ON EMBEDDING MATELESS LATIN SQUARES OF ORDERS 12 AND 16
IN A COMPLETE SET OF ORTHOGONAL F-SQUARES

by

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Abstract

A Latin square of order 16 that has no orthogonal Latin square is shown to have 210 mutually orthogonal F(16;8,8)-square mates. This is the complete set of F-squares for the mateless Latin square. Row and column operations are used to construct this complete set of F-squares from a Hadamard matrix and 15 OF(16;8,8)-squares into which the Latin square is decomposed. For a Latin square of order 12 that has no orthogonal Latin square, this technique fails to produce any F(12;6,6)-square mates. Tables of complete sets of mutually orthogonal F(4t;2t,2t)-square mates are given for t = 1, 2, and 4, i.e., for mateless Latin squares of orders 4, 8, and 16. The computer program used to generate these complete sets by row and column operations is given.

Introduction

A mateless Latin square of order four, [OL(4,1)], was embedded in a complete set of nine orthogonal F-squares of order four with two symbols, OF(4;2,2;6), by Mandeli [1975] and Mandeli and Federer [1981]. A mateless Latin square of order eight was embedded in a complete set of 49 orthogonal F-squares of order eight with two symbols, OF(8;4,4;42), by Federer, Mandeli, and Schwager [1981]. This paper demonstrates that the

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procedure used to obtain these embeddings fails for a mateless Latin square of order 12 but works for a mateless Latin square of order 16. This leads to the conjecture that the procedure works for all mateless Latin squares of order 2^n and possibly for s^n where $s = p^m$, p a prime number.

In order to study patterns of the mateless Latin squares of orders 4, 8, and 16, the complete sets of orthogonal F-squares for orders four and eight are given in Tables 1 and 2. Using the same procedure, the orthogonal F-squares of order 16 obtained by the row operations described by Federer, Mandeli, and Schwager [1981] are presented in Table 3; the number of orthogonal F(16;8,8)-squares obtained by this procedure is 140. An additional 85 mutually orthogonal F(16;8,8)-squares are required for a complete set of 210 orthogonal F-square mates for a mateless Latin square of order 16. These are obtained from the column operations. The complete set is presented in Table 4. The computer program used to obtain these squares for orders 8, 12, and 16 is given in the fourth section of this paper.

F-Squares of Order 12

A mateless Latin square of order 12,

1	2	3	4	5	6	7	8	9	10	11	12
12	1	2	3	4	5	6	7	8	9	10	11
11	12	1	2	3	4	5	6	7	8	9	10
10	11	12	1	2	3	4	5	6	7	8	9
9	10	11	12	1	2	3	4	5	6	7	8
8	9	10	11	12	1	2	3	4	5	6	7
7	8	9	10	11	12	1	2	3	4	5	6
6	7	8	9	10	11	12	1	2	3	4	5
5	6	7	8	9	10	11	12	1	2	3	4
4	5	6	7	8	9	10	11	12	1	2	3
3	4	5	6	7	8	9	10	11	12	1	2
2	3	4	5	6	7	8	9	10	11	12	1 ,

can be decomposed into the 11 mutually orthogonal F(12;6,6)-squares F_1 to F_{11} given in Table 5. The Hadamard matrix H_{12} of order 12 is also given in this table.

Table 1. Six Mutually Orthogonal F-Squares of Order 4, F_4 Through F_9, Obtained by Row and Column Operations Using H_4 and F_1 Through F_3

Original F-square	Row of H_4 operating on F_i, i = 1, 2, 3			Column of H_4 operating on F_i, i = 1, 2, 3		
	2	3	4	2	3	4
1	–	4	–	–	7	–
2	5	–	6	8	–	9
3	–	7	–	–	4	–

-: not an F-square

Number: orthogonal F-square

Number: duplicated orthogonal F-square

Table 2. Forty-two Mutually Orthogonal F-Squares of Order 8, F_8 Through F_{49}, Obtained by Row and Column Operations Using H_8 and F_1 Through F_7

Original F-square	Row of H_8 operating on F_i, $i=1,2,\cdots,7$							Column of H_8 operating on F_i, $i=1,2,\cdots,7$						
	2	3	4	5	6	7	8	2	3	4	5	6	7	8
1	8	14	19	–	–	–	–	11	*	*	–	–	–	–
2	9	–	–	24	27	30	33	12	–	–	38	41	44	47
3	10	15	20	–	–	–	–	13	*	*	–	–	–	–
4	–	16	21	25	28	31	34	–	36	37	39	42	45	48
5	11	17	22	–	–	–	–	8	*	*	–	–	–	–
6	12	–	–	26	29	32	35	9	–	–	40	43	46	49
7	13	18	23	–	–	–	–	10	*	*	–	–	–	–

