BU-841-M(?)

Using Prior Interval Information in Constructing Interval Estimates for a Gamma Mean

Walter W. Piegorsch
Biometry and Risk Assesment Program
National Institute of Environmental Health Sciences
Research Triangle Park, NC, 27709, USA

SUMMARY

Methods are presented for construction of interval estimates on the mean parameter of a gamma distribution when there is some prior interval information as to the location of this parameter. The methods produce Bayes credible regions by basing the prior distributions for the mean parameter on the prior interval information. Three differing approaches for the derivation of the priors are considered. A discussion of existing procedures is also given.

Key Words: Gamma distribution, Nuisance shape parameter, Bayes interval estimates.

Using Prior Interval Information in Constructing Interval Estimates for a Gamma Mean

1. Introduction

Experimental situations often arise where the investigator has some predetermined expectations as to the outcome of the experiment. One such example of this occurs when previous experiments, existing literature, or other knowledge suggests a reasonable interval within which some population parameter should lie. It may be known that the population mean, θ, should lie in the interval (L,U).

It will be of interest to justify (or possibly improve upon) the prior interval expectations by incorporating information in data into the construction of interval estimates for θ. For example, in biochemical laboratories the examination of blood serum level of many important compounds is performed with increasingly sophisticated technology, prompting the continual upgrading of laboratory machinery. When a new analyzing system is installed, the pathologist may wish to know if the capabilities of the new system have changed. Table 1 presents an example of such data for levels of serum magnesium in a random sample of healthy felines. It was of interest to see if the new chemical analyzer operated within some prior interval for the mean response.

A histogram of the data is presented in Figure 1. There, one sees that the departure from normality of the data appears great, and that an alternative model is called for. Indeed, the use of non-normal probability models for such a situation is common (Reed, Henry, and Mason, 1971; Elveback, 1972). For the serum magnesium (Mg) example, an alternative that provided a positive-valued, skewed frequency distribution was considered. These criteria led quite naturally to the gamma probability model (Linhart, 1965, Sec. 1). The gamma distribution gives a highly flexible, rich model

for consideration, and its use is common in many reliability and life testing situations (Martz and Waller, 1982; Bartolucci and Dickey, 1977).

The common parameterization for this gamma model is in terms of the shape parameter $r>0$ and scale parameter $\lambda>0$ [denoted $X \sim \Gamma(r,\lambda)$]. The probability density function (pdf) of X is

$$f(x;r,\lambda) = \frac{\lambda^r}{\Gamma(r)} x^{r-1}\exp[-\lambda x] \qquad 0 < x < \infty \qquad (1.1)$$

(where $\Gamma(\cdot)$ is the gamma function). However, since our interest is in inferences on the mean of X, we parameterize the pdf in terms of the shape parameter, r, and mean $E[X]=r/\lambda\equiv\theta$:

$$f(x;r,\theta) = \frac{r^r}{\Gamma(r)\theta^r} x^{r-1}\exp[-xr/\theta] \quad . \qquad (1.2)$$

It is unusual to see the mean and shape parameterization for the gamma model; most authors have used the scale and shape form in (1.1). For this scale and shape form, maximum likelihood (ML) parameter estimates were given by Choi and Wette (1969) and later (approximately) corrected for bias by Anderson and Ray (1975). For the mean and shape parameterization in (1.2), their unbiased estimators are $\hat{\theta}=\bar{X}$ and $\tilde{r} = (n-3)\hat{r}/n + 2/3n$ (n is the sample size and $\hat{r}$ is the ML estimate). The ML shape estimator, $\hat{r}$, has no closed form, and must be found by solving the equation

$$\ln r - \psi(r) = \ln[\bar{x}/\tilde{x}] \quad . \qquad (1.3)$$

Here $\psi(\cdot)$ is the digamma function (Abramowitz and Stegun, 1972, p.258) and $\tilde{x}$ is the geometric mean of the sample observations.

