AN EXTENSION OF MACNEISH'S THEOREM TO THE CONSTRUCTION OF SETS OF MUTUALLY ORTHOGONAL F-SQUARES OF COMPOSITE ORDER

BY JOHN P. MANDELI AND WALTER T. FEDERER
Virginia Commonwealth University and Cornell University

The purpose of this paper is to give a method of constructing sets of mutually orthogonal F-squares of composite order. The technique is an extension of MacNeish's theorem for latin squares. If $n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$ is the prime power decomposition of a composite number n then Macneish (1922) proved the existence of $\min(p_1^{e_1}, p_2^{e_2}, \ldots, p_m^{e_m}) - 1$ mutually orthogonal latin squares of order n. Our extension of MacNeish's result shows the existence of $\max(p_1^{e_1}, p_2^{e_2}, \ldots, p_m^{e_m}) - 1$ mutually orthogonal F-squares of order n. These sets of F-squares contain the sets of mutually orthogonal latin squares constructed by MacNeish and so the results may be looked upon as embedding the mutually orthogonal latin squares of MacNeish into a set of mutually orthogonal F-squares.

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1. Introduction and summary.

To conserve space, the reader is referred to Hedayat and Seiden (1970) for details and definitions concerning the theory and applications of F-squares and orthogonality of F-squares. We use the notation $F(n ; \lambda^t)$ to denote an F-square of order n with t symbols each replicated λ times in each row and column. Note that $n = \lambda t$. When $\lambda = 1$ and $t = n$ we have a latin square of order n. Hedayat, Raghavarao and Seiden (1975) have shown how to construct sets of mutually orthogonal F-squares when n is a prime or prime power number. Federer (1977) constructs sets of mutually orthogonal F-squares for $n = 4t$, t an integer, i.e. for n equal to a multiple of four. Kirton and Seberry (1978) show how to construct an F-square orthogonal to a latin square of order 10. No general theory however has been found for constructing sets of mutally orthogonal F-squares of order n when n is a composite number, except the case $n = 4t$. In this paper we show how to construct sets of mutually orthogonal F-squares of order equal to a composite number by using an extension of MacNeish's theorem. These sets of F-squares contain the sets of mutually orthogonal latin squares constructed by MacNeish and so the results may be looked upon as embedding the mutually orthogonal latin squares of MacNeish into a set of mutually orthogonal F-squares.

2. Extending MacNeish's theorem to F-square designs.

In 1922, MacNeish proved the following theorem:

THEOREM 2.1. *If we let the prime decomposition of a number* n *be* $n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$ *then there exists a set of* $\min(p_1^{e_1}, p_2^{e_2}, \ldots, p_m^{e_m}) - 1$ *mutually orthogonal latin squares of order* n.

We present the following extension to MacNeish's theorem for F-squares:

THEOREM 2.2. _If we let the prime decomposition of a number_ n _be_ $n = p_1^{e_1} p_2^{e_2} \dots p_m^{e_m}$ _and without loss of generality let_ $p_1^{e_1} < p_2^{e_2} < \dots < p_m^{e_m}$ _then there exists a set of_ $(p_m^{e_m}-1)$, _i.e._ $\max(p_1^{e_1}, p_2^{e_2}, \dots p_m^{e_m}) - 1$ _mutually orthogonal F-squares of order_ n. _This set contains_ $(p_{i+1}^{e_{i+1}} - p_i^{e_i})$ _mutually orthogonal F-squares of the form_ $F[n ; (p_1^{e_1} p_2^{e_2} \dots p_i^{e_i})^{p_{i+1}^{e_{i+1}} p_{i+2}^{e_{i+2}} \dots p_m^{e_m}}]$ _for_ $i = 0,1, \dots, (m-1)$, _where_ $e_0 = 0$ _and_ $p_0 = 1$.

