A simple construction procedure for resolvable incomplete block designs for any number of treatments

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SUMMARY

A simple, straightforward procedure, which requires no special tables or generators, is presented for constructing resolvable incomplete block designs for $v = pk$, $v = p^2k$, $\cdots$, treatments, for $k \leq p$, in incomplete blocks of size k. Also, it is shown how to obtain incomplete block designs for any v in blocks of size k and $k+1$. The procedure allows construction of balanced incomplete block designs for $p = k$ a prime number. For $p = n$ not a prime number, incomplete block designs can be obtained by the procedure, but are not balanced. However, for p_s being the smallest prime multiple of n, $p_s + 1$ for $v = n^2$, $p_s^2 + p_s + 1$ for $v = n^3$, $\cdots$, arrangements can be obtained for which the occurrence of any treatment pair in the blocks is either zero or one. This is called a zero-one concurrence design. Procedures are described for obtaining additional zero-one concurrence arrangements. It is shown that the efficiency of these designs is maximum. Both intra-block and inter-block analyses are described.

Some key words: Zero-one concurrence; Variety cutting; Successive diagonalizing; Efficiency.

1. INTRODUCTION

Each year, around the world several hundred experiments on varietal trials, pesticides, soil fumigants, medical trials, sensory difference tests, etc. are designed as incomplete block experiment designs. The number of entries in an experiment is often larger than can be accommodated in the available blocks of relatively uniform experimental units, and it is often desirable to have resolvable incomplete block designs, i.e., the incomplete blocks can be arranged in complete blocks for each replication of the entries. Any attempt to create a complete file, or catalogue, of experiment designs for all situations is doomed at the start, owing to the size of such a file and in having a method for finding a design, given that such a file existed. The main files now available are those of Bose, Clatworthy, and Shrikhande (1954), Clatworthy (1973), and Cochran and Cox (1957). Sources for constructing resolvable incomplete block designs with some files are Yates (1936) for square lattices, Harshbarger (1947, 1949, 1951) for rectangular lattices, Kempthorne (1952) and Federer (1955) for prime power lattices, Bose and Nair (1962) for two-replicate designs, and David (1967) and John, Wolock, and David (1972) on cyclic designs. Numerous other references on various aspects of incomplete block designs may be found in Federer and Balaam (1972) under categories E2 to E5.

Since it is not feasible to construct a complete file of incomplete block designs for all situations, some simple construction procedures usable by an experimenter would be desirable. Two such procedures are available in the literature, and a third one is presented herein. The first one is by Patterson and Williams (1976); it requires that a table of initial generating α-arrays be available; then, after some manipulations on the α-array to produce an intermediate α-array, the incomplete block experiment design may be easily obtained. The second procedure, given by Jarrett and Hall (1978), requires a set of initial blocks, but once these are obtained, it is a simple procedure to construct incomplete block designs.

The procedure presented here requires no tables, arrays, or generators, and it leads to a wide class of designs with efficiencies as high or higher than those given by Patterson and Williams (1976) or by Jarrett and Hall (1978). If block size and the number of treatments are specified, it may be necessary to use one of these three methods in order to construct an appropriate design. However, if the block size need only be in a range of values, say $a \leq k \leq b$, then the method proposed here may be suitable for most situations. This method involves first writing down numbers 1 to n^2 in a square array of n rows and n columns, numbers 1 to n^3 in a rectangular array of n^2 rows and n columns, etc. Then, use is successively made of main right diagonals, denoted as "successive diagonalizing" and certain numbers are deleted, i.e., "variety cutting", to reduce the total number of symbols to the desired level. Both equal and unequal block sizes are obtained. The method of "successive diagonalizing" produces a resolvable balanced incomplete block design for n a prime number; the number of times a pair of varieties occur together, a concurrence, in this design is $\lambda = 1$. From these designs, and with "variety cutting", either pairwise balanced incomplete block designs or designs with zero or one concurrence, i.e., a pair of varieties occurs together either zero or one time in the blocks, are constructed. When the number of entries deleted from n^2, n^3, ... is not a multiple of n, unequal block sizes result. Variance heterogeneity may or may not be encountered (see, e.g., Federer and Ladipo (1978) and Shafiq and Federer (1979)) when blocks are of unequal size, but it should be negligible for blocks of size k and $k+1$ in several types of experiments.