The alternative estimation approach provided by Bayes methodology has also been given a measure of attention. Canavos (1971), Lwin and Singh (1974), and Tummala and Sathe (1978) consider Bayesian estimation of the various gamma parameters when considering the gamma model as a failure time

distribution in reliability analysis (Martz and Waller, 1982, Sec. 9.5, provide a good review of this topic). Bartolucci and Dickey (1977) consider Bayesian inference with known shape parameter in survival analysis.

Because of the open-form nature of the shape parameter estimates, inference on θ can become complicated. Mehta and Srinivasan (1971) sidestep this problem by considering estimation of r when $\lambda=1$ (so that $\theta=r$). However, the flexibility of the gamma model suffers under this approach. Grice and Bain (1980) provide a more complete investigation by giving (one-sided) hypothesis tests and confidence intervals for θ for any r. However, their two-sided confidence interval for θ is only approximate, due effectively to the dependence of its probability statements on the nature of r (as we'll see below).

Many authors simplify this situation by assuming that the shape parameter is known (Woodward and Gray, 1975; Bartolucci and Dickey, 1977), and Lawless (1980) gives confidence intervals on θ and other functionals of the gamma pdf in such a case. It is easy to see why such an assumption simplifies the estimation problem. $\bar{X}$ is still an unbiased statistic for θ, and, when $X_i \sim$ i.i.d.$\Gamma(r, r/\theta)$, $\bar{X} \sim \Gamma(rn, rn/\theta)$. Thus $Q = 2rn\bar{X}/\theta \sim \chi^2(2rn)$ is a pivotal quantity in the sense that the pdf of Q does not depend upon θ. Then

$$P[\, q_1 < Q < q_2 \,] = P[\, 2rn\bar{X}q_2^{-1} < \theta < 2rn\bar{X}q_1^{-1} \,] = 1-\alpha \qquad (1.4)$$

Hence $(2rn\bar{x}q_2^{-1},\ 2rn\bar{x}q_1^{-1})$ is a conditional (on r) $1-\alpha$ confidence interval for θ whenever q_1 and q_2 are appropriately chosen quantiles from a $\chi^2(2nr)$ distribution.

For the case of unknown shape parameter, interval estimates on θ are much more elusive. In Section 2, the approach to this problem by Grice and Bain (1980) is discussed in greater detail. In Section 3, interval esti-

mation of θ is considered using Bayes techniques. In particular, the prior distribution parameters will be constructed by basing them on the prior interval information. Section 4 gives an example of this methodology using the serum Mg data from Table 1.

2. Approximate Confidence Intervals on θ

Obviously, the complication of moving to an unknown r when considering inference on θ deserves careful attention. For the traditional (confidence) interval estimates, Grice and Bain (1980) argue that a reasonable approximation to the confidence interval in (1.4) would replace r with $\hat{r}$. However, the ML estimator for r is biased (Choi and Wette, 1969). A perhaps better approximation would replace r with the (approximately) unbiased estimator, $\tilde{r}$, given by Anderson and Ray (1975).

Quickly summarized, use of $\hat{r}$ instead of r is justified by consideration of the traditional coverage probability. As with (1.4)

$$P[\ 2\hat{r}n\bar{X}q_2^{-1} < \theta < 2\hat{r}n\bar{X}q_1^{-1}] = P[\frac{\bar{X}}{\theta} < \frac{q_2}{2\hat{r}n} \text{ and } \frac{q_1}{2\hat{r}n} < \frac{\bar{X}}{\theta}\ ] . \quad (2.1)$$

From the Bonferroni inequality, this is greater than or equal to $P[\bar{X}/\theta < q_2/2\hat{r}n] + P[q_1/2\hat{r}n < \bar{X}/\theta] - 1$. Here, q_1 and q_2 are the β and $1-\beta$ quantiles, respectively, of a $\chi^2(2\hat{r}n)$ distribution. $\beta=\beta(n,\alpha)$ is chosen to make (2.1) approximately equal to $1-\alpha$ (Grice and Bain, 1980, Table 2). For example, for $\alpha=.05$ and $n=20$, $\beta=.0159$.