Note that when $i = 0$ we have the result of MacNeish (1922) on mutually orthogonal latin squares. The set of $(p_1^{e_1} - 1)$ mutually orthogonal latin squares of order n therefore is a subset of the set of $(p_m^{e_m} - 1)$ mutually orthogonal F-squares of order n of theorem 2.2. Hence theorem 2.2 may be looked upon as showing how to embed the set of $(p_1^{e_1} - 1)$ mutually orthogonal latin squares of order n of MacNeish into a set of $(p_m^{e_m} - 1)$ mutually orthogonal F-squares of order n.

PROOF. Let $n_i = p_i^{e_i}$ for $i = 1,2,\dots,m$. Therefore $n = n_1 n_2 \dots n_m$ where $n_1 < n_2 < \dots < n_m$. Let $\{L_{n_i}^1, L_{n_i}^2, \dots, L_{n_i}^{n_i-1}\}$ be the complete set of $(n_i - 1)$ mutually orthogonal latin squares of order n_i for $i = 1,2,\dots,m$. Define 0_1 to be the 1×1 matrix consisting of all zeros. Then trivially 0_1 is an $F(1 ; 1)$ - square, an F-square with one symbol. Also trivially 0_1 is orthogonal to any F-square of order 1.

Let A be a $p \times p$ matrix and denote it by $A = (a_{ij})$, $i,j = 1, \ldots ,p$. Let B be a $q \times q$ matrix and denote it by $B = (b_{ij})$, $i, j = 1, \ldots , q$. Form the direct sum of A and B, denoted by $A \oplus B$, by forming the $pq \times pq$ matrix:

$$A \oplus B = \begin{matrix} (a_{11},b_{11}) & \cdots & (a_{11},b_{1q}) & \cdots & (a_{1p},b_{11}) & \cdots & (a_{1p},b_{1q}) \\ (a_{11},b_{21}) & \cdots & (a_{11},b_{2q}) & \cdots & (a_{1p},b_{21}) & \cdots & (a_{1p},b_{2q}) \\ \vdots & & \vdots & & \vdots & & \vdots \\ (a_{11},b_{q1}) & \cdots & (a_{11},b_{qq}) & \cdots & (a_{1p},b_{q1}) & \cdots & (a_{1p},b_{qq}) \\ \vdots & & \vdots & & \vdots & & \vdots \\ (a_{p1},b_{11}) & \cdots & (a_{p1},b_{1q}) & \cdots & (a_{pp},b_{11}) & \cdots & (a_{pp},b_{1q}) \\ (a_{p1},b_{21}) & \cdots & (a_{p1},b_{2q}) & \cdots & (a_{pp},b_{21}) & \cdots & (a_{pp},b_{2q}) \\ \vdots & & \vdots & & \vdots & & \vdots \\ (a_{p1},b_{q1}) & \cdots & (a_{p1},b_{qq}) & \cdots & (a_{pp},b_{q1}) & \cdots & (a_{pp},b_{qq}) \end{matrix}$$

Using the direct sum, construct the following matrices:

$L^j_{n_1} \oplus L^j_{n_2} \oplus \cdots \oplus L^j_{n_m}$ for $j = 1,2, \ldots, (n_1 - 1)$
[total of $(n_1 - 1)$ matrices.]

$0_{n_1} \oplus L^j_{n_2} \oplus \cdots \oplus L^j_{n_m}$ for $j = n_1 , \ldots, (n_2 -1)$
[total of $(n_2 - n_1)$ matrices.]

$0_{n_1 n_2} \oplus L^j_{n_3} \oplus \cdots \oplus L^j_{n_m}$ for $j = n_2 , \ldots, (n_3 - 1)$
[total of $(n_3 - n_2)$ matrices.]

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$$O_{n_1 n_2 \ldots n_i} \oplus L^j_{n_{i+1}} \oplus \ldots \oplus L^j_{n_m} \quad \text{for } j = n_i , \ldots , (n_{i+1} -1)$$

[total of $(n_{i+1} - n_i)$ matrices.]

$$\vdots$$

$$O_{n_1 n_2 \ldots n_{m-1}} \oplus L^j_{n_m} \quad \text{for } j = n_{m-1} , \ldots , (n_m -1)$$

[total of $(n_m - n_{m-1})$ matrices.]

