BU-957-M

Distribution and Estimation Theory for a Single Server Queue With Random Arrivals and Complete Balking

G. Rubin
NY State Agricultural Experiment Station and Cornell University and
D.S. Robson
Cornell University

ABSTRACT

Investigation of a queueing system, which consists of Poisson input of customers, some of whom are lost to balking, and a single server working a shift of fixed length L and providing a service whose duration can vary from customer to customer but is always strictly less than L, was motivated by the behavior of fishermen encountered in the Great Lakes Creel Survey. In this study, an interviewer, working a shift of length L, is stationed at a marina and asks fishermen returning with their catch a fixed set of questions, requiring service time, w. Balking arises since a fisherman will leave immediately if the interviewer is occupied. If a service is in progress at the end of a shift, the server works overtime to complete the service. For the New York Creel Survey, the goal is to estimate the unknown number of fishermen returning to the marina during the shift, based on the known number of interviews. Once the number of arrivals is estimated, the number of fish caught can be estimated for the balkers, thereby improving the estimates of the total catch for each fish species. In this paper, we derive point and interval estimators of the number of arrivals (n) and the customer arrival rate (λ) for the queueing system with variable service time, as well as some specialized estimators for the case of all service times equal.

All estimators of the underlying parameters are based on observing a single realization of the queueing process during [0, L]. The target of inference and the associated array of relevant distributions may differ with the type of data that have been collected.

Point and interval estimators of the unknown number of arrivals (n) or the unknown rate (λ) of the Poisson arrival process can be derived from the conditional and unconditional distributions, respectively, of total idle time (T) or the number of services (X). Confidence limits are derived by applying the "statistical method" to the cdfs, while maximum likelihood estimators of n and λ are derived using the pdfs of T or X. For the case of equal service time, w, point estimators of n, based on T or X,

can be constructed to be unbiased over the restricted range of $n < 1/w$. Point and interval estimators of n can be derived using the conditional cdf of the number of balkers (Z) for the case of equal service time, as well. Estimators of n and λ could be derived from the marginal distributions of overtime, although only observing overtime seems unlikely for most applications.

In addition, the conditional distributions can be used to estimate shift length (L) or common service time (w) when n and X or n and T are observed. Both maximum likelihood estimators and confidence limits can be derived.

Distribution and Estimation Theory for a Single Server Queue With Random Arrivals and Complete Balking

1. Introduction

The queueing system to be considered consists of Poisson input of customers, some of whom are lost to balking, and a single server working a shift of fixed length L and providing a service whose duration, w_i, can vary from customer to customer but is always strictly less than L. Investigation of this queueing system was motivated by the behavior of fishermen encountered in the New York Great Lakes Creel Survey (Robson and Jones, 1988). In this study, an interviewer, working a shift of length L, is stationed at a marina or boat launch and asks fishermen returning with their catch a fixed set of questions, requiring service time, w. Balking arises since a fisherman will leave immediately if the interviewer is occupied. Although the model described above allows for variable service time, the length of the interview is virtually constant for all fishermen. If a service is in progress at the end of a shift, the server works overtime to complete the service. Consequently, no queue accumulates: the customer's waiting time is always zero and the customer's time in the system is either 0 or w_i. For the Creel Survey, the goal is to estimate the unknown number of fishermen returning to the marina during the shift, based on the known number of interviews. Once the number of arrivals is estimated, the number of fish caught can be estimated for the balkers, thereby improving the estimates of the total catch for each fish species.

The service times are assumed to be independent, identically distributed, strictly positive random variables. However, for purposes of estimation, it is useful to realize that the service time distribution also can be viewed in terms of predetermined sequences of w_i's that are known at least through the final service of the shift. The latter perspective allows one to estimate n or λ when only the sequence of service times through the final service has been recorded for a shift. The special case of constant service time, w, for all customers will be discussed where this simpler case affords results unavailable in general.

The distributional properties of the number of services performed (X), overtime (Y) and total server idle time (T) are derived both unconditionally (for Poisson arrivals) and conditionally on the number of arrivals per shift, assuming that arrival times are not recorded in the data. Estimation based on observation of a single shift from the queueing process is considered for several situations in which incomplete data are collected. We consider the distributional results for overtime more appropriate for measuring queueing system behavior rather than for estimating n or λ. Gross and Harris (1985), Prabhu (1980) and Borovkov (1976; 1984) provide background on queues and the distributions they generate. Assefi (1979) and Basawa and Prakasa Rao (1980) provide an overview of estimation and statistical inference for stochastic processes.

For notational convenience, variables, moments, probabilities and distributions, which are conditional on the realized number (n) of arrivals, will be denoted with a lowercase subscript (n), while their unconditional counterparts bear an uppercase subscript (N). We use the term probability density function (pdf) loosely, applying it to mixed distributions as well as to continuous distributions.

2. Joint Distribution of the Number of Services and Overtime

The joint pdf of the number of services and overtime can be used as the basis for deriving all other distributional results. Consequently, we will derive it for the general case of unequal service times. Note that the fixed shift length, L, can be set to unit length, without loss of generality.

Assume that the sequence of service times $w_1, w_2, \ldots$ is predetermined and is known at least through the final service of the shift or that these observed w_i are the given realization of independent, identically distributed (strictly positive) random variables. Let W_i denote the partial sums of this sequence, $W_0 \equiv 0$ and $W_i = W_{i-1} + w_i$. Let Z_i denote the number of balkers arriving during the i^{th} service and let η_i denote the partial sums, $\eta_0 \equiv 0$ and $\eta_i = \eta_{i-1} + Z_i$. If N customers arrive during (0, L) and exactly X services are performed then $\eta_X = N\text{-}X$.

Let I_j denote the duration of the server's idle period immediately preceding the start of the j^{th} service. Thus, I_1 is the arrival time of the first of the N customers and for $j>1$, I_j is the length of time between completion of the $(j\text{-}1)^{th}$ service and the arrival of the next customer. We note that the Poisson input assumption implies that $I_1, I_2, \ldots$ is a sequence of independent, identically distributed exponential random variables.

Let T denote the total idle time during the fixed interval (0, L) when the server is present, and define

$$T = L - W_X + Y \tag{1}$$

where Y then denotes the amount of overtime, $0 \leq Y < w_X$. Note that when X, or here, equivalently, when W_X is fixed then Y is equivalent to T through the definition (1); i.e., the joint distribution of X and Y is equivalent to the joint distribution of X and T when the service time sequence $w_1, w_2, \ldots$ is specified.

This joint distribution is obtainable by formally invoking the Poisson assumption to give, for $y>0$,

$$\begin{aligned}
&P\,(X=x, Y>y)\\
&\quad = P\,(I_1+w_1+I_2+w_2+\ldots+I_x \leq L < L+y < I_1+w_1+I_2+w_2+\ldots+I_x+w_x)\\
&\quad = P\,(I_1+I_2+\ldots+I_x \leq L-W_{x-1} < L-W_{x-1}+y < I_1+I_2+\ldots+I_x+w_x)\\
&\quad = P\,(L-W_x+y < I_1+I_2+\ldots+I_x \leq L-W_{x-1})\\
&\quad = \int_{L-W_x+y}^{L-W_{x-1}} g_x\,(t)\,dt\,.
\end{aligned}$$

Similarly, we formally obtain

$$\begin{aligned}
P\,(X=x,\ Y=0) &= P(I_1+I_2+\ldots+I_x \leq L-W_x < I_1+I_2+\ldots+I_{x+1})\\
&= G_x\,(L-W_x) - G_{x+1}\,(L-W_x)\\
&= \exp\left(-\lambda(L-W_x)\right)\left(\lambda(L-W_x)\right)^x/\,x!\,.
\end{aligned}$$

Equivalently,

$$P\,(X=x, T=L-W_x) = \exp\left(-\lambda(L-W_x)\right)\left(\lambda(L-W_x)\right)^x/\,x!$$

and for $L-W_x < t \leq L-W_{x-1}$

$$P\,(X=x, T>t) = \int_t^{L-W_x} g_x\,(u)\,du$$

or, differentiating,

$$P\,(X=x,\ T=L-W_x) = \exp\left(-\lambda(L-W_x)\right)\left(\lambda(L-W_x)\right)^x/\,x\,! = \left(1/\lambda\right)g_{x+1}\,(L-W_x)$$

$$P\,(X=x,\ T\in dt) = g_x\,(t)\,dt \qquad \text{for } L-W_x < t \leq L-W_{x-1}$$

and

$$P\,(X=x,\ Y=0) = \left(1/\lambda\right)g_{x+1}\,(L-W_x)$$

$$P\,(X=x,\ Y\in dy) = g_x\,(L-W_x+y)\,dy \qquad \text{for } L-W_x < y \leq w_x.$$

Thus, we have shown that the unconditional joint distribution of the number of services and overtime is given by:

Theorem 1: The unconditional joint density function of the number of services, X_N , and overtime, Y_N , when the customer arrivals form a Poisson process with rate parameter $\lambda>0$, is:

$$f_{X_N, Y_N}(x, y; \lambda, L) = \begin{cases} \exp(-\lambda(L-W_x))\left(\lambda(L-W_x)\right)^x / x! & \text{for } y=0,\ W_x < L \\ \lambda \exp(-\lambda(L-W_x+y))\left(\lambda(L-W_x+y)\right)^{x-1} / (x-1)! & \text{for } 0<y\leq w_x,\ W_x < L+y \\ 0 & \text{otherwise.} \end{cases}$$

Conditioning on X and T, the frequency distribution of the number of balkers arriving during the fixed interval (0, L), $\eta_X = N - X$, is the Poisson distribution of the number of arrivals during a period of total duration L-T. Hence,

$$P(N=n,\ X=x,\ T=L-W_X)$$
$$= \exp(-\lambda(L-W_x))\left(\lambda(L-W_x)\right)^x \exp(-\lambda W_x)\left(\lambda W_x\right)^{n-x} / (x!\,(n-x)\,!)$$

and for $L - W_X < t \leq L - W_{X-1}$

$$P(N=n,\ X=x,\ T\in dt)$$
$$= \lambda^x t^{x-1} \exp(-\lambda t)\left(\lambda(L-t)\right)^{n-x} \exp(-\lambda(L-t)) / ((x-1)!\,(n-x)!)\,.$$

Since N is Poisson distributed with parameter λL, we obtain the conditional distributions of (X_n, Y_n) or (X_n, T_n) as

Theorem 2: The joint density function of the number of services, X_n, and overtime, Y_n, conditional on N = n arrivals, is:

$$f_{X_n, Y_n}(x, y; n, L) = \begin{cases} \binom{n}{x}(L-W_x)^x W_x^{n-x}/L^n & \text{for } W_x < W_n < L,\ y = 0 \\ x\binom{n}{x}(L-W_x+y)^{x-1}(W_x-y)^{n-x}/L^n & \text{for } W_x < L+y,\ 0 < y < w_x \\ 0 & \text{otherwise.} \end{cases}$$

Exactly the same distributions for (X, Y) or (X, T) would be obtained under an alternative scenario in which service is instantaneous but the server's work shift is foreshortened by an amount w_i with the arrival of the i^{th} customer, subject only to the overtime constraint; i.e., the server gives up an amount of time off equal to y. We refer to this alternative scenario as the "instant service mode" in contrast to the earlier "real time mode". Probability calculations are sometimes more easily seen in one mode than the other (see Appendix B).

3. Marginal Distributions of Total Idle Time, Number of Services and Overtime

The unconditional marginal distributions of T, X, and Y can be derived as corollaries to Theorem 1, while their conditional counterparts can be derived as corollaries to Theorem 2. The marginal results for a given variable are derived by integrating over or summing out the other variable in the bivariate distributions.

