CONDITIONAL PROPERTIES OF INTERVAL ESTIMATORS OF THE NORMAL VARIANCE

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Abstract

Both one-sided and two-sided interval estimators are examined using conditional criteria, and we find that most common intervals have acceptable conditional properties. In the two-sided case we further examine three well known intervals, and find the shortest-unbiased (Neyman-shortest) possessing the strongest conditional properties, with the minimum-length interval a close second. In the one-sided case we have the somewhat surprising result that the lower confidence interval (which results from inverting the UMP test of $H_0: \sigma \leq \sigma_0$) has weaker conditional properties than the upper interval (where there does not exist an UMP test).

1. Introduction. The frequency theory of statistics provides a framework for making unconditional confidence statements about a random phenomenon. Recent research pertaining to frequentist set estimation has raised some disturbing questions about the validity of such unconditional statements. In particular, even though frequentist confidence statements are constructed unconditionally, there is a temptation (often fulfilled) to interpret them conditionally. In some cases this conditional confidence is very different from the unconditional confidence.

If $C(X)$ is a $1-\alpha$ frequentist confidence set for a parameter θ based on an observation X, that is,

$$P_\theta[\theta \in C(X)] \geq 1-\alpha \qquad \text{for all } \theta \tag{1.1}$$

but for some set $\mathcal{S}$ (in the sample space) and some positive ε

$$P_\theta[\theta \in C(X) | X \in \mathcal{S}] \leq 1-\alpha-\varepsilon \qquad \text{for all } \theta \quad , \tag{1.2}$$

then the statement that we are $1-\alpha$ confident with respect to the proposition $\theta \in C(X)$ seems questionable when $X \in \mathcal{S}$. Fisher (1956a) first called the subset $\mathcal{S}$ a recognizable subset and contended that one should be no more than $1-\alpha-\varepsilon$ confident in intervals based on outcomes in $\mathcal{S}$.

Since Fisher's first statements on conditional evaluations, many researchers have examined these properties. Buehler (1959) defined desirable conditional properties of confidence intervals. He based his definitions on a two-person game and showed how such a framework could be used to make conditional statements. Kiefer (1977) approached conditional inference from a slightly different viewpoint, working with a concept of estimated confidence. This approach, while addressing many conditional concerns, is somewhat tangential to what we consider here.

Robinson (1979a,b) further advanced and refined Buehler's theory. In particular, he logically laid out the definitions for a variety of betting procedures, different types of conditioning sets, and more general ideas.

We use the notation $\mathcal{X}$, x, and X to denote a sample space, a point in the space and a random variable, respectively. The symbols Θ and θ will denote a parameter space and a point in the space. The indicator function of a set A is denoted by $I_A(\cdot)$.

As in Buehler (1959) and Robinson (1979a), we state the conditional definitions and properties using the betting game setting for two players, Peter and Paul. The game proceeds as follows: Both players observe X and Peter quotes an interval estimator consisting of the set function C(X) and confidence function $1-\alpha(X)$, which states the degree of confidence in the proposition "$\theta\varepsilon C(X)$." Paul's goal is to bet for or against coverage, with a strategy that can be expressed as a real-valued function K(X), called a betting procedure, with the following properties:

(i) If $K(X) > 0$, Paul places a bet of size K(X) that $\theta\varepsilon C(X)$, risking $1-\alpha(X)$ to win $\alpha(X)$.

(ii) If $K(X) < 0$, Paul places a bet of size $-K(X)$ that $\theta\notin C(X)$, risking $\alpha(X)$ to win $1-\alpha(X)$.

(iii) If $K(X) = 0$, Paul places no bet.

Paul's expected gain from the above betting procedure can be written as

$$E_\theta\left[\{I_{C(X)}(\theta) - [1-\alpha(X)]\}K(X)\right] . \tag{1.3}$$

To evaluate the conditional properties of a statistical procedure is to determine whether or not Paul can find a winning betting procedure. For a given set function C(X) with confidence function $1-\alpha(X)$, denoted by $\langle C(X), 1-\alpha(X)\rangle$, the betting procedure K(X) is said to be:

Semirelevant, if

$$(1.4)\qquad E_\theta\left[\{I_{C(X)}(\theta) - [1-\alpha(X)]\}K(X)\right] \geq 0 \qquad \text{for all } \theta$$

and strictly positive for some θ,

Relevant, if for some $\varepsilon > 0$,

$$(1.5)\qquad E_\theta\left[\{I_{C(X)}(\theta) - [1-\alpha(X)]\}K(X) - \varepsilon|K(X)|\right] \geq 0 \qquad \text{for all } \theta \ .$$

A set estimator that admits a relevant or semirelevant betting procedure admits a winning strategy against it. The distinction between relevant and semirelevant is not trivial, however, and seems to closely correspond to a distinction between proper and generalized Bayes procedures. In general, only proper Bayes procedures are free of semirelevant betting. Generalized Bayes (or limits of Bayes) procedures tend to allow semirelevant but not relevant betting. There are exceptions to these statements however: a rigorous treatment of these relationships is given in Robinson (1979a,b).

There is little restriction on the form of $K(X)$, the function that defines Paul's betting procedure. The one major requirement is that $K(X)$ be bounded since, as Robinson (1979a) points out, unbounded betting is not statistically interesting (one can beat proper Bayes procedures).

We will be particularly concerned with betting procedures that have straightforward statistical interpretations: betting strategies that have either positive or negative bias. A positively biased strategy is one in which $0 \leq K(X) \leq 1$, and corresponds to always betting that the interval covers θ. In particular, if $K(X)$ defines a subset of the sample space, say $K(X) = I_S(X)$ for some set $S \subset \mathcal{X}$, then the relevance of $K(X)$ implies the existence of $\varepsilon > 0$ such that $P_\theta(\theta \varepsilon C(X) | X \varepsilon S) \geq 1-\alpha+\varepsilon \quad \forall\theta$. We have a corresponding definition for negatively biased betting procedure: $-1 \leq K(X) \leq 0$. In a similar manner, a negatively biased relevant set, say $K(X) = -I_S(X)$, gives us that $P_\theta(\theta \varepsilon C(X) | X \varepsilon S) \leq 1-\alpha-\varepsilon$ for some ε and all θ.

