GENERIC INITIAL IDEALS OF LOCALLY COHEN-MACAULAY SPACE CURVES

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GENERIC INITIAL IDEALS OF LOCALLY COHEN-MACAULAY SPACE CURVES

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We analyze the degree reverse lexicographic generic initial ideals of locally Cohen-Macaulay space curves and how they behave under biliaison. We provide a complete classification for both general members of componenets of the locally Cohen-Macaulay Hilbert scheme of degree three space curves for each genus and lower triangle diagrams with only one non-zero entry after a general change of coordinates. We also consider curves where a general hyperplane section has the same Hilbert function as a general hyperplane section of an extremal curve, giving special consideration to curves in double planes. We resolve the ideals of curves in double planes and give a symmetry condition on their triangle diagrams.

BIOGRAPHICAL SKETCH

Kristine grew up in Montara, California. She graduated from Half Moon Bay High School as Salutatorian in 2002. She attended the University of Chicago for her undergraduate studies, receiving the degree of Bachelor of Science in Mathematics with Specialization in Economics, with general and departmental honors, in 2006. She completed her Master of Science in Mathematics degree at Cornell University in 2010. Throughout graduate school, Kristine has been an active member of Phi Beta Kappa, Sigma Xi, the American Mathematical Society, and the Association for Women in Mathematics. She was a founding member of the Cornell student chapter of the Association for Women in Mathematics and served as the chapter's Vice President from 2011 to 2013.

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CHAPTER 1

INTRODUCTION

Locally Cohen-Macaulay space curves (LCM space curves) are equidimensional algebraic curves in projective three-space. They may be both reducible and non-reduced. Despite this, the class of LCM space curves is well structured.

Determining the degrees and genera where smooth space curves exist is a delicate question. The picture for LCM space curves is much clearer. Non-planar LCM space curves exist for all degrees $d \geq 2$ and genera $g \leq \binom{d-2}{2}$. Further, as we will see in Chapter 3, LCM space curves are the natural class of space curves for liaison theory. From each equidimensional space curve, liaison theory allows one to compute a class of LCM curves with isomorphic Rao modules.

Locally Cohen-Macaulay space curves also arise in questions about connectedness of Hilbert schemes. In his thesis ([22]), Hartshorne proved that the Hilbert scheme of space curves of degree d and genus g was connected for each (d, g) pair. The deformations used in this work require passing through non-equidimensional schemes, even, in some cases, when deforming between smooth curves. For this reason, it is natural to ask for which degrees and genera the Hilbert scheme of equidimensional space curves is connected. Further, each ideal on the Hilbert scheme of degree d and genus g space curves lies on the same irreducible component as its generic initial ideal (with respect to any term order). Determining these generic initial ideals has the potential to yield structural information on Hilbert schemes of space curves.

The primary question with which this dissertation is concerned is: which Borel-fixed monomial ideals occur as the generic initial ideal, with respect to degree reverse lexicographic order, of a LCM space curve? We have chosen this term order because of its computational strength with regard to computing regularity and hyperplane sections.

We give complete results for general members of components of degree three Hilbert schemes. We also give complete results when only a single element of the monomial ideal concerned is divisible by the third variable in the the term order. We give partial results in the case that the Hilbert function of a general hyperplane section of the concerned curve is the same as the Hilbert function of a general hyperplane section of an extremal curve.

Several other questions arose while studying these generic initial ideals, especially related to the possible generic initial ideals of hyperplane sections and to the structure of the ideals of LCM space curves in double planes. In regard to the first topic, there are several results restricting the possible Hilbert functions for hyperplane sections of smooth curves, which we discuss in Chapter 2. See, for example, [1, 9]. We show that these results do not transfer to the LCM case in Chapter 7. In particular, given any set of λ-invariants, there exists an LCM space curve for which they occur. In regard to the second topic, Hartshorne and Schlesinger perform a scheme-theoretic analysis in [29]. Our interest in an ideal-theoretic analysis arose from our study of LCM space curves where the Hilbert function of a general hyperplane section is the same as the Hilbert function of a general hyperplane section of an extremal curve.

In brief, the organization of this dissertation is as follows: Chapter 2 establishes notation, provides general background, and establishes terminology. Chapter 3 reviews the major concepts of liaison theory. Chapter 4 discusses the triangle diagrams of Liebling, ending with a result on the placement of non-zero entries in the triangle diagrams of curves in the same biliaison class. Chapter 5 discusses extremal and subextremal curves with a smattering of new proofs and computational results throughout. In Chapter 6, for each genus we compute the degree reverse lexicographic generic initial ideals of the general member of each component of the Hilbert scheme of degree three LCM space curves. In Chapter 7, we completely determine the degree reverse lexicographic generic initial ideals of LCM space curves where only one minimal generator is divisible by the third variable in the term order. We also show that every possible set of λ-invariants occurs for some LCM space curve. In Chapter 8, we analyze LCM space curves whose general hyperplane sections have the same Hilbert function as a general hyperplane section of an extremal curve. In Chapter 9, we provide an ideal-theoretic construction of LCM space curves in double planes. We also resolve their ideals and compute a symmetry condition on their triangle diagrams. We conclude with a chapter on questions for further study.

CHAPTER 2

PRELIMINARIES

We assume the reader is familiar with the material typically covered in introductory graduate coursework in computational algebra, commutative algebra, and algebraic geometry. We suggest [2, 11, 36, 47] as references for commutative and computational algebra and [12, 23] as references for algebraic geometry. Additionally, we suggest [21, 32, 51, 52, 57] as references for the construction and theory of Hilbert schemes.

Let k be an algebraically closed field of characteristic zero. Throughout we work in the polynomial rings $S = k[x, y, z, w]$ and $R = k[x_0, x_1, \ldots, x_n]$, each equipped with the standard grading and multi-grading. We let $\mathbb{P}^n$ denote $\mathbb{P}^n_k$. We assign R and S the degree reverse lexicographic term order (degrevlex) with $x > y > z > w$ and $x_0 > x_1 > \cdots > x_n$. We take all ideals, modules, Gröbner bases, and complexes to be graded. For a graded structure A, we let A_i denote the degree i graded component of A. An unfortunate consequence of this notation is that it is easily confused with subscripts used to enumerate lists, especially the modules in resolutions. For a graded structure A, we write $A(j)$ to mean the isomorphic structure that is shifted in degree by j, that is $A(j)_i = A_{i+j}$ for all i. For a scheme $V \subset \mathbb{P}^n$, we write I_V to denote the saturated ideal of V in R, or in S if $n = 3$, and $\mathcal{I}_V$ to denote the ideal sheaf of V. For a sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}^n}$ modules, we write $H^i_*(\mathcal{F})$ for the graded R-module $\bigoplus_{t\in\mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{F}(t))$ and $h^i(\mathcal{F})$ for $\dim_k(H^i(\mathbb{P}^n, \mathcal{F}))$.

In the first four sections of this chapter, we review some ideas from commutative algebra and the theory of algebraic curves, including generic initial ideals, primarily to establish notation and terminology. We close the chapter with a section on known restrictions on degrevlex generic initial ideals of locally Cohen-Macaulay space curves and some immediate consequences.

2.1 Notions from Commutative Algebra

In this section, we review some potentially less familiar concepts from commutative algebra, and then we give the statement of the Buchsbaum-Eisenbud Exactness Criterion.

Definition 2.1 (Hilbert Series). *Let V be a subscheme of $\mathbb{P}^n$. We define the* Hilbert series *of V to be the sum*

$$H_V(t) = \sum_{i=0}^{\infty} dim_k((R/I_V)_i)t^i.$$

That is, the Hilbert series of V is the Hilbert series of R/I_V.

Proposition 2.2 ([47], Proposition 16.7). *Let V be a subscheme of $\mathbb{P}^n$, and let $H_V(t)$ denote the Hilbert series of V. Let d denote the Krull dimension of R/I_V. Then we may write*

$$H_V(t) = \frac{P_V(t)}{(1-t)^d}$$

where $P_V(t) = 1 + c_1t + c_2t^2 + \cdots + c_rt^r$, with $c_i \in \mathbb{Z}$ for $1 \leq i \leq r$.

Note that if $dim(V) = 0$, then $H_V(t) = P_V(t)$.

Definition 2.3 (h-Vector). *Following the notation of Proposition 2.2, the sequence $(1, c_1, c_2, \ldots, c_r)$ is the* h-vector *of V. We will occasionally append zeroes to the end of this sequence for clearer exposition in some proofs.*

Definition 2.4 (Buchsbaum). *We say an R-module M is Buchsbaum if it is annihilated by $(x_0, \ldots, x_n)$.*

Notation 2.5 (Dual). *For a graded R-module M, let M^* denote the graded k-vector space*

$$\bigoplus_{i\in\mathbb{Z}} Hom_k(M_{-i}, k) = \bigoplus_{i\in\mathbb{Z}} (M^*)_i$$

with R-module structure given by $c \cdot f(m) = f(cm)$ where $c \in R, m \in M$, and $f \in M^*$.

For a locally free $\mathcal{O}_{\mathbb{P}^n}$ module $\mathcal{F}$ we use $\mathcal{F}^\vee$ to denote the dual sheaf $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^n})$.

Definition 2.6 (Socle Degree). *Let $X \subset \mathbb{P}^n$ be an arithmetically Gorenstein scheme of codimension c with minimal free resolution*

$$0 \to R(-t) \to F_{c-1} \to \cdots \to F_1 \to R.$$

We call the integer $r := t - c$ the socle degree *of R/I_X, or equivalently, the socle degree of X.*

Definition 2.7 (General Linear Form). *If M is a finite length graded S-module and l is a linear form, then we say l is* general *with respect to M if*

$$Length(M/l^r M) \leq Length(M/h^r M)$$

for any linear form h and any integer $r > 0$.

Lemma 2.8 ([31], Lemma 5.1.6). *Let M be a finite length graded S-module. There exists a Zariski open set of linear forms in S which are general with respect to M.*

Notation 2.9. *For a matrix A and an integer $i \geq 1$, let $I_i(A)$ denote the ideal generated by the $i \times i$ minors of A.*

We now present the Buchsbaum-Eisenbud Exactness Criterion.

Theorem 2.10 (Buchsbaum-Eisenbud Exactness Criterion; [7]). *Let*

$$\mathcal{F}:\ 0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

be a complex of finitely generated free R-modules. Then $\mathcal{F}$ is exact if and only if both:

1. *$rank(F_j) = rank(\phi_j) + \mathrm{rank}(\phi_{j+1})$ for $1 \leq j \leq n-1$, and*
2. *$I_{rank(\phi_j)}(\phi_j) = R$ or contains a regular sequence of length j.*

We note that we have not stated the Buchsbaum-Eisenbud Exactness Criterion in its full generality.

2.2 Generic Initial Ideals and General Coordinates

In this section, we review generic initial ideals and their properties, concluding with the statement of the Crystallization Principle.

Notation 2.11. *We write $GL(r,k)$ to denote the general linear group of $r \times r$ matrices over k.*

Definition 2.12 (Borel-Fixed Ideal)**.** *Let $\mathcal{B}$ denote the subgroup of upper-triangular matrices in $GL(n+1,k)$. If $g \in GL(n+1,k)$ is a matrix with $(i,j)^{th}$ entry denoted by $g_{i,j}$, we define an action on R by $g(x_j) = \sum_i g_{i+1,j+1}x_i$. Let $I \subset R$ be a monomial ideal. We say that I is Borel-fixed if for all $g \in \mathcal{B}$, $g \cdot I = I$.*

Equivalently, if $x_i m \in I$ for some monomial m, then $x_j m \in I$ for all $j < i$.

Proposition 2.13 ([4])**.** *If $I \subset R$ is Borel-fixed, then*

$$(I : x_j^\infty) = (I : (x_0, \ldots, x_j)^\infty)$$

and

$$(I : x_j^r) = (I : (x_0, \ldots, x_j)^r)$$

for all $r \geq 0$ *and* $0 \leq j \leq n$.

Theorem 2.14 ([5, 16])**.** *Let* $I \subset R$ *be an ideal. For any multiplicative term order* $\succ$ *on* R*, there is a non-empty Zariski open subset* $U \subset GL(n+1, k)$ *such that if* $g \in U$*, then the initial ideal* $in_{\succ}(g \cdot I)$ *is constant. If* $x_0 \succ x_1 \succ \cdots \succ x_n$*, then this ideal is Borel-fixed. We note that* U *depends on* I*.*

Definition 2.15 (Generic Initial Ideal)**.** *We call the fixed initial ideal from Theorem 2.14 the* generic initial ideal *of* I *with respect to* $\succ$ *and write* $gin_{\succ}(I)$*.*

Notation 2.16. *For an ideal* $I \subset R$*, we will write* $gin(I)$ *to mean the generic inital ideal of* I *with respect to degrevlex. From this point forward, when we say 'generic initial ideal,' we mean the generic initial ideal with respect to degrevlex. Similarly, for* $f \in R$*, we will write* $lt(f)$ *to mean the lead term of* f *with respect to degrevlex.*

Theorem 2.17 ([4])**.** *Let* $I \subset R$ *be an ideal.*

1. $gin(I^{sat}) = (gin(I) : x_n^{\infty})$.
2. *If* $X \subset \mathbb{P}^n$ *is a scheme and* H *is a general hyperplane, then*

$$gin(I_{X \cap H}) = (gin(I_X) : x_{n-1}^{\infty}) \cap k[x_0, \ldots, x_{n-1}].$$

3. *The Castelnuovo-Mumford regularity of* I *is equal to the highest degree of a minimal generator of* $gin(I)$*.*

Definition 2.18 (Double Saturation)**.** *Let* $I \subset R$ *be a monomial ideal. We call the ideal* $((I : x_n^{\infty}) : x_{n-1}^{\infty})$ *the* double saturation *of* I*.*

Definition 2.19 (General Coordinates)**.** *We say that a subscheme $V \subset \mathbb{P}^n$ is in* general coordinates *if $in(I_V) = gin(I_V)$.*

We will use this definition until Remark 4.23, where we will modify the definition for the remainder of the text.

We now present the Crystallization Principle and related concepts.

Definition 2.20 (Gap)**.** *Let $I \subset R$ be a monomial ideal. If I has no minimal generator in degree r, then we say that I has a* gap *in degree r.*

Notation 2.21. *Let $I \subset R$ be an ideal. We write $I_{<r}$ to denote the ideal generated by elements of I of degree less than r.*

Theorem 2.22 (Crystallization Principle; [19])**.** *If $gin(I)$ has a gap in degree r, then $gin(I_{<r}) = gin(I)_{<r}$.*

2.3 Space Curves and Deficiency Modules

In this section, we first define space curves and related notions, then follow with a discussion of the deficiency modules of schemes. We close the section with a brief discussion of the spectrum of a space curve.

Definition 2.23. *We say $C \subset \mathbb{P}^3$ is a* space curve *if C is a codimension two closed subscheme.*

Definition 2.24 (Arithmetically Cohen-Macaulay)**.** *Let $V \subset \mathbb{P}^n$ be a closed subscheme. V is* arithmetically Cohen-Macaulay (ACM) *if R/I_V is a Cohen-Macaulay ring.*

Definition 2.25 (Locally Cohen-Macaulay)**.** *Let $V \subset \mathbb{P}^n$ be a closed subscheme. V is* locally Cohen-Macaulay (LCM) *if $\mathcal{O}_{V,p}$ is a Cohen-Macaulay ring for each $p \in V$.*

We call a subscheme $C \subset \mathbb{P}^3$ a LCM space curve *if it is equidimensional of dimension one. This is equivalent to C being a dimension one subscheme of $\mathbb{P}^3$ having no isolated points and $\mathcal{O}_{C,p}$ being a Cohen-Macaulay ring for each $p \in C$.*

Theorem 2.26 ([25])**.** *A LCM space curve of degree d and genus g exists if and only if either*

1. *$d \geq 1$ and $g = \binom{d-1}{2}$ (plane curves), or*

2. *$d \geq 2$ and $g \leq \binom{d-2}{2}$.*

If $d \geq 2$ and $g = \binom{d-2}{2}$, then all LCM space curves are ACM.

We now discuss deficiency modules of schemes, giving some restrictions in the case of space curves.

Definition 2.27 (Deficiency Modules, Rao Module)**.** *Given a closed subscheme $V \subset \mathbb{P}^n$ of dimension $r \geq 1$, we define the i^{th}* deficiency module *of V to be the i^{th} cohomology module of the ideal sheaf of V. We write*

$$M_V^i = H_*^i(\mathcal{I}_V) \text{ for } 1 \leq i \leq r.$$

When V is of dimension one, we simply write M_V for the deficiency module and call this the Rao module *of V.*

Proposition 2.28 ([39], Lemma 1.2.3). *Given a closed subscheme $V \subset \mathbb{P}^n$ of dimension $r \geq 1$, V is ACM if and only if all the deficiency modules of V are equal to 0.*

Proposition 2.29 ([56], Theorem 9). *Let $V \subset \mathbb{P}^n$ be a closed subscheme. Assume that dim$(V) = r \geq 1$. Then V is LCM and equidimensional if and only if the all the deficiency modules of V have finite length.*

Remark 2.30. In particular, a closed one-dimensional subscheme of $\mathbb{P}^3$ is a LCM space curve if and only if its Rao module has finite length.

Corollary 2.31. *Let $C \subset \mathbb{P}^3$ be a codimension two subscheme. Then C is LCM if and only if the ideal of maximal minors of last map in its free resolution has codimension four.*

We can see this either by dualizing the resolution for S/I_C and applying Proposition 2.29 or by localizing the resolution and applying the Auslander-Buchsbaum formula.

Proposition 2.32 ([6]). *Given a collection $\{M^1, \ldots, M^r\}$ of graded R-modules of finite length and $n \geq r + 2$, there is a scheme $X \subset \mathbb{P}^n$ of dimension r with the following properties:*

1. *there is an integer d such that $M^i_X = M^i(d)$ for all $1 \leq i \leq r$,*
2. *if Y is a scheme of dimension r with $M^i_Y = M^i(e)$ for all $1 \leq i \leq r$, then $e \leq d$, and*
3. *for any integer e with $e \leq d$, there exists a scheme, Y, of dimension r with $M^i_Y = M^i(e)$ for all $1 \leq i \leq r$.*

Note that the integers d and e are the same for all i; that is, the modules in the collection are all shifted together.

In [54], Schlesinger develops the notion of the spectrum of a scheme.

Definition 2.33 (Spectrum)**.** *Let C be a LCM space curve disjoint from the line L defined by $z = w = 0$. Let $A_C = H^0_*(\mathbb{P}^3, \mathcal{O}_C)$. Define the* spectrum *of C to be the function*

$$h_C(n) = dim_k(A_C \otimes_S S/(z, w))_n.$$

That is, the spectrum of C gives the number of elements in a free basis of A_C over $k[x, y]$ in each degree.

We will use the notation

$$\{\ldots -2^{h_C(-2)}, -1^{h_C(-1)}, 0^{h_C(1)}, 1^{h_C(1)}, 2^{h_C(2)}, \ldots\}$$

to denote the spectrum of the curve C, omitting entries where $h_C(n) = 0$.

Remark 2.34. The definition of spectrum is independent of the choice of line L since it is determined by the Hilbert function of A_C. If L is a line in $\mathbb{P}^3$ defined by linear forms F_1 and F_2, and C is a LCM space curve disjoint from L, note that

$$h_C(n) = \dim_k(A_C \otimes_S S/(F_1, F_2))_n.$$

Proposition 2.35 ([54])**.** *Let C be a LCM space curve disjoint from the line L defined by $z = w = 0$. Then the spectrum of C is finitely supported.*

2.4 Hilbert Schemes

In this section, we establish notation and briefly discuss connectedness results.

Notation 2.36 (Hilbert Scheme)**.** *We write $Hilb_{d,g}$ to denote the Hilbert scheme of all closed codimension two subschemes of degree d and genus g in $\mathbb{P}^3$.*

We write $H_{d,g}$ to denote the Hilbert scheme of all LCM space curves of degree d and genus g in $\mathbb{P}^3$. We note that $H_{d,g}$ is an open subscheme of $Hilb_{d,g}$.

Theorem 2.37 ([22])**.** *$Hilb_{d,g}$ is connected for all pairs (d, g).*

Theorem 2.37 motivates the question of whether $H_{d,g}$ is connected. $H_{d,g}$ has two or more irreducible components when:

1. $d = 3$ and $g \leq -2$,
2. $d = 4$ and $g \leq 0$, or
3. $d \geq 5$ and $g \leq \binom{d-3}{2} + 1$.

$H_{d,g}$ is known to be connected for $d \leq 4$, for $d \geq 5$ and $g > \binom{d-3}{2} + 1$, and for several other specific pairs (d, g). For a more complete overview of connectedness results for $H_{d,g}$ see [27].

2.5 Known Restrictions

In this section, we present some known restrictions on generic initial ideals of LCM space curves. We close the section with a classification of the generic initial ideals of planar LCM space curves.

In his thesis ([54]), Schlesinger proves the following restriction on the possible spectrum of a LCM space curve.

Theorem 2.38 ([54], Proposition 1.7.1). *Let C be a LCM space curve with spectrum h_C. Let $e = max\{n \mid H^1(\mathcal{O}_C(n)) \neq 0\}$. Then $e + 2 = max\{n \mid h_C(n) \neq 0\}$. Suppose there exists an integer l with $1 \leq l \leq e + 2$ and $h_C(l) = 1$. Then for all n with $l \leq n \leq e + 2$ we have $h_C(n) = 1$.*

Example 2.39. The set $\{-1, 0, 1^2, 2, 3^2\}$ cannot be the spectrum of a LCM space curve because 2 appears in the spectrum only once, whereas 3 appears twice.

We now introduce the concept of λ-invariants for space curves and give a restriction on the λ-invariants of integral space curves.

Definition 2.40 (λ-Invariants). *Let $I \subset k[x, y, z]$ be a saturated, codimension two, Borel-fixed ideal. There exists an increasing sequence of positive integers $\lambda_1 < \lambda_2 < \cdots < \lambda_s$ such that*

$$in(I) = (x^s, x^{s-1}y^{\lambda_1}, x^{s-2}y^{\lambda_2}, \ldots, xy^{\lambda_{s-1}}, y^{\lambda_s}).$$

We call the set $\{\lambda_i \mid 1 \leq i \leq s\}$ the λ-invariants *of the ideal I. We use the convention $\lambda_0 = 0$.*

If X is a set of points in $\mathbb{P}^2$, we define the λ-invariants of X to be those of $gin(I_X)$. We use the notation λ_i^X to refer to the i^{th} λ-invariant of X, suppressing the exponent if the context is clear.

If C is a space curve in general coordinates we define the λ-invariants of C to be the λ-invariants of the saturated ideal of a general hyperplane section of C. More precisely, the λ-invariants of C are those of the double saturation of $gin(I_C)$. We use the notation λ_i^C to refer to the i^{th} λ-invariant of C, suppressing the exponent if the context is clear. We note that we have numbered the λ-invariants in the opposite direction from the usual convention.

Definition 2.41 (Connected)**.** *We say that a set of λ-invariants is* connected *if they satisfy $\lambda_{i+1} - 1 \geq \lambda_i \geq \lambda_{i+1} - 2$.*

Definition 2.42 (Uniform Position)**.** *Let $X \subset \mathbb{P}^2$ be as set of points. We say that X is in* uniform position *if all the subsets of X of the same cardinality have the same Hilbert function.*

Theorem 2.43 (Uniform Position Principle; [20])**.** *If $X \subset \mathbb{P}^2$ is a set of points in uniform position, then the λ-invariants of X are connected.*

Corollary 2.44 (Connectedness Criterion; [1], Chapter 3, Section 1)**.** *If C is an integral space curve, then the λ-invariants of C are connected.*

We note that Cook has proved a more general result in [9].

The following theorem of Hartshorne restricts the possible LCM space curves with general hyperplane section contained in a line.