3.1 Total idle time (T)

Notice that when the sequence of service times is known through the final service, T uniquely specifies X. Consequently, the pdf of idle time is given by:

Corollary 1.1: The pdf of total server idle time (T_N), when customer arrivals form a Poisson process with rate $\lambda>0$ and for a given sequence of service times $w_1, w_2, \ldots, w_x$, is:

$$f_{T_N}(t;\ \lambda, L) = \begin{cases} (\lambda t)^x \exp(-\lambda t)/x! & \text{for } t = \max(0, L\text{-}W_x) \\ \lambda(\lambda t)^{x-1}\exp(-\lambda t)/(x-1)! & \text{for } \max(0, L\text{-}W_x < t < L\text{-}W_x+w_x) \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \leq t < L$, $0 < w_x < L$ and $\lambda > 0$.

The cdf and upper tail probability follow directly from the pdf. The cdf has jump discontinuities corresponding to increases in the number of services.

Rubin (1987) discusses a generalized unconditional cdf of idle time for equal service times, which covers the scenario of a time homogeneous Poisson arrival process with parameter λ, but λ itself is the realization of a random variable with cdf G. Consequently, each of multiple observations of the process could have a different rate parameter for the Poisson arrival process, and with idle time observed for each replicate, the mixing distribution G becomes the estimation target.

The marginal distribution of total server idle time, conditional on the number of arrivals (n), can be derived from Theorem 2.

Corollary 2.1: The pdf of total server idle time (T_n), conditional on the number of arrivals (n) during a shift of length L, is given by:

$$f_{T_n}(t;\ n, L) = \begin{cases} \binom{n}{x^*} t^{x^*} (L-t)^{n-x^*} / L^n & \text{for } t = \max(0, L-W_x) \\ x^* \binom{n}{x^*} t^{x^*-1} (L-t)^{n-x^*} / L^n & \text{for } \max(0, L-W_x) < t < L-W_{x-1} \\ 0 & \text{otherwise,} \end{cases}$$

where $\max(0, L-W_n) \le t \le L$, $0 < w_x < L$ and

$$x^* = \begin{cases} x+1 & \text{for } L-t = W_x \\ x & \text{for } W_{x-1} < L-t < W_x . \end{cases}$$

The conditional cdf and conditional upper tail probability follow directly from the pdf; they bear superficial resemblance to an incomplete beta function, but the arguments depend on t. For $0 < t < L$, the cdf is strictly increasing unless $0 \le t < L-W_n$, when $F_{T_n}(t) = 0$, and the cdf jumps at $t = 1- W_j$, for $j = 1, \ldots, n$. The cdf approaches the limit one as t approaches one, but it is actually undefined at $t = L$.

Figure 1 illustrates the conditional cdf, F_{T_n}, for equal service times with $w = 0.1$ and $n = 20$, as well as the unconditional cdf, F_{T_N}, with $w = 0.1$ and Poisson rate parameter, $\lambda = 20$. The conditional and unconditional cdfs of idle time can be used to construct interval estimators for n and λ, respectively, while the pdfs of idle time allow for maximum likelihood estimation of n and λ.

The unconditional and conditional expected values of total server idle time do not have a compact form. For the case of equal service time, they are most easily calculated using the linearity property of expectations and substituting the appropriate expectations of X and Y.

3.2 Number of services (X)

The marginal distributions of the number of services (X) become relevant when the amount of overtime (Y) is unobservable. The unconditional marginal distribution of X_N can be written most succinctly as the upper tail probability, which is the following cumulative chi square probability.

Corollary 1.2: The unconditional upper tail probability of the number of services, X_N, is:

$$P\left(X_N \geq x; \lambda, L\right) = \int_0^{\lambda(L-W_{x-1})} q^{x-1} \exp(-q)\, dq \,/\, \Gamma(x)$$

$$= P\left(\chi^2_{2x} < 2\lambda\left(L-W_{x-1}\right)\right)$$

for $W_{x-1} < L$ and $\lambda > 0$.

Note that $P\ (X_N \geq 0;\ \lambda, L) = 1$. The expected value for the unconditional distribution is:

$$E\left(X_N\right) = \sum_{x=0}^{M} \int_0^{(L-W_{x-1})} \lambda^x q^{x-1} \exp\left(-\lambda q\right) dq \,/\, \Gamma(x)$$

for $0 \leq W_{x-1} < L$ and M: $W_{M-1} \leq L < W_M$.

For equal service time, the unconditional expected number of services can be approximated as:

$$E\left(X_N\right) \cong \lambda / \left(1+\lambda w\right) + 0.5\left(\lambda w / \left(1+\lambda w\right)\right)^2, \qquad (2)$$

for fixed λw and large λ. This approximation results from an argument of approximate stationarity of the queueing process for large λ and small w.

An intuitive rationale for (2) is that the average number of services should be (average number of arrivals) / (average number of arrivals per service), where the denominator is $1+\lambda w$, the served customer plus the expected number of balkers arriving during the ensuing period. If overtime occurs, then arrivals during the overtime period should be included in the numerator, since the denominator, $1+\lambda w$, treats all service periods as complete. Thus, the numerator is the average number of arrivals in an interval $(0, 1+Y)$ of random length $1+Y$. Conditional on Y_N, the expected number of arrivals is $\lambda(1+Y_N)$. Therefore, the ratio above becomes:

$$\lambda\,(1+E\,(Y_N))\,/\,(1+\lambda w). \tag{3}$$

As will be shown in Section 3.4, the unconditional expected value of overtime can be approximated by:

$$E\,(Y_N) \cong \lambda w^2 / \bigl(2\,(1+\lambda w)\bigr). \tag{4}$$

Substituting (4) into (3) gives (2).

Cox and Smith (1961, pp. 49, 65-68) note that stationary processes often provide good approximations to similar processes lacking stationarity. The approximation given in (2), like those which will be presented for overtime, can be shown numerically to behave very well for small to moderate w and small to large λ (see Table 1).

As for the unconditional marginal distribution, the marginal distribution of the number of services (X_n), conditional on the number of arrivals (n), can be written most succintly as the upper tail probability. Notice that the conditional upper tail probability corresponds to a cumulative beta probability.

<u>Corollary 2.2</u>: The upper tail probability of the number of services, X_n, conditional on the number of arrivals, is:

$$P(X_n \geq x) = x\binom{n}{x}\int_0^{1-(W_{x-1}/L)} u^{x-1}(1-u)^{n-x}\,du \qquad \text{for } W_{x-1} < L$$

$$= \begin{cases} \sum_{r=0}^{n-x}\binom{n}{r} W_{x-1}^{r}(L-W_{x-1})^{n-r}/L^{n} & \text{for } 0 \leq W_{x-1} \leq L \\ 0 & \text{otherwise.} \end{cases}$$

Recalling the connection between an incomplete beta integral and a truncated binomial sum provides the sum given above (Johnson and Kotz, 1969; 1970). The expected value for the conditional distribution is:

$$E(X_n) = \sum_{x=1}^{n}\sum_{r=0}^{n-x}\binom{n}{r} W_{x-1}^{r}(L-W_{x-1})^{n-r}/L^{n} \qquad \text{for } 0 \leq W_{x-1} < L\,.$$

For the special case of equal service time for all customers, approximate stationarity of the process occurs for large arrival rate (λ) and short service times (w), suggesting the approximation:

$$E(X_n) \cong n/(1+(n-1)(w/L)),$$

which is exact for $w = 0$, $w = L/(n-1)$ and $w = L$. Notice that the denominator of the approximation is the expected number of clients that balk or are served per service time w, while the numerator is the total number of arrivals in the shift. The approximation behaves well for a variety of n, L, w configurations (see Figure 2).

Moreover, for the case of equal service time, the conditional cdf of the number of services (X_n) and the conditional upper tail probability of the number of balkers ($n - X_n \equiv Z_n$) are equivalent (Rubin, 1987). This allows one to derive estimators of n when only the number of balkers (Z_n) is observed.

3.3 Overtime (Y)

The marginal distributions of overtime and approximations to relevant quantities are given in this section. The approximations for the probability of no overtime and for the mean and variance of overtime are useful in measuring the behavior of queueing systems. Simplified conditional results exist for the case of equal service times (w) with $n < [L/w]$, where $[\cdot]$ designates the integer part of the argument; they are given in Section 3.4.

The unconditional distribution of overtime, Y_N, is derived directly from the unconditional joint density function of the number of services and overtime given in Theorem 1.

Corollary 1.3: The unconditional pdf of overtime, Y_N, when customer arrivals form a Poisson process with rate $\lambda>0$, is:

$$f_{Y_N}(y;\lambda,L)=\begin{cases}\sum_{x=0}^{M}\exp\left(-\lambda(L-W_x)\right)\left(\lambda(L-W_x)\right)^{x}/\,x! & \text{for } y=0,\ W_x<L\\ \sum_{x=1}^{M}\lambda\exp\left(-\lambda(L-W_x+y)\right)\left(\lambda(L-W_x+y)\right)^{x-1}/(x-1)! & \text{for } 0<y\le w_x,\ W_x<L+y\\ 0 & \text{otherwise,}\end{cases}$$

where M: $W_{M-1} \le L < W_M$.

The exact unconditional mean and variance of overtime are quite messy, even for equal service times (Rubin, 1987). However, in the case of equal service time, many quantities calculated for the unconditional distribution of overtime can be approximated making use of the approximate stationarity of the process. The following approximations for the unconditional distribution of overtime have been shown empirically to be well-behaved for fixed λw and large λ:

$$P\left(Y_N = 0; \lambda, w\right) \cong 1 / \left(1+\lambda w\right) \tag{5}$$

$$f_{Y_N}\left(y; \lambda, w\right) \cong \lambda w / \left(1+\lambda w\right) \tag{6}$$

$$F_{Y_N}\left(y; \lambda, w\right) \cong \left(1+\lambda y\right) / \left(1+\lambda w\right) \tag{7}$$

$$E\left(Y_N\right) \cong \lambda w^2 / \left(2\left(1+\lambda w\right)\right) \tag{8}$$

$$Var\left(Y_N\right) \cong \lambda w^3 \left(4+\lambda w\right) / \left(12\left(1+\lambda w\right)^2\right) \tag{9}$$

The rationale for these approximations is as follows. Recall that a shift consists of pairs of busy and idle times for the server, and the occurrence of overtime depends on the random positioning of a shift termination point within the busy/idle pair. Thus, the probability of positive overtime is equivalent to the probability of the shift termination point occurring during a busy period (the constant service time, w). Applying the renewal theorem (Taylor and Karlin, 1984) to the limiting case of the queueing process gives:

$$\begin{aligned} P\left(\text{positive overtime}\right) &\cong \frac{E\left(\text{busy period}\right)}{E\left(\text{busy period}\right) + E\left(\text{idle period}\right)} \\ &= w / \left(w+\lambda^{-1}\right) = \lambda w / \left(1+\lambda w\right), \end{aligned}$$

which equals (6). Consequently, the probability of no overtime, for the limiting case, is:

$$\begin{aligned} P\left(Y_N = 0; \lambda, w\right) &= 1 - P\left(\text{positive overtime}\right) \\ &\cong 1 / \left(1+\lambda w\right), \end{aligned}$$

which equals (5).

Notice that the random positioning of a termination point implies that if the point <u>does</u> fall in a busy period, then overtime is uniformly distributed on the interval (0, w).

This observation, together with (5) and (6) give the cdf of overtime for the stationary process as:

$$F_{Y_N}(y;\lambda,w) \cong 1/(1+\lambda w) + (\lambda w/(1+\lambda w))\int_0^y (1/w)\,dt$$
$$= (1+\lambda y)/(1+\lambda w),$$

which equals (7). Similarly, the expected value of overtime for the stationary process is:

$$E(Y_N) \cong (1+\lambda w)^{-1}(0) + (\lambda w/(1+\lambda w))\,w^{-1}\int_0^w y\,dy,$$

while the second moment of Y_N is approximately:

$$\lambda w^3/(3(1+\lambda w))$$

Consequently, the variance of overtime can be approximated by (9).