Based on work to date, it has been agreed on by some authors (see, for example, Bondar 1977), that a confidence procedure that allows a negatively biased relevant betting procedure should not be endorsed. Existence of such a betting procedure means that the conditional confidence can be bounded strictly below the nominal level for all parameter values, casting doubts on the validity of the frequentist assertion. The existence of positively biased betting procedures is, we feel, not statistically troublesome, only pointing out that the confidence assertion may be conservative (a situation that has long been accepted, although see Seidenfeld, 1979 and 1981, and Mayo, 1981 for another view).

Between 1959 and 1979, a number of results were established. In 1963, Buehler and Fedderson showed that for two normal observations, the usual t-interval for the unknown mean μ allowed positively biased relevant subsets, and Brown (1967) extended this result to any size sample. However, Robinson (1976) showed that no negatively biased relevant subsets of any kind can exist for the t-interval.

Other major results deal with the t-interval and Scheffé's simultaneous confidence intervals. In 1959, Buehler showed that the t-interval, conditional upon rejecting the null hypothesis that the mean is zero, results in the existence of a negatively biased semirelevant subset. The same was shown to be true of Scheffé's simultaneous confidence intervals for the unknown treatment means in an analysis of variance (Olshen, 1973).

A few results are known about statistical procedures involving two normal populations. Fisher (1956b) showed that negatively biased relevant subsets exist for Welch's (1947) intervals. Robinson (1976) showed that no negatively biased relevant subsets exist for the Behrens-Fisher two-means interval.

Robinson's (1979b) paper dealing with Pitman estimators established results for the location parameter case. It was shown that the best invariant interval allows no relevant subsets. Thus, it has recommendable properties both conditionally and unconditionally. Since a scale parameter family of distributions can be transformed into a location parameter family of distributions by a log transformation, this case of an interval estimator of σ^2 reduces to that for a location parameter. That is, no relevant subsets exist for variance intervals if the mean is known.

In this paper we deal with the more practical situation of interval estimators for the normal variance when the mean is unknown, and consider estimators of the form

$$C(s^2) = \left\{\sigma^2 : \sigma^2 \in (\nu s^2/b_\nu, \nu s^2/a_\nu)\right\} \tag{1.6}$$

where ν is the degrees of freedom of $s^2 = \Sigma(X_i-\bar{X})^2/\nu$ and a_ν and b_ν satisfy $P(a_\nu < \chi^2_\nu < b_\nu) = 1-\alpha$, where χ^2_ν is a chi-squared random variable with ν degrees of freedom. Note that (1.6) can define either a one-sided or two-sided interval.

For the one-sided intervals, the constants are uniquely determined by the probability constraint, but the situation is more complicated for two-sided intervals, where a second constraint is needed to uniquely specify an interval (see, e.g., Tate and Klett (1959) for details).

Intervals of the form (1.6) depend on $\bar{x}$ only through its appearance in s^2, and more recent works (Cohen, 1972 and Shorrock, 1982) have shown that one can uniformly increase coverage probability by allowing a more explicit dependence on $\bar{x}$. These intervals will not be dealt with here, but will be the topic of future work.

In the next section we deal with the general interval estimator (1.6) and show that there are no relevant betting procedures against this interval. This result applies to any choice of a_ν and b_ν that satisfy $P(a_\nu < \chi^2_\nu < b_\nu) = 1-\alpha$ and, hence, shows that most commonly used intervals are free from major conditional defects.

The situation with respect to semirelevant betting is more complex, however, and in Section 3 we examine two-sided intervals in some detail. We show that a common interval, the equal-tailed interval, allows negatively biased semirelevant (NBSR) betting. However, two lesser-used intervals are free from this defect.

In Section 4 we consider one-sided confidence intervals, both upper and lower, where we show that no NBSR betting procedures exist for the one-sided upper confidence interval while for the one-sided lower confidence interval there does exist an NBSR betting procedure. This result is, perhaps, contrary to intuition, since the lower confidence interval results from inversion of the UMP test of $H_0 : \sigma \leq \sigma_0$, and no UMP test exists for the opposite case (although the test is UMPU). Section 5 contains some discussion and conclusions and there is also an Appendix with some technical details.

2. Relevant betting procedures. This section contains the main results for relevant betting procedures against confidence intervals of the form described in (1.6). We show that no relevant betting procedures exist for these intervals, showing that they are free of serious conditional defects.

As mentioned in Section 1, procedures that possess good conditional properties are those that are some type of Bayes procedures, either proper Bayes or limits of Bayes procedures. Thus, the goal of the proof is to show that the usual interval estimators of variance, those of the form in (1.6), are limits of Bayes procedures. We use a hierarchical structure of priors that allows us to accomplish our goal.

THEOREM 2.1. *Let* $X_1,\cdots,X_n$ *be* iid $N(\mu,\sigma^2)$ *with* μ *and* σ^2 *both unknown. Then for intervals of the form* (1.6), *no relevant betting procedures exist for the confidence procedure* $\langle C(s^2), 1-\alpha\rangle$, *where* $1-\alpha = P(a_\nu < \chi^2_\nu < b_\nu)$.

PROOF. Suppose that a relevant betting procedure exists, then, from (1.5) there exists an $\varepsilon > 0$ and a betting procedure, $K(\bar{X},s^2)$, such that

$$E\left\{\left[I_{C(s^2)}(\sigma^2) - (1-\alpha)\right]K(\bar{X},s^2)\right\} \geq \varepsilon\ E|K(\bar{X},s^2)| \tag{2.1}$$

for all μ and σ^2, with strict inequality for some (μ,σ^2), where the expectation is taken with respect to the joint density of $\bar{X}$ and s^2. Multiply both sides of (2.1) by the prior distribution

$$(2.2) \qquad \pi(\mu,\sigma^2|r,a) = \left(\frac{1}{2\pi r\sigma^2}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(\mu^2/2r\sigma^2)} \frac{1}{(\sigma^2)^a}$$

and integrate with respect to both μ and σ^2. If $K(\bar{X},s^2)$ is a relevant betting procedure, then it follows that

$$(2.3) \qquad \begin{aligned} &\int_0^\infty \int_{-\infty}^\infty E\left\{\left[I_{C(s^2)}(\sigma^2) - (1-\alpha)\right]\bar{K}(X,s^2)\right\} \pi(\mu,\sigma^2|r,a)d\mu d\sigma^2 \\ &\geq \varepsilon \int_0^\infty \int_{-\infty}^\infty E|K(\bar{X},s^2)|\pi(\mu,\sigma^2|r,a)d\mu d\sigma^2 \quad . \end{aligned}$$

