ON A COMPLETE SET OF ORTHOGONAL F-SQUARES OF ORDER 8
WITH A MATELESS LATIN SQUARE

by

W. T. Federer, J. P. Mandeli, and S. J. Schwager

BU-753-M* September, 1981

Abstract

A Latin square of order eight that has no orthogonal Latin
square is shown to have 42 mutually orthogonal F(8;4,4)-square mates.
This is the complete set of F-squares needed for the mateless Latin
square. A new F-square geometry of order eight and a new orthogonal
array involving three sets of eight symbols and 42 sets of two sym-
bols are then constructed.

*In the Mimeo Series of the Biometrics Unit, Cornell University, Ithaca,
New York 14853.
Mandeli (1975) and Mandeli and Federer (1981) showed how to construct six F-squares, denoted as $F(4;2,2)$-squares, that are mutually orthogonal and are orthogonal to a Latin square of order four that has no orthogonal Latin square mate, i.e., an [OL(4,1)] set. The method of construction follows. The [OL(4,1)]-square

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
4 & 1 & 2 & 3 \\
3 & 4 & 1 & 2 \\
2 & 3 & 4 & 1 \\
\end{bmatrix}
\]

is decomposed into three orthogonal F-squares, $F(4;2,2)$, as follows:

\[
F_1 = \begin{bmatrix}
++- & - & + & - \\
- & ++- & - & + \\
- & + & ++- & - \\
+ & - & - & + \\
\end{bmatrix}, \quad F_2 = \begin{bmatrix}
+ & - & + & - \\
- & + & + & - \\
+ & - & - & + \\
- & + & - & + \\
\end{bmatrix}, \quad \text{and} \quad F_3 = \begin{bmatrix}
+ & - & + \\
- & + & - \\
+ & - & + \\
- & + & - \\
\end{bmatrix}.
\]

Then, given

\[
H = \begin{bmatrix}
+++ \\
+-+ \\
+-+ \\
++- \\
\end{bmatrix},
\]

six orthogonal F-squares, $OF(4;2,2;6)$, are constructed as follows:

*In the Mimeo Series of the Biometrics Unit, Cornell University, Ithaca, New York 14853.
\[ F_4 = F_1 \times \text{row 3 of } H = F_3 \times \text{column 3 of } H, \]
\[ F_5 = F_2 \times \text{row 2 of } H, \]
\[ F_6 = F_2 \times \text{row 4 of } H, \]
\[ F_7 = F_3 \times \text{row 3 of } H = F_1 \times \text{column 3 of } H, \]
\[ F_8 = F_2 \times \text{column 2 of } H, \text{ and } \]
\[ F_9 = F_2 \times \text{column 4 of } H. \]

To illustrate the procedure, the F-square \( F_4 \) is obtained by multiplying each row of \( F_1 \) by row 3 of \( H \):

\[
\begin{array}{c|c|c}
\text{Row 3 of } H & + & + \\
\hline
F_1 & + & - \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c}
 & + & - & - \\
\hline
F_1 & + & + & - \\
 & - & + & + \\
 & - & - & + \\
 & + & - & + \\
\end{array}
\]

\[ \implies F_4 = \begin{bmatrix} + & - & + \\ - & + & + \\ + & + & - \end{bmatrix}. \]

and the F-square \( F_8 \) is obtained by multiplying each column of \( F_2 \) by column 2 of \( H \):

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\text{Column 2 of } H & F_2 & + & + & - & - & + & + & - \\
\hline
+ & + & - & - & + & + & - & + & - \\
+ & + & - & - & + & + & - & + & - \\
- & + & - & - & + & + & - & + & - \\
- & + & - & - & + & + & - & + & - \\
\end{array}
\]

\[ \implies F_8 = \begin{bmatrix} + & - & + \\ - & + & + \\ + & + & - \end{bmatrix}. \]

Note that \( F_1 \times \text{row 2 of } H \) does not produce an F-square:

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\text{Row 2 of } H & + & + & - \\
\hline
F_1 & + & + & - \\
 & - & + & + \\
 & - & - & + \\
 & + & - & + \\
\end{array}
\]

\[ \implies \begin{bmatrix} + & + & + \\ - & + & + \\ - & - & - \\
\end{bmatrix} \neq \text{F-square}. \]