Numerical results for $P(Y_N = 0)$, $E(Y_N)$ and $Var(Y_N)$ indicate that the approximations are good to at least eleven decimal places for $w < 0.2$ and moderate to large λ ($\lambda>6$) and are virtually exact for $0.001 < w < 0.1$ and $\lambda>1$. The approximations are good to at least six decimal places for $0.2 < w < 0.4$ and moderate to large λ ($\lambda>10$).

The conditional marginal distribution follows directly from the conditional joint distribution of the number of services and overtime given in Theorem 2.

Corollary 2.3: The pdf of overtime, Y_n, conditional on the number of arrivals, is:

$$f_{Y_n}(y;n,L) = \begin{cases} \displaystyle\sum_{x=1}^{\min(n,M)} \binom{n}{x}(L-W_x)^x\, W_x^{n-x}/L^n & \text{for } W_x < W_n < L,\ y=0 \\ \displaystyle\sum_{x=1}^{\min(n,M)} x\binom{n}{x}(L-W_x+y)^{x-1}(W_x-y)^{n-x}/L^n & \text{for } W_x < L+y,\ 0 < y \le w_x \\ 0 & \text{otherwise,} \end{cases}$$

for M: $W_{M-1} \le L < W_M$.

The conditional cdf and conditional expectation can be found by integration, but the latter does not have a compact form.

3.4 Conditional distribution of overtime for the case of n < [L/ w]

The conditional distribution of overtime simplifies greatly for the special case of equal service time for all customers and the number of arrivals less than [L/w], the maximum number of services possible in a shift of length L. Recursion formulae exist for the $P(Y_n = 0)$ and the $P(Y_n > y)$ in this special case. These are helpful in performing numerical and algebraic calculations. For typographic simplicity, we let the shift length equal unity ($L \equiv 1$) in this section.

Corollary 2.4: The probability of no overtime, conditional on the number of arrivals less than [1/w], is:

$$P(Y_n = 0) = 1 - nw\, P(Y_{n-1} = 0) \qquad \text{for } n < [1/w].$$

The recursion relation is proved by showing that

$$P(Y_n = 0) + nw\,P(Y_{n-1} = 0) = 1.$$

This requires several binomial expansions in conjunction with recombining of terms, which allows one to rewrite the resulting expression as a polynomial in w. Details of the proof are given in Appendix A.

<u>Corollary 2.5</u>:
$$P(Y_n = 0) = \sum_{i=1}^{n} (n)_i (-w)^i,$$
where $(n)_i = \prod_{k=1}^{i} (n-k+1)$ for $n < [1/w]$.

<u>Proof</u>: From Corollary 2.4 we know that

$$P(Y_n = 0) = 1 - nw \sum_{x=0}^{n-1} \binom{n-1}{x} (1-xw)^x (xw)^{n-x}.$$

Expanding the finite sum and rearranging terms give the above formula. □

<u>Corollary 2.6</u>: The probability that overtime exceeds y, conditional on the number of arrivals less than [1/w], is:

$$P(Y_n > y) = n\,(w - y)\,P(Y_{n-1} = 0)$$

for $n < [1/w]$ and $0 < y < w < 1$.

A sketch of the proof is given in Appendix B.

For $n < [1/w]$, $P(Y_n = 0)$ can be approximated by

$$1 / (1 + (n + 0.5)w).$$

The core of the approximation, $1/(1+ nw)$, results from the approximate stationarity argument for the queueing process. Empirical results indicate that the addition of

1/(2w) in the denominator improves the approximation. An approximation based on the same reasoning is available for $P(Y_n > y)$ with $n < [1/w]$:

$$P(Y_N > y) \cong n(w-y)/(1+(n-0.5)w).$$

As a consequence of Corollaries 2.4 and 2.6 we note that:

Corollary 2.7: The distribution of overtime, conditional on the number of arrivals less than [1/w] and overtime positive, is uniform on the interval (0, w).

Proof: Because we have Poisson arrival of customers, the arrival times are uniformly distributed on the interval (0, 1) for shift length equal to one. Positive overtime is generated by the arrival time of the last served customer occurring after 1-w but before 1. By conditioning on positive overtime, we are rescaling an arrival time distribution that is uniform on (0, 1) to be uniform on a shorter interval, (0, w). □

The expected value and variance for this overtime distribution are w /2 and $w^2/12$, respectively. Applying the results of Corollaries 2.4 and 2.6 gives $P(Y_n > y | Y_n > 0)$ equal to (w-y) / w. Thus, for $n < [1/w]$

$$F_{Y_n}(y \mid Y_n > 0) = y / w.$$

4. Role of Distributional Results in Estimation

Point and interval estimators of the unknown number of arrivals (n) or the unknown rate (λ) of the Poisson arrival process can be derived from the conditional and unconditional distributions, respectively, of total idle time (T) or the number of services (X) (see Rubin, 1987; Rubin and Robson, 1988). Confidence limits are derived by applying the "statistical method" to the cdfs (Mood, Graybill and Boes, 1974), while maximum likelihood estimators of n and λ are derived using the pdfs of T or X. For the case of equal service time, point estimators of n, based on T or X, can be constructed to be unbiased over the restricted range of $n < 1/w$. Point and interval estimators of n can be derived using the conditional cdf of the number of balkers (Z) for the case of equal service time, as well. Estimators of n and λ could be derived from the marginal distributions of overtime, although only observing overtime seems unlikely for most applications.

In addition, the conditional distributions can be used to estimate shift length (L) or common service time (w) when n and X or n and T are observed. Both maximum likelihood estimates and confidence limits can be derived.

Notice that the distributional results of Sections 2 and 3 provide the basis for estimation of the underlying parameters from a single realization of the queueing process during [0, L]. The target of inference and which distributions are relevant may differ in accordance with the type of data that have been collected.

5. Estimation when the Number of Services (X) and Overtime (Y) or Total Idle Time (T) are Observed

When the number of services and the amount of overtime required to complete the last service are both observed, the joint density functions of X and Y can be used as the basis for constructing estimators: the joint distribution, conditional on the number of arrivals, yields an estimator for n, while the unconditional joint distribution yields an estimator for λ. Recall that when the sequence of service times is known through the x^{th} service, then $T = L - W_X + y$ uniquely specifies X. In those cases, it will be found preferable to use the conditional and unconditional density functions of total server idle time (T) as a statistical basis for estimation of n and λ.

5.1 Maximum Likelihood Estimator of n

The maximum likelihood estimator (MLE) of the number of arrivals can be constructed either from the conditional density function of T_n or from the conditional joint density function of X_n and Y_n. Notice that one must know the sequence of w_i's through x to use the former, while one need know only the cumulative service time, W_x, to use the latter. In either case, setting the difference between the likelihood functions at n and n - 1 equal to zero yields:

$$\hat{n}_{ml}(T) \equiv x^* / T = X / (L - W_x + Y).$$

The ratio of the likelihoods,

$$f_n(t_x) / f_{n-1}(t_x) = (n/(n-x))(1-t_x) = (n/(n-x))(W_x - y),$$

where $0 < L - t_x = W_x - y < L$, is a strictly decreasing function of n, which passes through unity at a point $n = \hat{n}_{ml}(T)$ that is relatively close to x for small W_x. The ratio of adjacent ratios of the likelihoods is less than unity:

$$(f_{n+1}(t_x) / f_n(t_x)) / (f_n(t_x) / f_{n-1}(t_x)) = 1 - x/(n(n-x+1)) < 1$$

for $0 \le x \le n$, implying that the likelihood function is unimodal with its maximum near $\hat{n}_{ml}(T) = X / (L - W_x + Y)$. Numerical results indicate that the variance estimator:

$$\widehat{Var}(\hat{n}_{ml}(T) \mid N = n) = \hat{n}_{ml}(T)(\hat{n}_{ml}(T) - X) / X$$

(based on second differencing) is improved by replacing n with n+0.5.

5.2 Construction of a Restricted Unbiased Estimator of n

A point estimator, $\tilde{n}(T)$, which is unbiased over the restricted range of n less than

M: $W_{M-1} \le L < W_M$, can be constructed by noting that for such values of n neither the range of X_n nor the range of Y_n is dependent on L, and hence, the first derivative of the conditional log likelihood with respect to L must have zero expectation for all such values of n. This calculation can be implemented on the joint conditional likelihood of (X_n, Y_n), given in Theorem 2, or on the conditional pdf of T_n, given in Corollary 2.1. Expressed in terms of the busy time in [0, L], $S = W_x - Y$, and the number of services during the busy period,

$$x(S) = \begin{cases} x & \text{if } S = W_x \\ x-1 & \text{if } W_{x-1} < S < W_x \end{cases},$$

this gives:

$$E\left(d \ln f_{X_n, Y_n}(x, y) / dL\right) \equiv 0 = E\left(\{x(S)/(L-S)\} - n/L\right)$$

or

$$E\left(L\, x(S)/(L-S)\right) = n,$$

for nonnegative integer values of n less than M: $W_{M-1} \le L < W_M$. In terms of previous notation,

$$(L-S)/L = T/L,$$

which represents the fraction of the shift during which the server is idle. Thus, we have proved:

<u>Theorem 3</u>:

$$\tilde{n}(T) = (x^*-1)/(T/L) = L\, x(S)/(L-S)$$

is unbiased for $n < M$: $W_{M-1} \le L < W_M$, where

$$x^* = \begin{cases} x+1 & \text{for } L-t = W_x \\ x & \text{for } W_{x-1} < L-t < W_x . \end{cases}$$

An anomolous feature of $\tilde{n}(T)$ is that it estimates the number of arrivals to be zero when there is, in fact, a single service which occurs late enough to require overtime. The possible occurrence renders $\tilde{n}(T)$ inadmissible. Moreover, $\tilde{n}(T)$ exceeds M essentially whenever the idle time constitutes less than 50 percent of the shift. These features make $\tilde{n}(T)$ unacceptable for estimation of n.

Note, however, that if n is also an observed datum, then

$$\hat{L}_{ml}(S) = S/(1 - x(S)/n),$$

while if n is not observed and L is unknown then identifiability is lost. Estimation of L is discussed further in Section 8 .

5.3 Interval Estimation of n

The cdf of idle time, conditional on n arrivals, is an increasing function of n and can be used to construct confidence limits for n. A 1-α lower confidence limit for n can be constructed by solving for n in the equation:

$$\alpha = P(T_n \le t; n) = x^* \binom{n}{x^*} \int_0^t u^{x^*-1} (1 - u)^{n-x^*} du, \tag{12}$$

where $0 < t < L$, $0 < w_x / L$ and $x^* = \begin{cases} x+1 & \text{for } L\text{-}t = W_x \\ x & \text{for } W_{x-1} < L\text{-}t < W_x . \end{cases}$

We can transform the incomplete beta probability given in (12) to an F probability, so that lower and upper confidence limits for the number of arrivals (n) can be determined from the F - tables, using the appropriate confidence level. Applying the transformation

$$v = (n-x^*+1)\, t / x^* (L-t)$$

to (12) gives

$$P\left(V_n \le (n-x^*+1)\, t / x^* (L-t)\right) = \alpha,$$

where V_n has an F distribution with parameters $2x^*$ and $2(n - x^* + 1)$. Notice that $t / (L - t)$ represents the estimated odds for service of any given one of the n randomly arriving customers, as does the unobserved ratio $(x^*/n)/\{1 - (x^*-1)/n\}$. Thus, the $1 - \alpha$ lower confidence limit for n can be determined from the equation:

$$F_{2x^*,\,2(n-x^*+1)}(\alpha) = (n-x^*+1)\,t / x^*(L-t), \tag{13}$$

where $F_{a,\,b}$ is the critical value of the central F distribution with a and b degrees of freedom, for all t, $0 \leq t \leq L$.

