \mathbb{D}_{1,2}$, and $D_t F = 1_{[0,\infty]}(G) D_t G$. □

The Malliavin derivative of F when $\mathcal{F}_t$ is generated by only a Poisson random measure:

When F_t is generated by only the Poisson random measure, the discounted payoff becomes

$$F = e^{-\int_0^T r(s)ds} (r(T) - K)^+$$

$$= \quad Ce^{\int_0^T \int_{\mathbb{R}\setminus\{0\}} \Gamma(T)_u x N'(du,dx) - \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(T)_u x} - 1 - \Gamma(T)_u x\right) \nu'(du,dx)} \left(\int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T) x N'(du,dx) - K' \right)^+$$
$$= \quad (H)^+$$

where C, and K' are constants, and

$$\begin{aligned} H &= Ce^{\int_0^T \int_{\mathbb{R}\setminus\{0\}} \Gamma(T)_u x N'(du,dx) - \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(T)_u x} - 1 - \Gamma(T)_u x\right) \nu'(du,dx)} \left(\int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T) x N'(du,dx) \right) \\ &\quad - CK' e^{\int_0^T \int_{\mathbb{R}\setminus\{0\}} \Gamma(T)_u x N'(du,dx) - \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(T)_u x} - 1 - \Gamma(T)_u x\right) \nu'(du,dx)}. \end{aligned}$$

Lemma 2.2.3 *Assume $r(s)$ is bounded below. $H \in \mathbb{D}_{1,2}$, and*

$$\begin{aligned} D_t H &= H\left(e^{\Gamma(T)_t x} - 1\right) \\ &\quad + Ce^{\int_0^T \int_{\mathbb{R}\setminus\{0\}} \Gamma(T)_u x N'(du,dx) - \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(T)_u x} - 1 - \Gamma(T)_u x\right) \nu'(du,dx)} \left(\gamma(t,T) x e^{\Gamma(T)_t x}\right) \end{aligned}$$

In this case, $D_{t,x}F = (H + D_{t,x}H)^+ - F$.

Proof: From the previous part, $Z_T = e^{\int_0^T \int_{\mathbb{R}\setminus\{0\}} \Gamma(T)_u x N'(du,dx) - \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(T)_u x} - 1 - \Gamma(T)_u x\right) \nu'(du,dx)}$ is in $\mathbb{D}_{1,2}$, and its Malliavin derivative is equal to

$$D_{t,x} Z_T = Z_T \left(e^{\Gamma(T)_t x} - 1\right)$$

Therefore, the second term in H is in $\mathbb{D}_{1,2}$. Since $r(s)$ is bounded below, CZ_T is bounded, $0 < CZ_T \leq M_1$. Let $V = \int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T) x N'(du,dx)$. $\mathbb{D}^{\varepsilon}_{1,2}$, the set of linear combinations of exponentials $\{e^{\int_0^T \int_{\mathbb{R}\setminus\{0\}} h(x) x N'(du,dx)} | h \in L^2([0,T])\}$, is dense in $\mathbb{D}_{1_2}$. Let V_n be a sequence $\mathbb{D}^{\varepsilon}_{1,2}$ such that $V_n \to V$ in $L^2(\mathbb{Q})$. Using the product rule ($CZ_T \in \mathbb{D}^{\varepsilon}_{1,2}$)

$$D_{t,x}(V_n(CZ_T)) = V_n D_{t,x}(CZ_T) + CZ_T D_{t,x} V_n + D_{t,x} V_n D_{t,x}(CZ_T)$$

Note that, $0 < CZ_T \leq M_1$

$$\|V_n(CZ_T) - V(CZ_T)\|_{L^2(\mathbb{Q})} \leq M_1 \|V_n - V\|_{L^2(\mathbb{Q})} \to 0$$

which shows that $V_n(CZ_T) \to V(CZ_T)$ in $L^2(\mathbb{Q})$.

$$\|CZ_T D_{t,x} V_n - CZ_t D_{t,x} V\|_{L^2(\lambda\times\nu'\times\mathbb{Q})} \le M_1 \|D_{t,x} V_n - D_{t,x} V\|_{L^2(\lambda\times\nu'\times\mathbb{Q})} \to 0$$

$$\|V_n D_{t,x}(CZ_T) - V D_{t,x}(CZ_T)\|_{L^2(\lambda\times\nu'\times\mathbb{Q})} \le M_1 \|e^{\Gamma(T)_t x} - 1\|_{L^2(\lambda\times\nu')} \|V_n - V\|_{L^2(\mathbb{Q})} \to 0$$

Since jump sizes, and $\Gamma(T)$ are bounded, there exists a real number M_2 such that $|e^{\Gamma(T)_t x} - 1| \le M_2$.

$$\|D_{t,x} V_n D_{t,x}(CZ_T) - D_{t,x} V D_{t,x}(CZ_T)\|_{L^2(\lambda\times\nu'\times\mathbb{Q})} \le M_2 \|D_{t,x} V_n - D_{t,x} V\|_{L^2(\lambda\times\nu'\times\mathbb{Q})} \to 0$$

From the closability of the Malliavin derivative, $D_{t,x}(V(CZ_T)) \in \mathbb{D}_{1,2}$, and

$$D_{t,x}(V(CZ_T)) = V D_{t,x}(CZ_T) + CZ_T D_{t,x} V + D_{t,x} V D_{t,x}(Z_T)$$

and $D_{t,x} H \in \mathbb{D}_{1,2}$. Finally using the chain rule, $D_{t,x} F = (H + D_{t,x} H)^+ - F$. □

We can relax the assumption that $r(s)$ is bounded below.

Lemma 2.2.4 *$H \in \mathbb{D}_{1,2}$, and $D_{t,x} F = (H + D_{t,x} H)^+ - F$ where*

$$\begin{aligned} D_t H &= H\left(e^{\Gamma(T)_t x} - 1\right) \\ &\quad + C e^{\int_0^T \int_{\mathbb{R}\setminus\{0\}} \Gamma(T)_u x N'(du,dx) - \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(T)_u x} - 1 - \Gamma(T)_u x\right) \nu'(du,dx)} \left(\gamma(t,T) x e^{\Gamma(T)_t x}\right) \end{aligned}$$

Proof: From the previous part, $Z_T = e^{\int_0^T \int_{\mathbb{R}\setminus\{0\}} \Gamma(T)_u x N'(du,dx) - \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(T)_u x} - 1 - \Gamma(T)_u x\right) \nu'(du,dx)}$ is in $\mathbb{D}_{1,2}$, and its Malliavin derivative is equal to

$$D_{t,x} Z_T = Z_T \left(e^{\Gamma(T)_t x} - 1\right)$$

Therefore, the second term in H is in $\mathbb{D}_{1,2}$. Using the integration by parts formula for a Skorohod integral,

$$Z_T \int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T) z N'(du,dz) = \int_0^T \int_{\mathbb{R}\setminus\{0\}} (Z_T + D_{u,z} Z_T) \gamma(u,T) z N'(\delta u, dz)$$

$$+\int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T)(D_{u,z}Z_T)\nu'(dz)du$$

$$= \int_0^T \int_{\mathbb{R}\setminus\{0\}} Z_T e^{\Gamma(T)_u z}\gamma(u,T) z N'(\delta u, dz)$$

$$+\int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T) Z_T \left(e^{\Gamma(T)_u z} - 1\right)\nu'(dz)dt$$

$$= \int_0^T \int_{\mathbb{R}\setminus\{0\}} Z_T e^{\Gamma(T)_u z}\gamma(u,T) z N'(\delta u, dz)$$

$$+Z_T \int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T)\left(e^{\Gamma(T)_u z} - 1\right)\nu'(dz)dt$$

where $\int_0^T \int_{\mathbb{R}\setminus\{0\}} g(u,z)N'(\delta u, dz)$ is Skorohod integral. The second term is in $\mathbb{D}_{1,2}$ and its Malliavin derivative is equal to

$$D_{t,x}\left(Z_T \int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T)\left(e^{\Gamma(T)_u z} - 1\right)\nu'(dz)dt\right)$$

$$= (D_{t,x}Z_T) \int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T)\left(e^{\Gamma(T)_u z} - 1\right)\nu'(dz)dt$$

$$= Z_T\left(e^{\Gamma(T)_t x} - 1\right) \int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T)\left(e^{\Gamma(T)_u z} - 1\right)\nu'(dz)dt$$

Since jumps sizes, $\Gamma(T)_t$ and $\gamma(t,T)$ are bounded,

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \int_{\mathbb{R}\setminus\{0\}} Z_T^2 e^{2\Gamma(T)_u z}\gamma(u,T)^2 z^2 \nu'(dz)du\right] < \infty$$

Also, $Z_T e^{\Gamma(T)_u z}\gamma(u,T)z \in \mathbb{D}_{1,2}$ for every $(u,z) \in [0,T]\times\mathbb{R}$ and $D_{t,x}(Z_T e^{\Gamma(T)_u z}\gamma(u,T)z)$ is Skorohod integrable with

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(\int_0^T \int_{\mathbb{R}\setminus\{0\}} D_{t,x}\left(Z_T e^{\Gamma(T)_u z}\gamma(u,T)z\right) N'(\delta u, dz)\right)^2 \nu'(dx)dt\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(\int_0^T \int_{\mathbb{R}\setminus\{0\}} Z_T\left(e^{\Gamma(T)_t x} - 1\right) e^{\Gamma(T)_u z}\gamma(u,T) z N'(\delta u, dz)\right)^2 \nu'(dx)dt\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(T)_t x} - 1\right)^2 \left(\int_0^T \int_{\mathbb{R}\setminus\{0\}} Z_T e^{\Gamma(T)_u z}\gamma(u,T) z N'(\delta u, dz)\right)^2 \nu'(dx)dt\right]$$

$$= \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(T)_t x} - 1\right)^2 \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \int_{\mathbb{R}\setminus\{0\}} Z_T e^{\Gamma(T)_u z}\gamma(u,T) z N'(\delta u, dz)\right]^2 \nu'(dx)dt < \infty$$

From the fundamental theorem of calculus (Nunno et al. [2010])

$\int_0^T \int_{\mathbb{R}\setminus\{0\}} Z_T e^{\Gamma(T)_u z}\gamma(u,T)zN'(\delta u,dz) \in \mathbb{D}_{1,2}$ and

$$\begin{aligned}
D_{t,x} & \left(\int_0^T \int_{\mathbb{R}\setminus\{0\}} Z_T e^{\Gamma(T)_u z}\gamma(u,T)zN'(\delta u,dz)\right) \\
= & \int_0^T \int_{\mathbb{R}\setminus\{0\}} D_{t,x}\left(Z_T e^{\Gamma(T)_u z}\gamma(u,T)z\right)N'(\delta u,dz) + Z_T e^{\Gamma(T)_t x}\gamma(t,T)x \\
= & \left(e^{\Gamma(T)_t x}-1\right)\int_0^T \int_{\mathbb{R}\setminus\{0\}} Z_T e^{\Gamma(T)_u z}\gamma(u,T)zN'(\delta u,dz) + Z_T e^{\Gamma(T)_t x}\gamma(t,T)x \\
= & \left(e^{\Gamma(T)_t x}-1\right)\left(Z_T\int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T)zN'(du,dz) - Z_T\int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T)\left(e^{\Gamma(T)_u z}-1\right)\nu'(dz)dt\right) \\
& +Z_T e^{\Gamma(T)_t x}\gamma(t,T)x
\end{aligned}$$

Therefore, $H \in \mathbb{D}_{1,2}$ and

$$\begin{aligned}
D_{t,x}H &= CD_{t,x}\left(Z_T\int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T)zN'(du,dz)\right) - CK'D_{t,x}Z_T \\
&= CD_{t,x}\left(\int_0^T \int_{\mathbb{R}\setminus\{0\}} Z_T e^{\Gamma(T)_u z}\gamma(u,T)zN'(\delta u,dz)\right) + CD_{t,x}\left(Z_T\int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T)\left(e^{\Gamma(T)_u z}-1\right)\nu'(dz)\right. \\
&\quad -CK'Z_T\left(e^{\Gamma(T)_t x}-1\right) \\
&= C\left(e^{\Gamma(T)_t x}-1\right)\left(Z_T\int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T)zN'(du,dz) - Z_T\int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T)\left(e^{\Gamma(T)_u z}-1\right)\nu'(dz)dt\right) \\
&\quad +CZ_T e^{\Gamma(T)_t x}\gamma(t,T)x + CZ_T\left(e^{\Gamma(T)_t x}-1\right)\int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T)\left(e^{\Gamma(T)_u z}-1\right)\nu'(dz)dt \\
&\quad -CK'Z_T\left(e^{\Gamma(T)_t x}-1\right) \\
&= C\left(e^{\Gamma(T)_t x}-1\right)\left(Z_T\int_0^T \int_{\mathbb{R}\setminus\{0\}} \gamma(u,T)zN'(du,dz) - K'Z_T\right) + CZ_T e^{\Gamma(T)_t x}\gamma(t,T)x \\
&= C\left(e^{\Gamma(T)_t x}-1\right)H + CZ_T e^{\Gamma(T)_t x}\gamma(t,T)x
\end{aligned}$$

Finally using the chain rule, $D_{t,x}F = (H + D_{t,x}H)^+ - F$. □

We have found the Malliavin derivatives of a caplet. In hedging, we need to compute the conditional expectation of these derivatives given $\mathcal{F}_t$. Although, a closed form solution may not be possible, the conditional expectation can always be approximated by simulation.

2.3 Hedging in an HJM Market Driven By A Pure Jump Process

Hedging in an HJM model is the solution of the following integral equations:

$$\int_t^{T^*} \phi(t,T)P(T)_{t-}\Gamma(T)_t \sqrt{c_t}dT = \mathbb{E}^{\mathbb{Q}}\left[D_t\left(e^{-\int_0^{T^*} r(s,s,)ds}F\right)|\mathcal{F}_t\right] \tag{2.20}$$

$$\int_t^{T^*} \phi(t,T)P(T)_{t-}\left(e^{\Gamma(T)_t x} - 1\right)dT = \mathbb{E}^{\mathbb{Q}}\left[D_{t,x}\left(e^{-\int_0^{T^*} r(s,s,)ds}F\right)|\mathcal{F}_t\right] \tag{2.21}$$

In this section, we assume that $L(t)$ is a pure jump process under an equivalent martingale measure and the jump sizes are bounded by M, i.e. $\nu'((-\infty, M)) = 0$ and $\nu'((M,\infty)) = 0$. In this case, we only have the second equation:

$$\int_t^{T^*} \left(e^{\Gamma(T)_t x} - 1\right)\varphi_t(T)dT = f_t(x) \tag{2.22}$$

where $\phi(t,T)P(T)_{t-} = \varphi_t(T)$ and $f_t(x) = \mathbb{E}^{\mathbb{Q}}\left[D_{t,x}\left(e^{-\int_0^{T^*} r(s,s,)ds}F\right)|\mathcal{F}_t\right] \in L^2([0,T^*] \times [-M,M] \times \Omega, \lambda \times \nu' \times \mathbb{Q})$. Note that $P(T)_{t-} > 0$.

This equation is a Fredholm integral equation of the first kind. The integral is a linear operator from $L^2([t,T^*])$ to $L^2([-M,M])$ and the kernel of the operator is $\left(e^{\Gamma(T)_t x} - 1\right)$. Note that, $\Gamma(T)_t$ is a continuous function of T and the kernel is a continuous function of both T and x. The integral operator defined by a continous kernel is a compact operator (Hochstadt [1989]).

Proposition 2.3.1 *Let $K(x,y)$ be continuous for all $a \le x, y \le b$. The associated integral operator*

$$Kf = \int_a^b K(x,y)f(y)dy$$

is a compact operator on $L^2([a,b])$.

Without lost of generality, we can assume that the jump sizes are in $[-1,1]$ and by a change of variable, $T \in [-1,1]$. This proposition shows that the hedging problem is equivalent to the solution of an integral equation and the operator is compact.

A compact operator [2] cannot have a bounded inverse (Kress [1999]). This fact is an indicator of forthcoming difficulties in finding solutions for every contingent claim.

Definition 2.3.1 *Let X and Y be two normed spaces and $U \in X$ and $V \in Y$. A is an operator from U into V, $A : U \to V$. The equation*

$$A\varphi = f \tag{2.23}$$

is called well-posed if A is invertible and its inverse is continuous. Otherwise it is called ill-posed.

Furthermore, let X and Y be two normed spaces and $A : X \to Y$ is a compact operator and well-posed. It means that, A^{-1} is also bounded and $I = AA^{-1}$ is also a compact operator. The identity operator $I : X \to X$ is compact if and only if X has finite dimension. Therefore, X should have finite dimension (Kress [1999]). We have the following result:

Proposition 2.3.2 *Assume that X and Y are two normed spaces and $A : X \to Y$ is a compact operator. The equation*

$$A\varphi = f$$

is ill-posed if X is not of finite dimension.

[2]A linear operator $A : X \to Y$ from a normed space X into a normed space is called compact if it maps each bounded set in X into a relatively compact set in Y.

The domain of the integral operator is not finite dimensional and we conclude that hedging in HJM market is an ill-posed problem. In other words, the operator

$$A\varphi_t = f_t$$

is not invertible. This implies that, the HJM market is not complete.

On the other hand, the equation has a solution when f_t is in the range of the operator A. Existence of the solution depends on the singular values of A. For the definitions of singular value, eigenvalue, adjoint operator etc. please see (Kress [1999]). Using Picard's thorem, we can obtain necessary and sufficient conditions for the existence of the solution.

Definition 2.3.2 *Let X and Y be Hilbert spaces, $A : X \to Y$ be a compact linear operator, and $A^* : Y \to X$ be its adjoint. The non–negative square roots of the eigenvalues of the nonnegative self adjoint operator $AA^* : X \to X$ are called singular values of A.*

Before we state Picard's theorem, we need to understand singular values.

Proposition 2.3.3 *Let $\{\mu_n\}$ denote the sequence of the nonzero singular values of the compact linear operator A repeated according to their multiplicity. Then there exist orthonormal sequences $\{\varphi_n\}$ in X and $\{g_n\}$ in Y such that*

$$A\varphi_n = \mu_n g_n$$

$$A^* g_n = \mu_n \varphi_n$$

for all $n \in \mathbb{N}$. For each $\varphi \in X$ we have the singular value decomposition

$$\varphi = \sum_{n=1}^{\infty} (\varphi, \varphi_n)\varphi_n + Q\varphi$$

with the orthogonal projection operator $Q : X \to N(A)$ and

$$A\varphi = \sum_{n=1}^{\infty} \mu_n(\varphi, \varphi_n)g_n$$

where $(.,.)$ denotes the inner product in the Hilbert space X.