Also note that there are \( 3 \times 3 = 9 \) squares produced by row operations and \( 3 \times 3 = 9 \) squares produced by column operations. Of these, eight are F-squares and ten are not F-squares.
The question now arises as to whether or not the same operations produce $(8-2)(8-1) = 42$ F-squares of order 8 that are mutually orthogonal and are orthogonal to a mateless Latin square of order 8, i.e., an $[OL(8,1)]$-set. We first decompose the following mateless Latin square into seven mutually orthogonal $F(8;4,4)$-squares, $F_1$ to $F_7$. These together with a Hadamard matrix $H$ of order eight are given below:

<table>
<thead>
<tr>
<th>LS(8)</th>
<th>$F_1$</th>
<th>$F_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3 4 5 6 7 8</td>
<td>$++++-++++-+</td>
<td>$++++-++++-+$</td>
</tr>
<tr>
<td>8 1 2 3 4 5 6 7</td>
<td>$++-++++++-+</td>
<td>$++++-++++-+$</td>
</tr>
<tr>
<td>7 8 1 2 3 4 5 6</td>
<td>$++-++++++-+</td>
<td>$++++-++++-+$</td>
</tr>
<tr>
<td>6 7 8 1 2 3 4 5</td>
<td>$++-++++++-+</td>
<td>$++++-++++-+$</td>
</tr>
<tr>
<td>5 6 7 8 1 2 3 4</td>
<td>$+---++++++-</td>
<td>$++++-++++-+$</td>
</tr>
<tr>
<td>4 5 6 7 8 1 2 3</td>
<td>$+---++++++-</td>
<td>$++++-++++-+$</td>
</tr>
<tr>
<td>3 4 5 6 7 8 1 2</td>
<td>$++++-++++-+$</td>
<td>$++++-++++-+$</td>
</tr>
<tr>
<td>2 3 4 5 6 7 8 1</td>
<td>$++++-++++-+$</td>
<td>$++++-++++-+$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$F_3$</th>
<th>$F_4$</th>
<th>$F_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$+-++++++-+$</td>
</tr>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$+-++++++-+$</td>
</tr>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$+-++++++-+$</td>
</tr>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$+-++++++-+$</td>
</tr>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$+-++++++-+$</td>
</tr>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$+-++++++-+$</td>
</tr>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$+-++++++-+$</td>
</tr>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$+-++++++-+$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$F_6$</th>
<th>$F_7$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$++-++++++-+$</td>
</tr>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$++-++++++-+$</td>
</tr>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$++-++++++-+$</td>
</tr>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$++-++++++-+$</td>
</tr>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$++-++++++-+$</td>
</tr>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$++-++++++-+$</td>
</tr>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$++-++++++-+$</td>
</tr>
<tr>
<td>$++-++++++-+$</td>
<td>$+-++++++-+$</td>
<td>$++-++++++-+$</td>
</tr>
</tbody>
</table>

Examining the $7 \times 7 = 49$ squares produced by the row operations described above, we obtain 28 orthogonal F-squares as follows:
\[ F_8 = F_1 \times \text{row 2 of } H = F_5 \times \text{column 2 of } H, \]
\[ F_9 = F_2 \times \text{row 2 of } H = F_6 \times \text{column 2 of } H, \]
\[ F_{10} = F_3 \times \text{row 2 of } H = F_7 \times \text{column 2 of } H, \]
\[ F_{11} = F_5 \times \text{row 2 of } H = F_1 \times \text{column 2 of } H, \]
\[ F_{12} = F_6 \times \text{row 2 of } H = F_2 \times \text{column 2 of } H, \]
\[ F_{13} = F_7 \times \text{row 2 of } H = F_3 \times \text{column 2 of } H, \]
\[ F_{14} = F_1 \times \text{row 3 of } H, \]
\[ F_{15} = F_3 \times \text{row 3 of } H, \]
\[ F_{16} = F_4 \times \text{row 3 of } H, \]
\[ F_{17} = F_5 \times \text{row 3 of } H, \]
\[ F_{18} = F_7 \times \text{row 3 of } H, \]
\[ F_{19} = F_1 \times \text{row 4 of } H, \]
\[ F_{20} = F_3 \times \text{row 4 of } H, \]
\[ F_{21} = F_4 \times \text{row 4 of } H, \]
\[ F_{22} = F_5 \times \text{row 4 of } H, \]
\[ F_{23} = F_7 \times \text{row 4 of } H, \]
\[ F_{24} = F_2 \times \text{row 5 of } H, \]
\[ F_{25} = F_4 \times \text{row 5 of } H, \]
\[ F_{26} = F_6 \times \text{row 5 of } H, \]
\[ F_{27} = F_2 \times \text{row 6 of } H, \]
\[ F_{28} = F_4 \times \text{row 6 of } H, \]
\[ F_{29} = F_6 \times \text{row 6 of } H, \]
\[ F_{30} = F_2 \times \text{row 7 of } H, \]
\[ F_{31} = F_4 \times \text{row 7 of } H, \]
\[ F_{32} = F_6 \times \text{row 7 of } H, \]
\[ F_{33} = F_2 \times \text{row 8 of } H, \]
\( F_{34} = F_4 \times \text{row 8 of } H \), and
\( F_{35} = F_6 \times \text{row 8 of } H \).