Proposition 2.45 (Restriction Theorem; [25])**.** *Let K be a field and C be a locally Cohen-Macaulay space curve in $\mathbb{P}^3_K$. Assume that deg$(C) \geq 3$, that C is not contained in a plane, and that for all general planes H, the scheme intersection $C \cap H$ is contained in a line of H. Then K is of positive characteristic, the support of C is a line L, and for any $P \in L$ there exists a surface T containing C, which is smooth at P, so that in a neighborhood of P, the scheme C is just the divisor dL on T.*

Remark 2.46. The Restriction Theorem implies that if C is a LCM space curve of degree at least three and $xz^n \in \text{gin}(I_C)$, then $x \in \text{gin}(I_C)$. In particular, C is planar.

We make repeated use of the following two lemmas.

Lemma 2.47. *Let $I, J_1, J_2 \subset R$ be ideals with $(I : J_1) = J_2$. Then the associated primes of J_2 are associated primes of I.*

Proof. Suppose that $I = \cap_i Q_i$, where this intersection is a minimal primary decomposition of I. Then $J_2 = (I : J_1) = \cap_i (Q_i : J_1)$ is a primary decomposition of J_2 after removing copies of R appearing in the intersection. For each i, let P_i denote the radical of Q_i, so that the set of all the P_i is the set of associated primes of I.

If $J_1 \subset Q_i$, then $(Q_i : J_1) = R$. If $J_1 \subset P_i$ and $J_1 \not\subset Q_i$, then $Q_i \subset (Q_i : J_1) \subset P_i$; in particular, the radical of $(Q_i : J_1)$ is P_i. If J_1 is not contained in P_i, then the radical of $(Q_i : J_1)$ is P_i.

In particular, the associated primes of J_2 are all associated primes of I. □

Note that if C is a LCM space curve and $J_2 = (I_C : J_1)$ for some ideals J_1 and $J_2 \neq R$, then J_2 is also the ideal of a LCM space curve.

We use the following elementary fact throughout.

Lemma 2.48. *Let $C \subset \mathbb{P}^n$ be a equidimensional closed subscheme of codimension one. Then I_C is generated by a single element.*

Proof. Suppose that $I_C = (f_1, \ldots, f_r)$ for some $r \geq 2$, and that this is a minimal generating set. Since C has codimension one, the generators have a common factor. Let $H = \gcd\{f_i\}_{i=1}^r$. Let g_i denote f_i/H for each i. Then $(I_C : H) = (g_1, \ldots, g_r)$. This ideal has codimension two. Since none of the g_i are constants and I_C is saturated, this contradicts Lemma 2.47. We conclude that I_C is generated by a single element. □

Lemma 2.48 gives the following restriction for LCM space curves contained in a plane:

Proposition 2.49. *Let C be a LCM space curve of degree d. C is planar if and only if $gin(I_C) = (x, y^d)$.*

Proof. If $\text{gin}(I_C) = (x, y^d)$, then I_C contains a linear form, and C is planar. For the reverse implication, we apply Lemma 2.48 to see that if C is planar and in general coordinates, then $I_C = (L, f)$ for some linear form L with $\text{lt}(L) = x$. Without loss of generality, $\{L, f\}$ in a Gröbner basis for I_C. We conclude that

$$\text{gin}(I_C) = (x, \text{lt}(f)).$$

Since I_C has codimension two, $\text{lt}(f) = y^n$ for some $n \geq 1$. Since the leading coefficient of the Hilbert polynomial is d, we must have $n = d$. □

CHAPTER 3
LIAISON

In this chapter we discuss the concepts of linked schemes and biliaison classes, following the development in [39]. These concepts are useful tools in the study of LCM space curves. Given a LCM scheme, all schemes linked to it are also LCM; linkage provides a way to compute new LCM schemes from known ones. Further, if we know cohomological and geometric invariants for a given scheme, we can compute many of these invariants for schemes linked to it. In particular, space curves in the same biliaison class have isomorphic Rao modules, up to a twist. We close the chapter by presenting some results connecting the concept of liaison with deformations on $H_{d,g}$.

Several results in this section were originally proved only for the case of linked LCM schemes and were later extended to the case of linked equidimensional schemes in [40] and [42].

Definition 3.1 (Geometrically Linked Schemes)**.** *Let V_1 and V_2 be subschemes of $\mathbb{P}^n$ such that no component of V_1 is contained in any component of V_2 and conversely. If $X = V_1 \cup V_2$ is an arithmetically Gorenstein scheme, we say V_1 is* geometrically directly linked *to V_2 via X. The scheme V_1 is said to be* residual *to V_2 in X.*

We extend the notion of geometrically linked schemes to the notion of algebraically linked schemes in order to handle cases where V_1 and V_2 share components.

Definition 3.2 (Algebraically Linked Schemes)**.** *Let V_1 and V_2 be subschemes of $\mathbb{P}^n$ of codimension r. Let X be an arithmetically Gorenstein subscheme of $\mathbb{P}^n$ such* *that $I_X \subset I_{V_1} \cap I_{V_2}$. We say V_1 is* algebraically directly linked *to V_2 by X, and write $V_1 \overset{X}{\sim} V_2$, if $(I_X : I_{V_1}) = I_{V_2}$ and $(I_X : I_{V_2}) = I_{V_1}$. The scheme V_1 is said to be* residual *to V_2 in X.*

Remark 3.3. There is a notion of linkage by complete intersections, and some stronger results are true in that case. In codimension two, Gorenstein and complete intersection ideals coincide, so that in the situation of space curves we may apply results from either the Gorenstein or the complete intersection case.

Proposition 3.4 ([39], Proposition 5.2.2)**.** *Let V_1 and V_2 be subschemes of $\mathbb{P}^n$. If V_1 and V_2 are geometrically linked, then they are algebraically linked.*

Conversely, assume that $V_1 \overset{X}{\sim} V_2$. Then:

1. *as sets, $V_1 \cup V_2 = X$,*
2. *V_1 and V_2 are equidimensional of the same dimension, and have no embedded components, and*
3. *if V_1 and V_2 have no common components, then they are geometrically linked.*

Remark 3.5. From this point on, we will assume all our links are algebraic links.

Definition 3.6 (Liaison)**.** Liaison *is the equivalence relation generated by direct linkage. Let V_1 and V_2 be subschemes of $\mathbb{P}^n$. We say that V_1 is* linked *to V_2, denoted $V_1 \sim V_2$, if there is a sequence of schemes $W_1, \ldots, W_i$ and a sequence of arithmetically Gorenstein schemes $X_1, \ldots X_{i+1}$ such that*

$$V_1 \overset{X_1}{\sim} W_1 \overset{X_2}{\sim} \cdots \overset{X_i}{\sim} W_i \overset{X_{i+1}}{\sim} V_2.$$

The equivalence classes generated by liaison are called liaison classes.

Definition 3.7 (Biliaison)**.** *If the number $i+1$ in Definition 3.6 is even, then we say that V_1 and V_2 are* evenly linked. *We will also refer to evenly linked schemes* *as being* bilinked. Biliaison *is the equivalence relation generated by even linkage. We refer to the equivalence classes generated by this relation as* even liaison classes *or* biliaison classes.

As a consequence of elementary facts about colon ideals, linked schemes are always equidimensional. We will see a similar result for LCM schemes after a series of results relating the resolutions of linked schemes.

Proposition 3.8 ([39], Proposition 5.2.6)**.** *Let $V_1 \overset{X}{\sim} V_2$ where V_1 and V_2 are of codimension c in $\mathbb{P}^n$ and X is arithmetically Gorenstein with minimal free resolution*

$$0 \to R(-t) \to F_{c-1} \to \cdots \to F_1 \to I_X \to 0.$$

Then we have an exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_{V_1} \to \omega_{V_2}(n+1-t) \to 0.$$

Proposition 3.9 ([48])**.** *Let $V_1 \overset{X}{\sim} V_2$, where V_1 and V_2 are of codimension c in $\mathbb{P}^n$ and X is arithmetically Gorenstein with sheafified minimal free resolution*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^n}(-t) \arrow[r] & \mathcal{F}_{c-1} \arrow[r] & \cdots \arrow[r] & \mathcal{F}_1 \arrow[rr] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^n} \arrow[r] & \mathcal{O}_X \arrow[r] & 0. \\
& & & & & \mathcal{I}_X \arrow[ur] \arrow[dr] & & & \\
& & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Assume that V_1 is LCM and suppose that we are given a locally free resolution for $\mathcal{I}_{V_1}$ of the form

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}_{c-1} \arrow[r] & \cdots \arrow[r] & \mathcal{G}_1 \arrow[rr] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^n} \arrow[r] & \mathcal{O}_{V_1} \arrow[r] & 0. \\
& & & & & \mathcal{I}_{V_1} \arrow[ur] \arrow[dr] & & & \\
& & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Then there is a (non-minimal) locally free resolution of $\mathcal{I}_{V_2}$ of the form

$$
\begin{aligned}
0 \longrightarrow \mathcal{G}_1^\vee(-t) \longrightarrow \mathcal{F}_1^\vee \oplus \mathcal{G}_2^\vee(-t) \longrightarrow \mathcal{F}_2^\vee \oplus \mathcal{G}_3^\vee(-t) \to \cdots \\
\cdots \to \mathcal{F}_{c-1}^\vee(-t) \oplus \mathcal{G}^\vee \longrightarrow \mathcal{I}_{V_2} \longrightarrow 0.
\end{aligned}
$$

Remark 3.10. The resolutions in Proposition 3.9 can be constructed so that the $\mathcal{G}_i$ are free, and in this case, the assumption that V_1 is LCM is not necessary. See [40].

Corollary 3.11 ([48])**.** *Let $V_1 \overset{X}{\sim} V_2$, where $V_1, V_2 \subset \mathbb{P}^n$ and X is arithmetically Gorenstein. Then V_1 is LCM if and only if V_2 is LCM.*

The next two results follow from Proposition 3.8.

Proposition 3.12 ([39], Proposition 5.2.17)**.** *Let $V_1 \overset{X}{\sim} V_2$, where $V_1, V_2 \subset \mathbb{P}^n$, X is arithmetically Gorenstein, and $2 \leq codim(X) < n$. Let F be a general hypersurface of degree d. Then*

$$(V_1 \cap F) \overset{X \cap F}{\sim} (V_2 \cap F)$$

in $\mathbb{P}^n$.

Proposition 3.13 ([10])**.** *Let $V_1 \overset{X}{\sim} V_2$, where V_1 and V_2 are ACM and of codimension c in $\mathbb{P}^n$, and X is arithmetically Gorenstein with socle degree $r = t - c$. Let V_1, V_2, and X have h-vectors $(1, v_1^1, v_2^1, \ldots, v_j^1)$, $(1, v_1^2, v_2^2, \ldots, v_l^2)$, and $(1, c, c_2, \ldots, c_{r-2}, c, 1)$, respectively. Then*

$$c_i - v_i^1 = v_{t-c-i}^2$$

for all i.

Example 3.14. Let $C \subset \mathbb{P}^3$ be a complete intersection intersection of type $(2,3)$ contained in a complete intersection X of type $(3,3)$. The h-vector of C is given by $(1,2,2,1)$. The h-vector of X is given by $(1,2,3,2,1)$.

Define D by $C \overset{X}{\sim} D$. To compute the h-vector of D we first subtract the h-vector of C from that of X to get $(0,0,1,1,1)$, and then reverse the order to get (1,1,1).

This relationship between the Hilbert functions of linked schemes allows us to compute the degree of a scheme from that of a linked scheme and the genus of a curve from that of a linked curve.

Proposition 3.15 ([39], Corollary 5.2.13)**.** *Let $V_1 \overset{X}{\sim} V_2$, where $V_1, V_2 \subset \mathbb{P}^n$ and X is arithmetically Gorenstein. Then $deg(V_1) + deg(V_2) = deg(X)$.*

Proposition 3.16 ([40])**.** *Let $C_1 \overset{X}{\sim} C_2$, where C_1, $C_2 \subset \mathbb{P}^n$ are closed dimension one subschemes of genera g_1 and g_2, respectively, and X is arithmetically Gorenstein with minimal free resolution*

$$0 \to R(-t) \to F_{r-1} \to \cdots \to F_1 \to I_X \to 0.$$

Then

$$g_1 - g_2 = \frac{1}{2}(t-n-1)(deg(C_1) - deg(C_2)).$$

We now review the relationships between the deficiency modules of linked schemes.

Theorem 3.17 (Hartshorne-Schenzel Theorem; [8, 25, 38, 53]; cf. [49])**.** *Let $V_1 \overset{X}{\sim} V_2$ where $V_1, V_2 \subset \mathbb{P}^n$ are of dimension r and X is arithmetically Gorenstein. If X has sheafified minimal free resolution*

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-t) \to \mathcal{F}_{n-r-1} \to \cdots \to \mathcal{F}_1 \to \mathcal{I}_X \to 0,$$

then

$$M_{V_2}^{r-i+1} \cong (M_{V_1}^{i})^*(n+1-t) \text{ for } 1 \le i \le r.$$

The proof of the Hartshorne-Schenzel Theorem makes substantial use of Proposition 3.9.

Corollary 3.18. *Let V_1, $V_2 \subset \mathbb{P}^n$ of dimension r be evenly linked schemes. Then there is an integer p such that $M_{V_1}^i(p) \cong M_{V_2}^i$ for each $1 \le i \le r$.*

Remark 3.19. See Proposition 4.40 for a computation of the value of p in the case of space curves.

In [50], Rao proves a converse to the Hartshorne-Schenzel Theorem for space curves.

Theorem 3.20 ([50])**.** *Let C and C' be LCM space curves with Rao modules M_C and $M_{C'}$. Then C and C' are in the same biliaison class if and only if M_C is isomorphic to some shift of $M_{C'}$.*

We now define two types of links that are of particular use computationally. First we introduce basic double links, a concept of which we make substantial use in Chapter 7.

Definition 3.21 (Basic Double Link)**.** *Let $V_1 \subset \mathbb{P}^n$ be a scheme of dimension r. Choose a general polynomial $F_1 \in R$ of any degree greater than zero and $F_2, \ldots, F_{n-r} \in I_{V_1}$ such that $(F_1, F_2, \ldots, F_{n-r})$ forms a regular sequence. Let d_i denote the degree of F_i for $1 \le i \le n-r$.*

Choose a general polynomial $G \in I_{V_1}$ of sufficiently large degree so that $(G, F_2, \ldots, F_{n-r})$ forms a regular sequence.

Let X_1 denote the scheme defined by $(G, F_2, \ldots, F_{n-r})$ and let X_2 denote the scheme defined by $(GF_1, F_2, \ldots, F_{n-r})$. Define W and V_2 by

$$V_1 \overset{X_1}{\sim} W \overset{X_2}{\sim} V_2.$$

We say that V_2 is obtained from V_1 by a basic double link*, or equivalently, V_2 is a basic double link of V_1.*

Proposition 3.22 ([6, 17, 30])**.** *With notation as in Definition 3.21,*

$$I_{V_2} = F_1 I_{V_1} + (F_2, \ldots, F_{n-r})$$

and

$$M^i_{V_2} \cong M^i_{V_1}(-d_1)$$

for all i.

Definition 3.23 (Height)**.** *We call d_1 from Proposition 3.22 the* height *of the basic double link from V_1 to V_2.*

Notation 3.24. *Let V_1 be a codimension two subscheme of $\mathbb{P}^n$ and $F_2 \in I_{V_1}$ be a homogeneous polynomial of degree d_2. Choose $F_1 \in R_{d_1}$ such that F_1 and F_2 form a regular sequence. Let V_2 be the scheme obtained by performing the corresponding basic double link. Then we write:*

$$V_1 : (d_2, d_1) \to V_2.$$

We say the corresponding basic double link is of type (d_2, d_1).

Proposition 3.25 ([39], Lemma 6.4.10)**.** *Let $C : (a, b) \to C'$, where C and C' are both LCM space curves. Let $g(C)$ and $g(C')$ denote the arithmetic genera of C and C' respectively. Then*

$$g(C') = b \cdot deg(C) + \frac{1}{2}ab(a + b - 4) + g(C)$$

and

$$deg(C') = ab + deg(C).$$

This proposition is an immediate consequence of Propositions 3.15. and 3.16

Basic double links are a special case of elementary double links. We define elementary double links in the case of space curves. There are more general definitions of such links, but the one below is more clear for our circumstances. In [31], Liebling uses elementary double links in several classification results. For example, see Proposition 4.38.

Definition 3.26 (Elementary Double Link; cf. [31], Definition 3.4.13)**.** *Let C be a LCM space curve. Choose F, $G_1 \in I_C$ such that (F, G_1) is a complete intersection. Let X_1 denote the scheme defined by (F, G_1). Define D by $C \overset{X_1}{\sim} D$.*

Choose $G_2 \in I_D$ such that (F, G_2) is a complete intersection. Let X_2 denote the scheme defined by (F, G_2). Define C' by $D \overset{X_2}{\sim} C'$.

Let the integer a denote the degree of F, and define the integer $b := deg(G_2) - deg(G_1)$. We say that C' is obtained from C by an elementary double link *of type (a, b). The integer b is the* height *of elementary double link from C to C'. If $b > 0$ we call this an* ascending *elementary double link. If $b < 0$ we call this a* descending *elementary double link.*

Remark 3.27. Let C and C' be LCM space curves such that C' is obtained from C by an elementary double link of type (a, b). Let $g(C)$ and $g(C')$ denote the arithmetic genera of C and C' respectively. Using Propositions 3.15 and 3.16, we compute

$$g(C') = b \cdot \deg(C) + \frac{1}{2}ab(a + b - 4) + g(C)$$

and

$$\deg(C') = ab + \deg(C).$$

Proposition 3.28. *Let C and C' be LCM space curves such that C' is obtained from C by an elementary double link of height b. Then*

$$M_{C'} \cong M_C(-b).$$

Proposition 3.28 is a consequence of the Hartshorne-Schenzel Theorem.

In order to more clearly explain the connections between liaison and deformations on $H_{d,g}$, we introduce the notion of minimal elements in a biliaison class.

Definition 3.29 (Minimal in a Biliaison Class)**.** *Let $\mathcal{L}$ be an even liaison class. We will denote by $\mathcal{L}^0$ the set of schemes $V \in \mathcal{L}$ with deficiency modules generated in the smallest possible degrees among elements of $\mathcal{L}$ (see Proposition 2.32 and Corollary 3.18). The elements of $\mathcal{L}^0$ are the* minimal *elements of the even liaison class $\mathcal{L}$.*

Let $V_0 \in \mathcal{L}^0$. We will denote by $\mathcal{L}^h$ the set of schemes $V \in \mathcal{L}$ satisfying $M^i_V \cong M^i_{V_0}(-h)$ for $1 \leq i \leq r$.

Remark 3.30. By using basic double links of height one, we see that $\mathcal{L}^h$ is nonempty for all $h \geq 0$.

In [33], Martin-Deschamps and Perrin give an algorithm for computing minimal curves in a biliaison class.

Definition 3.31 (Lazarsfeld-Rao Property)**.** *Let $\mathcal{L}$ be an even complete intersection liaison class of dimension r subschemes in $\mathbb{P}^n$. We say that $\mathcal{L}$ has the* Lazarsfeld-Rao Property *if the following conditions hold:*

1. *If $V_1, V_2 \in \mathcal{L}^0$, then there is a deformation from one to the other through subschemes all in $\mathcal{L}^0$; in particular, all subschemes in the deformation are in the same even liaison class.*

2. *Given $V_0 \in \mathcal{L}^0$ and $V \in \mathcal{L}^h$ with $h \geq 1$, there exists a sequence of subschemes $V_0, V_1, \dots, V_t$ such that for $1 \leq i \leq t$, V_i is a basic double link of V_{i-1} and V is a deformation of V_t through subschemes all in $\mathcal{L}^h$.*

Theorem 3.32 ([3])**.** *Every biliaison class of non-ACM codimension two subschemes of $\mathbb{P}^n$ has the Lazarsfeld-Rao Property.*

Remark 3.33. Theorem 3.32 is trivially not true in the case of ACM space curves. The Rao module of any ACM curve is zero, but ACM curves exist for different degrees and genera.

CHAPTER 4

TRIANGLE DIAGRAMS

Lower triangle diagrams are a useful tool for studying Borel-fixed monomial ideals. These diagrams are well studied (cf. [19]).

In [14], Fløystad develops the notion of higher initial ideals of an ideal and uses these to define a complete triangle diagram of a space curve in general coordinates. In his Ph.D. thesis ([31]), Liebling modifies and generalizes this definition, allowing him to develop connections between the numerics of complete triangle diagrams and the Rao modules of space curves.

After reviewing lower triangle diagrams, we proceed to highlight some of Liebling's results. We follow the notation introduced in [31].

Definition 4.1 (Lower Triangle Diagram)**.** *A* lower triangle diagram Δ_0 *is defined to be a function*

$$\Delta_0 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$$

satisfying the following conditions:

1. $\Delta_0(0,0) = +\infty$,

2. for all $(i,j) \in \mathbb{N} \times \mathbb{N}$, *we have*

$$\Delta_0(i,j) \geq \Delta_0(i+1,j) \text{ and}$$

$$\Delta_0(i,j) \geq \Delta_0(i,j+1), \text{ and}$$

3. there exists $N > 0$ *such that if* $i + j \geq N$, *then* $\Delta_0(i,j) = 0$.

The set of ordered pairs (i,j) *such that* $0 < \Delta_0(i,j) < +\infty$ *will be denoted* $\mathcal{B}_{\Delta_0}$, *or simply* $\mathcal{B}$ *if the context is clear.*

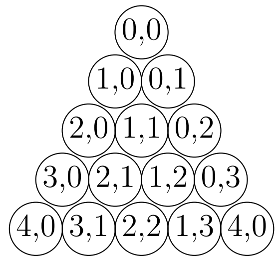


Figure 4.1: Diagram Indexing

We will write $(i, j) \in \Delta_0$ to mean $\Delta_0(i, j) < +\infty$. Similarly, when we refer to entries of Δ_0, we mean those pairs (i, j) such that $\Delta_0(i, j) < +\infty$.

For fixed i and j, we will refer to the indices of the form $(i, \cdot)$ as the i^{th} x-column of Δ_0 and the indices of the form $(\cdot, j)$ as the j^{th} y-column of Δ_0.

For fixed n, we will refer to the indices (i, j) such that $i + j = n$ as the n^{th} row of Δ_0, or equivalently, as the row of Δ_0 corresponding to degree n.

We may represent a lower triangle diagram Δ_0 by a triangular array of circles with coordinates as in the Figure 4.1. Subsequent rows are indexed similarly, as needed. We note that it is sufficient to display only the first N rows of a lower triangle diagram, where N is as in (3), above. For each pair (i, j), the $(i, j)^{th}$-entry will be left blank if $\Delta_0 = +\infty$ and will shaded otherwise. If $(i, j) \in \mathcal{B}_{\Delta_0}$, the $(i, j)^{th}$-entry will be labeled with the value of $\Delta_0(i, j)$.

Remark 4.2. Condition (2) in Definition 4.1 implies that along any column all indices that are entries in Δ_0 occur after indices that are not entries in Δ_0. Further, nonzero entries of Δ_0 are top justified within Δ_0.

The following well known result relates lower triangle diagrams to monomial ideals. We take our statement from [31], Proposition 2.3.19.

Proposition 4.3. *Suppose $I \subset k[x, y, z, w]$ is a monomial ideal and that the following conditions hold:*

1. *I is generated by monomials in x, y, and z, or equivalently, $(I : w) = I$,*
2. *there exist $s, t > 0$ such that $x^s, y^t \in I$, and*
3. *$z^r \notin I$ for any $r \geq 0$. In particular, $1 \notin I$.*

Define $\Delta_0(I) : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ by

$$\Delta_0(I)(i, j) = min\{n \mid x^i y^j z^n \in I\}$$

allowing $\Delta_0^I(i, j) = +\infty$ if $x^i y^j z^n \notin I$ for all n. Then $\Delta_0(I)$ is a lower triangle diagram.

