CHARACTERISTICS FOR DISTINGUISHING AMONG BALANCED INCOMPLETE BLOCK DESIGNS WITH REPEATED BLOCKS

by

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Abstract

The class ß of balanced incomplete block designs with parameters v, b, r, k, λ considered here includes designs with repeated blocks. Attention to date has centered on constructing the members of ß and on finding the minimum number of distinct blocks $d \leq b$. Under the classical model, the usual statistical characteristics do not distinguish among members of ß; however, effects related to block totals can be used as distinguishing characteristics. A competing effects model is introduced and is shown to distinguish among members of ß. This model is appropriate for intercropping, marketing, and survey investigations. The model is applied to the ten nonisomorphic BIB designs with parameters 7, 21, 9, 3, 3. A computer algorithm for obtaining the pairwise treatment by block incidence matrix of this model is given.

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1. <u>Introduction</u>. A balanced incomplete block (BIB) design has parameters v, b, r, k, and λ, where v is the number of treatments, b is the number of blocks, r is the number of times each of the v treatments is repeated, k is the block size, and λ is the number of times any pair of treatments occurs together in the b blocks. This paper treats the class $\beta = \beta(v,b,r,k,\lambda)$ of all BIB designs for given v, b, r, k, λ with k < v and the binary structure that the number of occurrences of the ith treatment in the jth block is restricted to be $n_{ij} = 0$ or 1. In such a design, there are $\binom{v}{k} = b^*$ possible distinct blocks. To consider a BIB design as a repeated block design, specify the additional parameter d that gives the number of distinct blocks present in the design. Then $d \le \min(b,b^*)$, and $\sum_{j=1}^{b^*} w_j = b$, where w_j denotes the number of times the jth possible block occurs in the design. A design in β is a <u>repeated block design</u> if and only if (iff) d < b; there are no repeated blocks iff d = b. The existence, construction, and applications of BIB designs with repeated blocks have been treated by Foody and Hedayat (1977), Van Lint (1973), and others.

The existence of repeated block designs in β raises the problem of obtaining criteria that distinguish among members of β. This paper examines several criteria and demonstrates which ones will distinguish among members of β and which will not. Section 2 discusses the estimation of treatment effect contrasts from the usual additive linear model and notes that the members of β cannot be distinguished by standard properties. However, the estimation of block effect contrasts in Section 3 reveals some distinguishing characteristics and some invariance properties. In Section 4, the class $\beta(7,21,9,3,3)$ is considered; nine of its ten members are repeated block designs. A competing effects model useful in such areas as intercropping and marketing studies is introduced in Section 5. This model

distinguishes among the members of class ß for which d assumes different values. A computer algorithm for obtaining the pairwise treatment by block incidence matrix for each pair of treatments and the ranks of the v submatrices of order v-1 is given in Section 6.

2. Estimating treatment effect contrasts for BIB designs with repeated blocks. From standard design theory (see Federer (1955, Ch. 11-13) or Kempthorne (1952, Ch. 26)), the following theorem can easily be verified by determining that the coefficient matrices of the reduced normal equations for estimating contrasts of treatment effects after eliminating general mean and block effects are identical for all members of ß.

THEOREM 2.1. All members of the class ß = ß(v,b,r,k,λ) have the following in common:

(i) the estimators for contrasts of treatment effects when the block effects are assumed fixed (or random), and the variances of the estimators,

(ii) the expected value of the blocks (eliminating treatment effects) mean square when the block effects are random, and

(iii) A-, D-, and E-optimality criteria for estimating treatment effects.

In the case of analysis with recovery of interblock information, some differences may arise in the estimation of σ_β^2, the block effects variance. Consider a design in class ß with d distinct blocks and let σ^2 be the intrablock error variance. Then the blocks eliminating treatments sum of squares can be partitioned further. Such a partitioning, along with the expected values of mean squares, is given in Table 2.1; the mean squares are derived as follows.

Table 2.1.

Partitioning of blocks (eliminating treatments) sum of squares
and expectations of mean squares

Source	d.f.	S.S.	M.S.	E(M.S.)
Blocks (elim. treat.)	$b-1$	SS_1	B_1	$\sigma^2 + (bk-v)\sigma_\beta^2/(b-1)$
Distinct blocks (elim. treat.)	$d-1$	SS_2	B_2	$\sigma^2 + (dk-v)\sigma_\beta^2/(d-1)$
Repeated blocks	$b-d$	SS_3	B_3	$\sigma^2 + k\sigma_\beta^2$