The $1-\alpha$ upper confidence limit of n can be obtained setting the upper tail probability of T_n equal to α, yielding the equation

$$F_{2x^*,\,2(n-x^*+1)}(1-\alpha) = (n-x^*+1)\,t / x^*(L-t). \tag{14}$$

It can be shown that for the conditional upper tail probability of T_n, $x^* = x+1$ for all t (Rubin, 1987). It is more convenient to apply this definition of x^* to upper and lower confidence limits, alike. Thus, $1- 2\alpha$ confidence limits for n can be constructed by holding the observed odds estimate, $T / (L-T)$, fixed and adjusting the unobserved odds estimate, $(x^*/n) /\{1 - (x^*-1) /n\}$, to achieve odds ratios equal to upper and lower critical values of V_n.

Integer-valued approximate solutions for equation (13) and (14) can be determined using F-tables. Exact solutions, which are noninteger, can be computed using the inverse F function and a derivative free method of estimating nonlinear functions.

5.4 Maximum Likelihood Estimator of λ

The MLE of the rate parameter λ of the Poisson arrival process, $\hat{\lambda}_{ml}(T)$, can be constructed from the unconditional density function of T_N or the unconditional joint density function of X_N and Y_N. Setting the first derivative with respect to λ of the unconditional likelihood function equal to zero yields:

$$\widehat{\lambda}_{ml}(T) = x^* / T = X / (L - W_x + Y).$$

The second derivative with respect to λ of the likelihood function evaluated at $\widehat{\lambda}_{ml}(T)$ is negative, indicating that $\widehat{\lambda}_{ml}(T)$ is a maximum. Notice that the MLE's of n and λ are identical.

For the case of equal service time for all customers and $L \equiv 1$, $\widehat{n}_{ml}(T)$ and $\widehat{\lambda}_{ml}(T)$ both are bounded above by:

$$[1/w] / (1 - w[1/w]) \equiv B(w) > [1/w](1 + [1/w]).$$

Thus, the MLE's must underestimate a parameter that exceeds B(w). However, if X and Y are replaced by their asymptotic or exact expected values in $\widehat{\lambda}_{ml}(T)$, an approximation to the expected value of $\widehat{\lambda}_{ml}(T)$ is obtained:

$$E\left(\widehat{\lambda}_{ml}(T)\right) \cong \left\{\lambda\left(1 + E(Y_N)\right) / \left(1 + \lambda w\right)\right\} / \left\{1 - \left(\lambda w\left(1 + E(Y_N)\right) / \left(1 + \lambda w\right)\right) + E(Y_N)\ \right\} = \lambda\,;$$

the approximation improves in accuracy as B(w) approaches infinity. $\widehat{\lambda}_{ml}(T)$ will underestimate parameter values exceeding B(w). However, B(w) is very large for small w.

The variance of $\widehat{\lambda}_{ml}(T)$ is approximated by the inverse of the information, the expectation of the negative of the second derivative with respect to λ of the log likelihood function:

$$\mathrm{Var}\left(\widehat{\lambda}_{ml}(T)\right) \cong \lambda^2 / E(X_N). \tag{14}$$

An estimator of the variance of $\widehat{\lambda}_{ml}(T)$ can be constructed by substituting $\widehat{\lambda}_{ml}(T)$ and X_N

into the relationship given in (14):

$$\widehat{\mathrm{Var}}\left(\hat{\lambda}_{ml}(T)\right) = \hat{\lambda}_{ml}(T)^2 / x^* = x^* / T^2 . \tag{15}$$

If n were observable, then $\hat{\lambda}_{ml}(T)$ would equal n and have variance λ. In the present circumstance, however,

$$\mathrm{Var}\left(\hat{\lambda}_{ml}(T)\right) = \mathrm{Var}\left(E\left(\hat{n}_{ml}(T) \mid N = n\right)\right) + E\left(\mathrm{Var}\left(\hat{n}_{ml}(T) \mid N = n\right)\right).$$

The term $E\left(\hat{n}_{ml}(T) \mid N = n\right)$ is approximately n, and the variance of N, a Poisson random variable, is λ. Hence, the variance of $\hat{\lambda}_{ml}(T)$ can be approximated as:

$$\mathrm{Var}\left(\hat{\lambda}_{ml}(T)\right) \cong \lambda + E\left(\mathrm{Var}\left(\hat{n}_{ml}(T) \mid N = n\right)\right).$$

Consequently, the estimated variance of $\hat{\lambda}_{ml}(T)$ is:

$$\widehat{\mathrm{Var}}\left(\hat{\lambda}_{ml}(T)\right) = \hat{\lambda}_{ml}(T) + \widehat{\mathrm{Var}}\left(\hat{n}_{ml}(T) \mid N = n\right).$$

Notice that combining (15) with the result above yields

$$\widehat{\mathrm{Var}}\left(\hat{n}_{ml}(T) \mid N = n\right) = x^* / T^2 - x^* / T = x^* (1-T) / T^2 ,$$

which is the same as the estimator derived using the second difference method.

5.5 Interval Estimation for λ

Confidence limits for λ can be constructed using the unconditional distribution of idle time. A $1-\alpha$ upper confidence limit for λ is found by solving for λ in the equation:

$$P\left(T_N \geq t; \lambda\right) = 1 - \int_0^t \lambda^{x^*} \exp(-\lambda u)\, u^{x^*-1}\, du / \Gamma(x^*) = \alpha ,$$

where $x^* = x+1$. One can use the inverse gamma function and solve for λ. Using the unconditional cdf of idle time, a $1-\alpha$ lower confidence limit for λ, is found by solving for

λ in the equation

$$P\left(T_N \leq t; \lambda\right) = \int_0^t \lambda^{x^*} \exp(-\lambda u)\, u^{x^*-1}\, du \,/\, \Gamma(x^*) = \alpha ,$$

where $0 < t < L$ and $x^* = \begin{cases} x+1 & \text{for } L\text{-}t = W_x \\ x & \text{for } W_{x-1} < L\text{-}t < W_x . \end{cases}$

As for confidence limits of n, one can adopt the convention of allowing x^* to equal x+1 for both upper and lower confidence limits of λ. Alternatively, one can use the Poisson form of the cdf or upper tail probability to solve for the lower and upper confidence limits of λ. It should be noted that the $1- 2\alpha$ confidence limits on λ derived here produce an open confidence interval, $\lambda_{lower} < \lambda < \lambda_{upper}$, with coverage probability of at least $1- 2\alpha$.

6. Estimation when Only the Number of Services (X) is Observed

When the number of services is the sole observation and the shift length, L, and the cumulative service time through the x^{th} service, W_x , are known constants, the distribution of X_n, conditional on the number of arrivals, can be used to derive point and interval estimators of n. A maximum likelihood estimator can be derived for the case of unequal service times. For the equal service time case, an approximate method of moments estimator (MOM) has been derived and an estimator constructed to be unbiased for n < L/w can be constructed. Confidence limits for n are based on the conditional distribution of X_n and are derived similarly to those based on the conditional distribution of T_n.

It is possible to use the unconditional distribution of the number of services to derive point and interval estimators of λ, the unknown rate parameter of the Poisson arrival process. Less emphasis has been placed on this, however; if balking is an unobservable feature, it seems more useful to estimate the number of arrivals that occurred rather than the arrival rate. Empirical results confirm that the 1-2α confidence intervals for λ are longer than the corresponding intervals for n.

6.1 Maximum Likelihood Estimator of n

The MLE of n, $\hat{n}_{ml}(X)$, does not have a closed form and is most easily found by calculating

$$P(X_n = x; n, L) = \sum_{r=0}^{n-x} \binom{n}{x} W_{x-1}^{r} (L-W_{x-1})^{n-r} / L^n - \sum_{r=0}^{n-x-1} \binom{n}{x+1} W_x^{r} (L-W_x)^{n-r} / L^n$$

for successive values of n until the probability decreases. The value of n, for which $P(X_n = x; n, L)$ is largest, is $\hat{n}_{ml}(X)$.

The finiteness of $\hat{n}_{ml}(X)$ is guaranteed because it is bounded above and below by the largest and smallest values, respectively, of

$$\hat{n}_{ml}(T) = \hat{n}_{ml}(X, Y) = X / (L-W_x+Y)$$

as a function of Y_n ; thus, the following theorem proves that $P(X_n = x; n, L)$ is maximum for n in the given interval. Using the established fact that $f_n(x, y)/f_{n-1}(x, y)$ is a decreasing function of n, which passes through unity at $n = X / (L-W_x+Y)$ for $0 \leq y < w_x$, we can prove by contradiction that the n-solution to $f_n(x) = f_{n-1}(x)$ belongs to the interval given in the theorem below.

Theorem 4: The integer-valued MLE of n, $\hat{n}_{ml}(X)$, satisfies

$$X / (L-W_{x-1}) \leq \hat{n}_{ml}(X) \leq X / (L-W_x)$$

for $W_x < L$ and $0 \leq y < w_x$.

6.2 Method of Moments Estimator of n for Equal Service Times

As shown in Section 3.2, the conditional mean number of services is well approximated by

$$E(X_n) \cong n/(1+(n-1)(w/L)),$$

for the case of equal service times; in fact, the approximation is exact for w = 0, L or L/(n-1). A point estimator of n based on this approximation can be derived using the method of moments technique:

$$\tilde{n}(X) = X(L-w)/(L-wX),$$

for $0 < w < L$ and $wX < L$. Notice that $\tilde{n}(X)$ is not necessarily integer-valued. It can be shown algebraically that the MOM estimator conforms to the bounds on $\hat{n}_{ml}(X)$ that were given in Theorem 4 (Rubin, 1987).

Table 2 can be used to compare $\hat{n}_{ml}(X)$ and $\tilde{n}(X)$ and their bounds for several values of w.

6.3 Construction of a Restricted Unbiased Estimator of n

When service times are equal, a unique unbiased estimator of n, $\hat{n}_u(X)$, can be constructed for n over the restricted range $0 \leq n < L/w$. The ability to construct such an estimator capitalizes on the recursive nature of the formula for expectations using the conditional density function of X_n when $n \leq L/w$. The existence and uniqueness of $\hat{n}_u(X)$ are proved in the following theorem. The form of the estimator is given as a corollary to the theorem. To simplify notation, the shift length has been set to unity ($L \equiv 1$).

Theorem 5: There exists a unique function of X, say $\hat{n}_u(X)$, which is unbiased for n over the restricted range $0 \le n < 1/w$.

Proof: We require $\hat{n}_u(X)$ to satisfy

$$E\left(\hat{n}_u(X)\right) \equiv E\left(\hat{n}_u(X) | N = n\right) = n;$$

i.e.,

$$n = \sum_{x=0}^{n} \hat{n}_u(x) P\left(X_N = x \mid N = n\right) \text{ for } n = 0, 1, \ldots, [1/w]. \tag{16}$$

At N = 0 we have $P(X_0 = 0 \mid N = 0) = 1$, so (16) becomes

$$0 = \hat{n}_u(0) \times 1 \Rightarrow \hat{n}_u(0) = 0\,.$$

At N = 1 we have $P(X_N \le 1 \mid N = 1) = 1$, so (16) becomes:

$$1 = \sum_{x=0}^{1} \hat{n}_u(x) P\left(X_N = x \mid N = 1\right)$$

$$= \binom{1}{0} 0^0 1^1 \left\{\hat{n}_u(1) - \hat{n}_u(0)\right\} = \hat{n}_u(1)\,.$$

$$\Rightarrow \hat{n}_u(1) = 1\,.$$

At N = 2 we have $P(X_N \le 2 \mid N = 2) = 1$, so (16) becomes:

$$2 = \sum_{x=0}^{2} \hat{n}_u(x) P\left(X_N = x \mid N = 2\right)$$

$$= 1 + \left(1-w^2\right)\left\{\hat{n}_u(2) - 1\right\}.$$

$$\Rightarrow \hat{n}_u(2) = 1 + 1/\left(1-w^2\right) = \sum_{r=0}^{1} 1/(1-rw)^2\,.$$

At N = k we have $P(X_N \le k \mid N = k) = 1$, so recursively solving (16) gives:

$$\hat{n}_u(k) = k - \left\{\sum_{x=0}^{k-1} \hat{n}_u(x) P\left(X_N = x \mid N = k\right)\right\} / P\left(X_N = k \mid N = k\right), \tag{17}$$

where, for $0 < x \le k \le 1/w$, $P(X_N = k \mid N = k) > 0$, ensuring the finiteness of (17).

