Conditional Properties of Generalized Bayes Variance Intervals

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SUMMARY

Generalized Bayes intervals for the variance of a normal distribution have previously been developed in a decision-theoretic framework. We show that these intervals are highest posterior density regions against a mixture prior, and examine the conditional properties of the intervals. It is shown that these intervals, which are also frequentist confidence intervals, have excellent conditional properties.

1. Introduction. Conditional evaluations of frequentist confidence procedures have become of increasing interest particularly after many authors have illustrated that usual frequentist procedures can have quite disturbing conditional, or post-data, evaluations [see Berger and Wolpert (1984) for an excellent review of many of these examples]. The idea of conditional evaluations was originally suggested by Fisher (1956). Buehler (1959) introduced a betting based theory, which was later formalized and extended by Robinson (1979a,b).

In Robinson's general formalization, we observe a random variable $X \sim f(x|\theta)$, where $f(x|\theta)$ is a known density function. Given $X=x$, we construct a confidence region, $C(x)$, for θ, with confidence coefficient $1-\alpha(x)$ denoted by $\langle C(x), 1-\alpha(x)\rangle$. After observing both x and $C(x)$, we must be willing to accept bets at odds $1-\alpha(x):\alpha(x)$ as to whether $C(x)$ covers θ. A bettor places bets according to a betting strategy, $k(x)$, which Robinson allows to be any bounded function. In a more straightforward statistical interpretation, we allow $k(x)$ to be a signed indicator function for some set S, $k(x) = I_S(x)$, with positive sign denoting a bet for coverage and a negative sign bet against coverage.

The conditional properties of the procedure, $C(x)$, can be evaluated by determining whether or not the bettor can find a strategy, $k(x)$, such that the expected gain of the bettor is positive. In particular, we are interested in two types of strategies: relevant and semirelevant.

A betting strategy $k(x)$, against confidence procedure $\langle C(X), 1-\alpha(X)\rangle$ is said to be

(i) *semirelevant*, if

$$E_\theta \left[\left\{ I_{C(X)}(\theta) - [1-\alpha(X)] \right\} k(X) \right] \geq 0 \qquad \text{for all } \theta \tag{1.1}$$

and strictly positive for some θ, and

(ii) *relevant*, if for some $\varepsilon > 0$

$$(1.2) \qquad E_\theta\left[\left\{I_{C(X)}(\theta) - [1-\alpha(X)]\right\}k(X)\right] \geq \varepsilon\, E_\theta\left|k(X)\right| \qquad \text{for all } \theta .$$

In terms of strategies that are signed indicator functions, the above can be interpreted as follows. A relevant betting strategy, $k(X) = I_s(X)$, against a confidence procedure $\langle C(X), 1-\alpha\rangle$, implies the existence of $\varepsilon>0$ such that $P_\theta[\theta\varepsilon C(X)|X\varepsilon S] \geq 1-\alpha + \varepsilon$ for every θ. The function $k(X)$ would be defined as a positively biased relevant betting strategy. An indicator function with a negative sign, $k(X) = -I_s(X)$ defines a negatively biased relevant strategy. Thus, a relevant betting strategy defines a subset of the sample space where conditional coverage probability can be bounded away from $1-\alpha$.

Note that a positively biased strategy is indicating a conservative procedure which, in most people's minds, is not a statistically worrisome problem. However a procedure that admits a negatively biased relevant strategy should be avoided. Avoidance of procedures with negatively biased *semirelevant* strategies, however, is too harsh a restriction since, for example, the t-interval and Scheffe's simultaneous intervals allow such strategies [see Buehler (1959) or Olshen (1973)].

A confidence procedure that admits a relevant or semirelevant betting strategy admits a winning strategy. The difference between relevant and semirelevant is subtle, and in general, only proper Bayes procedures are free of semirelevant (and therefore relevant) strategies (though these cannot yield a $1-\alpha$ frequentist guarantee). Generalized (or limits of) Bayes generally allow semirelevant but not relevant strategies. Thus, a generally agreed upon principle [see Robinson (1976), Bondar (1977)] is: use procedures which allow no negatively-biased relevant strategies. Such procedures will most probably be Bayes, or limits of Bayes, procedures, since they eliminate relevant betting [see Maatta and Casella (1987) for a more complete specification of definitions].