Picard's theorem gives us necessary and sufficient conditions for the existence of a solution and also it gives solution in terms of a singular value decomposition.

Proposition 2.3.4 *(Picard) Let $A : X \to Y$ be a compact linear operator with singular system (μ_n, φ_n, g_n). The Fredholm equation of the the first kind*

$$A\varphi = f$$

is solvable if and only if f belongs to $N(A^*)^\perp$, the orthogonal complement of $N(A^*)$, and satisfies

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^2}|(f, g_n)|^2 < \infty$$

In this case, a solution is given by

$$\varphi = \sum_{n=1}^{\infty} \frac{1}{\mu_n}(f, g_n)\varphi_n$$

In our HJM market, our operator A_t is

$$A_t\varphi_t = \int_t^{T^*} \left(e^{\Gamma(T)_t x} - 1\right)\varphi_t(T)dT \tag{2.24}$$

We realize that $e^{\Gamma(T)_t x} - 1 = \sum_{n=1}^{\infty} \Gamma(T)_t^n \frac{x^n}{n!}$ and it is a sum of all powers of x^n, $n \geq 1$. In L^2 space, polynomials are dense and $\{1, x, x^2, x^3, ...\}$ span $L^2([-M, M], \lambda)$ space. At this point our goal is to expand the range of the operator in terms of polynomials. Intuitively, we try to add a constraint on $\Gamma(T)_t$ such that our operator has the largest range.

Picard's thorem shows that, a necessary but not sufficient condition on f is $f \in N(A^*)^\perp$. We will try to add a constraint on Γ_t such that this condition will be satisfied for all x^n, $n \geq 1$.

Lemma 2.3.1 *Assume that for every fixed t, the functions $\{(\Gamma_t)^n\}$ are linearly independent. For all $n \geq 1$, x^n is in $N(A^*)^\perp$, the orthogonal complement the null space of the operator A^*.*

Proof: First note that, the jump sizes are bounded and there exists a constant M such that $-M \leq x \leq M$. Let $g \in N(A^*)$. Then $\int_{-M}^{M}\left(e^{\Gamma(T)_t x} - 1\right) g(x)dx = 0$. $\Gamma_t(T)$ is a continuous function of T and it is bounded, $|\Gamma_t| \leq C$ by a constant C.

$$\sum_{n=1}^{\infty}\left|\Gamma_t(T)^n \frac{x^n}{n!} g(x)\right| \leq e^{|\Gamma(T)_t x|}|g(x)| \leq e^{M \cdot C}|g(x)|$$

From the dominated convergence theorem,

$$\begin{aligned}
\int_{-M}^{M}\left(e^{\Gamma(T)_t x} - 1\right) g(x)dx &= \int_{-M}^{M}\sum_{n=1}^{\infty}\Gamma_t(T)^n \frac{x^n}{n!} g(x)dx \\
&= \sum_{n=1}^{\infty}\int_{-M}^{M}\Gamma_t(T)^n \frac{x^n}{n!} g(x)dx \\
&= \sum_{n=1}^{\infty}\frac{\Gamma_t(T)^n}{n!}\int_{-M}^{M} x^n g(x)dx = 0
\end{aligned}$$

Since the $\{(\Gamma_t)^n\}$ are linearly independent, this implies that $\int_{-M}^{M} x^n g(x)dx = 0$ for all $n \geq 1$. □

The additional constraint on $\Gamma_t(T)$ mentioned earlier is not a strong assumption. As an example, a polynomial satisfies this constraint. Also, any continous function that is not constant in a interval also satisfies this constraint.

Although, x^n are in $N(A^*)^\perp$, we still need to show that $\sum_{k=1}^{\infty}\frac{1}{\mu_k^2}|(x^n, g_k)|^2 < \infty$ and this is a non-trivial problem. We can directly show that x^n is in the range of the operator.

Lemma 2.3.2 *Assume that for every fixed t, the functions $\{(\Gamma_t)^n\}$ are linearly independent. For all $n \geq 1$, x^n is in the range of the operator A. If g_m is the solution for x^m then g_m and $\{(\Gamma_t)^n\}$ are orthogonal in L^2 for $m \neq n$.*

Proof: Let $V = \operatorname{span}\left\{(\Gamma_t)^n\} \;\middle|\; n \geq 1, n \neq m\right\}$. By a Gram–Schmidt orthogonalization, we can find an orthonormal sequence $\{v_n\}$ that spans V. We can write $(\Gamma_t)^m$ as a sum of the two functions $(\Gamma_t)^m = \phi_1 + \phi_2$ where $\phi_1 \in V$ and $\phi_2 \in V^\perp$. ϕ_1 can be written as a sum of its projections on each v_n. Since $\{(\Gamma_t)^n\}$ are linearly independent, $\phi_2 \neq 0$. Let a be a constant $a = m!\left(\int_t^{T^*} \Gamma_t(T)^m \phi_2(T)dT\right)^{-1} = m!\left(\int_t^{T^*} \phi_2(T)^2 dT\right)^{-1}$. From the dominated convergence theorem,

$$
\begin{aligned}
A_t(a\phi_2) &= \int_t^{T^*} \sum_{n=1}^{\infty} \frac{1}{n!} \Gamma_t(T)^n x^n a\phi_2(T)dT \\
&= a \sum_{n=1}^{\infty} \frac{1}{n!} \int_t^{T^*} \Gamma_t(T)^n x^n \phi_2(T)dT \\
&= a x^m \frac{1}{m!} \int_t^{T^*} \Gamma_t(T)^m \phi_2(T)dT = x^m
\end{aligned}
$$

□

We conclude that, for all polynomials with constant terms equal to zero, the equation has a solution. In other words, only one dimension, the ″constant function″ is missing in the range. An L^2 function is a limit of polynomials and can be approximated by a finite degree polynomials. In our case, we can approximate all except the ″constant part″ of f_t. The part that can not be approximated is in a one dimensional subspace and it is not necessary in the approximation of the Malliavin derivative.

Next, we will show that, HJM market is complete in the limit. We can explain this result by using power jump assets. Corcuera et al. [2005] enlarged a

Lévy market by a series of orthonormalized power-jump assets and the market becomes complete. They define power jump assets as follows:

Definition 2.3.3 *Assume that for some $\epsilon > 0$, and $\lambda > 0$*

$$\int_{(-\epsilon,\epsilon)} e^{(\lambda|x|)}\nu(dx) < \infty$$

This assumptions implies that all moments of the Lévy process exist. We define the compensated i-th-power-jump process ($i \geq 2$) as

$$Y^{(i)}(t) = Z^{i}(t) - \mathbb{E}^{\mathbb{Q}}[Z^{(i)}(t)]$$

where $Z^{(i)}(t) = \sum_{0<s\leq t} [L(s) - L(s-)]^{i}$.

In the extended market, they assume that power jump assets exist. Note that, when $L(t)$ is a pure jump process, it corresponds to a process with a Malliavin derivative equal to x. The Malliavin derivative of the compensated i-th-power-jump process is equal to x^{i} and the previous lemma shows that power jump processes can be perfectly hedged in HJM market. In chapter 1, we showed that if the power jump assets exist in the market, a square integrable contingent claim can be either perfectly or approximately hedged by a finite number of power jump assets. In other words, if the power-jump processes can be perfectly hedged, the market is complete in the limit, a dense subset of square integrable contingent claims can be hedged perfectly (proposition 1.3.4).

In practice, we always use a finite number of bonds to hedge a contingent claim and this gives us an approximation of the final payoff. Therefore, a complete market in the limit is still useful in practice.

Instead of using the power-jump assets approach, we will use properties of the Malliavin derivative to obtain a clear understanding of hedging. This approach will give us a better understanding of the set of contingent claims that can be hedged.

Proposition 2.3.5 *Assume that*

(i) For every fixed t, the functions $\{(\Gamma_t)^n\}$ are linearly independent,

(ii) Jump sizes are bounded.

Let F be a square integrable contingent claim in HJM market driven by a pure jump process. Then either

(1) F can be replicated prefectly or

(2) There exists a sequence of contingent claims F_n such that F_n can be perfectly hedged and $F_n \to F$ in L^2.

Proof: Since jump sizes are bounded, all moments of the Lévy process with degree larger than one exist. Therefore, for every $n \geq 1$, $Z^{(n)}(T^*) = \int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} x^n N'(dt,dx)$ is a square integrable random variable. From the previous lemma 2.3.2, x^n is in the range of the operator and $Z^{(n)}(T^*)$ can be replicated by a portfolio. Therefore $Z^{(n)}(T^*)$ is in $L^2(\mathbb{Q})$ and can be replicated.

Let h be a function in $L^2([0,T^*])$ and $G = e^{\int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} h(t)xN'(dt,dx)} \in \mathbb{D}^{\epsilon}_{1,2}$. The Malliavin derivative of G is equal to $D_{t,x}G = G\left(e^{h(t)x} - 1\right)$ and

$$\begin{aligned} G &= \mathbb{E}^{\mathbb{Q}}[G] + \int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} \mathbb{E}^{\mathbb{Q}}\left[D_{t,x}G|\mathcal{F}_t\right] N'(dt,dx) \\ &= \mathbb{E}^{\mathbb{Q}}[G] + \int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} \left(e^{h(t)x} - 1\right)\mathbb{E}^{\mathbb{Q}}\left[G|\mathcal{F}_t\right] N'(dt,dx) \end{aligned}$$

Let $F_n = \mathbb{E}^{\mathbb{Q}}[G] + \int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} \sum_{n=1}^{N} \frac{x^n}{n!} h(t)^n \mathbb{E}^{\mathbb{Q}}\left[G|\mathcal{F}_t\right] N'(dt,dx)$.

$$G - F_n = \int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} \sum_{n=N+1}^{\infty} \frac{x^n}{n!} h(t)^n \mathbb{E}^{\mathbb{Q}}\left[G|\mathcal{F}_t\right] N'(dt,dx)$$

From the Itó isometry (Medvegyev [2007]), Fubini theorem and Jensen's inequality:

$$\begin{aligned}
||G - F_n||^2 &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}}\left(\sum_{n=N+1}^{\infty}\frac{x^n}{n!}h(t)^n\mathbb{E}^{\mathbb{Q}}\left[G|\mathcal{F}_t\right]\right)^2 \nu'(dx)dt\right] \\
&= \int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}}\mathbb{E}^{\mathbb{Q}}\left[\left(\sum_{n=N+1}^{\infty}\frac{x^n}{n!}h(t)^n\mathbb{E}^{\mathbb{Q}}\left[G|\mathcal{F}_t\right]\right)^2\right]\nu'(dx)dt \\
&= \int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}}\left(\sum_{n=N+1}^{\infty}\frac{x^n}{n!}h(t)^n\right)^2\mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{E}^{\mathbb{Q}}\left[G|\mathcal{F}_t\right]\right)^2\right]\nu'(dx)dt \\
&\leq \int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}}\left(\sum_{n=N+1}^{\infty}\frac{x^n}{n!}h(t)^n\right)^2\mathbb{E}^{\mathbb{Q}}\left[G^2\right]\nu'(dx)dt \\
&= \mathbb{E}^{\mathbb{Q}}\left[G^2\right]\int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}}\left(\sum_{n=N+1}^{\infty}\frac{x^n}{n!}h(t)^n\right)^2\nu'(dx)dt
\end{aligned}$$

The last statement is bounded by $e^{|xh(t)|} - 1$ and by the dominated covergence theorem, F_n converges to G. Linear combinations of such exponentials are dense in $\mathbb{D}_{1,2}$ and $\mathbb{D}_{1,2}$ is dense in $L^2(\mathbb{Q})$ space. This completes the proof. □

Corollary : Assume that

(i) For every fixed t, there exists an interval $[T_t^a, T_t^b] \subset [t, T^*]$ such that restrictions of the functions $\{(\Gamma_t)^n\}$ on $[T_t^a, T_t^b]$ are linearly independent,

(ii) Jump sizes are bounded.

Let F be a square integrable contingent claim in an HJM market driven by a pure jump process. Then either

(1) F can be replicated prefectly by bonds in $[T_t^a, T_t^b]$ at time t or

(2) There exists a sequence of contingent claims F_n such that F_n can be perfectly hedged by bonds in $[T_t^a, T_t^b]$ at time t and $F_n \to F$ in L^2.

This result shows that, we can approximate a replicating portfolio by a finite

number of other portfolios in the following way:
First, we approximate $\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}\left(e^{-\int_0^{T^*} r(s,s,)ds}F\right)|\mathcal{F}_t\right]$ by a polynomial of x with degree n and whose constant term is 0. This can be done easily. Let V be subspace of $L^2([-M,M],\nu')$ spanned by polynomials $x, x^2, x^3, ..., x^n$. By Gram - Schmidt orthogonalization, we can find an orthonormal set $g_1(x), g_2(x), g_3(x), ..., g_n(x)$ that spans V. Each $g_i(x)$ is a linear combination of $x, x^2, ..., x^i$. We have a replicating portfolio for each g_i. Let $\varphi_t^{(i)}(T)$ be the replicating portfolio for g_i.
The next step is to find projection of $\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}\left(e^{-\int_0^{T^*} r(s,s,)ds}F\right)|\mathcal{F}_t\right]$ on each g_n. Let $c_i = \left\langle \mathbb{E}^{\mathbb{Q}}\left[D_{t,x}\left(e^{-\int_0^{T^*} r(s,s,)ds}F\right)|\mathcal{F}_t\right], g_i\right\rangle$
Finally, our approximating portfolio is given by $\sum_{n=1}^{N} c_n\varphi_t^{(n)}(T)$.

2.4 Hedging In An HJM Market Driven By A Lévy Process

When the probability space is generated by a Brownian motion and a Poisson random measure, there exist $\mathcal{F}_{T^*}$ measurable random variables that cannot be hedged approximately. A simple example is $F = e^{\int_0^{T^*} r(s,s,)ds}\int_0^{T^*}\int_{\mathbb{R}} xN(dt,dx)$. In this case, $\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}\left(e^{-\int_0^{T^*} r(s,s,)ds}F\right)|\mathcal{F}_t\right] = 1$ and $\mathbb{E}^{\mathbb{Q}}\left[D_t\left(e^{-\int_0^{T^*} r(s,s,)ds}F\right)|\mathcal{F}_t\right] = 0$. A sequence of self-financing trading strategies $\varphi^{(n)}(t,T)$ in the market should satisfy

$$\int_t^{T^*} \varphi^{(n)}(t,T)P(T)_{t-}\Gamma(T)_t\sqrt{c_t}dT \to 0$$

$$\int_t^{T^*} \varphi^{(n)}(t,T)P(T)_{t-}\left(e^{\Gamma(T)_t x} - 1\right)dT \to x$$

Let us assume that $c_t = 1$. Let $\epsilon > 0$. The first equation implies that there exists a function ϕ such that

$$-\epsilon < a = \int_t^{T^*} \phi(t,T)\Gamma(T)_t dT < \epsilon$$

From the dominated convergence theorem,

$$\begin{aligned}
\int_t^{T^*} \phi(t,T)\left(e^{\Gamma(T)_t x} - 1\right) dT &= \sum_{n=1}^{\infty} \frac{x^n}{n!} \int_t^{T^*} \phi(t,T)\Gamma(T)_t^n dT \\
&= ax + \sum_{n=2}^{\infty} \frac{x^n}{n!} \int_t^{T^*} \phi(t,T)\Gamma(T)_t^n dT
\end{aligned}$$

Note that, $ax + \sum_{n=2}^{\infty} \frac{x^n}{n!} \int_t^{T^*} \phi(t,T)\Gamma(T)_t^n dT$ is a polynomial with first degree ax, $-\epsilon < a < \epsilon$, and it can not be an approximate of x. This contradiction shows that, it cannot be hedged approximately.

On the other hand, such contingent claims are uninteresting for financial markets. The meaningful ones are the claims measurable with respect to the filtration generated by $L(t)$. To show that HJM market is complete in the limit, we only need to prove that power jump assets are implicitly available in the market.

2.4.1 Hedging A Variance Swap In An HJM Market Driven A Lévy Process

In this section, we will find a hedging strategy for a variance swap. We assume that, the HJM model is driven by a Lévy process $L(t)$ under $\mathbb{P}$, there exists an equivalent martingale measure $\mathbb{Q}$, and (u, Y) are non-random processes. Therefore, this measure is unique (Eberlein et al. [2005]). We also assume that $L(t)$ is a Lévy process under $\mathbb{Q}$.

Variance Swap

A variance swap is the cumulative volatility of some underlying product. We will assume that there exists a variance swap on a bond with maturity T^{sw}. The seller will pay the realized variance of the price changes of the bond until time t_{sw}. Let $P(T^{sw})_t$ be time t value of discounted price of this bond. The swap payoff is defined as

$$
\begin{aligned}
V_{sw}(t_{sw}) &= \left[\ln\left(\frac{e^{\int_0^t r_u du} P(T^{sw})_t}{P(T^{sw})_0}\right)\right]_{t_{sw}} \\
&= \left[\int_0^t r_u du + \ln\left(\frac{P(T^{sw})_t}{P(T^{sw})_0}\right)\right]_{t_{sw}} \\
&= \left[\int_0^t r_u du + \int_0^t A(T^{sw})_u du + \int_0^t \Gamma(T^{sw})_u dL_u\right]_{t_{sw}} \\
&= \left[-\int_0^t \Gamma(t)_u dL_u + \int_0^t A(T^{sw})_u du + \int_0^t \Gamma(T^{sw})_u dL_u\right]_{t_{sw}} \\
&= \left[\int_0^t \left(\Gamma(T^{sw})_u - \Gamma(t)_u\right) dL_u\right]_{t_{sw}} \\
&= \int_0^{t_{sw}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 c_u du + \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx)
\end{aligned}
$$

where [.] is quadratic variation.

