BEST LINEAR UNBIASED ESTIMATION IN MIXED MODELS OF
THE ANALYSIS OF VARIANCE

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Abstract

A broad definition is given of balanced data in mixed models. For all such models, it is shown that the BLUE (best linear unbiased estimator) of an estimable function of the fixed effects is the same as the ordinary least squares estimator (OLSE).

1. INTRODUCTION

a. Fixed effects models

Analysis of variance models are traditionally formulated in terms of additive main effects and additive interaction effects. For example, suppose \( y_{ijk} \) is the \( k \)'th observation on treatment \( i \) of variety \( j \) in a two-factor experiment concerned with fertilizer treatments and plant varieties. Then a usual analysis of variance model is of the form

\[
y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}
\]

where \( \mu \) is a general mean, \( \alpha_i \) is the effect on the response variable due to the \( i \)'th treatment, \( \beta_j \) is the effect due to the \( j \)'th variety, \( \gamma_{ij} \) is the interaction effect between treatment \( i \) and variety \( j \), and \( e_{ijk} \) is the residual error term defined as \( e_{ijk} = y_{ijk} - E(y_{ijk}) \) for

\[
E(y_{ijk}) = \mu + \alpha_i + \beta_j + \gamma_{ij}
\]
where $E$ denotes expectation over repeated sampling.

Models such as (1), where estimation of (and testing of hypotheses about) parameters are the features of interest, are known as fixed effects models, and in such models the customary assumptions about variances and covariances are that each observation has the same variance and that every pair of observations has zero covariance. The dispersion matrix $\Sigma$ of the vector of observations $\gamma$ then has the form

$$\text{var}(\gamma) = \Sigma = \sigma^2 I,$$  \hspace{1cm} (2)

$I$ being an identity matrix and $\sigma^2$ being the variance of every observation.

An assumption about $\Sigma$ more general than (2) is that it is simply a symmetric, positive semi-definite matrix; and in many cases that it be not just positive semi-definite but positive definite, and hence non-singular.

b. Mixed models

Variations of (1) are models where some or all of the $\alpha_i, \beta_j$ and $\gamma_{ij}$ terms are assumed not to be parameters to be estimated, but are modeled as being random variables with zero means and some assumed variance-covariance structure. For example, suppose in the no-interaction form of (1), with one observation $y_{ij}$ on treatment $i$ and variety $j$, namely

$$y_{ij} = \mu + \alpha_i + \beta_j + e_{ij},$$  \hspace{1cm} (3)

that the $\beta_j$ for $j = 1, \ldots, b$, are modeled as random variables with zero mean $E(\beta_j) = 0 \forall j$. The $\beta_j$ are then called random effects and, along with the random error terms $e_{ij}$, usually have the following variance-covariance structure attributed to them:

$$\text{var}(\beta_j) = \sigma^2 \beta \hspace{1cm} \text{cov}(\beta_j, \beta_{j'}) = 0 \forall j \neq j' \hspace{1cm} (4)$$

$$\text{var}(e_{ij}) = \sigma^2 e \hspace{1cm} \text{cov}(e_{ij}, e_{i',j'}) = 0 \text{ except for } i=i' \text{ and } j=j'$$

and

$$\text{cov}(\beta_j, e_{ij}) = 0 \forall i, j, j'. $$
Then with \( \mu \) and the \( \alpha_i \) in (3) being fixed effects and the \( \beta_j \) being random effects, (3) is known as a mixed model. And the variances \( \sigma^2_\beta \) and \( \sigma^2_e \) of (4) are the variance components. The structure of (4) then leads to \( Y \) having elements that are either zero, \( \sigma^2_\beta + \sigma^2_e \), or \( \sigma^2_\beta \); in general to elements that are either zero, or one of the variance components or a sum of them.