–: not an F-square

*: F-square but not orthogonal to preceding F-squares

Number: orthogonal F-square

Number: duplicated orthogonal F-square

Table 3. One Hundred Forty Mutually Orthogonal F-Squares of Order 16, F_{16} Through F_{155}, Obtained by Row Operations Using H_{16} and F_1 Through F_{15}

Original F-square	Row of H_{16} operating on F_1 through F_{15}: 2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	–	23	–	41	–	61	–	79	–	100	–	118	–	138	–
2	16	–	34	42	54	–	72	80	93	–	111	119	131	–	149
3	–	24	–	43	–	62	–	81	–	101	–	120	–	139	–
4	17	25	35	–	55	63	73	82	94	102	112	–	132	140	150
5	–	26	–	44	–	64	–	83	–	103	–	121	–	141	–
6	18	–	36	45	56	–	74	84	95	–	113	122	133	–	151
7	–	27	–	46	–	65	–	85	–	104	–	123	–	142	–
8	19	28	37	47	57	66	75	–	96	105	114	124	134	143	152
9	–	29	–	48	–	67	–	86	–	106	–	125	–	144	–
10	20	–	38	49	58	–	76	87	97	–	115	126	135	–	153
11	–	30	–	50	–	68	–	88	–	107	–	127	–	145	–
12	21	31	39	–	59	69	77	89	98	108	116	–	136	146	154
13	–	32	–	51	–	70	–	90	–	109	–	128	–	147	–
14	22	–	40	52	60	–	78	91	99	–	117	129	137	–	155
15	–	33	–	53	–	71	–	92	–	110	–	130	–	148	–

–: not an F-square

Number: orthogonal F-square

Number: duplicated orthogonal F-square

Table 4. Eighty-five Mutually Orthogonal F-Squares of Order 16, F_{156} Through F_{225}, Obtained by Column Operations Using H_{16} and F_1 Through F_{15}

Original F-square	Column of H_{16} operating on F_1 through F_{15}														
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	—	*	—	*	—	*	—	86	—	*	—	*	—	*	—
2	156	—	166	*	174	—	184	87	191	—	201	*	209	—	219
3	—	*	—	*	—	*	—	88	—	*	—	*	—	*	—
4	157	163	167	—	175	181	185	89	192	198	202	—	210	216	220
5	—	*	—	*	—	*	—	90	—	*	—	*	—	*	—
6	158	—	168	*	176	—	186	91	193	—	203	*	211	—	221
7	—	*	—	*	—	*	—	92	—	*	—	*	—	*	—
8	159	164	169	173	177	182	187	—	194	199	204	208	212	217	222
9	—	*	—	*	—	*	—	79	—	*	—	*	—	*	—
10	160	—	170	*	178	—	188	80	195	—	205	*	213	—	223
11	—	*	—	*	—	*	—	81	—	*	—	*	—	*	—
12	161	165	171	—	179	183	189	82	196	200	206	—	214	218	224
13	—	*	—	*	—	*	—	83	—	*	—	*	—	*	—
14	162	—	172	*	180	—	190	84	197	—	207	*	215	—	225
15	—	*	—	*	—	*	—	85	—	*	—	*	—	*	—

—: not an F-square

*: F-square but not orthogonal to preceding F-squares

Number: orthogonal F-square

Number: duplicated orthogonal F-square

Table 5. F-Squares F_1 Through F_{11} and Hadamard Matrix H_{12} Used for Order 12 Construction

F_1

1	-1	1	-1	1	-1	-1	1	-1	-1	1	1
1	1	-1	1	-1	1	-1	-1	1	-1	-1	1
1	1	1	-1	1	-1	1	-1	-1	1	-1	-1
-1	1	1	1	-1	1	-1	1	-1	-1	1	-1
-1	-1	1	1	1	-1	1	-1	1	-1	-1	1
1	-1	-1	1	1	1	-1	1	-1	1	-1	-1
-1	1	-1	-1	1	1	1	-1	1	-1	1	-1
-1	-1	1	-1	-1	1	1	1	-1	1	-1	1
1	-1	-1	1	-1	-1	1	1	1	-1	1	-1
-1	1	-1	-1	1	-1	-1	1	1	1	-1	1
1	-1	1	-1	-1	1	-1	-1	1	1	1	-1
-1	1	-1	1	-1	-1	1	-1	-1	1	1	1