The sum of squares for blocks eliminating treatments, SS_1, is obtained
as the sum of the products of the estimated block effects $\hat{\beta}_j$ and the
right-hand sides of the reduced normal equations:

$$SS_1 = \sum_{j=1}^{b} \hat{\beta}_j \left(Y_{.j} - \sum_{i=1}^{v} n_{ij} \bar{Y}_{i.} \right) \qquad (2.1)$$

where $Y_{.j}$ is the sum of responses for the k treatments in block j, $\bar{Y}_{i.}$ is
the mean response for the r occurrences of treatment i in the design, and
n_{ij} is as defined in Section 1. The jth term in this sum corresponds to
the jth block in the design. To write the sums of squares SS_2 and SS_3,
notation explicitly specifying the design's repeated block structure is
needed. Let $g=1,\cdots,d$ index the d distinct blocks that occur in the
design, and let w_g denote the number of times the gth of these block
patterns occurs. Each block in the design corresponds to a pair (g,h) with
$1 \le g \le d$, $1 \le h \le w_g$, which denotes the hth occurrence of the gth block
pattern in the design. Using this notation, (2.1) can be rewritten as

$$SS_1 = \sum_{g=1}^{d} \sum_{h=1}^{w_g} \hat{\beta}_{(g,h)} \left[Y_{.(g,h)} - \sum_{i=1}^{v} n_{i(g,h)} \bar{Y}_{i(\cdot,\cdot)} \right]$$

where $Y_{i(g,h)}$ is the response for treatment i in the hth occurrence of the
g^{th} block pattern; $\bar{Y}_{i(\cdot,\cdot)} = \bar{Y}_{i.}$; and $Y_{.(g,h)} = Y_{.j}$, $n_{i(g,h)} = n_{ij}$, and

$\hat{\beta}_{(g,h)} = \hat{\beta}_j$ for j chosen so that the jth block corresponds to the pair (g,h).

Define the estimated effects of the distinct block patterns,

$$\hat{\beta}_{(g,\cdot)} = (1/w_g) \sum_{h=1}^{w_g} \hat{\beta}_{(g,h)} \qquad \text{for } g=1,\cdots,d \quad .$$

Let $\underset{\sim}{W}$ be the d×d diagonal matrix with entries $w_1, w_2, \cdots, w_d$ on the diagonal, and let $\underset{\sim}{N}^*$ be the v×d matrix whose $(i,g)^{th}$ element is $\sum_{h=1}^{w_g} n_{i(g,h)}$. The estimated distinct block effects vector $\hat{\underset{\sim}{\beta}}^* = [\hat{\beta}_{(1,\cdot)}, \cdots, \hat{\beta}_{(d,\cdot)}]'$ is the solution of the reduced normal equations

$$[k\underset{\sim}{W} - r^{-1}\underset{\sim}{N}^{*'}\underset{\sim}{N}^*] \hat{\underset{\sim}{\beta}}^* = \begin{bmatrix} Y._{(1,\cdot)} \\ \vdots \\ Y._{(d,\cdot)} \end{bmatrix} - \underset{\sim}{N}^{*'} \begin{bmatrix} \bar{Y}_{1(\cdot,\cdot)} \\ \vdots \\ \bar{Y}_{v(\cdot,\cdot)} \end{bmatrix} \quad , \qquad (2.2)$$

where $Y._{(g,\cdot)}$ is the sum of the responses to the kw_g treatment occurrences in the w_g blocks having pattern g. The sum of squares for distinct blocks eliminating treatments, SS_2, is obtained as the sum of the products of the estimated distinct block effects and the right-hand sides of the reduced normal equations (2.2):

$$SS_2 = \sum_{g=1}^{d} \hat{\beta}_{(g,\cdot)} \left[Y._{(g,\cdot)} - \sum_{i=1}^{v} \sum_{h=1}^{w_g} n_{i(g,h)} \bar{Y}_{i(\cdot,\cdot)} \right] \quad .$$

The sum of squares for repeated blocks, SS_3, can be derived as $SS_3 = SS_1 - SS_2$. Mean squares are given by

$$B_1 = SS_1/(b-1), \qquad B_2 = SS_2/(d-1), \qquad B_3 = SS_3/(b-d) \quad .$$

The expected values of these mean squares are derived routinely by methods described in Federer (1955, Sec. 13.2) or Yates (1940).

The block effects variance σ_β^2 can be estimated using the intrablock error mean square and any one of B_1, B_2, or B_3. The estimated σ_β^2 will give different accuracies for the members of the class β if either B_2 or B_3 is used. However, it is inefficient to use B_2 or B_3 in estimating σ_β^2 because of the reduced degrees of freedom, and it is a common practice to use B_1 while estimating σ_β^2; in such cases, the members of β are indistinguishable as indicated in Theorem 2.1(ii).