Therefore, a unique unbiased estimator of n exists for $0 \le n < 1/w$.

Alternatively, one can prove Theorem 5 using a completeness argument. The form of $\hat{n}_u(X)$ is derived as a corollary to Theorem 5; the proof is an induction argument that uses the identity:

$$\sum_{x=0}^{n} \binom{n}{x} (xw)^{n-x}(1-xw)^{x-1} = 1 \qquad \text{for } 0 < w < 1/n.$$

For details, see Rubin (1987).

Corollary 5.1: The unique unbiased estimator of n, for $0 \le n < 1/w$, is

$$\hat{n}_u(X) = \sum_{r=0}^{X-1} 1/(1-rw)^2,$$

where $0 < w < 1$ and $x \ge 1$.

Numerical results indicate that a well-behaved, closed-form approximation to the unbiased estimator is found by integration, i.e.,

$$\hat{n}_u(X) = \sum_{r=0}^{X-1} 1/(1-rw)^2 \cong \int_{-0.5}^{X-0.5} (1-rw)^{-2}\,dr = X/\{(1+0.5\,w)(1-(X-0.5)\,w)\}.$$

The approach taken in Corollary 5.1 can be used to produce an estimator of the $\mathrm{Var}(\hat{n}_u(X))$ which is unbiased for $n < 1/w$. An unbiased estimator of $C_2^n = \binom{n}{2}$ will be constructed; from this we get an unbiased estimator of n^2. It can be shown that the estimated variance of $\hat{n}_u(X)$ is:

$$\mathrm{Var}(\hat{n}_u(X)) = (\hat{n}_u(X))^2 - 2\,\hat{C}_2^n(X) - \hat{n}_u(X),$$

where $\hat{C}_2^n(X)$ is an unbiased estimator of $\binom{n}{2}$ for $n < 1/w$.

The proof of the existence of a unique unbiased estimator of $\binom{n}{2}$ for $n < 1/w$ and the construction of that estimator follow the pattern established for $\hat{n}_u(X)$. The form of $\hat{C}_2^n(X)$ is given by an induction argument, requiring the identity:

$$n\binom{N}{n}\sum_{r=0}^{N-n}\binom{N-n}{r}(-1)^r(r+n+1)^{N-n}/(r+n) = 1 \qquad \text{for } n = 1, 2, \ldots, N,$$

which follows from Rubin (1987).

Corollary 5.2: The unique unbiased estimator of $\binom{n}{2}$, for $n = 2, \ldots, [1/w]$, is

$$\hat{C}_2^n(X) = \sum_{r=1}^{X-1} r(1-w)/(1-rw)^3,$$

where $0 < w < 1$ and $x \geq 2$.

Numerical results indicate that a well-behaved, closed-form approximation to $\hat{C}_2^n(X)$ is found by integration:

$$\hat{C}_2^n(X) \cong \int_{0.5}^{X-0.5} \frac{r(1-w)}{(1-rw)^3}\,dr = \left(\frac{1-w}{2w^2}\right)\left\{\frac{2w(X-0.5)-1}{(1-(X-0.5)w)^2} + \frac{1-w}{(1-0.5w)^2}\right\}.$$

Since $E(\hat{C}_2^n(X)) = n(n-1)/2$ and $E(\hat{n}_u(X)) = n$,

$$n^2 = 2E\left(\hat{C}_2^n(X)\right) + E\left(\hat{n}_u(X)\right).$$

Thus,

$$\widehat{n^2} = 2\hat{C}_2^n(X) + \hat{n}_u(X)$$

and

$$\operatorname{Var}\left(\hat{n}_u(X)\right) = \hat{n}_u(X)\left\{\hat{n}_u(X) - 1\right\} - 2\hat{C}_2^n(X).$$

The estimated variance of $\hat{n}_u(X)$ is guaranteed to be nonnegative for all values of X. Plotting $\hat{n}_u(X)$ versus $P(X_n \geq x) = P(\hat{n}_u(X) \geq \hat{n}_u(x))$ indicates that $\hat{n}_u(X)$ is nearly lognormal.

6.4 Interval Estimation for n

The upper tail probability of the number of services, conditional on the number of arrivals, given in Corollary 2.2, can be used to construct confidence limits for n. Proceeding as in the case of T_n, we transform the incomplete beta probability with parameters x+1 and n-x to an F probability with 2(x+1) and 2(n-x) degrees of freedom. Solving the equation

$$\frac{n-x}{x+1}\left(\frac{L-W_x}{W_x}\right) = F_{2(x+1),\,2(n-x)}(1-\alpha)\,. \tag{17}$$

for n yields the $1-\alpha$ upper confidence limit for n. For a $1-\alpha$ lower confidence limit for n, we solve for n using the equation:

$$\frac{n-x+1}{x}\left(\frac{L-W_{x-1}}{W_{x-1}}\right) = F_{2x,\,2(n-x+1)}(\alpha)\,. \tag{18}$$

Integer-valued solutions to (17) and (18) can be determined using F-tables. An exact solution, which is noninteger, can be computed using the inverse F function and a derivative free method of estimating nonlinear functions. Since $x \le n$, if $n_{lower} < x$, we replace n_{lower} with x. The given algorithm provides a $1-2\alpha$ confidence interval which is open $(\max(x, n_{lower}) < n < n_{upper})$ and has coverage probability of at least $1-2\alpha$.

6.5 Maximum Likelihood Estimator of λ

The unconditional distribution of X_N, given in Corollary 1.2, can be used to derive the MLE of λ, $\hat{\lambda}_{ml}(X)$. Notice that, for $x > 0$, the unconditional density function is the difference between two gamma cdfs with the same shape parameter but different location parameters:

$$\int_0^{L-W_{x-1}} \frac{\lambda^x}{\Gamma(x)} u^{x-1} \exp(-\lambda u)\, du - \int_0^{L-W_x} \frac{\lambda^{x+1}}{\Gamma(x+1)} u^x \exp(-\lambda u)\, du,$$

where $\lambda > 0$ and $W_{x-1} \le W_x < L$. For $x = 0$ and $\lambda > 0$,

$$P\left(X_N = 0; \lambda, L\right) = \exp\left(-\lambda\right),$$

since $P(X_N \ge 0) \equiv 1$ and $P(X_N \ge 1) = 1 - e^{-\lambda}$.

Setting the derivative of the density function with respect to λ equal to zero yields:

$$\frac{(L-W_x + w_x)^x}{\Gamma(x)} \exp\left(-\lambda(L-W_x + w_x)\right) - \frac{(L-W_x)^{x+1}}{\Gamma(x+1)} \lambda^x \exp\left(-\lambda(L-W_x)\right) = 0 .$$

The MLE of λ is the solution to the equation:

$$\ln(x / (L-W_x)) + x \ln(1 + w_x / (L-W_x)) - \ln(\lambda) - \lambda w_x = 0 .$$

Evaluating the second derivative of the likelihood function with respect to λ at the point for which the first derivative is equal to zero yields:

$$-\frac{(L-W_x)^{x+1}}{\Gamma(x+1)} \lambda^x \exp\left(-\lambda(L-W_x)\right)\left(w_x + 1/\lambda\right) < 0 ,$$

which implies that the likelihood is maximum at this point (and the maximum is unique.)

If one wishes to exclude the outcome N = 0, one can use a truncated Poisson distribution ($N \ge 1$) as the basis for deriving a maximum likelihood estimator of λ (Rubin, 1987).The MLE based on the truncated distribution is smaller than $\hat{\lambda}_{ml}(X)$ for all values of $X > 0$. Rubin (1987) also gives confidence limits of λ based on the truncated Poisson distribution.

Table 2 illustrates $\hat{n}_{ml}(X)$ and $\hat{\lambda}_{ml}(X)$ along with upper and lower confidence limits of n and λ, for several values of x with w = 0.1 and L = 1.

6.6 Interval Estimation for λ

The unconditional upper tail probability of X_N (Corollary 1.2) can be used to derive upper and lower confidence bounds for λ. Setting the upper tail probability, written in terms of the incomplete gamma function, equal to α and using the inverse gamma function with parameter x, allows us to solve for the 1- α lower confidence limit of λ. Likewise, the solution to the equation

$$1-\alpha = \int_0^{\lambda(L-W_x)} z^x \exp(-z)\, dz \,/\, \Gamma(x+1)$$

yields a 1 - α upper confidence limit of λ.

Table 2 illustrates the 1- α upper and lower confidence limits of λ, as well as the corresponding limits of n, for several values of x with w = 0.1 and L = 1. As one expects, the confidence intervals for λ are longer than the corresponding intervals for n, since the unconditional distribution incorporates more variability in X than does the conditional distribution.

7. Estimation when the Number of Balkers (Z=n-X) is Observed

Estimation of the number of arrivals or the arrival rate is not possible for the case of unequal service time, when only the number of balkers (Z = n - X) is observed, since the cumulative service time through the x^{th} service, W_x, is unknown.

If the service times are equal for all customers and the shift length is known, one can estimate the number of arrivals, or equivalently, the number of services, when only the number of balkers is observed. Notice that, conditional on the number of arrivals, Z_n is a simple transformation of X_n, and $P(Z_n \leq z) = P(X_n \geq n - z)$. Equating the cdf of

Z_n to α for a fixed z yields a 1- α upper confidence limit for n. As in the case for X_n, transforming the beta probability to an F probability gives an equation in terms of a cutoff point and a critical value of an F distribution, which must be solved iteratively for n to yield the 1- α upper confidence limit for n:

$$\frac{n-z}{z+1}\left(\frac{(n-z-1)\,w}{L-(n-z-1)\,w}\right) = F_{2(z+1),\,2(n-z)}(1-\alpha)\,. \qquad (20)$$

The inference target in this circumstance might be X rather than n, but since X = n - Z and Z is observed, any inference about n carries with it an inference about X. Thus, (20) can be written as:

$$\frac{x}{z+1}\left(\frac{(x-1)\,w}{L-(x-1)\,w}\right) = F_{2(z+1),\,2(x)}(1-\alpha)\,. \qquad (21)$$

A 1- α lower confidence limit of n is found by solving the following equation for n:

$$\frac{z}{(n-z+1)}\left(\frac{L-(n-z)\,w}{(n-z)\,w}\right) = F_{2(n-z+1),\,2z}(\alpha)\,. \qquad (22)$$

Integer-valued solutions to (20) and (22) can be determined from F-tables. An exact solution, which is noninteger, can be computed using the inverse F function and a derivative free method of estimating nonlinear functions.

Setting $\alpha = 0.5$ and solving for n in equation (20) or x in (21) yield median unbiased point estimators of n and X , respectively (Lehmann, 1983). Figure 3 shows the median unbiased estimators of X flanked by their corresponding 90% and 95% upper and lower confidence limits for w = 0.01 and L = 1.

An attempt was made to construct an unbiased estimator of n using the conditional distribution of Z_n. The procedure used was similar to that detailed in Section 6.3. Unfortunately, the estimator is badly-behaved. The unbiased estimator, $\hat{n}_u(Z)$, fluctuates wildly and even takes on values that are outside of the range of n. For example, with w = 0.1, L = 1, n = 3 and Z = 2, $\hat{n}_u(Z) = -92.23$. Therefore, $\hat{n}_u(Z)$ is unacceptable as an estimator and will not be considered further.