With $\hat{r}$ replacing r, the random quantities in (2.1) are independent of θ, but do depend on r and β (hence on n). For selected ranges of r and n (with α fixed), these values can be calculated via numerical integration. Grice and Bain do this, and tabulate some of their results in the 1980 paper (*their* Table 1). Table 2 of the present paper gives the associated coverage probabilities for some common values of r and n ($\alpha=.05$). As can

be seen, at n=10 the coverage probabilities stay above .95, thus the intervals are (minimally) 95% confidence regions. For larger n the values dip below the nominal .95 level, suggesting that the true 95% confidence intervals will actually be slightly longer than the reported intervals.

3. Bayes Interval Estimates for θ

With all of the problems encountered in the traditional approach to inference for the gamma model, one naturally questions whether the various tools of Bayesian analysis can provide appropriate alternatives. As noted, many authors have considered Bayesian analyses for the scale/shape parameterization (cf. Martz and Waller, 1982, Sec. 9.5). However, few have worked with the mean parameter, θ. Bartolucci and Dickey (1977) consider inference when the shape parameter is known, so that statements made on the scale parameter can alternatively be made on the mean. Tummala and Sathe (1978) consider point estimation under the mean/shape parameterization, but do not address our stated goal of interval estimation with prior (interval) knowledge. We will consider approaches taken by all of these authors, modifying them somewhat to provide inferences on θ.

When the shape parameter, r, is known, the Bayes approach using conjugate priors is relatively simple. A sufficient statistic for θ is $\bar{X}$ (cf. Berger, 1980, Sec. 4.2). Its distribution is $\bar{X} \sim \Gamma(nr,\theta/nr)$. A conjugate prior is the inverse gamma distribution, denoted $I\Gamma(\alpha,\beta)$, with density function

$$\pi(\theta) = \frac{\exp[-1/\theta\beta]}{\Gamma(\alpha)\beta^{\alpha}\theta^{\alpha+1}} \qquad \theta>0 \ , \qquad (3.1)$$

and parameters $\alpha>0$, $\beta>0$. The resulting conjugate posterior distribution is

$\theta|\bar{x} \sim I\Gamma(nr+a, [\bar{x}nr+\beta^{-1}])$.

Selection of the prior parameters α and β is usually the most difficult aspect of this Bayesian problem. In our setting, we suppose that the prior information is in interval form, say (L,U), such that there is $100p_0$% "prior assurance" that $L<\theta<U$ (for some known percentage p_0). Martz and Waller (1982, Sec. 6.6) suggest a method for selecting the parameters of a *gamma* prior involving this sort of prior information. Since the reciprocal of a gamma random variable has an inverse gamma distribution, the Martz-Waller algorithm is easily modified for our use:

STEP 1: The practitioner provides values L, U and p_0 such that $P[\theta<L] = P[\theta>U] = (1-p_0)/2$ (if the assumption of equal tail probabilities is inappropriate, Martz and Waller, 1982, p.240, suggest a more complex graphical method that is just as easily modified).

STEP 2: Calculate $\lambda = \log_{10}(U/L)$.

STEP 3: Given p_0 and λ, find α using Figure C1 of Martz and Waller (1982, Appendix C).

STEP 4: Given p_0 and α, find the intermediate value b_0 from Figure C2 or Table C3 of Martz and Waller (1982, Appendix C).

STEP 5: Calculate

$$\beta = L/b_0\tau$$

where τ is a "lower limit" for θ. One suggestion for this lower limit is 10^{-s}, where s is the number of decimal places being carried. For example, Martz and Waller give $\tau=10^{-6}$ in the context of a failure rate analysis (1982, p.239).

Notice that, although we have developed this approach for the case of known shape parameter, this selection algorithm requires no knowledge about r (other that its independence of θ). Moving to the case of unknown r, we can simply adopt this approach with the addition of a prior distribution on r.