**Example 1** Consider (3) and (4), where the \( \beta \) factor represents blocks in a randomized complete blocks experiment. Suppose there are 2 treatments and 3 blocks. Then for a zero element of a matrix being shown as a dot,

\[
\Sigma = \begin{bmatrix}
\sigma^2_\beta + \sigma^2_e & \cdot & \cdot & \cdot \\
\cdot & \sigma^2_\beta + \sigma^2_e & \cdot & \cdot \\
\cdot & \cdot & \sigma^2_\beta + \sigma^2_e & \cdot & \cdot \\
\sigma^2_\beta & \cdot & \cdot & \sigma^2_\beta + \sigma^2_e & \cdot \\
\cdot & \sigma^2_\beta & \cdot & \cdot & \sigma^2_\beta + \sigma^2_e \\
\cdot & \cdot & \sigma^2_\beta & \cdot & \sigma^2_\beta + \sigma^2_e \\
\cdot & \cdot & \cdot & \sigma^2_\beta + \sigma^2_e & \cdot \\
\end{bmatrix}
\]

c. **Estimation with balanced data**

Section 3 formulates a set of models that specifies a wide class of balanced data. First, though, we appeal to the general understanding that balanced data have equal numbers of observations in the subclasses. Model equations (1) and (3) are examples, having, for each treatment-variety combination, one observation and (with \( k = 1, 2, \ldots, n \)) \( n \) observations, respectively. In both cases the best linear unbiased estimator (BLUE) of a treatment difference is a well known, simple function of means. Thus when each of (1) and (3) are fixed effects models, the BLUE of \( \alpha_i - \alpha_i' \), is

\[
\text{BLUE}(\alpha_i - \alpha_i') = y(i) - y(i')
\]
where \( \bar{y}(i) \) is the mean of all observations on treatment \( i \). Moreover, the right-hand side of (5) is also the ordinary least squares estimator of \( a_i - a_i' \). Hence, for these examples

\[
\text{BLUE} (a_i - a_i') = \text{OLSE} (a_i - a_i') .
\]

(6)

Of additional importance is the fact that although (5) is true when (1) and (3) are fixed effects models, it is also true when (1) and (3) are mixed models with \( \alpha \) as fixed. The generalization of (6) is that for any estimable function of fixed effects in a mixed model with balanced data, \( \text{BLUE} = \text{OLSE} \). The utility of this result is that although a BLUE is a desirable estimator, its direct derivation generally involves inverting \( \bar{y} \), which can be tedious; in contrast, with balanced data, the OLSE is often easily derived as a simple function of observed means. Moreover, the equality \( \text{BLUE} = \text{OLSE} \) for balanced data is broad in scope. For example, (5) is true for (1) being not only a fixed effects model, but also a mixed model with \( \beta \)s, or \( \gamma \)s, or \( \beta \)s and \( \gamma \)s taken as random effects. Furthermore (5), as an example of (6) is also true if (1) is extended by the additional of other random effects: for example, in the model \( y_{ijkm} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \theta_k + \tau_m + \delta_{jk} + \epsilon_{ijkm} \), with \( \mu \) and \( \alpha \)'s being fixed effects and all other effects being random, (5) is still true.

We proceed to establish (6) for any mixed model with balanced data. To do so we first describe a general mixed model and then give a broad definition of balanced data.
2. A GENERAL MIXED MODEL

a. Description

The elements of the mixed model (3) are of two kinds: \( \mu \) and \( \alpha_i \) that are fixed effects, and \( \beta_1 \) and \( e_{ij} \) that are random variables. Recognizing the dichotomy of fixed and random effects in a mixed model, we write the model equation for a vector of observations \( \chi \) as

\[
\chi = X\beta + Zu
\]

where \( \beta \) is a vector of fixed effects and \( u \) is a vector of random effects, including error terms. The matrices and vectors of (7) are partitioned thus:

\[
X = \begin{bmatrix} X_1 & X_2 & \cdots & X_d & \cdots & X_f \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_q & \cdots & Z_r \end{bmatrix}
\]

\[
\beta = \begin{bmatrix} \beta'_1 & \beta'_2 & \cdots & \beta'_d & \cdots & \beta'_f \end{bmatrix}' \quad \text{and} \quad u = \begin{bmatrix} u'_1 & u'_2 & \cdots & u'_q & \cdots & u'_r \end{bmatrix}'
\]