Estimation of λ, the rate parameter of the Poisson arrival process, is intractable when only the number of balkers is observed. For instance, the distribution of Z_N, conditional on X_N, is given by:

$$P(Z_N = z \mid X_N = x)$$

$$= \exp(-\lambda L)\{(\lambda L)^z / z!\} \frac{\int_{1-xw/L}^{1-(x-1)w/L} u^{x-1}(1-u)^z \, du + \int_0^{1-xw/L} u^{x-1}(1-u)^{z-1}(1-xu-zu) \, du / x}{\int_{1-xw/L}^{1-(x-1)w/L} u^{x-1} \exp(-\lambda L u) \, du},$$

for $x > 0$, $\lambda > 0$ and $0 < w < L$. In addition to being complicated, this distribution depends on both x and λ. Notice that if both x and z are observed, then n is known and $\hat{\lambda}_{ml} = n$.

8. Estimation of Shift Length (L)

Estimation of shift length, L, is possible for the case of unequal service times when the data observed are (n, X), (n, X, Y), (n, T) or (n, S). Point and interval estimators are derived using distributional results that are conditional on the number of arrivals.

Recall that when the sequence of service times through the x^{th} service is known, the distributions of (X, Y), T and S are equivalent. For L and W_x known, T and S are still equivalent since $S = W_x - Y$ is a known univariate transformation of T that is one to one. It is not possible, however, to resurrect (X, Y) from T when only W_x is known.

8.1 Point Estimators of L Based on (n, T), (n, S) or (n, X, Y)

Differentiating the conditional likelihood of total idle time, T, or server busy time, S, with respect to L yields:

$$\hat{L}_{ml}(S, n) = n\,S/(n - x(S))$$

or

$$\hat{L}_{ml}(T, n) = n\,T/x^*$$

for $0 < S = L\text{-}T < W_n$ and $n \geq 1$ with

$$x(S) = \begin{cases} x & \text{if } S = W_x \\ x-1 & \text{if } W_{x-1} < S < W_x \end{cases}$$

or

$$x^* = \begin{cases} x+1 & \text{for } L\text{-}t = W_x \\ x & \text{for } W_{x-1} < L\text{-}t < W_x\,. \end{cases}$$

From the joint likelihood of X_n and Y_n, conditional on n arrivals, we find

$$\hat{L}_{ml}(X, Y, n) = \begin{cases} W_x/(1 - X/n) & \text{for } Y = 0 \\ (W_x - Y)/(1 - (X-1)/n) & \text{for } 0 < Y < w_x \end{cases}$$

for $X > 0$ and $n > 0$.

8.2 Confidence Limits for L Based on (n, T) or (n, S)

The construction of $1-\alpha$ confidence limits for L based on (n, T) or (n, S) is similar to that done for n based on observation of T when the sequence of service times through the x^{th} service is known. Equations (13) and (14) of Section 5.3 can be used to calculate upper and lower confidence limits of L, when (L-S)/L is substituted for T/(L-T) and $x^* = x(S)+1$. Consequently, confidence limits of L are obtained by solving for L in the following equation:

$$F_{2x^*,\,2(n-x^*+1)}(\alpha) = (n-x^*+1)\,t/x^*(L-t) = \{(L-S)/\hat{L}_{ml}(S, n)\}/\{x^*/n\}$$

$$\text{for } x^* = \begin{cases} x+1 & \text{for } L\text{-}t = W_x \\ x & \text{for } W_{x-1} < L\text{-}t < W_x . \end{cases}$$

Thus, the 1- α upper confidence limit of L is given by:

$$L_{upper}(S) = S + (x^* / n)\,\widehat{L}_{ml}(S, n)\; F_{2x^*, 2(n-x^*+1)}(1-\alpha).$$

Notice that the upper confidence limit is the sum of the observed busy time, S, and the estimated idle time in a workshift of unknown duration L, $(x^* / n)\,\widehat{L}_{ml}(S, n)$. Note that $\widehat{L}_{ml}(S, n)$ and $L_{upper}(S)$ are infinite for $S = W_n$, when the number of services equals the number of arrivals. The 1- α lower confidence limit of L is given by:

$$L_{lower}(S) = S + (S\, x(S) / (n-x(S)))\; F_{2x(S), 2(n-x(S)+1)}(\alpha),$$

where x(S) is the number of services during the busy period (= x^*-1). $L_{lower}(S)$ is always finite.

9. Estimation of Service Time (w) or Relative Service Time (w/L) when Service Times are Equal

For the case of equal service time, one can estimate the common service time (w) when the data are the number of arrivals, the number of services and the amount of overtime required to complete the last service (n, X, Y), if the shift length is known. Notice that, even for the equal service time case, one cannot resurrect (X, Y) from total idle time (T) or total busy time (S) when w is unknown. When both n and X are observed, one can estimate w if the shift length is known or the relative service time, w/L, if both w and L are unknown.

Interval estimation of w based on the joint cdf of X_n and Y_n, conditional on the number of arrivals, may be possible. It has not been attempted because the conditional joint cdf is cumbersome.

9.1 Maximum Likelihood Estimation of w Based on (n, X, Y)

The MLE of w based on (n, X, Y), $\hat{w}_{ml}(X, Y, n)$, can be derived from the joint density function of X_n and Y_n, conditional on n arrivals, when the shift length is known. Setting the first derivative of the likelihood function with respect to w equal to zero yields:

$$\hat{w}_{ml}(X, Y, n) = \begin{cases} (L/X) - (L/n) & \text{for } Y = 0,\ 0<X\le n \\ (L/X)((n-X)/(n-1) + Y) & \text{for } 0 < Y \le L(n-X)/(n-1)(X-L),\ L<X\le n \\ Y & \text{for } Y > L(n-X)/(n-1)(X-L),\ L<X\le n. \end{cases}$$

The possibility that $\hat{w}_{ml}(X, Y, n) = Y$ occurs only for values of X that are relatively large (i.e., X close to n).

9.2 Maximum Likelihood Estimation of w Based on (n, X)

The conditional distribution of the number of services can be used to derive the MLE of w, $\hat{w}_{ml}(X, n)$. From the upper tail probability of X_n given in Corollary 2.2, notice that the conditional density function is the difference between two beta probabilities and that only the upper limit of integration for each probability depends on w. Therefore, the first derivative of the likelihood function with respect to w is proportional to

$$x^{n-x-1} (L-xw)^x - w (x-1)^{n-x+1} (L-(x-1)w)^{x-1}/(n-x). \qquad (23)$$

Evaluating the second derivative of the likelihood function with respect to w at the point for which the first derivative (or, equivalently, (23)) is equal to zero yields:

$$-x\binom{n}{x} w^{n-x} (x-1)^{n-x+1} (L-(x-1)w)^{x-1}\{L/w(L-(x-1)w) + x^2/(L-xw)\} < 0,$$

indicating that the MLE of w is the unique maximum of the likelihood of X, conditional on n arrivals. Using Newton's Recursion Formula with

$$w_0 = (n-x)/n(x-1)$$

as a starting value, we find that the iteration converges rapidly to $\widehat{w}_{ml}(X, n)$.

The starting value was derived by setting equation (23) equal to zero and algebraically rearranging the equation to yield:

$$w(x-1)/(n-x) = \frac{(1-x^{-1})^{-(n-x)}(1+w/(L-xw))^{-x}}{(x/(L-(x-1)w))}$$

Note that the two terms in the numerator of the right hand side (RHS) of the equation above are approximately equal to one and that the denominator of the RHS is greater than one for x = 1, . . ., n, and is approximately equal to n, for x = n and small w. Thus,

$$1/w \cong n(x-1)/(n-x).$$

9.3 <u>Confidence Limits for w Based on (n, X)</u>

The conditional upper tail probability of the number of services is used to derive confidence limits for w when (n, X) are observed and the shift length is known. Since n and w are inversely related, solving equation (17) for w yields the $1-\alpha$ lower confidence limit, while solving equation (18) for w yields the $1-\alpha$ upper confidence limit. Note that equations (17) and (18) can be solved explicitly for w. Therefore, the $1-2\alpha$ confidence interval for w based on (n, X) is:

$$\frac{L(n-X)}{X((n-X)+(X+1)F_{2(X+1),2(n-X)}(1-\alpha))} < w < \frac{L(n-X+1)F_{2(n-X+1),2X}(1-\alpha)}{(X-1)(X+(n-X+1)F_{2(n-X+1),2X}(1-\alpha))}.$$

The $1-2\alpha$ confidence interval for w/L based on (n, X) is found by dividing through by L in the expression given above.

Recall that when equation (18) is used to calculate a lower bound on n, if $n_{lower}(x) < x$, we replace $n_{lower}(x)$ with x. Likewise, we replace the upper confidence limit of w with L if the upper confidence limit exceeds L. Therefore, when X = 1 both the MLE and the upper confidence limit of w are equal to L for all n, but the lower confidence limit of w depends on n. As n approaches infinity, the lower confidence limit of w approaches the limit L/X for all values of X > 1, while the upper confidence limit approaches L/(X-1) for X > 1.

10. Summary

The conditional and unconditional distributions of the number of services, X, and overtime, Y, and total server idle time, T, have been derived for a single server queueing model with Poisson arrivals, variable service time and complete balking. The conditional distributions may be easier to work with, because they do not require specification of λ, the parameter of the Poisson arrival distribution.

Estimation based on observation of a single shift from the queueing process has been considered for several situations in which incomplete data are collected. When total idle time is observed, point and interval estimators of the number of arrivals (n) and the arrival rate (λ) are derived. When only the number of services is observed, interval estimators and a variety of point estimators of n and λ are derived. For the case of equal service time, point estimators of n, based on T or X, can be constructed to be unbiased over the restricted range of n < L/w. When the number of balkers is observed, estimation of n or λ is possible only for the equal service time case; a median unbiased point estimator and interval estimators of n have been derived.

In addition, when n and X or n and T are observed, the distributions that are conditional on n can be used to estimate shift length (L) or common service time (w). Both maximum likelihood estimators and confidence limits can be derived.

Literature Cited

1. Assefi, T. (1979) Stochastic Processes and Estimation Theory with Applications. New York: John Wiley and Sons, 291 pp.

2. Basawa, I. V. and Prakas Rao, B. L. S. (1980) Statistical Inference for Stochastic Processes. London: Academic Press, Inc., 435 pp.

3. Borovkov, A. A. (1976) Stochastic Processes in Queueing Theory. New York: Springer-Verlag, 280 pp.

4. Borovkov, A. A. (1984) Asymptotic Methods in Queueing Theory. New York: John Wiley and Sons, 292 pp.

5. Cox, D. R. and Smith, W. L. (1961) Queues. London: Methuen and Co., Ltd., 180 pp.

6. Gradshteyn, I. S. and I. M. Ryzhik. (1980) Table of Integrals, Series, and Products (Corrected and Enlarged Edition). New York: Academic Press, Inc., 1160 pp.

7. Gross, D. and Harris, C. M. (1985) Fundamentals of Queueing Theory (second edition). New York: Wiley-Interscience, 587 pp.

8. Johnson, N. L. and Kotz, S. (1969) Distributions in Statistics: Discrete Distributions. New York: John Wiley and Sons, 328 pp.

9. Johnson, N. L. and Kotz, S. (1970) Distributions in Statistics: Continuous Univariate Distributions -- 2. New York: John Wiley and Sons, 306 pp.

10. Lehmann, E. L. (1983) Theory of Point Estimation. New York: John Wiley and Sons, 506 pp.

11. Mood, A. M., Graybill, F. A. and Boes, D. C. (1974) Introduction to the Theory of Statistics (third edition). New York: McGraw - Hill Book Co.,. 564 pp.

12. Prabhu, N. U. (1980) Stochastic Storage Processes: Queues, Insurance Risk, and Dams. New York: Springer- Verlag, 140 pp.

13. Robson, D. S. and Jones, C. (1988) Biometrics (in press)

14. Rubin, G. (1987) Statistical Distribution and Estimation Theory for a Single Server Queue with Fixed Service Time and Complete Balking. Ph.D. Dissertation, Cornell University, Ithaca, NY.

15. Rubin, G. and Robson, D. S. (1988) Interval Estimation for a Single Server Queue with Poisson Arrivals and Complete Balking. (Abstract) Namur, Belgium: Proceedings of the XIV International Biometric Conference, c.22.3.