Conversely, if Δ_0 is a lower triangle diagram, then the set of monomials

$$\{x^i y^j z^n \mid (i, j, n, 0) \in \mathbb{N}^4 \text{ and } \Delta_0(i, j) \leq n\}$$

generates a monomial ideal $I(\Delta_0)$ satisfying the conditions above, and $\Delta_0(I(\Delta_0)) = \Delta_0$.

If I is a monomial ideal satisfying the conditions above, then $I(\Delta_0(I)) = I$.

Remark 4.4. If $I = I_C$, the saturated ideal of a curve $C \subset \mathbb{P}^3$, we will write $\Delta_0(C)$ to refer to $\Delta_0(in(I_C))$, should $in(I_C)$ meet the conditions of the previous proposition. We note that this will be the case if C does not meet the line $z = w = 0$; in particular, this will be satisfied if C is in general coordinates.

Example 4.5. The lower diagram Δ_0 corresponding to the ideal $I = (x^2, xy^2, y^3z^2, y^4)$ is displayed in Figure 4.2.

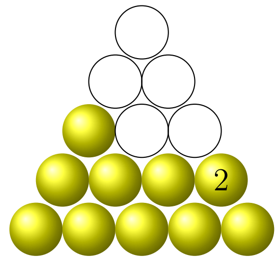


Figure 4.2: Diagram for Example 4.5

Definition 4.6 (Borel-fixed Lower Triangle Diagram)**.** *Let Δ_0 be a lower triangle diagram. We say that Δ_0 is* Borel-fixed *if it satisfies the following conditions:*

1. $\Delta_0(i,j) \leq \Delta_0(i-1, j+1)$ *whenever* $i > 0$*, and*
2. $\Delta_0(i,j) > \Delta_0(i, j+1)$ *unless both are* 0 *or both are* $+\infty$*.*

In particular, entries of Δ_0 are weakly increasing along rows from left to right and are strictly decreasing down x-columns until reaching 0*.*

Remark 4.7. Let Δ_0 be a Borel-fixed lower triangle diagram. Condition (1) implies that along any row, all indices that are entries in Δ_0 occur before indices that are not entries in Δ_0. Conditions (1) and (2) together imply that $\Delta_0(i,j) > \Delta_0(i+1,j)$ unless both are 0 or both are $+\infty$. In particular, entries of Δ_0 are strictly decreasing down y-columns until reaching 0. Further, nonzero entries in Δ_0 are right justified within Δ_0.

The following well-known proposition relates Borel-fixed ideals to Borel-fixed lower triangle diagrams.

Proposition 4.8. *If I is a monomial ideal satisfying the conditions of Proposition 4.3, then $\Delta_0(I)$ is Borel-fixed precisely when I is.*

We can also compute the λ-invariants of a space curve C from its lower triangle diagram. If C is in general coordinates and x^a is the smallest power of x appearing in the double saturation of I_C, then the λ-invariants of C can be computed from $\Delta_0(C)$ by noting that λ_i^C is equal to the number of indices along the $(a-i)^{th}$ x-column that are not entries in $\Delta_0(C)$.

We proceed with our discussion of Liebling's results.

Definition 4.9 (Upper Triangle Diagram; cf. [31], Definition 3.4.1)**.** *An* upper triangle diagram Δ_1 *is defined to be a function*

$$\Delta_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$$

satisfying the following conditions:

1. $\Delta_1(0,0) = 0$,
2. *for all* $(i,j) \in \mathbb{N} \times \mathbb{N}$*, we have*

$$\Delta_1(i,j) \leq \Delta_1(i+1,j) \text{ and}$$
$$\Delta_1(i,j) \leq \Delta_1(i,j+1), \text{ and}$$

3. *there exists* $N > 0$ *such that if* $i + j \geq N$ *then* $\Delta_1(i,j) = +\infty$.

The set of ordered pairs (i,j) *such that* $0 < \Delta_1(i,j) < +\infty$ *will be denoted* $\mathcal{A}_{\Delta_1}$*, or simply* $\mathcal{A}$ *if the context is clear.*

We will write $(i,j) \in \Delta_1$ *to mean* $\Delta_1(i,j) < +\infty$*. Similarly, when we refer to entries of* Δ_1*, we mean those pairs* (i,j) *such that* $\Delta_1(i,j) < +\infty$*.*

For fixed i *and* j*, we will refer to the indices of the form* $(i,\cdot)$ *as the* i^{th} x*-column of* Δ_1 *and the indices of the form* $(\cdot,j)$ *as the* j^{th} y*-column of* Δ_1*.*

For fixed n, we will refer to the indices (i, j) such that $i + j = n$ as the n^{th} row of Δ_1, or equivalently, as the row of Δ_1 corresponding to degree n.

We may represent an upper triangle diagram Δ_1 by a triangular array of circles with coordinates as in Figure 4.1. Subsequent rows are indexed similarly, as needed. We note that it is sufficient to display only the first N rows of an upper triangle diagram, where N is as in (3), above. For each pair (i, j), the $(i, j)^{th}$-entry will be shaded if $\Delta_1 = +\infty$ and will not be shaded otherwise. If $(i, j) \in \mathcal{A}_{\Delta_1}$, the $(i, j)^{th}$-entry will be labeled with the value of $\Delta_1(i, j)$.

Remark 4.10. Condition (2) in Definition 4.9 implies that along any column all indices that are entries in Δ_1 occur before indices that are not entries in Δ_1. Further, nonzero entries of Δ_1 are bottom justified within Δ_1.

Definition 4.11 (Complete Triangle Diagram; cf. [31], Definition 3.2.1)**.** *A* (complete) triangle diagram Δ *consists of a pair (Δ_0, Δ_1), where Δ_0 is a lower triangle diagram and Δ_1 is an upper triangle diagram, satisfying the following conditions:*

1. *$\Delta_1(i, j) = +\infty$ if and only if $\Delta_0(i, j) \neq +\infty$ and*
2. $\displaystyle\sum_{(i,j)\in\mathcal{B}} \Delta_0(i, j) = \sum_{(i,j)\in\mathcal{A}} \Delta_1(i, j).$

For fixed i and j, we will refer to the indices of the form $(i, \cdot)$ as the i^{th} x-column of Δ and the indices of the form $(\cdot, j)$ as the j^{th} y-column of Δ.

For fixed n, we will refer to the indices (i, j) such that $i + j = n$ as the n^{th} row of Δ, or equivalently, as the row of Δ corresponding to degree n.

Proposition 4.12 ([31], Proposition 3.4.4)**.** *Let C be a LCM space curve not meeting the line $z = w = 0$. Let X be any complete intersection curve containing*

C and having initial ideal $in(I_X) = (x^r, y^s)$ for some $r, s > 0$. Define D by $C \overset{X}{\sim} D$. Then D does not meet the line $z = w = 0$, so that $\Delta_0(D)$ is well defined. Further, $\Delta_0(D)$ is constant across all choices of X, r, and s. Define

$$\Delta_1^L(C)(i,j) = \begin{cases} \Delta_0(D)(r-1-i, s-1-j) & \text{if } i \le r-1 \text{ and } j \le s-1 \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\Delta^L(C) = (\Delta_0(C), \Delta_1^L(C))$ is a triangle diagram.

Definition 4.13 (Liebling Triangle Diagram)**.** *Let C be a LCM space curve not meeting the line $z = w = 0$. We call the triangle diagram*

$$\Delta^L(C) = (\Delta_0(C), \Delta_1^L(C))$$

from Proposition 4.12 the Liebling triangle diagram of C.

Proposition 4.14 ([31], Theorem 3.4.9)**.** *Let C be a LCM space curve not meeting the line $z = w = 0$. Let $\Delta^L = (\Delta_0, \Delta_1^L)$ denote the Liebling triangle diagram of C. Define two S-modules N_C and Q_C by the exact sequence*

$$0 \to N_C \to M_C(-1) \xrightarrow{w} M_C \to Q_C \to 0.$$

Then we have graded $k[z]$-module isomorphisms

$$N_C \cong_{k[z]} \bigoplus_{(i,j)\in\mathcal{B}} \frac{k[z]}{z^{\Delta_0(i,j)}}(-i-j)$$

and

$$Q_C \cong_{k[z]} \bigoplus_{(i,j)\in\mathcal{A}} \frac{k[z]}{z^{\Delta_1^L(i,j)}}(\Delta_1^L(i,j)-i-j).$$

Remark 4.15. Liebling defines Δ_1^L in terms of higher initial modules of an ideal, a concept that he develops in his thesis. He proves that the above definition is equivalent. The proofs of Proposition 4.12 and Proposition 4.14 are two of the main results of his thesis, requiring a thorough discussion of higher initial modules, which is beyond the scope of this dissertation. However, given Proposition 4.14, it should be unsurprising that the Liebling triangle diagram of a LCM space curve C is essentially the inversion of the Liebling triangle diagram of a linked curve D since their Rao modules are dual to each other.

Now we move to a discussion of triangle diagrams for curves in general coordinates, including a discussion of the triangle diagrams of Fløystad.

Definition 4.16 (Weakly Borel-Fixed Triangle Diagram). *We say a triangle diagram $\Delta = (\Delta_0, \Delta_1)$ is* weakly Borel-fixed *if Δ_0 is Borel-fixed and Δ_1 satisfies*

1. *$\Delta_1(i+1, j) > \Delta_1(i, j)$ unless both are zero or both are $+\infty$ and*
2. *$\Delta_1(i, j+1) > \Delta_1(i, j)$ unless both are zero or both are $+\infty$.*

That is, entries of Δ_1, once nonzero, are strictly increasing down both x-columns and y-columns.

Proposition 4.17 ([31], Proposition 5.1.10). *Let C be a LCM space curve. There exists a non-empty Zariski open subset $U \subset GL(4, k)$ such that for all $g \in U$ the Liebling triangle diagram of $g(I_C)$ exists and is weakly Borel-fixed.*

Proposition 4.18 ([31], Proposition 5.1.8). *Suppose that C is a LCM space curve disjoint from the line $z = w = 0$. Let $\Delta = (\Delta_0, \Delta_1^L)$ denote the Liebling triangle diagram of C. Let $\mathcal{A}_{\leq n}$ denote the elements $(i, j) \in \mathcal{A}$ with $i + j \leq n$. Define $\mathcal{B}_{\leq n}$ similarly. Suppose w is general with respect to M_C, as in Definition 2.7. Then*

$$\sum_{(i,j)\in\mathcal{A}_{\leq n}} \Delta_1^L(i, j) \geq \sum_{(i,j)\in\mathcal{B}_{\leq n}} \Delta_0(i, j).$$

In [14] Fløystad defines and develops the concept of higher initial ideals of an ideal. For a LCM space curve C in general coordinates, he defines an upper triangle diagram $\Delta_1^F(C)$ such that $(\Delta_0(C), \Delta_1^F(C))$ is a triangle diagram. Giving a precise definition of his diagram would require a thorough discussion of higher initial ideals, which is beyond the scope of this dissertation. Instead, we compare his diagrams to those of Liebling, giving an algorithm to obtain $\Delta_1^F(C)$ from $\Delta_1^L(C)$.

Notation 4.19 (Fløystad Triangle Diagram)**.** *Let C be a LCM space curve. Let $\Delta_1^F(C)$ denote the* Fløystad upper triangle diagram *of C, and let $\Delta^F(C) = (\Delta_0(C), \Delta_1^F(C))$ denote the* Fløystad triangle diagram *of C.*

Remark 4.20. For clarity, we will use superscripts 'L' and 'F' to refer to Liebling or Fløystad, respectively, when writing Δ, Δ_1, and $\mathcal{A}$. We will omit the superscript if the context is well-understood.

Definition 4.21 (Borel-Fixed Triangle Diagram)**.** *We say a triangle diagram $\Delta = (\Delta_0, \Delta_1)$ is* Borel-fixed *if Δ is weakly Borel-fixed and $\Delta_1(i, j) \leq \Delta_1(i-1, j+1)$ for $i \geq 1$. That is, Δ is weakly Borel-fixed and the function Δ_1 is weakly decreasing along rows from left to right.*

Proposition 4.22 ([14])**.** *Let C be a LCM space curve. There exists a non-empty Zariski open subset $U \subset GL(4, k)$ and a Borel-fixed triangle diagram Δ such that if $g \in U$, then $\Delta^F(g(I_C)) = \Delta$.*

Remark 4.23. From this point forward, given a LCM space curve C, when we say C is in general coordinates we will mean that C satisfies the following four conditions:

1. $in(I_C) = \text{gin}(I_C)$,
2. $\Delta^L(C)$ is weakly Borel-fixed,
3. $\Delta^F(C)$ is the diagram from Proposition 4.22, and

4. $\sum_{(i,j)\in\mathcal{A}^L_{\leq n}} \Delta_1^L(i,j) \geq \sum_{(i,j)\in\mathcal{B}_{\leq n}} \Delta_0(i,j)$.

Proposition 4.24. *Let C be a LCM space curve in general coordinates. The entries of $\Delta_1^F(C)$ along each row are a permutation of the entries of $\Delta_1^L(C)$ along the same row.*

Proof. Let $\Delta^F = (\Delta_0, \Delta_1^F)$ denote the Fløystad triangle diagram of C. Define two S-modules N_C and Q_C by the exact sequence

$$0 \to N_C \to M_C(-1) \xrightarrow{w} M_C \to Q_C \to 0.$$

In [15], Fløystad and Green show that

$$Q_C \cong_{k[z]} \bigoplus_{(i,j)\in\mathcal{A}^F} \frac{k[z]}{z^{\Delta_1^F(i,j)}}(\Delta_1^F(i,j) - i - j).$$

Applying Proposition 4.14, we conclude the result. □

Remark 4.25. Now we can easily compute the Fløystad triangle diagram of a LCM space curve C in general coordinates by simply reordering the entries in each row of $\Delta_1^L(C)$ into decreasing order. In particular, given $\Delta_1^L(C)$, there exists a way to permute the entries along each row so that after permutation the upper diagram is Borel-fixed and unique. For our purposes, the real strength of $\Delta^F(C)$ is that it places this restriction on $\Delta^L(C)$. We suspect that $\Delta_1^L(C) = \Delta_1^F(C)$, as did Fløystad and Liebling, although we have not proven the equivalence. We will see the importance of Liebling's ordering of entries in the upper diagram in Theorems 4.33 and 4.34.

Also, Proposition 4.24 implies that for a LCM space curve C in general coordinates with Fløystad triangle diagram (Δ_0, Δ_1^F)

$$\sum_{(i,j)\in\mathcal{A}^F_{\leq n}} \Delta_1^F(i,j) \geq \sum_{(i,j)\in\mathcal{B}_{\leq n}} \Delta_0(i,j).$$

We now move on to some classification and computational results. The following proposition describes the Liebling triangle diagram of an ACM space curve. The equivalences are all well known, with (4) being an immediate result of the definition of $\Delta_1^L(C)$.

Proposition 4.26 ([31], Corollary 3.1.14; cf. [39])**.** *Let C be a LCM space curve not meeting the line $z = w = 0$. Then the following are equivalent:*

1. *C is ACM,*
2. *$M_C = 0$,*
3. *$in(I)$ is generated by monomials involving only x and y,*
4. *all entries in $\Delta^L(C)$ are 0, and*
5. *the scheme defined by $in(I)$ has no embedded points.*

Remark 4.27. If C is in general coordinates, the conditions in Proposition 4.26 are equivalent to all the entries of $\Delta^F(C)$ being equal to zero.

Proposition 4.28 ([31], Propositions 3.5.1 and 3.6.3; cf. [19])**.** *Let $\Delta^L = (\Delta_0, \Delta_1^L)$ denote the Liebling triangle diagram associated to a LCM space curve C. Let*

$$A(n) = |\{(i,j) \in \Delta_1^L \mid i + j - \Delta_1^L(i,j) = n\}|$$

and

$$B(n) = |\{(i,j) \in \Delta_0 \mid i + j + \Delta_0(i,j) = n\}|.$$

1.

$$\begin{aligned} h^0(\mathcal{I}_C(n)) &= \sum_{k=0}^{n}(n-k+1)B(k) \\ h^1(\mathcal{I}_C(n)) &= \sum_{(i,j)\in\mathcal{A}} min(\Delta_1^L(i,j), max(n+1-i-j+\Delta_1^L(i,j),0)) - \\ &\quad \sum_{(i,j)\in\mathcal{B}} min(\Delta_0(i,j), max(n+1-i-j,0)) \\ h^2(\mathcal{I}_C(n)) &= \sum_{k=0}^{\infty}(k+1)A(n+2+k). \end{aligned}$$

2. The degree of C is given by the number of entries in Δ_1^L.

3. The arithmetic genus of C is given by

$$\sum_{\substack{(i,j)\in\Delta_1 \\ i+j\geq 2}} (i+j-1) \; - \sum_{(i,j)\in\Delta_1^L} \Delta_1^L(i,j).$$

4. The spectrum of C is given by the function $h_C(n) = A(n)$.

Remark 4.29. Proposition 4.24 implies that for a LCM space curve C in general coordinates, the formulas from Proposition 4.28 work for Fløystad triangle diagrams without modification.

Example 4.30. Consider the triangle diagram in Figure 4.3. We will see in the next chapter that this diagram occurs as the Liebling triangle diagram for so-called 'subextremal' curves of degree five and genus zero. Let C be such a curve, so that the triangle diagram in Figure 4.3 is $\Delta^L(C)$. $\Delta_1^L(C)$ has five entries, so the degree of C is five. $\Delta_1^L(C)$ has two entries in degree two, so the genus of C is $2-2=0$.

We compute

$$
\begin{aligned}
B(0) &= 0\\
B(1) &= 0\\
B(2) &= 1\\
B(3) &= 3\\
B(4) &= 5\\
B(5) &= 7\\
B(n) &= n+1 \text{ for } n \geq 6.
\end{aligned}
$$

We also compute

$$
\begin{aligned}
A(0) &= 2\\
A(1) &= 2\\
A(2) &= 1.
\end{aligned}
$$

We could compute $h^0(\mathcal{I}_C(n))$ and $h^2(\mathcal{I}_C(n))$ from these values. In particular, $h^2(\mathcal{I}_C(n)) = 0$ for $n \geq 1$. The spectrum of C is given by $\{0^2, 1^2, 2\}$. Of particular interest, for reasons to be discussed in the next chapter,

$$
h^1(\mathcal{I}_C(n)) = \begin{cases} 0 & \text{for } n \leq -1 \\ n+1 & \text{for } -1 \leq n \leq 1 \\ 4-n & \text{for } 2 \leq n \leq 4 \\ 0 & \text{for } n \geq 4 \end{cases}
$$

so that the regularity of I_C is five.

Proposition 4.31 (cf. [44])**.** *Let C be a LCM space curve of degree d. C is planar if and only if the spectrum of C is given by $\{0, 1, 2, \ldots, d-1\}$.*

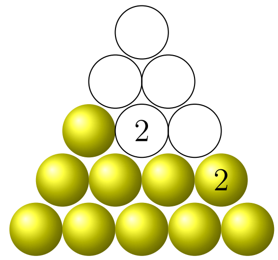


Figure 4.3: Diagram for Example 4.30

Proof. Without loss of generality, we may assume that C is in general coordinates.

First assume that the spectrum of C is given by $\{0, 1, 2, \ldots, d-1\}$. This implies $(0, d-1) \in \Delta_1(C)$, and hence $(0, j) \in \Delta_1(C)$ for $0 \leq j \leq d-1$. Since there are only d elements in the spectrum, $\{(0, j) \mid 0 \leq j \leq d-1\}$ are precisely the indices that are entries in $\Delta_1(C)$, and these entries are equal to 0. We conclude $gin(I_C) = (x, y^d)$; hence C is planar.

For the reverse implication, note that the generic initial ideal of a planar space curve is given by (x, y^d). In particular, the Liebling triangle diagram for a planar space curve in general coordinates has no non-zero entries. The entries of the upper diagram triangle occcur at indices $(0, j)$ for $0 \leq j \leq d-1$. Applying Proposition 4.28 gives the result. $\square$

Definition 4.32 (Shared Columns)**.** *Let $\Delta = (\Delta_0, \Delta_1)$ be a triangle diagram. We say that Δ_0 has* shared columns *if two or more elements of $\mathcal{B}$ occur along the same x-column or the same y-column. If Δ_0 does not satisfy this condition, we say that Δ_0 has no shared columns.*

Similarly, we say that Δ_1 has shared columns *if two or more elements of $\mathcal{A}$ occur along the same x-column or the same y-column. If Δ_1 does not satisfy this condition, we say that Δ_1 has no shared columns.*

We say that Δ has shared columns *if two or more elements of $\mathcal{A} \cup \mathcal{B}$ occur along the same x-column or the same y-column. If Δ does not satisfy this condition, we say that Δ has* no shared columns.

If the λ-invariants of a curve are known and its triangle diagram has no shared columns, the number of possible diagrams (and hence, possible generic initial ideals) is severely restricted. It makes sense, then, to find classes of curves whose triangle diagrams have no shared columns. Liebling finds two such classes of curves in [31], demonstrating a strength of Δ_1^L over Δ_1^F.

Theorem 4.33 ([31], Proposition 4.2.6). *Let C be a LCM space curve not meeting the line $z = w = 0$. If $Tor_1^S(M_C, S/(z,w))$ is Buchsbaum, then $\Delta^L(C)$ has no shared columns.*

Theorem 4.34 ([31], Corollary 4.2.8). *Let C be a LCM space curve not meeting the line $z = w = 0$. If either M_C or M_C^* is a principal $k[z,w]$-module, then $\Delta^L(C)$ has no shared columns.*

Equivalently, if $|\mathcal{A}| = 1$ or $|\mathcal{B}| = 1$, then Δ has no shared columns.

Proposition 4.35 ([31], Example 4.2.9). *The Liebling triangle diagram of a degree two, genus g, LCM space curve C in general coordinates is given by*

$$\Delta_0(C)(i,j) = \begin{cases} +\infty & \textit{for} \quad (i,j) = (0,0) \textit{ or } (0,1) \\ -g & \textit{for} \quad (i,j) = (1,0) \\ 0 & \textit{otherwise} \end{cases}$$

$$\Delta_1^L(C)(i,j) = \begin{cases} 0 & \textit{for} \quad (i,j) = (0,0) \\ -g & \textit{for} \quad (i,j) = (0,1) \\ +\infty & \textit{otherwise.} \end{cases}$$

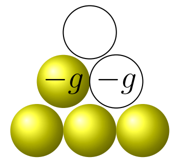


Figure 4.4: Triangle Diagram of a Degree 2 Curve

In particular, $gin(I_C) = (xz^{-g}, x^2, xy, y^2)$.

This result follows immediately from Theorem 4.34 and Proposition 4.28 after noting that the Liebling upper diagram for such a curve must have two entries and be weakly Borel-fixed.

Remark 4.36. Pictorially, the Liebling triangle diagram of a degree two, genus g, LCM space curve C in general coordinates is given in Figure 4.4.

Example 4.37. The ideal $I = (x^2, xy, y^2, xz^{-g} - yw^{-g}) \subset S$ is an ideal of a degree two, genus g, LCM space curve in general coordinates. It gives a multiplicity two structure on the line $x = y = 0$.

We close this chapter with two results concerning how triangle diagrams behave under certain types of liaison. The first is a result of Liebling presenting sufficient conditions on $\Delta^L(C)$ for a LCM space curve C to admit a descending elementary double link. The second result computes which numbers appear on which rows of a triangle diagram after a double link.