There are no distinguishing criteria among members of β in estimating treatment effects when individual treatment yields in each block are available and when only the treatment effects themselves are estimated.

3. Estimating block effect contrasts for BIB designs with repeated blocks. Let $\underset{\sim}{N}$ be the $v \times b$ incidence matrix of the design. Let $\underset{\sim m}{I}$ denote the identity matrix of order m, and $\underset{\sim m,m'}{J}$, the $m \times m'$ matrix with every entry equal to 1. The coefficient matrix $\underset{\sim}{D}$ in estimating the block effects vector $\underset{\sim}{\beta} = (\beta_1, \cdots, \beta_b)'$ is given by

$$\underset{\sim}{D} = k \underset{\sim b}{I} - r^{-1} \underset{\sim}{N}' \underset{\sim}{N} \quad .$$

The eigenvalues of $\underset{\sim}{D}$ are easily verified to be $\phi_0 = 0$, $\phi_1 = \lambda v/r$, and $\phi_2 = k$ with respective multiplicities $\alpha_0 = 1$, $\alpha_1 = v-1$, and $\alpha_2 = b-v$. Noting that if $\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_2, \cdots, \underset{\sim}{\xi}_{v-1}$ is a complete set of orthonormal eigen-

vectors corresponding to the eigenvalue $r-\lambda$ of $\underset{\sim}{N}\underset{\sim}{N}'$, then $\underset{\sim}{\eta}_1, \underset{\sim}{\eta}_2, \cdots, \underset{\sim}{\eta}_{v-1}$ is a complete set of orthonormal eigenvectors of $\underset{\sim}{N}'\underset{\sim}{N}$ corresponding to the eigenvalue $r-\lambda$, where

$$\underset{\sim}{\eta}_i = (r-\lambda)^{-\frac{1}{2}}\underset{\sim}{N}'\underset{\sim}{\xi}_i, \qquad i=1,2,\cdots,v-1 \quad ,$$

the orthogonal idempotent matrix corresponding to the eigenvalue ϕ_1 of $\underset{\sim}{D}$ is

$$\underset{\sim}{A}_1 = \sum_{i=1}^{v-1} \underset{\sim}{\eta}_i\underset{\sim}{\eta}_i' = (r-\lambda)^{-1}[\underset{\sim}{N}'\underset{\sim}{N} - (k^2/v)\underset{\sim}{J}_{b,b}] \quad .$$

The idempotent matrix corresponding to the eigenvalue ϕ_0 of $\underset{\sim}{D}$ is $\underset{\sim}{A}_0 = \frac{1}{b}\underset{\sim}{J}_{b,b}$, and hence the orthogonal idempotent matrix corresponding to the eigenvalue ϕ_2 of $\underset{\sim}{D}$ is

$$\underset{\sim}{A}_2 = \underset{\sim}{I}_b - \underset{\sim}{A}_0 - \underset{\sim}{A}_1 \quad .$$

Use

$$\underset{\sim}{D}^- = \frac{1}{\phi_1}\underset{\sim}{A}_1 + \frac{1}{\phi_2}\underset{\sim}{A}_2 = \frac{1}{\lambda vk}\underset{\sim}{N}'\underset{\sim}{N} + \frac{1}{k}\underset{\sim}{I}_b - \frac{\lambda v+rk}{\lambda v^2 r}\underset{\sim}{J}_{b,b}$$

as a generalized inverse of $\underset{\sim}{D}$ in solving the normal equations for $\underset{\sim}{\hat{\beta}}$ and finding the variances of estimated contrasts of block effects. Observe that the estimators for block effect contrasts, as well as their variances, may differ among members of the class β, because all the designs in β may not have the same $\underset{\sim}{N}'\underset{\sim}{N}$. In fact, if S_j and S_h are any two blocks of the design that have π treatments in common, the $(j,h)^{th}$ element of $\underset{\sim}{N}'\underset{\sim}{N}$ is π and

$$Var(\hat{\beta}_j-\hat{\beta}_h) = 2\sigma^2(\lambda v+k-\pi)/(\lambda vk) \quad . \tag{3.1}$$

This is a distinguishing characteristic of the members of the class β, which is stated in the following theorem.