Canavos (1971) considers use of an inverse gamma/uniform combination of priors for the scale/shape parameterization. We can, in similar fashion, retain our conjugate prior $\theta \sim I\Gamma(\alpha,\beta)$ [with α and β selected using the prior interval (L,U)], and place an independent uniform prior on r. Since r is, effectively, a nuisance parameter, we would probably say that r is uniform on (0,B), where B is some sufficiently large value. In some cases, it may be known that the gamma pdf being sampled from is non-increasing, so that B=1 [if $r \leq 1$, (1.2) has a mode at zero]. Or, in many cases it is known that the gamma pdf has a mode away from zero, so that one might chose $r \sim U(1,B)$. This is, in fact, the case with the serum Mg data of Figure 1, and will be the case developed in detail here.

The joint posterior of θ and r given $\bar{x}$ can be expressed as

$$\pi(\theta,r|\bar{x}) = \frac{(\bar{x}nr)^{nr}\exp[-\frac{1}{\theta}(\beta^{-1}+\bar{x}nr)]}{\Gamma(nr)\theta^{\alpha+nr+1}I^*} \qquad \begin{matrix} 1 < r < B, \\ 0 < \theta \end{matrix}, \tag{3.2}$$

where

$$I^* = \int_n^{Bn} \left(\frac{\beta\bar{x}y}{1+\beta\bar{x}y} \right)^y \frac{\Gamma(\alpha+y)}{\Gamma(y)} (1+\beta\bar{x}y)^{-\alpha} dy \quad . \tag{3.3}$$

Our interest is in θ, unconditional on r, so that we require the marginal posterior of θ given $\bar{x}$:

$$\pi(\theta|\bar{x}) = \frac{\exp[-1/\theta\beta]}{\theta^{\alpha}\bar{x}I^*} \int_{\bar{x}n/\theta}^{B\bar{x}n/\theta} \frac{u^{u\theta/\bar{x}}e^{-u}}{\Gamma(u\theta/\bar{x})} du \quad . \tag{3.4}$$

From this, a Bayes interval estimate on θ can be calculated: simply find the *credible region*, C, of the parameter space such that $P[\theta \in C|\bar{x}] \geq 1-\alpha$, where the probability content is taken from the marginal posterior distribution of $\theta|\bar{x}$. Although any interval (θ_ℓ, θ_u) that brackets $1-\alpha$ of probability content under $\pi(\theta|\bar{x})$ is an acceptable credible region, certain intervals are preferred. One such is a highest posterior density (HPD) region; i.e. a region of points with the "most likely values" of θ. It has the form

$$C_{HPD} = \{\theta : \pi(\theta|\bar{x}) \geq k_\alpha\}$$

where k_α is the largest value which gives $P[\theta \in C|\bar{x}] \geq 1-\alpha$. See Berger (1980, Sec. 4.3.2).

Just as with the Bayes approach to the scale/shape problem, the amount of numerical computing here is substantial, particularly for the integrals in (3.3) and (3.4). Some simplification is available via Stirling's approximation to the gamma function:

$$\Gamma(u) \approx e^{-u+1}(u-1)^{u-\frac{1}{2}}\sqrt{(2\pi)} \quad . \qquad (3.5)$$

This simplifies (3.4) to

$$\pi(\theta|\bar{x}) \approx \frac{\theta^{-\alpha}\exp[\alpha-(1/\theta\beta)]}{\bar{x}\sqrt{(2\pi)}} \frac{\int_{\bar{x}n/\theta}^{B\bar{x}n/\theta} \exp\{u[\frac{\theta}{\bar{x}} - 1]-1\}(\frac{\theta}{\bar{x}} - \frac{1}{u})^{-\theta u/\bar{x}} (\frac{\theta u}{\bar{x}} - 1)^{\frac{1}{2}} du}{\int_{n}^{Bn} \left(\frac{y+\alpha-1}{\bar{x}y+\beta^{-1}} \right)^{\alpha} \left(\frac{1+\alpha/(y-1)}{1+(\beta\bar{x}y)^{-1}} \right)^{y} \left(\frac{\alpha}{y-1} + 1 \right)^{-\frac{1}{2}} dy} . \qquad (3.6)$$

As an alternative to the inverse gamma prior, Martz and Waller (1982, Sec. 9.5.2) suggest a uniform prior on θ. They refer to this as "vague prior information [on] the range of each parameter." In the mean/shape parameterization we can utilize the same idea, i.e. take $\theta \sim U(0,A)$ and, as before, $r \sim U(1,B)$ (independent of θ). In practice, this would be useful when the practitioner's prior interval information is 100% assured, but involves only an upper bound on θ.