By MacNeish (1922) the $(n_1 -1)$ matrices $L^j_{n_1} \oplus L^j_{n_2} \oplus \ldots \oplus L^j_{n_m}$ for $j = 1,2, \ldots , (n_1 -1)$ are precisely a set of $(n_1 -1)$ mutually orthogonal latin squares of order $n = n_1 n_2 \ldots n_m$. In F-square notation, a set of $(n_1 -1)$ mutually orthogonal $F(n ; 1^{n_1 n_2 \ldots n_m})$ - squares.

We wish now to prove that the $(n_{i+1} - n_i)$ matrices $O_{n_1 n_2 \ldots n_i} \oplus L^j_{n_{i+1}} \oplus \ldots \oplus L^j_{n_m}$ for $j = n_i, \ldots , (n_{i+1} -1)$ and $i = 1,2, \ldots , (m -1)$ are exactly a set of $(n_{i+1} - n_i)$ mutually orthogonal $F(n ; (n\, n_2 \ldots n_i)^{n_{i+1} n_{i+2} \ldots n_m})$ - squares. By MacNeish (1922),

$$L^j_{n_{i+1}} \oplus L^j_{n_{i+2}} \oplus \ldots \oplus L^j_{n_m} = L^j_{n_{i+1} n_{i+2} \ldots n_m} .$$

Therefore,

$$O_{n_1 n_2 \ldots n_i} \oplus L^j_{n_{i+1}} \oplus L^j_{n_{i+2}} \oplus \ldots \oplus L^j_{n_m} = O_{n_1 n_2 \ldots n_i} \oplus L^j_{n_{i+1} n_{i+2} \ldots n_m}$$

for $j = n_i , \ldots ,(n_{i+1} -1)$ and $i = 1,2, \ldots , (m - 1)$. Now $O_{n_1 n_2 \ldots n_i} \oplus L^j_{n_{i+1} n_{i+2} \ldots n_m}$ is a $n_1 n_2 \ldots n_m \times n_1 n_2 \ldots n_m$, i.e. a $n \times n$ matrix of elements $(0 , l_{hk})$ where $l_{hk} \in L^j_{n_{i+1} n_{i+2} \ldots n_m}$

for h,k = 1,2, ... , $(n_{i+1}\, n_{i+2} \cdots n_m)$ of the following form:

$$
\begin{matrix}
(0,l_{11}) & \cdots & (0,l_{1,n_{i+1}\cdots n_m}) & \cdots & (0,l_{11}) & \cdots & (0,l_{1,n_{i+1}\cdots n_m}) \\
(0,l_{21}) & \cdots & (0,l_{2,n_{i+1}\cdots n_m}) & \cdots & (0,l_{21}) & \cdots & (0,l_{2,n_{i+1}\cdots n_m}) \\
\vdots & & \vdots & & \vdots & & \vdots \\
(0,l_{n_{i+1}\cdots n_m,1}) & \cdots & (0,l_{n_{i+1}\cdots n_m,n_{i+1}\cdots n_m}) & \cdots & (0,l_{n_{i+1}\cdots n_m,1}) & \cdots & (0,l_{n_{i+1}\cdots n_m,n_{i+1}\cdots n_m}) \\
\vdots & & \vdots & & \vdots & & \vdots \\
(0,l_{11}) & \cdots & (0,l_{1,n_{i+1}\cdots n_m}) & \cdots & (0,l_{11}) & \cdots & (0,l_{1,n_{i+1}\cdots n_m}) \\
(0,l_{21}) & \cdots & (0,l_{2,n_{i+1}\cdots n_m}) & \cdots & (0,l_{21}) & \cdots & (0,l_{2,n_{i+1}\cdots n_m}) \\
\vdots & & \vdots & & \vdots & & \vdots \\
(0,l_{n_{i+1}\cdots n_m,1}) & \cdots & (0,l_{n_{i+1}\cdots n_m,n_{i+1}\cdots n_m}) & \cdots & (0,l_{n_{i+1}\cdots n_m,1}) & \cdots & (0,l_{n_{i+1}\cdots n_m,n_{i+1}\cdots n_m})
\end{matrix}
$$