In this paper we consider the problem of interval estimation of the normal variance when the mean is unknown, that is, let $X_1,\cdots,X_n$ be iid normal random variables with mean μ and variances σ^2. The usual interval estimators are of the form

$$C(S^2) = \{\sigma^2 : \sigma^2 \in (S^2/b_\nu, S^2/a_\nu)\} , \tag{1.3}$$

where ν is the degrees of freedom of $S^2 = \sum(X_i - \bar{X})^2$ and a_ν and b_ν satisfy $P(a_\nu < \chi^2_\nu < b_\nu) = 1-\alpha$, where χ^2_ν is a chi-square random variable with ν degrees of freedom. For two-sided intervals, an additional constraint is needed to uniquely specify an interval. There are at least three well known intervals of the form (1.3): the equal-tailed interval, the minimum length interval, and the shortest-unbiased (or best invariant) interval, all with a specific second constraint that uniquely defines the interval [see Tate and Klett (1959) for details].

Maatta and Casella (1987) investigated the conditional properties of interval estimators of the form (1.3) and found that there are no relevant betting procedures against these intervals, and hence are free from major conditional defects. They also show that the two lesser known intervals, minimum length and shortest unbiased, do not allow negatively biased semirelevant (NBSR) strategies while the common interval allows NBSR strategies, clearly showing that the minimum length and shortest unbiased intervals have superior conditional properties.

Intervals of the form (1.3) depend on $\bar{X}$ only through its appearance in S^2 and more recent work [Cohen (1972) and Shorrock (1987)] show that one can uniformly increase coverage probability by allowing a more explicit dependency on $\bar{X}$. In this paper, we consider the form and nature of Shorrock's interval and investigate its conditional properties.

In the next section we introduce Shorrock's interval, including a brief description of its construction, coverage probability, and other properties. In Section 3, we specify a prior distribution on μ and σ^2 that leads to the constructed interval and also calculate the posterior distribution of σ^2 and thus get Shorrock's interval as a posterior interval. Section 4 contains the main conditional result: No relevant strategies against Shorrock's intervals. Section 5 contains discussion and comments.

2. Shorrock's Interval. Before we proceed with the specific details of the construction of Shorrock's interval, we recall the distributional assumptions. We assume that $X_1 \cdots X_n \sim$ iid $N(\mu, \sigma^2)$, with $\bar{X} = \Sigma X_i/n$ and $S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2$. We know that $n\bar{X}^2/\sigma^2 \sim \chi_1^2(\frac{1}{2}\frac{\mu^2}{\sigma^2})$, that is, a noncentral chi-square with one degree of freedom and noncentrality parameter, $\frac{1}{2}\mu^2/\sigma^2$. The distribution associated with S^2 is a chi-squared with ν degrees of freedom, i.e., $\frac{\nu S^2}{\sigma^2} \sim \chi_\nu^2$. Furthermore, since the density of a noncentral chi-square can be written as a Poisson mixture of central chi-squares if we define $W \sim \text{Poisson}\left(\frac{1}{2}\frac{\mu^2}{\sigma^2}\right)$, then the conditional distribution of $\bar{n}X^2/\sigma^2 | W$ is a central chi-square with $1 + 2W$ degrees of freedom. Shorrock made use of these facts when he constructed his improved intervals.

Following the lead of Cohen (1972), Shorrock constructed an interval that has fixed length and higher coverage probability than the intervals of the form (1.3). By fixing the length of his new interval to be that of the minimum length interval, Shorrock guaranteed that his interval will be no longer than the minimum length interval. In addition, he proved that his interval has uniformly higher probability of coverage.

