Decomposing a Latin Square of Order Six into Orthogonal Squares

by

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Abstract

A Latin square of order six is partitioned into conventional and unconventional F-squares in several ways. It is suggested that the definition of an F-square be broadened in such a way that the frequency of occurrence of a symbol in a row-column intersection could be 0, 1, 2, ..., rather than being confined to the number one. This would then include the unconventional F-squares obtained in the paper. Some geometrical questions are raised.

Introduction

Latin squares of prime power order can be decomposed into complete sets of F-squares (Mandeli, 1975). That is, a Latin square of order $s^2$, $s$ a prime power, can be decomposed into $(s^2-1)/(s-1)$ pairwise orthogonal F-square designs (POFSD) of order $s^2$ and with $s$ symbols. For example, a Latin square of order $4=2^2$ may be decomposed into a set of $(4-1)/(2-1)=3$ F-square designs of order 4, FSD(4;2,2), with two symbols, i.e., a POFSD(4;2,2;3) set. Now, the question

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arises as to how to decompose a Latin square design of nonprime power order into F-squares. We shall show how to decompose a Latin square of order six into squares, some of which are conventional F-squares and some are unconventional F-squares.

**Decomposition of a Latin Square of Order Six**

Consider the Latin square of order six

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
5 & 0 & 1 & 2 & 3 & 4 \\
4 & 5 & 0 & 1 & 2 & 3 \\
3 & 4 & 5 & 0 & 1 & 2 \\
2 & 3 & 4 & 5 & 0 & 1 \\
1 & 2 & 3 & 4 & 5 & 0 \\
\end{array}
\]

\[\text{LSD}(6) = \]

Let \(0, 1 = 0, 2, 3 = 1, 4, 5 = 2\) to construct

\[
\begin{array}{cccccc}
0 & 0 & 1 & 1 & 2 & 2 \\
2 & 0 & 0 & 1 & 1 & 2 \\
2 & 2 & 0 & 0 & 1 & 1 \\
1 & 2 & 2 & 0 & 0 & 1 \\
1 & 1 & 2 & 2 & 0 & 0 \\
0 & 1 & 1 & 2 & 2 & 0 \\
\end{array}
\]

\[\text{FSD}(6;2,2,2) = \]

Let \(0, 2, 4 = 0 \text{ and } 1, 3, 5 = 1\) to construct

\[
\begin{array}{cccccc}
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
\end{array}
\]

\[\text{FSD}(6;3,3) = \]

The six treatments 0, 1, 2, 3, 4, and 5 may be likened to a \(2 \times 3\) factorial, thus:

\[
\begin{array}{c|ccc}
\text{Factor a} & b_0 & b_1 & b_2 \\
\hline
a_0 & 0 & 2 & 4 \\
a_1 & 1 & 3 & 5 \\
\end{array}
\]
The contrast of $a_0$ with $a_1$ corresponds to the contrast of the two symbols 0 and 1 in FSD$(6;3,3)$ above. The contrasts among $b_0$, $b_1$, and $b_2$ correspond to the contrasts among the three symbols 0, 1, and 2 in FSD$(6;2,2,2)$ above. Note that FSD$(6;3,3)$ is orthogonal to FSD$(6;2,2,2)$. The frequency of occurrence of symbols 0, 1 in the first F-square with those in FSD$(6;2,2,2)$ is equal; thus

<table>
<thead>
<tr>
<th>Symbols in FSD$(6;3,3)$</th>
<th>Symbols in FSD$(6;2,2,2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6 6 6</td>
</tr>
<tr>
<td>1</td>
<td>6 6 6</td>
</tr>
</tbody>
</table>

An orthogonal contrast matrix for the six treatments in the Latin square is

<table>
<thead>
<tr>
<th>Contrast</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$C_2$</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$C_3$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$C_4$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>$C_5$</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$C_6$</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>-2</td>
<td></td>
</tr>
</tbody>
</table>

Contrast $C_1$ corresponds to the correction for the mean. Contrast $C_2$ corresponds to the contrast of symbols 0 and 1 in FSD$(6;3,3)$. Contrasts $C_3$ and $C_4$ correspond to contrasts among the three symbols in FSD$(6;2,2,2)$. Contrasts $C_5$ and $C_6$ are interaction contrasts for the $2 \times 3$ factorial.