Each \( \beta_d \) for \( d = 1, 2, \cdots, f \) has as its element the \( h_d \) effects corresponding to the \( h_d \) levels of the \( d \)'th fixed effect (main effect or interaction) factor, and \( X_d \) is the incidence matrix corresponding to \( \beta_d \). Similarly, \( u_q \) (of \( p_q \) elements) and \( Z_q \) for \( q = 1, 2, \cdots, r-1 \) are defined for the random effect (main effect or interaction) factors analogously to \( \beta_d \) and \( X_d \) for fixed effect factors. For \( q = r \), we define \( u_r = e \), the vector of error terms, and accordingly \( Z_r = 1_N \) where \( N \) is the total number of observations, and \( p_r = N \).

Example 2 Using (3) and (4) as the model for a randomized complete blocks experiment for \( a \) treatments in \( b \) blocks, \( \mu \) and \( [\alpha_1 \cdots \alpha_a]' \) would be \( \beta_1 \) and \( \beta_2 \) of (8), respectively, and \( [\beta_1 \cdots \beta_b] \) and the \( e_{ij} \)-terms of (3) would be \( u_1 \) and \( u_2 \) of (8), respectively.

The variance and covariance properties of (4) generalized to \( u \) are...
\[
\text{var}(u_q) = \sigma^2 I_{pq} \quad \text{for} \quad q = 1, 2, \ldots, r
\]
\[
\text{cov}(u_q^*, u_{q'}^*) = 0_{pq} \quad \text{for} \quad q \neq q' = 1, 2, \ldots, r.
\]

Hence from (7) the variance-covariance matrix of \( \gamma \) is
\[
\Sigma = \text{var}(\gamma) = \text{var}(Z\beta) = \sum_{q=1}^{r} \sigma^2 Z_q Z_q'.
\]

Thus (7) through (10) constitute a description of a general mixed model.

b. Estimation

The OLSE estimator of an estimable function \( \lambda' \chi \beta \) of the parameters in \( \beta \) in the model (7) will be denoted by \( \text{OLSE}(\lambda' \chi \beta) \) and is, as is well-known,
\[
\text{OLSE}(\lambda' \chi \beta) = \lambda' (\chi' \chi)^{-1} \chi' \beta
\]
where \( (\chi' \chi)^{-1} \) is a generalized inverse of \( \chi' \chi \), i.e., \( (\chi' \chi)^{-1} \) is any matrix satisfying
\[
\chi' \chi (\chi' \chi)^{-1} \chi' = \chi' \chi.
\]

Similarly the BLUE of that same estimable \( \lambda' \chi \beta \) is
\[
\text{BLUE}(\lambda' \chi \beta) = \lambda' (\chi' \chi^{-1} \chi)^{-1} \chi' \chi^{-1} \chi
\]
where \( \chi \) is assumed to be positive definite.

In fixed effects models, \( \Sigma = \sigma^2 I \), as in (2), whereupon (12) very simply reduces to (11), as is well known. An extension to \( \Sigma = [(1-\rho)I + \rho J_I] \sigma^2 \) is given by McElroy (1967) and, in complete generality, Zyskind (1967) has shown that these two estimators are equal, if and only if
\[
\chi \chi = \chi Q \quad \text{for some} \ Q.
\]
Graybill (1976, p. 209) also has this result, restricted to $X$ of full column rank. We use (13) to show for a broad definition of balanced data that for mixed models of the form (7) through (10) the BLUE of an estimable function of the fixed effects parameters is the same as the OLSE.