Shorrock's construction began by finding a function $\phi_w(r)$ that maximizes

$$P[\phi_w(r)S^2 \le \sigma^2 \le (\phi_w(r) + c)S^2 | Z^2 \le r, W=w] , \tag{2.1}$$

for each $r \in [0,1]$, W, and Z^2, $Z^2 = \dfrac{n\bar{X}^2}{n\bar{X}^2 + S^2}$. The constant $c = \frac{1}{a_\nu} - \frac{1}{b_\nu}$, with a_ν and b_ν satisfying

$$\int_{a_\nu}^{b_\nu} f_\nu(x) d_x = 1-\alpha \quad \text{and} \quad f_{\nu+4}(a_\nu) = f_{\nu+4}(b_\nu) \quad , \tag{2.2}$$

[$f_\nu(\cdot)$ representing the χ^2_ν density function], that is, the constraints of the minimum length interval. He continued his construction of his interval by using the smoothing technique first used by Brewster and Zidek (1974). By choosing a dense set, $\{r_0, r_1, \cdots, r_\ell, \cdots\}$ in $[0,1]$, then finding a maximizing $\phi_W(r_\ell)$ for each r_ℓ, and then taking the limit of the resulting intervals, he obtained as the limit

$$C_S(\bar{X}, S^2) = \left\{\sigma^2 : \sigma^2 \in [\phi_0(Z^2)S^2, (\phi_0(Z^2) + c)S^2]\right\} . \tag{2.3}$$

where $\phi_0(Z^2)$ is the solution to

$$\frac{f_{\nu+4}\left(\frac{1}{\phi}\right)}{f_{\nu+4}\left(\frac{1}{\phi+c}\right)} = \frac{F_1\left(\frac{1}{\phi+c}\,\frac{Z^2}{1-Z^2}\right)}{F_1\left(\frac{1}{\phi}\,\frac{Z^2}{1-Z^2}\right)} . \tag{2.4}$$

C_S of (2.3) has uniformly higher probability of coverage than does (1.3) by construction. The functions $\phi_w(r)$ were chosen to maximize (2.1) for each r and w and thus, in the limit, $\phi_0(r)$ maximizes the conditional coverage probability, which then results in a higher unconditional probability.

Shorrock calls his interval a generalized Bayes interval, which it is, but he does not show that it is generalized Bayes in the sense that it is the posterior credible region with respect to some improper prior. Shorrock derives a prior, similar to that of Brewster and Zidek, in which the Bayes risk, using a loss equal the probability of noncoverage, is minimized.

3. Prior and Posterior Distributions. In this section, we seek the prior distribution of μ and σ^2 that allows us to arrive at Shorrock's interval. In addition, we seek the resulting posterior distribution of σ^2 given $\bar{X}$ and S^2 from this prior. In fact, we get more general results by considering a family of priors, one of which leads to Shorrock's interval.

Since we are sampling from an $N(\mu, \sigma^2)$ distribution, we can reduce to the sufficient statistics $\bar{X}$ and S^2. Thus,

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \quad \text{and} \quad \frac{S^2}{\sigma^2} \sim \chi^2_\nu \quad .$$

The joint density of $\bar{X}$ and S^2 is proportional to

$$\left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} e^{-\frac{n}{2\sigma^2}(\bar{X}-\mu)^2} \frac{1}{\sigma^2}\left(\frac{S^2}{\sigma^2}\right)^{(\nu/2)-1} e^{-\frac{1}{2}(S^2/\sigma^2)} \quad .$$

In his thesis, Shorrock shows that C_S is generalized Bayes in the following sense: If the loss of a set estimator C is measured by

$$L(\sigma^2, C) = I(\sigma^2 \notin C) \ ,$$

then among all intervals of the form $[\phi(Z^2)S^2, (\phi(Z^2) + c)S^2]$, where c is a fixed constant, C_S minimizes the Bayes risk with respect to the prior

$$\pi(\eta) = \tfrac{1}{4} \int_0^\infty e^{-\eta u/2}(1+u)^{-\frac{1}{2}} du \ , \tag{3.1}$$

where $\eta = \mu^2/\sigma^2$. In investigating conditional properties of Shorrock's interval, however, we are interested in finding a prior for which C_S is a highest posterior density region. Building on (3.1) we arrive at the following class of priors:

$$\pi(\mu, \sigma^2 | d, a) \propto \left(\frac{1}{\sigma^2}\right)^{d/2} \int_0^\infty e^{-\frac{a\mu^2}{\sigma^2}\cdot\frac{u}{2}} \frac{1}{u^{\frac{1}{2}}(n+u)} du \ . \tag{3.2}$$