THEOREM 3.1. <u>Members of class ß differ in their estimators for block effect</u> <u>contrasts. If the jth and h^{th} blocks have π treatments in common, the</u> <u>variance of the estimated elementary contrast of the jth and h^{th} block</u> <u>effects is as given in (3.1).</u>

Because π can take at most k+1 values, namely, $0,1,\cdots,k$, there are at most k+1 possible variances for the estimated elementary block effect contrasts. The number of times a particular variance arises is determined by the number of pairs of blocks that provide the required intersection frequency in formula (3.1). Even when designs in ß differ in the distribution of the variances of estimated elementary block effect contrasts, the following holds:

THEOREM 3.2. <u>All members of class ß have identical values of (i) the</u> <u>average variance of all estimated elementary block effect contrasts, and</u> <u>(ii) the variance of all variances of estimated elementary block effect</u> <u>contrasts.</u>

PROOF. Let $\underset{\sim}{D}^{-} = (d^{jh})$. Then

$$\sum_{j=1}^{b} d^{jh} = 0, \qquad \text{for every } h=1,\cdots,b \ .$$

Analogous to equation (4.3.4) of Raghavarao (1971), the average variance of all estimated elementary contrasts of block effects is equal to

$$\frac{2\sigma^2}{(b-1)} \ \text{tr}(\underset{\sim}{D}^{-}) = \frac{2\sigma^2}{(b-1)} \ \frac{(\lambda vb+bk-\lambda v-rk)}{\lambda vk} \ ,$$

which is the same for all members of class ß. This proves (i).

To prove (ii), it is sufficient to prove that $\sum\limits_{\substack{j,h=1 \\ j<h}}^{b} \{\text{Var}(\hat{\beta}_j - \hat{\beta}_h)\}^2$ is

constant for all members of $\mathcal{B}$. Since $\bar{D}$ is symmetric, straightforward alge-
braic manipulations yield:

$$\sum_{\substack{j,h=1 \\ j<h}}^{b} \{\text{Var}(\hat{\beta}_j - \hat{\beta}_h)\}^2 = \sigma^4 \sum_{\substack{j,h=1 \\ j<h}}^{b} (d^{jj} + d^{hh} - 2d^{jh})^2$$

$$= \sigma^4 \left\{ b \sum_{j=1}^{b} (d^{jj})^2 + 2\text{tr}[(\underset{\sim}{D}^-)^2] + [\text{tr}(\underset{\sim}{D}^-)]^2 \right\}$$

$$= \frac{\sigma^4}{\lambda^2 v^2 k^2} \left\{ (bk + \lambda vb - \lambda v - rk)^2 + 2[r^2 k^2 (v-1) + (b-v)\lambda^2 v^2] \right.$$

$$\left. + [rk(v-1) + (b-v)\lambda v]^2 \right\} \quad ,$$

which is again constant for all members of class $\mathcal{B}$. □

The members of the class $\mathcal{B}$ differ in the estimability of linear
functions of treatment effects from block totals only and the degrees of
freedom, b-d, for an error mean square for treatment effects estimated from
block totals. This is the case in problems considered by Federer (1979)
and Raghavarao and Federer (1979).

4. <u>Repeated block BIB designs with v=7, b=21, r=9, k=3, λ=3</u>. The
complete class of balanced incomplete block designs with v=7, b=21, r=9,
k=3, λ=3 for which every n_{ij} = 0 or 1 is given in Table 4.1. It has been
shown by Seiden (1977) that any other such design must be isomorphic to one
of the ten designs given. Each of these designs is a repeated block design
except the last one, which has d=21.

There are four different variances possible for an estimated elemen-
tary contrast of block effects for this class of designs. These are
$42\sigma^2/63$, $44\sigma^2/63$, $46\sigma^2/63$, and $48\sigma^2/63$. The frequencies of occurrence of

Table 4.1.

Values of w_j for all possible values of d in the class ß
with parameters v=7, b=21, r=9, k=3, and λ=3*

