

THE MEAN AND DISPERSION OF QUADRATIC FORMS
WITH APPLICATIONS TO THE WISHART DISTRIBUTION

by

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ABSTRACT

The mean and dispersion (variance-covariance) matrix of $\tilde{x} \otimes \tilde{x}$, where \tilde{x} is a realized vector of random variables, is derived. Applications are made to quadratic forms using the relationship $\tilde{x}'A\tilde{x} = (\text{vec}A)'(\tilde{x} \otimes \tilde{x})$ and to the matrix of sums of squares and crossproducts $\tilde{S} = \tilde{X}\tilde{X}' = \sum_{i=1}^n \tilde{x}_i \tilde{x}_i'$ of a multivariate observation \tilde{X} , having i th column \tilde{x}_i , using $\text{vec}\tilde{S} = \sum_{i=1}^n (\tilde{x}_i \otimes \tilde{x}_i)$. An important special case is when \tilde{x} and the \tilde{x}_i 's are a linear transformation of independent scaled variables. Further under normality these two situations lead, respectively, to well-known expressions for the mean and variance of $\tilde{x}'A\tilde{x}$ and of $\text{vec}\tilde{S}$, where \tilde{S} is now the scale matrix of a noncentral Wishart distribution.

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1. INTRODUCTION

Sums of squares in analysis of variance and hypothesis testing can be expressed as quadratic forms. A great deal is known about estimates of the variance of these when the underlying distribution is normal. This chapter presents some results without normality assumptions. An application is made to the sample covariance matrix that plays a key role in multivariate statistical analysis.

2. NOTATION

It is convenient to begin by summarizing properties of and relationships among the vec and kindred operators used in the sequel.

The vector formed by stacking the columns of a matrix \underline{A} one under the other is denoted by $\text{vec} \underline{A}$. The vec of a product matrix \underline{ABC} ,

$$\text{vec}(\underline{ABC}) = (\underline{C}' \otimes \underline{A}) \text{vec} \underline{B} \quad , \quad (1)$$

involves the Kronecker (direct) product of matrices, defined as

$$\underline{A} \otimes \underline{B} = \{a_{ij}\} \quad . \quad (2)$$

The vec -permutation Matrix $\underline{I}_{m,n}$ is defined by Henderson and Searle [1979] as showing the relationship between $\text{vec} \underline{A}$ and $\text{vec} \underline{A}'$:

$$\text{vec} \underline{A}_{m \times n} = \underline{I}_{m,n} \text{vec} \underline{A}' \quad . \quad (3)$$

It is to be distinguished from \underline{I}_{mn} , the identity matrix of the same order. Properties of $\underline{I}_{m,n}$ include:

$$\underline{I}_{\underline{m}, \underline{n}} \underline{I}_{\underline{n}, \underline{m}} = \underline{I}_{\underline{mn}} \quad , \quad (4)$$

$$(\underline{I}_{\underline{m}, \underline{n}})^{-1} = (\underline{I}_{\underline{m}, \underline{n}})' = \underline{I}_{\underline{n}, \underline{m}} \quad (5)$$

and the reversing of the order of Kronecker products:

$$\underline{I}_{\underline{m}, \underline{p}} (\underline{A}_{\underline{m} \times \underline{n}} \otimes \underline{B}_{\underline{p} \times \underline{q}}) \underline{I}_{\underline{q}, \underline{n}} = \underline{B} \otimes \underline{A} \quad . \quad (6)$$

The i, j th submatrix of $\underline{I}_{\underline{m}, \underline{n}}$ is $\underline{e}_i \underline{e}_j'$ of order $m \times n$, where \underline{e}_i is the i th column of an identity matrix of appropriate order.

The trace of a square matrix is the sum of its diagonal elements. When \underline{A} and \underline{B} have the same order,

$$\text{tr}(\underline{A}' \underline{B}) = (\text{vec} \underline{A})' \text{vec} \underline{B} \quad . \quad (7)$$

Special cases for vectors $\underline{a}_{\underline{m} \times 1}$ and $\underline{b}_{\underline{n} \times 1}$ are

$$\text{vec} \underline{a} = \text{vec} \underline{a}' \quad , \quad (8)$$

$$\text{vec} \underline{a} \underline{b}' = \underline{b} \otimes \underline{a} \quad , \quad (9)$$

$$\underline{a} \otimes \underline{b}' = \underline{a} \underline{b}' = \underline{b}' \otimes \underline{a} \quad (10)$$

and

$$\underline{I}_{\underline{m}, \underline{n}} (\underline{a} \otimes \underline{b}) = \underline{b} \otimes \underline{a} \quad . \quad (11)$$

Finally, for a scalar s we have the useful and obvious identities

$$s = \text{vec} s = \text{vec} s' = (\text{vec} s)' = (\text{vec} s')' = \text{tr} s \quad . \quad (12)$$

3. THE MEAN AND DISPERSION OF $\underline{x} \otimes \underline{x}$

The expected value of $\underline{x} \otimes \underline{x}$ is easily and directly derived in terms of the first two central moments of \underline{x} denoted as $E(\underline{x}) = \underline{\mu}$

and $\text{var}(\underline{x}) = \underline{V}$, using (9) and $E(\underline{xx}') = \underline{V} + \underline{\mu}\underline{\mu}'$:

$$E(\underline{x} \otimes \underline{x}) = E\text{vec}(\underline{xx}') = \text{vec}E(\underline{xx}') = \text{vec}(\underline{V} + \underline{\mu}\underline{\mu}') \quad .$$

The dispersion matrix of $\underline{x} \otimes \underline{x}$ also involves the expected value of the matrix of fourth noncentral moments, which we denote by

$$\underline{K} = E[(\underline{x} \otimes \underline{x})(\underline{x} \otimes \underline{x})'] = E(\underline{xx}' \otimes \underline{xx}') \quad .$$

Then for arbitrary \underline{x}

$$\begin{aligned} \text{var}(\underline{x} \otimes \underline{x}) &= \underline{K} - E(\underline{x} \otimes \underline{x})[E(\underline{x} \otimes \underline{x})]' \\ &= \underline{K} - \text{vec}(\underline{V} + \underline{\mu}\underline{\mu}')[\text{vec}(\underline{V} + \underline{\mu}\underline{\mu}')]' \quad . \end{aligned} \quad (13)$$

We now consider the important special case where \underline{x} is a linear transformation of independent scaled variables, which reduces to well-known results under normality.