The proof proceeds by showing that (2.3) can be violated in the limit, as $r\to\infty$ and $a\to\frac{1}{2}$. We first normalize by multiplying both sides of (2.3) by $(nr+1)^{\frac{1}{2}}$, and consider the inequality

$$(2.4) \qquad \begin{aligned} &\lim_{\substack{r\to\infty \\ a\to\frac{1}{2}}} (nr+1)^{\frac{1}{2}} \int_0^\infty \int_{-\infty}^\infty E\left\{\left[I_{C(s^2)}(\sigma^2) - (1-\alpha)\right]K(\bar{X},s^2)\right\}\pi(\mu,\sigma^2|r,a)d\mu d\sigma^2 \\ &\geq \varepsilon \lim_{\substack{r\to\infty \\ a\to\frac{1}{2}}} (nr+1)^{\frac{1}{2}} \int_0^\infty \int_{-\infty}^\infty E|K(\bar{X},s^2)|\pi(\mu,\sigma^2|r,a)d\mu d\sigma^2 \quad . \end{aligned}$$

If we can contradict (2.4), this in turn will contradict (2.1) and, hence, no relevant betting procedure can exist.

We consider two separate cases, depending on whether

$$(2.5) \qquad \int_{-\infty}^\infty \int_0^\infty \frac{|K(\bar{x},s^2)|}{s^2} ds^2 d\bar{x}$$

is finite or infinite.

CASE I: If (2.5) is finite then we show that the LHS of (2.4) is identically equal to zero. Expanding and interchanging the order of integration allows us to write the LHS OF (2.4) as

$$\text{(2.6)} \qquad \lim_{\substack{r\to\infty \\ a\to\frac{1}{2}}} \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} \left[I_{C(s^2)}(\sigma^2) - (1-\alpha)\right] K(\bar{x},s^2)\pi(\mu|\bar{x},s^2,\sigma^2,r,a)$$
$$\times\ \pi(\sigma^2|\bar{x},s^2,r,a)m(\bar{x},s^2|r,a)d\mu d\sigma^2 ds^2 d\bar{x} \quad ,$$

where

$$\pi(\mu|\bar{x},s^2,\sigma^2,r,a) = N\left(\frac{nr\bar{x}}{nr+1}\ ,\ \frac{r\sigma^2}{nr+1}\right),$$

$$\pi(\sigma^2|\bar{x},s^2,r,a) = IG\left(\frac{\nu+1}{2} + a-1,\ \frac{2(nr+1)}{\nu(nr+1)s^2+n\bar{X}^2}\right) \quad ,$$

the inverted gamma distribution and,

$$m(\bar{x},s^2|r,a) = \left(\frac{n}{2\pi(nr+1)}\right)^{\frac{1}{2}} \frac{\nu^{\nu/2}(s^2)^{\nu/2-1}}{\Gamma\left(\frac{\nu}{2}\right)^{\nu/2}}\ \Gamma\left(\frac{\nu+1}{2} + a-1\right)\left(\frac{2(mr+1)}{\nu(nr+1)s^2+n\bar{x}^2}\right)^{\frac{\nu+1}{2}+a-1} .$$

We first integrate out the density of μ, then note that

$$\text{(2.7)} \qquad (nd+1)\ m(\bar{x},s^2|r,a) \le \frac{M}{(s^2)^{\frac{1}{2}+a}} \ ,$$

for some constant M. The finiteness of (2.5), together with (2.7) allows us to apply Lebesgue's Dominated Convergence to pull the limit inside the integral in (2.6). It is then straightforward to calculate

$$\int_{\nu s^2/b_\nu}^{\nu s^2/a_\nu} \lim_{\substack{d\to\infty \\ a\to\frac{1}{2}}} \pi(\sigma^2|\bar{x},s^2,r,a)d\sigma^2 = 1-\alpha \ ,$$

which gives us zero for the value of (2.6). This contradicts (2.4) and hence our assumption of a relevant betting procedure existing.

CASE II: If (2.5) is infinite then we only need show that the LHS of (2.4) remains finite as $r\to\infty$ and $a\to\frac{1}{2}$. To deal with this integral we split the range of integration into three pieces: $A_1 = \{(\bar{x},s^2):|x| \le N,\ s^2 < \delta\}$, $A_2 = \{(\bar{x},s^2):|\bar{x}| \le N,\ s^2 \ge \delta\}$, and $A_3 = \{(\bar{x},s^2):|\bar{x}| > N,\ s^2 > 0\}$, where $N<\infty$ and $\delta>0$ are constants.

The integral over A_2 can be dealt with by an argument similar to that used in Case I, hence, in the limit, it it equal to zero. For the integral over A_3, by using the bounds $|I_{C(s^2)}(\sigma^2) - (1-\alpha)| \le 1$, $|K(X,\bar{s}^2)| \le 1$, and integrating out μ, σ^2, and s^2, we have that the integral over A_3 is bounded above by

$$\int_N^\infty m(\bar{x}|r,a)d\bar{x} \quad ,$$

where $m(\bar{x}|r,a)$, the marginal of $\bar{X}$, is an improper density;

$$(2.8) \qquad m(\bar{x}|r,a) = \left(\frac{1}{\pi}\right)^{\frac{1}{2}}\left(\frac{2(nr+1)}{n}\right)^{a-1}\left(\frac{1}{\bar{x}^2}\right)^{\frac{2a-1}{2}} \Gamma\left(\frac{2a-1}{2}\right) .$$

Straightforward calculation gives

$$(2.9) \qquad \lim_{\substack{r\to\infty \\ a\to\frac{1}{2}}} (nr+1)^{\frac{1}{2}} \int_N^\infty m(\bar{x}|r,a)d\bar{x} = \lim_{\substack{r\to\infty \\ a\to\frac{1}{2}}} \frac{2a(nr+1)^{a-\frac{1}{2}}}{N^{2a}} .$$

By letting a go to $\frac{1}{2}$ faster than r goes to infinity (2.9) can be made finite and, hence, so can the limit of the integral over A_3.