In the second section an algorithm is given for constructing designs for $v = p^2$ treatments, and consequently, for $v = pk$ treatments in equal and unequal block sizes for p a prime number. Then we discuss the construction procedure for $v = n^2$ treatments, when n is a prime power as well as when n is not a prime power. Designs are constructed for any number of treatments $\leq n^2$. In the third section, a second algorithm, which makes use of the first algorithm, is given for constructing incomplete block designs for $v = p^3$ treatments, and consequently, for $v = p^2k$ treatments in equal and unequal block sizes, using the "variety cutting" method. The method is then extended for n equal any positive integer.

These construction procedures, using the two algorithms, may be used to obtain designs for $v = p^4, p^5, p^6, \cdots$, etc. treatments in blocks of size p, and consequently, for $v = p^3k, p^4k, p^5k$, etc. treatments in equal and unequal block sizes. Then we consider the efficiencies of some of the constructed designs, and compare them with designs available in the literature. It is also noted that additional replicates for $n^2, n^3, \cdots$ treatments, n any integer, may be obtained for all situations for which t orthogonal latin squares of order n exist. It is possible to obtain $t+2$ replicates of a zero-one concurrence design in this case.

Statistical analyses for the constructed designs follow directly from published theory. For completeness, we have included the normal equations and solutions of effects in matrix form. Both intra-block and inter-block equations and solutions are given.

2. CONSTRUCTION PROCEDURES FOR A ZERO-ONE CONCURRENCE CLASS OF INCOMPLETE BLOCK DESIGNS FOR v TREATMENTS

In this section, we give an algorithm for constructing resolvable balanced and partially balanced incomplete block designs for $v = pk$ treatments in p blocks of size k and for which the concurrences of pairs of treatments are either zero or one. Then, we present a class of pairwise balanced (i.e., every pair of treatments occurs together exactly $\lambda = 1$ time and there is only one concurrence type) incomplete block designs for $v = pk$ treatments in p^2 blocks of size k and in k blocks of size p . Also, designs with two block sizes k and $k+1$, for non-prime numbers, and additional plans for $v = n^2$ designs are presented in the last part of this section.

The procedure involves first constructing a balanced incomplete block design for $v = p^2$ treatments in blocks of p by a method called "successive diagonalizing" and then using a method called "variety cutting" (an early reference to the method is given by Rao (1947); also see Federer (1955), page 424)) which involves deleting treatments from the set $v = p^2$ to obtain the resulting incomplete block designs.

2.1. For equal block sizes k

The "successive diagonalizing" method and consequent "variety cutting" method has been used and taught by the second author since he was a graduate student at Iowa State University in the late 1940's. It is a method for constructing resolvable balanced incomplete block (BIB) designs for $v = p^2$, p a prime number, in $b = p(p+1)$ blocks of size p, for the number of replicates $r = p+1$, and for $\lambda = 1$. This method is formalized below in Algorithm 2·1 and is exemplified in Example 2·1.

ALGORITHM 2·1. The steps in constructing BIB designs with parameters $v=p^2$, $k=p$, $b=p(p+1)=p^2+p$, $r=p+1$, and $\lambda=1$ for p a prime number, by the method of "successive diagonalizing", are:

(1) Write the numbers 1, 2, ···, p^2 consecutively in a square array of p rows and p columns beginning in the left-hand corner of the first row and subsequently continuing at the beginning of each row. This is square 1 with rows being the blocks.

(2) Transpose the rows and columns of square 1 to obtain square 2 .

(3) Take the main right diagonal of square 2 as the first row of square 3 . Then, write the elements of each column of square 2 in a cyclic order in the same column for square 3 .

(4) Repeat the process in step 3 on square 3 to obtain square 4 .

⋮

(p+1) Repeat the process in step 3 on square p to obtain square p + 1 .

(p+2) As a check on the previous steps, repeat the process of step 3 on the (p+1)st square, and square 2 should result. The rows of the p + 1 squares form the (p^2+p) blocks of p treatments.