3.1. The standardized variable, \underline{z}

Let \underline{z} be a vector of n independent identically distributed (i.i.d.) standardized random variables z_i with

$$Ez_i = 0, \quad Ez_i^2 = 1, \quad Ez_i^3 = \alpha_i \quad \text{and} \quad Ez_i^4 = 3 + \gamma_i \quad . \quad (14)$$

When \underline{z} has the multivariate normal distribution, $\underline{z} \sim N(\underline{0}, \underline{I}_n)$, the skewness, α_i , and the kurtosis parameter, γ_i (to be distinguished from the kurtosis, $\gamma_i + 3$), are both zero.

3.2. The translated variable, \underline{w}

Further, let $\underline{w} = \underline{z} + \underline{m}$ for constant vector \underline{m} . From (14) the independence of the elements of \underline{w} and their first four central

moments follow, which we denote by

$$w_i \sim \text{ind}(m_i, 1, \alpha_i, \gamma_i + 3) \quad , \quad (15)$$

a straightforward extension to higher moments of the representation $w_i \sim (Ew_i, \text{var}w_i)$.

3.3. The transformed variable, \tilde{x}

Finally, consider for symmetric nonnegative definite

$$\tilde{V} = \tilde{V}^{\frac{1}{2}} \tilde{V}^{\frac{1}{2}} = \tilde{V}' ,$$

$$\tilde{x} = \tilde{V}^{\frac{1}{2}} \tilde{z} + \tilde{\mu} = \tilde{V}^{\frac{1}{2}} \tilde{w} \quad . \quad (16)$$

Here, in contrast to the z_i 's and w_i 's, the x_i 's are correlated, with $\text{var}\tilde{x} = \tilde{V}$. When \tilde{V} is the diagonal matrix of σ_i^2 's we have $x_i \sim [\mu_i, \sigma_i^2, \alpha_i \sigma_i^3, (\gamma_i + 3)\sigma_i^4]$.

3.4. An expression for $\text{var}(\tilde{z} \otimes \tilde{z})$

$\tilde{z} \otimes \tilde{z}$ is a vector of order $n^2 \times 1$ so that $\text{var}(\tilde{z} \otimes \tilde{z})$ is an $n^2 \times n^2$ matrix. Partition $\text{var}(\tilde{z} \otimes \tilde{z})$ into $n \times n$ submatrices, $\tilde{\Sigma}_{ij}$, of order $n \times n$; with $\text{cov}(z_i z_k, z_j z_l)$ being the k, l th element of $\tilde{\Sigma}_{ij}$. Then the elements of the i th diagonal block $\tilde{\Sigma}_{ii}$ are:

$$\text{var}z_i^2 = Ez_i^4 - (Ez_i^2)^2 = 3 + \gamma_i - 1 = 2 + \gamma_i \quad \text{for } i = j = k = l \quad ,$$

$$\text{var}z_i z_k = Ez_i^2 z_k^2 = Ez_i^2 Ez_k^2 = 1 \quad \text{for } i = j \neq k = l \quad ,$$

and

$$\text{cov}(z_i z_k, z_i z_l) = 0 \quad \text{for } i = j \neq k \neq l \quad .$$

The j, i th element of $\tilde{\Sigma}_{ij}$ for $i \neq j$ is $\text{cov}(z_i z_j, z_j z_i) = \text{var}z_i z_j = 1$, and all other elements are zero.

An example of these expressions with $n=3$ (and dots denoting zeros), is

$$\text{var}(\underline{z} \otimes \underline{z}) = \text{var} \begin{bmatrix} z_1^2 \\ z_1 z_2 \\ z_1 z_3 \\ z_2 z_1 \\ z_2^2 \\ z_2 z_3 \\ z_3 z_1 \\ z_3 z_2 \\ z_3^2 \end{bmatrix} = \begin{bmatrix} 2+\gamma_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2+\gamma_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2+\gamma_3 \end{bmatrix} .$$

In general the scalar elements of Σ_{ij} may be assembled in matrix form as

$$\Sigma_{ij} = \underline{e}_j \underline{e}_i' + \delta_{ij} (\underline{I}_n + \gamma_i \underline{e}_i \underline{e}_i') \quad , \quad (18)$$

where δ_{ij} is the Kronecker delta, so that

$$\text{var}(\underline{z} \otimes \underline{z}) = \underline{I}_{n,n} + \underline{I}_{n^2} + \bigoplus_{i=1}^n \gamma_i \underline{e}_i \underline{e}_i' \quad , \quad (19)$$

with $\bigoplus_{i=1}^n A_i$ denoting the direct sum $A_1 \oplus \dots \oplus A_n$. Also, using $\bigoplus_{i=1}^n \gamma_i \underline{e}_i \underline{e}_i' = \text{dg}\{\text{vec}[\text{dg}(\underline{\gamma})]\}$, where $\text{dg}(\underline{\gamma})$ denotes a diagonal matrix of the elements of $\underline{\gamma} = (\gamma_1, \dots, \gamma_n)'$, provides the alternative expression

$$\text{var}(\underline{z} \otimes \underline{z}) = \underline{I}_{n,n} + \underline{I}_{n^2} + \text{dg}\{\text{vec}[\text{dg}(\underline{\gamma})]\} \quad .$$

3.5. A generalization to $\text{var}(\underline{w} \otimes \underline{w})$

An expression for $\text{var}(\underline{w} \otimes \underline{w})$ is now derived for $\underline{w} = \underline{z} + \underline{m}$ of (15):

$$\begin{aligned}
 \text{var}(\underline{\underline{w}} \otimes \underline{\underline{w}}) &= \text{var}[(\underline{\underline{z}} + \underline{\underline{m}}) \otimes (\underline{\underline{z}} + \underline{\underline{m}})] \\
 &= \text{var}(\underline{\underline{z}} \otimes \underline{\underline{z}} + \underline{\underline{z}} \otimes \underline{\underline{m}} + \underline{\underline{m}} \otimes \underline{\underline{z}} + \underline{\underline{m}} \otimes \underline{\underline{m}}) \\
 &= \text{var}[\underline{\underline{z}} \otimes \underline{\underline{z}} + (\underline{\underline{I}}_{n^2} + \underline{\underline{I}}_{n,n})(\underline{\underline{z}} \otimes \underline{\underline{m}})] \quad , \quad (20)
 \end{aligned}$$