16. Taylor, H. M. and Karlin, S. (1984) An Introduction to Stochastic Modeling. New York: Academic Press, Inc., 399 pp.

Table 1 - The exact and approximate unconditional expected number of services, $E(X_N)$, for several values of λ and w with λw held constant and L=1. The % relative error of the approximation has been calculated as $\frac{\text{exact - approximate}}{\text{exact}} \times 100$.

			$E(X_N)$		% Relative
λ	w	λw	Exact	Approximate	Error
2.0	0.5000	1.0	1.12890583	1.12500000	0.34598
3.0	0.3333	1.0	1.62450848	1.62500000	-0.03026
4.0	0.2500	1.0	2.12484783	2.12500000	-0.00716
5.0	0.2000	1.0	2.62503016	2.62500000	0.00115
6.0	0.1667	1.0	3.12500570	3.12500000	0.00018
7.0	0.1429	1.0	3.62499832	3.62500000	-0.00005
8.0	0.1250	1.0	4.12499832	4.12500000	-0.00000
9.0	0.1111	1.0	4.62500009	4.62500000	0.00000
10.0	0.1000	1.0	5.12500000	5.12500000	0
25.0	0.0400	1.0	12.62500000	12.62500000	0
50.0	0.0200	1.0	25.12500000	25.12500000	0
100.0	0.0100	1.0	50.12500000	50.12500000	0
2.0	0.2500	0.5	1.38889234	1.38888889	0.00025
3.0	0.1667	0.5	2.05555549	2.05555556	0.00000
4.0	0.1250	0.5	2.72222222	2.72222222	0
5.0	0.1000	0.5	3.38888889	3.38888889	0
6.0	0.0833	0.5	4.05555556	4.05555556	0
7.0	0.0714	0.5	4.72222222	4.72222222	0
8.0	0.0625	0.5	5.38888889	5.38888889	0
9.0	0.0556	0.5	6.05555556	6.05555556	0
10.0	0.0500	0.5	6.72222222	6.72222222	0
25.0	0.0200	0.5	16.72222222	16.7222222	0
50.0	0.0100	0.5	33.38888889	33.3888889	0
100.0	0.0050	0.5	66.72222222	66.72222222	0

Table 2 - Two point estimators of n, $\hat{n}_{ml}(X)$ and $\tilde{n}(X)$, and their lower and upper bounds, $\hat{n}_{ml}(X, Y = w^-)$ and $\hat{n}_{ml}(X, Y = 0)$, respectively, calculated for several values of w and X with L = 1.

w	X	$\hat{n}_{ml}(X, Y = w^-)$	$\hat{n}_{ml}(X)$	$\tilde{n}(X)$	$\hat{n}_{ml}(X, Y = 0)$
0.60	1	1.0000	2	1.0000	2.5000
0.40	1	1.0000	1	1.0000	1.6667
0.40	2	3.3333	6	6.0000	10.0000
0.20	1	1.0000	1	1.0000	1.2500
0.20	2	2.5000	3	2.6667	3.3333
0.20	3	5.0000	6	6.0000	7.5000
0.20	4	10.0000	19	16.0000	20.0000
0.20	5	25.0000	∞	∞	∞
0.10	1	1.0000	1	1.0000	1.0000
0.10	2	2.2222	2	2.2500	2.5000
0.10	3	3.7500	4	3.8571	4.2857
0.10	4	5.7143	6	6.0000	6.6667
0.10	5	8.3333	9	9.0000	10.0000
0.10	6	12.0000	13	13.5000	15.0000
0.10	7	17.5000	21	21.0000	23.3333
0.10	8	26.6667	33	36.0000	40.0000
0.10	9	45.0000	65	81.0000	90.0000
0.10	10	100.0000	∞	∞	∞

Table 3 - Maximum likelihood estimators of n and λ, $\hat{n}_{ml}(X)$ and $\hat{\lambda}_{ml}(X)$, and the corresponding 95 % upper and lower confidence limits for n and λ, when w = 0.1 and L = 1.

X	$\lambda_L(X)$	$n_L(X)$	$\hat{\lambda}_{ml}(X)$	$\hat{n}_{ml}(x)$	$\lambda_U(X)$	$n_U(X)$
1	0.05	1	1.11	1	5.271	2.71
2	0.40	2	2.47	2	7.870	5.12
3	1.02	3	4.20	4	11.077	8.18
4	1.95	4	6.47	6	15.256	12.24
5	3.28	5	9.56	9	21.026	17.91
6	5.27	6.79	14.05	13	29.606	26.40
7	8.21	9.86	21.13	21	43.827	40.54
8	13.27	14.99	34.05	33	72.173	68.82
9	23.48	25.27	65.55	65	157.052	153.60
10	54.25	56.11	∞	∞	∞	∞

Appendix A - Proof of Corollary 2.4

Corollary 2.4: The probability of no overtime, conditional on the number of arrivals less than [1/w], is:

$$P(Y_n = 0) = 1 - nw\, P(Y_{n-1} = 0) \qquad \text{for } n < [1/w].$$

Proof: Using the pdf of Y_n for the case $n < [1/w]$ yields:

$$P(Y_n = 0) = \sum_{x=1}^{n} \binom{n}{x} (1-xw)^x (xw)^{n-x} \qquad \text{for } nw < 1.$$

The recursion relation is proved by showing that

$$P(Y_n = 0) + nw\, P(Y_{n-1} = 0) = 1$$

if $0 < nw < 1$, which is equivalent to showing that

$$-1 + \sum_{x=1}^{n} \binom{n}{x} (1-xw)^x (xw)^{n-x} + nw \sum_{x=1}^{n-1} \binom{n-1}{x} (1-xw)^x (xw)^{n-1-x} = 0. \quad \text{(i)}$$

Noting that

$$nw \sum_{x=1}^{n-1} \binom{n-1}{x} (1-xw)^x (xw)^{n-1-x} = \sum_{x=1}^{n} \binom{n}{x} (1-xw)^x (xw)^{n-x} (n-x)/x,$$

we find that the left hand side (LHS) of (i) is equal to

$$-1 + (1-nw)^n + \sum_{x=1}^{n-1} \binom{n}{x} (1-xw)^x (xw)^{n-x-1}.$$

Applying the Binomial Theorem to $(1-nw)^n$ and $(1-xw)^x$ and combining these terms yields:

$$\text{LHS} = nw \left\{ \sum_{x=1}^{n} \binom{n}{x} (-1)^x (nw)^{x-1} + \sum_{x=1}^{n-1} \binom{n}{x} \sum_{v=0}^{x} \binom{x}{v} (-1)^v (xw)^{n-x-1+v} \right\}.$$

Making the transformation $k = x-v$ and noting that

$$\binom{n}{x}\binom{x}{x-k} = \binom{n}{k}\binom{n-k}{x-k}$$

give:

$$\text{LHS} = nw \sum_{k=0}^{n-1} \binom{n}{k} (w)^{n-1-k} \sum_{x=k}^{n} \binom{n-k}{x-k} (-1)^{x-k} (x)^{n-k-1}.$$

Making the transformation $t = x\text{-}k$ gives:

$$\text{LHS} = nw \sum_{k=0}^{n-1} \binom{n}{k} (w)^{n-1-k} \sum_{t=0}^{n-k} \binom{n-k}{t} (-1)^{t} (t+k)^{n-k-1}.$$

Letting $m = n\text{-}k\text{-}1$ and applying the Binomial Theorem to $(t+k)^{n-k-1=m}$ give:

$$\text{LHS} = nw \sum_{k=0}^{n-1} \binom{n}{k} (w)^{n-1-k} \sum_{t=0}^{m+1} \binom{m+1}{t} (-1)^{t} \sum_{v=0}^{m} \binom{m}{v} k^{v} t^{m-v}.$$

Switching the order of summation for t and v gives:

$$\text{LHS} = nw \sum_{k=0}^{n-1} \binom{n}{k} (w)^{n-1-k} \sum_{v=0}^{m} \binom{m}{v} k^{v} \sum_{t=0}^{m+1} \binom{m+1}{t} (-1)^{t} \; t^{m-v}. \tag{ii}$$

Note that

$$\sum_{t=0}^{m+1} \binom{m+1}{t} (-1)^{t} \; t^{m-v} = 0 \qquad \text{for } v = 0, 1, 2, \ldots, m$$

(Gradshteyn and Ryzhik, 1980, p.4). Since (ii) is a polynomial in k with the coefficient of each term equal to zero, then (ii) is equal to zero. Therefore, (i) is equal to zero and

$$P(Y_n = 0) = 1 - nw\, P(Y_{n-1} = 0). \qquad \square$$

Appendix B - Proof of Corollary 2.6

Corollary 2.6: The probability that overtime exceeds y, conditional on the number of arrivals less than [1/w], is:

$$P(Y_n > y) = n\,(w - y)\,P(Y_{n-1} = 0)$$

for $n < [1/w]$ and $0 < y < w < 1$.

Proof: Let $D = w - y$.

(Note that $P(Y_n > y) = P(w - Y_n < w - y \equiv D)$.)

$$P(Y_N > y) = \sum_{x=0}^{n-1} P(Y_N > y, X_N = x+1)$$

$$= P(Y_N > y, X_N = 1) + \sum_{x=1}^{n-1} P(Y_N < y, X_N = x+1). \qquad \text{(i)}$$

For n arrivals and only one service, overtime exceeds y only when all n arrivals occur in the interval (1-w+y, 1). Consequently, $P(Y_n > y, X_n = 1) = (w-y)^n = \Delta^n$. The second term in (i) is most easily calculated in instant service mode as

$$\sum_{x=1}^{n-1} \sum_{a,b,c} \frac{n!}{a!\,b!\,c!}\left(1-xw-\Delta\right)^a \Delta^b (xw)^c,$$

where $a = 0, 1, \ldots, x$, $b = 1, 2, \ldots, n$ and $c = n - (a+b)$. With $r \equiv a+b$, applying the Binomial Theorem to $(1-xw-\Delta)^a$ gives:

$$\sum_{x=1}^{n-1} \sum_{r=x+1}^{n} \binom{n}{r}(xw)^{n-r} \sum_{b=n-x}^{r} \binom{r}{b} \sum_{v=0}^{r-b} \binom{r-b}{v}(-1)^v \Delta^{b+v} (1-xw)^{r-(b+v)}.$$

With $k \equiv b+v$, we must calculate the coefficients of Δ^k for the polynomial in the equation above. Applying the procedure developed in the proof of Corollary 2.4, we find that the coefficient of Δ^n in (i) is:

$$\sum_{a=0}^{n-1} \binom{n}{a} (-1)^a (n-a) = 0$$

and that for Δ^k with $1 < k < n$ the coefficient is:

$$\sum_{x=1}^{n-1} \sum_{r=x+1}^{x+k} \binom{n}{r} (xw)^{n-r} (1-xw)^{n-k} \sum_{b=n-x}^{k} \binom{r}{b} \binom{r-b}{k-b} (-1)^{k-b} . \qquad \text{(ii)}$$