The Malliavin Derivative of a Variance Swap

Before we calculate the Malliavin derivative of the discounted variance swap, we need to define the discount process. The discount process is defined as $e^{-\int_0^t r(s)ds}$ where $r(s)$ is the risk free spot rate.

$$e^{-\int_0^t r(s)ds} = e^{-\int_0^t f(0,s)ds - \int_0^t \int_0^s \alpha(u,s)duds - \int_0^t \int_0^s \gamma(u,s)dL_u ds}$$

By changing the order of integrations and using Fubini's theorem, it becomes

$$e^{-\int_0^t r(s)ds} = e^{\int_0^t h_u du + \int_0^t \Gamma(t)_u \sqrt{c_u} dW'_u} e^{\int_0^t \int_{\mathbb{R}\setminus\{0\}} \Gamma(t)_u x N'(du,dx) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^{\Gamma(t)_u x} - 1 - \Gamma(t)_u x) \nu'(du,dx)}$$

where h_t is a deterministic function of time.

The discounted variance swap becomes

$$\begin{aligned}
e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw}) &= e^{-\int_0^{t_{sw}} r_u du} \int_0^{t_{sw}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 c_u du \\
&\quad + e^{-\int_0^{t_{sw}} r_u du} \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx) \\
&= HFG \int_0^{t_{sw}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 c_u du \\
&\quad + HFG \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx)
\end{aligned}$$

where

$$H = e^{\int_0^{t_{sw}} h_u du}$$

$$F = e^{\int_0^{t_{sw}} \Gamma(t_{sw})_u \sqrt{c_u} dW'_u}$$

$$G = e^{\int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \Gamma(t_{sw})_u x N'(du,dx) - \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} (e^{\Gamma(t_{sw})_u x} - 1 - \Gamma(t_{sw})_u x) \nu'(du,dx)}$$

Note that, the discounted variance swap payoff is in the product space $\Omega_{W'} \times \Omega_{N'}$, F is in $\Omega_{W'}$ and G is $\Omega_{N'}$. H is a deterministic constant. Now, we can compute the Malliavin derivative of swap's discounted payoff.

Lemma 2.4.1

$$D_t\left(e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw})\right) = \Gamma(t_{sw})_t \sqrt{c_t} e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw})$$

$$\mathbb{E}^{\mathbb{Q}}\left[D_t\left(e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw})\right) | \mathcal{F}_t\right] = \Gamma(t_{sw})_t \sqrt{c_t} e^{-\int_0^{t} r_u du} V_{sw}(t)$$

$$\begin{aligned}
D_{t,x}\left(e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw})\right) &= \left(e^{\Gamma(t_{sw})_t x} - 1\right) e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw}) \\
&\quad + \left(\Gamma(T^{sw})_t - \Gamma(t)_t\right)^2 x^2 e^{\Gamma(t_{sw})_t x} e^{-\int_0^{t_{sw}} r_u du}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}\left(e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw})\right) | \mathcal{F}_t\right] &= \left(e^{\Gamma(t_{sw})_t x} - 1\right) e^{-\int_0^{t} r_u du} V_{sw}(t) \\
&\quad + x^2 e^{\Gamma(t_{sw})_t x} \left(\Gamma(T^{sw})_t - \Gamma(t)_t\right)^2 P(t_{sw})_t
\end{aligned}$$

Proof:

First method:

$$
\begin{aligned}
D_t\left(e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw})\right) &= H(D_t F)G \int_0^{t_{sw}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 c_u du \\
&\quad + H(D_t F)G \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx) \\
&= H\Gamma(t_{sw})_t \sqrt{c_t} FG \int_0^{t_{sw}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 c_u du \\
&\quad + H\Gamma(t_{sw})_t \sqrt{c_t} FG \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx) \\
&= \Gamma(t_{sw})_t \sqrt{c_t} e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw})
\end{aligned}
$$

$$
\mathbb{E}^{\mathbb{Q}}\left[D_t\left(e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw})\right) | \mathcal{F}_t\right] = \Gamma(t_{sw})_t \sqrt{c_t} e^{-\int_0^{t} r_u du} V_{sw}(t)
$$

$D_{t,x}\left(e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw})\right)$ is more complicated. We will obtain it by using two different methods. From the previous part, G is in $\mathbb{D}_{1,2}$, and its Malliavin derivative is equal to

$$
D_{t,x} G = G\left(e^{\Gamma(t_{sw})_t x} - 1\right)
$$

$$
\begin{aligned}
D_{t,x}\left(e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw})\right) &= HF(D_{t,x} G) \int_0^{t_{sw}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 c_u du \\
&\quad + HF\left(D_{t,x}\left(G \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx)\right)\right) \\
&= HFG\left(e^{\Gamma(t_{sw})_t x} - 1\right) \int_0^{t_{sw}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 c_u du \\
&\quad + HF\left(D_{t,x}\left(G \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx)\right)\right)
\end{aligned}
$$

From an integration by parts formula for Skorohod integrals and using the Lévy-Skorohod isometry (Nunno et al. [2010]),

$$
\begin{aligned}
& G \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx) \\
= & \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} (G + D_{u,x} G) \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(\delta u, dx) \\
& + \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 (D_{u,x} G) \nu'(dx) du
\end{aligned}
$$

$$\begin{aligned}
&= \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} Ge^{\Gamma(t_{sw})_u x} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(\delta u, dx) \\
&\quad + \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 G \left(e^{\Gamma(t_{sw})_u x} - 1\right) \nu'(dx)du \\
&= \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} Ge^{\Gamma(t_{sw})_u x} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(\delta u, dx) \\
&\quad + G \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 \left(e^{\Gamma(t_{sw})_u x} - 1\right) \nu'(dx)du
\end{aligned}$$

where $\int_0^T \int_{\mathbb{R}\setminus\{0\}} g(u,z)N'(\delta u, dz)$ is a Skorohod integral. The second term is in $\mathbb{D}_{1,2}$ and its Malliavin derivative is equal to

$$\begin{aligned}
& D_{t,x}\left(G \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 \left(e^{\Gamma(t_{sw})_u z} - 1\right) \nu'(dz)du\right) \\
&= (D_{t,x}G) \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 \left(e^{\Gamma(t_{sw})_u z} - 1\right) \nu'(dz)du \\
&= G\left(e^{\Gamma(t_{sw})_t x} - 1\right) \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 \left(e^{\Gamma(t_{sw})_u z} - 1\right) \nu'(dz)du
\end{aligned}$$

Since jumps sizes, $\Gamma(t_{sw})_t$ and $\left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2$ are bounded,

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \int_{\mathbb{R}\setminus\{0\}} G^2 e^{2\Gamma(t_{sw})_u x} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^4 x^4 \nu'(dx)du\right] < \infty$$

Also, $G\left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 \in \mathbb{D}_{1,2}$ for every $(u,x) \in [0,T] \times \mathbb{R}$ and $D_{t,x}\left(G\left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2\right)$ is Skorohod integrable with

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(\int_0^T \int_{\mathbb{R}\setminus\{0\}} D_{t,x}\left(Ge^{\Gamma(t_{sw})_u z} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2\right) N'(\delta u, dz)\right)^2 \nu'(dx)dt\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(\int_0^T \int_{\mathbb{R}\setminus\{0\}} G\left(e^{\Gamma(t_{sw})_t x} - 1\right) e^{\Gamma(T)_u z} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 N'(\delta u, dz)\right)^2 \nu'(dx)dt\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(t_{sw})_t x} - 1\right)^2 \left(\int_0^T \int_{\mathbb{R}\setminus\{0\}} Ge^{\Gamma(t_{sw})_u z} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 N'(\delta u, dz)\right)^2 \nu'(dx)dt\right] \\
&= \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(t_{sw})_t x} - 1\right)^2 \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \int_{\mathbb{R}\setminus\{0\}} Ge^{\Gamma(t_{sw})_u z} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 N'(\delta u, dz)\right]^2 \nu'(dx)dt < \infty
\end{aligned}$$

From the fundamental theorem of Malliavin calculus (Nunno et al. [2010]) $\int_0^T \int_{\mathbb{R}\setminus\{0\}} Ge^{\Gamma(t_{sw})_u z} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 N'(\delta u, dz) \in \mathbb{D}_{1,2}$ and

$$
\begin{aligned}
& D_{t,x}\left(\int_0^T \int_{\mathbb{R}\setminus\{0\}} Ge^{\Gamma(t_{sw})_u z} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 N'(\delta u, dz)\right) \\
= \; & \int_0^T \int_{\mathbb{R}\setminus\{0\}} D_{t,x}\left(Ge^{\Gamma(t_{sw})_u z} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2\right) N'(\delta u, dz) + Ge^{\Gamma(t_{sw})_t x} \left(\Gamma(T^{sw})_t - \Gamma(t_{sw})_t\right)^2 x^2 \\
= \; & \left(e^{\Gamma(t_{sw})_t x} - 1\right) \int_0^T \int_{\mathbb{R}\setminus\{0\}} Ge^{\Gamma(t_{sw})_u z} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 N'(\delta u, dz) \\
& + Ge^{\Gamma(t_{sw})_t x} \left(\Gamma(T^{sw})_t - \Gamma(t_{sw})_t\right)^2 x^2 \\
= \; & \left(e^{\Gamma(t_{sw})_t x} - 1\right) G \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 N'(du, dz) \\
& - \left(e^{\Gamma(t_{sw})_t x} - 1\right) G \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 \left(e^{\Gamma(T)_u z} - 1\right) \nu'(dz) dt \\
& + Ge^{\Gamma(t_{sw})_t x} \left(\Gamma(T^{sw})_t - \Gamma(t_{sw})_t\right)^2 x^2
\end{aligned}
$$

Therefore, $G \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx) \in \mathbb{D}_{1,2}$ and

$$
\begin{aligned}
& D_{t,x}\left(G \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx)\right) \\
= \; & D_{t,x}\left(\int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} Ge^{\Gamma(t_{sw})_u x} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(\delta u, dx)\right) \\
& + D_{t,x}\left(G \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 \left(e^{\Gamma(t_{sw})_u x} - 1\right) \nu'(dx) du\right) \\
= \; & \left(e^{\Gamma(t_{sw})_t x} - 1\right) G \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 N'(du, dz) \\
& - \left(e^{\Gamma(t_{sw})_t x} - 1\right) G \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 \left(e^{\Gamma(T)_u z} - 1\right) \nu'(dz) dt \\
& + Ge^{\Gamma(t_{sw})_t x} \left(\Gamma(T^{sw})_t - \Gamma(t_{sw})_t\right)^2 x^2 \\
& + G\left(e^{\Gamma(t_{sw})_t x} - 1\right) \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 \left(e^{\Gamma(t_{sw})_u z} - 1\right) \nu'(dz) du \\
= \; & \left(e^{\Gamma(t_{sw})_t x} - 1\right) G \int_0^T \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 z^2 N'(du, dz) \\
& + Ge^{\Gamma(t_{sw})_t x} \left(\Gamma(T^{sw})_t - \Gamma(t_{sw})_t\right)^2 x^2
\end{aligned}
$$

Finally,

$$
D_{t,x}\left(e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw})\right) \;=\; HFG\left(e^{\Gamma(t_{sw})_t x} - 1\right) \int_0^{t_{sw}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 c_u du
$$

$$
\begin{aligned}
&+HF\left(D_{t,x}\left(G\int_0^{t_{sw}}\int_{\mathbb{R}\setminus\{0\}}\left(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u\right)^2 x^2 N'(du,dx)\right)\right)\\
=\ & HFG\left(e^{\Gamma(t_{sw})_t x}-1\right)\int_0^{t_{sw}}\left(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u\right)^2 c_u du\\
&+HFG\left(e^{\Gamma(t_{sw})_t x}-1\right)\int_0^{T}\int_{\mathbb{R}\setminus\{0\}}\left(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u\right)^2 z^2 N'(du,dz)\\
&+HFGe^{\Gamma(t_{sw})_t x}\left(\Gamma(T^{sw})_t-\Gamma(t_{sw})_t\right)^2 x^2\\
=\ & \left(e^{\Gamma(t_{sw})_t x}-1\right)e^{-\int_0^{t_{sw}} r_u du}V_{sw}(t_{sw})\\
&+\left(\Gamma(T^{sw})_t-\Gamma(t)_t\right)^2 x^2 e^{\Gamma(t_{sw})_t x}e^{-\int_0^{t_{sw}} r_u du}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}\left(e^{-\int_0^{t_{sw}} r_u du}V_{sw}(t_{sw})\right)|\mathcal{F}_t\right] &= \left(e^{\Gamma(t_{sw})_t x}-1\right)e^{-\int_0^{t} r_u du}V_{sw}(t)\\
&\quad +e^{\Gamma(t_{sw})_t x}\left(\Gamma(T^{sw})_u-\Gamma(t)_u\right)^2 x^2 P(t_{sw})_t
\end{aligned}
$$

Second method:

First, we will find the discounted variance swap price and then apply Itô's formula.

$$
\begin{aligned}
e^{-\int_0^{t_{sw}} r_u du}V_{sw}(t_{sw}) &= e^{-\int_0^{t_{sw}} r_u du}\int_0^{t_{sw}}\left(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u\right)^2 c_u du\\
&\quad +e^{-\int_0^{t_{sw}} r_u du}\int_0^{t_{sw}}\int_{\mathbb{R}\setminus\{0\}}\left(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u\right)^2 x^2 N'(du,dx)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{sw}} r_u du}V_{sw}(t_{sw})|\mathcal{F}_t\right]\\
=\ & \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{sw}} r_u du}\int_0^{t_{sw}}\left(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u\right)^2 c_u du|\mathcal{F}_t\right]\\
&+\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{sw}} r_u du}\int_0^{t_{sw}}\int_{\mathbb{R}\setminus\{0\}}\left(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u\right)^2 x^2 N'(du,dx)|\mathcal{F}_t\right]\\
=\ & P(T_{sw})_t\int_0^{t_{sw}}\left(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u\right)^2 c_u du\\
&+P(T_{sw})_t\int_0^{t}\int_{\mathbb{R}\setminus\{0\}}\left(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u\right)^2 x^2 N'(du,dx)\\
&+e^{-\int_0^{t} r_u du}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_{sw}} r_u du}\int_t^{t_{sw}}\int_{\mathbb{R}\setminus\{0\}}\left(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u\right)^2 x^2 N'(du,dx)|\mathcal{F}_t\right]
\end{aligned}
$$

Since a Lévy process has independent increments,

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_{sw}} r_u du} \int_t^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx) | \mathcal{F}_t\right] \\
= & \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_{sw}} r_u du} \int_t^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx)\right] \\
= & e^{\int_t^{t_{sw}} h_u du} \mathbb{E}^{\mathbb{Q}}\left[e^{\int_t^{t_{sw}} \Gamma(t)_u \sqrt{c_u} dW'_u}\right] \\
& \cdot E\left[e^{\int_t^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \Gamma(t)_u x N'(du,dx) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^{\Gamma(t)_u x} - 1 - \Gamma(t)_u x) \nu'(du,dx)} \int_t^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx)\right] \\
= & e^{\int_t^{t_{sw}} h_u du} e^{\frac{1}{2}\int_t^{t_{sw}} \Gamma(t)_u^2 c_u d_u} \\
& \cdot \mathbb{E}^{\mathbb{Q}}\left[e^{\int_t^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \Gamma(t)_u x N'(du,dx) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^{\Gamma(t)_u x} - 1 - \Gamma(t)_u x) \nu'(du,dx)} \int_t^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx)\right] \\
= & e^{\int_t^{t_{sw}} h_u du} e^{\frac{1}{2}\int_t^{t_{sw}} \Gamma(t)_u^2 c_u d_u} \cdot E\left[\sum_{n=0}^{\infty} I_n(g_n) \int_t^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx)\right] \\
= & e^{\int_t^{t_{sw}} h_u du} e^{\frac{1}{2}\int_t^{t_{sw}} \Gamma(t)_u^2 c_u d_u} \cdot \int_t^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} g_1(x, u) \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 \nu'(dx) du \\
= & e^{\int_t^{t_{sw}} h_u du} e^{\frac{1}{2}\int_t^{t_{sw}} \Gamma(t)_u^2 c_u d_u} \cdot \int_t^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(t_{sw})_u x} - 1\right) \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 \nu'(dx) du
\end{aligned}$$

where $\sum_{n=0}^{\infty} I_n(g_n)$ is the chaos expansion of $e^{\int_t^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \Gamma(t)_u x N'(du,dx) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^{\Gamma(t)_u x} - 1 - \Gamma(t)_u x) \nu'(du,dx)}$ and $g_1(x, u) = \left(e^{\Gamma(t_{sw})_u x} - 1\right)$.

