A Note on Confidence Bands in Bilinear Segmented Regression

by

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Abstract

Much of the literature on segmented regression has focused on the two-line (bilinear) case with one join point. Yet very little has been done concerning confidence band construction -- even in the bilinear setting -- due to the non-linear nature of the problem. By placing certain conditions on the situation, a result equating the abscissae of the confidence band intersections and join point is developed.

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A Note on Confidence Bands in Bilinear Segmented Regression

Over the past 20 years the literature on segmented, or splined, regression has grown significantly[1]. Of particular interest has been the two-line, one join point case, upon which much of this literature focuses. Yet very little has been suggested in the way of confidence band construction about the prediction function, even in the (relatively) simple bilinear case. After wrestling with the equations for only a short time one can easily understand why: the non-linear structure of the least squares procedure in this setting produces problems which can, at the very least, be described as nerve-racking. However, by placing various conditions* on the model and the state of the solution one can achieve some particularly interesting (or at least mathematically exciting) results.

The general bilinear model can be expressed as

$$\mu(x_i) = \begin{cases} \alpha_1 + \beta_2 x_i & i = 1, \ldots, \tau \\ \alpha_2 + \beta_2 x_i & i = \tau + 1, \ldots, n, \end{cases} \tag{1}$$

with the constraint that the lines meet at the join point with abscissa value

$$J = \frac{\alpha_1 - \alpha_2}{\beta_2 - \beta_1} . \tag{2}$$

Blischke [2] has shown that the least squares estimate of J for a given value of τ, $\hat{J}_\tau$, is the intersection of the two lines formed by performing least squares regression independently on the first τ, then the next $n-\tau$, sets of points,

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* which are, in many cases, not all-that-unusually restrictive.

when this value is in the interval $(x_\tau, x_{\tau+1})$. If the intersection is outside of this interval, then $\hat{J}_\tau$ is taken to be whichever of the two values, x_τ or $x_{\tau+1}$, minimizes the residual sum of squares. The final least squares estimate is that $\hat{J}_\tau$ which minimizes the residual sum of squares over all $\tau = 1, \ldots, n-1$. If however τ is known, as is many times the case when some previous knowledge of the subject matter is at hand, the least squares estimate is simply $\hat{J} = \hat{J}_\tau$. We will take this as our first "restrictive" supposition.

Let us now further suppose that, with τ known, the data produce a value of $\hat{J}$ such that

$$\hat{J} \in (x_\tau, x_{\tau+1}) \tag{3}$$

(which is also not all-that-unusual in practice), i.e.

$$\hat{J} = \frac{\hat{\alpha}_1 \quad \hat{\alpha}_2}{\hat{\beta}_2 \quad \hat{\beta}_1} \; . \; * \tag{4}$$

Considering each line independently then, the (hyperbolic) $1 - \frac{\alpha}{2}$ confidence bands are given by Snedecor and Cochran as [3],

$$\mu(x) \in \left[\hat{\alpha}_1 + \hat{\beta}_1 x - t_{\alpha/2}(\tau-2)\hat{\sigma}_1\sqrt{\frac{1}{\tau} + \frac{(x-\bar{x}_1)^2}{c_1}}\right., \tag{5}$$
$$\left.\hat{\alpha}_1 + \hat{\beta}_1 x + t_{\alpha/2}(\tau-2)\hat{\sigma}_1\sqrt{\frac{1}{\tau} + \frac{(x-\bar{x}_1)^2}{c_1}}\right],$$

and

$$\mu(x) \in \left[\hat{\alpha}_2 + \hat{\beta}_2 x - t_{\alpha/2}(n-\tau-2)\hat{\sigma}_2\sqrt{\frac{1}{n-\tau} + \frac{(x-\bar{x}_2)^2}{c_2}}\right., \tag{6}$$
$$\left.\hat{\alpha}_2 + \hat{\beta}_2 x \; t_{\alpha/2}(n-\tau-2)\hat{\sigma}_2\sqrt{\frac{1}{n-\tau} + \frac{(x-\bar{x}_2)^2}{c_2}}\right] \; ;$$

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* Hudson [4] has shown that with this supposition, the least squares estimates for the parameters $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}_1$, $\hat{\beta}_2$ are the usual (unconstrained) least squares estimates.