Block composition	Number of distinct blocks d									
	7	11	13	14	15	17	18	19	20	21
1 2 3	3	3	3	2	3	2	2	3	2	1
1 2 4	-	-	-	1	-	1	-	-	-	1
1 2 5	-	-	-	-	-	-	-	-	-	1
1 2 6	-	-	-	-	-	-	-	-	1	-
1 2 7	-	-	-	-	-	-	1	-	-	-
1 3 4	-	-	-	-	-	-	-	-	1	-
1 3 5	-	-	-	-	-	-	-	-	-	-
1 3 6	-	-	-	1	-	1	1	-	-	1
1 3 7	-	-	-	-	-	-	-	-	-	1
1 4 5	3	3	-	2	1	2	2	1	1	1
1 4 6	-	-	1	-	1	-	-	1	-	-
1 4 7	-	-	2	-	1	-	1	1	1	1
1 5 6	-	-	2	-	1	-	1	1	1	1
1 5 7	-	-	1	1	1	1	-	1	1	-
1 6 7	3	3	-	2	1	2	1	1	1	1
2 3 4	-	-	-	-	-	-	-	-	-	1
2 3 5	-	-	-	-	-	1	1	-	-	-
2 3 6	-	-	-	-	-	-	-	-	-	-
2 3 7	-	-	-	1	-	-	-	-	1	1
2 4 5	-	-	1	-	1	-	-	1	1	-
2 4 6	3	2	2	2	2	1	2	1	1	1
2 4 7	-	1	-	-	-	1	1	1	1	-
2 5 6	-	1	-	1	-	1	1	1	1	1
2 5 7	3	2	2	2	2	1	1	1	1	1
2 6 7	-	-	1	-	1	1	-	1	-	1
3 4 5	-	-	2	1	1	-	1	1	1	1
3 4 6	-	1	-	-	-	1	1	1	1	1
3 4 7	3	2	1	2	2	2	1	1	-	-
3 5 6	3	2	1	2	2	1	-	1	1	1
3 5 7	-	1	-	-	-	1	1	1	1	1
3 6 7	-	-	2	-	1	-	1	1	1	-
4 5 6	-	-	-	-	-	1	-	-	-	-
4 5 7	-	-	-	-	-	-	-	-	-	1
4 6 7	-	-	-	1	-	-	-	-	1	1
5 6 7	-	-	-	-	-	-	1	-	-	-

* Value w_j = 0 is indicated by -.

these variances are given in Table 4.2. In most cases, they differ for various values of d. For d=14 and d=15, the frequencies are identical, and the same is true for d=18 and d=19. Thus, the frequencies of occurrence of the four types of variances may be used to distinguish among most members of the class. The frequency of variance $42\sigma^2/63$ decreases from 21 to zero as the number of distinct blocks increases from 7 to 21. The reverse holds for the variance $48\sigma^2/63$. For any given number d of distinct blocks, the sum of the frequencies of these two variances is 21, and the frequency of the variance $44\sigma^2/63$ is three times the frequency of $48\sigma^2/63$. The frequency of the variance $46\sigma^2/63$ is simply $\binom{21}{2}$ = 210 minus the sum of frequencies of the other three variances.

A statistical model will now be discussed in which the structure of the individual designs in ß plays an important role. The composition of the blocks and the number of distinct blocks in a design will influence the analysis under this competing effects model.

Table 4.2.

Frequency of occurrence of variance of the estimated elementary contrast of block effects

Number of distinct blocks	Variance			
	$42\sigma^2/63$	$44\sigma^2/63$	$46\sigma^2/63$	$48\sigma^2/63$
7	21	0	189	0
11	13	24	165	8
13	9	36	153	12
14	7	42	147	14
15	7	42	147	14
17	4	51	138	17
18	3	54	135	18
19	3	54	135	18
20	1	60	129	20
21	0	63	126	21

5. A competing effects model: estimability of effects under various repeated block designs. It was seen in Section 2 that the intra- and interblock analyses do not differentiate in the class of repeated block designs. In this section, a new model that accounts for competing effects is introduced; such a model can find wide applications in marketing, agriculture, and other areas. For example, each of b stores may be willing to stock k of v items, but not all v items. Sales for each item would be available. Note that the k items would be in competition. In agriculture, k of v crops could compose a cropping system where responses of each crop would be available. If the k crops are grown in alternate rows or inter-mingled in the same row, the crops would be competing for food, water, and light. Note that the competition may increase the sales or yields over situations with no competition, in which case competition would be highly desirable.

Consider a BIB design with parameters v, b, r, k, λ, with or without repeated blocks. Assume that there are b experimental units and that the jth unit receives all the treatments of the jth block at the same time. Assume further that all units are homogeneous. Since each unit receives several treatments simultaneously, the observation resulting from any unit receiving a particular treatment will have components of competing effects of other treatments in that unit in addition to the usual treatment effect of that particular treatment. Let S_j denote the jth block of the design, let i denote a treatment in S_j, and let $y_{i(S_j)}$ denote the observation of the ith treatment used on the jth experimental unit. Then specify the competing effects model

$$y_{i(S_j)} = \mu + \tau_i + \sum_{\substack{\ell \varepsilon S_j \\ \ell \neq i}} \gamma_{i(\ell)} + e_{i(S_j)} , \qquad (5.1)$$

