COMPLETE SETS OF PAIRWISE MUTUALLY ORTHOGONAL LATIN RECTANGLES

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BU-769-M February 1982

Paper No. BU-769-M in the Biometrics Unit Mimeo Series.

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ABSTRACT. The theory and construction of sets of pairwise mutually orthogonal latin squares and F-squares have been known for many years. The idea of mutually orthogonal sets of simple change-over designs, which are latin rectangles, appears not to have been studied. Complete sets of pairwise mutually orthogonal latin squares of order n, denoted by MOL(n,n-1), are available for all prime powers, as well as for classes of F-squares. We demonstrate that complete sets of pairwise mutually orthogonal latin rectangles of v rows, v symbols, and $c = v^n$ columns are available for v equal a prime power.

1. Introduction and Summary.

The existence of complete sets of pairwise mutually orthogonal latin squares of order n, a prime power, has been known for 60 years; see, e.g., MacNeish (1922). The existence of complete sets of F-squares of order $n = s^m$ with s symbols for s a prime power was demonstrated by Hedayat, *et al.* (1975), while the existence of complete sets of F-squares of order 4t, t=1,2,···, with two symbols was proved by Federer (1977). Mandeli (1975) showed how to construct complete sets of pairwise mutually orthogonal F-squares with a variable number of symbols for prime powers. Mandeli, *et al.* (1981) showed how to construct sets of pairwise mutually orthogonal F-squares of order $n = 2s^m$ with s symbols and for s a prime power. The set was not complete, but became asymptotically complete as s and/or m approached infinity.

Instead of using a square row by column design, one might wish to use an r row by c column design for an experiment, and one might wish to make the design orthogonal to a previous or simultaneous set of t-1 other experiments on the same rc experimental units and with v treatments (symbols). A simple change-over design (SCOD) is a latin rectangle design wherein $r = v$, $c = \lambda v$, the v treatments occur once in each of the c columns, and the v treatments each occur λ times in each row; see, e.g., Kershner and Federer (1981).

DEFINITION 1.1. Two SCOD's with v symbols, v rows and λv columns are said to be mutually orthogonal if the v symbols in one SCOD appear λ times with each and every symbol of the second SCOD. A set of t SCOD's are said to be pairwise mutually orthogonal if each and every pair of SCOD's are mutually orthogonal. Denote this set by MOSCOD$(v,\lambda v;t)$.

Note that t has a maximum value, as defined in the following theorem.

THEOREM 1.1. *The maximum value of t is* $v\lambda-1$.

PROOF. In a v row by $v\lambda$ column design, there are $(v-1)(v\lambda-1)$ degrees of freedom associated with the row by column interaction. Since each set of v symbols of a SCOD is associated with v-1 degrees of freedom, and since all t sets of v-1 degrees of freedom for the t SCOD's must come from the interaction degrees of freedom in order to be orthogonal to row and column contrasts, there are at most $(v-1)(v\lambda-1)/(v-1) = \lambda v-1$ sets. Therefore, t cannot exceed $v\lambda-1$.

DEFINITION 1.2. When $t = v\lambda-1$, the set MOSCOD$(v,\lambda v;v\lambda-1)$ is said to be complete.

2. Simple Change-Over Designs for v = 2.

In a simple change-over design with $v = 2$ symbols, there are two rows and 2λ columns. Now, when $2\lambda = 4k$, $k=1,2,\cdots$, a complete set of pairwise mutually orthogonal SCOD's exists as described below.

THEOREM 2.1. *A MOSCOD(2,4k;4k-1) set exists for all k for which a Hadamard matrix exists.*

PROOF. In a SCOD(2,4k), there are two sequences of symbols, namely $\begin{matrix}1\\2\end{matrix}$ and $\begin{matrix}2\\1\end{matrix}$ in the 4k columns. Denote one of the sequences as +1 and the other as -1. When a Hadamard matrix is normalized there are 4k plus ones in the first column and in the first row. In the second through the $4k^{th}$ row, there are 2k plus ones and 2k minus ones, and every row is orthogonal to every other row. Now construct 4k-1 SCOD's from the last 4k-1 rows of the Hadamard matrix where a plus one indicates the sequence $\begin{matrix}1\\2\end{matrix}$ and a minus one indicates the sequence $\begin{matrix}2\\1\end{matrix}$. Since any two rows of the Hadamard matrix are orthogonal, any two corresponding two SCOD's will be orthogonal. Since $4k-1 = t$ is the maximum number of SCOD's that can be constructed, the set is complete. Hence, a MOSCOD(2,4k;4k-1)-set results.