Next consider the integral over A_1. Making the transformation $u = s^2/\bar{x}^2$ $t = \bar{x}^2$, this integral can be expressed as

$$(2.10) \quad \left[\int_0^{\delta/N^2}\int_0^{N^2}\int_0^\infty + \int_{\delta/N^2}^\infty\int_0^{\delta/u}\int_0^\infty\right]\left[I_{C(s^2)}(\sigma^2) - (1-\alpha)\right] \pi(\sigma^2|t,u,r,a)K(t,u)$$
$$\times\ m(t,u|r,a)d\sigma^2 dt du ,$$

where

$$\pi(\sigma^2|t,u,r,a) = IG\left(\frac{\nu+1}{2} + a-1,\ \frac{2(nr+1)}{t[\nu(nr+1)u+n]}\right) ,$$

and

$$m(t,u|r,a) = M(nr+1)^{\frac{\nu}{2}+a-1} \frac{1}{t^a} \frac{u^{\nu/2-1}}{[\nu(nr+1)u+n]^{\frac{\nu+1}{2}+a-1}} ,$$

and M is a positive constant that can be ignored. We consider the second triple integral first. Since $|I_{C(s^2)}(\sigma^2)-(1-\alpha)| \le 1$ and $|K(\bar{X},s^2)| \le 1$, substitute these bounds and integrate out σ^2 and t in the second term of (2.10). Multiplying by $(nr+1)^{\frac{1}{2}}$, the limit of this second term is finite if

$$(2.11) \qquad \lim_{\substack{r\to\infty \\ a\to\frac{1}{2}}} \int_{\delta/N^2}^{\infty} M \frac{u^{\frac{\nu}{2}+a-2}}{\left(\nu u + \frac{n}{nr+1}\right)^{\frac{\nu+1}{2}+a-1}} du < \infty .$$

The denominator of the integrand can be bounded below by $(\nu u)^{\frac{\nu}{2}+a-\frac{1}{2}}$ and again letting $a \to \frac{1}{2}$ faster than $r \to \infty$ shows that the limit in (2.11) is finite.

Thus, we are left to show that the first triple integral in (2.10) is finite. We will show that as $r\to\infty$ and $a\to\frac{1}{2}$, the integral goes to zero. Using the transformation $\omega = \nu ut/\sigma^2$, and writing $1-\alpha$ as a chi-squared integral, the first integral in (2.10) can be written as

$$(2.12) \qquad \int_0^{\delta/N^2} \int_0^{N^2} \left[\int_{a_\nu}^{b_\nu} \frac{\beta^{\frac{\nu+1}{2}+a-1}}{\Gamma\left(\frac{\nu+1}{2}+a-1\right)} \omega^{\frac{\nu+1}{2}+a-2} e^{-d\omega} d\omega - \int_{a_\nu}^{b_\nu} \frac{1}{\Gamma\left(\frac{\nu}{2}\right) 2^{\nu/2}} \omega^{\frac{\nu}{2}-1} e^{-\frac{1}{2}\omega} d\omega \right]$$

$$\times K(t,u)m(t,u|r,a)dtdu ,$$

where $\beta=\frac{1}{2}[\nu(nr+1)u+n]/[\nu(nr+1)u]$. Finally, we note that

$$\lim_{\substack{r\to\infty \\ a\to\frac{1}{2}}} \beta^{\frac{\nu+1}{2}+a-1} e^{-\beta\omega} = \frac{e^{-\omega/2}}{2^{\nu/2}} ,$$

and since this is a bounded continuous function of u on the compact set $0 \le u \le \delta/N^2$, the convergence is uniform in u. Therefore, for sufficiently large r and a close to $\frac{1}{2}$, the difference of the two integrals in (2.12) can be made arbitrarily small independent of u. Therefore, as $r\to\infty$ and $a\to\frac{1}{2}$, (2.12) goes to zero.

Combined with all previous results, we find that the LHS of (2.4) is finite, which contradicts our original assumption. Therefore, a relevant betting procedure cannot exist for the variance interval estimator of the form (1.6). ||

Though we have not yet addressed the subject of whether semirelevant betting procedures exist for these intervals, certainly the absence of relevant betting procedure is a favorable property of these intervals, and shows that the variance intervals of (1.6) have acceptable conditional properties. In particular, the absence of negatively biased relevant procedures implies that the conditional confidence levels cannot be bounded strictly below $1-\alpha$ for all parameter values.

3. Semirelevant betting against two-sided intervals. In this section we focus on two-sided intervals, starting with the equal-tailed interval, I_{ET}, for the normal variance. The interval is given by (1.6) with the additional requirement that a_ν and b_ν satisfy

$$(3.1) \qquad P(\chi^2_\nu \le a_\nu) = \alpha/2, \qquad P(\chi^2_\nu \ge b_\nu) = \alpha/2 \quad .$$

The results of the preceding section show that there are no relevant betting procedures against this interval. However, in this section we show the existence of a negatively biased semirelevant subset. That is, we find a subset A of the sample space satisfying

$$(3.2) \qquad P[\sigma^2 \in I_{ET} | (\bar{X}, s^2) \in A] \le 1-\alpha \quad \forall \quad \mu, \sigma^2 \quad .$$

The set A identifies a portion of the sample space in which the conditional probability of coverage can be bounded below 1-α, showing that I_{ET} is not the most desirable interval based on conditional evaluations. Define

$$(3.3) \qquad \begin{aligned} K(\bar{X}, s^2) &= -1 \qquad \text{if} \qquad \bar{X}^2/s^2 < q_0 \\ &= \;\; 0 \qquad \text{otherwise} \quad . \end{aligned}$$

There exists a $q_0 > 0$ such that

$$(3.4) \qquad E\left\{ \left[I_{ET}(s^2) - (1-\alpha) \right] K(\bar{X}, s^2) \right\} \ge 0 \quad \forall \mu, \sigma^2 \quad ,$$

with strict inequality for some parameter values, showing that (3.3) defines a negatively biased semirelevant betting procedure. The details are given in the Appendix.