Since $L^{j}_{n_{i+1}\, n_{i+2}\cdots n_m}$ is a latin square, the above matrix has only $n_{i+1}\, n_{i+2}\cdots n_m$ symbols. Each symbol can be seen to be replicated $n_1 n_2 \cdots n_i$ times in each row and each column. Hence $0_{n_1 n_2 \cdots n_i} \oplus L^{j}_{n_{i+1}\, n_{i+2}\cdots n_m}$ is an $F(n\,;\,(n_1 n_2 \cdots n_i)^{n_{i+1}\, n_{i+2}\cdots n_m})$ - square for $j = n_i, \ldots, (n_{i+1}-1)$. We need now show that these $(n_{i+1} - n_i)$ F-squares are mutually orthogonal. Let

$F_j = 0_{n_1 n_2 \cdots n_i} \oplus L^{j}_{n_{i+1}\, n_{i+2}\cdots n_m}$ and $F_{j'} = 0_{n_1 n_2 \cdots n_i} \oplus L^{j'}_{n_{i+1}\, n_{i+2}\cdots n_m}$,

for $j \neq j'$, be two F-squares from this set. By MacNeish (1922) $L^{j}_{n_{i+1}\, n_{i+2}\cdots n_m}$ and $L^{j'}_{n_{i+1}\, n_{i+2}\cdots n_m}$ are mutually orthogonal.

Let $\mathcal{L}^{j}_{n_{i+1}\, n_{i+2} \cdots\, n_m} = \{(0\ ,\ l_{hk})\}$ where $L^{j}_{n_{i+1}\, n_{i+2} \cdots n_m} = \{l_{hk}\}$

and let $\mathcal{L}^{j'}_{n_{i+1}\, n_{i+2} \cdots n_m} = \{(0, l'_{hk})\}$ where $L^{j}_{n_{i+1}\, n_{i+2} \cdots n_m} = \{l'_{hk}\}$.

Then F_j and $F_{j'}$ have the following forms:

$$F_j = \begin{pmatrix} \mathcal{L}^{j}_{n_{i+1} \cdots n_m} & \mathcal{L}^{j}_{n_{i+1} \cdots n_m} & \cdots & \mathcal{L}^{j}_{n_{i+1} \cdots n_m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}^{j}_{n_{i+1} \cdots n_m} & \mathcal{L}^{j}_{n_{i+1} \cdots n_m} & \cdots & \mathcal{L}^{j}_{n_{i+1} \cdots n_m} \end{pmatrix}$$

$$F_{j'} = \begin{pmatrix} \mathcal{L}^{j'}_{n_{i+1} \cdots n_m} & \mathcal{L}^{j'}_{n_{i+1} \cdots n_m} & \cdots & \mathcal{L}^{j'}_{n_{i+1} \cdots n_m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}^{j'}_{n_{i+1} \cdots n_m} & \mathcal{L}^{j'}_{n_{i+1} \cdots n_m} & \cdots & \mathcal{L}^{j'}_{n_{i+1} \cdots n_m} \end{pmatrix}$$

Since $L^{j}_{n_{i+1} \cdots n_m}$ and $L^{j'}_{n_{i+1} \cdots n_m}$ are mutually orthogonal so are $\mathcal{L}^{j}_{n_{i+1} \cdots n_m}$ and $\mathcal{L}^{j'}_{n_{i+1} \cdots n_m}$ because of their form. Hence F_j and $F_{j'}$ are mutually orthogonal by looking at their above forms. Thus the $(n_{i+1} - n_i)$ $F(n\ ;\ (n_1 n_2 \ldots n_i)^{n_{i+1}\, n_{i+2} \cdots n_m})$ - squares constructed are a mutually orthogonal set.