Now we can also prove the following.

THEOREM 2.2. $t = 0$ *or* 1 *for all* $2\lambda \neq 4k$, $k=1,2,\cdots$.

PROOF. When $2\lambda = 4k-1$ or $4k-3$, $k=1,2,\cdots$, the number of columns is equal to an odd number, and hence no SCOD exists, i.e., $t = 0$. When $2\lambda = 4k-2$, $k=1,2,\cdots$, one can easily construct a SCOD; hence, t is at least one. Now, in constructing +1 and -1 $(4k-2) \times (4k-2)$ contrast matrices containing (2k-1) plus ones and (2k-1) minus ones, one may construct the first row with all plus ones and the second row with (2k-1) plus ones and (2k-1) minus ones. Now it is impossible to construct a third row of the matrix which has (2k-1) plus ones and (2k-1) minus ones and which is orthogonal to each of the first two rows of the matrix. This is so because it is impossible to divide an odd number, 2n-1, into two equal parts. Since this is not possible, $t = 1$ for all 4k-2. Note that when $k = 1$, we have a 2×2 latin square, and we know that it is mateless.

3. Simple Change-Over Designs for $v = s^m$, s a Prime Power

Prior to presenting the general result for complete sets of pairwise mutually orthogonal simple change-over designs with s symbols (s a prime power), s rows, and λs columns, let us consider a MOSCOD(3,9;8)-set. To construct this set we use the MOL(3,2)-set and the orthogonal array OA(9,4,3,2)-set which are:

MOL(3,2) set		OA(9,4,3,2)		
L_1	L_2	000	111	222
012	012	012	012	012
120	201	012	120	201
201	120	012	201	120

Now use L_1 and associate the symbols 0, 1, 2 in the OA with the columns of L_1. Using the four rows of the OA, we obtain the following four SCOD's:

L_1

Row 1 of OA			Row 2 of OA			Row 3 of OA			Row 4 of OA		
000	111	222	012	012	012	012	120	201	012	201	120
111	222	000	120	120	120	120	201	012	120	012	201
222	000	111	201	201	201	201	012	120	201	120	012

Now use L_2 in the same manner to obtain four more SCOD's:

L_2

Row 1 of OA			Row 2 of OA			Row 3 of OA			Row 4 of OA		
000	111	222	012	012	012	012	120	201	012	201	120
222	000	111	201	201	201	201	012	120	201	120	012
111	222	000	120	120	120	120	201	012	120	012	201

We now have $v\lambda-1 = 8$ pairwise mutually orthogonal SCOD's, and the set is complete.

Now consider a MOSCOD(3,27;26)-set. To construct this set use L_1 and L_2 above and the OA(27,13,3,2) which is

000	000	000	111	111	111	222	222	222
000	111	222	000	111	222	000	111	222
000	111	222	111	222	000	222	000	111
000	222	111	111	000	222	222	111	000
012	012	012	012	012	012	012	012	012
012	012	012	120	120	120	201	201	201
021	021	021	102	102	102	201	201	201
012	120	201	012	120	201	012	120	201
021	102	210	021	102	210	021	102	210
012	120	201	120	201	012	201	012	120
021	102	210	102	210	021	210	021	102
012	201	120	120	012	201	201	120	012
021	210	102	102	021	210	210	102	021

Thus, the MOL(3,2)-set and the OA(27,13,3,2) may be used to construct the $3^3-1 = 26$ MOSCOD's which is the complete set.

Following the above procedure we may state the following theorem.

THEOREM 3.1. <u>A complete set of pairwise mutually orthogonal simple change-over designs exists for</u> v <u>a prime power and</u> λv <u>equal to</u> v^n, <u>that is a</u> MOSCOD($v,v^n;v^n-1$)-<u>set</u>.