We obtain

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw}) | \mathcal{F}_t\right] \\
= & P(T_{sw})_t \int_0^{t_{sw}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 c_u du \\
& + P(T_{sw})_t \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx) \\
& + e^{-\int_0^t r_u du} e^{\int_t^{t_{sw}} h_u du} e^{\frac{1}{2}\int_t^{t_{sw}} \Gamma(t)_u^2 c_u d_u} \\
& \cdot \int_t^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(t_{sw})_u x} - 1\right) \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 \nu'(dx) du \\
= & P(T_{sw})_t \int_0^{t_{sw}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 c_u du \\
& + P(T_{sw})_t \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx)
\end{aligned}$$

$$
\begin{aligned}
&+e^{\int_0^{t_{sw}} h_u du} e^{\frac{1}{2}\int_t^{t_{sw}} \Gamma(t)_u^2 c_u d_u} \cdot \int_t^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(t_{sw})_u x} - 1\right) \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 \nu'(dx)du \\
&\cdot e^{\int_0^t \Gamma(t)_u \sqrt{c_u} dW'_u} e^{\int_0^t \int_{\mathbb{R}\setminus\{0\}} \Gamma(t)_u x N'(du,dx) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^{\Gamma(t)_u x} - 1 - \Gamma(t)_u x)\nu'(du,dx)} \\
= \; & P(T_{sw})_t \int_0^{t_{sw}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 c_u du \\
&+P(T_{sw})_t \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx) \\
&+Y_t e^{\int_0^t \Gamma(t)_u \sqrt{c_u} dW'_u} e^{\int_0^t \int_{\mathbb{R}\setminus\{0\}} \Gamma(t)_u x N'(du,dx)}
\end{aligned}
$$

where

$$
\begin{aligned}
Y_t \; = \; & e^{\int_0^{t_{sw}} h_u du + \frac{1}{2}\int_t^{t_{sw}} \Gamma(t)_u^2 c_u d_u - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^{\Gamma(t)_u x} - 1 - \Gamma(t)_u x)\nu'(dx,du)} \\
& \cdot \int_t^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(t_{sw})_u x} - 1\right) \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 \nu'(dx)du
\end{aligned}
$$

Since jump sizes are bounded and $\Gamma(T)_u$ is a continuous function of u, from the dominated convergence theorem $\int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(t_{sw})_u x} - 1\right) \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 \nu'(dx)$ is a continuous function of u. This implies that the outer integral in $\int_t^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(e^{\Gamma(t_{sw})_u x} - 1\right) \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 \nu'(dx)du$ is a Riemann integral with respect to u and Y_t is a deterministic differentiable function of time t. We can use Itô's formula for $\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw}) | \mathcal{F}_t\right]$.
$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw}) | \mathcal{F}_t\right]$ is in closed form and it is a martingale. Therefore, we only need to calculate dW'_t and $N'(dx, du)$ terms in Itô's formula.

$$
d\left(e^{-\int_0^t r_u du} V_{sw}(t)\right) = g(t) dW'_t + \int_{\mathbb{R}\setminus\{0\}} q(t, x) N'(dt, dx)
$$

where

$$
\begin{aligned}
g(t) \; = \; & \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 \nu'(dx) du P(T_{sw})_{t-} \Gamma(t^{sw})_t \sqrt{c_t} \\
& + \int_0^{t_{sw}} \int_{\mathbb{R}\setminus\{0\}} \left(\Gamma(T^{sw})_u - \Gamma(t_{sw})_u\right)^2 x^2 N'(du, dx) P(T_{sw})_{t-} \Gamma(t^{sw})_t \sqrt{c_t} \\
& + Y_t e^{\int_0^t \Gamma(t)_u \sqrt{c_u} dW'_u} e^{\int_0^t \int_{\mathbb{R}\setminus\{0\}} \Gamma(t)_u x N'(du,dx)} \Gamma(t)_t \sqrt{c_t} \\
= \; & \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw}) | \mathcal{F}_t\right] \Gamma(t)_t \sqrt{c_t} \\
= \; & \Gamma(t_{sw})_t \sqrt{c_t} e^{-\int_0^t r_u du} V_{sw}(t)
\end{aligned}
$$

$$
\begin{aligned}
q(t,x) &= \left(\int_0^{t_{sw}}\int_{\mathbb{R}\backslash\{0\}}(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u)^2 z^2\nu'(dz)du\right)P(T_{sw})_{t-}\left(e^{\Gamma(t^{sw})_t x}-1\right)\\
&\quad+\Big[\left(P(T_{sw})_{t-}+P(T_{sw})_{t-}\left(e^{\Gamma(t^{sw})_t x}-1\right)\right)\\
&\qquad\cdot\left(\int_0^{t_{sw}}\int_{\mathbb{R}\backslash\{0\}}(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u)^2 z^2N'(du,dz)+(\Gamma(T^{sw})_t-\Gamma(t_{sw})_t)^2x^2\right)\\
&\qquad-P(T_{sw})_{t-}\int_0^{t_{sw}}\int_{\mathbb{R}\backslash\{0\}}(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u)^2 z^2N'(du,dz)\Big]\\
&\quad+Y_te^{\int_0^t\Gamma(t)_u\sqrt{c_u}dW'_u}e^{\int_0^t\int_{\mathbb{R}\backslash\{0\}}\Gamma(t)_uzN'(du,dz)+\Gamma(t)_tx}-Y_te^{\int_0^t\Gamma(t)_u\sqrt{c_u}dW'_u}e^{\int_0^t\int_{\mathbb{R}\backslash\{0\}}\Gamma(t)_uzN'(du,dz)}\\
&= \left(\int_0^{t_{sw}}\int_{\mathbb{R}\backslash\{0\}}(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u)^2 z^2\nu'(dz)du\right)P(T_{sw})_{t-}\left(e^{\Gamma(t^{sw})_t x}-1\right)\\
&\quad+P(T_{sw})_{t-}(\Gamma(T^{sw})_t-\Gamma(t_{sw})_t)^2x^2\\
&\quad+P(T_{sw})_{t-}\left(e^{\Gamma(t^{sw})_t x}-1\right)\left(\int_0^{t_{sw}}\int_{\mathbb{R}\backslash\{0\}}(\Gamma(T^{sw})_u-\Gamma(t_{sw})_u)^2 z^2N'(du,dz)\right)\\
&\quad+P(T_{sw})_{t-}\left(e^{\Gamma(t^{sw})_t x}-1\right)(\Gamma(T^{sw})_t-\Gamma(t_{sw})_t)^2x^2\\
&\quad+Y_te^{\int_0^t\Gamma(t)_u\sqrt{c_u}dW'_u}e^{\int_0^t\int_{\mathbb{R}\backslash\{0\}}\Gamma(t)_uzN'(du,dz)}\left(e^{\Gamma(t^{sw})_t x}-1\right)\\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{sw}}r_udu}V_{sw}(t_{sw})|\mathcal{F}_t\right]\left(e^{\Gamma(t^{sw})_t x}-1\right)+P(T_{sw})_{t-}(\Gamma(T^{sw})_t-\Gamma(t_{sw})_t)^2e^{\Gamma(t^{sw})_t x}x^2\\
&= e^{-\int_0^t r_udu}V_{sw}(t)\left(e^{\Gamma(t^{sw})_t x}-1\right)+P(T_{sw})_{t-}(\Gamma(T^{sw})_t-\Gamma(t_{sw})_t)^2e^{\Gamma(t^{sw})_t x}x^2
\end{aligned}
$$

□

Hedging

In this section, we will construct two separate admissible and self-financing trading strategies to hedge the jump risk of a variance swap. Note that $\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}\left(e^{-\int_0^{t_{sw}}r_udu}V_{sw}(t_{sw})\right)|\mathcal{F}_t\right]$ has two terms. The first term $\left(e^{\Gamma(t_{sw})_tx}-1\right)e^{-\int_0^t r_udu}V_{sw}(t)$ can be hedged by holding $\frac{e^{-\int_0^t r_udu}V_{sw}(t)}{P(t_{sw})_{t-}}$ number of bonds with maturity t_{sw}. This portfolio has a Brownian motion component $e^{-\int_0^t r_udu}V_{sw}(t)\Gamma(t_{sw})_t\sqrt{c_t}dW'_t$ and this component hedges the Brownian motion part.

The second component is a function of the "jump size", $e^{\Gamma(t_{sw})_tx}x^2$. For a given small error, we can choose a large K_t such that

$$e^{\Gamma(t_{sw})_t x} \approx \sum_{k=0}^{K_t} \frac{\Gamma(t_{sw})_t^k x^k}{k!}$$

Since jump sizes are bounded, we can choose K_t such that

$$P(t_{sw})_t \left(\Gamma(T^{sw})_t - \Gamma(t)_t\right)^2 x^2 \left| e^{\Gamma(t_{sw})_t x} - \sum_{k=0}^{K_t} \frac{\Gamma(t_{sw})_t^k x^k}{k!} \right| < x^2 \epsilon_t$$

where $\epsilon_t > 0$.

In this case, we can construct a second admissible self-financing trading strategy and that perfectly replicates a payoff H with

$$D_{t,x}H = P(t_{sw})_t \left(\Gamma(T^{sw})_t - \Gamma(t)_t\right)^2 x^2 \sum_{k=0}^{K_t} \frac{\Gamma(t_{sw})_t^k x^k}{k!}$$

From the previous part, the second trading strategy doesn't have any Brownian motion component ($D_{t,x}$ is a polynomial and coefficient of x is zero).

After we sum these portfolios, let X_t be the value of this final portfolio. This final portfolio approximately replicates the jump risk of a variance swap and it perfectly replicates the additional Brownian motion component. The approximation error for the jump component can be as small as desired. If we choose a different approximation error the for the jump risk, we have a different portfolio with a different approximation error, but we hedge the Brownian motion component in the same way.

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[\left(e^{-\int_0^{t_{sw}} r_u du} X_{t_{sw}} - e^{-\int_0^{t_{sw}} r_u du} V_{sw}(t_{sw})\right)^2\right] \\
= \; & \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^{t_{sw}} \int_{\mathbb{R}\backslash\{0\}} P(t_{sw})_t \left(\Gamma(T^{sw})_t - \Gamma(t)_t\right)^2 x^2 \left(e^{\Gamma(t_{sw})_t x} - \sum_{k=0}^{K_t} \frac{\Gamma(t_{sw})_t^k x^k}{k!}\right) N'(dt, dx)\right)^2\right] \\
= \; & \mathbb{E}^{\mathbb{Q}}\left[\int_0^{t_{sw}} \int_{\mathbb{R}\backslash\{0\}} \left(P(t_{sw})_t \left(\Gamma(T^{sw})_t - \Gamma(t)_t\right)^2 x^2 \left(e^{\Gamma(t_{sw})_t x} - \sum_{k=0}^{K_t} \frac{\Gamma(t_{sw})_t^k x^k}{k!}\right)\right)^2 \nu'(dx)dt\right]
\end{aligned}$$

$$< \quad \mathbb{E}^{\mathbb{Q}}\left[\int_0^{t_{sw}}\int_{\mathbb{R}\setminus\{0\}} x^4\epsilon_t^2 v'(dx)dt\right] = \int_0^{t_{sw}}\epsilon_t^2 dt\int_{\mathbb{R}\setminus\{0\}} x^4 v'(dx)$$

where we use the fact that the jump sizes are bounded, $\int_{\mathbb{R}\setminus\{0\}}(1\wedge x^2)v'(dx) < \infty$ so $\int_{\mathbb{R}\setminus\{0\}} x^4 v'(dx) < \infty$.

We conclude of this section with the following lemma:

Lemma 2.4.2 *Assume that an HJM market is driven by a Lévy process such that :*
(i) For every fixed t, the functions $\{(\Gamma_t)^n\}$ are linearly independent,
(ii) The jump sizes are bounded.
and for all t, $\Gamma(T^{sw})_t - \Gamma(t)_t \neq 0$. Then for a given $\epsilon > 0$, there exists an admissible self-financing trading strategy X_t to hedge a variance swap such that total hedging error is less than ϵ.

Proof: From the previous discussion, we can build an admissible self-financing trading strategy $X(t)$ such that

$$\mathbb{E}^{\mathbb{Q}}\left[\left(e^{-\int_0^{t_{sw}} r_u du}X_{t_{sw}} - e^{-\int_0^{t_{sw}} r_u du}V_{sw}(t_{sw})\right)^2\right] < \int_0^{t_{sw}}\epsilon_t^2 dt\int_{\mathbb{R}\setminus\{0\}} x^4 v'(dx)$$

Choose ϵ_t such that $\int_0^{t_{sw}}\epsilon_t^2 dt\int_{\mathbb{R}\setminus\{0\}} x^4 v'(dx) < \epsilon$. □

The above example gives us a natural way to hedge approximately a variance swap in an HJM market. The hedging strategy we built is based on using ″power-jump assets″.

2.4.2 Hedging In An HJM Market When $L(t)$ Is A Lévy Process

In the previous section, we constructed a self-financing strategy to hedge a variance swap. We will generalize this idea for any contingent claim in an HJM market that is measurable with respect to the σ-algebra generated by $L(t)$. We assume that $L(t)$ is a Lévy process under $\mathbb{P}$ and $Y(t,x)$ is deterministic and Y is only a function of x. So $L(t)$ is a Lévy process under $\mathbb{Q}$.

Proposition 2.4.1 *Assume that an HJM market is driven by a Lévy process such that :*

(i) For every fixed t, the functions $\{(\Gamma_t)^n\}$ are linearly independent,

(ii) The jump sizes are bounded.

Let X be a square integrable contingent claim (under both $\mathbb{P}$ and $\mathbb{Q}$), $\mathcal{G}_{T^*}$ measurable payoff at T^*. Then for a given $\epsilon > 0$, there exists admissible self-financing trading strategy $\phi(t,T) \in \mathcal{A}_{\mathcal{G}}$ to hedge X such that total hedging error is less than ϵ.

Proof: We need to show that power jump assets can be hedged in the market. Let $V = \text{span}\{\{(\Gamma_t), (\Gamma_t)^2, (\Gamma_t)^3, ...\} \backslash \{(\Gamma_t)^m\}$. Using a Gram - Schmidt orthogonalization, we can find an orthonormal sequence $\{v_n\}$ that spans V. We can write $(\Gamma_t)^m$ as a sum of the two functions $(\Gamma_t)^m = \phi_1 + \phi_2$ where $\phi_1 \in V$ and $\phi_2 \in V^{\perp}$. ϕ_1 can be written as a sum of projections of $(\Gamma_t)^m$ on each v_n. Since $\{(\Gamma_t)^n\}$ are linearly independent, $\phi_2 \neq 0$. Let a be a constant $a = m!\left(\int_t^{T^*} \Gamma_t(T)^m \phi_2(T)dT\right)^{-1} = m!\left(\int_t^{T^*} \phi_2(T)^2 dT\right)^{-1}$. From the dominated convergence theorem,

$$\begin{aligned} A_t(a\phi_2) &= \int_t^{T^*} \sum_{n=1}^{\infty} \frac{1}{n!} \Gamma_t(T)^n x^n a\phi_2(T)dT \\ &= a\sum_{n=1}^{\infty} \frac{1}{n!} \int_t^{T^*} \Gamma_t(T)^n x^n \phi_2(T)dT \\ &= ax^m \frac{1}{m!} \int_t^{T^*} \Gamma_t(T)^m \phi_2(T)dT = x^m \end{aligned}$$

Assume $i \geq 2$. Let $\varphi^{(i)}(t,T) = \phi_2(T)/P(T)_t$.

$$\int_0^{T^*} \varphi^{(i)}(t,T)P(T)_{t-}\left(e^{\Gamma(T)_t x} - 1\right) dT = x^i$$

$$\int_0^{T^*} \varphi^{(i)}(t,T)P(T)_{t-}\Gamma(T)_t\sqrt{c_t}dT = \sqrt{c_t}\int_0^{T^*} \phi_2(T)\Gamma(T)_t dT = 0$$

Therefore, $\varphi^{(i)}(t,T)$ perfectly hedges the i-th jump process.

Assume $i = 1$. Let $\varphi^{(1)}(t,T) = \phi_2(T)/P(T)_t$.

$$\int_0^{T^*} \varphi^{(1)}(t,T)P(T)_{t-}\left(e^{\Gamma(T)_t x} - 1\right)dT = x \tag{2.25}$$

$$\int_0^{T^*} \varphi^{(1)}(t,T)P(T)_{t-}\Gamma(T)_t\sqrt{c_t}dT = \sqrt{c_t}\int_0^{T^*} \phi_2(T)\Gamma(T)_t dT = \sqrt{c_t} \tag{2.26}$$

Therefore, $\varphi^{(1)}(t,T)$ perfectly hedges the first jump process. We conclude that market is complete in the limit. □

In the next section, we will generalize this result for non-homogenous Lévy process.

2.4.3 Hedging In An HJM Market When $L(t)$ Is A Non-homogenous Lévy Process

In the previous section, we have shown that the HJM market is complete in the limit when $L(t)$ is a Lévy process under both $\mathbb{P}$ and $\mathbb{Q}$. In this section we relax the assumption that $L(t)$ is a Lévy process under $\mathbb{Q}$. We assume only that $L(t)$ is a Lévy process under $\mathbb{P}$.

In the first part of this section, we also assumed that the (u, Y) pair in the change of measure was non-random. Balland [2002], extended the representation property for regular Levy martingales to regular martingales with independent increments. If power jump assets exist in a market driven by a process with independent increments, and there exists an equivalent martingale measure then the market is complete in the limit. The main result of this section is the following proposition:

Proposition 2.4.2 *Assume that an HJM market is driven by a Lévy process such that :*

(i) For every fixed t*, the functions* $\{(\Gamma_t)^n\}$ *are linearly independent,*

(ii) The jump sizes are bounded,

(iii) (u, Y) *pair is non-random.*

Let X *be a square integrable contingent claim (under both* $\mathbb{P}$ *and* $\mathbb{Q}$*),* $\mathcal{G}_{T^*}$ *measurable payoff at* T^**. Then for a given* $\epsilon > 0$*, there exists an admissible self-financing trading strategy* $\phi(t, T) \in \mathcal{A}_{\mathcal{G}}$ *to hedge* X *such that total hedging error is less than* ϵ*.*

Proof:

First, we need to show that power jump assets can be hedged in the market. Using a generalized Clark-Ocone theorem (Nunno et al. [2010]), we have the following representation for a contingent claim X:

$$
\begin{aligned}
e^{-\int_0^{T^*} r(s,s)ds} F &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{T^*} r(s,s)ds} X\right] + \\
&+ \int_0^{T^*} \mathbb{E}^{\mathbb{Q}}\left[D_t\left(e^{-\int_0^{T^*} r(s,s)ds} X\right) - e^{-\int_0^{T^*} r(s,s)ds} X \int_t^{T^*} D_t u(s) dW'_s \middle| \mathcal{F}_t\right] dW'_t \\
&+ \int_0^{T^*} \int_{\mathbb{R}\setminus\{0\}} \mathbb{E}^{\mathbb{Q}}\left[H D_{t,x}\left(e^{-\int_0^{T^*} r(s,s)ds} X\right) + e^{-\int_0^{T^*} r(s,s)ds} X(H-1) \middle| \mathcal{F}_t\right] N'(dt, dx)
\end{aligned}
$$

where W'_t is a Brownian motion under $\mathbb{Q}$, $N'(dt, dx)$ is a compensated Poisson random measure under under $\mathbb{Q}$,

$$
\begin{aligned}
H(t, x) &= \exp\left(\int_0^t \int_{\mathbb{R}\setminus\{0\}} \left[D_{t,x} Y(s, y) + \ln\left(1 - \frac{D_{t,x}(Y(s, y))}{1 - Y(s, y)}\right)(1 - Y(s, y)\right] \nu(dy) ds\right) \\
&\quad \cdot \exp\left(\ln\left(1 - \frac{D_{t,x}(Y(s, y))}{1 - Y(s, y)}\right) N'(ds, dy)\right)
\end{aligned}
$$

Note that, if (u, Y) pair is non-random, $D_t u_s = 0$ and $H = 1$. The hedging equation becomes

$$e^{-\int_0^{T^*} r(s,s)ds} X = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{T^*} r(s,s)ds} X\right] + \int_0^{T^*} \mathbb{E}^{\mathbb{Q}}\left[D_t\left(e^{-\int_0^{T^*} r(s,s)ds} X\right) dW'_s \middle| \mathcal{F}_t\right] dW'_t$$
$$+ \int_0^{T^*} \int_{\mathbb{R}\setminus\{0\}} \mathbb{E}^{\mathbb{Q}}\left[D_{t,x}\left(e^{-\int_0^{T^*} r(s,s)ds} X\right) \middle| \mathcal{F}_t\right] N'(dt, dx)$$

We have the same hedging integral equations as mentioned in the previous section. The rest of the proof is the same. □

Now, in the second part of this section, we assume that the (u, Y) pair is random. Using the Clark-Ocone theorem, we have the following representation for a contingent claim X:

$$e^{-\int_0^{T^*} r(s,s)ds} X = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{T^*} r(s,s)ds} X\right]$$
$$+ \int_0^{T^*} \mathbb{E}^{\mathbb{Q}}\left[D_t\left(e^{-\int_0^{T^*} r(s,s)ds} X\right) - e^{-\int_0^{T^*} r(s,s)ds} X \int_t^{T^*} D_t u(s) dW'_s \middle| \mathcal{F}_t\right] dW'_t$$
$$+ \int_0^{T^*} \int_{\mathbb{R}\setminus\{0\}} \mathbb{E}^{\mathbb{Q}}\left[H D_{t,x}\left(e^{-\int_0^{T^*} r(s,s)ds} X\right) + e^{-\int_0^{T^*} r(s,s)ds} X(H-1) \middle| \mathcal{F}_t\right] dN'(dt, dx)$$

The hedging integral equations becomes

$$\int_0^{T^*} \phi(t,T) P(T)_{t-} \Gamma(T)_t \sqrt{c_t} dT = \mathbb{E}^{\mathbb{Q}}\left[D_t\left(e^{-\int_0^{T^*} r(s,s)ds} X\right) - e^{-\int_0^{T^*} r(s,s)ds} X \int_t^{T^*} D_t u(s) dW'_s \middle| \mathcal{F}_t\right]$$

$$\int_0^{T^*} \phi(t,T) P(T)_{t-} \left(e^{\Gamma(T)_t x} - 1\right) dT = \mathbb{E}^{\mathbb{Q}}\left[H D_{t,x}\left(e^{-\int_0^{T^*} r(s,s)ds} X\right) + e^{-\int_0^{T^*} r(s,s)ds} X(H-1) \middle| \mathcal{F}_t\right]$$

In general, $L(t)$ is not a non-homogenous Lévy process under $\mathbb{Q}$ and the market is not complete in the limit.