It is tempting to apply some intuition as to why (3.3) should produce a negatively biased betting procedure for I_{ET}. The procedure bets against coverage if $\bar{X}^2/s^2$ < constant, which can be interpreted as betting against coverage if s^2 is "large." Other authors (Brown, 1967; Robinson, 1979b) have noted that s^2 has a tendency to "overestimate" σ^2, so we might interpret the working of the betting procedure (3.3) as follows: If s^2 is large, it is an overestimate of σ^2, and the interval I_{ET} is too far away from zero to cover σ^2.

Other common intervals, for example the minimum length interval, are closer to zero than the equal-tailed interval, and do not suffer from the same conditional defects as I_{ET}. We consider the performance of two other two-sided intervals: I_{ML}, the minimum length interval, and I_{SU}, the shortest unbiased interval.

The interval I_{ML} is constructed by minimizing the length of the interval, $a_\nu^{-1} - b_\nu^{-1}$, subject to the constraint that $P(a_\nu < \chi_\nu^2 < b_\nu) = 1-\alpha$. The solution is to choose a_ν and b_ν to satisfy $f_{\nu+4}(a_\nu) = f_{\nu+4}(b_\nu)$, where we still use the notation f_q for the density of a χ_q^2 random variable. The shortest unbiased (or Neyman-shortest) interval is constructed by minimizing length over all unbiased confidence intervals, and results in the constraint $f_{\nu+2}(a_\nu) = f_{\nu+2}(b_\nu)$.

These two constraints for I_{ML} and I_{SU} are similar in that they define highest density regions from chi-squared distributions, giving us an interpretation of I_{ML} and I_{SU} as Bayes highest posterior density regions against (improper) priors which result in $\chi^2_{\nu+4}$ and $\chi^2_{\nu+2}$ posterior densities, respectively. It will turn out that this property is enough to insure the nonexistence of NBSR betting. We now show that no NBSR betting procedures exist for I_{ML} and I_{SU}. Further, we show that no positively biased semirelevant betting procedures exist for I_{SU}.

The method of proof mimics the proof of Theorem 2.1. We assume that an NBSR betting procedure exists and then we reach a contradiction. If there exists an NBSR betting procedure, then by definition there exists

$K(\bar{X},s^2) < 0$ such that

$$(3.5) \qquad E\left\{\left[I_{C(s^2)}(\sigma^2) - (1-\alpha)\right]K(\bar{X},s^2)\right\} \geq 0 \quad \forall\mu,\sigma^2 \;,$$

with strict inequality for at least one (μ,σ^2). We will show that the expression in (3.5) must be negative for some parameter values, contradicting our assumption of the existence of an NBSR procedure.

THEOREM 3.1. *Let* $X_1,\cdots,X_n \sim$ iid $N(\mu,\sigma^2)$ *with* μ *and* σ^2 *both unknown. For the intervals* I_{ML} *and* I_{SU} *no negatively biased semirelevant betting procedures exist.*

PROOF. Assume that a negatively biased semirelevant betting procedure exists, that is, assume there exists a function $K(\bar{X},s^2)$ with $-1 \leq K(\bar{X},s^2) \leq 0$ satisfying (3.5). Multiply (3.5) by

$$(3.6) \qquad \pi(\mu,\sigma^2\,|\,r) = \frac{1}{(\sigma^2)^r}\,d\mu d\sigma^2$$

and integrate with respect to both μ and σ^2. If $K(\bar{X},s^2)$ is an NBSR betting procedure, then

$$(3.7) \qquad \int_0^\infty \int_{-\infty}^\infty E\left\{\left[I_{C(s^2)}(\sigma^2) - (1-\alpha)\right]K(\bar{X},s^2)\right\}\pi(\mu,\sigma^2\,|\,r)d\mu d\sigma^2 \geq 0 \quad .$$

Manipulating the densities in a manner similar to that used in the proof of Theorem 2.1, and integrating out μ, will establish that the LHS of (3.7) can be written as

$$(3.8) \qquad \int_{-\infty}^\infty \int_0^\infty K(\bar{x},s^2)\left[\int_{\nu s^2/b_\nu}^{\nu s^2/a_\nu} \pi(\sigma^2\,|\,\bar{x},s^2,r)d\sigma^2 - (1-\alpha)\right]m(\bar{x},s^2\,|\,r)ds^2 d\bar{x} \quad ,$$

where

$$\pi(\sigma^2|\bar{x},s^2,r) = IG\left(\frac{\nu}{2} + r-1,\ \frac{2}{\nu s^2}\right) \tag{3.9}$$

$$m(\bar{x},s^2|r) = \left(\frac{\nu}{2}\right)^{\frac{\nu}{2}} \frac{\Gamma\left(\frac{\nu}{2}+r-1\right)}{\Gamma\left(\frac{\nu}{2}\right)} (s^2)^{\frac{\nu}{2}-1} \left(\frac{2}{\nu s^2}\right)^{\frac{\nu}{2}+r-1}$$

Each of the intervals I_{ML} and I_{SU} will be handled separately, as each will have a specific value that r must approach. Both I_{ML} and I_{SU} are defined by

$$f_{\nu+k}(a_\nu) = f_{\nu+k}(b_\nu) \tag{3.10}$$

where k=4 for I_{ML} and k=2 for I_{SU}. We want to specify the values of r so that the marginal distribution of $\sigma^2|\bar{X},s^2,r$ of (3.9) has its first parameter, $\nu/2 + r-1$, equal to $\nu+k$. In case of I_{ML} we need r=3 , while for I_{SU}, r=2.

The same limiting forms of posterior densities exist in Theorem 3.1 as in Theorem 2.1. Therefore, we can pass to the limit in equation (3.8). The key step in the proof is to show that the limit of the bracketed term in (3.8) is greater than or equal to zero. If we show this, then we contradict our assumption of a negatively biased semirelevant betting procedure existing.

To show that the limit of the bracketed term is greater than or equal to zero, we use the following lemma.