where
$$\bar{x}_1 = \frac{1}{\tau} \sum_{i=1}^{\tau} x_i$$

$$c_1 = \sum_{i=1}^{\tau} (x_i - \bar{x}_1)^2$$

$$\hat{\sigma}_1^2 = \frac{1}{\tau-2} \sum_{i=1}^{\tau} (y_i - \hat{\alpha}_1 - \hat{\beta}_1 x_i)^2$$

$$\bar{x}_2 = \frac{1}{n-\tau} \sum_{i=\tau+1}^{n} x_i$$

$$c_2 = \sum_{i=\tau+1}^{n} (x_i - \bar{x}_2)^2$$

$$\hat{\sigma}_2^2 = \frac{1}{n-\tau-2} \sum_{i=\tau+1}^{n} (y_i - \hat{\alpha}_2 - \hat{\beta}_2 x_i)^2 \quad ,$$

and t_p (df) is the (positive) value of the t-distribution with df degrees of freedom which encloses area = p symmetrically about zero. For convenience's sake, we will henceforth denote $t_{\alpha/2}(\tau-2)$ as t_1 and $t_{\alpha/2}(n-\tau-2)$ as t_2 for some fixed $\alpha\epsilon(0,1)$. Thus the first set of bands has the functional form

$$f_1(x) = \hat{\alpha}_1 + \hat{\beta}_1 \pm t_1\hat{\sigma}_1\sqrt{\frac{1}{\tau} + \frac{(x-\bar{x}_1)^2}{c_1}} \qquad (7)$$

and the second set has the form

$$f_2(x) = \hat{\alpha}_2 + \hat{\beta}_2 \pm t_2\hat{\sigma}_2\sqrt{\frac{1}{n-\tau} + \frac{(x-\bar{x}_2)^2}{c_2}} \qquad (8)$$

Now, using a simplified formulation of the Bonferroni inequality, namely that

$$P(A \cap B) \geq P(A) + P(B) - 1 \quad , \qquad (9)$$

we can construct a band about the prediction function with maximum confidence level = $1 - \alpha$ by intersecting the two $1 - \frac{\alpha}{2}$ confidence bands about each line.

One problem that immediately confronts us is that one or both of the bands may have no intersection. For example, setting the upper band equations equal produces

$$(\hat{\beta}_1-\hat{\beta}_2)x + t_1\hat{\sigma}_1\sqrt{\frac{1}{\tau} + \frac{(x-\bar{x}_1)^2}{c_1}} - t_2\hat{\sigma}_2\sqrt{\frac{1}{n-\tau} + \frac{(x-\bar{x}_2)^2}{c_2}} + \hat{\alpha}_1 - \hat{\alpha}_2 = 0. \qquad (10)$$

This is of the form

$$U(x) = 0,$$

and, being a difference of two rather complicated hyperbolas, will in general only be solved by some numerical algorithm. Even then, the solution may not be real-valued. The situation is similar for the lower intersection, where the equation corresponding to (10) is

$$(\hat{\beta}_1-\hat{\beta}_2)x - t_1\hat{\sigma}_1\sqrt{\frac{1}{\tau} + \frac{(x-\bar{x}_1)^2}{c_1}} + t_2\hat{\sigma}_2\sqrt{\frac{1}{n-\tau} + \frac{(x-\bar{x}_2)^2}{c_2}} + \hat{\alpha}_1 - \hat{\alpha}_2 = 0. \qquad (11)$$

If we now suppose that both sets of bands do meet, one might question if there exists some x where the upper and lower intersections coincide. The answer is yes (and since the bands are hyperbolas it is, in this setting, unique). Yet perhaps what is most interesting about this point of intersection is the value of x at which it occurs:

<u>Theorem</u>: Under the above suppositions, the point of this intersection occurs at $x=\hat{J}$.

<u>Proof</u>: From (4), take

$$x = \hat{J} = \frac{\alpha_2 - \alpha_1}{\hat{\beta}_1 - \hat{\beta}_2}$$

in (10). This yields, after a bit of algebra and the assumption that $\hat{\beta}_1 \neq \hat{\beta}_2$ (again, not all that unreasonable, since if $\hat{\beta}_1 = \hat{\beta}_2$ we would have good reason to suspect that the join point analysis is unwarrented),

$$t_1\sigma_1\sqrt{\frac{1}{\tau} + \frac{(x-\bar{x}_1)^2}{c_1}} = t_2\sigma_2\sqrt{\frac{1}{n-\tau} + \frac{(x-\bar{x}_2)^2}{c_2}} \tag{12}$$

But combining (12) and (4) shows that (11) holds. Now, retracing our steps and applying (4) to (11) again produces (12), so we can conclude that, given (4) and the other suppositions, (10) $\Leftrightarrow$ (11). Q.E.D.

In practice however, even when all these conditions are met, the numerical procedure (or utilized computers) which produce the values of the intersections and join may not be precise enough to indicate this situation. The experimenter may therefore never realize that his join and band intersections have the same abscissa. Still, in a (sub)topic where much more work needs to be done, it is a fascinating little result, and perhaps a glimmer of hope for the future.

References

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