After several transformations we find that (ii) equals

$$\binom{n}{k} \sum_{g=0}^{n-k} ((n-k)!/g!)(-w)^{n-k-g} \sum_{b=0}^{k-1} \binom{k-1}{b} (-1)^{k-b} ,$$

where

$$\sum_{b=0}^{k-1} \binom{k-1}{b} (-1)^{k-b} = 0 .$$

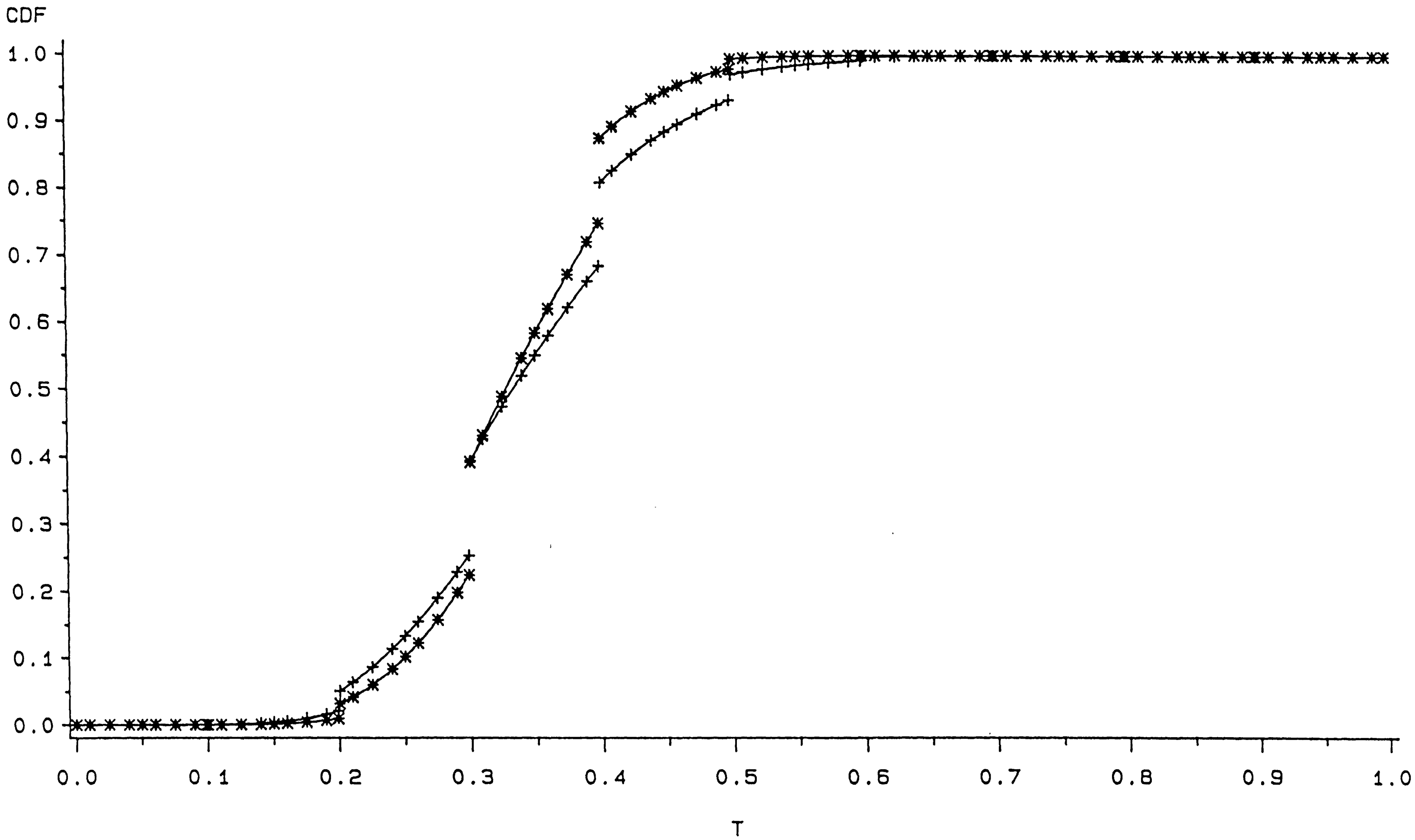
Therefore, the coefficient of Δ^k is zero for $1 < k < n$. Since (ii) holds for $k = 1$, the coefficient for Δ is:

$$\sum_{x=1}^{n-1} \binom{n}{x+1} (xw)^{n-x-1} (1-xw)^x \binom{x+1}{1} = n \sum_{x=1}^{n-1} \binom{n-1}{x} (1-xw)^x (xw)^{n-1-x}$$

$$= n\, P(Y_{n-1} = 0) .$$

Therefore,

$$P(Y_n > y) = n\,(w-y)\,P(Y_{n-1} = 0). \qquad \square$$

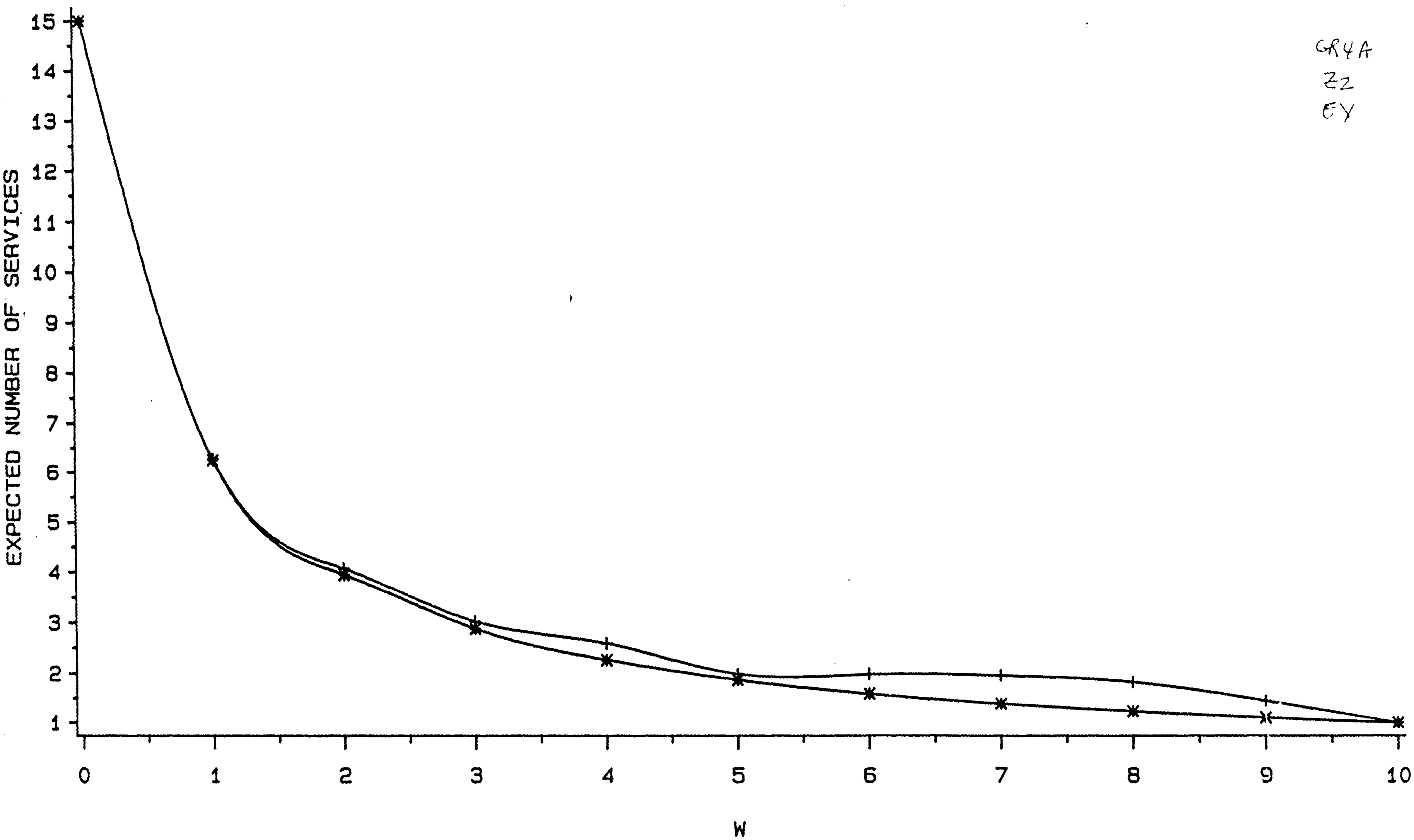


98

FIGURE 2.11 CDF OF IDLE TIME (T) WITH W=0.1, UNCONDITIONAL (+) WITH LAMBDA=20
AND CONDITIONAL (*) WITH N=20.

Z2 X → Δ 95%
+ → □ 90%
* → ◇ Median

GR4A
Z2
EY



#2

FIGURE 2.5 EXPECTED NUMBER OF SERVICES (CONDITIONAL ON N=15) VS. W
FOR SHIFT LENGTH=10 (+=EXACT; *=APPROXIMATE).

Δ □

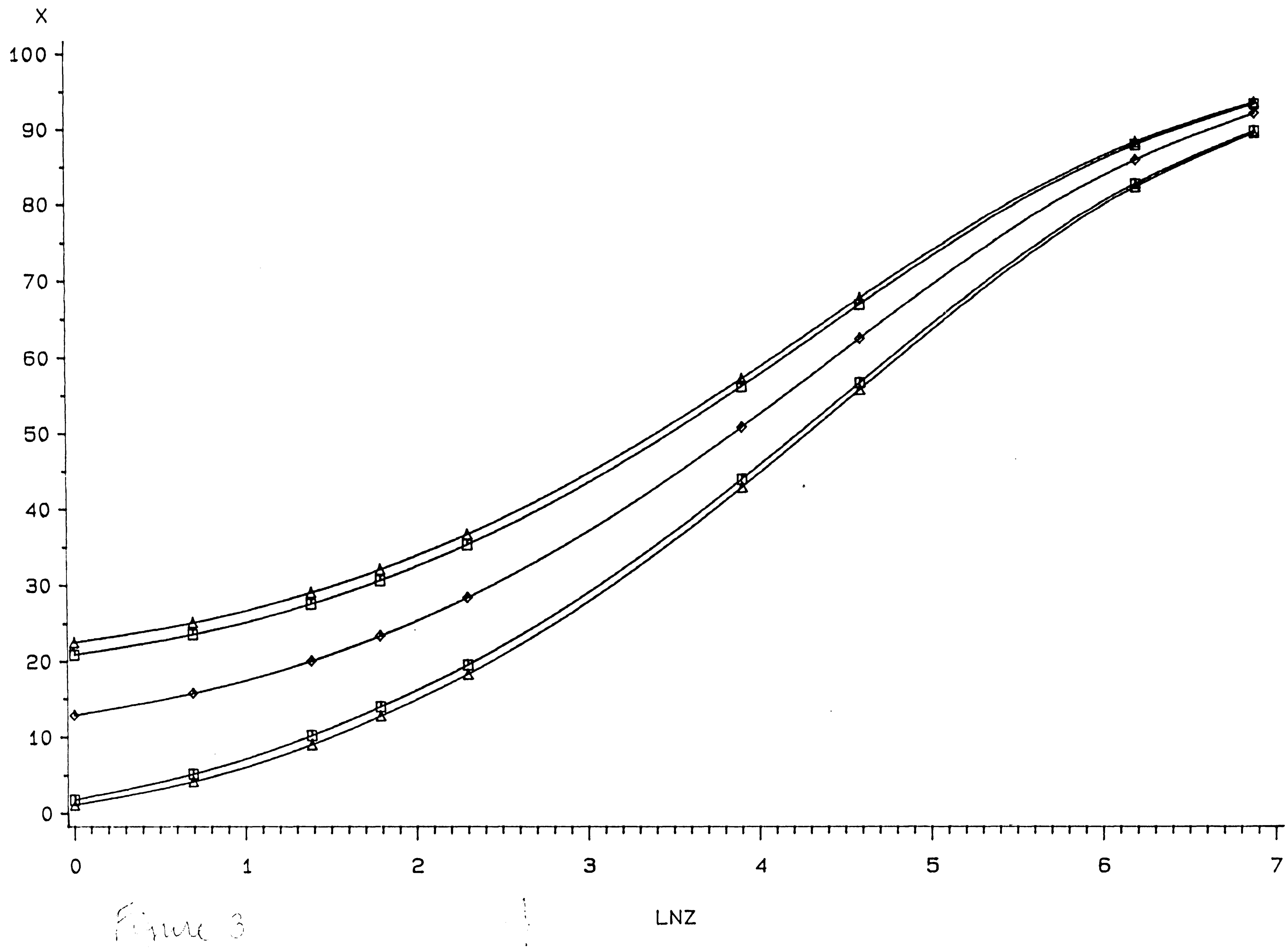


Figure 3