2.5 Conclusion

In this chapter, we explained the hedging problem for different cases. First, we assumed that $L(t)$ is a Lévy process under both $\mathbb{P}$ and $\mathbb{Q}$. If $L(t)$ is a pure jump process and there exists an equivalent martingale measure then an HJM model is complete in the limit. In this case, the hedging strategy obtained is a solution of a Fredholm integral equation of first kind. Second, we showed that, power jump assets can be hedged perfectly in the model. If the Malliavin derivative of a given contingent claim is known then the hedging strategy can be obtained by numerical methods and simulation techniques. We obtained closed form Malliavin derivatives for a variance swap and a caplet.

Third, we considered the same problem when $L(t)$ is a general Lévy process under both $\mathbb{P}$ and $\mathbb{Q}$. Further, we showed that market is complete in the limit and obtained similar results as in Eberlein et al. [2005]. Finally, we proved that market is complete when $L(t)$ is a non-homogenous Lévy process under $\mathbb{Q}$.

Our approach is based on power jump assets and Malliavin calculus. With this approach, we not only obtained conditions for a complete in the limit market but also we developed hedging strategies. Björk et al. [1997] proved that the HJM market can not be ”complete” but can be at most ”complete in the limit”. A unique measure is a necessary and sufficient condition for a complete in the limit HJM market. Using our approach, all we need to do is to check whether power jump assets can be hedged in the market. If we can replicate them, the market is complete in the limit and the equivalent martingale measure is unique.

CHAPTER 3

HEDGING IN A EXPONENTIAL LÉVY MARKETS

3.1 Introduction

Analysis of financial times series has shown that skewness and a kurtosis of assets returns are different from the Gaussian distribution. In general, we observe a negative skewness and kurtosis which is greater than three (for a Gaussian distribution, kurtosis is three). The excess kurtosis indicates that the tails of the observed returns are thicker than the tails of the Gaussian distribution (Figueroa-López [2012]). This evidence shows that extreme daily returns in financial markets have higher probability than when asset returns follow a Gaussian distribution. Equivalently, this translates to the fact that geometric Brownian motion is not a correct model for asset price processes. For this reason, an extension of the classical Black-Scholes model, the exponential Lévy model in which assets are driven by a Lévy process is proposed. Such an extension keeps important properties of Brownian motion, a Lévy process has also independent increment, it is a stationary, and a Markov process.

When we depart from the Black-Scholes framework, and use a Lévy process, our market is either incomplete or complete in the limit. In the finance industry, investors assume that the markets are arbitrage free but not necessarily complete. An incomplete market is not the only difficulty of a Lévy model. Another very important limitation, unlike the Black Scholes framework, is that a closed form solution for call option price cannot be derived using the Lévy model. Under certain assumptions, Carr et al. [1999] suggested a general pricing scheme for European call options in an exponential Lévy markets using Fourier analysis. The authors provide a formula in terms of the inverse Fourier transform to evaluate vanilla options prices under exponential Lévy models that has an analytical characteristic function for the return of underlying asset. Although their approach gives us a numerical method to obtain the option price when the asset price is driven by an exponential Lévy process, the question of hedging continues to be unresolved and open.

In incomplete markets, an equivalent martingale is not unique. When the market is driven by a Lévy process and has finite number of assets, the market is not complete. We will extend such a market by a continuum of call options to obtain a complete market in the limit. Our first step is to define the extended market. Throughout, we work under an equivalent martingale measure and assume that there are infinitely many call options in the market. We also develop a hedging strategy by using Malliavin calculus. Here we show that, under certain conditions, a dense subset of square integrable contingent claims can be hedged perfectly and this shows that market is complete in the limit. Further, we combine this result with the result in Jarrow et al. [1999], and conclude that this martingale measure is also unique.

In this chapter, we focus on hedging in an exponential Lévy market enlarged by call options. The rest of the chapter is organized as follows. First, we provide a review of exponential Lévy markets and pricing. Next, we enlarge the market by using call options. Further, we develop hedging strategies in terms of Malliavin calculus and Fourier transforms. We operate under the assumption that the market is driven by a pure jump process. Finally, we extend our results for the market driven by a general Lévy process.

3.1.1 Exponential Lévy Markets and Pricing

We assume that under an equivalent martingale measure $\mathbb{Q}$, the asset price S_t is driven by a pure jump process L_t. We assume that the market has a finite time horizon $[0, T^*]$.

Definition 3.1.1 *Let S_t be the spot price of some underlying asset driven by a Lévy process L_t under the equivalent martingale measure $\mathbb{Q}$*

$$S_t = S_0 e^{L_t} \tag{3.1}$$

where L_t is a pure jump Lévy process with bounded jump sizes. Assume that the jump sizes are in $[M_1, M_2]$ where M_1 and M_2 are both finite. Also, we assume that the risk free rate r is a constant.

Initially, we assume that the market consists of the asset and a money marketing account. Later, we enlarge it to obtain a complete market in the limit. In general, it is not possible to find a closed form for the price of a vanilla call option. Under the equivalent martingale measure, the price of a call option at time t is given by

$$C_t(K) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^+ \mid \mathcal{F}_t\right] \tag{3.2}$$

where K is the strike price and T is the maturity. Unless we know the density of S_T, only simulation or numerical methods can be used to approximate this conditional expectation.

Carr et al. [1999] proposed a formula to obtain the Fourier transform of a call option price under certain conditions. The advantage of their approach is that one can use one of many of algorithms developed for inverse Fourier transforms. These methods are also used extensively in various areas of engineering and science.

Without loss of generality, they assume that $S_0 = 1$. Let k be equal to $\ln(K)$, the logarithm of the strike price, and $q(u)$ be the density of L_T under $\mathbb{Q}$. The call price can be expressed as

$$C_0(K) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}(S_T - K)^+\right] = \int_K^{\infty} e^{-rT}\left(e^u - e^k\right)q(u)du$$

We define a new function $C_0(e^k)$, the call price with respect to k. $C_0(e^k)$ is not integrable since

$$\lim_{k\to-\infty} C_0(e^k) = \lim_{K\to 0} C_0(K) = S_0$$

and its Fourier transform does not exist.

Carr et al. [1999] solved this problem by using a coefficient $\alpha > 0$ such that $e^{\alpha k}C_0(e^k)$ is integrable and moderately decreasing (Stein et al. [2003]). Let

$$c(k) = e^{\alpha k}C_0(e^k) \tag{3.3}$$

We assume that its Fourier transform exists and

$$\Psi(v) = \mathfrak{I}(e^{\alpha k}c(k))(v) = \int_{-\infty}^{\infty} e^{ivk}c(k)dk = \Psi_{Re}(v) + i\,\Psi_{Im}(v) \tag{3.4}$$

where

$$\Psi_{Re}(v) = \int_{-\infty}^{\infty} \cos(vk)c(k)dk \tag{3.5}$$

$$\Psi_{Im}(v) = \int_{-\infty}^{\infty} \sin(vk)c(k)dk \tag{3.6}$$

Carr et al. [1999] express the Fourier transform of the modified price in terms of the characteristic function of L_T under $\mathbb{Q}$

$$\begin{aligned}\Psi(v) &= e^{-rT}\int_{-\infty}^{\infty}\int_{k}^{\infty} e^{ivk}e^{\alpha k}(e^u - e^k)q_T(u)dudk \\ &= e^{-rT}\int_{-\infty}^{\infty}\int_{-\infty}^{u} e^{ivk}e^{\alpha k}(e^u - e^k)q_T(u)dkdu\end{aligned}$$

$$
\begin{aligned}
&= e^{-rT}\int_{-\infty}^{\infty} \frac{e^{ivu+\alpha u+u}}{\alpha+iv} - \frac{e^{ivu+\alpha u+u}}{\alpha+iv+1})q_T(u)du \\
&= e^{-rT}\int_{-\infty}^{\infty} \frac{e^{ivu+\alpha u+u}}{\alpha^2+\alpha-v^2+iv(2\alpha+1)} q_T(u)du
\end{aligned}
$$

where we use Fubini's theorem and change the order of integrals. They obtain the following formula:

$$
\Psi(v) = \frac{e^{-rT}\Phi(v-i(\alpha+1))}{\alpha^2+\alpha-v^2+iv(2\alpha+1)} \tag{3.7}
$$

where

$$
\Phi(v) = \mathbb{E}^{\mathbb{Q}}\left[e^{L_T}\right] \tag{3.8}
$$

Note that, we use $e^{\alpha k}$ $\alpha > 0$ to obtain a moderately decreasing function for large negative k values. We need to add a constraint on α such that $c(k)$ is also a moderately decreasing function for large positive k values. Otherwise, the Fourier transform doesn't exist. In other words, $\Psi(0) < \infty$ which is equivalent to

$$
\mathbb{E}^{\mathbb{Q}}\left[S_T^{\alpha+1}\right] < \infty \tag{3.9}
$$

From the inverse Fourier transform, we obtain the modified call price

$$
c(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ivk}\Psi(v)dv = c_l(k) + c_r(k) \tag{3.10}
$$

where

$$
\begin{aligned}
c_l(k) &= \frac{1}{2\pi}\int_{-\infty}^{0} e^{-ivk}\Psi(v)dv = \frac{1}{2\pi}\int_{0}^{\infty} e^{ivk}\Psi(-v)dv \\
&= \frac{1}{2\pi}\int_{0}^{\infty}\Big[\cos(vk)\Psi_{Re}(v) + \sin(vk)\Psi_{Im}(v) + i\Big(\sin(vk)\Psi_{Re}(v) - \cos(vk)\Psi_{Im}(v)\Big)\Big]dv
\end{aligned}
$$

$$
\begin{aligned}
c_r(k) &= \frac{1}{2\pi}\int_{0}^{-\infty} e^{-ivk}\Psi(v)dv \\
&= \frac{1}{2\pi}\int_{0}^{\infty}\Big[\cos(vk)\Psi_{Re}(v) + \sin(vk)\Psi_{Im}(v) - i\Big(\sin(vk)\Psi_{Re}(v) - \cos(vk)\Psi_{Im}(v)\Big)\Big]dv
\end{aligned}
$$

We see that $c_l(k)$ and $c_r(k)$ are complex conjugates of each other and that $c(k)$ is a real-valued function. Therefore,

$$c(k) = \Re\mathrm{e}\{c(k)\} = \Re\mathrm{e}\{c_l(k)\} + \Re\mathrm{e}\{c_r(k)\} = 2\Re\mathrm{e}\{c_r(k)\}$$

We can compute the call price as follows:

$$C_0(e^k) = \frac{e^{-\alpha k}}{\pi}\Re\mathrm{e}\Big[\int_0^\infty e^{-ivk}\Psi(v)dv\Big] \tag{3.11}$$

In general, the density of L_T under the equivalent martingale measure is unknown, but the Lévy Khintchine representation (Sato [1999]) gives us the equation for the Fourier transform.

Before, we complete this section, we prove a result about the absolute value of the Fourier transform of $c_t(k)$.

Lemma 3.1.1 $\Psi(v) = \frac{e^{-rT}\Phi(v-i(\alpha+1))}{\alpha^2+\alpha-v^2+iv(2\alpha+1)}$ *is never zero where*

$$\Phi(v - i(\alpha + 1)) = \mathbb{E}^{\mathbb{Q}}\left[e^{(iv+\alpha+1)L_T}\right]$$

Proof: Let $f_t(v) = \mathbb{E}^{\mathbb{Q}}\left[e^{(iv+\alpha+1)L_t}\right]$. As L_t has stationary and independent increments

$$\begin{aligned}
f_{t+s}(v) &= \mathbb{E}^{\mathbb{Q}}\left[e^{(iv+\alpha+1)L_{t+s}}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{(iv+\alpha+1)(L_{t+s}-L_t)} \cdot e^{(iv+\alpha+1)L_t}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{(iv+\alpha+1)(L_{t+s}-L_t)}\right] \cdot \mathbb{E}^{\mathbb{Q}}\left[e^{(iv+\alpha+1)L_t}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{(iv+\alpha+1)L_s}\right] \cdot \mathbb{E}^{\mathbb{Q}}\left[e^{(iv+\alpha+1)L_t}\right] \\
&= f_t(v) \cdot f_s(v)
\end{aligned}$$

Therefore, $|f_{t+s}(v)| = |f_t(v)| \cdot |f_s(v)|$ and $f_0(v) = 1$. From Cauchy's functional equation, $f_t(v) = e^{tg(v)}$ for some function g. We conclude that $f_t(v)$ is never zero. □

3.1.2 Hedging In An Exponential Lévy Market - Pure Jump Case

In this section, we assume that L_t is a pure jump process under $\mathbb{P}$. $\mathcal{F}_t$, $t \in [0, T^*]$ is the augmented filtration generated by L_t. Therefore, $\mathcal{F}_t$ is a right continuous filtration that satisfies the usual conditions (Medvegyev [2007]). We assume that all moments of L_t exist. Then L_t admits the following The LévyItô decomposition (Sato [1999]):

$$L_t = at + \int_0^t \int_{\mathbb{R}\backslash\{0\}} xN(ds, dx) \tag{3.12}$$

where $N(dt, dx)$ is a compensated Poisson random measure under $\mathbb{P}$.
We enlarge the market by European call options with the same maturity T^*. Assume that for every strike $K \in [K_1, K_2]$, there is a call option trading in the market. Let $C_t(S_t, K)$ be the price of a call option at time t. We will construct self-financing trading strategies by using these call options to hedge contingent claims. First, we review Girsanov's and Clark–Ocone theorems.

We use Girsanov's theorem to define an equivalent martingale measure:

Proposition 3.1.1 *(Girsanov's theorem for pure jump processes) Let* $Y(t, x) \leq 1$, $t \in [0, T^*]$, $x \in \mathbb{R}$

0 *such that*

$$\int_0^T \int_{\mathbb{R}\backslash\{0\}} \Big(|\ln(1 + Y(t, x))| + Y^2(t, x)\Big)\, \nu(dx)dt < \infty \ \ \mathbb{P}\ a.s.$$

Let $\mathbb{Q}$ *be a measure on* $\mathcal{F}_{T^*}$ *defined as*

$$d\mathbb{Q}(w) = Z(w, T^*)d\mathbb{P}(w)$$

where

$$Z(t) = \exp\left(\int_0^t \int_{\mathbb{R}\setminus\{0\}} [\ln(1 - Y(s,x)) + Y(s,x)]\, \nu(dx)ds + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \ln(1 - Y(s,x)N(ds,dx)\right)$$

and assume that $Z(T^*)$ satisfies the Novikov condition

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_0^{T^*} \int_{\mathbb{R}\setminus\{0\}} [(1 - Y(t,x))\ln(1 - Y(t,x)) + Y(t,x)]\, \nu(dx)dt\right)\right] < \infty$$

Define

$$N'(dt,dx) = Y(t,x)\nu(dx)dt + N(dt,dx)$$

then $N'(dt,dx)$ is a compensated random measure under $\mathbb{Q}$.

Later, in the derivation of a hedging strategy, we invoke the Clark-Ocone theorem, which allows us to represent a contingent claim as an integral. We also use it to represent call option value as an integral.

Proposition 3.1.2 *(Clark–Ocone theorem under change of measure for pure jump Lévy processes) Let $F \in L^2(\mathbb{P} \cap \mathbb{Q})$ be a $\mathcal{F}_{T^*}$ measurable random variable. Then the representation of F with respect $N'(dt,dx)$ is as follows:*

$$F = \mathbb{E}^{\mathbb{Q}}[F] + \int_0^{T^*} \int_{\mathbb{R}\setminus\{0\}} \mathbb{E}^{\mathbb{Q}}\left[F(H-1) + HD_{t,x}F \mid \mathcal{F}_t\right] N'(dt,dx)$$

where $N'(dt,dx)$ is a compensated Poisson random measure under $\mathbb{Q}$ and

$$\begin{aligned} H = \exp \quad \Bigg(& \int_0^{T^*} \int_{\mathbb{R}\setminus\{0\}} \left[D_{t,x}Y(s,x) + \ln\left(1 - \frac{D_{t,x}Y(s,x)}{1 - Y(s,x)}\right)(1 - Y(s,x))\right] \nu(dx)ds \\ + & \int_0^{T^*} \int_{\mathbb{R}\setminus\{0\}} \ln\left(1 - \frac{D_{t,x}Y(s,x)}{1 - Y(s,x)}\right) N'(ds,dx)\Bigg) \end{aligned}$$

In the representation of F, $Y(t,x)$ is the process in Girsanov's theorem to obtain an equivalent measure $\mathbb{Q}$.