Example 2·1. The steps of Algorithm 2·1 for $v = p^2 = 9$ are:

Square

	1		2		3		4=p+1
block 1	1 2 3	block 4	1 4 7	block 7	1 5 9	block 10	1 6 8
2	4 5 6	5	2 5 8	8	2 6 7	11	2 4 9
3	7 8 9	6	3 6 9	9	3 4 8	12	3 5 7

The construction method has a built in check for either a clerk or a computer program. If it is carried through the (p+2)nd step, it produces square 2. As illustrated below, p + 1 resolvable squares have been obtained to produce a BIB design with parameter $v = 9$, $k = 3$, $b = 12$, $r = 4$, and $\lambda = 1$, for which statistical analyses are readily available. If fewer than p + 1 replicates are desired, one may use square 1, square 2, ···, square n for $n = 2, 3, \cdots, p+1$ replicates. If more than p + 1 replicates are desired, multiples of p + 1 squares plus n squares can be obtained for $n < p+1$. It is not necessary to proceed in any specific order to obtain the n squares. Statistical analyses for such designs (square lattice designs) are readily available (see Federer (1955), chapters XI, XIII, and Kempthorne (1952), chapters 22, 23).

We now consider the case where $v = pk$, $k < p$. From the BIB design constructed by Algorithm 2.1, consider squares 2 to $p+1$. In these squares, "cut out" (delete) the treatment numbers from $pk+1$ to p^2 . This "variety cutting" deletes treatments and reduces the block size to $k < p$. In Example 2.1, for $v = 6 = 3(2)$, the numbers 7, 8, and 9 are deleted, resulting in 3 rectangles with 3 incomplete blocks of size 2. When $p > 3$ additional multiples of p treatments may be deleted. These designs have been called rectangular lattice designs (see Harshbarger (1947, 1949, 1951); Robinson and Watson (1949)). Both Kempthorne (1952) and Federer (1955) have used a different method of construction than presented here, but Kempthorne (1952) in chapter 25, essentially gives the above designs. Both intra-block and inter-block analyses have been presented.

A computer program for using Algorithm 2.1 to construct the $p+1$ squares and the (p+2)nd square as a check is given in Appendix C of the Cornell University Masters Thesis by M. Khare. The treatment numbers up to 31^2, the block size k, and the replication number are given in Table II.1 of this thesis. The possible treatment numbers up to $v = 150$ for $p > 31$ are also included in the table.

2.2. <u>For unequal block sizes</u>

Using all the $p+1$ squares from the p^2 design and the method of "variety cutting" on these squares, one obtains p squares for $v = pk$ treatments with p incomplete blocks of equal size k as described in subsection 2.1. If in addition, square 1 with k incomplete blocks of size p is included, then every pair of treatments will occur together $\lambda = 1$ times in the incomplete blocks. For instance, in Example 2.1 all 4 squares can be used for $v = 3(2) = 6$ treatments with square 1 having 2 incomplete blocks of size 3 and others having incomplete blocks of size 2.

Omitting the first square and using "variety cutting", one can construct designs having any number of treatments not equal to kp in incomplete blocks of size k and $k+1$ in the remaining p squares. To illustrate, resolvable blocks for $v = 7 = 3(2)+1$ treatments in blocks of size 2 and 3, from Example 2.1 are obtained by deleting (i.e., using "variety cutting" method) treatments 8 and 9 in squares 2, 3, and 4. There are $(pt) = 3(1) = 3$ blocks of size $k+1=3$ and $p^2 - pt = 9 - 3(1) = 6$ blocks of size $k = 2$ in the resulting plan.

2.3. Designs for $v=n^2$ treatments, where $n=mp_s$, p_s is the smallest prime in n and m is any integer, or where $n=p^m$, a prime power

Using the first p_s+1 squares from the $n+1$ squares obtained by using the "successive diagonalizing" method on the square of n^2 treatments in blocks of size n, one can obtain incomplete block designs with zero and one concurrence. The concurrence number increases in groups of p_s, if more squares are included. Therefore, if more squares are desired, one could duplicate the (p_s+1) squares or one could use the additional squares obtained from "successive diagonalizing". For example, at least three squares can be obtained for all the even treatment numbers $(n=2m)$, four for $n=3m$ treatments, and six for $n=5m$ treatments with (0,1) concurrences. The same methods can be used for the $n=p^m$, a prime power. For example, for $v = (3^2)^2 = 9^2 = 81$ treatments, 1 to 4 replicates are obtained with a zero or one concurrence, 5 to 7 with a zero, one or two concurrence, and 8 to 10 with a zero, one, two or three concurrence. These concurrences can be checked from the $(NN')_{v\times v}$ matrices obtained for the normal equations, N' being the transpose of the incidence matrix N .