F_2

1	-1	-1	1	1	1	1	-1	-1	-1	-1	1
1	1	-1	-1	1	1	1	1	-1	-1	-1	-1
-1	1	1	-1	-1	1	1	1	1	-1	-1	-1
-1	-1	1	1	-1	-1	1	1	1	1	-1	-1
-1	-1	-1	1	1	-1	-1	1	1	1	1	-1
-1	-1	-1	-1	1	1	-1	-1	1	1	1	1
1	-1	-1	-1	-1	1	1	-1	-1	1	1	1
1	1	-1	-1	-1	-1	1	1	-1	-1	1	1
1	1	1	-1	-1	-1	-1	1	1	-1	-1	1
1	1	1	1	-1	-1	-1	-1	1	1	-1	-1
-1	1	1	1	1	-1	-1	-1	-1	1	1	-1
-1	-1	1	1	1	1	-1	-1	-1	-1	1	1

F_3

1	1	-1	-1	-1	1	-1	-1	1	-1	1	1
1	1	1	-1	-1	-1	1	-1	-1	1	-1	1
1	1	1	1	-1	-1	-1	1	-1	-1	1	-1
-1	1	1	1	1	-1	-1	-1	1	-1	-1	1
1	-1	1	1	1	1	-1	-1	-1	1	-1	-1
-1	1	-1	1	1	1	1	-1	-1	-1	1	-1
-1	-1	1	-1	1	1	1	1	-1	-1	-1	1
1	-1	-1	1	-1	1	1	1	1	-1	-1	-1
-1	1	-1	-1	1	-1	1	1	1	1	-1	-1
-1	-1	1	-1	-1	1	-1	1	1	1	1	-1
-1	-1	-1	1	-1	-1	1	-1	1	1	1	1
1	-1	-1	-1	1	-1	-1	1	-1	1	1	1

F_4

1	-1	1	1	-1	1	-1	-1	-1	1	1	-1
-1	1	-1	1	1	-1	1	-1	-1	-1	1	1
1	-1	1	-1	1	1	-1	1	-1	-1	-1	1
1	1	-1	1	-1	1	1	-1	1	-1	-1	-1
-1	1	1	-1	1	-1	1	1	-1	1	-1	-1
-1	-1	1	1	-1	1	-1	1	1	-1	1	-1
-1	-1	-1	1	1	-1	1	-1	1	1	-1	1
1	-1	-1	-1	1	1	-1	1	-1	1	1	-1
-1	1	-1	-1	-1	1	1	-1	1	-1	1	1
1	-1	1	-1	-1	-1	1	1	-1	1	-1	1
1	1	-1	1	-1	-1	-1	1	1	-1	1	-1
-1	1	1	-1	1	-1	-1	-1	1	1	-1	1

F_5

1	1	-1	1	-1	-1	-1	1	-1	1	-1	1
1	1	1	-1	1	-1	-1	-1	1	-1	1	-1
-1	1	1	1	-1	1	-1	-1	-1	1	-1	1
1	-1	1	1	1	-1	1	-1	-1	-1	1	-1
-1	1	-1	1	1	1	-1	1	-1	-1	-1	1
1	-1	1	-1	1	1	1	-1	1	-1	-1	-1
-1	1	-1	1	-1	1	1	1	-1	1	-1	-1
-1	-1	1	-1	1	-1	1	1	1	-1	1	-1
-1	-1	-1	1	-1	1	-1	1	1	1	-1	1
1	-1	-1	-1	1	-1	1	-1	1	1	1	-1
-1	1	-1	-1	-1	1	-1	1	-1	1	1	1
1	-1	1	-1	-1	-1	1	-1	1	-1	1	1

F_6

1	-1	1	-1	-1	-1	1	-1	1	1	-1	1
1	1	-1	1	-1	-1	-1	1	-1	1	1	-1
-1	1	1	-1	1	-1	-1	-1	1	-1	1	1
1	-1	1	1	-1	1	-1	-1	-1	1	-1	1
1	1	-1	1	1	-1	1	-1	-1	-1	1	-1
-1	1	1	-1	1	1	-1	1	-1	-1	-1	1
1	-1	1	1	-1	1	1	-1	1	-1	-1	-1
-1	1	-1	1	1	-1	1	1	-1	1	-1	-1
-1	-1	1	-1	1	1	-1	1	1	-1	1	-1
-1	-1	-1	1	-1	1	1	-1	1	1	-1	1
1	-1	-1	-1	1	-1	1	1	-1	1	1	-1
-1	1	-1	-1	-1	1	-1	1	1	-1	1	1

F_7

1	1	1	1	1	-1	-1	-1	1	-1	-1	-1
-1	1	1	1	1	1	-1	-1	-1	1	-1	-1
-1	-1	1	1	1	1	1	-1	-1	-1	1	-1
-1	-1	-1	1	1	1	1	1	-1	-1	-1	1
1	-1	-1	-1	1	1	1	1	1	-1	-1	-1
-1	1	-1	-1	-1	1	1	1	1	1	-1	-1
-1	-1	1	-1	-1	-1	1	1	1	1	1	-1
-1	-1	-1	1	-1	-1	-1	1	1	1	1	1
1	-1	-1	-1	1	-1	-1	-1	1	1	1	1
1	1	-1	-1	-1	1	-1	-1	-1	1	1	1
1	1	1	-1	-1	-1	1	-1	-1	-1	1	1
1	1	1	1	-1	-1	-1	1	-1	-1	-1	1