where μ is the general mean, τ_i is the ith treatment effect, $\gamma_{i(\ell)}$ is the competing effect of the ℓth treatment on the ith treatment, and $e_{i(S_j)}$ is a random error term. For the present analysis, it will be assumed that the random errors are independently distributed $N(0,\sigma^2)$. It may be desirable to account for other variabilities related to more complicated competing effects by partitioning the $e_{i(S_j)}$ into component parts. Let G be the grand total, T_i the total of all observations receiving the ith treatment, and $P_{i\ell}$ the total of all observations on the ith treatment on those units where the ℓth treatment is also present. Define the v×1 vector $\underset{\sim}{T} = (T_1,T_2,\cdots,T_v)'$, the (v-1)×1 vector $\underset{\sim}{P}_{(i)} = (P_{i1},P_{i2},\cdots,P_{i,i-1},P_{i,i+1},\cdots,P_{iv})'$ for $i=1,\cdots,v$, and the v(v-1)×1 vector

$$\underset{\sim}{P} = \left(\underset{\sim}{P}'_{(1)}, \underset{\sim}{P}'_{(2)}, \cdots, \underset{\sim}{P}'_{(v)} \right)'$$

$$= \left(P_{12},P_{13},\cdots,P_{1v},P_{21},P_{23},\cdots,P_{2v},\cdots,P_{v1},P_{v2},\cdots,P_{v,v-1} \right)' .$$

Also define the v×1 parameter $\underset{\sim}{\tau} = (\tau_1,\tau_2,\cdots,\tau_v)'$, the (v-1)×1 vector $\underset{\sim}{\chi}_{(i)} = (\gamma_{i(1)},\gamma_{i(2)},\cdots,\gamma_{i(i-1)},\gamma_{i(i+1)},\cdots,\gamma_{i(v)})'$ for $i=1,\cdots,v$, and the v(v-1)×1 vector $\underset{\sim}{\chi} = (\underset{\sim}{\chi}'_{(1)},\underset{\sim}{\chi}'_{(2)},\cdots,\underset{\sim}{\chi}'_{(v)})'$. Let $\underset{\sim}{N}$ be the usual incidence matrix of the BIB design. In addition, introduce v pairwise treatment by block incidence matrices $\underset{\sim}{M}_1,\underset{\sim}{M}_2,\cdots,\underset{\sim}{M}_v$, each of order (v-1)×b. The rows of $\underset{\sim}{M}_i$ correspond to the treatment pairs (i,1), (i,2), $\cdots$, (i,i-1), (i,i+1), $\cdots$, (i,v) and columns of $\underset{\sim}{M}_i$ to blocks. Put 1 or 0 in the (i,ℓ),j position of $\underset{\sim}{M}_i$ according to whether the pair of treatments (i,ℓ) is or is not in S_j.

The normal equations for the competing effects model (5.1) can then be obtained as:

$$
\begin{bmatrix}
bk & r\underset{\sim}{J}_{1,v} & \lambda\underset{\sim}{J}_{1,v(v-1)} \\[2ex]
r\underset{\sim}{J}_{v,1} & r\underset{\sim}{I}_{v} & \underset{\sim}{I}_{v} \otimes \lambda\underset{\sim}{J}_{1,v-1} \\[2ex]
\lambda\underset{\sim}{J}_{v(v-1),1} & \underset{\sim}{I}_{v} \otimes \lambda\underset{\sim}{J}_{v-1,1} & \underset{\sim}{D}(\underset{\sim}{M}_1\underset{\sim}{M}_1',\underset{\sim}{M}_2\underset{\sim}{M}_2',\cdots,\underset{\sim}{M}_v\underset{\sim}{M}_v')
\end{bmatrix}
\begin{bmatrix}
\hat{\mu} \\[2ex]
\hat{\underset{\sim}{\tau}} \\[2ex]
\hat{\underset{\sim}{\chi}}
\end{bmatrix}
=
\begin{bmatrix}
G \\[2ex]
\underset{\sim}{T} \\[2ex]
\underset{\sim}{P}
\end{bmatrix} ,
$$

where '$\hat{}$' over a parameter indicates its least squares estimator, $\underset{\sim}{D}(\underset{\sim}{M}_1\underset{\sim}{M}_1',\cdots,\underset{\sim}{M}_v\underset{\sim}{M}_v')$ is a block-diagonal square matrix of order $v(v-1)$ with matrices $\underset{\sim}{M}_i\underset{\sim}{M}_i'$ on the diagonal and zeros elsewhere, and $\otimes$ denotes the Kronecker product of matrices. The reduced normal equations for estimating $\hat{\underset{\sim}{\chi}}_{(i)}$ are

$$
\underset{\sim}{F}_i\hat{\underset{\sim}{\chi}}_{(i)} = \underset{\sim}{P}_{(i)} - (\lambda/r)T_i\underset{\sim}{J}_{v-1,1} \qquad \text{for } i=1,\cdots,v ,
$$

where the $(v-1)\times(v-1)$ matrix $\underset{\sim}{F}_i$ is defined by

$$
\underset{\sim}{F}_i = \underset{\sim}{M}_i\underset{\sim}{M}_i' - (\lambda^2/r)\underset{\sim}{J}_{v-1,v-1} . \tag{5.2}
$$

In addition,

$$
\hat{\tau}_i = (1/r)T_i - [1/(vr)]G \qquad \text{for } i=1,\cdots,v .
$$

The ANOVA table can then be computed and is shown in Table 5.1.