This prior can, perhaps, be more easily understood in stages. Conditional on u, the prior is

$$\pi(\mu,\sigma^2\,|\,d,a,u) \propto \left(\frac{1}{\sigma^2}\right)^{d/2} e^{-\frac{1}{2}\frac{a\mu^2}{\sigma^2}u} \;,$$

a "normal-type" prior on μ and a "noninformative-type" prior on σ^2. This conditional prior is then mixed over u to give the Shorrock prior. Notice that, if a=0, then this prior gives rise to some of the more common intervals that are based only on S^2.

From now on we take a=1, as Shorrock does, and calculate the posterior density of σ^2. The posterior density of σ^2 is given by

$$\pi(\sigma^2\,|\,\bar{X},S^2) \propto \int_{-\infty}^{\infty} \left(\frac{1}{\sigma^2}\right)^{\frac{\nu+d+1}{2}-1} e^{-\frac{S^2}{2\sigma^2}} e^{-\frac{n}{2\sigma^2}(\bar{X}-\mu)^2} \int_0^{\infty} e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}u} \frac{1}{(n+u)u^{\frac{1}{2}}}\, du\, d\mu \;.$$

If we now interchange the order of integration and complete the square in μ, we obtain

$$\text{(3.3)} \qquad \pi(\sigma^2\,|\,\bar{X},S^2) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu+d+1}{2}-1} e^{-\frac{S^2}{2\sigma^2}} \int_0^{\infty} e^{-\frac{n\bar{X}^2}{2\sigma^2}\frac{u}{n+u}} \frac{1}{(n+u)u^{3/2}u^{\frac{1}{2}}}\, du \;.$$

Now make the substitution $w = \frac{n\bar{X}^2}{\sigma^2}\left(\frac{u}{n+u}\right)$ to get

$$\pi(\sigma^2\,|\,\bar{X},S^2) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu+d-1}{2}-1} e^{-\frac{S^2}{2\sigma^2}} \int_0^{n\bar{X}^2/\sigma^2} e^{-\frac{1}{2}w} w^{-\frac{1}{2}} dw \;.$$

If we now let $f_q(\cdot)$ denote the pdf of a χ^2_q random variable, we can write

$$\pi(\sigma^2\,|\,\bar{X},S^2) \propto f_{\nu+d-1}(S^2/\sigma^2)P(\chi^2_1 < n\bar{X}^2/\sigma^2) \;.$$

Now use the identity

$$\int_0^{\infty} f_q(a/x)P(\chi^2_1 < at/x)dx = \frac{a}{(q-2)(q-4)}\, P\left(F_{1,q-4} < (q-4)t\right) \;,$$

where $F_{m,n}$ is an F random variable with m and n degrees of freedom, and we have

$$(3.4)\qquad \pi(\sigma^2|\bar{X},S^2) = \frac{f_{\nu+d-1}(S^2/\sigma^2)P(\chi_1^2 < n\bar{X}^2/\sigma^2)}{\left(\frac{S^2}{\nu+d-3}\right)P\left(F_{1,\nu+d-5}<(\nu+d-5)(n\bar{X}^2/S^2)\right)} .$$