3. BALANCED DATA

We deal with data categorized by a number of factors, each of which is either a main effects factor (including the possibility of nested main effects factors), or an interaction factor representing the interaction of two or more main effects factors. Suppose there are $m$ main effects factors, with the $t$'th one having $N_t$ levels, for $t = 1, 2, \cdots, m$. Then the $k$'th observation in the "cell" defined by the $i_t$'th level (for $i_t = 1, \cdots, N_t$) of the $t$'th main effect for $t = 1, \cdots, m$, where there are $n_{i_1 i_2 \cdots i_m}$ such observations, is $y_{i_1 i_2 \cdots i_m}$ for $k = 1, 2, \cdots, n_{i_1 i_2 \cdots i_m}$. On defining $i = [i_1 i_2 \cdots i_m]$, a typical observation can then be denoted as $y_{i_k}$ for $k = 1, 2, \cdots, n_i$. Furthermore, the total number of observations is

$$N = p_r = \sum_{i=1}^{N'} n_i$$

for $N' = [N_1 N_2 \cdots N_t \cdots N_m]$.

($1_m$ is a row vector of $m$ unities.)

A tight, rigorous, formal and complete definition of balanced data is elusive. Development of such a definition would, as Cornfield and Tukey (1956) write, involve "\cdots systematic algebra [which] can take us deep into the forest of notation. But the detailed manipulation will, sooner or later, blot out any understanding we may have started with." Nevertheless, one formulation of a model that yields a wide class of balanced data situations is as follows. It is similar to that used by Smith and Hocking.
The balanced data models we consider are those that have \( n \triangleq n \forall \varphi_1 \). They also have each \( X_d \) and each \( Z_q \) of (8) being a Kronecker product (KP, for brevity) of \( m + 1 \) matrices, each of which is either an \( \frac{1}{2} \)-matrix or a \( \frac{1}{2} \)-vector; i.e.,

\[
each X_d \text{ and each } Z_q \text{ is a KP of } m+1 \text{ matrices that are each } I \text{ or } \frac{1}{2} . \quad (14)
\]

The occurrence of the \( \frac{1}{2} \)-matrices and \( I \)-vectors in these KPs is as follows. First, corresponding to the scalar parameter \( \mu \) in the model is \( X_1 \) which is \( \frac{1}{2} \)-normal, and so every matrix in its KP is a \( I \):

\[
X_1 = \frac{1}{2}_N = \frac{1}{2}_N \ast \frac{1}{2}_N \ast \cdots \ast \frac{1}{2}_N \ast \cdots \ast \frac{1}{2}_N \ast \frac{1}{2}_n ,
\]

where \( \ast \) represents the operation of Kronecker multiplication. Second, corresponding to \( \varphi_r = e \) is \( \frac{1}{2}_N \), and so each of the \( m + 1 \) matrices in the KP that is \( Z_r = I_N \) is an \( \frac{1}{2} \)-matrix:

\[
Z_r = \frac{1}{2}_N = \frac{1}{2}_N \ast \frac{1}{2}_N \ast \cdots \ast \frac{1}{2}_N \ast \cdots \ast \frac{1}{2}_N \ast \frac{1}{2}_n .
\]

Finally, in the KP for each \( X_d \) and \( Z_q \) (other than \( X_1 \) and \( Z_r \)), the \( t \)'th matrix corresponds to the \( t \)'th main effects factor and is \( \frac{1}{2}_N \) when that factor is part of the definition of the factor corresponding to \( X_d \) or \( Z_q \); otherwise it is \( \frac{1}{2}_N \). This is for \( t = 1, \cdots, m \). And for all \( X_d \) and \( Z_q \), other than \( Z_r \), the \((m+1)\)'th matrix in the KP is \( \frac{1}{2}_n \).