F_8

1	-1	-1	-1	1	1	-1	1	1	1	-1	-1
-1	1	-1	-1	-1	1	1	-1	1	1	1	-1
-1	-1	1	-1	-1	-1	1	1	-1	1	1	1
1	-1	-1	1	-1	-1	-1	1	1	-1	1	1
1	1	-1	-1	1	-1	-1	-1	1	1	-1	1
1	1	1	-1	-1	1	-1	-1	-1	1	1	-1
-1	1	1	1	-1	-1	1	-1	-1	-1	1	1
1	-1	1	1	1	-1	-1	1	-1	-1	-1	1
1	1	-1	1	1	1	-1	-1	1	-1	-1	-1
-1	1	1	-1	1	1	1	-1	-1	1	-1	-1
-1	-1	1	1	-1	1	1	1	-1	-1	1	-1
-1	-1	-1	1	1	-1	1	1	1	-1	-1	1

F_9

1	1	-1	-1	1	-1	1	-1	-1	1	1	-1
-1	1	1	-1	-1	1	-1	1	-1	-1	1	1
1	-1	1	1	-1	-1	1	-1	1	-1	-1	1
1	1	-1	1	1	-1	-1	1	-1	1	-1	-1
-1	1	1	-1	1	1	-1	-1	1	-1	1	-1
-1	-1	1	1	-1	1	1	-1	-1	1	-1	1
1	-1	-1	1	1	-1	1	1	-1	-1	1	-1
-1	1	-1	-1	1	1	-1	1	1	-1	-1	1
1	-1	1	-1	-1	1	1	-1	1	1	-1	-1
-1	1	-1	1	-1	-1	1	1	-1	1	1	-1
-1	-1	1	-1	1	-1	-1	1	1	-1	1	1
1	-1	-1	1	-1	1	-1	-1	1	1	-1	1

F_{10}

1	-1	-1	1	-1	-1	1	1	1	-1	1	-1
-1	1	-1	-1	1	-1	-1	1	1	1	-1	1
1	-1	1	-1	-1	1	-1	-1	1	1	1	-1
-1	1	-1	1	-1	-1	1	-1	-1	1	1	1
1	-1	1	-1	1	-1	-1	1	-1	-1	1	1
1	1	-1	1	-1	1	-1	-1	1	-1	-1	1
1	1	1	-1	1	-1	1	-1	-1	1	-1	-1
-1	1	1	1	-1	1	-1	1	-1	-1	1	-1
-1	-1	1	1	1	-1	1	-1	1	-1	-1	1
1	-1	-1	1	1	1	-1	1	-1	1	-1	-1
-1	1	-1	-1	1	1	1	-1	1	-1	1	-1
-1	-1	1	-1	-1	1	1	1	-1	1	-1	1

F_{11}

1	1	1	-1	-1	1	1	1	-1	-1	-1	-1
-1	1	1	1	-1	-1	1	1	1	-1	-1	-1
-1	-1	1	1	1	-1	-1	1	1	1	-1	-1
-1	-1	-1	1	1	1	-1	-1	1	1	1	-1
-1	-1	-1	-1	1	1	1	-1	-1	1	1	1
1	-1	-1	-1	-1	1	1	1	-1	-1	1	1
1	1	-1	-1	-1	-1	1	1	1	-1	-1	1
1	1	1	-1	-1	-1	-1	1	1	1	-1	-1
-1	1	1	1	-1	-1	-1	-1	1	1	1	-1
-1	-1	1	1	1	-1	-1	-1	-1	1	1	1
1	-1	-1	1	1	1	-1	-1	-1	-1	1	1
1	1	-1	-1	1	1	1	-1	-1	-1	-1	1

H_{12}

1	1	1	1	1	1	1	1	1	1	1	1
1	-1	1	-1	1	-1	-1	1	-1	-1	1	1
1	-1	-1	1	1	1	1	-1	-1	-1	-1	1
1	1	-1	-1	-1	1	-1	-1	1	-1	1	1
1	-1	1	1	-1	1	-1	-1	-1	1	1	-1
1	1	-1	1	-1	-1	-1	1	-1	1	-1	1
1	-1	1	-1	-1	-1	1	-1	1	1	-1	1
1	1	1	1	1	-1	-1	-1	1	-1	-1	-1
1	-1	-1	-1	1	1	-1	1	1	1	-1	-1
1	1	-1	-1	1	-1	1	-1	-1	1	1	-1
1	-1	-1	1	-1	-1	1	1	1	-1	1	-1
1	1	1	-1	-1	1	1	1	-1	-1	-1	-1