Proposition 3.1.3 *Assume that Y is deterministic. Then the representation of F with respect to $N'(dt, dx)$ is as follows:*

$$F = \mathbb{E}^{\mathbb{Q}}[F] + \int_0^{T^*} \int_{\mathbb{R}\setminus\{0\}} \mathbb{E}^{\mathbb{Q}} \left[D_{t,x}F \mid \mathcal{F}_t\right] N'(dt, dx) \tag{3.13}$$

In this section, we assume that $Y(t, x)$ is deterministic. We proceed to the Malliavin derivative of a call option.

Proposition 3.1.4 *Let F be discounted payoff of a call option*

$$F = e^{-rT^*} [S_{T^*} - K]^+$$

then $F \in \mathbb{D}_{1,2}$, its Malliavin derivative is equal to

$$D_{t,x}F = e^{-rT^*} \left([e^x S_{T^*} - K]^+ - [S_{T^*} - K]^+\right) \tag{3.14}$$

and

$$\mathbb{E}^{\mathbb{Q}} \left[D_{t,x}F|\mathcal{F}_t\right] = e^{-rt} \left(e^x C_t(S_t, Ke^{-x}) - C_t(S_t, K)\right) \tag{3.15}$$

Proof: First note that r is constant and e^{-rT^*} is a constant. Let $H = S_{T^*} - K$. F is equal to $e^{-rT^*}(H)^+$ and it is a continuous function of H where $H \in \mathbb{D}_{1,2}$. Using the chain rule (Nunno et al.[2010]),

$$\begin{aligned}
D_{t,x}F &= e^{-rT^*} \left([H + D_{t,x}H]^+ - [H]^+\right) \\
&= e^{-rT^*} \left([S_{T^*} - K + S_{T^*} (e^x - 1))]^+ - [S_{T^*} - K]^+\right) \\
&= e^{-rT^*} \left([e^x S_{T^*} - K]^+ - [S_{T^*} - K]^+\right) \\
&= e^{-rT^*} \left(e^x [S_{T^*} - Ke^{-x}]^+ - [S_{T^*} - K]^+\right)
\end{aligned}$$

□

Before we derive a hedging strategy, we define the set of admissible self-financing trading strategies.

Definition 3.1.2 *A stochastic process $\phi : [0, T^*] \times [K_1, K_2] \times \Omega \to \mathbb{R}$ is called an admissible strategy if*

- *ϕ is $\mathcal{B}([0, T^*]) \otimes \mathcal{B}[K_1, K_2]) \otimes \mathcal{F}_{T^*}$ measurable.*
- *$\phi(\cdot, K)$ is a predictable and adapted process (with respect to $\mathcal{F}$) for every $K \in [K_1, K_2]$.*
- *$\mathbb{E}^{\mathbb{Q}}\left[\int_0^{T*} \phi(t, K)^2 \int_{\mathbb{R}\setminus\{0\}} (e^x C_t(S_t, Ke^{-x}) - C_t(S_t, K))^2\, Y(x) F(dx) dt\right] < \infty$ for all $K \in [K_1, K_2]$.*
- *$\int_{K_1}^{K_2} \left| \int_0^{T^*} \int_{\mathbb{R}\setminus\{0\}} \phi(t, K) e^{-rt} (e^x C_t(S_t, Ke^{-x}) - C_t(S_t, K))\, N'(dt, dx) \right| dK < \infty$ almost surely.*

The set of all $\mathcal{F}$-admissible self-financing trading strategies [1] *is denoted by $\mathcal{A}_{\mathcal{F}}$.*

The portfolio at time t can be approximated by a sum of n portfolios $X_{t,1}, X_{t,2}, X_{t,3}, \ldots, X_{t,n}$. Each portfolio $X_{t,i}$ has only one call option (strike $K_i = K_1 + i\Delta K$, $\Delta K = (K_2 - K_1)/n$), and its quantity in the portfolio is $\phi(t, K_i)\Delta K$.

Proposition 3.1.5 *Assume that Y is deterministic. Suppose F is a square integrable (under both $\mathbb{P}$ and $\mathbb{Q}$), $\mathcal{F}_{T^*}$ measurable. The self-financing strategy $\phi(t, K) \in \mathcal{A}_{\mathcal{F}}$ perfectly hedges the final payoff F if and only if*

$$\int_{K_1}^{K_2} \phi(t, K) \left(e^x C_t(S_t, Ke^{-x}) - C_t(S_t, K)\right) dK = e^{-r(T^*-t)} \mathbb{E}^{\mathbb{Q}} \left[D_{t,x}(F) \,|\mathcal{F}_t\right] \qquad (3.16)$$

for all $t \in [0, T^*]$.

Proof: From the generalized Clark–Ocone theorem (Nunno et al. [2010]), the discounted payoff can be written as:

$$e^{-rT^*} F = \mathbb{E}^{\mathbb{Q}}[e^{-rT^*} F] + e^{-rT^*} \int_0^{T^*} \int_{\mathbb{R}\setminus\{0\}} \mathbb{E}^{\mathbb{Q}} \left[D_{t,x} F \mid \mathcal{F}_t\right] N'(dt, dx)$$

[1] Let $\phi(t, K)$, $t \in [0, T^*]$ and $K \in [K_1, K_2]$ be the number of call options with strike price K in the portfolio at time t, and V_t be the value of the portfolio at time t. A trading strategy $\phi(t, K)$, $t \in [0, T^*]$ and $K \in [K_1, K_2]$, is a self-financing trading strategy if $d(e^{-rt} V_t) = \int_{K_1}^{K_2} \phi(t, K) d(e^{-rt} C_t(S_t, K)) dK$.

The discounted portfolio value at maturity is equal to

$$
\begin{aligned}
e^{-rT^*}X(T^*) &= X(0) + \int_{K_1}^{K_2}\left(\int_0^{T^*} \phi(t,K)d(e^{-rt}C_t(S_t,K))\right)dK \\
&= X(0) + \int_{K_1}^{K_2}\left(\int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} \phi(t,K)\mathbb{E}^{\mathbb{Q}}\left[e^{-rT^*}D_{t,x}\left((S_{T^*}-K)^+\right) \mid \mathcal{F}_t\right]N'(dt,dx)\right)dK \\
&= X(0) + \int_{K_1}^{K_2}\left(\int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} \phi(t,K)e^{-rt}\left(e^xC_t(S_t,Ke^{-x}) - C_t(S_t,K)\right)N'(dt,dx)\right)dK
\end{aligned}
$$

Since $\int_{K_1}^{K_2}\left|\int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} \phi(t,K)e^{-rt}\left(e^xC_t(S_t,Ke^{-x}) - C_t(S_t,K)\right)N'(dt,dx)\right|dK < \infty$ almost surely, order of the integrals can be changed

$$
\begin{aligned}
&\int_{K_1}^{K_2}\left(\int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} \phi(t,K)e^{-rt}\left(e^xC_t(S_t,Ke^{-x}) - C_t(S_t,K)\right)N'(dt,dx)\right)dK \\
&= \int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}}\left(\int_{K_1}^{K_2} \phi(t,K)e^{-rt}\left(e^xC_t(S_t,Ke^{-x}) - C_t(S_t,K)\right)dK\right)N'(dt,dx)
\end{aligned}
$$

Then the discounted portfolio value becomes

$$
e^{-rT^*}X(T^*) = X(0) + \int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}}\left(\int_{K_1}^{K_2} \phi(t,K)e^{-rt}\left(e^xC_t(S_t,Ke^{-x}) - C_t(S_t,K)\right)dK\right)N'(dt,dx)
$$

The self-financing strategy $\phi(t,K) \in \mathcal{A}_{\mathcal{F}}$ should satisfy the following integral equation:

$$
\int_{K_1}^{K_2} \phi(t,K)\left(e^xC_t(S_t,Ke^{-x}) - C_t(S_t,K)\right)dK = e^{-r(T^*-t)}\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}(F)\,|\mathcal{F}_t\right]
$$

□

Let $h(K,x) = e^xC_t(S_t,Ke^{-x}) - C_t(S_t,K)$ be the kernel in the integral equation. A call option's value is a continuous monotone decreasing function of its strike price and the kernel is a continuous function of both x and K. Therefore, it is a compact operator. A compact operator cannot have a bounded inverse and the integral equation 3.16 doesn't have a solution for all contingent claims (Kress [1999]). In other words, the market is either incomplete or complete in the limit.

The next proposition is the main result of this section. We show that, if there is a call option in the enlarged market with strike e^x where x is a possible jump size, every contingent claim can be approximately hedged. In other words, for every possible jump size, we need a call option to obtain a hedging strategy.

Proposition 3.1.6 *Assume that Y is deterministic, jump sizes are bounded and in $[M_1, M_2] \subseteq [\ln(K_1), \ln(K_2)]$.*

Let F be a square integrable contingent claim. Then either

(1) F can be replicated perfectly or

(2) There exists a sequence of contingent claims F_n such that F_n can be perfectly hedged and $F_n \to F$ in L^2.

Proof: Since jump sizes are bounded, all moments of the Lévy process with degree larger than or equal to one exist. Therefore, for every $n \geq 1$, $Z^{(n)}(T^*) = \int_0^{T^*} \int_{\mathbb{R}\setminus\{0\}} x^n N'(dt, dx)$ is a square integrable random variable. From the previous proposition 3.1.5, $Z^{(n)}(T^*)$ can be hedged if and only if

$$\int_{K_1}^{K_2} \phi(t, K) \left(e^x C_t(S_t, Ke^{-x}) - C_t(S_t, K)\right) dK = e^{-r(T^*-t)} x^n$$

for all $x \in [M_1, M_2]$. Change the variable from K to $k = \ln(K)$.

$$\int_{\ln(K_1)}^{\ln(K_2)} \phi(t, e^k) \left(e^x C_t(S_t, e^{k-x}) - C_t(S_t, e^k)\right) e^k dk = e^{-r(T^*-t)} x^n$$

Let $h(x) = \int_{\ln(K_1)}^{\ln(K_2)} \phi(t, e^k) e^x C_t(S_t, e^{k-x}) e^k dk$. This integral equation can be written as

$$h(x) - h(0) = e^{-r(T^*-t)} x^n$$

and $h(x) = e^{-r(T^*-t)} x^n + \eta_t$ where η_t is a constant or an $\mathcal{F}_t$ measurable random variable. Let

$$c_t(k) = C_t(S_t, e^k)$$

$$\varphi_t(k) = e^k \phi(t, e^k)$$

$$f_t(x) = \begin{cases} e^{-x}\left[e^{-r(T^*-t)}x^n + \eta_t\right] & \text{if } x \text{ is in } [M_1, M_2] \\ 0 & \text{otherwise} \end{cases}$$

The solution to the hedging problem is the solution of the first order Fredholm integral equation

$$\int_{\ln(K_1)}^{\ln(K_2)} \varphi_t(k)c_t(k-x)dk = f_t(x) \tag{3.17}$$

We take the cosine and sine transform of $c_t(k)$ with respect to k. As $k \to -\infty$, $c_t(k) \to S_t$ the Fourier transform of $c_t(k)$ doesn't exist. In order to overcome this problem, Carr et al. [1999] multiply $c_t(k)$ by an exponential $e^{\alpha k}$ where α is a positive constant. In this proof, we do not follow their approach. The integral equation variable k is in $[\ln(K_1), \ln(K_2)]$ and $x \in [M_1, M_2]$. We can modify or truncate $c_t(u)$ for large and small u values such that the Fourier transform of c_t exists and we have the same integral equation. We assume that

$$c_t(k) = A(k)\, C_t(S_t, e^k)$$

where $A(k)$ is a function of k such that the Fourier transform of c_t exists and $A(k) = 1$ for $k \in [\ln(K_1) - M_2, \ln(K_2) - M_1]$. We can choose $A(k)$ with the compact support

$$A(k) = \begin{cases} 0 & \text{if } k \leq a-1 \\ k-(a-1) & \text{if } a-1 < k < a \\ 1 & \text{if } a \leq k \leq b \\ -k+b+1 & \text{if } b < k < b+1 \\ 0 & \text{if } k \geq b+1 \end{cases}$$

where $a \leq \ln(K_1) - M_2$ and $\ln(K_2) - M_1 \leq b$. Therefore, without loss of generality, we can assume that $c_t(k)$ is a moderately decreasing function of k (Stein et al. [2003]).

Note that, this integral equation is also a convolution type integral equation. At time t, the kernel of the operator is $c_t(k-x)$ and it is a continuous function of both k and x. In other words, the operator is compact and the integral equation 3.17 cannot have solution for any right hand side. This shows that, our market cannot be complete, it can be complete in the limit at most. Let

$$\Im_c(c_t)(v) = \int_{-\infty}^{\infty} c_t(u)cos(\pi vu)du$$

$$\Im_s(c_t)(v) = \int_{-\infty}^{\infty} c_t(u)sin(\pi vu)du$$

be the Fourier cosine and sine transforms of c_t.

Multiplying both sides of the integral equation by $cos(\pi v(x-z)f_t(x)$ and taking the integral:

$$\begin{aligned}
&\int_{-\infty}^{\infty} cos(\pi v(x-z))f_t(x)dx = \int_{\ln(K_1)}^{\ln(K_2)} cos(\pi v(x-z))f_t(x)dx \\
&\int_{-\infty}^{\infty} cos(\pi v(x-z)) \int_{\ln(K_1)}^{\ln(K_2)} \varphi_t(k)c_t(k-x)dkdx \\
= &\int_{-\infty}^{\infty}\int_{\ln(K_1)}^{\ln(K_2)} c_t(y)cos(\pi v(k-y-v))\varphi_t(k)dkdy \\
= &\int_{-\infty}^{\infty}\int_{\ln(K_1)}^{\ln(K_2)} c_t(y)\left[cos(\pi v(k-v))cos(\pi vy) + sin(\pi v(k-v))sin(\pi vy)\right]\varphi_t(k)dkdy \\
= &\ \Im_c(c_t)(v)\int_{\ln(K_1)}^{\ln(K_2)} cos(\pi v(k-v))\varphi_t(k)dk + \Im_s(c_t)(v)\int_{\ln(K_1)}^{\ln(K_2)} sin(\pi v(k-v))\varphi_t(k)dk
\end{aligned}$$

Multiplying both sides of the integral equation by $sin(\pi v(x-z)f_t(x)$ and taking the integral:

$$\begin{aligned}
&\int_{-\infty}^{\infty} sin(\pi v(x-z))f_t(x)dx = \int_{\ln(K_1)}^{\ln(K_2)} sin(\pi v(x-z))f_t(x)dx \\
&\int_{-\infty}^{\infty} sin(\pi v(x-z)) \int_{\ln(K_1)}^{\ln(K_2)} \varphi_t(k)c_t(k-x)dkdx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{\ln(K_1)}^{\ln(K_2)} c_t(y) sin(\pi v(k-y-v))\varphi_t(k) dk dy \\
&= \int_{-\infty}^{\infty}\int_{\ln(K_1)}^{\ln(K_2)} c_t(y)\left[sin(\pi v(k-v))cos(\pi v y) - cos(\pi v(k-v))sin(\pi v y)\right]\varphi_t(k) dk dy \\
&= \Im_s(c_t)(v)\int_{\ln(K_1)}^{\ln(K_2)} sin(\pi v(k-v))\varphi_t(k)dk - \Im_s(c_t)(v)\int_{\ln(K_1)}^{\ln(K_2)} cos(\pi v(k-v))\varphi_t(k)dk
\end{aligned}
$$

We obtain

$$
\int_{\ln(K_1)}^{\ln(K_2)} cos(\pi v(k-z))\varphi_t(k)dk = \int_{\ln(K_1)}^{\ln(K_2)} \frac{\Im_c(c_t)(v)cos(\pi v(x-z)) - \Im_s(c_t)(v)sin(\pi v(x-z)))}{(\Im_c(c_t)(v))^2 + (\Im_s(c_t)(v))^2} f_t(x)dx
$$

From Fourier's theorem,

$$
\int_{-\infty}^{\infty}\int_{\ln(K_1)}^{\ln(K_2)} cos(\pi v(k-z)\varphi_t(z)dz dv = \begin{cases} \varphi_t(k) & \text{if } k \in [\ln(K_1), \ln(K_2)] \\ 0 & \text{otherwise} \end{cases}
$$

We conclude that,

$$
\varphi_t(k) = \int_{-\infty}^{\infty}\int_{\ln(K_1)}^{\ln(K_2)} \frac{\Im_c(c_t)(v)cos(\pi v(x-k)) - \Im_s(c_t)(v)sin(\pi v(x-k)))}{(\Im_c(c_t)(v))^2 + (\Im_s(c_t)(v))^2} f_t(x)dx dv
$$

and the integral is zero for all $k \notin [\ln(K_1), \ln(K_2)]$. □

In the proof of the previous proposition 3.1.5, we use a general function $A(k)$ to modify $c(k)$. This function $A(k)$ can be a function with compact support or a function such that $A(k) \cdot C_t(S_t, e^k)$ is moderately decreasing. In the next proposition, we use $e^{\alpha k}$ as $A(k)$ and obtain a solution in terms of the characteristic function of the Lévy process L_t.