Table 5.1.

An analysis of variance for one replicate for a competing effects design
of v treatments in groups of size k

Source of variation	Degrees of freedom	Sum of squares
Treatment effects	$v-1$	$\sum_{i=1}^{v} T_i^2/r - G^2/vr$
Competing effects of pairs	$s = \sum_{i=1}^{v} \text{Rank}(\underset{\sim}{F}_i)$	$\sum_{i=1}^{v} \hat{\ell}'_{(i)} \underset{\sim}{P}_{(i)}$
Remainder	$vr-v-s$	by subtraction
Total	$vr-1$	$\sum y_{i(S_j)}^2 - G^2/vr$

An analysis of variance for m replicates of the above in a
randomized complete block design (rcbd)

Source of variation	Degrees of freedom	Sum of squares
Total	mvr	
Correction for mean	1	usual rcbd computation
Blocks	$m-1$	
Treatment effects = T	$v-1$	
Competing effects of pairs = C	s	see above
Remainder = R	$vr-v-s$	
Blocks × T		
Blocks × C	$(vr-1)(m-1)$	by subtraction
Blocks × R		

The ten designs listed in Table 4.1 differ in providing the degrees of freedom for competing effects shown in Table 5.2. The total degrees of freedom s are identical for d = 14 and 19, and these two designs are the exceptions to the pattern that s decreases as d decreases. The ranks of $F_i = M_i M_i' - J_{6,6}$ are identical for all $i=1,2,\cdots,7$ when d = 7, 14, 19, and 21.

For d = 7, two estimable contrasts among the v-1 = 6 parameters $\gamma_{1(\ell)}$ are

$$\gamma_{1(2)} + \gamma_{1(3)} - \gamma_{1(4)} - \gamma_{1(5)}$$

and

$$\gamma_{1(2)} + \gamma_{1(3)} + \gamma_{1(4)} + \gamma_{1(5)} - 2\gamma_{1(6)} - 2\gamma_{1(7)} \quad .$$

Estimable contrasts of the same form can be constructed from the six terms $\gamma_{i(\ell)}$ with ℓ varying from 1 to 7 ($\ell \neq i$) and i a fixed integer between 2 and 7. In order to obtain the v(v-2) = 35 estimable contrasts among the $\gamma_{i(\ell)}$'s, it is necessary to have d = 21 in this class of designs. For d = 20, only four linearly independent estimable contrasts exist among the $\gamma_{5(\ell)}$; one such set is $\gamma_{5(1)} - \gamma_{5(2)}$, $\gamma_{5(1)} - \gamma_{5(3)}$, $\gamma_{5(4)} - \gamma_{5(6)}$, and $\gamma_{5(4)} - \gamma_{5(7)}$. Thus contrasts like $\gamma_{5(1)} - \gamma_{5(4)}$ are not estimable. Consequently, there are 34 estimable contrasts among the terms $\gamma_{i(\ell)}$. Similar results hold for smaller values of d.

6. **An algorithm for obtaining the rank of F_i.** The ranks of the matrices F_i defined in (5.2) are needed for determining the estimability of contrasts and the ANOVA tables of the designs treated in Section 5. Their importance is shown by Tables 5.1 and 5.2 and the accompanying discussion. Analysis of competing effects models by computer calculations will now be described. An algorithm for finding the rank of a matrix F_i

Table 5.2.

Ranks of F_i for designs of Table 4.1

i	number d of distinct blocks									
	7	11	13	14	15	17	18	19	20	21
1	2	2	3	4	4	4	4	4	5	5
2	2	3	3	4	3	5	4	4	5	5
3	2	3	3	4	3	4	5	4	5	5
4	2	3	4	4	4	4	4	4	5	5
5	2	3	4	4	4	5	5	4	4	5
6	2	3	4	4	4	5	5	4	5	5
7=v	2	3	4	4	4	4	5	4	5	5
sum = s	14	20	25	28	26	31	32	28	34	35

is implemented in the computer programs in the APL language appearing in Figure 6.1. The values of the BIB design parameters V,B,R,K, and LAMBDA (=v,b,r,k, and λ) are required as inputs.