It remains to be shown that the $(n_{i+1} - n_i)$ F-squares constructed from $0_{n_1 n_2 \ldots n_i} \oplus L^{j}_{n_{i+1}} \oplus \cdots \oplus L^{j}_{n_m}$ for $j = n_i\ , \ldots ,\ (n_{i+1} - 1)$ are orthogonal to the $(n_{i'+1} - n_{i'})$ F-squares constructed from

$$0_{n_1 n_2 \ldots n_{i'}} \oplus L^{j'}_{n_{i'+1}} \oplus \ldots \oplus L^{j'}_{n_m} \quad \text{for} \quad j' = n_{i'}, \ldots, (n_{i'+1} - 1)$$

where $i > i'$. We have that

$$0_{n_1 n_2 \ldots n_i} \oplus L^{j}_{n_{i+1}} \oplus \ldots \oplus L^{j}_{n_m} = 0_{n_1 n_2 \ldots n_i} \oplus L^{j}_{n_{i+1} \ldots n_m}$$

and $0_{n_1 n_2 \ldots n_{i'}} \oplus L^{j'}_{n_{i'+1}} \oplus \ldots \oplus L^{j'}_{n_m}$

$$= 0_{n_1 n_2 \ldots n_{i'}} \oplus L^{j'}_{n_{i'+1}} \oplus \ldots \oplus L^{j'}_{n_i} \oplus L^{j'}_{n_{i+1}} \oplus \ldots \oplus L^{j'}_{n_m}$$

$$= 0_{n_1 n_2 \ldots n_{i'}} \oplus L^{j'}_{n_{i'+1} \ldots n_i} \oplus L^{j'}_{n_{i+1} \ldots n_m} .$$

By MacNeish (1922), $L^{j}_{n_{i+1} \ldots n_m}$ is orthogonal to $L^{j'}_{n_{i+1} \ldots n_m}$. Also, trivially $0_{n_1 n_2 \ldots n_i}$ is orthogonal to $0_{n_1 n_2 \ldots n_{i'}} \oplus L^{j'}_{n_{i'+1} \ldots n_i}$. These facts together with looking at the form of the matrix $0_{n_1 n_2 \ldots n_i} \oplus L^{j}_{n_{i+1} \ldots n_m}$ and the matrix $0_{n_1 n_2 \ldots n_{i'}} \oplus L^{j'}_{n_{i'+1} \ldots n_i} \oplus L^{j'}_{n_{i+1} \ldots n_m}$ show that these F-squares are orthogonal. By a similar argument, we can see that the $(n_1 - 1)$ latin squares formed from $L^{j}_{n_1} \oplus L^{j}_{n_2} \oplus \ldots \oplus L^{j}_{n_m}$ for $j = 1, 2, \ldots, (n_1 - 1)$ mutually orthogonal to the $(n_{i+1} - n_i)$ F-squares formed from

$0_{n_1 n_2 \ldots n_i} \oplus L^{j}_{n_{i+1}} \oplus \ldots \oplus L^{j}_{n_m}$ for $j = n_i, \ldots, (n_{i+1} - 1)$ and $i = 1, 2, \ldots, (m - 1)$.

Finally, we establish that the total number of mutually orthogonal F-squares is $(p_m^{e_m} - 1)$:

$$\sum_{i=0}^{m-1} \left(p_{i+1}^{e_{i+1}} - p_i^{e_i} \right) = \sum_{i=1}^{m} p_i^{e_i} - \sum_{i=0}^{m-1} p_i^{e_i}$$

$$= p_m^{e_m} - p_o^{e_o}$$

$$= p_m^{e_m} - 1 .$$

This completes the proof.

In closing, we present the following two examples of the use of theorem 2.2.

EXAMPLE. 2.1. For order $n = 5 \cdot 2 = 10$, there exists a set of $5 - 1 = 4$ mutually orthogonal F-squares. The set contains one $F(10 ; 1^{10})$ -square, that is a latin square of order 10, and three $F(10 ; 2^5)$ -squares.

EXAMPLE. 2.2. For order $n = 25 \cdot 3 = 75$, there exists a set of $25 - 1 = 24$ mutually orthogonal F-squares. The set contains two $F(75 ; 1^{75})$ - squares, that is two latin squares of order 75, and twenty two $F(75 ; 3^{25})$ - squares.

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