on using (11). Observe that $\underline{\underline{z}} \otimes \underline{\underline{m}} = (\underline{\underline{I}}_n \otimes \underline{\underline{m}})(\underline{\underline{z}} \otimes \underline{\underline{1}}) = (\underline{\underline{I}}_n \otimes \underline{\underline{m}})\underline{\underline{z}}$, so that (20), on expanding the variance of a sum, becomes

$$\text{var}(\underline{\underline{w}} \otimes \underline{\underline{w}}) = \text{var}(\underline{\underline{z}} \otimes \underline{\underline{z}}) + \text{var}[(\underline{\underline{I}}_{n^2} + \underline{\underline{I}}_{n,n})(\underline{\underline{I}}_n \otimes \underline{\underline{m}})\underline{\underline{z}}] + \underline{\underline{C}} + \underline{\underline{C}}' \quad , \quad (21)$$

where $\underline{\underline{C}}$ is the $n^2 \times n^2$ covariance matrix

$$\underline{\underline{C}} = \text{cov}[(\underline{\underline{I}}_{n^2} + \underline{\underline{I}}_{n,n})(\underline{\underline{I}}_n \otimes \underline{\underline{m}})\underline{\underline{z}} \quad , \quad \underline{\underline{z}} \otimes \underline{\underline{z}}] \quad .$$

Using the general result $\text{cov}(\underline{\underline{Bx}}, \underline{\underline{y}}) = \underline{\underline{B}}\text{cov}(\underline{\underline{x}}, \underline{\underline{y}})$ and (11) $\underline{\underline{C}}$ becomes

$$\underline{\underline{C}} = (\underline{\underline{I}}_n \otimes \underline{\underline{m}} + \underline{\underline{m}} \otimes \underline{\underline{I}}_n) \text{cov}(\underline{\underline{z}} \quad , \quad \underline{\underline{z}} \otimes \underline{\underline{z}}) \quad . \quad (22)$$

The term $\text{cov}(\underline{\underline{z}} \quad , \quad \underline{\underline{z}} \otimes \underline{\underline{z}})$ in (22) is equal to $E[\underline{\underline{z}}(\underline{\underline{z}} \otimes \underline{\underline{z}})']$ because $E\underline{\underline{z}} = \underline{\underline{0}}$ which, together with the independence of the z_i 's and $Ez_i^2 = \alpha_i$ of (14), yields

$$\begin{aligned}
 \text{cov}(\underline{\underline{z}} \quad , \quad \underline{\underline{z}} \otimes \underline{\underline{z}}) &= \{\alpha_i \text{vece}_{\underline{\underline{z}}_i} e_i'\}_{i=1, \dots, n} \\
 &= \{\alpha_i e_i' \otimes e_i'\}_{i=1, \dots, n} \quad , \quad (23)
 \end{aligned}$$

which is an $n \times n^2$ matrix. For example, with $n=3$,

$$\text{cov}(\underline{\underline{z}} \quad , \quad \underline{\underline{z}} \otimes \underline{\underline{z}}) = \left[\begin{array}{ccc|ccc} \alpha_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \alpha_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_3 \end{array} \right] \quad .$$

Collecting terms in (21), using $\text{var}(\underline{\underline{z}} \otimes \underline{\underline{z}})$ of (19) and $\text{var}(\underline{\underline{Az}}) = \underline{\underline{A}}\text{var}\underline{\underline{z}}\underline{\underline{A}}' = \underline{\underline{A}}\underline{\underline{I}}_n\underline{\underline{A}}' = \underline{\underline{A}}\underline{\underline{A}}'$, gives

$$\begin{aligned} \text{var}(\underline{w} \otimes \underline{w}) &= \underline{I}_{n^2} + \underline{I}_{n,n} + \bigoplus_{i=1}^n \gamma_i \underline{e}_i \underline{e}_i' \\ &\quad + (\underline{I}_{n^2} + \underline{I}_{n,n}) (\underline{I}_n \otimes \underline{m}) (\underline{I}_n \otimes \underline{m})' (\underline{I}_{n^2} + \underline{I}_{n,n})' \\ &\quad + \underline{C} + \underline{C}' \quad . \end{aligned}$$

Finally, using (6) and $(\underline{I}_{n^2} + \underline{I}_{n,n}) \underline{I}_{n,n} = \underline{I}_{n^2} + \underline{I}_{n,n}$, which follows from (4),

$$\begin{aligned} \text{var}(\underline{w} \otimes \underline{w}) &= (\underline{I}_{n^2} + \underline{I}_{n,n}) (\underline{I}_{n^2} + \underline{I}_n \otimes \underline{m} \underline{m}' + \underline{m} \underline{m}' \otimes \underline{I}_n) \\ &\quad + \bigoplus_{i=1}^n \gamma_i \underline{e}_i \underline{e}_i' + \underline{C} + \underline{C}' \quad , \end{aligned} \quad (24)$$

where from substituting (23) into (22)

$$\underline{C} = (\underline{I}_n \otimes \underline{m} + \underline{m} \otimes \underline{I}_n) \{ \alpha_i \underline{e}_i' \otimes \underline{e}_i' \}_{i=1, \dots, n} \quad . \quad (25)$$

The matrix $\underline{I}_n \otimes \underline{m} + \underline{m} \otimes \underline{I}_n$ in (25) is a special case of the Kronecker sum of \underline{A} and \underline{B} defined as $\underline{A}_{\underline{m} \times \underline{m}} \otimes \underline{I}_n + \underline{I}_m \otimes \underline{B}_{n \times n}$ (Lancaster [1969, p. 260]) which we denote by $\underline{A} \underline{\text{K}} \underline{B}$. The consequence of the multiplication in (25) is that the i th column in the $n^2 \times n$ Kronecker sum $\underline{m} \underline{\text{K}} \underline{m}$, multiplied by α_i , becomes column $(i-1)n + i$ in $\underline{C}_{n^2 \times n}$, with all other columns zero.