LEMMA 3.1.

$$P(a_\nu < \chi^2_\nu < b_\nu) = 2[f_{\nu+2}(b_\nu) - f_{\nu+2}(a_\nu)] + P(a_\nu < \chi^2_{\nu+2} < b_\nu) \tag{3.11}$$

where χ^2_ν represents a chi-squared random variable with ν degrees of freedom and density $f_\nu(\cdot)$.

PROOF. Integrate the LHS of (3.11) by parts. ||

Using Lemma 3.1, we see that for I_{SU}

(3.12) $$(1-\alpha) = P(a_{SU} < \chi^2_\nu < b_{SU}) = P(a_{SU} < \chi^2_{\nu+2} < b_{SU}) \quad ,$$

for a_{SU} and b_{SU} satisfying $f_{\nu+2}(a_{SU}) = f_{\nu+2}(b_{SU})$. For I_{ML}, applying Lemma 3.1 twice, we see that

(3.13) $$\begin{aligned}(1-\alpha) = P(a_{ML} < \chi^2_\nu < b_{ML}) &= 2[f_{\nu+2}(b_{ML}) - f_{\nu+2}(a_{ML})] \\ &\quad + P(a_{ML} < \chi^2_{\nu+4} < b_{ML})\end{aligned}$$

for a_{ML} and b_{ML} satisfying $f_{\nu+4}(a_{ML}) = f_{\nu+4}(b_{ML})$.

It is also easy to verify (again integrating by parts) that

(3.14) $$P(a_{ML} < \chi^2_{\nu+4} < b_{ML}) > P(a_{ML} < \chi^2_\nu < b_{ML}) = 1-\alpha \quad .$$

Reconsider the bracketed term of (3.8) where a_ν and b_ν are defined to yield I_{ML} or I_{SU}. If we let $t = \nu s^2/\sigma^2$ then this term is equivalent to

(3.15) $$P(a_\nu \le \chi^2_{\nu+2r-1} \le b_\nu) - (1-\alpha) \ .$$

If we let r go to 3 for I_{ML}, and to 2 for I_{SU}, it follows that the limit of (3.15) is greater than or equal to zero. Using (3.14), we have for I_{ML} that the limit of (3.15), as $r \to 3$ is strictly less than zero. For I_{SU}, using (3.12), the limit of (3.15), as $r \to 2$ is identically equal to zero. Therefore, we have reached a contradiction. No negatively biased betting procedures exist for the intervals I_{ML} and I_{SU}. ||

The absence of NBSR betting procedure for I_{ML} is clearly a plus for this interval. However, our result is actually stronger for I_{SU}. Since we have shown that the limit of (3.15) is identically zero, this implies that no *positively* biased semirelevant betting procedures exist for I_{SU}.

4. Semirelevant betting against one-sided confidence intervals. We now turn our attention to one-sided confidence intervals, which are slightly easier to deal with. We consider one-sided upper confidence intervals

$$C_1(s^2) = \left\{\sigma^2 : \sigma^2 \in (0, \nu s^2/a_\nu)\right\} \tag{4.1}$$

and one-sided lower confidence intervals

$$C_2(s^2) = \left\{\sigma^2 : \sigma^2 \in (\nu s^2/b_\nu, \infty)\right\}$$

where s^2 is the sample variance, ν equals the degrees of freedom associated with s^2, and a_ν and b_ν are the appropriate $1-\alpha$ cutoff points of the χ^2_ν distribution. Note that in the one-sided case, the requirement of confidence level $1-\alpha$ uniquely determines the constant a_ν or b_ν.

Theorem 2.1 applies to C_1 and C_2, and hence no relevant betting procedures exist. However, an argument similar to that in Section 3 produces an NBSR betting procedure for $C_2(s^2)$, the lower confidence interval, and a modification of the proof of Theorem 3.1 shows that no NBSR procedures exist for $C_1(s^2)$, the upper confidence interval.

THEOREM 4.1. *Let* $X_1, \cdots, X_n \sim$ iid $N(\mu, \sigma^2)$ *with both mean and variance unknown. Then there exists* $q^* > 0$ *such that*

$$\begin{aligned} K(\bar{X}, s^2) &= -1 \qquad \text{if } \bar{X}^2/s^2 < q^* \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{4.3}$$

is a negatively biased semirelevant betting procedure for the one-sided lower confidence interval $C_2(s^2)$.

PROOF. The proof follows the argument given in the Appendix with the following changes. Expression (A1) must be modified to

$$(1-\alpha)P\left(\frac{\bar{X}^2}{s^2} < q^*\right) - P\left(\frac{\bar{X}^2}{s^2} < q^*, \frac{\nu s^2}{a_\nu} < \sigma^2 < \infty\right)$$

$$= (1-\alpha)P\left(\frac{\bar{X}^2}{\sigma^2} < q\chi^2_\nu\right) - P\left(\frac{\bar{X}^2}{\sigma^2} < q\chi^2_\nu, 0 < \chi^2_\nu < b_\nu\right)$$

where $q = q^*n/\nu$ with n and ν being the degrees of freedom associated with $\bar{X}$ and s^2, respectively. With similar modifications to the subsequent equations in the Appendix, we can show the existence of an NSBR betting procedure if we show that we can choose $q > 0$ so that

$$P\left(0 < \chi^2_{\nu+1+2w} < b_\nu(q+1)\right) - (1-\alpha) \leq 0$$

for $w=0,1,\cdots$. Since $P(\chi^2_\nu < b_\nu) = (1-\alpha)$, and χ^2 is stochastically increasing in w, it follows that for $w=0,1,\cdots$,

$$P(\chi^2_{\nu+1+2w} < b_\nu) \leq P(\chi^2_{\nu+1} < b_\nu) < 1-\alpha \quad .$$

This implies there exists a $q' > 0$ such that

$$P\left(\chi^2_{\nu+1} \leq b_\nu(q'+1)\right) \leq 1-\alpha \qquad \text{for } 0 < q < q' \quad .$$

Therefore, (4.3) is an NSBR betting procedure. ‖

THEOREM 4.2. *Let* $X_1,\cdots,X_m \sim$ iid $N(\mu,\sigma^2)$ *with both mean and variance unknown. No negatively biased semirelevant betting procedures exist for the one-sided upper confidence interval* $C_1(s^2)$.