None of the other 21 squares obtained by row operations are F-squares; there are no duplications of F-squares, either.

Proceeding next to the 49 squares produced by column operations, we obtain the following 14 mutually orthogonal \( F(8;4,4) \)-squares:

\[
\begin{align*}
F_{36} &= F_4 \times \text{column 3 of } H, \\
F_{37} &= F_4 \times \text{column 4 of } H, \\
F_{38} &= F_2 \times \text{column 5 of } H, \\
F_{39} &= F_4 \times \text{column 5 of } H, \\
F_{40} &= F_6 \times \text{column 5 of } H, \\
F_{41} &= F_2 \times \text{column 6 of } H, \\
F_{42} &= F_4 \times \text{column 6 of } H, \\
F_{43} &= F_6 \times \text{column 6 of } H, \\
F_{44} &= F_2 \times \text{column 7 of } H, \\
F_{45} &= F_4 \times \text{column 7 of } H, \\
F_{46} &= F_6 \times \text{column 7 of } H, \\
F_{47} &= F_2 \times \text{column 8 of } H, \\
F_{48} &= F_4 \times \text{column 8 of } H, \quad \text{and} \\
F_{49} &= F_6 \times \text{column 8 of } H.
\end{align*}
\]

When added to the orthogonal F-squares \( F_1 \) to \( F_{35} \), these 14 additional F-squares give the complete set. Of the other 35 squares obtained by column operations, 21 are not F-squares, eight are F-squares but are not orthogonal to one or more of \( F_1 \) to \( F_{35} \), and six are duplicates of squares in the first set, namely \( F_8 \) to \( F_{13} \).
A complete set of 49 orthogonal F(8;4,4)-squares has been obtained, seven, F₁ to F₇, of which can be used to compose a mateless Latin square of order eight. Thus, for orders 4 and 8, a complete set has been obtained. The question arises as to whether or not this F-square geometry and the construction method holds for all 4t or only for 4t = 2ⁿ. Using an extension of the computer program at the end of this paper, one could check for F-squares of order 12 and 16. This is being done.

A proof that a complete set exists for F-squares of order 2ⁿ should be possible. This is because of closure under multiplication of the elements in a Galois Field of 2ⁿ elements. There are 2²ⁿ effects, 2ⁿ - 1 are row effects and 2ⁿ - 1 are column effects. Without loss of generality, the same H₂ⁿ×2ⁿ used for row and column operations can be used to decompose the mateless Latin square of order 2ⁿ into 2ⁿ - 1 F(2ⁿ;2ⁿ⁻¹,2ⁿ⁻¹)-squares. Note that the interaction of the rows and columns has (2ⁿ - 1)(2ⁿ - 1) single degree of freedom contrasts and these may be used to construct the (2ⁿ - 1)(2ⁿ - 2) OF(2ⁿ;2ⁿ⁻¹,2ⁿ⁻¹)-squares that are orthogonal to a mateless Latin square of order 2ⁿ. This means that a new Projective Geometry is available with one axis of dimension 2ⁿ - 1 and the other (2ⁿ - 1)(2ⁿ - 2) axes of dimension one.