3.6. An expression for $\text{var}(\underline{x} \otimes \underline{x})$

It is now a straightforward task to derive $\text{var}(\underline{x} \otimes \underline{x})$ for $\underline{x} = \underline{V}^{\frac{1}{2}} \underline{z} + \underline{\mu} = \underline{V}^{\frac{1}{2}} \underline{w}$ of (16),

$$\begin{aligned} \text{var}(\underline{x} \otimes \underline{x}) &= \text{var}(\underline{V}^{\frac{1}{2}} \underline{w} \otimes \underline{V}^{\frac{1}{2}} \underline{w}) = \text{var}[(\underline{V}^{\frac{1}{2}} \otimes \underline{V}^{\frac{1}{2}}) (\underline{w} \otimes \underline{w})] \\ &= (\underline{V}^{\frac{1}{2}} \otimes \underline{V}^{\frac{1}{2}}) \text{var}(\underline{w} \otimes \underline{w}) (\underline{V}^{\frac{1}{2}} \otimes \underline{V}^{\frac{1}{2}})' \quad . \end{aligned}$$

On substituting (24), this becomes

$$\begin{aligned} \text{var}(\underline{x} \otimes \underline{x}) &= (\underline{V}^{\frac{1}{2}} \otimes \underline{V}^{\frac{1}{2}})(\underline{I}_{n^2} + \underline{I}_{n,n})(\underline{I}_{n^2} + \underline{I}_n \otimes \underline{m}m' + \underline{m}m' \otimes \underline{I}_n)(\underline{V}^{\frac{1}{2}} \otimes \underline{V}^{\frac{1}{2}})' \\ &+ (\underline{V}^{\frac{1}{2}} \otimes \underline{V}^{\frac{1}{2}}) \left(\bigoplus_{i=1}^n \gamma_i e_i e_i' (\underline{V}^{\frac{1}{2}} \otimes \underline{V}^{\frac{1}{2}})' \right) + (\underline{V}^{\frac{1}{2}} \otimes \underline{V}^{\frac{1}{2}})(\underline{C} + \underline{C}')(\underline{V}^{\frac{1}{2}} \otimes \underline{V}^{\frac{1}{2}})' . \end{aligned}$$

By (4) and (6) the first two products in the first term commute, and on substituting $\underline{V}^{\frac{1}{2}} \underline{m} = \underline{\mu}$ the form of $\text{var}(\underline{x} \otimes \underline{x})$ becomes

$$\text{var}(\underline{x} \otimes \underline{x}) = \underline{S}(\underline{\mu}, \underline{V}) + \underline{T}(\underline{\mu}, \underline{V}, \underline{\alpha}) + \underline{T}'(\underline{\mu}, \underline{V}, \underline{\alpha}) + \underline{F}(\underline{V}, \underline{\gamma}) \quad , \quad (26)$$

where

$$\begin{aligned} \underline{S}(\underline{\mu}, \underline{V}) &= (\underline{I}_{n^2} + \underline{I}_{n,n})(\underline{V} \otimes \underline{V} + \underline{V} \otimes \underline{\mu}\underline{\mu}' + \underline{\mu}\underline{\mu}' \otimes \underline{V}) \quad , \\ \underline{T}(\underline{\mu}, \underline{V}, \underline{\alpha}) &= (\underline{V}^{\frac{1}{2}} \otimes \underline{\mu} + \underline{\mu} \otimes \underline{V}^{\frac{1}{2}}) \{ \alpha_i \underline{V}^{\frac{1}{2}} e_i \otimes \underline{V}^{\frac{1}{2}} e_i \}'_{i=1, \dots, n} \end{aligned} \quad (27)$$

and

$$\underline{F}(\underline{V}, \underline{\gamma}) = (\underline{V}^{\frac{1}{2}} \otimes \underline{V}^{\frac{1}{2}}) \left(\bigoplus_{i=1}^n \gamma_i e_i e_i' (\underline{V}^{\frac{1}{2}} \otimes \underline{V}^{\frac{1}{2}})' \right)$$

involve up to the second, third and fourth moments, respectively.

3.7. Special cases

The general expressions (26) and (27) for $\text{var}(\underline{x} \otimes \underline{x})$ are now applied to obtain special cases available in the literature. Pukelsheim [1977, p. 327], Anderson et al. [1977] and Anderson [1978, p. 70] consider $\underline{\mu} = \underline{0}$, in which case the third moment terms in (26) are zero, i.e., $\underline{T}(\underline{0}, \underline{V}, \underline{\alpha}) = \underline{0}$, irrespective of the skewness α_i in the underlying distribution. Magnus [1978] and Magnus and Neudecker [1979], generalize Neudecker's [1968] expressions, with $\underline{x} \sim N(\underline{\mu}, \underline{I}\sigma^2)$ to $\underline{x} \sim N(\underline{\mu}, \underline{V})$, for which $\alpha_i = \gamma_i = 0$ and so give only the first term in (26), $\underline{S}(\underline{\mu}, \underline{V})$.

4. APPLICATIONS TO QUADRATIC FORMS

Quadratic forms are of interest in statistics as they constitute succinct representations of sums of squares and products. A quadratic form in \underline{x} is $\underline{x}'\underline{A}\underline{x} = \sum_{ij} a_{ij} x_i x_j$ where, without loss of generality, \underline{A} is symmetric. The following identities, which follow from (12) with $s = \underline{x}'\underline{A}\underline{x}$, are particularly useful in deriving moments of $\underline{x}'\underline{A}\underline{x}$: Firstly,

$$\underline{x}'\underline{A}\underline{x} = \text{vec}(\underline{x}'\underline{A}\underline{x}) = (\underline{x}' \otimes \underline{x}') \text{vec} \underline{A} = (\text{vec} \underline{A})' (\underline{x} \otimes \underline{x}) \quad (28)$$

transforms a quadratic form in \underline{x} into a linear form in $\underline{x} \otimes \underline{x}$ and secondly, because $\text{tr} \underline{A}\underline{B} = \text{tr} \underline{B}\underline{A}$,

$$\underline{x}'\underline{A}\underline{x} = \text{tr}(\underline{x}'\underline{A}\underline{x}) = \text{tr}(\underline{A}\underline{x}\underline{x}') \quad , \quad (29)$$

expresses the quadratic form $\underline{x}'\underline{A}\underline{x}$ in terms of the matrix of products $\underline{x}\underline{x}'$.