The first program, BLPAT, generates the B × K matrix BL whose rows are the blocks that actually occur in the design being analyzed. It uses the values of V, B, and K; a BSTAR × K matrix BPOS whose rows are all possible blocks of size K, e.g., the 35 × 3 matrix in the left margin of Table 4.1; and a vector BF of length BSTAR whose elements are the frequencies with which the blocks given by rows of BPOS occur, e.g., any column in the body of Table 4.1, where dashes represent zeros. The block pattern matrix BL is created by entering each possible block as a row of BL a number of times equal to the frequency of the block's occurrence in the design.

The second program, INCID, computes the pairwise treatments-block incidence matrix MI(=M_i) and obtains the rank of F_i for I=i=1,$\cdots$,V. It begins by initializing MI as a V × B matrix of 0's. Inspecting each block

```
      ∇ BLPAT BF;J
[1]   ⍝ INPUTS: V,K,B; BPOS[B*×K]; BF[B*]  (SEE TEXT)
[2]   ⍝ OUTPUTS: BL[B×K]; BSTAR (SEE TEXT)
[3]   →ST×ι∧/(B,ρBPOS)=(+/BF),(BSTAR←(!V)÷×/!K,V-K),K
[4]   →0,ρ□←'** NO GO: B≠SUM OF BF ENTRIES OR RANK OF BPOS IS BAD'
[5]   ST:BL←(0,K)ρJ←1
[6]   LP:BL←BL,[1](BF[J],K)ρBPOS[J;]
[7]   →LP×ιBSTAR≥J←J+1
      ∇
```

```
      ∇ INCID I;J
[1]   ⍝ INPUTS: V,B,R,LAMBDA; BL (SEE TEXT)
[2]   ⍝ OUTPUTS: MI; MM; RK=RANK(FI) (SEE TEXT)
[3]   MI←(V,B)ρ1-J←1
[4]   CON:→UP×ι0=I∈BL[J;]
[5]   MI[BL[J;];J]←1
[6]   UP:→CON×ιB≥J←J+1
[7]   'INCIDENCE MATRIX MI FOR I = ',▼I
[8]   □←MI←(φIφ0,(V-1)ρ1)/[1] MI
[9]   'MI×MI'' IS '
[10]  →RNK×ι∧/(2×LAMBDA)=(+/MM),+/□←MM←MI+.×⍉MI
[11]  →0,ρ□←'** NO GO: SOME ROW OR COLUMN SUM ≠ 2×LAMBDA'
[12]  RNK:RK←+/1E¯10≤|(EIGENR MM-(LAMBDA*2)÷R)[1;]
[13]  'RANK OF MI×MI'' - (LAMBDA*2÷R)×J = ',▼RK
      ∇
```

```
      ∇ INCIDALL BF;I
[1]   ⍝ INPUTS: V,B,R,K,LAMBDA; BPOS,BF (SEE TEXT)
[2]   ⍝ OUTPUTS: BL; MI, MI×MI', RK(FI) FOR I=1 TO V (SEE TEXT)
[3]   BLPAT BF
[4]   'BLOCK PATTERN IS'
[5]   I←ρρρ□←BL
[6]   IN:INCID I
[7]   →IN×ιV≥I←I+1
      ∇
```

Figure 6.1. APL programs for computing M_i, $M_i M_i'$, and rank(F_i).

of BL, it changes the (L,J) entry of MI to 1 if block J contains both treatments I and L. When it has gone through all the blocks in BL, it deletes row I of MI, producing the desired MI of dimension (V-1) × B. Matrices $M_{\sim i}$ and $M_{\sim i} M'_{\sim i}$ are then displayed, and the program EIGENR from APL public library workspace 321 EIGENV finds the eigenvalues of $F_{\sim i}$, which give the rank of $F_{\sim i}$. (Any program that finds these eigenvalues could be used in place of EIGENR.)

The last program, INCIDALL, is a main program that calls the first two. Using BF and BPOS, it creates the block pattern matrix BL by invoking BLPAT, displays BL, and obtains $M_{\sim i}$, $M_{\sim i} M'_{\sim i}$, and the rank of $F_{\sim i}$ for each I=i=1,···,V. The programs in Figure 6.1 can be used to determine the degrees of freedom in the ANOVA of Table 5.1 for any values of the BIB design parameters v, b, r, k, λ and any design in class ß. These programs or equivalent programs in other languages supply a practical method for distinguishing among different repeated block designs, as well as the design with no repeated blocks, having identical BIB design parameters.

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