Upon performing the row and column operations using H_{12} and the F-squares F_1 through F_{11}, no F-squares are obtained. This could mean that the row and column procedure will not result in embedding a mateless Latin square of order $4t \neq 2^n$ in a complete set of OF(4t;2t,2t)-squares. It is possible, however, that use of the row and column procedure with H_{12} replaced by a different Hadamard matrix or some permutation of H_{12} would produce F-squares. From the nature of the geometries for $4t \neq 2^n$, it would appear that the former statement holds and that a different procedure from the one used here is required. Federer [1977] showed how to construct complete sets of OF(4t;2t,2t)-squares. Perhaps one could find a procedure that would combine $4t-1$ F(4t;2t,2t)-squares, resulting in a mateless Latin square of order $4t$. In any event, the attempt on order 12 resulted in a complete failure.

F-Squares of Order 16

The mateless Latin square of order 16,

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
16	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
15	16	1	2	3	4	5	6	7	8	9	10	11	12	13	14
14	15	16	1	2	3	4	5	6	7	8	9	10	11	12	13
13	14	15	16	1	2	3	4	5	6	7	8	9	10	11	12
12	13	14	15	16	1	2	3	4	5	6	7	8	9	10	11
11	12	13	14	15	16	1	2	3	4	5	6	7	8	9	10
10	11	12	13	14	15	16	1	2	3	4	5	6	7	8	9
9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
8	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7
7	8	9	10	11	12	13	14	15	16	1	2	3	4	5	6
6	7	8	9	10	11	12	13	14	15	16	1	2	3	4	5
5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4
4	5	6	7	8	9	10	11	12	13	14	15	16	1	2	3
3	4	5	6	7	8	9	10	11	12	13	14	15	16	1	2
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1 ,

can be decomposed into the 15 OF(16;8,8)-squares F_1 to F_{15} given in Table 6. The Hadamard matrix H_{16} is also given in this table. Upon performing the row and column operations using H_{16} and F_1 through F_{15}, the complete set of 210

Table 6. F-Squares F_1 Through F_{15} and Hadamard Matrix H_{16} Used for Order 16 Construction

F_1

1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1
-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1
-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1
-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1
-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1
-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1
1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1
1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1
1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1
1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1
1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1
1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1
1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1

F_2

1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1
-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1
-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1
-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1
1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1
1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1
1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1
1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1
-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1
-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1
-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1
1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1
1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1
1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1

F_3

1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1
1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1
1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1
1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1
1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1
-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1
-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1
-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1
-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1
-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1
1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1
1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1
1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1

F_4

1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1
-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1
1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1
-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1
1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1
-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1
1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1
-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1
1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1

F_5

1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1
1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1
1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1
-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1
-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1
1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1
1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1
-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1
-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1
-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1
-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1
1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1
1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1
-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1
-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1
1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1

F_6

1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1
1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1
1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1
-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1
-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1
-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1
1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1
1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1
1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1
1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1
-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1
-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1
-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1
1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1

F_7

1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1
-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1
-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1
1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1
1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1
1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1
1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1
-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1
-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1
1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1
1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1
-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1
-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1
-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1
-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1
1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1

F_8

1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1

Table 6 (continued)

F_9

1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1
1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1
-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1
1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1
-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1
1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1
-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1
1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	-1
-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1
-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1
1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1
-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1
1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1
-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1
1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1
-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1

F_{10}

1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1
1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1
-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1
1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1
-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1
-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1
1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1
-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1
1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1
1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1
-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1
1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1
-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1
-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1
1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1
-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1

F_{11}

1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1
-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1
1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1
-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1
1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1
1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1
-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1
1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1
-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1
1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1
-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1
1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	-1
-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1
-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1
1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1
-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1

F_{12}

1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1
-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1
1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1
-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1
1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1
-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1
1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1
-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1

F_{13}

1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1
-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1
1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1
1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1
-1	1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1
-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1
1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1
1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1	-1
-1	1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1
1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1
-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1
-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1	1
1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1
1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1
-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1
-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1

F_{14}

1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1
-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1
1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1
1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1
-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1
1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1
-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1
-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1
1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1
-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1
1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1
1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1
-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1
1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1
-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1
-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1

F_{15}

1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1
1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1
-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1
-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1
1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1
-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1
1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1
1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1
-1	1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1
-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1
1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1
1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1	-1
-1	1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1
1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1
-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1
-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1	1

H_{16}

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1
1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1
1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1
1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1
1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1
1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1
1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1
1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1
1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1
1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1

orthogonal F-square mates, F_{16} to F_{225}, is obtained. The original F-square and the row or column of H_{16} used to obtain each of these 210 F-squares are described in Tables 3 and 4.