If we set d=5 in (3.4), we obtain the posterior that yields Shorrock's interval. We seek a function $\phi(t)$ that maximizes

$$(3.5)\qquad \int_{\phi S^2}^{(\phi+c)S^2} \pi(\sigma^2|\bar{X},S^2)d\sigma^2 = \int_{\phi S^2}^{(\phi+c)S^2} \frac{f_{\nu+4}\left(\frac{S^2}{\sigma^2}\right)P\left(\chi_1^2<\frac{S^2}{\sigma^2}t\right)}{\frac{S^2}{\nu+2}P(F_{1,\nu}\le\nu t)}\, d\sigma^2 ,$$

where c again is the minimum length value, and $t = \frac{n\bar{X}^2}{S^2}$. If we differentiate (3.5) with respect to ϕ and set the result equal zero we obtain the determining value of ϕ to be the solution to

$$(3.6)\qquad f_{\nu+\phi}\left(\frac{1}{\phi}\right)P\left(\chi_1^2 < \frac{1}{\phi}t\right) - f_{\nu+\phi}\left(\frac{1}{\phi+c}\right)P\left(\chi_1^2 < \frac{1}{\phi+c}\, t\right) = 0 .$$

This is equivalent to (2.4) with $t = \frac{Z^2}{1-Z^2}$. Recall that $Z^2 = n\bar{X}^2/(n\bar{X}^2 + S^2)$, so that $Z^2/(1-Z^2) = \frac{n\bar{X}^2}{S^2}t$. Thus the $\phi(\cdot)$ that maximizes the conditional probability coverage (for fixed c) is equivalent to the $\phi(\cdot)$ that maximizes posterior probability of coverage (for fixed c) with respect to prior (3.2).

It is of interest to note that for a=0 and d=5, the posterior is independent of $\bar{X}$ and is

$$\pi(\sigma^2|S^2) = \frac{S^2}{\sigma^4}\, f_{\nu+4}\left(\frac{S^2}{\sigma^2}\right) .$$

The ϕ that would maximize the posterior probability with respect to prior (3.2), with a=0 and d=5, turns out to yield exactly the a_ν and b_ν of the minimum length specification [see (2.2)].

It can be shown that the function ϕ is an increasing function of r [see Shorrock (1987)]. And also that as r goes from 0 to 1, $\phi(r)$ goes from $\frac{1}{b_{\nu+5}} \longrightarrow \frac{1}{b_{\nu+4}}$ where b_ν is the solution to $f_\nu(b_\nu) = f_\nu\left(\frac{1}{b_\nu^{-1}+c}\right)$. That That is, $b_{\nu+4}$ is the solution to the minimum length specification with corresponding length c.

4. Conditional Results: Relevant Betting Procedures. This section contains the main results for the conditional properties of Shorrock's intervals, (2.3). The proof of the main theorem mimicks that of Theorem 2.1, in Maatta and Casella (1987), concerning relevant betting procedures for intervals of the form (1.3). The goal of Theorem 2.1 was to show that the confidence procedure of interest was close to some type of limiting Bayes procedure. In Section 3, we have shown that Shorrock's interval is a posterior interval with respect to the prior given in (3.2). Thus, our theorem should follow with ease.

THEOREM 4.1. Let $X_1, \cdots, X_n$ be iid $N(\mu, \sigma^2)$ with μ and σ^2 both unknown. Then for the interval (2.3) with ϕ defined to be the solution to (2.4), no relevant betting procedures exist for the confidence procedure $\langle C_S(\bar{X}, S^2), \gamma_S(\bar{X}, S^2)\rangle$ where

$$\gamma_S(\bar{X}, S^2) = \int_{\phi S^2}^{(\phi+c)S^2} \pi(\sigma^2 | \bar{X}, S^2) d\sigma^2 , \tag{4.1}$$

and $\pi(\sigma^2 | \bar{X}, S^2)$ is given in (3.4).