Proposition 4.38 ([31], Proposition 3.4.15)**.** *Let C be a LCM space curve not meeting the line $z = w = 0$. Let $\Delta = (\Delta_0, \Delta_1^L)$ be the Liebling triangle diagram of C.*

If for all i we have $(i, 0) \notin \mathcal{A}_{\Delta_1^L} \cup \mathcal{B}_{\Delta_0}$, then C admits a descending elementary double link. The Liebling triangle diagram $\Delta' = (\Delta_0', \Delta_1^{L'})$ of the resulting curve is *given by*

$$\Delta_0'(i,j) = \Delta_0(i,j+1)$$
$$\Delta_1^{L'}(i,j) = \Delta_1^L(i,j+1).$$

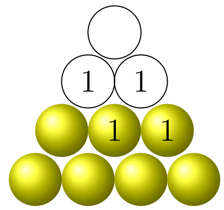
Similarly, if for all j *we have* $(0,j) \notin \mathcal{A}_{\Delta_1^L} \cup \mathcal{B}_{\Delta_0}$*, then* C *admits a descending elementary double link. The triangle diagram* $\Delta' = (\Delta_0', \Delta_1^{L'})$ *of the resulting curve is given by*

$$\Delta_0'(i,j) = \Delta_0(i+1,j)$$
$$\Delta_1^{L'}(i,j) = \Delta_1^L(i+1,j).$$

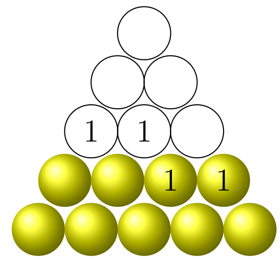
Liebling proves the first part of this result by linking C to another curve D via X where $in(I_X) = (x^r, y^s)$ and r denotes the smallest power of x appearing in $in(I_C)$. He then shows there is an element of $in(I_D)$ with initial term y^{s-1}. The second part is proved similarly.

The conditions given in Proposition 4.38 are sufficient, but not necessary. We will see this in the next example by considering the case of three disjoint lines.

Example 4.39. Although we use material presented in a later chapter for this example, that content does not depend on this example in any way. Let the curve $C \subset \mathbb{P}^3$ be a union of three disjoint lines, and assume that C is in general coordinates. Using Corollary 6.6, we see that the Liebling triangle diagram of C is given as in Figure 4.5. Let C' be obtained from C by a basic double link of type $(3,1)$. The Liebling triangle diagram of C' is also given in Figure 4.5 and was computed using *Macaulay2* ([18]). Although C' admits a descending elementary double link, neither the leftmost y-column nor the rightmost x-column of its Liebling triangle diagram is identically zero.



Liebling Triangle Diagram of C



Liebling Triangle Diagram of C'

Figure 4.5: Diagram for Example 4.39

Proposition 4.40. *Let $C \overset{X_1}{\sim} D \overset{X_2}{\sim} C'$, with $C, D, C', X_1, X_2 \subset \mathbb{P}^3$ of dimension one. If X_1 is a complete intersection of type (a, b) with $a \leq b$ and X_2 is a complete intersection of type (a', b') with $a' \leq b'$, then*

$$M_C \cong M_{C'}(a' - a + b' - b).$$

In particular, if C and C' are both in general coordinates, then the non-zero values in the i^{th} row of $\Delta_0(C)$ are precisely the non-zero values in the $(i+(a'-a+b'-b))^{th}$ row of $\Delta_0(C')$, and the order of the entries along each row is preserved.

Similarly, the non-zero values in the i^{th} row of $\Delta_1^L(C)$ are precisely the non-zero values in the $(i+(a'-a+b'-b))^{th}$ row of $\Delta_1^L(C')$, up to permutation of the order of the entries along each row.

The non-zero values in the i^{th} row of $\Delta_1^F(C)$ are precisely the non-zero values in the $(i+(a'-a+b'-b))^{th}$ row of $\Delta_1^F(C')$, and the order of the entries along each row is preserved.

Proof of Proposition 4.40. We note that the twist in the last step of the free resolution of X_1 is $a+b$ and the twist in the last step of the free resolution of X_2 is $a'+b'$. Applying Theorem 3.17 twice, we compute that $M_C \cong M_{C'}(a'-a+b'-b)$.

We note that if C and C' are in general coordinates, then we may define the modules N_C, Q_C, $N_{C'}$ and $Q_{C'}$ by the following exact sequences:

$$0 \to N_C \to M_C(-1) \xrightarrow{w} M_C \to Q_C \to 0$$

$$0 \to N_{C'} \to M_{C'}(-1) \xrightarrow{w} M_{C'} \to Q_{C'} \to 0.$$

In particular, $N_C \cong N_{C'}(a'-a+b'-b)$, and $Q_C \cong Q_{C'}(a'-a+b'-b)$. We conclude that the non-zero values in the i^{th} row of $\Delta_0(C)$ (respectively, $\Delta_1^L(C)$, $\Delta_1^F(C)$) are precisely the non-zero values in the $(i+(a'-a+b'-b))^{th}$ row of $\Delta_0(C')$ (respectively, $\Delta_1^L(C')$, $\Delta_1^F(C)$). Since $\Delta_0(C')$ and $\Delta_1^F(C')$ are Borel-fixed, the order of the entries along each row is preserved. $\square$

CHAPTER 5

EXTREMAL AND SUBEXTREMAL CURVES

There has been much work on computing the dimensions of the graded components of Rao modules of space curves, giving strong and surprising bounds (cf. [13, 34, 35, 41, 44]). In particular, for many degrees and genera, there is a notion of 'largest' Rao modules. The set of curves with such a Rao module are called extremal curves. Such curves form an irreducible component of $H_{d,g}$. A key strategy in connectedness results for $H_{d,g}$ has been to try to connect curves to this component, cf. [26, 28, 55]. It is not the case, however, that every component of $H_{d,g}$ intersects the component of extremal curves, as Nollet and Schlesinger show in [46].

Theorem 5.1 ([34])**.** *Let C be a non-planar LCM space curve of degree d and genus g. Then*

$$h^1(\mathcal{I}_C(n)) \leq \begin{cases} 0 & \text{for } n \leq g - \binom{d-2}{2} \\ \binom{d-2}{2} - g + n & \text{for } g - \binom{d-2}{2} < n < 0 \\ \binom{d-2}{2} - g & \text{for } 0 \leq n \leq d-2 \\ \binom{d-1}{2} - g - n & \text{for } d-2 < n \leq \binom{d-1}{2} - g \\ 0 & \text{for } n > \binom{d-1}{2} - g. \end{cases}$$

Definition 5.2 (Extremal Curve)**.** *A LCM space curve C is called* extremal *if it achieves the bounds in Theorem 5.1 and is not ACM.*

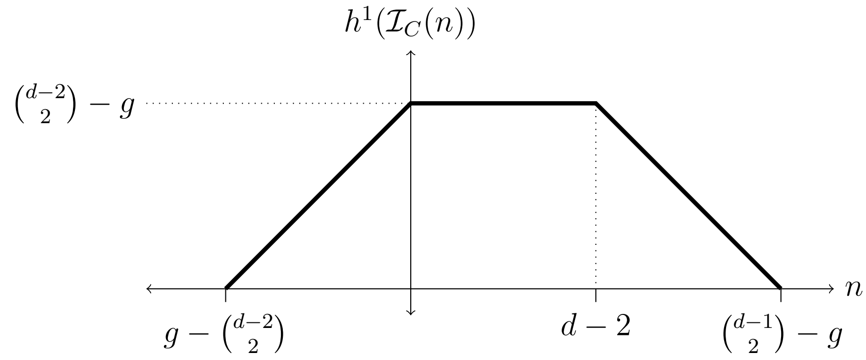


Figure 5.1: Extremal Cohomological Bounds

Remark 5.3. In Figure 5.1 we graph the cohomological bounds for an extremal curve of degree d and genus g.

Theorem 5.4 ([13, 35])**.** *Fix a positive integer $d \geq 2$. Then:*

1. *extremal curves of degree d and genus g exist precisely when $g < \binom{d-2}{2}$,*
2. *the set of all extremal curves of degree d and genus g form an irreducible component $E_{d,g}$ of $H_{d,g}$, and*
3. *for all $d \geq 6$ and $g \leq \binom{d-3}{2} + 1$, this component is non-reduced on the Hilbert scheme.*

Remark 5.5. We note that Hartshorne and Schlesinger use nonstandard definitions of extremal and subextremal curves in [29], expanding the definition to include ACM curves. While this is useful in the setting of their work, it deemphasizes the cohomology and the structure of $H_{d,g}$.

We move on to several results discussing the structure and numerical invariants of extremal curves.

Proposition 5.6 ([29, 35, 44])**.** *Let C be a non-planar LCM space curve of degree $d \geq 2$ and genus g. Then the following conditions are equivalent:*

1. *C is an extremal curve.*

2. *There exists an integer $a \leq 0$ such that C has spectrum*

$$\{a\} \cup \{0, 1, 2, \ldots, d-2\}.$$

3. *C is a non-planar curve which contains a plane curve of degree $d-1$ and is not ACM.*

Corollary 5.7 (cf. [41])**.** *Let C be an extremal LCM space curve of degree $d \geq 3$ and genus g. Let $a = g - \binom{d-2}{2} + 1$. Then,*

$$gin(I_C) = (x^2, xy, y^{d-1}z^{1-a}, y^d).$$

In particular, the spectrum of an extremal curve is given by

$$\{a\} \cup \{0, 1, 2, \ldots, d-2\}.$$

Conversely, if C is a LCM space curve with $gin(I_C) = (x^2, xy, y^{n-1}z^{1-a}, y^n)$, for some $a \leq 0$, $n \geq 3$, then C is extremal.

Remark 5.8. Note that all degree two LCM space curves that are not ACM are extremal. We discuss their generic initial ideals in Proposition 4.35.

Proof of Corollary 5.7. Without loss of generality, assume that C is in general coordinates. Let $\Delta^L = (\Delta_0, \Delta_1^L)$ denote the Liebling triangle diagram of C.

We first assume that C is extremal. First consider the case where $d = 3$. Since C is not planar, the Restriction Theorem implies $(0, 1)$ and $(1, 0) \in \Delta_1^L$. For the number one to occur in the spectrum of C, one of these entries must be 0. In particular, Δ_1^L has only one non-zero entry, forcing Δ^L to have no shared columns. This means that only one entry in Δ_0 is non-zero. Since Δ^L is weakly

Borel-fixed, $\Delta_0(0,2)$ is nonzero, and equals $-g$ by Proposition 4.28. We conclude $\text{gin}(I_C) = (x^2, xy, y^{d-1}z^{1-a}, y^d)$, where $a = g - \binom{d-2}{2} + 1$. The spectrum of C is given by $\{g - \binom{d-2}{2} + 1\} \cup \{0, 1, 2, \ldots, d-2\}$.

Now take $d \geq 4$. Since C is not planar, the Restriction Theorem implies $(0,1)$ and $(1,0) \in \Delta_1^L$. Since $d-2$ occurs in the spectrum of C, at least one entry of Δ_1^L occurs in degree $d-2$. Since Δ^L is weakly Borel-fixed, $(0, d-2) \in \Delta_1^L$, forcing $(0,j) \in \Delta_1^L$ for $0 \leq j \leq d-2$. This accounts for all the entries of Δ_1^L, and hence $\Delta_1^L(0,j) = 0$ for $0 \leq j \leq d-2$. The only non-zero entry in Δ_1^L occurs at index $(1,0)$, and Proposition 4.28 implies $\Delta_1^L(1,0) = \binom{d-2}{2} - g$. The spectrum of C is given by $\{g - \binom{d-2}{2} + 1\} \cup \{0, 1, 2, \ldots, d-2\}$.

Since Δ_1^L has only one non-zero entry, Δ^L has no shared columns. Since Δ^L is weakly Borel-fixed, the only non-zero entry in Δ_0 occurs at index $(0, d-1)$, which implies $\Delta_0(0, d-1) = \binom{d-2}{2} - g$. We conclude $\text{gin}(I_C) = (x^2, xy, y^{d-1}z^{1-a}, y^d)$, where $a = g - \binom{d-2}{2} + 1$.

Now, conversely, assume $\text{gin}(I_C) = (x^2, xy, y^{n-1}z^{1-a}, y^n)$, for some $a \leq 0$, $n \geq 3$. Since Δ_0 has only one non-zero entry, Δ^L has no shared columns. This forces $\Delta_1^L(1,0) = 1 - a$, with all other entries of Δ_1^L being equal to 0. Using Proposition 4.28, the spectrum of C is given by $\{a\} \cup \{0, 1, 2, \ldots, d-2\}$. Using Proposition 5.6, we conclude that C is extremal. $\square$

Corollary 5.9 (cf. [41])**.** *If C is an extremal LCM space curve of degree d and genus g, then*

$$\mathit{reg}(I_C) = \binom{d-1}{2} + 1 - g.$$

Proof. The regularity of I_C is equal to the maximal degree of a minimal generator of $\text{gin}(I_C)$.

We conclude that $\operatorname{reg}(I_C) = \binom{d-2}{2} + d - 1 - g = \binom{d-1}{2} + 1 - g$. $\square$

The following result is well known. We present a proof using triangle diagrams.

Corollary 5.10 (cf. [35])**.** *If C is an extremal LCM space curve, then C is minimal in its biliaison class.*

Proof. Without loss of generality, assume that C is in general coordinates. Since $\Delta_0(C)$ has only one entry, $\Delta^L(C)$ has no shared columns and $\Delta_1^L(1, 0) \neq 0$. Assume that C is not minimal in its biliaison class. Let $\mathcal{L}$ denote the biliaison class of C, and write $C \in \mathcal{L}^h$. By Theorem 3.32, we may deform C to a curve $C' \in \mathcal{L}^h$ through curves all in $\mathcal{L}^h$ such that C' is obtained from a curve D by a basic double link. We may assume that C' is in general coordinates. Since C and C' have isomorphic Rao modules, $\Delta_1^L(C')$ has a non-zero entry in degree one. We apply Proposition 4.40 to see that $\Delta_1^L(D)$ has a non-zero entry at index $(0, 0)$, a contradiction. We conclude that C is minimal in its biliaison class. $\square$

Corollary 5.11 (cf. [35, 41])**.** *Let C be a LCM space curve. If C is extremal, then M_C is principal.*

Proof. Without loss of generality, assume that C is in general coordinates. Define modules N_C and Q_C by the following exact sequence:

$$0 \to N_C \to M_C(-1) \xrightarrow{w} M_C \to Q_C \to 0.$$

The proof of Corollary 5.7 gives that $\Delta_1^L(C)$ has only one nonzero entry. Proposition 4.14 implies that Q_C is principal. By Nakayama's Lemma, M_C is principal. $\square$

In fact, we can say much more about the Rao module of an extremal curve.

Proposition 5.12 ([35, 41])**.** *Let C be an extremal LCM space curve of degree d and genus g. Then M_C is isomorphic to a complete intersection module of type $(\binom{d-1}{2} - g, \binom{d-2}{2} - g, 1, 1)$ with twist $\binom{d-2}{2} - g - 1$.*

LCM space curves that are not extremal satisfy much stronger cohomological bounds.

Theorem 5.13 ([44])**.** *Let C be a non-planar LCM space curve of degree $d \geq 4$ and genus g that is neither ACM nor extremal. Then*

$$h^1(\mathcal{I}_C(n)) \leq \begin{cases} 0 & \text{for } n < g - \binom{d-3}{2} + 1 \\ \binom{d-3}{2} + n - g & \text{for } g - \binom{d-3}{2} + 1 \leq n < 1 \\ \binom{d-3}{2} + 1 - g & \text{for } 1 \leq n \leq d-3 \\ \binom{d-2}{2} + 1 - g - n & \text{for } d-3 < n \leq \binom{d-2}{2} - g \\ 0 & \text{for } n > \binom{d-2}{2} - g. \end{cases}$$

Remark 5.14. Theorem 5.13 implies that if C is a LCM space curve of degree $d \geq 4$ and genus g that is not ACM, then $g \leq \binom{d-3}{2}$.

Definition 5.15 (Subextremal Curve)**.** *A non-planar LCM space curve C is called* subextremal *if it achieves the bounds in Theorem 5.13 and is neither ACM nor extremal.*

Remark 5.16. In Figure 5.2 we graph the cohomological bounds for a subextremal curve of degree d and genus g.

Remark 5.17. In Figure 5.3 we graph together the cohomological bounds for an extremal curve and a subextremal curve, both of degree d and genus g. The

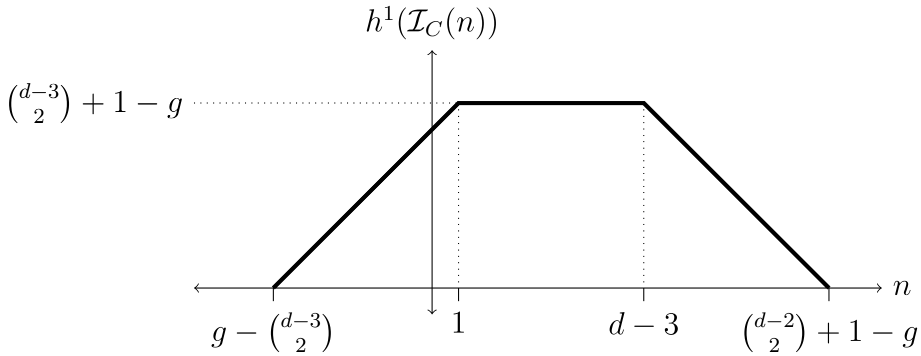


Figure 5.2: Subextremal Cohomological Bounds

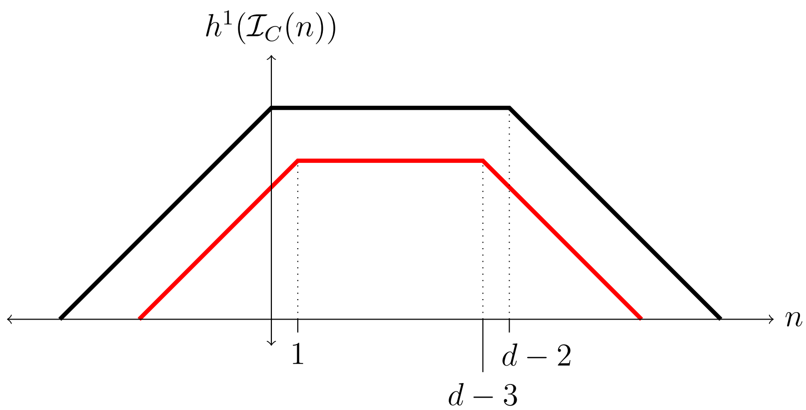


Figure 5.3: Extremal and Subextremal Cohomological Bound Comparison

bounds for the subextremal curve are shown in red and those for the extremal curve are shown in black. For clarity, some of the data from the earlier figures has been removed. If C is a LCM space curve of degree $d \geq 4$ and genus g that is not extremal, then $h^1(\mathcal{I}_C(n))$ is contained within the red trapezoid, possibly touching its boundary in places.

We continue with a discussion of the existence, structure, and numerical invariants of subextremal curves.

Theorem 5.18 ([44])**.** *Let C be a LCM space curve of degree d and genus g. Then C is subextremal if and only if C is obtained from an extremal curve by an* *elementary double link of type* $(2,1)$*. If one (and hence both) of these conditions holds, then* C *has spectrum*

$$\{a\} \cup \{0, 1^2, 2, \ldots, d-3\}$$

for some $a \leq 1$*.*

Corollary 5.19 (cf. [29])**.** *Fix* $d \geq 4$*. For all* $g \leq \binom{d-3}{2}$*, there exist subextremal curves of degree* d *and genus* g*.*

Proof. Let C' be an extremal curve of genus $g' \geq 2$ and degree $d' = d-2$ in general coordinates. Let C be obtained from C' by a basic double link of type $(2,1)$. By Theorem 5.18, C is subextremal. Applying Proposition 3.25, $\deg(C) = d' + 2 = d$.

Let g denote the genus of C. Applying Proposition 3.25, $g = d - 1 + g' = d - 3 + g'$. If $g' = \binom{d'-2}{2} - 1$, then $g = d - 4 + \binom{d-4}{2} = \binom{d-3}{2}$. Since extremal curves exist for each $g' \leq \binom{d'-2}{2} - 1$, we conclude that subextremal curves exist for each $g \leq \binom{d-3}{2}$. □

Example 5.20. A basic double link of height one on a quadric surface of a degree two LCM space curve of genus g gives a degree four LCM space curve of genus $g+1$ that is subextremal.

Corollary 5.21. *Let* C *be a subextremal LCM space curve of degree* d *and genus* g*. Let* $a = g - \binom{d-3}{2} + 1$*. If* $d = 4$*, then*

$$gin(I_C) = (x^2, xyz^{2-a}, xy^2, y^3).$$

If $d \geq 5$*, then*

$$gin(I_C) = (x^2, xy^2, y^{d-2}z^{2-a}, y^{d-1}).$$

In either case, the spectrum of C *is given by*

$$\{a\} \cup \{0, 1^2, 2, \ldots, d-3\}.$$

Conversely, if C is a LCM space curve of degree $d = 4$ and genus g with $gin(I_C) = (x^2, xyz^n, xy^2, y^3)$ for some $n \geq 1$, then C is subextremal. If C is a LCM space curve of degree $d \geq 5$ and genus g with $gin(I_C) = (x^2, xy^2, y^{d-2}z^n, y^{d-1})$, for some $n \geq 1$ then C is subextremal.

Proof. Assume, without loss of generality, that C is in general coordinates. Let $\Delta^L = (\Delta_0, \Delta_1^L)$ be the Liebling triangle diagram of C.

We first assume that C is subextremal.

By Theorem 5.18, C is obtained from an extremal curve C' by an elementary double link of type $(2,1)$. Note that $\deg(C') = d - 2$.

If $d = 4$, we apply Lemma 7.4 (2) to see that $\text{gin}(I_C) = (x^2, xyz^{1-g}, xy^2, y^3)$.

If $d \geq 5$, we apply Lemma 7.4 (1) to see that $\text{gin}(I_C) = (x^2, xy^2, y^{d-2}z^a, y^{d-1})$.

In either case, by Propositions 4.40 and 4.28, and the proof of Corollary 5.7, the spectrum of C is given by

$$\left\{g - \binom{d-3}{2} + 1\right\} \cup \{0, 1^2, 2, \ldots, d-3\}.$$

Now asuume $d = 4$ and $\text{gin}(I_C) = (x^2, xyz^n, xy^2, y^3)$ for some $n \geq 1$. The Liebling triangle diagram of C has no shared columns and is weakly-Borel fixed. We conclude that the only entry in Δ_1^L occurs at the index $(0,2)$. Proposition 4.38 implies that C is obtained by an elementary double link from a curve D with initial ideal given by (xz^n, x^2, xy, y^2). We compute that this elementary double link is of type $(2,1)$. In particular, C is obtained from an extremal curve by an elementary double link of height one on a quadric surface. We conclude that C is subextremal.

Finally, assume $d \geq 5$ and $\text{gin}(I_C) = (x^2, xy^2, y^{d-2}z^n, y^{d-1})$, for some $n \geq 1$.The

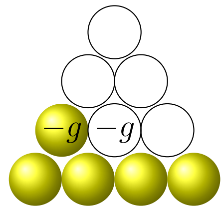


Figure 5.4: Fløystad Triangle Diagram of D, as in Example 5.23

Liebling triangle diagram of C has no shared columns and is weakly-Borel fixed. We conclude that the only entry in Δ_1^L occurs at the index $(1, 1)$. Proposition 4.38 implies that C is obtained by an elementary double link from a curve D with initial ideal given by $(x^2, xy, y^{d-3}z^n, y^{d-2})$. We compute that this elementary double link is of type $(2, 1)$. The only nonzero entry in $\Delta_1^L(D)$ occurs at index $(1, 0)$ and equals n. Computing $h^1(\mathcal{I}_D(n))$ for all n from Proposition 4.28 implies that D is an extremal curve. In particular, C is obtained from an extremal curve by an elementary double link of height one on a quadric surface. We conclude that C is subextremal. □

Remark 5.22. The above proof uses material that appears in a later chapter. The content used does not rely on Corollary 5.21 in any way.