PROOF. The proof of Theorem 3.1 will apply to Theorem 4.2 if we can show that the limit of

$$\left[(1-\alpha) - \int_0^{\nu s^2/a_\nu} \pi(\sigma^2 \mid \bar{x}, s^2, r)\, d\sigma^2\right] \tag{4.4}$$

is negative, where $\pi(\sigma^2|\bar{x},s^2,r)$ is defined in (3.9). If we make the transformation $t = \nu s^2/\sigma^2$, then $t \sim \chi^2_{\nu+2r-1}$. With this transformation (4.4) can be written as

$$(4.5) \qquad (1-\alpha) - P(\chi^2_{\nu+2r-1} > a_\nu) \quad .$$

Since $P(\chi^2_\nu > a_\nu) = (1-\alpha)$, and χ^2_ν is stochastically increasing in ν, (4.5) is negative for any $r \geq 1$. With this modification, we have shown that the limit of (4.4) as $r \to 1$, is negative. Thus, the proof of Theorem 3.1 applies in this case and no NBSR betting procedures exist for $C_1(s^2)$. ‖

The same intuition applied to the equal-tailed interval can be applied here. The form of the negatively biased semirelevant betting procedure, (4.3), implies that we are betting against $C_2(s^2)$, the lower confidence interval, when s^2 is large. This suggests that the interval is overestimating the lower confidence limit. This lower limit is, again, too far from zero and we can win by betting against the interval when s^2 is large. For the one-sided upper confidence interval our intuition is consistent. For large observed s^2, the interval $C_1(s^2)$ guards against overestimation of σ^2 just by its form: it covers everything from zero to the upper limit.

We have a somewhat remarkable dichotomy between usual Neyman-Pearson optimality and conditional properties in the one-sided case. The lower confidence interval, $C_2(s^2)$, can be derived by inverting the UMP test of $H_0 : \sigma \leq \sigma_0$ (making it uniformly most accurate), but no UMP test exists for $H_0 : \sigma \geq \sigma_0$, so $C_1(s^2)$ cannot be derived in a similar manner (although it is uniformly most accurate *unbiased*). So, the interval with the stronger classical property, $C_2(s^2)$, actually has a weaker conditional property. We have no rigorous explanation for this, other than reiterating that good conditional properties are possessed by intervals with some type of Bayesian interpretation (as I_{ML} and I_{SU}) and classical, unconditional optimality gives no conditional guarantees.

5. Discussion. Faced with a choice of which interval estimator to use, an experimenter must take many factors into consideration, including both conditional and unconditional performance. The results of Section 2 show that all intervals of the form (1.6) are free from major conditional defects, so a choice will be based on more subtle conditional performance and unconditional performance.

In the one-sided case, of course, the choice of interval is dictated by the problem at hand; however, the two-sided case is more complex. The equal-tailed interval has the advantage of ease of construction: tables in common textbooks will often suffice. However, ease of construction seems to be its only advantage. The equal-tailed interval possesses no optimality properties, either conditional or unconditional.

Both I_{ML} and I_{SU} have attractive unconditional properties. In addition to being minimum length, Cohen (1972) shows that I_{ML} is strongly admissible [among procedures of the form of (1.6)], using the definitions of Joshi (1970). This result also implies that I_{ML} is minimax among these procedures. The interval I_{SU}, on the other hand, is shortest unbiased (all the others are biased). It can be constructed by inverting the uniformly most powerful unbiased test and is therefore uniformly most accurate unbiased, hence minimizing the probability of covering false values (Lehmann 1959, Chapter 5). I_{SU} can also be obtained in two other manners. It can be derived by assuming invariance of the confidence limits with respect to the usual translation group and considering uniformly most powerful invariant tests. In addition, I_{SU} results if we minimize the ratio of the right endpoints to the left endpoint. This particular measure of size may be more appropriate than the usual length for the scale parameter case (see, e.g., Berger 1980, Chapter 6).

When choosing a two-sided confidence interval we have to weigh both unconditional and conditional characteristics. If overall length of the interval is of importance (as it is in most location parameter cases), then the combination of strong admissibility and the nonexistence of negatively biased semirelevant betting procedures would suggest I_{ML}. However, if the minimization of the ratio of right endpoint to left endpoint (the natural measure of size for scale parameters) and unbiasedness are important, the exceptionally good conditional properties suggests that I_{SU} is the superior interval.

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Appendix: A negatively biased semirelevant betting procedure against the equal-tailed interval. Using the definition of $K(\bar{X},s^2)$ in (3.3), we can write the expectation in (3.4) as

$$\text{(A1)} \qquad (1-\alpha)\ P\left(\frac{\bar{X}^2}{\sigma^2} < q\chi^2_\nu\right) - P\left(\frac{\bar{X}^2}{\sigma^2} < q\chi^2_\nu,\ a_\nu < \chi^2_\nu < b_\nu\right) \quad ,$$

where $q = q_0 n/\nu$ with n equaling the number of observations in $\bar{X}$ and ν equaling the degrees of freedom of s^2 as before.

The distribution of $\bar{X}^2/\sigma^2$ is a non-central χ^2 with one degree of freedom and non-centrality parameter $\frac{1}{2}\mu^2/\sigma^2$. Let $\tau = \frac{1}{2}\mu^2/\sigma^2$ and let $\chi^2_1(\tau) = \bar{X}^2/\sigma^2$. Then (A1) can be written as

$$\text{(A2)} \qquad (1-\alpha)\ P\left(\chi^2_1(\tau) < q\chi^2_\nu\right) - P\left(\chi^2_1(\tau) < q\chi^2_\nu,\ a_\nu < \chi^2_\nu < b_\nu\right) .$$

Expression (A2) can be evaluated further by making use of the well-known fact that if $Z|W \sim \chi^2_{k+2W}$ and $W \sim$ Poisson (θ), then $Z \sim \chi^2_k(2\theta)$. We can now write (A2) as

$$\text{(A3)} \quad \sum_{w=0}^{\infty}\left[(1-\alpha)\ P\left(\chi^2_{1+2w} < q\chi^2_\nu\right) - P\left(\chi^2_{1+2w} < q\chi^2_\nu,\ a_\nu < \chi^2_\nu < b_\nu\right)\right] P(W=w)$$

$$\overset{\text{def'n.}}{=} \quad M(\tau) \ .$$

Since $M(\tau) = 0$ when $q = 0$, if we can show that $q_0 > 0$ can be chosen independent of w so that the quantity in square brackets is nonnegative for all w and $0 < q < q_0$, then it will follow that $M(\tau) \geq 0$ for all τ. To meet this end, we will show that q_0 can be chosen, independent of w, such that the derivative of $M(\tau)$ is nonnegative, uniformly in τ, for $0 < q < q_0$.