For example, the expectation of $\underline{x}'\underline{A}\underline{x}$ when $\underline{x} \sim (\underline{\mu}, \underline{V})$, regardless of distribution and higher moments, is derived as by Searle [1971, p. 55] from (29) as

$$\begin{aligned} E(\underline{x}'\underline{A}\underline{x}) &= E \text{tr}(\underline{A}\underline{x}\underline{x}') = \text{tr} \underline{A} E(\underline{x}\underline{x}') = \text{tr} \underline{A} (\underline{V} + \underline{\mu}\underline{\mu}') \\ &= \text{tr}(\underline{A}\underline{V}) + \underline{\mu}' \underline{A} \underline{\mu} \quad . \end{aligned}$$

The variance of $\underline{x}'\underline{A}\underline{x}$ is more difficult. We deal firstly with \underline{x} normally distributed, $\underline{x} \sim N(\underline{\mu}, \underline{V})$.

4.1. Quadratic forms of normal variables

The central moments of quadratic forms of normal variables are well-known and are discussed, for example, by Searle [1971, pp. 54-57]. He derives the rth cumulant of $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$, for $\underset{\sim}{x} \sim N(\underset{\sim}{\mu}, \underset{\sim}{V})$, as

$$K_r(\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}) = 2^{r-1}(r-1)! [\text{tr}(\underset{\sim}{A}\underset{\sim}{V})^r + r\underset{\sim}{\mu}'\underset{\sim}{A}(\underset{\sim}{V}\underset{\sim}{A})^{r-1}\underset{\sim}{\mu}] \quad (30)$$

When $r=2$ this yields

$$\text{var}(\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}) = 2\text{tr}(\underset{\sim}{A}\underset{\sim}{V})^2 + 4\underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{V}\underset{\sim}{A}\underset{\sim}{\mu} \quad (31)$$

Magnus [1978, pp. 203-204] derives (30) anew for the special case of $\underset{\sim}{\mu} = 0$, i.e., for $\underset{\sim}{w} \sim N(0, \underset{\sim}{V})$, to develop results for the noncentral moments $E(\underset{\sim}{w}'\underset{\sim}{A}\underset{\sim}{w})^r$ and $E \prod_{i=1}^r \underset{\sim}{w}'\underset{\sim}{A}_i \underset{\sim}{w}$.

Searle [1971, p. 60] also has an expression for the covariance of two quadratic forms:

$$\text{cov}(\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}, \underset{\sim}{x}'\underset{\sim}{B}\underset{\sim}{x}) = 2\text{tr}\underset{\sim}{A}\underset{\sim}{V}\underset{\sim}{B}\underset{\sim}{V} + 4\underset{\sim}{\mu}'\underset{\sim}{A}\underset{\sim}{V}\underset{\sim}{B}\underset{\sim}{\mu} \quad (32)$$

He derives this directly from (31) on using

$$2\text{cov}(\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}, \underset{\sim}{x}'\underset{\sim}{B}\underset{\sim}{x}) = \text{var}(\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{x}'\underset{\sim}{B}\underset{\sim}{x}) - \text{var}\underset{\sim}{x}'(\underset{\sim}{A} + \underset{\sim}{B})\underset{\sim}{x} \quad (33)$$

Also, on p. 66, he has a more general result for the covariance of two bilinear forms, of which (32) is then a special case.

4.2. The variance of $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$

To derive $\text{var}\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ for a general $\underset{\sim}{x}$, we adopt a different approach from that used for normal $\underset{\sim}{x}$ in (31). From (28),

$$\text{var}(\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}) = (\text{vec}\underset{\sim}{A})'\text{var}(\underset{\sim}{x} \otimes \underset{\sim}{x})(\text{vec}\underset{\sim}{A}) \quad (34)$$

itself a quadratic form in $\text{vec} \underline{\underline{A}}$, involving the matrix $\text{var}(\underline{\underline{x}} \otimes \underline{\underline{x}})$ of (13).

The special case for $\underline{\underline{x}}$ of (16), being a linear transformation of independent scaled variables, has $\text{var}(\underline{\underline{x}} \otimes \underline{\underline{x}})$ given in (26). Substituting (26) into (34) gives four terms which are now evaluated separately. The first term involves $\underline{\underline{S}}(\underline{\underline{\mu}}, \underline{\underline{V}})$, and is the only non-zero term under normality when α_i and γ_i are zero, and so (34) reduces as it should do to (31).

$$\begin{aligned} & (\text{vec} \underline{\underline{A}})' \underline{\underline{S}}(\underline{\underline{\mu}}, \underline{\underline{V}}) \text{vec} \underline{\underline{A}} \\ &= (\text{vec} \underline{\underline{A}})' (\underline{\underline{I}}_{\underline{\underline{n}}^2} + \underline{\underline{I}}_{\underline{\underline{n}}, \underline{\underline{n}}}) (\underline{\underline{V}} \otimes \underline{\underline{V}} + \underline{\underline{V}} \otimes \underline{\underline{\mu}} \underline{\underline{\mu}}' + \underline{\underline{\mu}} \underline{\underline{\mu}}' \otimes \underline{\underline{V}}) \text{vec} \underline{\underline{A}} \\ &= 2(\text{vec} \underline{\underline{A}})' \text{vec} (\underline{\underline{V}} \underline{\underline{A}} \underline{\underline{V}} + \underline{\underline{\mu}} \underline{\underline{\mu}}' \underline{\underline{A}} \underline{\underline{V}} + \underline{\underline{V}} \underline{\underline{A}} \underline{\underline{\mu}} \underline{\underline{\mu}}') \quad , \end{aligned}$$

using (1), (3) and the symmetry of $\underline{\underline{A}}$ and $\underline{\underline{V}}$. Finally, applying (8) and $\text{tr} \underline{\underline{A}} \underline{\underline{B}} = \text{tr} \underline{\underline{B}} \underline{\underline{A}}$ yields

$$(\text{vec} \underline{\underline{A}})' \underline{\underline{S}}(\underline{\underline{\mu}}, \underline{\underline{V}}) \text{vec} \underline{\underline{A}} = 2 \text{tr} (\underline{\underline{A}} \underline{\underline{V}})^2 + 4 \underline{\underline{\mu}}' \underline{\underline{A}} \underline{\underline{V}} \underline{\underline{\mu}} \quad , \quad (35)$$

as expected.