PROOF. We suppose a relevant betting procedure exists and then reach a contradiction. If such a betting procedure exists then, from (1.2), there exists $\varepsilon > 0$ and a betting procedure $|k(\bar{X}, S^2)| \le 1$ such that

$$E\left\{\left[I_{C_S(\bar{X}, S^2)}(\sigma^2) - \gamma_S(\bar{X}, S^2)\right] k(\bar{X}, S^2)\right\} > \varepsilon\, E\left|k(\bar{X}, S^2)\right|$$

for all μ and σ^2, with strict inequality for some (μ, σ^2), where the expectation is taken with respect to the joint density of $\bar{X}$ and S^2. Multiply both sides of (4.2) by the prior distribution given by (3.2), with $d=5$ and $a=1$, and integrate with respect to both μ and σ^2. If $k(\bar{X}, S^2)$ is a relevant betting procedure then

$$(4.3) \qquad \int_0^\infty \int_{-\infty}^\infty E\left\{\left[I_{C_S(\bar{X},S^2)}(\sigma^2) - \gamma_S(\bar{X},S^2)\right] k(\bar{X},S^2)\right\}\pi(\mu,\sigma^2)d\mu d\sigma^2$$

$$\geq \varepsilon \int_0^\infty \int_{-\infty}^\infty E\left|k(\bar{X},S^2)\right|\pi(\mu,\sigma^2)d\mu d\sigma^2 \ .$$

It follows from the proof of Theorem 2.1 of Maatta and Casella (1987) that we are able to interchange order of integration, and the LHS of (4.3) is equal to

$$(4.4) \qquad \int_{-\infty}^\infty \int_0^\infty \left\{\left[\int_{\phi S^2}^{(\phi+c)S^2} \pi(\sigma^2|\bar{X},S^2)d\sigma^2 - \gamma_S(\bar{X},S^2)\right] k(\bar{X},S^2)m(\bar{X},S^2)\right\}dS^2 d\bar{X},$$

where $\pi(\sigma^2|\bar{X},S^2)$ is given by (3.4) and $m(\bar{X},S^2)$ is the resulting marginal distribution of $\bar{X}$ and S^2. It's clear from the results in Section 3 that the square-bracketed term in (4.4) is identically zero, which produces the needed contradiction. ||

With the establishment of Theorem 4.1, it has been demonstrated that Shorrock's intervals are free from any major conditional defects when our confidence statementuses the posterior coverage probability, $\gamma_S(\bar{X},s^2)$. More generally, frequentists like to make pre-experimental confidence statements, statements not dependent on the data. The following theorem summarizes the conditional properties of Shorrock's interval with respect to the confidence level $1-\alpha$, where $1-\alpha$ is the confidence level of the minimum length interval.

THEOREM 4.2. Let $X_1,\cdots,X_n$ be iid $N(\mu,\sigma^2)$ with μ and σ^2 both unknown. Then for the interval (2.3) with ϕ defined to be the solution to (2.4), no negatively biased relevant (NBR) betting procedures exist for the confidence procedure $\langle C_S(\bar{X},s^2),1-\alpha\rangle$.

PROOF. Assume that a negatively biased relevant betting procedure exists, that is, assume there exists a function $k(\bar{X},S^2) \le 0$ satisfying (1.2). Multiply (1.2) by $\pi(\mu,\sigma^2)$ of (3.2) (again taking d=5 and a=1) and integrate with respect to both μ and σ^2. If $k(\bar{X},S^2)$ is an NBR betting procedure, then

$$(4.5) \qquad \int_0^\infty \int_{-\infty}^\infty E\left\{\left[I_{C_S(\bar{X},s^2)}(\sigma^2) - (1-\alpha)\right] k(\bar{X},S^2)\right\}\pi(\mu,\sigma^2)d\mu,d\sigma^2$$

$$\ge \epsilon \int_0^\infty \int_{-\infty}^\infty E\left|k(\bar{X},S^2)\right|\pi(\mu,\sigma^2)d\mu d\sigma^2 \ .$$

Manipulating the densities as before and integrating out μ, will establish that the LHS of (4.5) can be written as

$$(4.6) \qquad \int_0^\infty \int_{-\infty}^\infty k(\bar{X},S^2)\left[\int_{\phi S^2}^{(\phi+c)S^2} \pi(\sigma^2|\bar{X},S^2)d\sigma^2 - (1-\alpha)\right] m(\bar{X},S^2)dS^2 d\bar{X},$$

where $\pi(\sigma^2|\bar{X},S^2)$ and $m(\bar{X},S^2)$ are as in Theorem 4.1. If we can establish that the bracketed term in (4.6) is positive then we are done, since this would supply the needed contradiction. The following Lemma establishes this fact, completing the proof. ||