In $C_5$, let -1 be zero and let +1 be a one. In $C_6$ let -1 or -2 be zero and let 1 or 2 be 1 with 1 or 2 being the frequency with which the symbol occurs in the row-column intersection. Doing this, we obtain the following two "F-squares". (The quotes are used as these are not conventional F-
squares which have one symbol in each row-column intersection.)

\[
\begin{array}{ccc}
0 & 1 & 1 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
\end{array}
\]

and

\[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
\end{array}
\]

Both of the above are orthogonal to \(FSD(6;3,3)\) and \(FSD(6;2,2,2)\), and to each other. They are also orthogonal to rows and columns.

Note that \(FSD(6;2,2,2)\) decomposes into

\[
\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

and

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]
FSD₃(6;2,2,0) corresponds to C₃ and FSD₄(6;4,4) corresponds to C₄.

Simple addition of FSD₃(6;2,2,0) and FSD₄(6;4,4) [that is, FSD₃(6;2,2,0) is superimposed on top of FSD₄(6;4,4)] produces FSD(6;2,2,2) as follows:

\[
\begin{array}{cccc}
0 & 0 & 1 & 1 \\
2 & 0 & 0 & 1 \\
2 & 0 & 0 & 1 \\
1 & 2 & 0 & 0 \\
1 & 1 & 2 & 0 \\
0 & 1 & 1 & 2 \\
\end{array}
\]

FSD(6;2,2,2) =

Addition of symbols in FSD₁(6;2,2,0) and FSD₂(6;4,4) produces:

\[
\begin{array}{cccc}
0 & 2 & 1 & 1 \\
0 & 0 & 2 & 1 \\
2 & 0 & 0 & 1 \\
1 & 2 & 0 & 1 \\
1 & 1 & 2 & 0 \\
2 & 1 & 1 & 2 \\
\end{array}
\]

FSD₁(6;2,2,2) =

However, the above square is not orthogonal to FSD(6;2,2,2), as shown by the following occurrences of symbols in the two squares:

<table>
<thead>
<tr>
<th>Symbols in FSD₁(6;2,2,2)</th>
<th>Symbols in FSD(6;2,2,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 1 2</td>
</tr>
<tr>
<td>1</td>
<td>6 0 6</td>
</tr>
<tr>
<td>2</td>
<td>0 12 0</td>
</tr>
<tr>
<td></td>
<td>6 0 6</td>
</tr>
</tbody>
</table>

FSD₁(6;2,2,2) is orthogonal to FSD(6;3,3). However, the squares FSD(6;3,3), FSD(6;2,2,2), FSD₁(6;2,2,0), and FSD₂(6;4,4) form a complete set in that the
sum of the sums of squares among the symbols in the four squares is equal to
the sum of squares among the six symbols in the original Latin square.
Squares $FSD_1(6;2,2,0)$ and $FSD_2(6;4,4)$ meet the requirement of $F$-squares
that symbols occur with equal frequency in rows and in columns, but not the
requirement that each row-column intersection contains one symbol. Broaden-
ing the definition of $F$-squares to include $FSD_1(6;2,2,0)$ and $FSD_2(6;4,4)$
would allow any Latin square of order $n$ to be decomposed into $(n-1)^2$ squares
with two symbols.

Some Questions

If one could somehow compose an $FSD_1(6;2,2,2)$ square which would be
orthogonal to $FSD(6;2,2,2)$ and $FSD(6;3,3)$, one would have a geometrical inter-
action in a $2 \times 3$ factorial. This author has been unable to compose an
$FSD_1(6;2,2,2)$ from $FSD_1(6;2,2,0)$ and $FSD_2(6;4,4)$ for which the sum of squares
among the three symbols would correspond to the interaction sum of squares.
Can it be done?

Note that one could decompose the Latin square of order six into the set

$$
FSD_5(6;2,2,0,0,0,0) = \begin{array}{cccc}
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{array}
,$$

$$
FSD_6(6;0,0,2,2,0,0) = \begin{array}{cccc}
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{array}
,$$


and FSD(6;2,2,2). This set is complete. Note that FSD(6;3,3) is not a member of this complete set. Adding the three squares F_5, F_6, and F_7 together produces the FSD(6;3,3) square.

Some other questions relate to the geometrical aspects of F-squares. Does it make sense to consider FSD_1(6;2,2,0) and FSD_2(6;4,4) in a geometrical sense? If it does not make sense using present geometries, would it be advisable to construct a new geometry based on single degree of freedom contrasts and use squares of the form of FSD_1(6;2,2,0) and FSD_2(6;4,4)? If this route were pursued, then a complete F-square geometry exists for every square of order n. There would be (n-1)^2 squares composed of two symbols, and this would be the complete set. There would be many ways to construct the complete set.

Reference