Example 5.23. Unlike in the extremal case, it is possible for a curve to have the same spectrum as a subextremal curve but not be subextremal. Let C denote a degree two LCM space curve of genus g. Let D be obtained from C by a basic double link of type $(3, 1)$.

Using Propositions 3.13 and 4.40, and the fact that $\Delta^F(D)$ is Borel-fixed, we see $\Delta^F(D)$ is given in Figure 5.4. Note that D does not have the generic initial ideal of a subextremal curve. The genus of D is $2 + g$ and the degree of D is five. The spectrum of D is given by $\{2+g\} \cup \{0, 1^2, 2\}$. This is the same as the spectrum of a degree five, genus $2 + g$ subextremal curve, but D is not subextremal. We note

$$h^1(\mathcal{I}_D(n)) = \begin{cases} 0 & \text{for} \quad n \le g+1 \\ n-1-g & \text{for} \quad g+1 \le n \le 1 \\ -n+1-g & \text{for} \quad 1 \le n \le 1-g \\ 0 & \text{for} \quad 1-g \le n. \end{cases}$$

If C' is a subextremal curve of degree five and genus $2+g$, then

$$h^1(\mathcal{I}_{C'}(n)) = \begin{cases} 0 & \text{for} \quad n \le g+1 \\ n-1-g & \text{for} \quad g+1 \le n \le 1 \\ -g & \text{for} \quad g+1 \le n \le 2 \\ -n+2-g & \text{for} \quad 2 \le n \le 2-g \\ 0 & \text{for} \quad 2-g \le n. \end{cases}$$

Corollary 5.24. *Let C be a LCM space curve of degree d and genus g that is not ACM. If C is not extremal, then*

$$reg(I_C) \le \binom{d-2}{2} + 2 - g,$$

with equality if C is subextremal.

Proof. Without loss of generality, assume that C is in general coordinates.

The regularity of the generic initial ideal of a degree two curve is $1-g$, and we conclude that the result holds in this case.

Now assume that C is of degree three. Since C is not planar, the entries of $\Delta_1^L(C)$ occur at indices $(0,0),(1,0)$, and $(0,1)$. We may compute the regularity of I_C by computing the highest degree of a minimal generator of $\text{gin}(I_C)$. This quantity is maximized if $\Delta_0(C)$ has only one non-zero entry since the triangle diagram of C is weakly Borel-fixed. This can only occur for extremal degree three curves. In particular, the regularity of I_C is less than that of an extremal degree three curve of the same genus. In particular, $\operatorname{reg}(I_C) \le 2 - g$.

Now assume that the degree of C is at least four. From Theorem 5.13,

$$h^1(\mathcal{I}_C(n)) = 0 \text{ for } n \ge \binom{d-2}{2} + 1 - g.$$

Let A be defined as in Proposition 4.28. If $n \ge d-4$, we compute $A(n+2+k) = 0$ for $k \ge 0$, since the largest value in the spectrum of C is at most $d-3$. Then,

$$h^2(\mathcal{I}_C(n)) = \sum_{k=0}^{\infty} (k+1)A(n+2+k) = 0 \text{ for } n \ge d-4.$$

The regularity of I_C is at most $\max\{\binom{d-2}{2} + 2 - g,\ d-2\}$. Since $g \le \binom{d-3}{2}$, we have that $d - 2 \le \binom{d-2}{2} + 2 - g$. We conclude that $\operatorname{reg}(I_C) \le \binom{d-2}{2} + 2 - g$. If C is subextremal, then

$$h^1(\mathcal{I}_C(n)) \ne 0 \text{ for } n = \binom{d-2}{2} - g,$$

and we conclude $\operatorname{reg}(I_C) \le \binom{d-2}{2} + 2 - g$. □

Corollary 5.25. *Let C be a LCM space curve. If C is subextremal, then M_C is principal.*

Proof. Without loss of generality, assume that C is in general coordinates. Define modules N_C and Q_C by the following exact sequnece:

$$0 \to N_C \to M_C(-1) \xrightarrow{w} M_C \to Q_C \to 0.$$

The proof of Corollary 5.21 gives that $\Delta_1^L(C)$ has only one nonzero entry. Proposition 4.14 implies that Q_C is principal. By Nakayama's Lemma, M_C is principal. □

In fact, we can say much more about the Rao module of a subextremal curve.

Proposition 5.26. *Let C be an extremal LCM space curve of degree d and genus g. Then M_C is isomorphic to a complete intersection module of type $(\binom{d-1}{2} - g, \binom{d-2}{2} - g, 1, 1)$ with twist $\binom{d-2}{2} - g - 2$.*

Proof. Apply Proposition 3.28 to the result of Proposition 5.12, considering the fact that subextremal curves are obtained from extremal curves by an elementary double link of type $(2, 1)$. □

We close this chapter with some results relating extremal and subextremal curves to questions about the connectedness of $H_{d,g}$.

Theorem 5.27 ([45])**.** *Let $d \geq 4$ and $g \leq \binom{d-3}{2}$. Then the closure of the family of subextremal curves in $H_{d,g}$ contains an extremal curve.*

Theorem 5.28 ([26])**.** *Let C be an extremal LCM space curve of degree d and genus g. Let C' be an ACM space curve of degree d and genus g. Then C can be connected within $H_{d,g}$ to C'.*

Theorem 5.29 ([55])**.** *Let C be an extremal LCM space curve of degree d and genus g. Let C' be an space curve of degree d and genus g that is in the same biliaison class as an extremal curve. Then C can be connected within $H_{d,g}$ to C'.*

Theorem 5.30 ([28])**.** *Let C be a smooth LCM space curve of degree d and genus g. Then any component of $H_{d,g}$ containing C intersects $E_{d,g}$.*

There are many more specialized results dealing with deforming LCM space curves to the extremal component of $H_{d,g}$, for example see [26]. Recent work of Hartshorne, Lella, and Schlesinger in [28] gives necessary and sufficient conditions for a component of $H_{d,g}$ to intersect $E_{d,g}$.

CHAPTER 6

DEGREE THREE CURVES

In [43], Nollet shows that $H_{3,g}$ is connected for all g by identifying the general element on each component and showing that such elements specialize to extremal curves. We compute the generic initial ideals of such elements. We note that degree three LCM space curves that are not ACM exist precisely when $g \leq -1$.

We first consider the case $g = -1$, and then we proceed with some of Nollet's classifications and our computations of generic initial ideals.

Proposition 6.1. *Let C be a degree three, genus negative one, LCM space curve. Then C is extremal.*

Proof. We may assume that C is in general coordinates. Let Δ_0 denote the lower triangle diagram of C. Since C is not planar, the indices (i, j) that are entries in Δ_0 are precisely those for which $i + j \geq 2$. Using Proposition 4.28 and the fact that Δ_0 is Borel-fixed, we see that $\Delta_0(0, 2) = 1$ and that all other entries of Δ_0 are zero. We conclude that $\text{gin}(I_C) = (x^2, xy, y^2z, y^3)$ and that C is extremal. $\square$

Proposition 6.2 ([43])**.** *Let Y be the line $x = y = 0$ in $\mathbb{P}^3$. Let f and g be forms of degree $a + 1$ having no common zeroes along Y. Let $Z \subset \mathbb{P}^3$ be the triple line with ideal $I_Z = (I_Y^3, xg - yf)$. Let p and q be forms of degrees b and $3a + b + 2$, respectively, having no common zeroes along Y. Then p and q define a surjection $\phi : \mathcal{I}_Z \to \mathcal{O}_Y(2a + b)$ by $x \mapsto pf^2$, $xy \mapsto pfg$, $y^2 \mapsto pg^2$, and $xg - yf \mapsto q$. The kernel of ϕ is the ideal sheaf of a multiplicity three structure W on Y. Further, we have:*

1. *The arithmetic genus of W is $-2 - 3a - b$.*

2. $I_W = (I_Y^3, x(xg - yf), y(xg - yf), p(xg - yf) - Ax^2 - Bxy - Cy^2)$ *for some polynomials* $A, B,$ *and* C *chosen so that* $q = Af^2 + Bfg + Cg^2 \mod I_Y$.

3. *If* p' *and* q' *define the multiplicity three structure* W', *then* $W = W'$ *if and only if there exists* $\alpha \in k^\times$ *such that* $p' = \alpha p \mod I_Y$ *and* $q' = \alpha q \mod I_Y$.

Definition 6.3 (Triple Line of Type (a, b))**.** *If* $W \subset \mathbb{P}^3$ *is constructed as in Proposition 6.2 up to change of coordinates, we say that* W *is a* triple line of type (a, b).

Proposition 6.4 ([43])**.** *Let* $g \leq -2$. *Then:*

1. *the family of curves formed by taking the union of a double line* Z *of genus* $g + 1$ *and a disjoint line* L *forms an irreducible family of dimension* $7 - 2g$, *and*

2. *the family of cuves which are triple lines of type* $(0, -2 - g)$ *forms an irreducible family of dimension* $6 - 2g$.

The curves above are all minimal, and each curve in the second family is obtained from curves in the first by a deformation that preserves cohomology.

Proposition 6.5 ([43])**.** *The LCM Hilbert scheme* $H_{3,-2}$ *consists of the following pair of irreducible components:*

1. *the extremal component* $E_{3,-2}$ *and*

2. *the closure* H_0 *of the irreducible family of sets of three disjoint lines.*

The closed family in (2) contains the curves from Proposition 6.4.

Corollary 6.6. *Let* C *be a general element of* H_0 *as in Proposition 6.5. Then the generic initial ideal of* C *is given by* $(x^2, xyz, y^2z, xy^2, y^3)$.

Proof. Let Δ_0 denote the lower triangle diagram of C. Since C is not planar, the indices (i, j) that are entries in Δ_0 are precisely those for which $i+j \geq 2$. Since C is not extremal, $\Delta_0(0, 2) \neq 2$. Since $\Delta_0(C)$ is Borel-fixed, $\Delta_0(1, 1) = \Delta_0(0, 2) = 1$. We conclude $\text{gin}(I_C) = (x^2, xyz, y^2z, xy^2, y^3)$. $\square$

Theorem 6.7 ([43]). *Let $g \leq -3$. Then $H_{3,g}$ consists of the following irreducible components:*

1. *the extremal component $E_{3,g}$,*
2. *the closure of the irreducible family of dimension $7 - 2g$ from Proposition 6.4, which we now denote H_0, and*
3. *for each $0 < a < (-2 - g)/3$, the closure of the irreducible family H_a, consisting of triple lines of type $(a, -2 - 3a - g)$.*

Corollary 6.8. *Let C be a general element of $H_0 \subset H_{3,g}$ for some $g \leq -3$ as in Theorem 6.7. Then the generic initial ideal of I_C is given by*

$$gin(I_C) = (x^2z, xyz, y^2z^{-2-g}, x^3, x^2y, xy^2, y^3).$$

Proof. We may assume C is a double line Z of genus $g + 1$ union a disjoint line Y. Let L and M be linear forms such that $I_Z = (L^2, LM, M^2, Lf - Mg)$ for some polynomials f and g of degree $-1 - g$. Let N and Q be linear forms such that $I_Y = (N, Q)$. Note that the set $\{L, M, N, Q\}$ is linearly independent.

We need to compute the generic initial ideal of $I_Y \cap I_Z$. There are no quadrics in the intersection. There are six cubics in the intersection: L^2N, L^2Q, LMN, LMQ, M^2N, and M^2Q. Let Δ_0 denote the lower diagram of C. Since C is not planar, the indices (i, j) that are entries in Δ_0 are precisely those for which $i + j \geq 2$. We also have that $\Delta_0(i, j) = 1$ for two indices (i, j) with $i + j = 2$.

Since C is of genus g, we apply Proposition 4.28 to see that the last entry along the degree 2 row of Δ_0 is $-2-g$. Since Δ_0 is Borel-fixed, we conclude that $\mathrm{gin}(I_C) = (x^2z, xyz, y^2z^{-2-g}, x^3, x^2y, xy^2, y^3)$. $\square$

Corollary 6.9. *Let C be a general element of $H_a \subset H_{3,g}$ for some $g \leq -3$ and some integer a with $0 < a < (-2-g)/3$ as in Theorem 6.7. Then the generic initial ideal of I_C is given by*

$$gin(I_C) = (x^2z^{a+1}, xyz^{a+1}, y^2z^{-2-2a-g}, x^3, x^2y, xy^2, y^3).$$

Proof. We may assume C is a triple line of type $(a, -2-3a-g)$. Let $b = -2-3a-g$. Let Δ_0 denote the lower diagram of C. Since C is not planar, the indices (i, j) that are entries in Δ_0 are precisely those for which $i + j \geq 2$. We see from Proposition 6.2 that I_C contains $(L, M)^3$ for some linear forms L and M. Under general change of coordinates, $(x, y)^3 \subset \mathrm{gin}(I_C)$, so that $\Delta_0(i, j) = 0$ for $i + j \geq 3$. Then, looking at the degrees of the generators in 6.2, we see that the entries of Δ_0 in degree two are $a + 1$, $a + 1$, and $a + b = -2-2a-g$. Since $0 < a < (-2-g)/3$, we conclude

$$\mathrm{gin}(I_C) = (x^2z^{a+1}, xyz^{a+1}, y^2z^{-2-2a-g}, x^3, x^2y, xy^2, y^3).$$

$\square$

CHAPTER 7

DIAGRAMS WITH ONE NONZERO ENTRY IN Δ_0

In this chapter we identify all the lower triangle diagrams with one non-zero entry that occur as $\Delta_0(C)$ for a LCM space curve C in general coordinates. In fact, all such diagrams can be obtained as $\Delta_0(C)$ for a LCM space curve C in the biliaison class of an extremal curve. The primary technique used in this classification is identifying λ-invariants of curves obtained from certain types of double links. Section 7.1 provides several lemmas along these lines, with the proof of the main result following in section 7.2.

7.1 Movement of Triangle Diagrams

We now begin a discussion of how the triangle diagram of a LCM curve C behaves under certain types of double links, making extensive use of Proposition 3.13.

Lemma 7.1. *Let C be a LCM space curve in general coordinates, and let F be a general hyperplane. Let $(c_0, c_1, \cdots, c_m)$ denote the h-vector of $C \cap F$, and let Λ_i denote the number of entries in the h-vector of $C \cap F$ which are greater than or equal to i. Let N denote the largest value appearing in the h-vector of $C \cap F$. Then the λ-invariants of I_C are given by*

$$\lambda_i = \Lambda_{N+1-i} \text{ for } 0 \leq i \leq N.$$

Proof. For each i, the entry c_i in the h-vector of C is equal to the number of entries of $\Delta_1^L(C)$ in the row corresponding to degree i. Since these entries are right justified, c_i contributes one to each of the last c_i λ-invariants of C. The result is immediate. $\square$

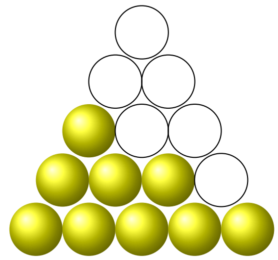


Figure 7.1: Triangle Diagram of a Complete Intersection of Type $(2, 3)$

Example 7.2. Let $C \subset \mathbb{P}^3$ be a complete intersection of type $(2, 3)$ in general coordinates. Its Liebling triangle diagram is shown in Figure 7.1. The h-vector of a general hyperplane section of C is given by $(1, 2, 2, 1)$. The λ-invariants of C are $\lambda_1 = 2$ and $\lambda_2 = 4$.

Proposition 7.3. *Let C be a LCM space curve. Suppose that $C \overset{X_1}{\sim} D \overset{X_2}{\sim} C'$, where X_1 is a complete intersection of type (a, b) with $a \leq b$ and X_2 is a complete intersection of type (a', b') with $a' \leq b'$. Then $gin(I_{C'})$ is uniquely determined by a, b, a', b', and $gin(I_C)$.*

Proof. This is immediate from Proposition 3.13, Proposition 4.40, and Lemma 7.1. $\square$

Lemma 7.4. *Let C be a LCM space curve. Suppose that x^a is the smallest power of x occurring in the double saturation of $gin(I_C)$.*

1. *Suppose that $x^a \in gin(I_C)$. If $C \overset{X_1}{\sim} D \overset{X_2}{\sim} C'$, where X_1 is a complete intersection of type (a, b) with $a \leq b$ and X_2 is a complete intersection of type $(a, b + n)$ for some $n \geq 1$, then*

$$gin(I_{C'}) = (x^a) + y^n gin(I).$$

In particular, if C and C' are in general coordinates,

$$\lambda_i^{C'} = \lambda_i^C + n \text{ for } 1 \leq i \leq a.$$

We note that an elementary double link of type (a, n) satisfies these conditions.

2. *Suppose that $x^{a+1} \in gin(I_C)$. If $C \overset{X_1}{\sim} D \overset{X_2}{\sim} C'$, where X_1 is a complete intersection of type $(a+1, b)$ with $a+1 \leq b$ and X_2 is a complete intersection of type $(a+1, b+n)$ for some $n \geq 1$, then*

$$gin(I_{C'}) = (x^{a+1}) + y^n gin(I).$$

In particular, if C and C' are in general coordinates,

$$\lambda_i^{C'} = \lambda_{i-1}^{C} + n \text{ for } 1 \leq i \leq a+1.$$

We note that an elementary double link of type $(a+1, n)$ satisfies these conditions.

Remark 7.5. In the situation of Lemma 7.4 (1), it is best to picture the triangle diagram of C sliding down n rows along x-columns to produce the triangle diagram of C', as depicted in Figure 7.2. In the depiction of the triangle diagram of C, the black region represents entries corresponding to $\Delta_1^L(C)$. In the depiction of the triangle diagram of C', the black and gray regions together represent entries corresponding to $\Delta_1^L(C')$, with the gray region also representing the shift down the x-columns. Note that the placement of $\Delta_0(C)$ below the dashed line is meant to be indicative of the placement of non-zero entries below the dashed line and is not the actual function $\Delta_0(C)$.

Remark 7.6. In the situation of Lemma 7.4 (2), it is best to picture the triangle diagram of C' in relation to that of C as depicted in Figure 7.3. In the depiction of the triangle diagram of C, the black region represents entries corresponding to $\Delta_1^L(C)$. In the depiction of the triangle diagram of C', the black and gray regions together represent entries corresponding to $\Delta_1^L(C')$, with the gray region

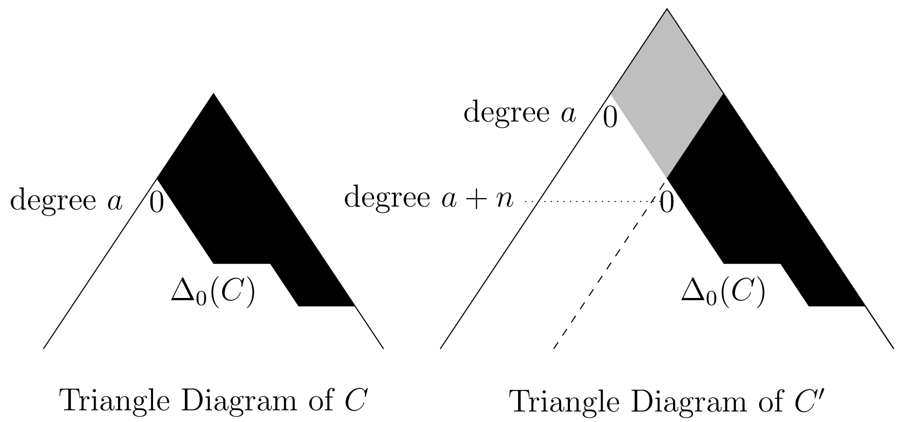


Figure 7.2: Shift of Lower Diagram, as in Lemma 7.4 (1)

also representing the shift down the x-columns of $\Delta(C)$ together with an additional x-column. Note that the placement of $\Delta_0(C)$ below the lower dashed line is meant to be indicative of the placement of non-zero entries below the dashed line and is not the actual function $\Delta_0(C)$.

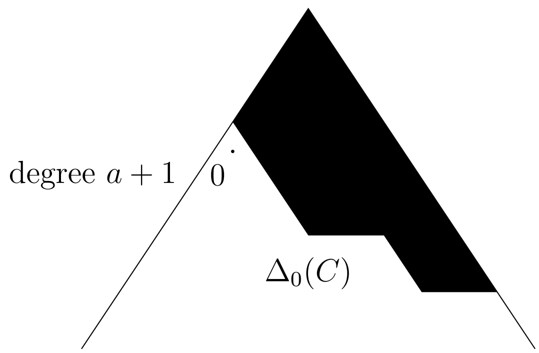
Figure 7.4 emphasizes the situation of Lemma 7.4 (2) when $n = 1$. Figure 7.5 emphasizes the situation of Lemma 7.4 (2) when $a = 1$. In this figure, r denotes negative one times the genus of C and the black regions represent the entries corresponding to the upper triangle diagrams.

Proof of Lemma 7.4. We assume throughout that C, D, C', X_1, and X_2 are in general coordinates. Note that given any C, D, C', X_1, and X_2 satisfying the conditions of either (1) or (2), we may achieve this by applying the same general change of coordinates to all of them.

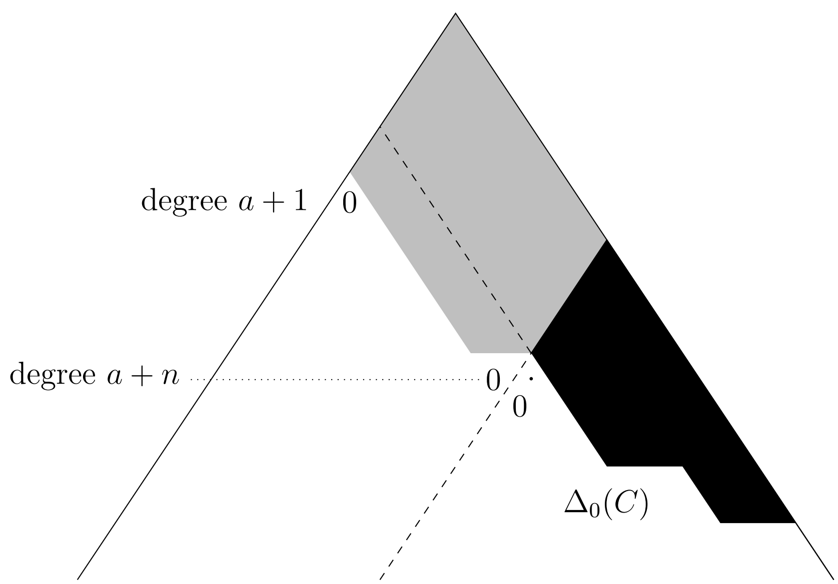
Let F be a general hyperplane. Then

$$(C \cap F) \overset{X_1 \cap F}{\sim} (D \cap F) \overset{X_2 \cap F}{\sim} (C' \cap F).$$

We now proceed with the two situations of the lemma.