Proposition 3.1.7 *Assume that Y is deterministic, jump sizes are bounded and in $[M_1, M_2] \subseteq [\ln(K_1), \ln(K_2)]$. Let F be a square integrable contingent claim, and f_t be a function of x defined as*

$$
f_t(x) = \begin{cases} e^{-x(\alpha+1)}\left[e^{-r(T^*-t)}\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}(F)\,|\mathcal{F}_t\right] + \eta_t\right] & \textit{if } x \textit{ is in } [\ln(K_1), \ln(K_2)] \\ 0 & \textit{otherwise} \end{cases} \tag{3.18}
$$

and η_t is a constant. Assume that f_t satisfies Dirichlet's conditions. Then F can be replicated perfectly and the quantity of the call options in the hedging strategy at time t, $\phi(t,K) = \phi(t,e^k)$, is given by

$$\phi(t,e^k) = e^{(\alpha-1)k}\int_{-\infty}^{\infty}\int_{\ln(K_1)}^{\ln(K_2)} \frac{\Psi_{Re}(\pi v)cos(\pi v(x-k)) - \Psi_{Im}(\pi v)sin(\pi v(x-k)))}{|\Psi(\pi v)|^2} f_t(x)dxdv \tag{3.19}$$

where

$$\Psi(v) = (S_t)^{\alpha+1}\cdot e^{iv\ln(S_t)}\cdot\frac{e^{-rT}\Phi(v-i(\alpha+1))}{\alpha^2+\alpha-v^2+iv(2\alpha+1)} \tag{3.20}$$

$$\Phi(v) = \mathbb{E}^{\mathbb{Q}}\left[e^{ivL_{T-t}}\right] \tag{3.21}$$

$$\mathbb{E}^{\mathbb{Q}}[S_T^{(\alpha+1)}] < \infty$$

Proof: We do not repeat all the steps. We sketch an outline of the proof. Hedging is the solution of

$$\int_{K_1}^{K_2}\phi(t,K)\left(e^x\cdot C_t(S_t,Ke^{-x}) - C_t(S_t,K)\right)dK = e^{-r(T^*-t)}\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}(F)\,|\mathcal{F}_t\right]$$

It is equivalent to

$$\int_{\ln(K_1)}^{\ln(K_2)}\phi(t,e^k)e^x\cdot C_t(S_t,e^{k-x})\cdot e^k dk = e^{-r(T^*-t)}\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}(F)\,|\mathcal{F}_t\right] + \eta_t$$

Let

$$c_t(k) = e^{\alpha k}\cdot C_t(S_t,e^k)$$

$$\varphi_t(k) = e^{(1-\alpha)k}\phi(t,e^k)$$

$$f_t(x) = \begin{cases} e^{-x(\alpha+1)}\left[e^{-r(T^*-t)}\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}(F)\,|\mathcal{F}_t\right]+\eta_t\right] & \text{if } x \text{ is in } [\ln(K_1),\ln(K_2)] \\ 0 & \text{otherwise} \end{cases}$$

where we choose α such that $\mathbb{E}^{\mathbb{Q}}[S_T^{(\alpha+1)}] < \infty$ (Carr et al. [1999]).

The integral equation becomes

$$\int_{\ln(K_1)}^{\ln(K_2)}\varphi_t(k)c_t(k-x)dk = f_t(x)$$

and its solution is equal to

$$\varphi_t(k) = \int_{-\infty}^{\infty}\int_{\ln(K_1)}^{\ln(K_2)} \frac{\Psi_{Re}(\pi v)cos(\pi v(x-k)) - \Psi_{Im}(\pi v)sin(\pi v(x-k)))}{|\Psi(\pi v)|^2} f_t(x)dxdv$$

and

$$\phi(t, e^k) = e^{(\alpha-1)k}\int_{-\infty}^{\infty}\int_{\ln(K_1)}^{\ln(K_2)} \frac{\Psi_{Re}(\pi v)cos(\pi v(x-k)) - \Psi_{Im}(\pi v)sin(\pi v(x-k)))}{|\Psi(\pi v)|^2} f_t(x)dxdv$$

where

$$\begin{aligned}
\Psi(v) &= \Im(c(k))(v) \\
&= \Im\Big(e^{\alpha k}\cdot C_t(S_t, e^k)\Big)(v) \\
&= \Im\Big(e^{\alpha k}\cdot S_t\cdot C_t(1, e^{k-\ln(S_t)})\Big)(v) \\
&= S_t\cdot e^{\alpha\ln(S_t)}\cdot\Im\Big(e^{\alpha(k-\ln(S_t))}\cdot C_t(1, e^{k-\ln(S_t)})\Big)(v) \\
&= (S_t)^{\alpha+1}\cdot e^{iv\ln(S_t)}\cdot\Im\Big(e^{\alpha k}\cdot C_t(1, e^k)\Big)(v) \\
&= (S_t)^{\alpha+1}\cdot e^{iv\ln(S_t)}\cdot\frac{e^{-rT}\Phi(v - i(\alpha+1))}{\alpha^2+\alpha-v^2+iv(2\alpha+1)}
\end{aligned}$$

In the last expression, we use the shift property of the Fourier transform and the equation 3.7. □

On the other hand, if the jump sizes are not bounded above, we need more call options in the enlarged market to hedge any contingent claim. In the next proposition, we extend the previous results in the proposition 3.1.2 for a Lévy process with jump sizes not bounded below. Since the Lévy measure does not have a compact support, we add an additional assumption: all moments of the Lévy process L_t exist.

Proposition 3.1.8 *Assume that Y is deterministic, jump sizes are bounded below only and in $[M_1, \infty) \subseteq [\ln(K_1), \infty)$. Assume also that all moments of L_t exist under $\mathbb{Q}$. Let F*

be a square integrable contingent claim, and f_t be a function of x defined as

$$f_t(x) = \begin{cases} e^{-x}\left[e^{-r(T^*-t)}\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}(F)\,|\mathcal{F}_t\right] + \eta_t\right] & \text{if } x \geq \ln(K_1) \\ 0 & \text{otherwise} \end{cases}$$

and η_t is a constant. Assume that f_t satisfies Dirichlet's conditions. Then F can be replicated perfectly and the quantity of the call options in the hedging portfolio at time t, $\phi(t,K) = \phi(t,e^k)$ $k \geq \ln(K_1)$ is given by

$$\phi(t,e^k) = e^{-k}\int_0^{\infty}\int_{\ln(K_1)}^{\infty}\frac{\Psi_{Re}(\pi v)cos(\pi v(x-k)) - \Psi_{Im}(\pi v)sin(\pi v(x-k)))}{|\Psi(\pi v)|^2}f_t(x)dxdv$$

where

$$\Psi(v) = e^{-r(T-t)}S_t \cdot e^{iv\ln(S_t)}\left[\frac{\Phi(v-i) - e^{-iv\ln(S_t)}\Phi(-i)}{iv} - \frac{\Phi(v-i) - e^{-(iv+1)\ln(S_t)}}{1+iv}\right]$$

$$\Phi(v) = \mathbb{E}^{\mathbb{Q}}\left[e^{ivL_{T-t}}1_{(L_{T-t}\geq -\ln(S_t))}\right]$$

Proof: We do not repeat all the steps. We sketch an outline of the proof. Hedging is the solution of

$$\int_{K_1}^{\infty}\phi(t,K)\left(e^x \cdot C_t(S_t, Ke^{-x}) - C_t(S_t,K)\right)dK = e^{-r(T^*-t)}\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}(F)\,|\mathcal{F}_t\right]$$

It is equivalent to

$$\int_{\ln(K_1)}^{\infty}\phi(t,e^k)e^x \cdot C_t(S_t, e^{k-x}) \cdot e^k dk = e^{-r(T^*-t)}\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}(F)\,|\mathcal{F}_t\right] + \eta_t$$

We will take the Fourier transform of $C_t(S_t, e^k)$ $k \geq 0$ and the Fourier integral exists for $C_t(S_t,e^k)1_{k\geq\ln(K_1)}$. Therefore, we do not need a function $A(k)$ to modify $C_t(S_t,e^k)$. Let

$$c_t(k) = C_t(S_t, e^k)$$

$$\varphi_t(k) = e^k\phi(t,e^k)$$

$$f_t(x) = \begin{cases} e^{-x}\left[e^{-r(T^*-t)}\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}(F)\,|\mathcal{F}_t\right] + \eta_t\right] & \text{if } x \text{ is in } [\ln(K_1),\infty) \\ 0 & \text{otherwise} \end{cases}$$

The integral equation becomes:

$$\int_{\ln(K_1)}^{\infty} \varphi_t(k) c_t(k-x) dk = f_t(x)$$

Let

$$\Im_c(c_t)(v) = \int_0^{\infty} c_t(u) cos(\pi v u) du$$
$$\Im_s(c_t)(v) = \int_0^{\infty} c_t(u) sin(\pi v u) du$$

be the Fourier cosine and sine transforms of c_t. The integrals are from 0 to ∞ and $c_t(k)$ is integrable under this domain. The Fourier transform of $c_t(k) \cdot 1_{k \geq 0}$ in terms of the characteristic function of L_{T-t} under $\mathbb{Q}$ is equal to:

$$\begin{aligned}
\Psi(v) &= \Im(c_t(k) \cdot 1_{k\geq 0})(v) \\
&= \Im\Big(C_t(S_t, e^k) \cdot 1_{k\geq 0}\Big)(v) \\
&= S_t \cdot \Im\Big(C_t(1, e^{k-\ln(S_t)} \cdot 1_{k\geq 0}\Big)(v) \\
&= S_t \cdot e^{iv\ln(S_t)} \cdot \Im\Big(C_t(1, e^k) \cdot 1_{k\geq -\ln(S_t)}\Big)(v) \\
&= e^{-r(T-t)} S_t \cdot e^{iv\ln(S_t)} \int_{-\ln(S_t)}^{\infty} \int_k^{\infty} e^{ivk}(e^u - e^k) q_T(u) du dk \\
&= e^{-r(T-t)} S_t \cdot e^{iv\ln(S_t)} \int_{-\ln(S_t)}^{\infty} \int_{-\ln(S_t)}^{u} e^{ivk}(e^u - e^k) q_T(u) dk du \\
&= e^{-r(T-t)} S_t \cdot e^{iv\ln(S_t)} \int_{-\ln(S_t)}^{\infty} \Big(\frac{e^{(iv+1)u} - e^u e^{-iv\ln(S_t)}}{iv} - \frac{e^{(iv+1)u} - e^{-(iv+1)\ln(S_t)}}{iv+1}\Big) q_T(u) du \\
&= e^{-r(T-t)} S_t \cdot e^{iv\ln(S_t)} \left[\frac{\Phi(v-i) - e^{-iv\ln(S_t)}\Phi(-i)}{iv} - \frac{\Phi(v-i) - e^{-(iv+1)\ln(S_t)}}{1+iv}\right]
\end{aligned}$$

where we use the Fubini theorem to change the order of integrals and

$$\Phi(v) = \mathbb{E}^{\mathbb{Q}}\left[e^{ivL_{T-t}} 1_{(L_{T-t} \geq -\ln(S_t))}\right]$$

Multiplying both sides of the integral equation by $cos(\pi v(x-z) f_t(x)$ and taking the integral:

$$\int_{-\infty}^{\infty} cos(\pi v(x-z)) f_t(x) dx = \int_{\ln(K_1)}^{\infty} cos(\pi v(x-z)) f_t(x) dx$$

$$
\begin{aligned}
&= \int_{\ln(K_1)}^{\infty} cos(\pi v(x-z)) \int_{\ln(K_1)}^{\infty} \varphi_t(k) c_t(k-x) dk dx \\
&= \int_{0}^{\infty} \int_{\ln(K_1)}^{\infty} c_t(y) cos(\pi v(k-y-v)) \varphi_t(k) dk dy \\
&= \int_{0}^{\infty} \int_{\ln(K_1)}^{\infty} c_t(y) \left[cos(\pi v(k-v)) cos(\pi v y) + sin(\pi v(k-v)) sin(\pi v y) \right] \varphi_t(k) dk dy \\
&= \mathfrak{I}_c(c_t)(v) \int_{\ln(K_1)}^{\infty} cos(\pi v(k-v)) \varphi_t(k) dk + \mathfrak{I}_s(c_t)(v) \int_{\ln(K_1)}^{\infty} sin(\pi v(k-v)) \varphi_t(k) dk
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_{-\infty}^{\infty} sin(\pi v(x-z)) f_t(x) dx \\
&= \mathfrak{I}_s(c_t)(v) \int_{\ln(K_1)}^{\infty} sin(\pi v(k-v)) \varphi_t(k) dk - \mathfrak{I}_s(c_t)(v) \int_{\ln(K_1)}^{\infty} cos(\pi v(k-v)) \varphi_t(k) dk
\end{aligned}
$$

We obtain

$$
\int_{\ln(K_1)}^{\infty} cos(\pi v(k-z)) \varphi_t(k) dk = \int_{\ln(K_1)}^{\infty} \frac{\mathfrak{I}_c(c_t)(v) cos(\pi v(x-z)) - \mathfrak{I}_s(c_t)(v) sin(\pi v(x-z)))}{(\mathfrak{I}_c(c_t)(v))^2 + (\mathfrak{I}_s(c_t)(v))^2} f_t(x) dx
$$

From Fourier's theorem, we conclude that,

$$
\begin{aligned}
\varphi_t(k) &= \int_{0}^{\infty} \int_{\ln(K_1)}^{\infty} \frac{\mathfrak{I}_c(c_t)(v) cos(\pi v(x-k)) - \mathfrak{I}_s(c_t)(v) sin(\pi v(x-k)))}{(\mathfrak{I}_c(c_t)(v))^2 + (\mathfrak{I}_s(c_t)(v))^2} f_t(x) dx dv \\
&= \int_{0}^{\infty} \int_{\ln(K_1)}^{\infty} \frac{\Psi_{Re}(\pi v) cos(\pi v(x-k)) - \Psi_{Im}(\pi v) sin(\pi v(x-k)))}{|\Psi(\pi v)|^2} f_t(x) dx dv
\end{aligned}
$$

and the integral is zero for all $k \notin [\ln(K_1), \infty)$. The solution is equal to

$$
\phi(t, e^k) = e^{-k} \int_{0}^{\infty} \int_{\ln(K_1)}^{\infty} \frac{\Psi_{Re}(\pi v) cos(\pi v(x-k)) - \Psi_{Im}(\pi v) sin(\pi v(x-k)))}{|\Psi(\pi v)|^2} f_t(x) dx dv
$$

□

It is not difficult to show that the Malliavin derivative of a variance swap is the second order power jump asset. Therefore, a variance swap can be perfectly hedged. Before we finish this section, we obtain the Malliavin derivative of an Asian option and show that an Asian option can be hedged perfectly. An Asian option is a contract that depends on the path of the underlying asset's price at the time of exercise. Asian options are traded in financial markets very frequently and they reduce the risk of market manipulation of the underlying assets. The payoff to an Asian option is the positive part of the difference between the average of the underlying asset's price and a strike value. Because of the averaging feature, volatility of an Asian option is less than the volatility of the underlying assets.

An Asian call option payout is defined as

$$V(T,K) = \left[\frac{1}{N}\sum_{n=1}^{N} S_{t_n} - K\right]^+ = \left[\frac{1}{N}\sum_{n=1}^{N} S_0 e^{L_{t_n}} - K\right]^+ \tag{3.22}$$

where $t_1, t_2, ..., t_N$ are discrete monitoring times. Note that, $S_0 e^{L_{t_n}} - K$ is in $\mathbb{D}_{1,2}$. From the chain rule (Nunno et al. [2010]),

$$\begin{aligned}
D_{t,x}\Big(e^{-r(T-t)}V(T)\Big) &= e^{-r(T-t)}\left[\frac{1}{N}\sum_{n=1}^{N} S_0 e^{L_{t_n}} - K + D_{t,x}\Big(\frac{1}{N}\sum_{n=1}^{N} S_0 e^{L_{t_n}} - K\Big)\right]^+ \\
&\quad -e^{-r(T-t)}\left[\frac{1}{N}\sum_{n=1}^{N} S_0 e^{L_{t_n}} - K\right]^+ \\
&= e^{-r(T-t)}\left[\frac{1}{N}\sum_{n=1}^{N} S_0 e^{L_{t_n}} - K + \frac{1}{N}\sum_{n=1}^{N} S_0 (e^x - 1) e^{L_{t_n}}\right]^+ - e^{-r(T-t)}V(T,K) \\
&= e^{-r(T-t)}\left[\frac{1}{N}\sum_{n=1}^{N} e^x S_{t_n} - K\right]^+ - e^{-r(T-t)}V(T,K) \\
&= e^{-r(T-t)} e^x V(T, e^{-x}K) - e^{-r(T-t)}V(T,K)
\end{aligned}$$

We find the conditional expectation at time t:

$$\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}\Big(e^{-r(T-t)}V(T)\Big) \mid \mathcal{F}_t\right] = e^x V(t, e^{-x}K) - V(t,K) \tag{3.23}$$

where $V(t,K)$ is the price of an Asian option witk strike K at time t. The price of an Asian option is a continuous function of its strike price and the jump sizes are bounded (i.e. x is in a bounded set). Therefore, the Dirichlet conditions are satisfied and an Asian option can be hedged perfectly.

3.1.3 Hedging In An Exponential Lévy Market

In this section, we assume that L_t is a general Lévy process under $\mathbb{P}$, $\mathcal{F}_t$ $t \in [0, T^*]$ is the augmented filtration generated by L_t. Therefore, $\mathcal{F}_t$ is a right continuous filtration and satisfies the usual conditions (Medvegyev [2007]). We assume that all moments of L_t exist. Then L_t admits the following The LévyItô decomposition (Sato [1999]):

$$L_t = at + \sigma W_t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} xN(ds, dx) \tag{3.24}$$

where $N(dt, dx)$ is a compensated Poisson random measure and W_t is a Brownian motion under $\mathbb{P}$.

Definition 3.1.3 *Let S_t be spot price of some underlying asset driven by a Lévy process L_t under the probability measure $\mathbb{P}$*

$$S_t = S_0 e^{L_t}$$

where L_t is a Lévy process with bounded jump sizes. Assume that jump sizes are in $[M_1, M_2]$ where M_1 and M_2 are both finite. Also, we assume that the risk free rate r is a constant. The market consists of the asset, a money marketing account, and European call options with the same maturity T^*. Assume that for every strike $K \in [K_1, K_2]$, there is a call option in the market. Let $C_t(S_t, K)$ be the price of the call option at time t with strike K.

We will use the call options in hedging. First, we find the Malliavin derivative of a call option.