The joint posterior distribution of θ and r given $\bar{x}$ becomes

$$\pi(\theta,r|\bar{x}) = n\frac{(nr)^{nr}\bar{x}^{nr-1}e^{-\bar{x}nr/\theta}}{\theta^{nr}\Gamma(nr)J^*} \qquad \begin{matrix} 1 < r < B, \\ 0 < \theta < A. \end{matrix} \quad (3.7)$$

Here

$$J^* = \int_n^{Bn} \frac{y}{y-1}\left[\, 1 - \frac{\gamma(y-1,y\bar{x}/A)}{\Gamma(y-1)} \,\right] dy \quad , \quad (3.8)$$

and

$$\gamma(a,x) = \int_0^x e^{-t}t^{a-1}dt \quad (3.9)$$

is the incomplete gamma function (Abramowitz and Stegun, 1972, Equ. 6.5.2). The marginal on θ given $\bar{x}$ is

$$\pi(\theta|\bar{x}) = \frac{1}{J^*\bar{x}}\int_n^{Bn}\left(\frac{\bar{x}y}{\theta}\right)^y\frac{\exp\{-\bar{x}y/\theta\}}{\Gamma(y)}\,dy \quad . \quad (3.10)$$

Again, Stirling's approximation in (3.5) simplifies (3.10) to

$$\pi(\theta|\bar{x}) \approx \frac{\theta}{\bar{x}^2J^*\sqrt{(2\pi)}}\int_{\bar{x}n/\theta}^{B\bar{x}n/\theta}\exp\left\{u\left[\frac{\theta}{\bar{x}}-1\right]-1\right\}\left(\frac{\theta}{\bar{x}}-\frac{1}{u}\right)^{-\theta u/\bar{x}}\left(\frac{\theta u}{\bar{x}}-1\right)^{\frac{1}{2}}du. \quad (3.11)$$

Before exemplifying these two approaches to interval estimation on θ, it should be noted that there are many other ways to approach Bayesian estimation of θ under the gamma model. One, in particular, that provides some numerical relief over (3.6) or (3.11) places a discrete prior distribution on r (as in Lwin and Singh, 1974). Take $P[r=r_i] = p_i$ $(i=1,\ldots,m)$ such that $\sum_1^m p_i = 1$ and $r_i>1$ (or, alternatively, $r_i>0$, or $0<r_i\leq 1$) for all $i\leq m$. Let $\theta|r_i \sim I\Gamma(\alpha_i,\beta_i)$ where α_i and β_i depend on r_i only through their dependence on the index, i. Under (1.2) this gives $\theta|r_i,\bar{x} \sim I\Gamma(nr_i+\alpha_i,$

$[\bar{x}nr_i+\beta_i^{-1}]^{-1}$). Now,

$$\pi(r_i|\bar{x}) = p_i f(\bar{x}|r_i)/\sum_1^m p_i f(\bar{x}|r_i) \quad .$$

Since $f(\bar{x}|r_i) = \Gamma(nr_i+\alpha_i)/[\bar{x}+(\beta_i nr_i)^{-1}]^{nr_i}(1+\beta_i\bar{x}nr_i)^{\alpha_i}\Gamma(nr_i)\Gamma(\alpha_i)$, we find

$$\pi(r_i|\bar{x}) = \frac{p_i\Gamma(nr_i+\alpha_i)[(\beta_i nr_i)^{-1}+\bar{x}]^{-nr_i}(\beta_i\bar{x}nr_i+1)^{-\alpha}}{\Gamma(nr_i)\Gamma(\alpha_i)S^*} \quad ,$$

where

$$S^* = \sum_{i=1}^m p_i\Gamma(nr_i+\alpha_i)[(\beta_i nr_i)^{-1}+\bar{x}]^{-nr_i}(\beta_i\bar{x}nr_i+1)^{-\alpha}/\Gamma(nr_i)\Gamma(\alpha) \quad . \qquad (3.12)$$

