On Embedding a Mateless Latin Square In a Complete Set of Orthogonal F-Squares

John P. Mandeli
Virginia Commonwealth University

Walter T. Federer
Cornell University

This paper gives an example of a latin square which does not belong in any finite field that can be embedded in a complete set of orthogonal F-squares.

AMS Subject Classification: Primary 62K99, 62K15

Key Words and Phrases: Latin square design, F-square design, mateless latin square, complete set of orthogonal F-squares, decomposing latin squares into F-squares.

1. Introduction and Definitions

To save space the reader is referred to Hedayat and Seiden (1970) and Hedayat, Raghavarao, and Seiden (1975) for the definition of an F-square and a complete set of orthogonal F-squares.

Euler proved in the eighteenth century that a cyclic latin square of even order has no orthogonal latin square mate. Bose (1938) showed the one-to-one correspondence between finite fields and a complete set of orthogonal latin squares. Hence cyclic latin squares do not belong in any finite field. We will show that despite this the cyclic latin square of order 4 has orthogonal F-square mates and can in fact be embedded in a complete set of orthogonal F-squares of order 4.

2. The Cyclic Latin Square of Order 4

Consider the following cyclic 4 × 4 latin square:

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

We first find the 3 orthogonal F-squares that this latin square decomposes into. This involves decomposing the 3 degrees of freedom for treatments of the latin square into 3 single degree of freedom orthogonal contrasts. We know that the following decomposition is possible:

	d.f.
Treatments on latin square	3
$C_1 = 1 + 2 - 3 - 4$	1
$C_2 = 1 - 2 + 3 - 4$	1
$C_3 = 1 - 2 - 3 + 4$	1

These contrasts can of course be obtained from the Hadamard matrix of order 4:

$$H = \begin{pmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{pmatrix} .$$

To obtain the first F-square that the latin square decomposes into from contrast one we map symbols 1 and 2 from the latin square into the symbol "+" and map symbols 3 and 4 into the symbol "-".

We obtain

$$F_1 = \begin{matrix} + & + & - & - \\ - & + & + & - \\ - & - & + & + \\ + & - & - & + \end{matrix}$$

which is an F(4; 2,2) - square. To obtain F_2 from contrast 2 we map symbols 1 and 3 into "+" and symbols 2 and 4 into "-".

$$F_2 = \begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix} .$$

Similarly we obtain F_3 from C_3 .

$$F_3 = \begin{matrix} + & - & - & + \\ + & + & - & - \\ - & + & + & - \\ - & - & + & + \end{matrix} .$$

We can check and see that F_1 is orthogonal to F_2, F_1 is orthogonal to F_3, and F_2 is orthogonal to F_3. Hence the cyclic latin square of order 4 decomposes into three orthogonal F(4; 2,2) - squares.

By Hedayat, Raghavarao, and Sieden (1975) a complete set of 9 orthogonal F(4; 2,2) - squares exist. Therefore to embedd the cyclic latin square of order 4 in a complete set of orthogonal F-squares we need to find 6 orthogonal F-squares that are orthogonal to F_1, F_2, and F_3 so that we will have the complete set of 9 orthogonal F-squares.

To obtain our F-squares we form the multiplication tables of the rows of the Hadamard matrix H with F_1, F_2, and F_3. The multiplication tables of the rows of H with F_1 are:

·	+	+	−	−
	+	+	−	−
	−	+	+	−
	−	−	+	+
	+	−	−	+

·	+	−	+	−
	+	+	−	−
	−	+	+	−
	−	−	+	+
	+	−	−	+

·	+	−	−	+
	+	+	−	−
	−	+	+	−
	−	−	+	+
	+	−	−	+

=

+	+	+	+
−	+	−	+
−	−	−	−
+	−	+	−

=

+	−	−	+
−	−	+	+
−	+	+	−
+	+	−	−

=

+	−	+	−
−	−	−	−
−	+	−	+
+	+	+	+

F_4

The multiplication tables of the rows of H with F_2 are:

·	+	+	−	−
	+	−	+	−
	−	+	−	+
	+	−	+	−
	−	+	−	+

·	+	−	+	−
	+	−	+	−
	−	+	−	+
	+	−	+	−
	−	+	−	+

·	+	−	−	+
	+	−	+	−
	−	+	−	+
	+	−	+	−
	−	+	−	+

$$
= \begin{array}{cccc} + & - & - & + \\ - & + & + & - \\ + & - & - & + \\ - & + & + & - \end{array}
\qquad
= \begin{array}{cccc} + & + & + & + \\ - & - & - & - \\ + & + & + & + \\ - & - & - & - \end{array}
\qquad
= \begin{array}{cccc} + & + & - & - \\ - & - & + & + \\ + & + & - & - \\ - & - & + & + \end{array}
$$

F_5 F_6

The multiplication tables of the rows of H with F_3 are:

•	+	+	−	−
	+	−	−	+
	+	+	−	−
	−	+	+	−
	−	−	+	+

•	+	−	+	−
	+	−	−	+
	+	+	−	−
	−	+	+	−
	−	−	+	+

•	+	−	−	+
	+	−	−	+
	+	+	−	−
	−	+	+	−
	−	−	+	+

$$
= \begin{array}{cccc} + & - & + & - \\ + & + & + & + \\ - & + & - & + \\ - & - & - & - \end{array}
\qquad
= \begin{array}{cccc} + & + & - & - \\ + & - & - & + \\ - & - & + & + \\ - & + & + & - \end{array}
\qquad
= \begin{array}{cccc} + & + & + & + \\ + & - & + & - \\ - & - & - & - \\ - & + & - & + \end{array}
$$

F_7

Likewise we form the multiplication tables of the columns of H with F_1, F_2, and F_3:

•				
+	+	+	−	−
+	−	+	+	−
−	−	−	+	+
−	+	−	−	+

•				
+	+	+	−	−
−	−	+	+	−
+	−	−	+	+
−	+	−	−	+

•				
+	+	+	−	−
−	−	+	+	−
−	−	−	+	+
+	+	−	−	+

=

+	+	−	−
−	+	+	−
+	+	−	−
−	+	+	−

=

+	+	−	−
+	−	−	+
−	−	+	+
−	+	+	−

=

+	+	−	−
+	−	−	+
+	+	−	−
+	−	−	+

F_7

•				
+	+	−	+	−
+	−	+	−	+
−	+	−	+	−
−	−	+	−	+

=

+	−	+	−
−	+	−	+
−	+	−	+
+	−	+	−

•				
+	+	−	+	−
−	−	+	−	+
+	+	−	+	−
−	−	+	−	+

=

+	−	+	−
+	−	+	−
+	−	+	−
+	−	+	−

•				
+	+	−	+	−
−	−	+	−	+
−	+	−	+	−
+	−	+	−	+

=

+	−	+	−
+	−	+	−
−	+	−	+
−	+	−	+

F_8 F_9

•				
+	+	−	−	+
+	+	+	−	−
−	−	+	+	−
−	−	−	+	+

=

+	−	−	+
+	+	−	−
+	−	−	+
+	+	−	−

•				
+	+	−	−	+
−	+	+	−	−
+	−	+	+	−
−	−	−	+	+

=

+	−	−	+
−	−	+	+
−	+	+	−
+	+	−	−

•				
+	+	−	−	+
−	+	+	−	−
−	−	+	+	−
+	−	−	+	+

=

+	−	−	+
−	−	+	+
+	−	−	+
−	−	+	+

F_4

We see that six F(4; 2,2) - squares are obtained F_4, F_5, F_6, F_7, F_8, and F_9 (with F_4 and F_7 being constructed twice). We can check to see that the 6 F-squares are mutually orthogonal and also are orthogonal to F_1, F_2, and F_3. Hence the cyclic latin square of order four is orthogonal to the six orthogonal F-squares $F_4, F_5, \ldots, F_9$. We have therefore shown by construction that there exists a latin square, not belonging in any finite field, which is a member of a complete set of orthogonal F-squares.

3. Mateless Latin Squares of Order Other Than 4.

The existence and construction of complete sets of orthogonal F-squares obtained from mateless latin squares of order not equal to 4 is still an open problem. The above procedure can only be used of course when there exists a Hadamard matrix of order $n = 4t$. Unfortunately only $n = 4$ yields a Hadamard matrix H that has the following desired property. Any cyclic permutation of any row of H gives a row of H or the negative of a row of H . For example cyclicly permuting row two of H we get

$$\begin{array}{ccccl} + & + & - & - & = P_1 \\ - & + & + & - & = P_2 \\ - & - & + & + & = P_3 \\ + & - & - & + & = P_4 \end{array}$$

Note that P_1 is row two of H, P_2 is the negative of row four of H, P_3 is the negative of row two of H, and

P_4 is row four of H. The importance of this property can be seen if we look at the construction procedure used. If this property did not hold, the squares obtained from the multiplication tables will not meet the F-square definition. One can check to see that Hadamard matrices of order $n \neq 4$ do not have this property hence some construction procedure other than the one above needs to be developed.

References

[1] Bose, R. C. (1938). On the application of the properties of Galois Fields to the problem of construction of hyper-Graeco-Latin squares. Sankhya 3, 323-338.

[2] Euler, L. (1783). Recherches sur une nouvelle espêce de quarrés magiques. Mémoire de la Societé de Flessingue, Commentationes arithmetica collectae (elogé St. Petersburg 1783), 2 (1849), 302-361.

[3] Hedayat, A., Raghavarao, D., and Seiden, E. (1975). Further contributions to the theory of F-square designs. Ann. Statist. 3, 712-716.

[4] Hedayat, A. and Seiden, E. (1970). F-square and orthogonal F-square design: A generalization of latin squares and orthogonal latin squares design. Ann. Math. Statist. 41, 2035-2044.