Denoting the quantity in square brackets in (A3) as M_w, we have after a little algebra

$$M_w = (1-\alpha)\int_0^\infty P(t/q < \chi^2_\nu) f_{1+2w}(t)dt - (1-\alpha)\int_0^{qa_\nu} f_{1+2w}(t)dt$$

$$- \int_{qa_\nu}^{qb_\nu} P(t/q < \chi^2_\nu < b_\nu) f_{1+2w}(t)dt \ .$$

where $f_{1+2w}(t)$ is the central χ^2_{1+2w} density. Taking the derivative of M_w with respect to q and making the transformation $y = t/q$ yields

$$\text{(A4)} \qquad \frac{\partial}{\partial q} M_w = (1-\alpha) \int_0^\infty f_\nu(y) y f_{1+2w}(qy)dy - \int_{a_\nu}^{b_\nu} f_\nu(y) y f_{1+2w}(qy)dy \ .$$

Since we are only interested in the sign of the derivative, the positive constants involved in the density functions in (A4) don't matter. (They are the same for each integral.) We can also normalize appropriately so that $f_\nu(y)yf_{1+2w}(qy)$ is a gamma density with parameters $\nu+1+2w$ and $\frac{1}{2}(1+q)$, denoted by $g_w(y)$. With this normalization, the first integral in (A4) is equal to one and we have the sign of (A4) equal to the sign of

$$(1-\alpha) - \int_{a_\nu}^{b_\nu} g_w(y)dy \ .$$

It follows that if $T \sim g_w(y)$, then $(1+q)T \sim \chi^2_{\nu+1+2w}$, so that the sign of $(\partial/\partial q)\ M_w$ is equal to the sign of

$$\text{(A5)} \qquad (1-\alpha) - P\left(a_\nu(q+1) < \chi^2_{\nu+1+2w} < b_\nu(q+1)\right) \ .$$

To show that the derivative of $M(\tau)$ is nonnegative, we want to find a value of q, say q_0, to make (A5) negative for all $w=0,1,\cdots$. First, choose a value q', then find the smallest value of w for which $P(\chi^2_{\nu+1+2w}<b_\nu(q'+1))<1-\alpha$. Call this value w_0. We have that

$$P[a_\nu(q+1)<\chi^2_{\nu+1+2w}<b_\nu(q+1)]< P\left(\chi^2_{\nu+1+2w}< b_\nu(q+1)\right) < 1-\alpha \qquad \text{for all } w > w_0,\ q<q',$$

since $\chi^2_{\nu+1+2w}$ is stochastically increasing in w.

Therefore, we now have a finite problem, which can be solved numerically. This is, for $w=0,\cdots,w_0$, find a value $q_0 < q'$ such that

$$\text{(A6)} \qquad P\left(a_\nu(q_0+1) < \chi^2_{\nu+1+2w} < b_\nu(q_0+1)\right) < 1-\alpha \ .$$

If we can find such a q_0, this means that the sign of the derivative of M_w is nonnegative for all integers w, and implies the derivative of $M(\tau)$ remains above zero as a function of q, that is $M(\tau) \geq 0$ for all τ and q such that $0 \leq q \leq q_0$. Thus (3.3) is a negatively biased semirelevant betting procedure, or in terms of conditional probabilities,

$$P[\sigma^2 \in I_{ET} | X^2/s^2 < q_0] \leq 1-\alpha \ , \qquad \text{for all } \mu, \sigma^2 \ .$$

Table 1 contains the numerical verification that NBSR subsets exist for I_{ET} of (2.2). For $\nu = 4$, 15, and 28 degrees of freedom and $1-\alpha = .95$, we give the value of $P[a_\nu(q_0+1) < \chi^2_{\nu+1+2w} < b_\nu(q_0+1)]$ for $q_0 = .01$, .005, and .001 (respectively for the degrees of freedom) as w goes from zero to ten. Each of these values is less than $1-\alpha$, which shows that (A6) holds.

TABLE A1

Numerical verification that NSBR betting procedures exist for I_{ET},

$$p = P[a_\nu(q_0+1) < \chi^2_{\nu+1+2w} < b_\nu(q_0+1)].$$

$\nu = 4$, $1-\alpha = .95$, $a_\nu = .484$, $b_\nu = 11.143$, $q_0 = .01$

w	p	w	p
0	.945	6	.156
1	.871	7	.085
2	.741	8	.042
3	.577	9	.019
4	.410	10	.008
5	.265		

For $w_0 = 2$, $q' = .5$

$P[\chi^2_{\nu+1+2w_0} < b_\nu(q'+1)] = .946$

$\nu = 15$, $1-\alpha = .95$, $a_\nu = 6.262$, $b_\nu = 27.488$, $q_0 = .005$

w	p	w	p
0	.949	6	.515
1	.926	7	.409
2	.879	8	.312
3	.811	9	.228
4	.723	10	.159
5	.622		

For $w_0 = 6$, $q' = .5$

$P[\chi^2_{\nu+1+2w_0} < b_\nu(q'+1)] = .948$

$\nu = 28$, $1-\alpha = .95$, $a_\nu = 15.308$, $b_\nu = 44.461$, $q_0 = .001$

w	p	w	p
0	.949	6	.673
1	.936	7	.591
2	.909	8	.507
3	.868	9	.423
4	.814	10	.344
5	.748		

For $w_0 = 11$, $q' = .5$

$P[\chi^2_{\nu+1+2w_0} < b_\nu(q'+1)] = .930$