The computer program for finding the set of mutually orthogonal F(8;4,4)-squares consists of the following four APL functions:

v INITIALIZES
[1] a WRITTEN BY S J SCHWAGER, BIOMETRICS UNIT, CORNELL UNIV
[2] a 8/22/81
[3] a INITIALIZES FSQUARE MATRICES F[7x8x8] AND H[8x8]
[4] a USES FN 'POSN8' TO SET DIAGONALS OF F[I;;] TO -1
[5] F+ 7 8 8 pH+ 8 8 p1
[7] F[1;;]+H-2*POSN8 4 3 2 1 4 5 6 7
[8] F[2;;]+H-2*POSN8 6 5 2 1 7 3 6 7
[10] F[4;;]+H-2*POSN8 7 5 3 1 1 3 5 7
[12] F[6;;]+H-2*POSN8 7 6 3 2 1 2 5 6
[13] F[7;;]+H-2*POSN8 7 6 4 1 1 2 4 7

V
\[ V \times \text{POSN} V \]

1. Written by S.J. Schwager, Biometrics Unit, Cornell Univ
2. 8/22/81
3. V is a vector of integers
4. Input matrix \( IM[J[8 \times 8] \) comes from 'INITIALIZ8'
5. \( X + ([1][((pV),8,8)pIMJ] = 3, 2, 1 \quad [8,8,pV]pV \)

\[ V \times \text{FSQROW8} \]

1. Written by S.J. Schwager, Biometrics Unit, Cornell Univ
2. 8/26/81
3. Further optimization possible through loop avoidance
4. Program 'INITIALIZ8' must be run before 'FSQROW8'
5. Program 'FSQCOL8' must be run after 'FSQROW8'
6. Input variables \( H[8 \times 8], P[7 \times 8 \times 8] \) come from 'INITIALIZ8'
7. J+2
8. JLP: I+1
9. ILP: FT + F[I[:]] \times [8,8] \quad pH[J[:]]
10. '------------------------'
11. 'F-SQUARE AND COLUMN OF H ARE'
12. [a+b]
13. +ST2 \times 1 = \alpha / O = [+(/FT), +/([FT]
14. +LP, pH + 'COLUMN AND ROW SUMS ARE NOT ALL ZERO'
15. ST2: +ST3 \times 1 = \alpha / 16 = [1]/2 + /[3] \quad (pF) \times (pF) \quad pFT > 0
16. +LP, pH + 'F-SQUARE, BUT NOT ORTHOGONAL'
17. ST3: FT + ((1-D) \quad [1] \quad 8 \quad 8 \quad pFT + (D + ((1+D) \quad p1), 0) \quad [1] \quad F
18. '*** ABOVE IS AN ORTHOGONAL F-SQUARE ***'
19. LP: +ILP \times [72 I+I+1]
20. +JLP \times [82 I+J+1]
21. '------------------------'
22. 'END OF F-SQUARE ROW-WISE RUN'

\[ V \times \text{FSQCOL8} \]

1. Written by S.J. Schwager, Biometrics Unit, Cornell Univ
2. 8/26/81
3. Further optimization possible through loop avoidance
4. Program 'INITIALIZ8' must be run before 'FSQROW8'
5. Program 'FSQCOL8' must be run after 'FSQROW8'
6. Input variables \( H[8 \times 8], P[7 \times 8 \times 8] \) come from 'INITIALIZ8'
7. J+2
8. JLP: I+1
9. ILP: FT + F[I[:]] \times [8,8] \quad pH[J[:]]
10. '------------------------'
11. 'F-SQUARE AND COLUMN OF H ARE'
12. [a+b]
13. +ST2 \times 1 = \alpha / O = [+(/FT), +/([FT]
14. +LP, pH + 'COLUMN AND ROW SUMS ARE NOT ALL ZERO'
15. ST2: +ST3 \times 1 = \alpha / 16 = [1]/2 + /[3] \quad (pF) \times (pF) \quad pFT > 0
16. +LP, pH + 'F-SQUARE, BUT NOT ORTHOGONAL'
17. ST3: FT + ((1-D) \quad [1] \quad 8 \quad 8 \quad pFT + (D + ((1+D) \quad p1), 0) \quad [1] \quad F
18. '*** ABOVE IS AN ORTHOGONAL F-SQUARE ***'
19. LP: +ILP \times [72 I+I+1]
20. +JLP \times [82 I+J+1]
21. '------------------------'
22. 'END OF F-SQUARE COLUMN-WISE RUN'

\[ V \]
References Cited