2.4. Additional (0,1) concurrence plans for $v=n^2$ designs

The method of "successive diagonalizing" results in the smallest prime of n, say p_s, plus one resolvable zero-one concurrence replicate for $v=n^2$

treatments. Then, by "variety cutting", one obtains $p_s + 1$ replicates of a zero-one concurrence design for $v = nk$ treatments. In order to obtain additional replicates, it is necessary that there be more than $p_s - 1$ orthogonal latin squares. Denoting a set of t mutually orthogonal latin squares of order n as OL(n,t), we can construct $t + 2$ replicates for $v = n^2$ treatments, and the resulting design will be a zero-one concurrence design. For example, for $n = 10$, there is an OL(10,2) set which means that we can obtain $t + 2 = 4$ replicates for a zero-one concurrence design. For $n = 12$, there exists an OL(12,5) set which means that we can obtain $5 + 2 = 7$ replicates of a zero-one concurrence design. The numbers of orthogonal latin squares for various $n \neq$ a prime power are tabulated in Raghavarao (1971), Table 3.7.1. For n = a prime power, there is a complete set, $t = (n - 1)$, of orthogonal latin squares. Thus, for n equal to a prime or prime power, $n + 1$ replicates of a zero-one concurrence design are available. Incomplete block designs with unequal block sizes may also be obtained as described in subsection 2.2.

3. CONSTRUCTION PROCEDURE FOR A TWO-CONCURRENCE CLASS OF INCOMPLETE BLOCK DESIGNS FOR $v = p^2k$ TREATMENTS, p A PRIME NUMBER AND $k \leq p$

We first give an algorithm for constructing resolvable incomplete block designs, making use of Algorithm 2.1, for $v = p^2k$ treatments in p^2 blocks of size k and for which the concurrences of pairs of treatments are either zero or one. Secondly, we present a class of pairwise balanced, but not variance balanced, incomplete block designs for $v = p^2k$ treatments in blocks of unequal size p and k. Designs for $v = pk + t$, $t = 1, 2, \cdots, (p - 1)$, in unequal blocks of size k and $k + 1$ are also presented in this section. In the third subsection, designs for non-prime numbers (n), including prime powers, are presented for $v = n^3$ treatments, n an integer. Then, additional plans for n^3 designs are given.

The procedure involves first constructing a balanced incomplete block (BIB) design for $v = p^3$ treatments, p a prime number, in blocks of size p using the "successive diagonalizing" method as described in Algorithm 3.1 and then applying the method of "variety cutting" to reduce the treatment numbers. Thus, the parameters of the resolvable BIB design are $v = p^3$, $k = p$, $r = p^2 + p + 1$, $b = rp^2$ and $\lambda = 1$. Federer (1955) and Kempthorne (1952) give the analyses for partially balanced and balanced designs when $v = s^3$, $k = s$, and $r = 3, 4, \cdots, s^2 + s + 1$, $s = p^m$ a prime power, and for $v = n^3$, $k = n$, and $r = 3$, n an integer greater than one. Using their procedures, one may obtain statistical analyses for $r = t + 2$ replicates whenever an OL(n, t) set exists for any integer n.

3.1. *For equal block size k*

The procedure for constructing resolvable balanced incomplete block (BIB) designs for $v = p^2$ treatments with parameters $v = p^3$, $k = p$, $r = p^2 + p + 1$, is formalized in Algorithm 3.1.

ALGORITHM 3·1. *The steps in constructing BIB designs with parameters* $v = p^3$, $k = p$, $r = p^2 + p + 1$, $b = p^2(p^2 + p + 1)$ *and* $\lambda = 1$, *for* p *a prime number are as follows:*

(0) *Write the* $v = p^3$ *treatments consecutively in* p^2 *blocks of size* p *as described in step 1 of Algorithm 2.1. Then assign numbers* (1) *through* (p^2) *to* p^2 *blocks (i.e., rows) and arrange the block numbers in* $p + 1$ *groups as obtained using Algorithm 2.1 on* p^2 *treatments, considering treatment numbers* 1 *through* p^2 *as block numbers* 1 *through* p^2. *Partition the* p^2 *blocks in* p *sets of* p *blocks.*

