# POST-DATA PROBABILITY OF CORRECT SELECTION

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Abstract

The BKS sequential procedure for selecting the best of k Koopman-Darmois populations terminates when the conditional probability P(CS|order statistic of sample means) first exceeds a preassigned lower bound $P^*$ at the LF configuration of the natural parameters. The same expression for the post-data P(CS) applies irrespective of the stopping rule, provided only that the vector observations are independently distributed with natural parameters which differ additively.

We follow the notation of Bechhofer-Kiefer-Sobel (BKS) (1968) in characterizing their sequential likelihood procedure as applied to Indifference Zone Selection of the best (largest $\tau$) of k Koopman-Darmois populations

$$f(x_j;\tau_j) = \exp\{x_j\tau_j + r(x_j) + S(\tau_j)\}, \qquad j=1,2,\cdots,k \quad . \tag{1}$$

Letting $J_k = (j_1,\cdots j_k)$ denote a permutation of the integers $(1,2,\cdots,k)$, we form the m'th stage likelihood statistic

$$L_m = \exp\left(m\bar{x}_{[k]}\tau_{[k]}\right) \frac{\frac{1}{k!}\sum_{J_{k-1}} \exp\left(m\sum_{i=1}^{k-1}\bar{x}_{[i]}\tau_{[j_i]}\right)}{\frac{1}{k!}\sum_{J_k} \exp\left(m\sum_{i=1}^{k}\bar{x}_{[i]}\tau_{[j_i]}\right)} \tag{2}$$

In BKS this statistic is interpreted as representing the *a posteriori* likelihood that the population indexed by $\tau_{[k]}$ is associated with the largest m'th stage sample mean $\bar{x}_{[k]}$, when the *a priori* probability distribution over the k! possible orderings of the populations is uniform. They prove that for a fixed $\delta = \tau_{[k]} - \tau_{[k-1]}$

$$L_m \geq L_m(\delta) = \frac{\exp\left(m\delta\bar{x}_{[k]}\right)}{\sum_{i=1}^{k} \exp\left(m\delta\bar{x}_{[i]}\right)} = \frac{1}{1 + \sum_{i=1}^{k-1} \exp\left[-m\delta\left(\bar{x}_{[k]} - \bar{x}_{[i]}\right)\right]} \qquad (3)$$

and since (3) is increasing in $\delta$ then $L_m(\delta) \geq L_m(\delta^*)$ for $\delta \geq \delta^*$, where $0 \leq \delta \leq \delta^*$ is the predetermined indifference zone. Their basic sequential ranking procedure terminates at the earliest stage (n) for which $L_n(\delta^*) \geq P^*$, where $P^*$ is the predetermined lower bound on the probability of correct selection (PCS) when $\delta \geq \delta^*$.

We now observe that although (2) and (3) were originally constructed to represent a Bayesian *a posteriori* probability of correct selection, the ratio $L_m$ also has a frequentist interpretation as a conditional probability of correct selection. Given the order statistic $(\bar{x}_{[1]},\cdots,\bar{x}_{[k]})$ at the m'th stage, $L_m$ in (2) is the conditional probability that $\bar{x}_{[k]}$ is indeed associated with the largest parameter $\tau_{[k]}$; and the inequality at termination

$$P\left\{CS\middle|\left(\bar{x}_{[1]},\cdots,\bar{x}_{[k]}\right)\right\} = L_n \geq L_n(\delta) \geq L_n(\delta^*) \qquad \text{for } \delta \geq \delta^* \qquad (4)$$

is therefore a useful data-analytic result when $\delta^*$ is prespecified for the particular exponential family at hand. Since the post-data probability $L_n(\delta^*) \geq P^*$ under the BKS stopping rule, then the pre-data PCS $= E(L_n) \geq P^*$. Sobel and Wu (1984) generalize the BKS rule to include the problem of retaining the best (RB) by selecting the subset determined by the s largest means; (4) then generalizes to

$$P\left\{RB \mid \left(\bar{x}_{[1]},\ldots,\bar{x}_{[k]}\right)\right\} \geq \frac{1 + \sum_{i=k-s+1}^{k-1} \exp\left[-n\delta^*\left(\bar{x}_{[k]} - \bar{x}_{[i]}\right)\right]}{1 + \sum_{i=1}^{k-1} \exp\left[-n\delta^*\left(\bar{x}_{[k]} - \bar{x}_{[i]}\right)\right]} \geq P^* .$$

The utility of the result (4) is substantially enhanced by noting that (4) holds when $\delta \geq \delta^*$ *irrespective of the stopping rule*, provided the vector-at-a-time sampling rule is followed. A further generalization is achieved by noting that blocking in the experimental design has no effect on (2) — (4) provided the n block effects are additive with $\tau$. In the Bernoulli case, for example, $\tau$ is the logit of p so block effects must be logistically additive with treatment effects if (2) — (4) are to hold, while in the Poisson case log-linearity is required.

An especially ubiquitous special case is the situation of the paired (k=2) or balanced t-test where

$$P\left\{CS \left| \left(\bar{x}_{[1]},\bar{x}_{[2]}\right)\right.\right\} = \frac{1}{1 + \exp\left(-2|\bar{x}_1-\bar{x}_2||\mu_1-\mu_2|/\sigma^2_{\bar{x}_1-\bar{x}_2}\right)} \tag{5}$$

represents the post-data probability that the sign of $\mu_1 - \mu_2$ has been correctly estimated. (Here, as in (2) — (4), the conditioning event may be reduced from the order statistic to the spacings.) Note that an assertion of $\text{sgn}(\mu_1 - \mu_2)$ is not sanctioned by a Z- or t-test, or by a corresponding confidence interval for $\mu_1 - \mu_2$, even though a statistically significant difference may be strongly indicative that $\text{sgn}(\bar{x}_1 - \bar{x}_2) = \text{sgn}(\mu_1 - \mu_2)$. A $P^*$ as well as an indifference zone are often specified, in effect, at the pre-data stage as input required for sample size calculations. Such data-independent parameter specifications remain applicable at the post-data stage, and may be substituted into (5) as a data-analytic measure. Analogous measures of post-data probability of correct ranking or correct subset selection may be constructed in the balanced k-sample setting of multiple comparisons.

In multistage selection the conditional PCS or PRB for the next stage provides an adaptive basis for sample size determination. Cecce (1985) adopted this approach in two-stage selection.

## References

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