There are $(2^n - 1)^2$ squares of order 2^n obtained for $n = 2$, 3, and 4 by the row operations, and the same number is obtained by the column operations. The $2(2^n - 1)^2$ squares obtained may be partitioned into various categories (Tables 7 and 8). Upon performing the row operations first, $2(2^n - 1)(2^n - 2)/3$ orthogonal F-squares are obtained. An additional $(2^n - 1)(2^n - 2)/3$ orthogonal F-squares are obtained by the column operations. Note that column operations could have been used first and then row operations, and the numbers of orthogonal F-squares obtained would have been reversed. In this case, a different set of F-squares would have been obtained. This set includes those squares marked with *'s in Tables 2 and 4, and some of the OF-squares obtained by row operations would be omitted. The omitted ones should be in the same positions in the tables if the row and column labels are reversed in Tables 2, 3, and 4.

Some characteristics of Tables 1 through 4 are summarized in Tables 7 and 8. In the last rows of these two tables, it is conjectured what the results would be for squares of order 2^n .

In Table 7, it is interesting to note that no duplicates or F-squares nonorthogonal to the preceding set, NOFS, were obtained using row operations first. The number of squares that are not F-squares apppears to be $(2^n - 1)^2 - 2(2^n - 1)(2^n - 2)/3 = (2^{2n} - 1)/3$. As n increases, the ratio of this number to the number of orthogonal F-squares obtained by row permutations approaches one-half, and the ratio of this number to the number of additional orthogonal F-squares obtained by column operations approaches unity, since

$$[(2^{2n} - 1)/3]/[2(2^n - 1)(2^n - 2)/3] = \frac{1}{2} \cdot \frac{2^n+1}{2^n-2} \quad .$$

Table 7. Classification of Types of Squares Obtained

by Row and Column Operations

	Types of squares and number of OF(16;8,8)-squares obtained by					
	Row Operations		Column Operations			
Order	OFS	N	OFS	N	DUPS	NOFS
4	4	5	2	5	2	0
8	28	21	14	21	6	8
16	140	85	70	85	14	56
2^n	$2(2^n-1)(2^n-2)/3$	$(2^{2n}-1)/3$	$(2^n-1)(2^n-2)/3$	$(2^{2n}-1)/3$	2^n-2	$(2^{2n}-6(2^n)+8)/3$

OFS: Orthogonal F-squares

N: Not an F-square

DUPS: Operation duplicated a previous F-square in the set

NOFS: F-square was produced but was not orthogonal to the previous set of F-squares

Table 8. **Frequency of Squares that Were Not F-Squares in Columns (and in Rows) in Tables 1 to 4**

	Number of squares that were not F-squares in column (or row)				
Order	1	2	4	8	2^{n-1}
4	1	2	0	0	0
8	1	2	4	0	0
16	1	2	4	8	0
2^n	2^0	2^1	2^2	2^3	$2^{n} - 1$