Notice that some calculations in (3.12) can be simplified by using the *beta function*, $\beta(w,z) = \Gamma(z)\Gamma(w)/\Gamma(w+z)$. The posterior on θ given $\bar{x}$ is then

$$\begin{aligned}\pi(\theta|\bar{x}) &= \sum_1^m \pi(r_i|\bar{x})\pi(\theta|r_i,\bar{x}) \\ &= \frac{1}{S^*}\sum_1^m \frac{p_i\exp\left\{-\dfrac{\beta_i^{-1}+\bar{x}nr_i}{\theta}\right\}(nr_i)^{nr_i}\beta_i^{-\alpha_i}}{\Gamma(\alpha_i)\Gamma(nr_i)\,\theta^{nr_i+\alpha_i+1}} \quad .\end{aligned} \qquad (3.13)$$

From (3.13) point estimates (given some loss function) and interval estimates may be constructed.

4. Example

To exemplify the use of this methodology, consider again the serum Mg data in Table 1. The sample size is n=32 and $\bar{x}$=20.5625. The clinical pathologist's prior expectations gave a prior interval of (L,R)=(10,30),

with a prior assurance of about 80%, hence $p_0 = 0.8$.

To see how the approximate approach proposed by Grice and Bain (1980) would have performed, consider again the conditional interval in (1.4). Grice and Bain suggest replacing r with its ML estimate, $\hat{r}$, and then using corrected χ^2 quantiles to account for this use. However, $\hat{r}$ is biased for r, and a perhaps more appropriate value to use is the Anderson and Ray (1975) approximately unbiased estimate $\tilde{r} = (n-3)\hat{r}/n + 2/3n$. The approximate interval for θ is then

$$\left(\frac{2\tilde{r}n\bar{x}}{\chi^2_{1-\beta}(2n\tilde{r})} , \frac{2\tilde{r}n\bar{x}}{\chi^2_{\beta}(2n\tilde{r})} \right) , \tag{4.1}$$

where $0 \leq \beta \leq \alpha$ is a function of n and α (given by Grice and Bain, 1980, Table 2).

For the serum Mg data, the geometric mean is $\tilde{x}$=2.0026, yielding $\tilde{r}$=19.54248. For n=32 and $\alpha/2$=.025, the Grice and Bain tables give $\beta \approx .0177$. The resulting interval is then (18.9367, 22.40669). Thus the data suggest a much lower inherent variability than is evidenced by the prior interval. In such a case one would expect the Bayes credible regions produced from (3.6) or (3.11) to mirror this lower variability and improve upon the prior interval information. If this were not the case, then the procedures could be criticized as being too dependent on the prior intervals.

In order to investigate this expectation, intervals from (3.6) were calculated by employing the modified Martz-Waller algorithm in Section 3. The values of the prior parameters, α and β (δ=4 decimal places were carried), and 95% credible regions are given for a selection of prior intervals in Table 3. The value B=25 was used as an upper bound on r. In all cases, improvement over the prior interval is evidenced, particularly

when this interval is very spread out. Also, all the credible regions reflect the location and spread of the data adequately, allaying any fears that the prior information would overcome the information (in terms of low variability) in the data.

The alternative assumption that $\theta \sim U(0,35)$ suggests the use of the posterior in (3.11). Since this was intended to reflect vague prior information, one expects that the corresponding 95% credible region will be larger than those in Table 3. However, the result shows an HPD region of (18.1176,23.4602), smaller than any of the inverse-gamma based regions. This may come about because the uniform prior assumption "cuts off" the tail probability available with the inverse gamma density. At the very least, this result warns against always associating the concept of vague prior knowledge with uniform distributions.