The phrase "part of the definition" demands explanation. It is exemplified in the 2-factor model (1), wherein the two main effects factors are each part of the definition of the interaction factor. Similarly, if nested within an \( \alpha \)-factor there is a \( \beta \)-factor then the \( \alpha \)-factor is part of
the definition of that \( \beta \)-factor. (See also, comments B and C which follow the examples.)

Each \( h_d \) and \( p_q \) (number of levels in the \( d \)'th fixed factor and the \( q \)'th random factor, respectively) in the balanced data we have defined is the product of the numbers of columns in the \( I \) and \( Z \) terms in the KP (14) that is \( X_d \) and \( Z_q \). Hence \( h_d \) is the product of the \( N_t \) values for the main effects factors that are part of the definition of the \( d \)'th fixed effect factor; \( p_q \) is a similar product for the \( q \)'th random effects factor.

**Examples** We give four examples that are each in terms of those of the following vectors that are appropriate: \( \alpha = [\alpha_1, \ldots, \alpha_a]' \), \( \beta = [\beta_1, \ldots, \beta_b]' \) or \( \beta_+ = [\beta_1 \cdots \beta_{1b} \beta_{21} \cdots \beta_{2b} \cdots \beta_{al} \cdots \beta_{ab}]' \), \( \gamma = [\gamma_{11} \cdots \gamma_{1b} \gamma_{21} \cdots \gamma_{2b} \cdots \gamma_{al} \cdots \gamma_{ab}]' \), and \( \varepsilon \), the vector of error terms, the same order as \( \gamma \). Determination of which KPs are \( X \)-matrices and which are \( Z \)-matrices is governed by which factors are defined as fixed effects and which are random. This is illustrated for only example (iii).

(i) **One-way classification:** \( y_{ij} = \mu + \alpha_i + \epsilon_{ij} \) with \( i=1, \ldots, a \) and \( j=1, \ldots, n \).

\[
\chi = (I_a * I_n) \mu + (I_a * I_n) \alpha + (I_a * I_n) \epsilon \quad . \tag{15}
\]

(ii) **Two-way crossed classification, no interaction, and one observation per cell:** \( y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij} \) for \( i=1, \ldots, a \) and \( j=1, \ldots, b \).

\[
\chi = (I_a * I_b) \mu + (I_a * I_b) \alpha + (I_a * I_b) \beta + (I_a * I_b) \epsilon \quad . \tag{16}
\]

(iii) **Two-way crossed classification, with interaction and \( n \) observations per cell:** \( y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk} \) with \( i=1, \ldots, a \), \( j=1, \ldots, b \), and \( k=1, \ldots, n \).

\[
\chi = (I_a * I_b * I_n) \mu + (I_a * I_b * I_n) \alpha + (I_a * I_b * I_n) \beta + (I_a * I_b * I_n) \gamma + (I_a * I_b * I_n) \epsilon \quad . \tag{17}
\]
Suppose in (17) that elements of $\beta$ and $\chi$ were taken to be random effects. Then the terms of (8) for the general mixed model would have the following values:

$m=3$, $f=2$ with $h_1 = N_1 = 1$ and $x_1 = I_a * I_b * I_n$ for $\beta_1 = \mu$,
and $h_2 = N_2 = a$ and $x_2 = I_a * I_b * I_n$ for $\beta_2 = \alpha$;

$r=3$ with $p_1 = N_3 = b$ and $z_1 = I_a * I_b * I_n$ for $\gamma_1 = \delta$,
and $p_2 = N_2N_3 = ab$ and $z_2 = I_a * I_b * I_n$ for $\gamma_2 = \chi$,
and $p_3 = N_2N_3n = abn$ and $z_3 = I_a * I_b * I_n$ for $\gamma_3 = \epsilon$.