Lemma 3.1.2 *Let F be discounted payoff of a call option*

$$F = e^{-rT^*} [S_{T^*} - K]^+ \tag{3.25}$$

then $F \in \mathbb{D}_{1,2}$*, its Malliavin derivative is equal to*

$$D_t F = 1_{[K,\infty]}(S_{T^*})e^{-rT^*}\sigma S_{T^*} \tag{3.26}$$

$$D_{t,x}F = e^{-rT^*}\left([e^x S_{T^*} - K]^+ - [S_{T^*} - K]^+\right) \tag{3.27}$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}F|\mathcal{F}_t\right] = e^{-rt}\left(e^x C_t(S_t, Ke^{-x}) - C_t(S_t, K)\right) \tag{3.28}$$

Proof: $D_{t,x}F$ is the same as in the proposition 3.1.4. From Malliavin calculus, $D_t F = 1_{[K,\infty]}(S_{T^*})e^{-rT^*}S_{T^*}$. □

For the completeness of this section, we review Girsanov's and Clark–Ocone theorems for a general Lévy process:

Proposition 3.1.9 *(Girsanov's theorem for a Lévy process) Let* $Y(t,x) \leq 1$*,* $t \in [0, T^*]$*,*
$x \in \mathbb{R}$

0*,* u_t *is a predictable process such that*

$$\int_0^T \int_{\mathbb{R}\setminus\{0\}} \Big(|\ln(1 + Y(t,x))| + Y^2(t,x)\Big)\, v(dx)dt < \infty \;\; \mathbb{P}\; a.s.$$

$$\int_0^T u_t^2 dt < \infty \;\; \mathbb{P}\; a.s.$$

Let $\mathbb{Q}$ *be a measure on* $\mathcal{F}_{T^*}$ *defined as*

$$d\mathbb{Q}(w) = Z(w, T^*)d\mathbb{P}(w)$$

where

$$\begin{aligned} Z(t) \;\; = \;\; \exp\Bigg(& \int_0^t |u_s| dW_s - \int_0^t u_s^2 ds \\ & + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left[\ln(1 - Y(s,x)) + Y(s,x)\right] v(dx)ds \\ & + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \ln(1 - Y(s,x)N(ds,dx) \Bigg) \end{aligned}$$

and assume that $Z(T^*)$ satisfies the Novikov condition

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^{T^*} u_s^2 ds + \int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} [(1-Y(t,x))\ln(1-Y(t,x)) + Y(t,x)]\, v(dx)dt\right)\right] < \infty$$

Define

$$N'(dt,dx) = Y(t,x)v(dx)dt + N(dt,dx)$$

and

$$dW'_t = u_t dt + dW_t$$

then $N'(dt,dx)$ is a compensated random measure under $\mathbb{Q}$ and W' is a Brownian motion under $\mathbb{Q}$.

Proposition 3.1.10 *(The generalized Clark-Ocone theorem under the change of measure for Lévy processes) Let $F \in L^2(\mathbb{P}\cap\mathbb{Q})$ be a $\mathcal{F}_{T^*}$ measurable random variable. Then the representation of F with respect to W' and $N'(dt,dx)$ is as follows:*

$$\begin{aligned} F = \quad & \mathbb{E}^{\mathbb{Q}}[F] + \int_0^{T^*} \mathbb{E}^{\mathbb{Q}}\left[D_t F - F\int_t^{T^*} D_t u_s dW'_s \mid \mathcal{F}_t\right] dW'_t \\ & + \int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} \mathbb{E}^{\mathbb{Q}}\left[F(H-1) + HD_{t,x}F \mid \mathcal{F}_t\right] N'(dt,dx) \end{aligned}$$

where

$$\begin{aligned} H = \quad & exp\left(\int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} \left[D_{t,x}Y(s,x) + \ln\left(1 - \tfrac{D_{t,x}Y(s,x)}{1-Y(s,x)}\right)(1-Y(s,x))\right] v(dx)ds \right. \\ & \left. + \int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} \ln\left(1 - \tfrac{D_{t,x}Y(s,x)}{1-Y(s,x)}\right) N'(ds,dx)\right) \end{aligned}$$

In this section, we assume that both u and Y are deterministic.

Proposition 3.1.11 *Assume that u and Y are deterministic. Then the representation of F with respect to W' and $N'(dt,dx)$ is as follows:*

$$F = \mathbb{E}^{\mathbb{Q}}[F] + \int_0^{T^*} \mathbb{E}^{\mathbb{Q}}\left[D_t F \mid \mathcal{F}_t\right] dW'_t + \int_0^{T^*}\int_{\mathbb{R}\setminus\{0\}} \mathbb{E}^{\mathbb{Q}}\left[D_{t,x}F \mid \mathcal{F}_t\right] N'(dt,dx) \quad (3.29)$$

When L_t is not a pure jump processes, we need additional integration constraints to define the set of admissible strategies:

Definition 3.1.4 *A stochastic process $\phi : [0,T^*] \times [K_1,K_2] \times \Omega \to \mathbb{R}$ is called an admissible strategy if*

- *ϕ is $\mathcal{B}([0,T^*]) \otimes \mathcal{B}[K_1,K_2]) \otimes \mathcal{F}_{T^*}$ measurable.*
- *$\phi(\cdot,K)$ is a predictable and adapted process (with respect to $\mathcal{F}$) for every $K \in [K_1,K_2]$.*
- *$\mathbb{E}^{\mathbb{Q}}\left[\int_0^{T*} \phi(t,K)^2 \mathbb{E}^{\mathbb{Q}}\left[e^{-rT^*} D_t(S_{T^*}-K)^+ | \mathcal{F}_t\right]^2 dt\right] < \infty$ for all $K \in [K_1,K_2]$.*
- *$\mathbb{E}^{\mathbb{Q}}\left[\int_0^{T*} \phi(t,K)^2 \int_{\mathbb{R}\setminus\{0\}} (e^x C_t(S_t, Ke^{-x}) - C_t(S_t,K))^2\, Y(x) F(dx)dt\right] < \infty$ for all $K \in [K_1,K_2]$.*
- *$\int_{K_1}^{K_2} \left|\int_0^{T^*} \phi(t,K)\mathbb{E}^{\mathbb{Q}}\left[e^{-rT^*} D_t(S_{T^*}-K)^+ | \mathcal{F}_t\right] dW'_t\right| dK < \infty$ almost surely.*
- *$\int_{K_1}^{K_2} \left|\int_0^{T^*} \int_{\mathbb{R}\setminus\{0\}} \phi(t,K) e^{-rt} (e^x C_t(S_t, Ke^{-x}) - C_t(S_t,K))\, N'(dt,dx)\right| dK < \infty$ almost surely.*

The set of all $\mathcal{F}$-admissible self-financing trading strategies [2] is denoted by $\mathcal{A}_{\mathcal{F}}$.

Proposition 3.1.12 *Assume that u and Y are deterministic. Suppose F is a square integrable (under both $\mathbb{P}$ and $\mathbb{Q}$), $\mathcal{F}_{T^*}$ measurable. The self-financing strategy $\phi(t,K) \in \mathcal{A}_{\mathcal{F}}$ perfectly hedges the final payoff F if and only if*

$$\int_{K_1}^{K_2} \phi(t,K)\mathbb{E}^{\mathbb{Q}}\left[e^{-rT^*} D_t(S_{T^*}-K)^+ | \mathcal{F}_t\right] dK = e^{-rT^*}\mathbb{E}^{\mathbb{Q}}\left[D_t(F)|\mathcal{F}_t\right] \tag{3.30}$$

$$\int_{K_1}^{K_2} \phi(t,K)\left(e^x C_t(S_t, Ke^{-x}) - C_t(S_t,K)\right) dK = e^{-r(T^*-t)}\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}(F)|\mathcal{F}_t\right] \tag{3.31}$$

for all $t \in [0,T^*]$.

Proof: From the generalized Clark–Ocone theorem, Fubini theorem and previous section, we get the above result. □

Proposition 3.1.13 *Assume that u and Y are deterministic, jump sizes are bounded and in $[M_1,M_2] \subseteq [\ln(K_1),\ln(K_2)]$.*

[2] Let $\phi(t,K)$, $t \in [0,T^*]$ and $K \in [K_1,K_2]$ be the number of call options with strike price K in the portfolio at time t, and V_t be the value of the portfolio at time t. A trading strategy $\phi(t,K)$, $t \in [0,T^*]$ and $K \in [K_1,K_2]$, is a self-financing trading strategy if $d(e^{-rt}V_t) = \int_{K_1}^{K_2} \phi(t,K) d(e^{-rt} C_t(S_t,K))dK$.

Let F be a square integrable contingent claim. Then either

(1) F can be replicated perfectly or

(2) There exists a sequence of contingent claims F_n such that F_n can be perfectly hedged and $F_n \to F$ in L^2.

Proof: Since the jump sizes are bounded, all moments of the Lévy process with degree larger than or equal to one exist. Therefore, for every $n \geq 1$, $Z^{(n)}(T^*) = \int_0^{T^*} \int_{\mathbb{R}\setminus\{0\}} x^n N'(dt, dx)$ is a square integrable random variable. From the previous section proposition 3.1.6, we can find $\phi(t, K)$ such that

$$\int_{K_1}^{K_2} \phi(t, K) \left(e^x C_t(S_t, Ke^{-x}) - C_t(S_t, K)\right) dK = e^{-r(T^*-t)} \mathbb{E}^{\mathbb{Q}} \left[D_{t,x} \left(Z^{(n)}(T^*) \right) | \mathcal{F}_t \right]$$

Note that, in the proof of Proposition 3.1.6, we have an $\mathcal{F}_t$ measurable free variable η_t. We can choose η_t such that

$$\int_{K_1}^{K_2} \phi(t, K) \mathbb{E}^{\mathbb{Q}} \left[e^{-rT^*} D_t (S_{T^*} - K)^+ | \mathcal{F}_t \right] dK = 0$$

is satisfied. For a given $\epsilon > 0$, there exists a solution $\phi(t, K)$ such that

$$\left\| \int_{K_1}^{K_2} \phi(t, K) \left(e^x C_t(S_t, Ke^{-x}) - C_t(S_t, K)\right) dK - e^{-r(T^*-t)} \mathbb{E}^{\mathbb{Q}} \left[D_{t,x} (F) | \mathcal{F}_t \right] \right\|_{L^2(\mathbb{Q})} < \epsilon$$

We can choose η_t such that

$$\int_{K_1}^{K_2} \phi(t, K) \mathbb{E}^{\mathbb{Q}} \left[e^{-rT^*} D_t (S_{T^*} - K)^+ | \mathcal{F}_t \right] dK = e^{-rT^*} \mathbb{E}^{\mathbb{Q}} \left[D_t (F) | \mathcal{F}_t \right]$$

□

In the last proposition, we only used the call options in the market. The idea behind our approach is to hedge Brownian motion risk and the jump risk at the same time. We calibrate the parameter η_t to accomplish simultaneous hedging of the two risk factors. We extend the results in the previous section:

Proposition 3.1.14 *Assume that Y is deterministic, jump sizes are bounded and in $[M_1, M_2] \subseteq [\ln(K_1), \ln(K_2)]$. Let F be a square integrable contingent claim, and f_t be a function of x defined as*

$$f_t(x) = \begin{cases} e^{-x(\alpha+1)} \left[e^{-r(T^*-t)} \mathbb{E}^{\mathbb{Q}} \left[D_{t,x}(F) | \mathcal{F}_t \right] + \eta_t \right] & \text{if } x \text{ is in } [\ln(K_1), \ln(K_2)] \\ 0 & \text{otherwise} \end{cases} \tag{3.32}$$

and η_t is a constant. Assume that f_t satisfies Dirichlet's conditions. Then F can be replicated perfectly and the quantity of the call options in the hedging strategy at time t, $\phi(t, K) = \phi(t, e^k)$, is given by

$$\phi(t, e^k) = e^{(\alpha-1)k} \int_{-\infty}^{\infty} \int_{\ln(K_1)}^{\ln(K_2)} \frac{\Psi_{Re}(\pi v) cos(\pi v(x-k)) - \Psi_{Im}(\pi v) sin(\pi v(x-k)))}{|\Psi(\pi v)|^2} f_t(x) dx dv \tag{3.33}$$

where

$$\Psi(v) = (S_t)^{\alpha+1} \cdot e^{iv \ln(S_t)} \cdot \frac{e^{-rT} \Phi(v - i(\alpha+1))}{\alpha^2 + \alpha - v^2 + iv(2\alpha+1)} \tag{3.34}$$

$$\Phi(v) = \mathbb{E}^{\mathbb{Q}} \left[e^{ivL_{T-t}} \right] \tag{3.35}$$

$$\mathbb{E}^{\mathbb{Q}}[S_T^{(\alpha+1)}] < \infty$$

and η_t is chosen such that

$$\int_{\ln(K_1)}^{\ln(K_2)} \phi(t, e^k) \mathbb{E}^{\mathbb{Q}} \left[1_{[e^k, \infty]}(S_{T^*}) \sigma S_{T^*} | \mathcal{F}_t \right] e^k dk = \mathbb{E}^{\mathbb{Q}} \left[D_t(F) | \mathcal{F}_t \right]$$

Proof: From equation 3.26, the hedging equation 3.30

$$\int_{\ln(K_1)}^{\ln(K_2)} \phi(t, e^k) \mathbb{E}^{\mathbb{Q}} \left[e^{-rT^*} D_t(S_{T^*} - e^k)^+ | \mathcal{F}_t \right] e^k dk = e^{-rT^*} \mathbb{E}^{\mathbb{Q}} \left[D_t(F) | \mathcal{F}_t \right]$$

is equal to

$$\int_{\ln(K_1)}^{\ln(K_2)} \phi(t, e^k) \mathbb{E}^{\mathbb{Q}} \left[1_{[e^k, \infty]}(S_{T^*}) e^{-rT^*} \sigma S_{T^*} | \mathcal{F}_t \right] e^k dk = e^{-rT^*} \mathbb{E}^{\mathbb{Q}} \left[D_t(F) | \mathcal{F}_t \right]$$

By using proposition 3.1.7, we obtain the result. □

On the other hand, if the jump sizes are not bounded above, we need more call options in the enlarged market to hedge any contingent claim. In the next proposition, we extend the previous results in the proposition for a Lévy process with jump sizes not bounded below.

Proposition 3.1.15 *Assume that Y is deterministic, jump sizes are bounded below only and in $[M_1, \infty) \subseteq [\ln(K_1), \infty)$. Assume also that all moments of L_t exist under $\mathbb{Q}$. Let F be a square integrable contingent claim, and f_t be a function of x defined as*

$$f_t(x) = \begin{cases} e^{-x}\left[e^{-r(T^*-t)}\mathbb{E}^{\mathbb{Q}}\left[D_{t,x}(F)\,|\mathcal{F}_t\right] + \eta_t\right] & \text{if } x \geq \ln(K_1) \\ 0 & \text{otherwise} \end{cases}$$

and η_t is a constant. Assume that f_t satisfies Dirichlet's conditions. Then F can be replicated perfectly and the quantity of the call options in the hedging portfolio at time t, $\phi(t, K) = \phi(t, e^k)$ $k \geq \ln(K_1)$ is given by

$$\phi(t, e^k) = e^{-k}\int_0^\infty \int_{\ln(K_1)}^\infty \frac{\Psi_{Re}(\pi v)cos(\pi v(x-k)) - \Psi_{Im}(\pi v)sin(\pi v(x-k)))}{|\Psi(\pi v)|^2} f_t(x)dxdv$$

where

$$\Psi(v) = e^{-r(T-t)}S_t \cdot e^{iv\ln(S_t)}\left[\frac{\Phi(v-i) - e^{-iv\ln(S_t)}\Phi(-i)}{iv} - \frac{\Phi(v-i) - e^{-(iv+1)\ln(S_t)}}{1+iv}\right]$$

$$\Phi(v) = \mathbb{E}^{\mathbb{Q}}\left[e^{ivL_{T-t}}1_{(L_{T-t} \geq -\ln(S_t))}\right]$$

and η_t is chosen such that

$$\int_{\ln(K_1)}^\infty \phi(t, e^k)\mathbb{E}^{\mathbb{Q}}\left[1_{[e^k,\infty]}(S_{T^*})\sigma S_{T^*}|\mathcal{F}_t\right]e^k dk = \mathbb{E}^{\mathbb{Q}}\left[D_t(F)\,|\mathcal{F}_t\right]$$

Proof: From equation 3.26, the hedging equation 3.30

$$\int_{\ln(K_1)}^\infty \phi(t, e^k)\mathbb{E}^{\mathbb{Q}}\left[e^{-rT^*}D_t(S_{T^*} - e^k)^+|\mathcal{F}_t\right]e^k dk = e^{-rT^*}\mathbb{E}^{\mathbb{Q}}\left[D_t(F)\,|\mathcal{F}_t\right]$$

is equal to

$$\int_{\ln(K_1)}^{\infty} \phi(t, e^k)\mathbb{E}^{\mathbb{Q}}\left[1_{[e^k,\infty]}(S_{T^*})e^{-rT^*}\sigma S_{T^*}|\mathcal{F}_t\right]e^k dk = e^{-rT^*}\mathbb{E}^{\mathbb{Q}}\left[D_t(F)|\mathcal{F}_t\right]$$

By using proposition 3.1.8, we obtain the result. □

In practice, there are finitely many call options in financial markets and market is not complete in the limit. In this case, an approximate hedging is possible and our method is an alternative to the minimum variance hedging method developed by Nunno et al. [2010].

In this chapter, we assumed that jump sizes are bounded. A Lévy process may have countably infinitely many jumps in a finite time interval if the jump sizes are not bounded by a positive number. In finance theory, Lévy processes with infinitely many jumps in finite time intervals have been of interest to researchers to explain asset prices. In our approach, we assume that jump sizes are in $[M_1, M_2]$ where M_1 can be negative (in this case there are infinitely many jumps in a finite time interval) and our method provides a solution for a financial market driven by a Lévy process with infinitely many jumps in finite intervals.

3.2 Conclusion

In this chapter, we develop hedging strategies for contingent claims in exponential Lévy markets. Our approach is based on Fourier analysis and Malliavin calculus. Our results show that, the Malliavin derivative can be approximated by continuous functions and this approximation can be hedged perfectly in the market. We also show that, an Asian call option can be hedged perfectly.

Our results are consistent with the results in Jarrow et al. [1999] where the authors show that exponential Lévy market is either incomplete or complete in the limit. Our approach explains the hedging problem in terms of the integral equations. The integral equation is a convolution type integral operator and it is possible to use our approach in practice. Any derivative product that has a Malliavin derivative satisfying the Drichlet conditions can be hedged perfectly and the hedging strategy is defined in terms of the Fourier transform of the characteristic function of the Lévy process. Carr et al. [1999] developed a pricing method for call options in exponential Lévy markets. But their research does not address the problems of hedging the contingent claims. Our method fills this important gap by the use of Malliavin calculus.

One final remark is about the Malliavin derivative. Asian options are an example of sophisticated derivative products whose Malliavin derivatives have closed form results. However, for many other sophisticated products, like barrier options where the payoff depends on stopping times, closed form expressions for Malliavin derivative may not exist. We believe this is an most important limitation of our approach.

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