ACKNOWLEDGMENTS

Thanks go to Drs. G. Casella and C. E. M^{c}Culloch for their helpful suggestions during the preparation of this manuscript. This paper was prepared while the author was with the Biometrics Unit, Cornell University, Ithaca, NY, and is paper no. BU-841-M in the Biometrics Unit Series.

REFERENCES

Abramowitz, M. and Stegun, I.R. (1972), eds. *Handbook of Mathematical Functions*. New York:Dover.

Anderson, C.W. and Ray, W.D. (1975). Improved maximum likelihood estimators for the gamma distribution. *Communications in Statistics* **4**, 437-448.

Bartolucci, A.A. and Dickey, J.M. (1977). Comparative Bayesian and traditional inference for gamma-modeled survival data. *Biometrics* **33**, 343-354.

Berger, J.O. (1980). *Statistical Decision Theory*. New York:Springer-Verlag.

Canavos, G.C. (1971). A Bayesian Approach to Parameter and Reliability Estimation in Failure Time Distributions. Ph.D. Dissertation, Virginia Polytechnic Institute and State University, Blacksburg, Virginia.

Choi, S.C. and Wette, R. (1969). Maximum likelihood estimation of the parameters of the gamma distribution and their bias. *Technometrics* **11**, 683-690.

Elveback, L.R. (1972). How high is high? *Mayo Clinic Proceedings* **47**, 93-97.

Grice, J.V. and Bain, L.J. (1980). Inferences concerning the mean of the gamma distribution. *Journal of the American Statistical Association* **75**, 409-419.

Linhart, H. (1965). Approximate confidence limits for the coefficient of variation of gamma distributions. *Biometrics* **21**, 733-738.

Lwin, T. and Singh, N. (1974). Bayesian analysis of the gamma distribution model in reliability analysis. *IEEE Trans. Reliability* **R-23**, 314-319.

Martz, H.F. and Waller, R.A. (1982). *Bayesian Reliability Analysis*. New York:Wiley.

Mehta, J.S. and Srinivasan, R. (1971). Estimation of the mean by shrinkage to a point. *Journal of the American Statistical Association* **66**, 86-90.

Reed, A.H., Henry, R.J., and Mason, W.B. (1971). Influence of statistical method used on the resulting estimate of normal range. *Clinical Chemistry* **17**, 275-284.

Tummala, V.M.R. and Sathe, P.T. (1978). Minimum expected loss estimators of reliability and parameters of certain lifetime distributions. *IEEE Trans. Reliability* **R-27**, 283-285.

Woodward, W.A. and Gray, H.L. (1975). Minimum variance unbiased estimation in the gamma distribution. *Communications in Statistics* **4**, 907-922.

Table 1
Serum Mg levels in mg/dℓ×10 for healthy felines

19	21	21	21	26	23	23	23
23	21	22	23	22	20	23	23
24	26	21	20	17	18	18	18
19	18	18	18	17	17	18	17

Table 2
Grice and Bain (1980) coverage probabilities; α=.05.

		r		
n	β(n,.05)	2	4	8
10	.0086	.9613	.9613	.9590
20	.0159	.9399	.9371	.9385
40	.0203	.9335	.9318	.9318

Table 3

95% credible regions for inverse gamma prior on θ given (L,U); $p_o = .8$

L	U	λ	α	β	95% limits lower	upper
5	30	.7782	2.330	.0071	18.3779	24.1519
5	40	.9031	1.660	.0135	18.1542	23.5880
10	20	.3010	13.166	.0098	17.0764	22.5158
10	25	.3979	8.330	.0204	17.3010	22.7391
10	30	.4771	6.000	.0317	17.5067	22.8924
10	40	.6021	3.667	.0654	17.7018	23.0241
15	25	.2218	16.510	.0132	16.9019	23.1781
15	30	.3010	13.166	.0148	16.8334	22.9994
15	40	.4260	7.4176	.0356	17.3239	22.7619