LEMMA 4.1. The posterior probability of Shorrock's interval, $\gamma_S(\bar{X},S^2)$, satisfies

$$(4.7) \qquad \gamma_S(\bar{X},S^2) = \int_{\phi S^2}^{(\phi+c)S^2} \pi(\sigma^2|\bar{X},S^2)d\sigma^2 > 1-\alpha \ , \quad \forall\ \bar{X},S^2$$

where $\pi(\sigma^2|\bar{X},S^2)$ is given by (3.4).

PROOF: By letting $y = S^2/\sigma^2$ in LHS of (4.7), the posterior probability can be written as

$$(4.8) \qquad \int_{\frac{1}{\phi+c}}^{\frac{1}{\phi}} \frac{P(\chi_1^2 < yt)f_\nu(y)}{P(F_{1,\nu} < \nu t)}\, dy \ ,$$

where f_ν is the pdf of a χ^2_ν, $F_{1,\nu}$ is a random variable with 1 and ν degrees of freedom, and $t = n(\bar{X}/S)^2$. We want to show that (4.8) is greater than $1-\alpha$, which is equivalent to showing

$$(4.9) \qquad \int_{\frac{1}{\phi+c}}^{\frac{1}{\phi}} P(\chi^2_1 < yt) f_\nu(y)dy - (1-\alpha)P(F_{1,\nu} < \nu t) > 0 .$$

Differentiate (4.9), keeping in mind the constraint on ϕ. After some algebra, and making the transformation $w = (1+t)y$, we find that the first derivative of the LHS of (4.9) is

$$(4.10) \qquad \nu f_{1,\nu}(\nu t) \left[\int_{\frac{1+t}{\phi+c}}^{\frac{1+t}{\phi}} f_{\nu+1}(w)dw - (1-\alpha) \right].$$

Note that the sign of the derivative is given by the bracketed term in (4.10). We will now show that the function

$$(4.11) \qquad g(t) = \int_{\frac{1+t}{\phi+c}}^{\frac{1+t}{\phi}} f_{\nu+1}(w)dw$$

is decreasing in t. This will imply that the only possible derivative sign change is from positive to negative and hence that the only possible interior extrema is a maximum. Checking the endpoints $t=0$ and $t=\infty$, of (4.9), would then establish the result.

Return to (4.11), and make the transformation $x = (1/w)$ to get

$$(4.12) \qquad \int_{\frac{\phi}{1+t}}^{\frac{\phi+c}{1+t}} \frac{1}{x^2} f_{\nu+1}\left(\frac{1}{x}\right) dx .$$

Note that as t increases, the length of the integration interval decreases, since the length is equal to $c/(1+t)$. Define $b^*(t)$ as the value of ϕ^{-1} that would maximize (4.12). As t increases, the interval length gets shorter, so that the best $\phi = [b^*(t)]^{-1}$ must get smaller. Hence $b^*(t) > b^*(0) = b_{\nu+5}$. But $b_{\nu+5} > \phi^{-1} > b_{\nu+4}$ and $\phi(t)$ increases from $1/b_{\nu+5}$ to $1/b_{\nu+4}$ as $t:0 \to \infty$ (by the properties of ϕ). Since as t increases the interval length decreases, for fixed ϕ (4.11) decreases in t. Thus $\phi(t)$ moves further from the optimum value, and using the fact that $f_{\nu+1}(\cdot)$ is unimodal (Shorrock, 1987), we have that (4.11) decreases in t.