(iv) Two-way nested classification: $y_{ij} = \mu + a_i + b_{ij} + e_{ijk}$ for $i=1, \ldots, a$, $j=1, \ldots, b$ and $k=1, \ldots, n$.

$$
\chi = (I_a * I_b * I_n)\mu + (I_a * I_b * I_n)\alpha + (I_a * I_b * I_n)\beta_+ + (I_a * I_b * I_n)d.
$$

Comments on the examples. Several comments are in order. (A) In every case $x_1$ for $\mu$ is $I$, a KP of $I$-vectors; and $z_1$ for $\epsilon$ is $I$, a KP of $I$-matrices. (B) In every case the KP that is the coefficient of $\chi$ has only one $I$-matrix in it, namely $I_a$. This is so because, obviously, the definition of $\chi$ involves only $a$. The same is true of the coefficient of $\beta$ in (16) and (17). (C) In contrast, the KP that is the coefficient of $\beta_+$ in (18) has two $I$-matrices, $I_a$ and $I_b$. This is because $\beta_+$ has elements that represent the nesting of the $\beta$-factor within the $\alpha$-factor. Thus the $\alpha$-factor is involved in the definition of $\beta_+$ and so the coefficient of $\beta_+$ contains $I_a$ and $I_b$. Thus the coefficient of $\beta_+$ in (18) is the same as that of $\chi$, the interaction term, in (17). Judged solely by their coefficients, $\beta_+$ and $\chi$ would therefore appear to be the same. What makes $\chi$ an interaction term is that both main effect factors

...
that go into defining it are also present on their own in (17), but with $\beta_+$, only one factor that goes into defining it is present on its own in (18), and so $\beta_+$ represents nesting. In other words, a factor that looks like an interaction factor is such when all of its associated main effects factors are present in the model; otherwise it is a nested factor. (D) Equation (16) is a special case of (17) with $\chi$ omitted and $n=1$ and hence, for example, $1_a \ast 1_b \ast 1_n = 1_a \ast 1_b \ast 1 = 1_a \ast 1_b$.

A final observation concerns $V = \sum q^2 q_{q}q'_{q}$ of (10), based on the general result that $(A \ast B)(P \ast Q) = AP \ast BQ$, given the necessary conformity requirements. Thus, for $1_{n}1'_{n} = J_{n}$ being a square matrix of order $n$ with every element unity, we have from (14) that every $Z_{q}Z'_{q}$ is a KP of $I_{n}$ and $J_{n}$ matrices. Hence we rewrite (10) as

$$V = \sum q^2 (the\ KP\ of\ I\ and\ J\ matrices\ that\ is\ Z_{q}Z'_{q}) . \quad (19)$$

4. ESTIMATION FROM BALANCED DATA

We now show for mixed models as specified in (7) - (10), with balanced data as defined in the preceding section, that the BLUE of (12) equals the OLSE of (11). We do this by showing that (13) is satisfied for $V$ of (19) and $X = \{X_d\}$, $d = 1, \ldots, f$ of (14) with $X_d$ being a KP of $I$-matrices and $J$-vectors.

Writing $W_q$ for $Z_{q}Z'_{q}$ of (19) we have

$$W_q = Z_{q}Z'_{q} = (W_q1 \ast W_q2 \ast \ldots \ast W_qt \ast \ldots \ast W_q,m+1) = \ast W_{qt} \quad (20)$$

where, from (19) each $W_{qt}$ is either an $I$ or a $J$ matrix. Similarly, from (8),

$$X = [X_1 \ X_2 \ \ldots \ X_d \ \ldots \ X_f] \ with \ X_d = \ast X_{dt} \quad (21)$$

where each $X_{dt}$ is either $I_{N_t}$ or $J_{N_t}$. Then from (19)
where, by the curly braces notation, we mean that $V_X$ is partitioned into a row of $f$ sub-matrices. Thus

$$V_X = \left\{ \frac{r}{\sum_{q=1}^{q=d} Z_i X_j} \right\}_{d=1}^{d=f}$$

(22)

$$V_X = \left\{ \frac{r}{\sum_{q=1}^{q=d} W_X} \right\}_{d=1}^{d=f}$$

(23)