Computer Programs for Constructing Squares

A computer program given in Federer et al. [1981] obtains all mutually orthogonal F(8;4,4)-squares resulting from row and column operations on a Hadamard matrix H of order 8 and F-squares F_1 to F_7 of order 8 . The program has been generalized to obtain all mutually orthogonal F(4t;2t,2t)-squares resulting from row and column operations on a Hadamard matrix H of order 4t and F-squares F_1 to F_{4t-1} of order 4t . The program consists of the following four functions in the language APL:

```
    ∇ INITIALIZE R;I;J;IMJ
  ⍝ WRITTEN BY S J SCHWAGER, BIOMETRICS UNIT, CORNELL UNIV
  ⍝ 9/18/81
  ⍝ INITIALIZES NF, KT, AND FSQUARE MATRICES F[NF×R×R], H[R×R]
  ⍝ USES FN 'POSN' TO SET DIAGONALS OF F[I;;] TO ¯1(↔0)
  ⍝ ¯1↔0 FOR STORAGE IN 0-1 FORM
  ⍝ ARITHMETIC HAS BEEN MODIFIED ACCORDINGLY
  ⍝ INPUT VARIABLES NEEDED:  R = SCALAR RANK OF FSQUARES;
  ⍝    M = NF×(R÷2) MATRIX OF ¯1(↔0) POSITIONS FOR F DIAGONALS,
  ⍝        WHERE NF IS THE NUMBER OF INITIAL FSQUARES
  ⍝    HMINUS = R×(R÷2) MATRIX OF ¯1(↔0) POSITIONS FOR H
  ⍝        WHOSE FIRST ROW CAN BE SET TO 0 SINCE H[1;] IS 1
  F←((KT←NF←1↑⍴M),R,R)⍴H←(R,R)⍴1
  IMJ←I-J←⍉I←⍉(R,R)⍴⍳R
  I←1
 LP1:F[I;;]←~POSN M[I;],M[I;]-R
  →LP1×⍳NF≥I←I+1
  I←2
 LP2:H[I;HMINUS[I;]]←0
  →LP2×⍳R≥I←I+1
  'INITIALIZATION OF F AND H HAS BEEN COMPLETED FOR RANK ',⍕R
  'NUMBER OF INITIAL F-SQUARES IS ',⍕NF
    ∇
```

```
    ∇ X←POSN V
  ⍝ WRITTEN BY S J SCHWAGER, BIOMETRICS UNIT, CORNELL UNIV
  ⍝ 8/31/81
  ⍝ V IS INTEGER VECTOR TELLING WHICH F DIAGONALS BECOME ¯1
  ⍝ INPUTS R, IMJ[R×R] COME FROM CALLING PROGRAM 'INITIALIZE'
  X←+/[1](((⍴V),R,R)⍴IMJ)= 3 2 1 ⍉(R,R,⍴V)⍴V
    ∇
```

```
     ∇ FSQROWS R;RD2;R2D4;J;I;FT;SUMS;ORCT;NORTH
[1]   ⍝ WRITTEN BY S J SCHWAGER, BIOMETRICS UNIT, CORNELL UNIV
[2]   ⍝ 9/18/81
[3]   ⍝ FURTHER OPTIMIZATION POSSIBLE THROUGH LOOP AVOIDANCE
[4]   ⍝ PROGRAMS MUST BE RUN IN THIS ORDER: INITIALIZE, FSQROWS, FSQCOLS
[5]   ⍝ VARIABLES NF, KT, H[R×R], F[NF×R×R] COME FROM 'INITIALIZE'
[6]    R2D4←(RD2←R÷J←2)*2
[7]    F←F,[1]((NF×R-1),R,R)⍴0
[8]   JLP:I←1
[9]   ILP:FT←~2|F[I;;]+(R,R)⍴H[J;]
[10]   '----------------'
[11]   'F-SQUARE AND ROW OF H ARE  ',⍕I,J
[12]   →ST2×⍳1=∧/RD2=SUMS←(+⌿FT),+/FT
[13]   →LP,⍴⎕←'COLUMN AND ROW SUMS ARE NOT ALL ZERO: ',⍕(2×SUMS)-R
[14]  ST2:→ST3×⍳1=∧/~NORTH←R2D4≠ORCT←+/[2]+/[3](F[⍳KT;;]>0)×(KT,R,R)⍴FT>0
[15]   →LP,⍴⎕←'F-SQUARE, BUT NOT ORTHOGONAL TO: ',⍕NORTH/NORTH×⍳⍴NORTH
[16]  ST3:F[(KT←KT+1);;]←FT
[17]   '*** THIS IS AN ORTHOGONAL F-SQUARE ***'
[18]  LP:→ILP×⍳NF≥I←I+1
[19]   →JLP×⍳R≥J←J+1
[20]   F←F[⍳KT;;]
[21]   '----------------'
[22]   'END OF F-SQUARE ROW-WISE RUN; NUMBER OF F-SQUARES IS  ',⍕KT
     ∇
```

```
     ∇ FSQCOLS R;RD2;R2D4;J;I;FT;SUMS;ORCT;NORTH
[1]   ⍝ WRITTEN BY S J SCHWAGER, BIOMETRICS UNIT, CORNELL UNIV
[2]   ⍝ 9/18/81
[3]   ⍝ FURTHER OPTIMIZATION POSSIBLE THROUGH LOOP AVOIDANCE
[4]   ⍝ PROGRAMS MUST BE RUN IN THIS ORDER: INITIALIZE, FSQROWS, FSQCOLS
[5]   ⍝ VARIABLES NF, KT, H[R×R], F[NF×R×R] COME FROM 'INITIALIZE'
[6]    R2D4←(RD2←R÷J←2)*2
[7]    F←F,[1]((NF×R-1),R,R)⍴0
[8]   JLP:I←1
[9]   ILP:FT←~2|F[I;;]+⍉(R,R)⍴H[;J]
[10]   '----------------'
[11]   'F-SQUARE AND COLUMN OF H ARE  ',⍕I,J
[12]   →ST2×⍳1=∧/RD2=SUMS←(+⌿FT),+/FT
[13]   →LP,⍴⎕←'COLUMN AND ROW SUMS ARE NOT ALL ZERO: ',⍕(2×SUMS)-R
[14]  ST2:→ST3×⍳1=∧/~NORTH←R2D4≠ORCT←+/[2]+/[3](F[⍳KT;;]>0)×(KT,R,R)⍴FT>0
[15]   →LP,⍴⎕←'F-SQUARE, BUT NOT ORTHOGONAL TO: ',⍕NORTH/NORTH×⍳⍴NORTH
[16]  ST3:F[(KT←KT+1);;]←FT
[17]   '*** THIS IS AN ORTHOGONAL F-SQUARE ***'
[18]  LP:→ILP×⍳NF≥I←I+1
[19]   →JLP×⍳R≥J←J+1
[20]   F←F[⍳KT;;]
[21]   '----------------'
[22]   'END OF F-SQUARE COLUMN-WISE RUN; NUMBER OF F-SQUARES IS  ',⍕KT
     ∇
```