(1) *On the first grouping of the* p^2 *blocks from step 0, apply Algorithm 2.1 to the* p^2 *treatments in each set of* p *blocks separately to obtain* $p + 1$ *of the possible* $p^2 + p + 1$ *replicates (Reps) for* $v = p^3$ *treatments in blocks of size* p. *As a check, if the procedure is continued to the* $(p + 2)$nd *step, Rep* 2 *will result.*

(2) For the second grouping of p^2 blocks from step 0 with p^3 treatments written properly in each block, apply Algorithm 2.1 to each set of p blocks to obtain $p+1$ new arrangements for p^3 treatments. The first arrangement must be omitted as it is identical to Rep 1 from step 1, except for a permutation of the p^2 blocks of size p. This gives a set of p additional replicates.

(3) Repeat step 2 on the third grouping of p^3 treatments from step 0 to obtain an additional set of p replicates, omitting the first arrangement.

⋮

(p+1) Repeat step 2 on the (p+1)st grouping of p^3 treatments from step 0 to obtain a new (p+1)th set of p replicates, omitting the first arrangement.

Thus, there are $p+1$ replicates from step 1 and p from each of the additional p steps, resulting in p^2+p+1 replicates to form a BIB design. If fewer than p^2+p+1 replicates are desired, one may use replicates 1, 2, $\cdots$, r for $r=3, 4, \cdots, p^2+p+1$. If more than p^2+p+1 replicates are desired, the additional replicates can be obtained by selecting any n arrangements, $n \leq p^2+p+1$, from the p^2+p+1 arrangements. Statistical analyses are available using the procedures in Federer (1955) and Kempthorne (1952).

We now consider the case where $v=p^2k$, $k < p$. From the BIB design constructed by Algorithm 3.1, consider replicates $(p+2)$ to p^2+p+1 from step 2 through step $p+1$. On these arrangements, use "variety cutting" and delete treatment numbers p^2k+1 to p^3. This "variety cutting" reduces the block size to $k < p$. The resulting design is a zero-one concurrence one.

3.2. For unequal block sizes

Using all but the first $p+1$ replicates from p^3 design and the method of "variety cutting" on them, one obtains p^2 replicates for $v=p^2k$ treatments, each with p^2 incomplete blocks of equal size k as described in subsection 3.1. If, in addition, replicates 1 through $p+1$ from group 1, each with pk incomplete blocks of size p, are included, then every pair of treatments will occur together once in the incomplete blocks. Omitting replicates 1 through $p+1$ from Group 1, one can construct designs having any number of treatments not equal to p^2k, in incomplete blocks of size k and $k+1$ in the remaining p^2 replicates. The concurrence of pairs of treatments is either zero or one for this design.

3.3. Designs for $v = n^3$ treatments, where $n = mp_s$, p_s is the smallest prime in n and m is any integer, or $n = p^m$, a prime power

Obtain the first $p_s + 1$ squares with zero-one concurrence from an n^2 design as described in subsection 2.3, for the arrangement of n^2 incomplete blocks in $p_s + 1$ groupings for n^3 designs. Then, apply Algorithm 3.1 on those $p_s + 1$ groupings to obtain $p_s + 1$ replicates from the first grouping, p_s from the second grouping omitting the first one, p_s from the third grouping omitting the first one, ..., p_s from $(p_s + 1)$st grouping omitting the first one. Thus, there are $p_s(p_s) + (p_s + 1) = p_s^2 + p_s + 1$ replicates for $v = n^3$ treatments in incomplete blocks of n with zero-one concurrences. The concurrence of pairs of treatments increases as the number of replicates increases beyond $p_s^2 + p_s + 1$.

Using the method of "variety cutting", one may obtain designs for $v = n^2k$ treatments in unequal blocks of size k and $k+1$. To illustrate, consider $n = 2^2$, i.e., $v = 4^3 = 64$ treatments where there are $r = 7$ replicates with incomplete blocks of size $n = 4$ which have zero-one concurrences of treatment pairs. For $v = n^2k = 16(3) = 48$ in equal blocks of size 3, omit the 3 replicates from the first grouping and use the method of "variety cutting" on the rest of the replicates, i.e., delete treatments 49 through 64 in replicates 4, 5, 6 and 7. For $v = 16(3) = 48$ treatments in block sizes of 4 and 3, include the replicates from the first grouping and delete the blocks with treatments 49 through 64 and thus, there are $12(3) = 36$ incomplete blocks of size 4 and 64 incomplete blocks of size 3. And finally, for $v = 16(2) + 7 = 39$ treatments, omit the 3 replicates from the first grouping and use the "variety cutting" method on treatments 40 through 64 in replicates 4 through 7, resulting in blocks of size 2 and 3.