Now in (20), $W_{q_d}$ is either $I$ or $J$, and in (21) each $X_{d_t}$ is either $I$ or $J$, all of order $N_t$. Therefore the four possible values of the product $W_{q_d} X_{d_t}$, together with the definition of a matrix $M_{d_q}$ such that $W_{q_d} X_{d_t} = X_{d_t} M_{d_q}$ in each case, are as follows:

<table>
<thead>
<tr>
<th>$W_{q_d}$</th>
<th>$X_{d_t}$</th>
<th>$W_{q_d} X_{d_t} = X_{d_t} M_{d_q}$</th>
<th>$M_{d_q}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>$I$</td>
<td>$I = I$</td>
<td>$I$</td>
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<tr>
<td>$I$</td>
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<td>$J$</td>
<td>$J$</td>
<td>$N_{t^2} = N_{t^2}$</td>
<td>$N_{t^2}$</td>
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</tbody>
</table>

Therefore from (23)

$$V_X = \left\{ \frac{r}{\sum_{q=1}^{q=d+1} W_{q_d} X_{d_t} M_{d_q}} \right\}_{d=1}^{d=f}$$

(24)

$$V_X = \left\{ \frac{r}{\sum_{q=1}^{q=d+1} W_{q_d} X_{d_t} M_{d_q}} \right\}_{d=1}^{d=f}$$

(25)

for

$$M_{q_d} = M_{q_d 1} M_{q_d 2} * \ldots * M_{q_d t} * \ldots * M_{q_d, d+1}$$

(26)

Derivation both of (23) from (22) and of (25) from (24) is based both on $X_{d_q}$ and $M_{d_q}$ each being a KP, and on the product rule for KP quoted earlier.
The conformability requirements of the regular products in (24) might seem to be lacking because, from the preceding table, two forms of $M_{qdt}$ are scalars. However, both regular and Kronecker products of matrices do exist when one or more of the matrices is a scalar; e.g., for scalar $\theta$, both $\theta \Theta$ and $(\Theta \Theta)(\Theta \Theta) = \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta \Theta 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Example Suppose in (1) and (17) that the $\beta$s and $\gamma$s are random effects. Then

$$X = \begin{bmatrix} \frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n} & \frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n} \end{bmatrix}$$

and

$$V = \sigma_\beta^2 (\frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n}) + \sigma_\gamma^2 (\frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n}) + \sigma_e^2 (\frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n}) .$$

Hence in $VX$ the first sub-matrix is

$$V(\frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n}) = \sigma_\beta^2 (\frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n}) + \sigma_\gamma^2 (\frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n}) + \sigma_e^2 (\frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n})$$

$$= (\frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n}) [\sigma_\beta^2 (1 \ast 1 \ast 1) + \sigma_\gamma^2 (1 \ast 1 \ast 1) + \sigma_e^2 (1 \ast 1 \ast 1)] .$$

Similarly, the second sub-matrix of $VX$ is

$$V(\frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n}) = \sigma_\beta^2 (\frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n}) + \sigma_\gamma^2 (\frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n}) + \sigma_e^2 (\frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n})$$

$$= (\frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n}) [\sigma_\beta^2 (1 \ast 1 \ast 1) + \sigma_\gamma^2 (1 \ast 1 \ast 1) + \sigma_e^2 (1 \ast 1 \ast 1)] .$$

Hence

$$VX = \begin{bmatrix} \frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n} & \frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n} \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

for $M_1$ and $M_2$ being the matrices in square braces in (28) and (29), respectively, namely

$$M_1 = a \sigma_\beta^2 + n \sigma_\gamma^2 + \sigma_e^2 \quad \text{and} \quad M_2 = n \sigma_\beta^2 J + n \sigma_\gamma^2 I + \sigma_e^2 I \quad \text{for} \quad \frac{1}{a} \ast \frac{1}{b} \ast \frac{1}{n} .$$
References