PROOF. The proof follows the construction method outlined in the first part of this section. Use an OL(v,v-1)-set and the orthogonal array $OA\left(v^n,(v^n-1)/(v-1),v,2\right)$. Take the first latin square, L_1, from the OL(v,v-1)-set and the first row of $OA\left(v^n,(v^n-1)/(v-1),v,2\right)$ to form the first SCOD(v,v^n). Take L_1 and the second row of OA to form a second SCOD(v,v^n). Continue using rows of OA until $(v^n-1)/(v-1)$ SCOD(v,v^n)'s have been formed. These $(v^n-1)/(v-1)$ SCOD's are mutually orthogonal since the rows of the OA are orthogonal. Now take a second latin square from the MOL(v,v-1)-set and form an additional set of $(v^n-1)/(v-1)$ SCOD's. This set forms a pairwise mutually orthogonal set, and is pairwise mutually orthogonal to the first set of $(v^n-1)/(v-1)$ SCOD's. Continue this process until the last latin square in the MOL(v,v-1)-set has been used. There will be $(v-1)(v^n-1)/(v-1) = v^n-1$ MOSCOD's. Since v^n-1 is the maximum number, the set is complete.

4. Formation of Orthogonal Arrays and Codes

Just as MOL(v,t)-sets may be used to construct orthogonal arrays, the MOSCOD$(v,v^n;v^n-1)$-set may also be used to construct arrays of the OA$(v^{n+1},v^n,v,2)$ + OA$(v^{n+1},1,v^n,2)$ type. That is, one row of the array contains v^n symbols and v^n rows contain v symbols. Since widths of codes for v symbols is often important, the width of a code may be increased simply by increasing n. Of course, the length of the code can always be increased by repeating the v^{n+1} columns of the array.

5. Other Sets of MOSCOD's

It is not known what values of t are available when v is not a prime power and when $\lambda v \neq v^n$. For example, consider the following three row by six column SCOD's:

SCOD1			SCOD2		
00	11	22	00	11	22
22	00	11	11	22	00
11	22	00	22	00	11

Can t be greater than two in a MOSCOD(6,3;t)-set? What other values are possible for λ and t.

Consider the following pair of mutually orthogonal SCOD's:

SCOD(6,12)		SCOD(6,12)	
1 2 3 4 5 6	1 2 3 4 5 6	1 3 5 2 4 3	5 1 6 4 6 2
6 1 2 3 4 5	6 1 2 3 4 5	4 2 4 6 3 5	3 6 2 1 5 1
5 6 1 2 3 4	5 6 1 2 3 4	6 5 3 5 1 4	2 4 1 3 2 6
4 5 6 1 2 3	4 5 6 1 2 3	5 1 6 4 6 2	1 3 5 2 4 3
3 4 5 6 1 2	3 4 5 6 1 2	3 6 2 1 5 1	4 2 4 6 3 5
2 3 4 5 6 1	2 3 4 5 6 1	2 4 1 3 2 6	6 5 3 5 1 4

Note that a latin square of order six has no orthogonal mate, but here is a pair of orthogonal SCOD's for six symbols. Are there any more? What is the value of t in MOSCOD(v,λ,v;t) for v a nonprime power?

For any v we can state the following:

THEOREM 5.1. Given a MOL(v,r)-set and an OA(v^n,t,v,2), the method of construction for theorem 3.1 produces rt pairwise mutually orthogonal simple change-over designs, i.e., the MOSCOD(v,v^n;rt)-set.

Now, can one produce more than rt pairwise mutually orthogonal simple change-over designs by some other method? Do complete sets exist for any v not a prime power?

6. Generalizations

Consider generalized latin rectangles wherein each symbol appears λ times in each of the πv rows and π times in each of the λv columns. The maximum number t of sets will be the integer part of (πv-1)(λv-1)/(v-1). Under what conditions do complete sets exist? What values does t take on for values of v? Note that when π = λ an F-square results, and when λ = π = 1, a latin square results. Also, when π and/or λ is a power of v, (πv-1)(λv-1)/(v-1) becomes an integer, and a complete set is possible.

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