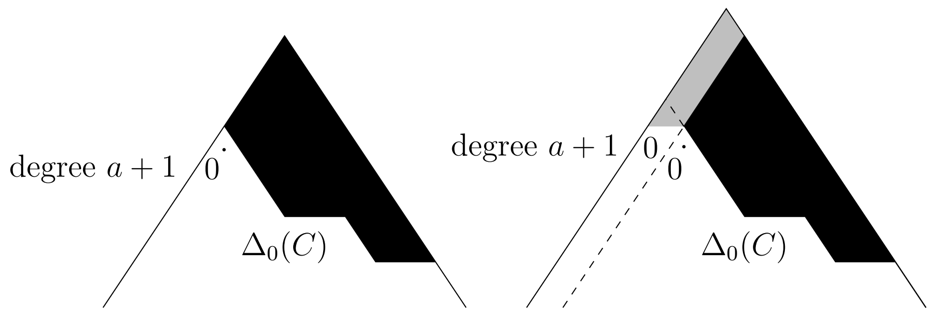


Triangle Diagram of C



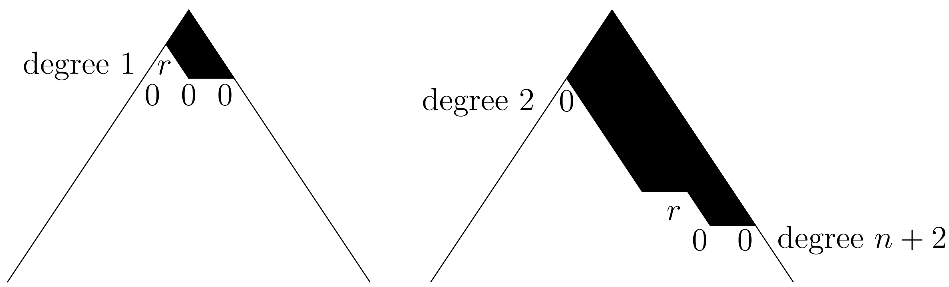
Triangle Diagram of C'

Figure 7.3: Shift of Lower Diagram, as in Lemma 7.4 (2)



Triangle Diagram of C Triangle Diagram of C', $n = 1$

Figure 7.4: Shift of Lower Diagram, as in Lemma 7.4 (2), $n = 1$



Triangle Diagram of C Triangle Diagram of C'

Figure 7.5: Shift of Lower Diagram, as in Lemma 7.4 (2), $a = 1$

1. We denote the h-vectors of $X_1 \cap F$ and $X_2 \cap F$ by $(1, x_1^1, x_2^1, \ldots, x_{b+a-2}^1)$ and $(1, x_1^2, x_2^2, \ldots, x_{b+n+a-2}^2)$ respectively. Note that

$$x_i^1 = \begin{cases} i+1 & \text{for} \quad 0 \le i \le a-2 \\ a & \text{for} \quad a-1 \le i \le b-1 \\ a+b-i-1 & \text{for} \quad b \le i \le b+a-2 \\ 0 & \text{for} \quad i \ge b+a-1 \end{cases}$$

and

$$x_i^2 = \begin{cases} i+1 & \text{for} \quad 0 \le i \le a-2 \\ a & \text{for} \quad a-1 \le i \le b+n-1 \\ a+b+n-i-1 & \text{for} \quad b+n \le i \le b+n+a-2 \\ 0 & \text{for} \quad i \ge b+n+a-1. \end{cases}$$

We denote the h-vectors of $C \cap F$, $D \cap F$, and $C' \cap F$ by $(1, c_1, \ldots, c_k)$, $(1, d_1, \ldots, d_l)$, and $(1, c'_1, \ldots, c'_m)$, respectively.

Using Proposition 3.13,

$$c'_{a+b+n-2-i} = x_i^2 - d_i.$$

Again using Proposition 3.13,

$$d_{a+b-2-i} = x_i^1 - c_i.$$

Noting that $c_i = x_i^1 = x_i^2$ for $0 \le i \le a-1$, we conclude that

$$c'_i = \begin{cases} c_i & \text{for} \quad 0 \le i \le a-1 \\ a & \text{for} \quad a-1 \le i \le a+n-1 \\ c_{i-n} & \text{for} \quad a+n \le i \le k+n \\ 0 & \text{for} \quad k+n+1 \le i \end{cases}$$

and that x^a is the smallest power of x in $I_{C' \cap F}$. Applying Lemma 7.1, we conclude that

$$\lambda_i^{C'} = \lambda_i^C + n.$$

By Proposition 4.40, we know that the values of the nonzero entries in $\Delta_0(C')$ are the same as those in $\Delta_0(C)$ and occur n rows lower in the diagram. Further, along each row these entries are increasing and right-justified. We conclude

$$\mathrm{gin}(I_{C'}) = (x^a) + y^n \mathrm{gin}(I).$$

2. This proof follows much the same form as the proof of (1). We denote the h-vectors of $X_1 \cap F$ and $X_2 \cap F$ by $(1, x_1^1, x_2^1, \ldots, x_{b+a-1}^1)$ and

$(1, x_1^2, x_2^2, \ldots, x_{b+n+a-1}^2)$, respectively. Note that

$$x_i^1 = \begin{cases} i+1 & \text{for} \quad 0 \leq i \leq a-1 \\ a+1 & \text{for} \quad a \leq i \leq b-1 \\ a+b-i & \text{for} \quad b \leq i \leq b+a-1 \\ 0 & \text{for} \quad i \geq b+a \end{cases}$$

and

$$x_i^2 = \begin{cases} i+1 & \text{for} \quad 0 \leq i \leq a-1 \\ a+1 & \text{for} \quad a \leq i \leq b+n-1 \\ a+b+n-i & \text{for} \quad b+n \leq i \leq b+n+a-1 \\ 0 & \text{for} \quad i \geq b+n+a. \end{cases}$$

We denote the h-vectors of $C \cap F$, $D \cap F$, and $C' \cap F$ by $(1, c_1, \ldots, c_k)$, $(1, d_1, \ldots, d_l)$, and $(1, c'_1, \ldots, c'_m)$, respectively.

Using Proposition 3.13,

$$c'_{a+b+n-1-i} = x_i^2 - d_i.$$

Again using Proposition 3.13,

$$d_{a+b-1-i} = x_i^1 - c_i.$$

Noting that $c_i = x_i^1 = x_i^2$ for $0 \leq i \leq a-1$, we conclude that

$$c'_i = \begin{cases} c_i & \text{for} \quad 0 \leq i \leq a-1 \\ a+1 & \text{for} \quad a \leq i \leq a+n-1 \\ c_{i-n} & \text{for} \quad a+n \leq i \leq k+n \\ 0 & \text{for} \quad k+n+1 \leq i \end{cases}$$

and that x^{a+1} is the smallest power of x in $I_{C' \cap F}$. Applying Lemma 7.1, we conclude

$$\lambda_i^{C'} = \lambda_{i-1}^{C} + n \text{ for } 1 \leq i \leq a+1.$$

By Proposition 4.40, we know that the values of the nonzero entries in $\Delta_0(C')$ are the same as those in $\Delta_0(C)$ and occur n rows lower in the diagram. Further, along each row these entries are increasing and right-justified. We conclude

$$\text{gin}(I_{C'}) = (x^{a+1}) + y^n \text{gin}(I).$$

□

Lemma 7.7. *Let C be a LCM space curve. Suppose that x^n is the smallest power of x occurring in the double saturation of gin(I_C). If $C \overset{X_1}{\sim} D \overset{X_2}{\sim} C'$, where X_1 is a complete intersection of type (a, b) with $\lambda_n^C + 1 \leq a \leq b$ and X_2 is a complete intersection of type $(a, b+1)$, and C and C' are in general coordinates, then*

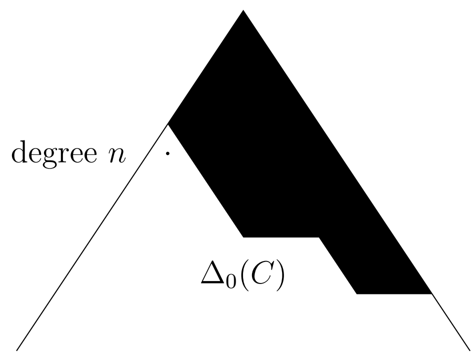
$$\lambda_i^{C'} = \lambda_i^C \textit{ for } 1 \leq i \leq n$$

and

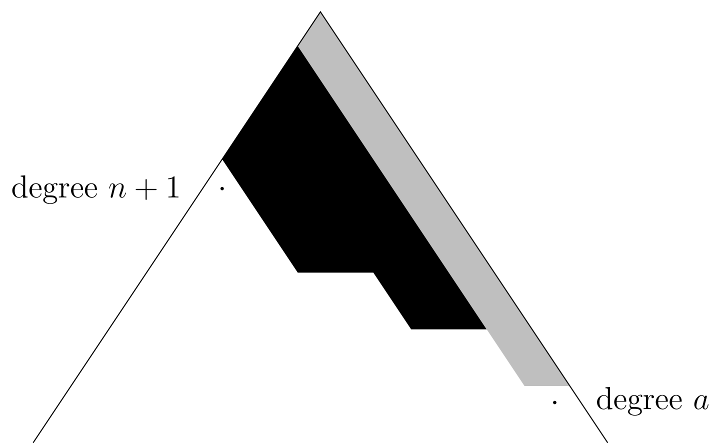
$$\lambda_{n+1}^{C'} = a.$$

We note that an elementary double link of type $(a, 1)$ satisfies these conditions when there is an element of degree a in I_C.

Remark 7.8. In the situation of Lemma 7.7, it is best to picture the triangle diagram of C' in relation to that of C as depicted in Figure 7.6. Essentially, the shape of the lower diagram of C' is that of C shifted one entry down the y-columns, with some entries added along then rightmost x-column. In the depiction of the triangle diagram of C, the black region represents entries corresponding to $\Delta_1^L(C)$. In the depiction of the triangle diagram of C', the black and gray regions together represent entries corresponding to $\Delta_1^L(C')$, with the gray region also representing the shift down the y-columns of $\Delta(C)$ and additional entries along the righmost x-column. Figure 7.7 emphasises the case where $a = \lambda_n^C + 1$.



Triangle Diagram of C



Triangle Diagram of C'

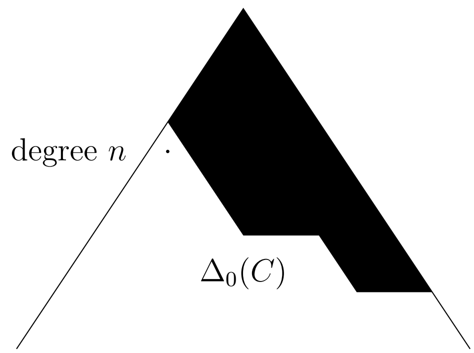
Figure 7.6: Shift of Lower Diagram, as in Lemma 7.7

Proof of Lemma 7.7. We assume throughout that C, D, C', X_1, and X_2 are in general coordinates. Note that given any C, D, C', X_1, and X_2 satisfying the conditions of the theorem, we may achieve this by applying the same general change of coordinates to all of them.

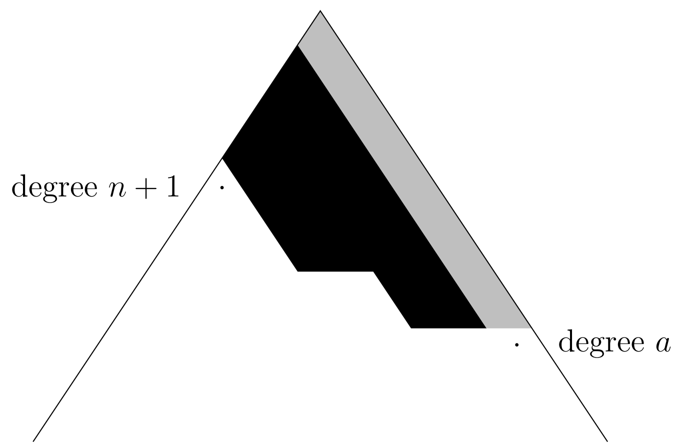
Let F be a general hyperplane. Then

$$(C \cap F) \overset{X_1 \cap F}{\sim} (D \cap F) \overset{X_2 \cap F}{\sim} (C' \cap F).$$

We denote the h-vectors of $X_1 \cap F$ and $X_2 \cap F$ by $(1, x_1^1, x_2^1, \ldots, x_b^1)$ and



Triangle Diagram of C



Triangle Diagram of C', $a = \lambda_n^C + 1$

Figure 7.7: Shift of Lower Diagram, as in Lemma 7.7, $a = \lambda_n^C + 1$

$(1, x_1^2, x_2^2, \ldots, x_{b+n}^2)$ respectively. Note that

$$x_i^1 = \begin{cases} i+1 & \text{for} \quad 0 \leq i \leq a-2 \\ a & \text{for} \quad a-1 \leq i \leq b-1 \\ a+b-i-1 & \text{for} \quad b \leq i \leq a+b-2 \\ 0 & \text{for} \quad i \geq a+b-1 \end{cases}$$

and

$$x_i^2 = \begin{cases} i+1 & \text{for} \quad 0 \le i \le a-2 \\ a & \text{for} \quad a-1 \le i \le b \\ a+b-i & \text{for} \quad b+1 \le i \le a+b-1 \\ 0 & \text{for} \quad i \ge a+b. \end{cases}$$

We denote the h-vectors of $C \cap F$, $D \cap F$ and $C' \cap F$ by $(1, c_1, \ldots, c_k)$, $(1, d_1, \ldots, d_l)$, and $(1, c'_1, \ldots, c'_m)$, respectively. Note that $k+1 = \lambda_n^C$ by Lemma 7.1. Using Proposition 3.13,

$$c'_{a+b-1-i} = x_i^2 - d_i.$$

Again using Proposition 3.13,

$$d_{a+b-2-i} = x_i^1 - c_i.$$

In particular, the h-vector of $D \cap F$ is given by

$$d_i = \begin{cases} i+1 & \text{for} \quad 0 \le i \le a-2 \\ a & \text{for} \quad a-1 \le i \le b-1 \\ a+b-i-1 & \text{for} \quad b \le i \le a+b-2-\lambda_n^C \\ a+b-i-1-c_{a+b-i-2} & \text{for} \quad a+b-\lambda_n^C-1 \le i \le a+b-2 \\ 0 & \text{for} \quad i \ge a+b-1 \end{cases}$$

and we may conclude that the h-vector of $C' \cap F$ is given by

$$c'_i = \begin{cases} 1 & \text{for} \quad i = 0 \\ c_{i-1}+1 & \text{for} \quad 1 \le i \le \lambda_n^C \\ 1 & \text{for} \quad \lambda_n^C + 1 \le i \le a-1 \\ 0 & \text{for} \quad i \ge a. \end{cases}$$

Applying Lemma 7.1, we conclude

$$\lambda_i^{C'} = \lambda_i^C \text{ for } 1 \le i \le n$$

and

$$\lambda_{n+1}^{C'} = a.$$

$\square$

Remark 7.9. Proposition 4.40 implies that the nonzero entries of $\Delta_0(C')$ in Proposition 7.7 are those of $\Delta_0(C)$ shifted down one row and right-justified. The generic initial ideal of C' depends on the number of these entries, their degrees, and the value of a. Note that $\mathrm{gin}(C')$ is unique after having fixed $\mathrm{gin}(C)$ and a, although it is complicated, and not particularly elucidating, to write a formula for it.

We note that when there is only one entry in $\Delta_0(C)$ and $a > \lambda_n^C + 1$, then $\mathrm{gin}(I_{C'}) = x \cdot \mathrm{gin}(I_C) + (y^a)$. Similarly, when there is only one entry in $\Delta_0(C)$, and it does not occur in the farthest right x-column, then $\mathrm{gin}(I_{C'}) = x \cdot \mathrm{gin}(I_C) + (y^a)$. We make use of these facts in the proof of Theorem 7.11.

7.2 Diagrams with One Nonzero Entry in Δ_0

We first prove a technical lemma regarding which lower triangle diagrams with one nonzero entry could possibly occur for LCM space curves in general coordinates and then proceed to show that all diagrams allowed by the lemma do, in fact, occur.

Lemma 7.10. *Let C be a LCM space curve of degree d and genus g in general coordinates. Suppose that $\Delta_0(C)$ has precisely one nonzero entry. Let $\lambda_1, \ldots, \lambda_s$ denote the λ-invariants of C and $\Delta^L = (\Delta_0, \Delta_1^L)$ denote the Liebling triangle diagram of C.*

1. *$\Delta_0(s-i, \lambda_i)$ is nonzero for exactly one integer i with $0 \leq i \leq s$.*

2. *If* $s = 1$*, then* $\Delta_0(1, 0) \neq 0$*.*

3. *If* $\Delta_0(s - i, \lambda_i) \neq 0$*, then* $i = s$ *or* $\lambda_{i+1} \geq \lambda_i + 2$*.*

4. *If* $\Delta_0(s, 0) \neq 0$*, then* $\lambda_1 = 2$*.*

5. *If* $\Delta_0(s - 1, \lambda_1) \neq 0$*, then* $\lambda_2 = \lambda_1 + 2$*.*

Proof. 1. This follows because one entry in Δ_0 is nonzero and such an entry must be top and right justified because Δ_0 is Borel-fixed.

2. If $s = 1$, then the general hyperplane section of C lies on a line. Since Δ_0 has a nonzero entry, the Restriction Theorem implies $d = 2$. Applying Proposition 4.35, $\Delta_0(1, 0) \neq 0$.

3. This follows because the single nonzero entry must be right justified.

4. We know that $\lambda_1 \geq 2$ in this case from (3). Since

$$\sum_{(i,j) \in \mathcal{B}_{\leq s}} \Delta_0(i, j) \leq \sum_{(i,j) \in \mathcal{A}_{\leq s}} \Delta_1^L(i, j),$$

all the nonzero entries in the upper diagram must occur in degree at most s. Entries in Δ_1^L must increase down columns once they are nonzero, so it must be the case that some column in the upper diagram ends in degree at most s, forcing the x-column corresponding to λ_1 to end in degree at most s. Hence, $\lambda_1 \leq 2$. We conclude $\lambda_1 = 2$.

5. Note that $s > 1$. Since there is only one nonzero entry in the lower diagram, Δ^L has no-shared-columns. In particular, no entry along the x-column corresponding to λ_1 in Δ_1^L can be nonzero. Since

$$\sum_{(i,j) \in \mathcal{B}_{\leq s-1+\lambda_1}} \Delta_0(i, j) \leq \sum_{(i,j) \in \mathcal{A}_{\leq s-1+\lambda_1}} \Delta_1^L(i, j),$$

all the nonzero entries in the upper diagram must occur in degree at most $s - 1 + \lambda_1$. Δ_1^L is weakly Borel-fixed, so it must be the case that some x-column in Δ_1^L ends in degree at most $s-1+\lambda_1$. Thus the x-column corresponding to λ_2 end in degree at most $s-1+\lambda_1$. Hence, $\lambda_2 \leq \lambda_1 + 2$. Using (3), we conclude $\lambda_2 = \lambda_1 + 2$.

$\square$

Theorem 7.11. *Let Δ_0 be a Borel-fixed lower triangle diagram with one nonzero entry satisfying the conditions given in Lemma7.10. Then there is a LCM space curve C', obtainable through a series of basic double links from an extremal curve C, such that $\Delta_0 = \Delta_0(C')$. Further, we may assume C and C' are in general coordinates.*

Proof. Let $\lambda_1^{\Delta_0}, \ldots, \lambda_s^{\Delta_0}$ denote the λ-invariants of Δ_0.

First consider the case where $\Delta_0(s,0) = r \neq 0$. Then $\lambda_1^{\Delta_0} = 2$. Let C be a degree two LCM space curve in general coordinates with $\Delta_0(C)(1,0) = r$. Since $\lambda_1^{\Delta_0} < \lambda_2^{\Delta_0} < \cdots < \lambda_s^{\Delta_0}$, Lemma 7.7 implies that after a general change of coordinates we may perform a series of basic double links of types $(\lambda_2^{\Delta_0}, 1)$, $(\lambda_3^{\Delta_0}, 1)$, $\ldots$, $(\lambda_s^{\Delta_0}, 1)$ to obtain a LCM space curve C' in general coordinates with $\Delta_0(C') = \Delta_0$.

Now consider the case where $\Delta_0(s-1, \lambda_1^{\Delta_0}) = r \neq 0$. Then $\lambda_2^{\Delta_0} = 2 + \lambda_1^{\Delta_0}$. Let C be a degree two LCM space curve in general coordinates with $\Delta_0(C)(1,0) = r$. By Lemma 7.4, after a general change of coordinates, we may perform a basic double link of type $(2, \lambda_1^{\Delta_0})$ to obtain a LCM space curve D where the λ-invariants of D are given by $\lambda_1^D = \lambda_1^{\Delta_0}$ and $\lambda_2^D = \lambda_2^{\Delta_0}$. Since $\lambda_1^{\Delta_0} < \lambda_2^{\Delta_0} < \cdots < \lambda_s^{\Delta_0}$, Lemma 7.7 implies that after a general change of coordinates we may perform a series of basic double links of types $(\lambda_3^{\Delta_0}, 1)$, $(\lambda_4^{\Delta_0}, 1)$, $\ldots$, $(\lambda_s^{\Delta_0}, 1)$ starting from D to obtain a curve C' in general coordinates with $\Delta_0(C') = \Delta_0$.

Finally, consider the case where $\Delta_0(s-i, \lambda_i^{\Delta_0}) = r \neq 0$ for some $i \geq 2$. By Lemma 7.7, it is enough to consider the case when $i = s \geq 2$. To see this, suppose that C is a LCM space curve in general coordinates such that the λ-invariants of C are given by $\lambda_j^C = \lambda_j^{\Delta_0}$ for $1 \leq j \leq i$, and $\Delta_0(C)(0, \lambda_i) = r$. Since $\lambda_1^{\Delta_0} < \lambda_2^{\Delta_0} < \cdots < \lambda_s^{\Delta_0}$, Lemma 7.7 implies that after a general change of coordinates we may perform a series of basic double links of types $(\lambda_{i+1}^{\Delta_0}, 1)$, $(\lambda_{i+2}^{\Delta_0}, 1)$, $\ldots$, $(\lambda_s^{\Delta_0}, 1)$ starting from C to obtain a LCM curve C' in general coordinates with $\Delta_0(C') = \Delta_0$.

We return to the case $\Delta_0(0, \lambda_s^{\Delta_0}) = r \neq 0$ and $s \geq 2$. Let C denote a degree $d = \lambda_s^{\Delta_0} - \lambda_{s-1}^{\Delta_0} + 2$ extremal curve with $\Delta_0(C)(0, d-1) = r$. By Lemma 7.4 (1), after a general change of coordinates we may perform a basic double link of type $(2, \lambda_{s-1}^{\Delta_0} - \lambda_{s-2}^{\Delta_0} - 1)$ to obtain a LCM curve D in general coordinates with λ-invariants $\lambda_1^D = \lambda_{s-1}^{\Delta_0} - \lambda_{s-2}^{\Delta_0}$ and $\lambda_2^D = \lambda_s^{\Delta_0} - \lambda_{s-1}^{\Delta_0}$. By Lemma 7.4 (2), after a general change of coordinates, we may then perform a series of basic links of types $(3, \lambda_{s-2}^{\Delta_0} - \lambda_{s-3}^{\Delta_0})$, $\ldots$, $(s-1, \lambda_2^{\Delta_0} - \lambda_1^{\Delta_0}), (s, \lambda_1^{\Delta_0})$ starting with D to obtain a curve C' in general coordinates with $\Delta_0(C') = \Delta_0$. $\square$

Remark 7.12. Theorem 7.11 implies that given any set of positive integers $\{\lambda_i\}_{i=1}^s$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_s$ where $s \geq 2$, there is a LCM space curve C that is not ACM such that the λ-invariants of $\mathrm{gin}(I_C)$ are given by $\{\lambda_i\}_{i=1}^s$. In particular, there is no connectedness criterion on the λ-invariants of LCM space curves.

Example 7.13. Note that the curve C' is not in general unique, even up to change of coordinates, as this example will illustrate. We exhibit two curves C_1' and C_2' with the same lower diagram, but distinct upper diagrams, and hence distinct Rao modules. C_1' is obtained from a series of basic double links from a degree two curve, and C_2' is obtained from a series of basic double links from an extremal degree three curve.