The terms involving $\underline{\underline{T}}(\underline{\underline{\mu}}, \underline{\underline{V}}, \underline{\underline{\alpha}})$ are

$$\begin{aligned} & (\text{vec} \underline{\underline{A}})' [\underline{\underline{T}}(\underline{\underline{\mu}}, \underline{\underline{V}}, \underline{\underline{\alpha}}) + \underline{\underline{T}}'(\underline{\underline{\mu}}, \underline{\underline{V}}, \underline{\underline{\alpha}})] \text{vec} \underline{\underline{A}} \\ &= 2(\text{vec} \underline{\underline{A}})' \underline{\underline{T}}'(\underline{\underline{\mu}}, \underline{\underline{V}}, \underline{\underline{\alpha}}) \text{vec} \underline{\underline{A}} \\ &= 2(\text{vec} \underline{\underline{A}})' \{ \alpha_j \underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{e}}_j \otimes \underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{e}}_j \}_{j=1, \dots, n} (\underline{\underline{V}}^{\frac{1}{2}} \otimes \underline{\underline{\mu}}' + \underline{\underline{\mu}}' \otimes \underline{\underline{V}}^{\frac{1}{2}}) \text{vec} \underline{\underline{A}} \\ &= 2 \{ \alpha_j \text{vec} \underline{\underline{e}}_j \underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{e}}_j \}_{j=1, \dots, n} [\text{vec} (\underline{\underline{\mu}}' \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}}) + \text{vec} \underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{\mu}}] \quad , \end{aligned}$$

on repeated application of (1). Then by (8) and (12) the vecs may be dropped to give

$$\begin{aligned}
 2(\text{vec} \underline{\underline{A}})' \underline{\underline{T}}'(\underline{\underline{\mu}}, \underline{\underline{V}}, \underline{\underline{\alpha}}) \text{vec} \underline{\underline{A}} &= 4\{\alpha_j e_j' \underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}} e_j\}_{j=1, \dots, n} \underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{\mu}} \\
 &= 4[\text{vecd}(\underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}})]' \text{dg}(\underline{\underline{\alpha}}) \underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{\mu}} \quad , \quad (36)
 \end{aligned}$$

where $\text{vecd} \underline{\underline{A}}$ is the column vector of diagonal elements of $\underline{\underline{A}}$, as defined by Henderson and Searle [1979].

Finally, the term involving $\underline{\underline{F}}(\underline{\underline{V}}, \underline{\underline{\gamma}})$ is

$$\begin{aligned}
 (\text{vec} \underline{\underline{A}})' \underline{\underline{F}}(\underline{\underline{V}}, \underline{\underline{\gamma}}) \text{vec} \underline{\underline{A}} &= (\text{vec} \underline{\underline{A}})' (\underline{\underline{V}}^{\frac{1}{2}} \otimes \underline{\underline{V}}^{\frac{1}{2}}) \left(\bigoplus_{i=1}^n \gamma_i e_i e_i' \right) (\underline{\underline{V}}^{\frac{1}{2}} \otimes \underline{\underline{V}}^{\frac{1}{2}})' \text{vec} \underline{\underline{A}} \\
 &= [\text{vec}(\underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}})]' \left(\bigoplus_{i=1}^n \gamma_i e_i e_i' \right) \text{vec}(\underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}}) \\
 &= [\text{vecd}(\underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}})]' \text{dg}(\underline{\underline{\gamma}}) \text{vecd}(\underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}}) \quad , \quad (37)
 \end{aligned}$$

because $\text{vecd} \underline{\underline{A}} = \underline{\underline{D}} \text{vec} \underline{\underline{A}}$ with $\underline{\underline{D}} = \left(\bigoplus_{i=1}^n e_i e_i' \right)$ and $\underline{\underline{D}}' \text{dg}(\underline{\underline{\gamma}}) \underline{\underline{D}} = \left(\bigoplus_{i=1}^n \gamma_i e_i e_i' \right)$.

This term, for $\gamma_i = \gamma$ for all i , is analogous to $\gamma \text{tr}(\text{diag} \underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}})^2$ of Rao [1971, p. 447] and to $\gamma(\text{sum of squares of diagonal elements of } \underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}})$ of Anderson [1978, p. 72].

Collecting the terms in (35), (36) and (37) gives the variance of a quadratic form in $\underline{\underline{x}}$ of (16):

$$\begin{aligned}
 \text{var}(\underline{\underline{x}}' \underline{\underline{A}} \underline{\underline{x}}) &= 2\text{tr}(\underline{\underline{A}} \underline{\underline{V}})^2 + 4\underline{\underline{\mu}}' \underline{\underline{A}} \underline{\underline{V}} \underline{\underline{A}} \underline{\underline{\mu}} \\
 &\quad + 4[\text{vecd}(\underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}})]' \text{dg}(\underline{\underline{\alpha}}) \underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{\mu}} \\
 &\quad + [\text{vecd}(\underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}})]' \text{dg}(\underline{\underline{\gamma}}) \text{vecd}(\underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}}) \quad . \quad (38)
 \end{aligned}$$

4.3. Special cases

A number of special cases of (38) are worthy of note.

(i) When the underlying standardized distribution of the z_i 's has common skewness and kurtosis, i.e., $\alpha_i = \alpha$ and $\gamma_i = \gamma$, (38) becomes

$$\begin{aligned} \text{var}(\underline{x}'\underline{Ax}) &= 2\text{tr}(\underline{AV})^2 + 4\underline{\mu}'\underline{AV}\underline{A}\underline{\mu} \\ &+ 4\alpha[\text{vecd}(\underline{V}^{\frac{1}{2}}'\underline{AV}^{\frac{1}{2}})]'\underline{V}^{\frac{1}{2}}\underline{A}\underline{\mu} \\ &+ \gamma[\text{vecd}(\underline{V}^{\frac{1}{2}}'\underline{AV}^{\frac{1}{2}})]'\text{vecd}(\underline{V}^{\frac{1}{2}}'\underline{AV}^{\frac{1}{2}}) \quad , \quad (39) \end{aligned}$$

(ii) We dealt, at some length in Section 4.1, with $\text{var}(\underline{x}'\underline{Ax})$ when $\underline{x} \sim N(\underline{\mu}, \underline{V})$, where $\alpha_i = \gamma_i = 0$ as a consequence of normality. Here (38) reduces to the simple expression as discussed for $\text{var}(\underline{x}'\underline{Ax})$ given in (31).