It remains to show that the posterior probability is greater than $1-\alpha$ as t=0 and ∞. At $t=\infty$, the posterior probability is exactly $1-\alpha$ since $\phi(\infty)$ is the minimum length specification and thus gives $1-\alpha$. The limit of the posterior probability as t approaches zero is, by L'Hospital's rule, equal to

$$\int_{\frac{1}{\phi+c}}^{\frac{1}{\phi}} f_{\nu+1}(w)\,dw \tag{4.13}$$

with $\phi(0) = b_{\nu+5}$. But this is equal to (4.11) at t=0 and we know that (4.11) is decreasing in t, so (4.13) is greater than $1-\alpha$. ||

5. Discussion. The generalized Bayes intervals constructed by Shorrock are shown to be good performers on a conditional basis. Coupled with their known good frequentist properties, these intervals provide excellent alternatives to the usual interval estimators of σ^2.

The procedure $\langle C_S(\bar{X},S^2),1-\alpha\rangle$ is a frequentist confidence procedure that is free from negatively biased betting. The interpretation is that the post-data assertion of $1-\alpha$ is conservative, being a lower bound on a Bayesian posterior probability. An alternative procedure, that also has interesting interpretations, is $\langle C_S(\bar{X},S^2),\gamma_S(\bar{X},S^2)\rangle$. This procedure is free of *all* relevant betting, and yields a post-data inference that is everywhere greater than $1-\alpha$. On an intuitive level, it might be expected, even desired, that post-data confidence be everywhere greater than pre-data (frequentist) confidence, since the addition of the data provides more information, and more information should lead to greater confidence. Although Shorrock's interval behaves in this intuitive way, in that $\gamma_S(\bar{X},S^2) > 1-\alpha \ \forall\ \bar{X},S^2$, this is not a general property of Bayes intervals.

With some more work, we should be able to further improve the intervals. For example, we could get the highest probability density region using the posterior distribution of (3.6), without using invariance considerations. Shorrock first the interval length of invariant intervals and then found the maximizing ϕ. One could maximize the posterior probability without constraint thus getting a procedure with uniformly higher probability than intervals (1.3). Another tactic is to fix the posterior probability at (say) $1-\alpha$, and calculate the resulting interval length. This latter approach may lead to shorter frequentist $1-\alpha$ confidence intervals.

It is also easily seen that we could make the prior distribution (3.2) proper by making the σ^2 portion an inverted gamma distribution showing that Shorrock's procedure is also a limit of proper Bayes procedures.

REFERENCES

BONDAR, J. V. (1977). A conditional confidence principle. *Ann. Statist.* **5** 881-891.

BERGER, J. O. and WOLPERT, R. W. (1984). *The Likelihood Principle.* Institute of Mathematical Statistics, Lecture Notes - Monograph Series.

BREWSTER, J. F. and ZIDEK, J. V. j(1974). Improving on equivariant estimators. *Ann. Statist.* **2** 21-38.

BUEHLER, R. J. (1959). Some validity criteria for statistical inference. *Ann. Math. Statist.* **30** 845-863.

COHEN, A. (1972). Improved confidence intervals for the variance of a normal distribution. *J. Amer. Statist. Assoc.* **67** 382-387.

FISHER, R. A. (1956). *Statistical Methods and Scientific Inference,* 3rd ed. Hafner, New York.

MAATTA, J. M. and CASELLA, G. (1987). Conditional properties of interval estimators of the normal variance. *Ann. Statist.* **15**, December.

OLSHEN, R. A. (1973). The conditional level of the F-test. *J. Amer. Statist. Assoc.* **68** 692-698.

ROBINSON, G. K. (1976). Properties of Student's t and of Behrens-Fisher solution to the two means problems. *Ann. Statist.* **4** 963-971.

ROBINSON, G. K. (1979a). Conditional properties of statistical procedures. *Ann. Statist.* **7** 742-755.

ROBINSON, G. K. (1979b). Conditional properties of statistical procedures for location and scale parameters. *Ann. Statist.* **7** 756-771.

SHORROCK, G. (1987). Improved confidence intervals for a normal variance. Technical Report, Université du Quebec à Montréal.

TATE, R. F. and KLETT, G. W. (1959). Optimal confidence intervals for the variance of a normal distribution. *J. Amer. Statist. Assoc.* **54** 674-682.