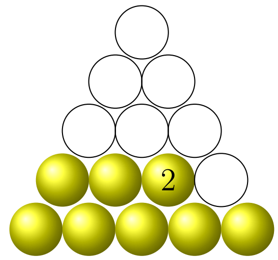
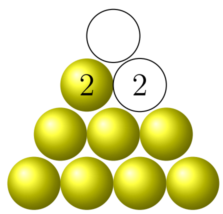
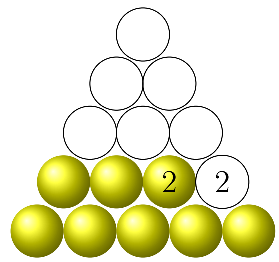


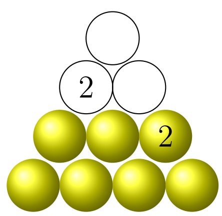
Figure 7.8: Lower Triangle Diagram, as in Example 7.13



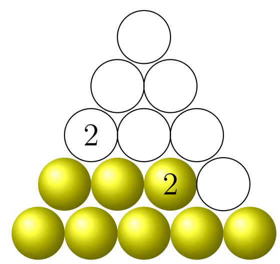
Liebling Triangle Diagram of C_1



Liebling Triangle Diagram of C_1'



Liebling Triangle Diagram of C_2



Liebling Triangle Diagram of C_2'

Figure 7.9: Triangle Diagrams for Curves in Example 7.13

Consider the lower triangle diagram in Figure 7.8. We may obtain this lower diagram as the result of two basic double links from a degree 2 genus -2 LCM space curve C_1. The first double link is of type (2, 2) and the second of type (3,1). We may also obtain this lower diagram as the result of a basic double link of type (4,1) from a degree 3 genus -2 extremal curve C_2. Triangle diagrams for these curves are presented in Figure 7.9.

We close this chapter by providing a complete list of the four-generated degrevlex generic initial ideals of LCM space curves that are not ACM.

Corollary 7.14. *Let C be a LCM space curve in general coordinates that is not ACM. If $in(I_C)$ is generated by four elements, then it is of one of the following forms:*

1. (xz^c, x^2, xy, y^2)
2. $(x^2, xy^a z^b, xy^{a+1}, y^{a+2})$
3. $(x^2, xy^a, y^b z^c, y^{b+1})$*, with* $b \geq a$

for some positive integers a, b, and c.

Remark 7.15. All of the ideals in Corollary 7.14 occur as the generic initial ideal of LCM space curves. Ideals of the form (xz^c, x^2, xy, y^2) occur as the generic initial ideals of degree two LCM space curves. Ideals of the form $(x^2, xy^a z^b, xy^{a+1}, y^{a+2})$ occur as the generic initial ideals of bilinks up from degree two LCM space curves. Ideals of the form $(x^2, xy^a, y^b z^c, y^{b+1})$ occur as the generic initial ideals of extremal LCM space curves and bilinks up from them. These are the only ways these ideals arise as the generic initial ideals of LCM space curves. This is immediate after application of Theorem 4.34 and Proposition 4.38.

Proof of Corollary 7.14. Let C be a LCM space cuve in general coordinates with $\mathrm{gin}(I_C)$ generated by four elements. Then C has at most three λ-invariants.

If C has exactly one λ-invariant, then the Restriction Theorem implies C is either planar, a contradiction, or degree two. Degree two curves have generic initial ideals of the form (xz^c, x^2, xy, y^2).

If C has exactly two λ-invariants, then since its initial ideal has only four generators, $x^2 \in \mathrm{gin}(I_C)$ and $\Delta_0(C)$ has only one entry. Lemma 7.10 implies that the generic initial ideal of I_C takes the form $(x^2, xy^a z^b, xy^{a+1}, y^{a+2})$ or the form $(x^2, xy^a, y^b z^c, y^{b+1})$.

If C has exactly three λ-invariants, then C is ACM since its initial ideal has only four generators. $\square$

CHAPTER 8

HOOK DIAGRAMS

Definition 8.1 (Hook Diagram)**.** *Let C be a LCM space curve of degree $d \geq 3$ in general coordinates that is not ACM. We say a triangle diagram of C is a* hook diagram *if* $(gin(I_C) : z^\infty) = (x^2, xy, y^{d-1})$.

Hook diagrams are so named because of their shape, as seen in Figure 8.1. In this figure, the black represents the upper diagram.

Hook diagrams are of interest because the triangle diagrams of extremal curves of degree at least three are hook diagrams. A natural question is: if C is a LCM space curve in general coordinates such that $\Delta^L(C)$ is a hook diagram, is C extremal? In this chapter provide an affirmative answer to the question if d, the degree of C, is at least five and the genus of C is at least $\binom{d-4}{2}$. There is no similar result in degree three or degree four because in these degrees the Liebling triangle diagram of any non-planar LCM curve is a hook diagram.

We also show that if the triangle diagram of a LCM curve of degree at least five is a hook diagram, then that curve is minimal in its biliaison class. Again, there

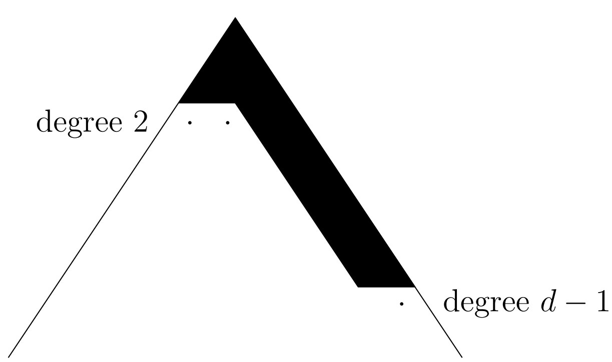


Figure 8.1: Hook Diagram

is no corresponding result in degree four since the Liebling triangle diagram of a degree four subextremal curve is a hook diagram and these curves can be obtained as ascending elementary double links from degree two curves. It is true, however, that all non-planar LCM degree three space curves that are not ACM are minimal in their biliaison classes.

We refer the reader to [43] and [46] for a full treatment of degree three and degree four LCM space curves.

We begin with a general lemma about equidimensional ideals.

Lemma 8.2. *If $I \subset R$ is an equidimensional ideal of codimension r and degree d, then $(x_0, ..., x_{r-1})^d \subset gin(I)$.*

Proof. Let $T = k[x_{r-1}, x_r, \ldots, x_n]$. Suppose that $I = \cap_i Q_i$, where this intersection is a minimal primary decomposition of I. There is an open subset U of $\mathrm{GL}(n+1, k)$ such that if $g \in U$, $(g \cdot Q_i) \cap T$ has codimension one in T for each i. Without loss of generality, we assume $(Q_i \cap T)$ has codimension one in T for each i. We note $(Q_i \cap T)$ is primary.

Define $J = I \cap T = \cap_i (Q_i \cap T)$. Then J is equidimensional of codimension one in T. By Lemma 2.48, $J = (f)$ for some f in T.

Letting gin_{lex} and lt_{lex} denote 'generic initial ideal' and 'lead term' with respect to lexicographic order, we see that $\mathrm{gin}_{lex}(I) \cap T = (\mathrm{lt}_{lex}(f))$. We conclude that $\mathrm{lt}_{lex}(f) = x_{r-1}^j$ for some j and that $\mathrm{lt}_{lex}(f) = \mathrm{lt}(f)$.

That $\deg(f) \leq d$ follows immediately from a comparison of the Hilbert polynomials of R/I and T/J. In particular, $x_{r-1}^d \in \mathrm{gin}(I)$. Since $\mathrm{gin}(I)$ is Borel-fixed, we conclude $(x_0, ..., x_{r-1})^d \subset \mathrm{gin}(I)$. □

We now proceed with our discussion of hook diagrams.

Lemma 8.3. *Let C be a LCM space curve of degree d in general coordinates that is not ACM. If $\Delta^L(C)$ is a hook diagram, x^2 and $xy \in gin(I_C)$, then C is extremal.*

Remark 8.4. This lemma follows from more general work by Hartshorne in [24], as noted in [31], Proposition 5.2.6. In his thesis, Liebling asks if there is an elementary proof of this lemma. We present such a proof below.

Proof. We apply Lemma 8.2 to see that the only possible index where a non-zero entry can occur in $\Delta_0(C)$ is $(0, d-1)$. Since C is not ACM, $\Delta_0(C)(0, d-1) \neq 0$. We apply Proposition 5.7 to see that C is extremal. $\square$

We provide a second proof of Lemma 8.3 that reveals additional structure and does not make use of Lemma 8.2. In particular, this proof parallels Construction 9.1.

Alternate Proof of Lemma 8.3. We will show that $\mathrm{gin}(I_C) = (x^2, xy, y^{d-1}z^a, y^d)$ for some $a \geq 1$, and then apply Corollary 5.7 to conclude that C is extremal.

We may write $I_C = (g_1, g_2, f_1, \ldots, f_r)$, where $lt(g_1) = x^2$, $lt(g_2) = xy$, $lt(f_i) = y^{d_i}z^{e_i}$ for some $d_i \geq d-1, e_i \geq 0$ with $d_i + e_i \geq 4$ for all i, and $\{g_1, g_2, f_1, \ldots, f_r\}$ forms a reduced, minimal (homogeneous) Gröbner basis for I_C.

Since no element of of $\{g_1, g_2, f_1, \ldots, f_r\}$ has degree three, the Crystallization Principle implies $\{g_1, g_2\}$ forms a Gröbner basis for (g_1, g_2). This ideal has codimension one, and hence g_1 and g_2 have a common factor, L. L is linear, and $lt(L) = x$.

Factoring, $g_1 = LM$, $g_2 = LN$, where M and N are linear forms with $lt(M) = x$ and $lt(N) = y$. It may be the case that $L = M$. Note that

$$(I_C : L) = (M, N)$$

and

$$(I_C : (M, N)) = (L, h)$$

for some form $h \in k[y, z, w]$.

Noting that $f_i \in (I_C : (M, N))$ for each i, we may write

$$f_i = \alpha_i L + \beta_i h$$

for some $\alpha_i \in k[z, w]$ and $\beta_i \in k[y, z, w]$. To see this, first note that by collecting terms we may assume $\beta_i \in k[y, z, w]$. Now suppose that α_i had a term divisible by x. In this case, f_i would have a term divisible by x^2 because $lt(L) = x$. No terms of $\beta_i h$ are divisible by x, causing the Gröbner basis to be non-reduced, a contradiction. Similarly, if α_i had a term divisible by y, then f_i would have a term divisible by xy, again a contradiction.

Also note,

$$\beta_i = b_i N + c_i$$

with $b_i \in k[y, z, w]$ and $c_i \in k[z, w]$.

We now show that for each i, $lt(f_i) = lt(h(b_i N + c_i))$. The monomial $lt(h(b_i N + c_i))$ cannot cancel with any term of $\alpha_i L$. To see this, first note that $x \mid lt(\alpha_i L)$ but $x \nmid lt(h(b_i N + c_i))$, so these two terms do not cancel. If $lt(h(b_i N + c_i))$ were to cancel with a smaller term of $\alpha_i L$, then $lt(f_i) = lt(\alpha_i L)$, which is divisible by x, a contradiction. In particular,

$$lt(f_i) = lt(h(b_i N + c_i)) = lt(h) \cdot lt(b_i N + c_i).$$

We now show that for each i, $lt(f_i) = lt(h) \cdot lt(b_i N)$ or $lt(f_i) = lt(h) \cdot lt(c_i)$. The monomial $lt(c_i)$ cannot cancel with any term of $b_i N$. To see this, first note that $y | lt(b_i N)$ but $y \nmid lt(c_i)$, so these two terms do not cancel. If $lt(c_i)$ were to cancel with a smaller term of $b_i N$, then $lt(b_i N + c_i) = lt(b_i N)$. In particular, $lt(h(b_i N + c_i)) = lt(h) \cdot lt(b_i N)$ or $lt(h) \cdot lt(c_i)$, and $lt(f_i) = lt(h) \cdot lt(b_i N)$ or $lt(f_i) = lt(h) \cdot lt(c_i)$.

By construction, $hN \in I_C$. Note that $lt(hN) = y \cdot lt(h)$ is not divisible by $lt(LM)$ or $lt(LN)$, and so must be divisible by $lt(f_i)$ for some i. The monomial $y \cdot lt(h)$ is only divisible by $lt(hc_i)$ if $c_i \in k^\times$. We conclude that either $c_j \in k^\times$ for some j or $lt(hb_j N)$ divides $lt(hN)$ for some j (i.e., $b_j \in k^\times$).

If $c_j \in k^\times$ for some j, then $b_j = 0$ by homogeneity. In this case, $lt(f_j) = lt(h)$, and hence $I_C = (g_1, g_2, \alpha_j L + h)$. This implies that I_C is ACM, a contradiction.

We conclude that $b_j \in k^\times$ for some j and $lt(hb_j N) = lt(f_j)$ divides $lt(hN)$. Without loss of generality, $j = 1$ and

$$f_1 = \alpha_1 L + h(N + c_1)$$

for some linear $c_1 \in k[z, w]$. By minimality of the specified Gröbner basis for I_C, we conclude that $lt(f_i) = lt(hc_i)$ for $i \geq 2$. Since I_C is saturated and the specified Gröbner basis is minimal, we have $lt(hN) = y^{d_1}$ and $lt(hc_i) = y^{d_1 - 1} z^{e_i}$ for $i \geq 2$. Since the specified Gröbner basis was minimal and reduced, $r = 2$ and $\mathrm{gin}(I_C) = (x^2, xy, y^{d_1+1} z^{e_2}, y^{d_1})$ for some $a \geq 1$. Applying Corollary 5.7 we conclude that C is extremal. $\square$

Lemma 8.5. *Let C be a LCM space curve of degree d in general coordinates that is not ACM. If C is not extremal, $d \geq 4$, and $\Delta^L(C)$ is a hook diagram, then $\Delta^L_1(0, d-2) \geq 1$.*

Proof. Let $\Delta^L = (\Delta_0, \Delta_1^L)$ denote the Liebling triangle diagram of C. Since C is not ACM, Δ_1^L must have at least one non-zero entry. Suppose that the only non-zero entry of Δ_1^L is $\Delta_1^L(1,0)$. We apply Theorem 4.34 to see that x^2 and $xy \in \mathrm{gin}(I_C)$. By Lemma 8.3, C is extremal, a contradiction. Since Δ_1^L is weakly Borel-fixed, we conclude that $\Delta_1^L(0, d-2)$ is nonzero. □

Proposition 8.6. *Let C be a LCM space curve of degree $d \geq 5$ in general coordinates that is not ACM. Suppose that $\Delta^L(C)$ is a hook diagram. Then C is minimal in its biliaison class.*

Proof. First consider the case where $\Delta_0(C)(2,0) = \Delta_0(C)(1,1) = 0$. Then by Lemma 8.3, C is extremal, and hence minimal in its biliaison class.

Now consider the case where $\Delta_0(C)(1,1) \neq 0$. Since C is not extremal, $\Delta_1^L(C)(0, d-2) \neq 0$. Assume that C is not minimal in its biliaison class. Let $\mathcal{L}$ denote the biliaison class of C, and write $C \in \mathcal{L}^h$. By Theorem 3.32, we may deform C to a curve $C' \in \mathcal{L}^h$ such that C' is obtained from a curve D by a basic double link. We may assume that C' is in general coordinates. Since C and C' have isomorphic Rao modules, $\Delta_0(C')$ has a non-zero entry in degree two. We apply Proposition 4.40 to see that $\Delta_0(D)$ has a non-zero entry in degree one. This forces the basic double link from D to C' to be of height one, and hence $\Delta_1^L(D)$ has a nonzero entry in degree $d-3$. By the Restriction Theorem, no such curve D exists. We conclude that C is minimal in its biliaison class. □

Remark 8.7. The technique used in Proposition 8.6 can be used to show that any degree three LCM space curve that is not ACM is minimal in its biliaison class.

Lemma 8.8. *Let C be a LCM space curve of degree d in general coordinates that is not ACM. Let $\Delta^L = (\Delta_0, \Delta_1^L)$ denote the Liebling triangle diagram of C. If C*

Table 8.1: Data for Lemma 8.8

i	upper bound for $\Delta_1^L(0,i)$	lower bound for $\Delta_1^L(0,i)$	range of $i-\Delta_1^L(0,i)$
$d-2$	$d-4$	1	2 to $d-3$
$d-3$	$d-5$	0	2 to $d-3$
$d-4$	$d-6$	0	2 to $d-4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
3	1	0	2 to 3
2	0	0	2
1	0	0	1
0	0	0	0

is not extremal, Δ^L *is a hook diagram, and* $d \geq 5$*, then either* $\Delta_1^L(1,0) = 0$ *or* $\Delta_1^L(0, d-2) \geq d-3$.

Proof. Assume $\Delta_1^L(1,0) > 0$. By way of contradiction, assume $\Delta_1^L(0, d-2) \leq d-4$. Since C is not extremal, Lemma 8.5 implies $\Delta_1^L(0, d-2) \geq 1$. Since the Liebling triangle diagram of C is weakly Borel-fixed,

$$\Delta_1^L(0,i) \leq max\{i-2, 0\}$$

for $0 \leq i \leq d-2$. We summarize this date in Table 8.1.

Note that

$$|\{i \mid 2 \leq i \leq d-2\}| = d-3,$$

while

$$|\{i - \Delta_1^L(0,i) \mid 2 \leq i \leq d-2\}| \leq d-4,$$

forcing one of $2, \ldots, d-3$ to occur in the spectrum of C more than once. Also note that $1 - \Delta_1^L(1,0) \leq 0$, so that one occurs in the spectrum of C only once.

By Theorem 2.38, the spectrum of C is inadmissible. We conclude that $\Delta_1^L(0, d-2) \geq d-3$. $\square$

Lemma 8.9. *Let C be a LCM space curve of degree d and genus g in general coordinates. If $\Delta^L(C)$ is a hook diagram, $d \geq 5$, and the minimal generators of $gin(I_C)$ can be partitioned into two sets B_1 and B_2 such that all of the following hold:*

1. *every element of B_1 is divisible by x,*
2. *no element of B_2 is divisible by x,*
3. *the degree of every element of B_1 is $\leq d-2$, and*
4. *the degree of every element of B_2 is $\geq d$,*

then x^2, $xy \in gin(I_C)$.

Remark 8.10. In the situation of Lemma 8.9, we may represent the lower triangle diagram of C as in Figure 8.2. The black region represents indices that are not entries in $\Delta_0(C)$. All entries in the white region must be zero. If a non-zero entry occurs in the yellow region at index (i, j), then we must have $i+j+\Delta_0(i, j) \leq d-2$. The topmost entry in the gray region must be nonzero. Elements of B_1 correspond to generators in the yellow region and along row $d-2$. Elements of B_2 correspond to generators in the gray region.

Proof of Lemma 8.9. Let $\{g_1, \ldots, g_s, f_1, \ldots, f_r\}$ be a minimal Gröbner basis for I_C, such that the elements $g_1, \ldots, g_s$ correspond to the elements of B_1 and the elements $f_1, \ldots, f_r$ correspond to the elements of B_2.

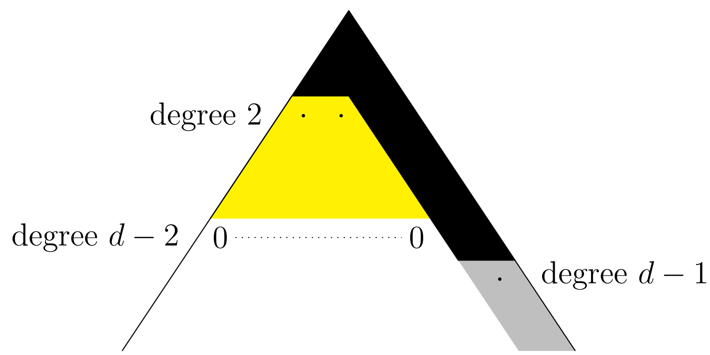


Figure 8.2: Lower Triangle Diagram of C as in Lemma 8.9

By the Crystallization Principle, $\{g_1, \ldots, g_s\}$ is a Gröbner basis for the ideal $(g_1, \ldots, g_s)$. This ideal has codimension 1, as $codim(B_1) = 1$. Hence, $g_1, \ldots, g_s$ have a common factor, L. L is linear, and $lt(L) = x$.

Consider $(I_C : L)$.

$$(I_C : L) = (\frac{g_1}{L}, \ldots, \frac{g_s}{L}, \text{forms of degree} \geq d-1),$$

where $\frac{g_1}{L}, \ldots, \frac{g_s}{L}$ each have degree at most $d-3$. Since C is LCM, $(I_C : L)$ is equidimensional of codimension 2. Further,

$$(x, y) \subset (in(I_C : L) : z^\infty)$$

because $x^2 z^{k_1}$, $xyz^{k_2} \in B_1$ for some k_1, k_2, and $lt(L) = x$. This implies $deg(I_C : L) = 1$, forcing $in(I_C : L) = (x, y)$. This implies that x^2, $xy \in \text{gin}(I_C)$, as desired. □

We are now ready to prove the main result of this chapter.

Theorem 8.11. *Let C be a LCM space curve of degree d and genus g in general coordinates that is not ACM. Let $\Delta^L = (\Delta_0, \Delta_1^L)$ denote the Liebling triangle diagram of C. Suppose that $d \geq 5$, and $g \geq \binom{d-4}{2}$. If Δ^L is a hook diagram, then C is extremal.*

Proof. By way of contradiction, assume that C is not extremal. We proceed with the aim to partition the generators of $\mathrm{gin}(I_C)$ in the manner of Lemma8.9.

By Proposition 4.28,

$$g = \binom{d-2}{2} - \sum \Delta_0(i,j),$$

which implies

$$\sum \Delta_0(i,j) \leq \binom{d-2}{2} - \binom{d-4}{2} = 2d-7.$$

By Lemma8.8, $\Delta_1^L(1,0) = 0$ or $\Delta_1^L(0,d-2) \geq d-3$. If $\Delta_1^L(0,d-2) \leq d-4$ (and hence, $\Delta_1^L(1,0) = 0$), then $\Delta_1^L(0,2) = \Delta_1^L(0,1) = 0$, since the diagram is weakly Borel-fixed. By Proposition 4.18, $\Delta_0(2,0) = \Delta_0(1,1) = 0$. By Lemma, 8.3, C is extremal.

We are reduced to the case $\Delta_1^L(0,d-2) \geq d-3$. In this case,

$$\sum_{i+j<d-2} \Delta_0(i,j) \leq 2d-7-(d-3) = d-4$$

and

$$\sum_{i+j \geq d-2} \Delta_0(i,j) \geq d-3.$$

We now show that $\Delta_0(1,d-3) = 0$. By way of contradiction, assume $\Delta_0(1,d-3) \geq 1$. This implies $\Delta_0(1,d-i) \geq i-2$ for $3 \leq i \leq d-1$ since $\mathrm{gin}(I_C)$ is Borel fixed. Then $\Delta_0(1,1) = d-3$, a contradiction. So $\Delta_0(1,d-3) = 0$, which implies that $\Delta_0(i,j) = 0$ when both $i \geq 1$ and $i+j \geq d-2$.

Since $\Delta_1^L(0,d-2) \neq 0$ and $\Delta_0(i,j) = 0$ when both $i \geq 1$ and $i+j \geq d-2$, it must be the case that $\Delta_0(0,d-1) \geq 1$ by Proposition 4.18. In particular, if m is a minimal generator for $\mathrm{gin}(I_C)$ that is not divisible by x, then the degree of m is at least d.

Note:

$$\begin{aligned}
\Delta_0(1,1) &\leq d-4 \\
\Delta_0(1,2) &\leq d-5 \\
&\vdots \\
\Delta_0(1,d-4) &\leq 1 \\
\Delta_0(1,d-3) &= 0.
\end{aligned}$$

Since Δ_0 is Borel-fixed, this implies $\Delta_0(i,j) \leq \max\{0,\, d-2-i-j\}$ for all pairs $(i,j) \in \Delta_0$ with $i \neq 0$. In particular, if m is a minimal generator for $\mathrm{gin}(I_C)$ that is divisible by x, then the degree of m is at most $d-2$.