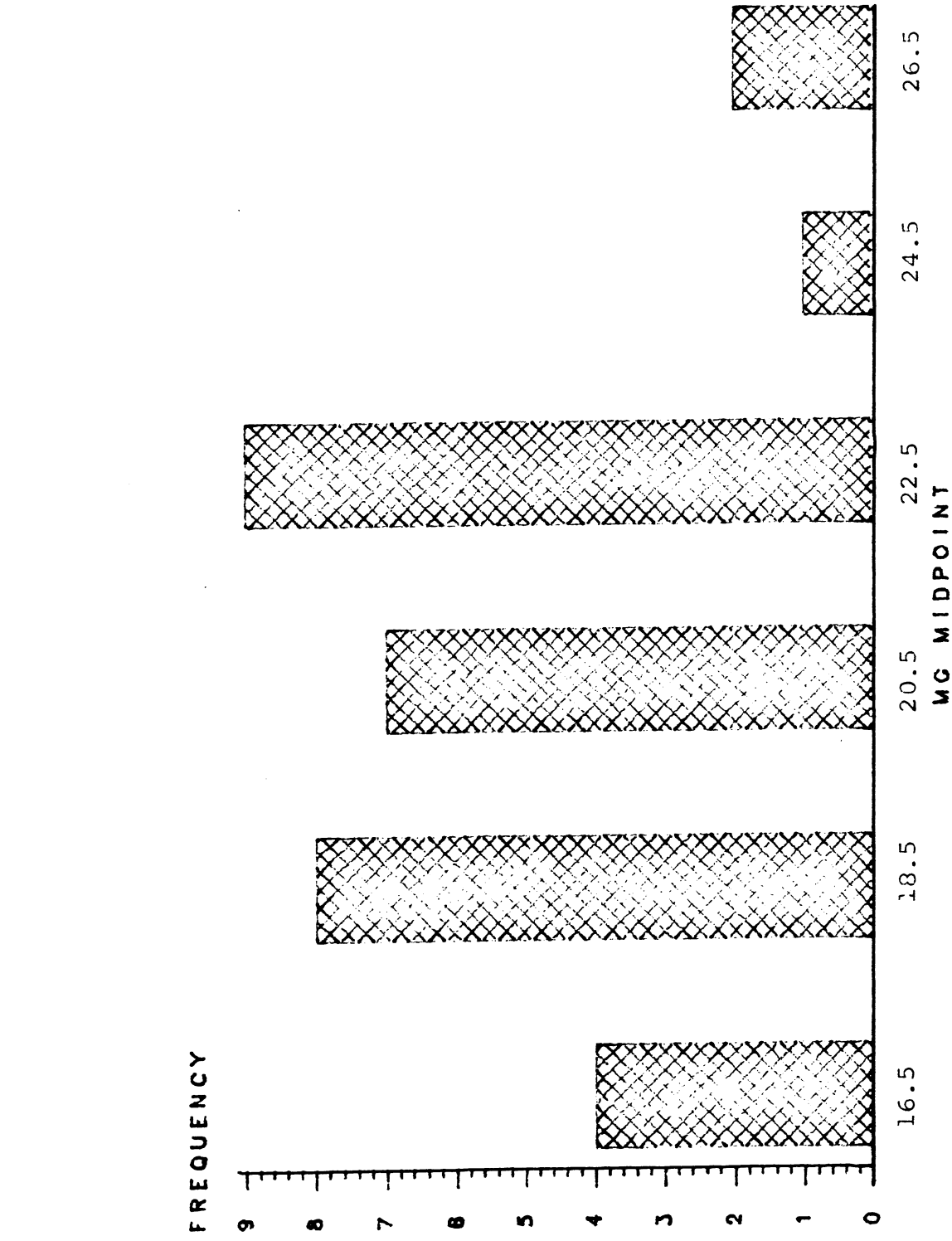

Figure 1. Frequency histogram for blood Mg data.