(iii) Atiqullah [1962, p. 84] states without proof a result for $\text{var}(\underline{x}'\underline{Ax})$ when $x_i \sim \text{ind}(\mu_i, \sigma^2, \alpha\sigma^3, \gamma\sigma^4)$, available from (9) on substituting $\underline{V} = \sigma^2\underline{I}$,

$$\text{var}(\underline{x}'\underline{Ax}) = 2\sigma^4 \text{tr}\underline{A}^2 + 4\sigma^2\underline{\mu}'\underline{A}\underline{\mu} + 4\alpha\sigma^3 (\text{vecd}\underline{A})'\underline{A}\underline{\mu} + \gamma\sigma^4 (\text{vecd}\underline{A})'\text{vecd}\underline{A} \quad . \quad (40)$$

Seber [1977, pp. 14-16] gives a proof for this and writes \underline{a} for our $\text{vecd}\underline{A}$.

(iv) Translation invariant estimators of variance components are invariant to the fixed effects in the model. Such an estimator is $\underline{x}'\underline{Ax}$ with $\underline{AX} = \underline{0}$, where \underline{X} is the model matrix for the fixed effects, $\underline{\beta}$. This invariance condition implies $\underline{A}\underline{\mu} = \underline{0}$, where $\underline{\mu} = \underline{X}\underline{\beta}$, and so without loss of generality (to have $E(\underline{x}'\underline{Ax}) = \text{tr}(\underline{AV}) + \underline{\mu}'\underline{A}\underline{\mu}$ free of $\underline{\mu}$) we take $\underline{\mu} = \underline{0}$. In this context and when $\underline{V} = \sigma^2\underline{I}$, Hsu [1938], Atiqullah [1962], Drygas [1972] and Drygas and Hupet

[1977, p. 333] give (40) with $\underline{\underline{\mu}} = \underline{\underline{0}}$. Rao [1971, p. 447], Pukelsheim [1977, p. 327] and Anderson [1978, p. 72] present analogous expressions for $\text{var}(\underline{\underline{x}}' \underline{\underline{A}} \underline{\underline{x}})$ for general $\underline{\underline{V}}$, which we obtain from (39) with $\underline{\underline{\mu}} = \underline{\underline{0}}$:

$$\text{var}(\underline{\underline{x}}' \underline{\underline{A}} \underline{\underline{x}}) = 2 \text{tr}(\underline{\underline{A}} \underline{\underline{V}})^2 + \gamma \text{vecd}(\underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}})' \text{vecd}(\underline{\underline{V}}^{\frac{1}{2}} \underline{\underline{A}} \underline{\underline{V}}^{\frac{1}{2}}) \quad .$$

5. APPLICATIONS TO THE WISHART DISTRIBUTION

Suppose $\underline{\underline{X}}_{p \times n}$ is a matrix of observations in a multivariate situation with columns $\underline{\underline{x}}_j \sim \text{ind}(\underline{\underline{\mu}}_j, \underline{\underline{V}})$ having the skewness vector $\underline{\underline{\alpha}} = (\alpha_1 \cdots \alpha_p)'$ and the vector of kurtosis parameters $\underline{\underline{\gamma}} = (\gamma_1 \cdots \gamma_p)'$, as in (14)-(16). In the special case when $\underline{\underline{x}}_j \sim N(\underline{\underline{\mu}}_j, \underline{\underline{V}})$, with its consequent $\underline{\underline{\alpha}}_j = \underline{\underline{\gamma}}_j = \underline{\underline{0}}$, $\underline{\underline{S}} = \underline{\underline{X}} \underline{\underline{X}}'$ follows the p -dimensional noncentral Wishart distribution. The mean and dispersion matrix of this are discussed by Magnus and Neudecker [1979] and for the central case, i.e., with $\underline{\underline{\mu}}_j = \underline{\underline{0}}$, by Henderson and Searle [1979].

We now remove the restriction of normality and consider the mean and variance (of elements) of $\underline{\underline{S}}$ for $\underline{\underline{x}}_j$ of arbitrary distribution.

The mean of $\underline{\underline{S}} = \underline{\underline{X}} \underline{\underline{X}}'$ is easily derived using $\underline{\underline{S}} = \sum_{j=1}^n \underline{\underline{x}}_j \underline{\underline{x}}_j'$:

$$E(\underline{\underline{S}}) = \sum_{j=1}^n E \underline{\underline{x}}_j \underline{\underline{x}}_j' = \sum_{j=1}^n (\underline{\underline{V}} + \underline{\underline{\mu}}_j \underline{\underline{\mu}}_j') = n \underline{\underline{V}} + \underline{\underline{M}} \underline{\underline{M}}' \quad , \quad (41)$$

on defining $\underline{\underline{M}} = [\underline{\underline{\mu}}_1 \cdots \underline{\underline{\mu}}_n]$ as the matrix of means $E(\underline{\underline{X}})$ (rather than $\underline{\underline{M}}'$ as used by Magnus and Neudecker [1979]).

The vec operator conveniently arranges elements of $\underline{\underline{S}}$ so as to

provide a concise expression for their variances and covariances; namely $\text{var}(\text{vec}\underline{\underline{S}})$, which we then refer to as the variance of the matrix $\underline{\underline{S}}$. In particular for $\underline{\underline{S}} = \underline{\underline{X}}\underline{\underline{X}}'$, $\text{var}(\text{vec}\underline{\underline{S}})$ is obtainable from $\text{var}(\underline{\underline{x}} \otimes \underline{\underline{x}})$ of (13), through using (9) and the independence of the $\underline{\underline{x}}_j$'s in $\underline{\underline{X}}$ as follows:

$$\begin{aligned} \text{var}(\text{vec}\underline{\underline{S}}) &= \text{var} \sum_{j=1}^n \text{vec}(\underline{\underline{x}}_j \underline{\underline{x}}_j') = \sum_{j=1}^n \text{var}(\underline{\underline{x}}_j \otimes \underline{\underline{x}}_j) \\ &= \sum_{j=1}^n \{ \underline{\underline{K}}_j - \text{vec}(\underline{\underline{V}} + \underline{\underline{\mu}}_j \underline{\underline{\mu}}_j') [\text{vec}(\underline{\underline{V}} + \underline{\underline{\mu}}_j \underline{\underline{\mu}}_j')] \}' \quad . \quad (42) \end{aligned}$$