The conclusions of the preceding two paragraphs allow us to partition the minimal generators of $\mathrm{gin}(I_C)$ into two sets B_1 and B_2 that satisfy the requirements of Lemma8.9. We conclude that x^2, $xy \in \mathrm{gin}(I_C)$. By Lemma8.3, C is extremal.

$\square$

CHAPTER 9

CURVES IN THE DOUBLE PLANE

In [29], Hartshorne and Schlesinger provide an in-depth analysis of LCM space curves in double planes. Our interest in their analysis arose out of a desire to limit the types of curves that could have hook diagrams. In this chapter, we provide an ideal-theoretic construction, mirroring their scheme-theoretic one, and use this construction to resolve the saturated ideals of such curves. Hartshorne and Schlesinger construct three other schemes from a LCM curve C in the double plane. We find the saturated ideals for these schemes. We close this chapter by applying one of the results in [29] to derive a symmetry condition on the triangle diagrams of curves in double planes. This condition is especially limiting in the case that a LCM curve C has a hook diagram and sits in a double plane.

Construction 9.1. Let L be a linear form in S. Let C be a LCM curve in the double plane defined by L^2 that is not ACM. Note that $C \not\subset V(L)$. We will assume the degree of C is at least three. Without loss of generality, C and $V(L)$ are in general coordinates.

Let Y denote the residual scheme to the intersection of C with $V(L)$. Let P denote the largest equidimensional curve in $V(L)$ that is contained in C. Let Z denote the residual scheme of P in $C \cap V(L)$. By construction, Y and P are curves, and Z is a zero-dimensional scheme. The schemes Y, P, and Z are those constructed by Hartshorne and Schlesinger, and $Z \subset Y \subset P$.

Then $I_Y = (I_C : L) = (L, F)$ for some form F, $I_P = \mathrm{top}(L + I_C)$ and $I_Z = ((L+I_C) : I_P)^{\mathrm{sat}}$, where the function 'top' returns the top dimensional part of the ideal.

We may write $I_C = (L^2, LF, f_1, \dots f_r)$, where the given generators are homogeneous and $r \geq 1$. Since $I_C \subset I_Y$, we may write $f_i = \alpha_i L + \beta_i' F$ for some forms α_i and β_i'. We may assume that α_i and β_i' have no terms divisible by L in $k[L, y, z, w]$, for all i.

Note that if $\beta_i' = 0$ for some i, then $\alpha_i \in (I_C : L)$. In this case, α_i is divisible by F and f_i is not necessary to generate I_C. We will assume that $\beta_i' \neq 0$ for all i.

Let $H = \gcd\{\beta_i'\}_{i=1}^r$, so that $f_i = \alpha_i L + \beta_i HF$ and $\gcd\{\beta_i\} = 1$.

Note, $(L + I_C) = (L, \beta_1 HF, \dots, \beta_r HF)$, so that $\mathrm{top}(L + I_C) = (L, HF)$, which is saturated. In particular, $I_P = (L, HF)$ and $I_Z = (L, \beta_1, \dots, \beta_r)^{\mathrm{sat}}$.

Lemma 9.2. *With notation as in Construction 9.1, define α by*

$$\alpha(\beta_i) = \alpha_i \ \mathit{mod}\ F.$$

Then, $\alpha \in Hom((\beta_1, \dots, \beta_r), S/(L, F))$

Proof. Suppose we have a syzygy $\sum_{i=1}^r \delta_i \beta_i = 0$. Without loss of generality, we may assume that δ_i has no terms divisible by L in $k[L, y, z, w]$. Note, $\sum_{i=1}^r \delta_i \alpha_i L \in I_C$, and hence $\sum_{i=1}^r \delta_i \alpha_i \in (I_C : L)$. Then $\sum_{i=1}^r \delta_i \alpha_i$ is divisible by F, and there exists $\alpha \in \mathrm{Hom}((\beta_1, \dots, \beta_r), S/(L, F))$ defined by $\alpha(\beta_i) = \alpha_i$ mod F. □

Lemma 9.3. *With notation as in Construction 9.1, if $\epsilon L + \delta FH \in I_C$ for some forms $\epsilon \in S$ and $\delta \in k[y, z, w]$, then $\delta \in (\beta_1, \dots, \beta_r)$ and $\epsilon = \alpha(\delta)$ in $S/(L, F)$, where α is as in Lemma 9.2.*

Proof. Write

$$\begin{aligned}\epsilon L + \delta FH &= \mu_1 L^2 + \mu_2 LF + \sum_{i=1}^{r} \gamma_i f_i \\ &= L\Big(\mu_1 L + \mu_2 F + \sum_{i=1}^{r} \gamma_i \alpha_i\Big) + FH \sum_{i=1}^{r} \gamma_i \beta_i,\end{aligned}$$

for some forms $\mu_1, \mu_2,$ and γ_i for $1 \leq i \leq r$. Without loss of generality, γ_i has no terms divisible by L in $k[L, y, z, w]$ for $1 \leq i \leq r$. Then $\delta \in (\beta_1, \ldots, \beta_r)$.

Consider $(\epsilon L + \delta FH) - (\alpha(\delta)L + \delta FH) = (\epsilon - \alpha(\delta))L$. Then $\epsilon - \alpha(\delta) \in (I_C : L) = (L, F)$. We conclude the result. $\square$

Lemma 9.4. *With notation as in Construction 9.1, the ideal $(L, \beta_1, \ldots, \beta_r)$ is saturated.*

Proof. It is sufficient to show that $(\beta_1, \ldots, \beta_r)$ is saturated in S/L. Throughout, let T denote S/L. Consider a form $\delta \in T$ such that $y\delta, z\delta$, and $w\delta$ are all elements of $(\beta_1, \ldots, \beta_r)$. We will show that $\delta \in (\beta_1, \ldots, \beta_r)$.

There exist forms α_I, α_{II} and α_{III} in S such that the polynomials

$$f_I = \alpha_I L + y\delta HF$$

$$f_{II} = \alpha_{II} L + z\delta HF$$

$$f_{III} = \alpha_{III} L + w\delta HF$$

are all elements of I_C. Further, we may assume α_I, α_{II} and $\alpha_{III} \in k[y, z, w]$.

By Lemma 9.2, we conclude that $z\alpha_I - y\alpha_{II}$, $w\alpha_I - y\alpha_{III}$, and $w\alpha_{II} - z\alpha_{III}$ are elements of (F).

The Koszul complex resolves k over T, so that the complex

$$\mathcal{F}:\ 0 \longrightarrow T \xrightarrow{\phi_3} T^3 \xrightarrow{\phi_2} T^3 \xrightarrow{\phi_1} T \longrightarrow k \longrightarrow 0$$

where $\phi_1 = [w, -z, y]$,

$$\phi_2 = \begin{bmatrix} -z & y & 0 \\ -w & 0 & y \\ 0 & -w & z \end{bmatrix},$$

and

$$\phi_3 = \begin{bmatrix} y \\ z \\ w \end{bmatrix},$$

is exact. Note that we have not chosen the standard basis.

We tensor the entire complex with T/F to obtain the complex

$$\mathcal{F} \otimes_T (T/F):\ 0 \longrightarrow T/F \xrightarrow{\bar{\phi}_3} (T/F)^3 \xrightarrow{\bar{\phi}_2} (T/F)^3 \xrightarrow{\bar{\phi}_1} T/F \longrightarrow k \longrightarrow 0.$$

Note $\mathrm{Tor}_i^T(k, T/F) = \mathrm{Tor}_i^T(T/F, k)$ and $\mathrm{Tor}_2^T(T/F, k) = 0$.

Then

$$\begin{bmatrix} \bar{\alpha}_I \\ \bar{\alpha}_{II} \\ \bar{\alpha}_{III} \end{bmatrix} \in \ker(\bar{\phi}_2) = \mathrm{im}(\bar{\phi}_3)$$

so that

$$\begin{bmatrix} \alpha_I \\ \alpha_{II} \\ \alpha_{III} \end{bmatrix} = \epsilon \begin{bmatrix} y \\ z \\ w \end{bmatrix} + F \begin{bmatrix} \lambda_I \\ \lambda_{II} \\ \lambda_{III} \end{bmatrix}$$

for some forms $\epsilon, \lambda_I, \lambda_{II}$ and λ_{III}.

We have

$$f_I = \epsilon y L + \lambda_I F L + \delta y F H,$$

$$f_{II} = \epsilon z L + \lambda_{II} F L + \delta z F H, \text{ and}$$

$$f_{III} = \epsilon w L + \lambda_{III} F L + \delta w F H.$$

Noting that $\lambda_i F L \in I_C$ for each $i \in \{I, II, III\}$, we have

$$f'_I = \epsilon y L + \delta y F H,$$

$$f'_{II} = \epsilon z L + \delta z F H, \text{ and}$$

$$f'_{III} = \epsilon w L + \delta w F H$$

are elements of I_C. Also, $L(\epsilon L + \delta F H) = \epsilon L^2 + \delta L F H \in I_C$. Since I_C is saturated, $\epsilon L + \delta F H \in I_C$.

We apply Lemma 9.3 to see that $\delta \in (\beta_1, \ldots, \beta_r)$ and conclude that $(\beta_1, \ldots, \beta_r)$ is saturated. □

Remark 9.5. We note that $I_Z = (L, \beta_1, \ldots, \beta_r)$. Hartshorne and Schlesinger show that C is ACM if and only if Z is empty. If $\deg(\beta_i) = 0$ for some i, then C is ACM. We conclude that $\deg(\beta_i) \geq 1$ for all i in Construction 9.1, and that $I_C = (L^2, LF, f_1, \ldots f_r)$ can be taken to be a minimal generating set.

Before we continue with the construction of the free resolution of S/I_C, we review some notation.

Notation 9.6. *For any integer $i \geq 1$, let $\mathbb{I}_i$ denote the $i \times i$ identity matrix.*

Theorem 9.7. *With notation as in Construction 9.1, let $\beta = [\beta_1, \ldots, \beta_r]$ and $\alpha = [\alpha_1, \ldots, \alpha_r]$. Let T denote S/L. Define the matrix M by the free resolution of T/I_Z given by*

$$0 \longrightarrow T^{r-1} \xrightarrow{M} T^r \xrightarrow{\beta} T$$

so that the $(r-1) \times (r-1)$ *minors of* M *are given by* $(-1)^{i-1}\beta_i$ *for* $1 \leq i \leq r$. *Define an* $1 \times (r-1)$ *matrix* $\gamma = [\gamma_1, \ldots, \gamma_{r-1}]$ *by* $F\gamma = \alpha M$. *Note that* γ *is well-defined by Lemma 9.2.*

Define the maps Φ_1, Φ_2, *and* Φ_3 *by the following matrices (matrix dimensions are annotated along the top and right of each matrix):*

$$\Phi_1 = \begin{array}{cc} \begin{array}{ccc} 1 & 1 & r \end{array} & \\ \begin{bmatrix} L^2 & LF & \alpha L + \beta H F \end{bmatrix} & 1 \end{array}$$

$$\Phi_2 = \begin{array}{cc} \begin{array}{ccc} 1 & r & r-1 \end{array} & \\ \begin{bmatrix} -F & -\alpha & 0 \\ L & -\beta H & -\gamma \\ 0 & L \cdot \mathbb{I}_r & M \end{bmatrix} & \begin{array}{c} 1 \\ 1 \\ r \end{array} \end{array}$$

$$\Phi_3 = \begin{array}{cc} r-1 & \\ \begin{bmatrix} \gamma \\ -M \\ L \cdot \mathbb{I}_{r-1} \end{bmatrix} & \begin{array}{c} 1 \\ r \\ r-1 \end{array} \end{array}$$

Then

$$\mathcal{F}: \ 0 \longrightarrow S^{r-1} \xrightarrow{\Phi_3} S^{2r} \xrightarrow{\Phi_2} S^{r+2} \xrightarrow{\Phi_1} S$$

is a free resolution of S/I_C.

Proof. The fact that $\mathcal{F}$ is a complex is immediate from block multiplication of the concerned matrices.

Since $\operatorname{rank}(\Phi_1) = 1$, $\operatorname{rank}(\Phi_2) = r + 1$, and $\operatorname{rank}(\Phi_3) = r - 1$, the Buchsbaum-Eisenbud Exactness Criterion implies that to check exactness it is sufficient to check the following three conditions:

1. $\operatorname{codim}(I_1(\Phi_1)) \geq 1$,
2. $\operatorname{codim}(I_{r+1}(\Phi_2)) \geq 2$, and
3. $\operatorname{codim}(I_{r-1}(\Phi_3)) \geq 3$.

Condition (1) is immediate, so we proceed with checking condition (2). The determinant of the submatrix of Φ_2 obtained by taking the first $(r+1)$ rows and the first $(r+1)$ columns of Φ_2 is L^{r+1}. It is sufficient to check that one $(r+1) \times (r+1)$ minor of the matrix

$$\begin{bmatrix} -F & -\alpha & 0 \\ 0 & -\beta H & -\gamma \\ 0 & 0 & M \end{bmatrix},$$

obtained by replacing L by 0 in Φ_2, is nonzero. Indeed, the determinant of the submatrix obtained by taking the first, second, and last $(r-1)$ columns and rows of Φ_2 is, up to sign, $\beta_1^2 HF \neq 0$.

We now check condition (3). The determinant of the submatrix obtained by taking the last $(r-1)$ rows of Φ_3 is L^{r-1}. It is sufficient to check that the codimension of the ideal generated by the $(r-1) \times (r-1)$ minors of the matrix

$$\begin{bmatrix} \gamma \\ -M \end{bmatrix}$$

has codimension at least two. The ideal of the $(r-1)\times(r-1)$ minors of M is $(\beta_1,\dots\beta_r)$, which has codimension two because I_Z defines a set of points in $\mathbb{P}^3$.

We conclude that $\mathcal{F}$ is a free resolution of S/I_C. $\square$

We compute the remaining maximal minors of

$$\begin{bmatrix} \gamma \\ -M \end{bmatrix}$$

up to sign.

Lemma 9.8. *In the situation of Theorem 9.7: Let A denote the matrix*

$$\begin{bmatrix} \gamma \\ -M \end{bmatrix}.$$

Then $I_{r-1}(A) = \left(\beta_1,\dots,\beta_r,\left\{\frac{\alpha_j\beta_i-\alpha_i\beta_j}{F}\right\}_{i\neq j}\right)$.

Proof. We let $m_{i,j}$ denote the entry in the i^{th} row and j^{th} column of M. Let M_i denote M with the i^{th} row removed. Let $M_{(i,j)}$ denote M with the i^{th} and j^{th} rows removed. Let $M_{(i,j),l}$ denote M with the i^{th}row, j^{th} row and l^{th} column removed.

We note that $\gamma_i = \sum_{j=1}^r \alpha_j m_{j,i}/F$.

Let M_i^j denote M with i^{th} row removed and γ replacing the j^{th} row.

Fix $1 < i < j < r$. We compute:

$$\begin{aligned}
\det(M_i) &= \sum_{l=1}^{r-1}(-1)^{j-1+l}m_{j,l}\det(M_{(i,j),l}) = (-1)^{i-1}\beta_i \\
\det(M_j) &= \sum_{l=1}^{r-1}(-1)^{i-1+l}m_{i,l}\det(M_{(i,j),l}) = (-1)^{j-1}\beta_j \\
\det(M_i^j) &= \sum_{l=1}^{r-1}(-1)^{j-1+l}\gamma_l\det(M_{(i,j),l}) \\
&= \sum_{l=1}^{r-1}(-1)^{j-1+l}\left(\frac{\sum_{n=1}^{r}\alpha_n m_{n,l}}{F}\right)\det(M_{(i,j),l}) \\
&= \sum_{n=1}^{r}\frac{\alpha_n}{F}\sum_{l=1}^{r-1}(-1)^{j-1+l}\det(M_{(i,j),l})m_{n,l} \\
&= \frac{\alpha_j}{F}(-1)^{i-1}\beta_i + \sum_{n\neq j}\frac{\alpha_n}{F}\sum_{l=1}^{r-1}(-1)^{j-1-l}m_{n,l}\det(M_{(i,j),l}) \\
&= \frac{\alpha_j}{F}(-1)^{i-1}\beta_i + \frac{\alpha_i}{F}(-1)^{j-1+j-1-i}\beta_j + \sum_{n\neq i,\,j}\frac{\alpha_n}{F}\sum_{l=1}^{r-1}(-1)^{j-1-l}m_{n,l}\det(M_{(i,j),l}) \\
&= \frac{\alpha_j}{F}(-1)^{i-1}\beta_i + \frac{\alpha_i}{F}(-1)^{i}\beta_j
\end{aligned}$$

The case for $i > j$ is symmetric. We conclude $I_{r-1}(A) = \left(\beta_1, \beta_2, \ldots, \beta_r, \left\{\frac{\alpha_j\beta_i-\alpha_i\beta_j}{F}\right\}_{i\neq j}\right)$. □

Corollary 9.9. *In the situation of Theorem 9.7, the ideal*

$$\left(L, \beta_1, \beta_2, \ldots, \beta_r, \left\{\frac{\alpha_j\beta_i-\alpha_i\beta_j}{F}\right\}_{i\neq j}\right)$$

has codimension four.

Proof. Since C is LCM, $\operatorname{codim}(I_{r-1}(\Phi_3)) = 4$, where Φ_3 is as in Theorem 9.7. □

Remark 9.10. Given a linear form $L \in S$, a form $H \in S$, a saturated codimension two ideal $(\beta_1, \ldots \beta_r) \subset S/L$, a form $F \in (\beta_1, \ldots, \beta_r)$ and $\alpha \in$

$Hom((\beta_1, \ldots, \beta_r), S/(L, F))$ of degee $\deg(H) + \deg F - 1$ such that the codimension of

$$\Big(L, \beta_1, \beta_2, \ldots, \beta_r, \Big\{\frac{\alpha(\beta_j)\beta_i - \alpha(\beta_i)\beta_j}{F}\Big\}_{i \neq j}\Big)$$

equals four, the ideal

$$I = (L^2, LF, \{\alpha(\beta_i)L + \beta_i HF\}_{i=1}^r)$$

is the ideal of a LCM curve in the double plane defined by L^2 for any lift of the elements $\alpha(\beta_i)$. All LCM curves in a double plane are of this form.

Hartshorne and Schlesinger show that

$$\Big(L, \beta_1, \beta_2, \ldots, \beta_r, \Big\{\frac{\alpha(\beta_j)\beta_i - \alpha(\beta_i)\beta_j}{F}\Big\}_{i \neq j}\Big)$$

having codimension four is equivalent to $(\beta_1, \ldots, \beta_r)$ being a locally complete intersection and α being general.

We now move to a discussion of the Rao modules and triangle diagrams of curves in a double plane.

Proposition 9.11 ([29])**.** *Let C be a curve of degree d in a double plane. Then $M_C^* \cong M_C(d-2)$, and in particular, $h^1(\mathcal{I}_C(n)) = h^1(\mathcal{I}_C(d-2-n))$ for all integers n.*

Proposition 9.12. *Let C be as in Construction 9.1. Let d denote the degree of C. Then the nonzero entries occuring along the row of $\Delta_0(C)$ corresponding to degree i are the same as the nonzero entries occuring along the row of $\Delta_1^L(C)$ corresponding to degree $d-i$.*

Proof. Let a be greater than or equal to the smallest power of y occuring in $\mathrm{gin}(I_C)$. Let $C \overset{X}{\sim} D$, where the initial ideal of X is given by (x^2, y^a). Then $M_D \cong M_C^*(2-a) \cong M_C(d-a)$.

By Proposition 4.12, the nonzero entries occuring along the row of $\Delta_0(C)$ corresponding to degree i are the same as the nonzero entries occuring along the row of $\Delta_1^L(D)$ corresponding to degree $a-i$.

Since $M_D \cong M_C(d-a)$, the nonzero entries occuring along the row of $\Delta_1^L(D)$ corresponding to degree i are the same as the nonzero entries occuring along the row of $\Delta_1^L(C)$ corresponding to degree $d-a+i$.

We conclude the result. $\square$

CHAPTER 10

QUESTIONS FOR FURTHER STUDY

Question 10.1. If C is a LCM space curve that is not extremal, subextremal, or ACM, then is

$$\operatorname{reg}(I_C) < \binom{d-2}{2} + 2 - g \ ?$$

As we have seen, $h^1(\mathcal{I}_C(n))$ is bounded away from $h^1(\mathcal{I}_D(n))$ when C and D are LCM space curves of the same degree and genus and D is extremal but C is not. The same is not true if D is subextremal. This question asks if we can identify subextremal curves solely by their degree, genus, and regularity.

Question 10.2. If C is a LCM space curve that is not minimal in its biliaison class, does it always admit a descending bilink? That is, is C directly bilinked to a curve D with $M_C \cong M_D(h)$ for some $h > 0$?

We know that there is a series of bilinks so that C is linked to a such a curve D, but we don't know if it is the case that we don't have to make ascending bilinks before descending ones. The Lazarsfeld-Rao Property says that we can deform and then perform a descending bilink, but not that we can necessarily perform a descending bilink on any non-minimal curve.

Migliore and Nagel address a variant of this question in [37].

Question 10.3. Given the Liebling triangle diagram for a LCM space curve C, can we determine if C is minimal in its biliaison class?

As we have seen in Proposition 4.38, Liebling gives a sufficient but not necessary condition for a LCM space curve to admit a descending bilink. Intuitively, the

Liebling upper triangle diagrams of non-minimal LCM curves should have lots of zeros because any minimal curve can be linked up to a curve of arbitrarily high degree. Such links preserve the number of non-zero entries in the Liebling upper triangle diagram.

Question 10.4. If C is a LCM space curve where $\Delta_0(C)$ has only one nonzero entry, is it always the case that C is in the biliaison class of an extremal curve? If not, what are the restrictions on $\Delta_1^L(C)$?

This would complete the classification of Chapter 7.

Question 10.5. For $H_{d,g}$ can we identify (restrictions on) the generic inital ideal of the general member of each component?

Specializations of the general member on each component of $H_{d,g}$ can be tricky to analyze and may result in a very large class of monomial ideals that can occur as the generic initial ideal for an LCM space curve. It is possible that the restrictions on the generic initial ideals of the general members of each component would reveal more geometric information.

Question 10.6. Does there exist a non-extremal, non-ACM, LCM space curve C of degree at least five where $\Delta^L(C)$ is a hook diagram? If so, can we classify all such curves?

The bound proved in Theorem 8.11 is not tight. It is not difficult to see that the only LCM space curves of degree five and genus negative one with hook diagrams are extremal by using Lemma 8.9, the Restriction Theorem, and spectrum considerations. In fact, we have not yet found a non-extremal non-ACM LCM space curve of degree at least five with a hook diagram as its Liebling triangle diagram.

Question 10.7. What can we say about lexicographic generic initial ideals of LCM space curves? For a LCM space curve C, what restrictions can we place on its lexicographic generic initial ideal given $\mathrm{gin}(I_C)$ and vice versa?

Degree reverse lexicographic generic initial ideals are very useful with regard to hyperplane sections. Lexicographic generic initial ideals are very interesting algebraically and useful with regard to projection. It would be illuminating to see how the two interact, and in particular, how the geometry informs the algebra in the lexicographic case.

Question 10.8. Can we determine another class of minimal curves that occurs for many degrees and genera? More generally, can we determine components of $H_{d,g}$ with 'nice' computational properties that occur for many degree and genera?

The most prevelant, and fruitful, strategy for analyzing the connectedness of $H_{d,g}$ has been to try to deform LCM curves to extremal curves. Finding other classes of minimal curves or computationally 'nice' components of $H_{d,g}$ that occur for many degrees and genera would promote the use of similar techniques across more components of the Hilbert scheme.

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