Input matrices M and HMINUS are needed to initialize the matrix F of F-squares and the Hadamard matrix H, respectively. The dimension of F is initially $NF \times R \times R$, where $NF = 4t - 1$ is the number of original F-squares and $R = 4t$ is the order of the F-squares. The NF orthogonal F-squares form the $R \times R$ layers of F . Each row of M indicates which diagonals of the corresponding F-square have elements equal to -1, which is written in APL output as ¯1 . When integer k appears in row ℓ of M, the ℓth original F-square contains -1 in the kth position below the upper left-hand element. Since F-squares are circular matrices [see Press, 1972, p. 14], the entire diagonal k levels below and 16 - k levels above the main diagonal consists of -1's.

The dimension of HMINUS is $R \times \frac{1}{2}R$. Each row of HMINUS indicates which elements in the corresponding row of H are equal to -1 . The first row is not used. The input matrices used to obtain Table 5 are:

M

11	9	7	6	4	3
11	10	5	4	3	2
10	9	8	6	5	3
11	8	6	5	4	1
10	8	7	6	4	2
11	9	8	7	5	2
7	6	5	3	2	1
11	10	9	6	2	1
10	9	7	5	4	1
11	10	8	7	3	1
9	8	4	3	2	1

HMINUS

0	0	0	0	0	0
2	4	6	7	9	10
2	3	8	9	10	11
3	4	5	7	8	10
2	5	7	8	9	12
3	5	6	7	9	11
2	4	5	6	8	11
6	7	8	10	11	12
2	3	4	7	11	12
3	4	6	8	9	12
2	3	5	6	10	12
4	5	9	10	11	12

The input matrices used to obtain Table 6 are:

M

8	7	6	5	4	3	2	1
12	11	10	9	4	3	2	1
12	11	10	9	8	7	6	5
14	13	10	9	6	5	2	1
14	13	10	9	8	7	4	3
14	13	12	11	6	5	4	3
14	13	12	11	8	7	2	1
15	13	11	9	7	5	3	1
15	13	11	9	8	6	4	2
15	13	12	10	7	5	4	2
15	13	12	10	8	6	3	1
15	14	11	10	7	6	3	2
15	14	11	10	8	5	4	1
15	14	12	9	7	6	4	1
15	14	12	9	8	5	3	2

HMINUS

0	0	0	0	0	0	0	0
9	10	11	12	13	14	15	16
5	6	7	8	13	14	15	16
5	6	7	8	9	10	11	12
3	4	7	8	11	12	15	16
3	4	7	8	9	10	13	14
3	4	5	6	11	12	13	14
3	4	5	6	9	10	15	16
2	4	6	8	10	12	14	16
2	4	6	8	9	11	13	15
2	4	5	7	10	12	13	15
2	4	5	7	9	11	14	16
2	3	6	7	10	11	14	15
2	3	6	7	9	12	13	16
2	3	5	8	10	11	13	16
2	3	5	8	9	12	14	15

The function INITIALIZE creates H and F, calling the function POSN while calculating F . The functions FSQROWS and FSQCOLS do the heavy work: they generate new squares by row and column operations, respectively, using H and the NF original F-squares; they check whether each new square is an F-square; and they determine whether each new F-square is orthogonal to all mutually orthogonal F-squares previously found. As new orthogonal F-squares are obtained, they are concatenated onto the matrix F as additional $R \times R$ layers.

To take advantage of APL's extremely efficient storage of 0 - 1 matrices, -1's are replaced by 0's in all calculations of these functions. This allows the manipulation and storage of large matrices, e.g., F of dimension $225 \times 16 \times 16$, without requiring large amounts of core. To implement this switch, multiplication of +1's and -1's is replaced by addition of 1's and 0's followed by taking residues modulo 2 and then negating, i.e., changing 1 to 0 and vice versa. Returning H and F to the form consisting of -1's and +1's is easily accomplished by subtracting 1 from twice the 0 - 1 form.

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