5.1. Special cases

When each $\underline{\underline{x}}_j$ in $\underline{\underline{X}}$, akin to $\underline{\underline{x}}$ of (16), is a linear transformation of independent scaled variables (42), on substituting $\text{var}(\underline{\underline{x}}_j \otimes \underline{\underline{x}}_j)$ of (26), becomes

$$\begin{aligned} \text{var}(\text{vec}\underline{\underline{S}}) &= \sum_{j=1}^n [\underline{\underline{S}}(\underline{\underline{\mu}}_j, \underline{\underline{V}}) + \underline{\underline{T}}(\underline{\underline{\mu}}_j, \underline{\underline{V}}, \underline{\underline{\alpha}}) + \underline{\underline{T}}'(\underline{\underline{\mu}}_j, \underline{\underline{V}}, \underline{\underline{\alpha}}) + \underline{\underline{F}}(\underline{\underline{V}}, \underline{\underline{\gamma}})] \\ &= (\underline{\underline{I}}_{n^2} + \underline{\underline{I}}_{n, n}) (n \underline{\underline{V}} \otimes \underline{\underline{V}} + \underline{\underline{V}} \otimes \underline{\underline{M}} \underline{\underline{M}}' + \underline{\underline{M}} \underline{\underline{M}}' \otimes \underline{\underline{V}}) \\ &\quad + (\underline{\underline{V}}^{\frac{1}{2}} \otimes \underline{\underline{M}} \underline{\underline{I}}_n + \underline{\underline{M}} \underline{\underline{I}}_n \otimes \underline{\underline{V}}^{\frac{1}{2}}) \{ \underline{\underline{\alpha}}_i \underline{\underline{V}}^{\frac{1}{2}} e_i \otimes \underline{\underline{V}}^{\frac{1}{2}} e_i \}'_{i=1, \dots, p} \\ &\quad + \{ \underline{\underline{\alpha}}_j \underline{\underline{V}}^{\frac{1}{2}} e_j \otimes \underline{\underline{V}}^{\frac{1}{2}} e_j \}_{j=1, \dots, p} (\underline{\underline{V}}^{\frac{1}{2}} \otimes \underline{\underline{M}} \underline{\underline{I}}_n + \underline{\underline{M}} \underline{\underline{I}}_n \otimes \underline{\underline{V}}^{\frac{1}{2}})' \\ &\quad + n (\underline{\underline{V}}^{\frac{1}{2}} \otimes \underline{\underline{V}}^{\frac{1}{2}}) \left(\bigoplus_{i=1}^p \underline{\underline{\gamma}}_i e_i e_i' (\underline{\underline{V}}^{\frac{1}{2}} \otimes \underline{\underline{V}}^{\frac{1}{2}}) \right)' \quad . \quad (43) \end{aligned}$$

When $\underline{\underline{x}}_j \sim N(\underline{\underline{\mu}}_j, \underline{\underline{V}})$, $\underline{\underline{\alpha}}_i = \underline{\underline{\gamma}}_i = 0$, so that (43) reduces to

$$\text{var}(\text{vec}\underline{\underline{S}}) = (\underline{\underline{I}}_{n^2} + \underline{\underline{I}}_{n, n}) (n \underline{\underline{V}} \otimes \underline{\underline{V}} + \underline{\underline{V}} \otimes \underline{\underline{M}} \underline{\underline{M}}' + \underline{\underline{M}} \underline{\underline{M}}' \otimes \underline{\underline{V}}) \quad ,$$

so giving an expression when $\underline{\underline{S}}$ follows a noncentral Wishart distribution. The central Wishart distribution has $\underline{\underline{M}} = 0$ in which case

$\tilde{S} \sim W_p(n, V)$ with

$$\text{var}(\text{vec}\tilde{S}) = n(\tilde{I}_{n^2} + \tilde{I}_{n,n})(\tilde{V} \otimes \tilde{V}) \quad .$$

The χ_n^2 distribution is the univariable Wishart with $p=1$, so that

with $x_j \sim N(\mu_j, 1)$

$$E s = E\left(\sum_{j=1}^n x_j^2\right) = n + \lambda \quad \text{and} \quad \text{vars} = 2n + 4\lambda \quad ,$$

where $\lambda = \sum x_j^2$ is the noncentrality parameter.

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REFERENCES

- Anderson, R. D. [1978]. Studies on the estimation of variance components. Ph.D. thesis, Cornell University. 136 pp.
- Anderson, R. D., Quaas, R. L., and Searle, S. R. [1977]. Fourth moments in the general linear model; and the variance of translation invariant quadratic forms. Paper No. BU-630-M in the Biometrics Unit, Cornell University. 11 pp.
- Atiqullah, M. [1962]. The estimation of residual variance in quadratically balanced least-squares problems and the robustness of the F-test. Biometrika, 49, 83-91.
- Drygas, H. [1972]. The estimation of residual variance in regression analysis. Math. Operationsforsch. Statist., 3, 373-388.
- Drygas, H. and Hupet, G. [1977]. A new proof of Hsu's theorem in regression analysis—a coordinate free approach. Math. Operationsforsch. Statist. Ser. Statist., 8, 333-335.
- Henderson, H. V. and Searle, S. R. [1979]. Vec and vech operators for matrices, with some uses in Jacobians and multivariate statistics. Canad. J. Statist., 7, No. 1 (in press).
- Hsu, P. L. [1938]. On the best quadratic estimate of variance. Statist. Res. Mem., 2, 91-104.
- Magnus, J. R. [1978]. The moments of products of quadratic forms in normal variables. Statistica Neerlandica, 32, 201-210.
- Magnus, J. R. and Neudecker, H. [1979]. The commutation matrix: Some properties and applications. Ann. Statist., 7, 381-394.
- Neudecker, H. [1968]. The Kronecker matrix product and some applications in Econometrics. Statistica Neerlandica, 22, 69-82.

- Pukelsheim, F. [1977]. On Hsu's model in regression analysis.
Math. Operationsforsch. Statist. Ser. Statist., 8, 323-331.
- Rao, C. R. [1971]. Minimum variance quadratic unbiased estimation of variance components. J. Multivariate Anal., 1, 445-456.
- Searle, S. R. [1971]. Linear Models. John Wiley and Sons, New York.
- Seber, G. A. F. [1977]. Linear Regression Analysis. John Wiley and Sons, New York.