3.4. Additional (0,1) concurrence plans for $v = n^3$ designs

The method of "successive diagonalizing" results in $p_s^2 + p_s + 1$ resolvable zero-one concurrence replicates for $v = n^3$ treatments, p_s being the smallest prime of n . Then by "variety cutting", one obtains $p_s^2 + p + 1$ replicates of a zero-one concurrence design for $v = n^2k$ treatments. In order to obtain additional replicates for a zero-one concurrence design, one may use a set of t mutually orthogonal latin squares of order n resulting in $t + 2$ replicates for a zero-one concurrence design. Using these replicates for the $t + 2$ arrangements of blocks, as described in step 0 of Algorithm 3.1, one may obtain the replicates for $v = n^3$ treatments from these groupings when n is not equal a prime power. There is a complete set of $n^2 + n + 1$ replicates available for $n =$ a prime power. For example, for $n = 2^2 = 4$, there exists $t = 3$ orthogonal latin squares and therefore a complete set of $n^2 + n + 1 = 4^2 + 4 + 1 = 21$ replicates are available for a zero-one concurrence design. To obtain $n^2 = 16$ replicates for $v = n^2k$ $= 16(3) = 48$ treatments in blocks of size 3, delete treatment numbers 49 through 64 from replicates in groupings 2 through 5. Incomplete block designs with unequal block sizes may also be obtained as described in subsection 3.2.

4. EXTENSION TO $v \leq p^4, p^5, \cdots, p^m$ TREATMENTS, m ANY INTEGER

Using Algorithms 2.1 and 3.1, one may obtain resolvable balanced incomplete designs for $v = p^4, p^5, \cdots, p^m$ treatments in blocks of size $k = p$. There are $(p^m - 1)/(p - 1) = p^{m-1} + p^{m-2} + \cdots + p^2 + p + 1$ arrangements available for balanced designs, using appropriate arrangements of p^{m-1} blocks in a replicate. Then replicates for $v = p^{m-1}k$ treatments may be obtained in equal and unequal block sizes for (0,1) concurrence by using the method of "variety cutting" on $kp^{m-1}+1$ through p^m treatments in the balanced design. There are $p^{m-1}+p^{m-2}+\cdots+p^3+p^2$ arrangements available for obtaining replications with equal block size k .

For $v \leq p^m$, $m=1, 2, 3, \ldots$ treatments in blocks of size $k \leq p^s$, $s=2, 3, \ldots, m-1$, one may obtain designs using the "successive diagonalizing" method and consequently the "variety cutting method", but these designs will have concurrences more than (0,1). To obtain designs with (0,1,2) concurrence, one may obtain available designs for $v=p^m$ treatments in blocks of size $k=p^s$ from the literature, and then use "variety cutting" to reduce the number of treatments. For example, Federer and Robson (1952) give 10 arrangements for $v=2^5$ treatments in blocks of size 2^2 and for $v=3^5$ in blocks of size 3^2, and the method of "variety cutting" may be used to reduce the number of treatments. Using their procedure, one could construct additional plans for other p and additional replicates.

5. EFFICIENCY FACTORS FOR THE CONSTRUCTED DESIGNS

The efficiency factor for a design is calculated as the harmonic mean of the non-zero eigenvalues of the matrix C, as described e.g., by Jarrett and Hall (1978) and by Raghavarao (1971) pp. 58-59. The matrix C for an equi-replicated and unequal block size design is obtained as

$$C_{v\times v} = (rI - NK^{-1}N')/r \quad , \tag{5.1}$$

where N' is the transpose of the incidence matrix N and $K=\mathrm{Diag}(k_1,k_2,\cdots,k_b)$; $k_1,k_2,\cdots,k_b$ are the block sizes for blocks 1, 2, $\cdots$, b, respectively. When $k_i=k$, K^{-1} is replaced by the scalar $1/k$. The efficiency factor is $E = (v-1)/\sum_{i=1}^{v-1} 1/\lambda_i$, where the λ_i's are the non-zero eigenvalues of the matrix C .

The proposed designs with unequal block size k and $k+1$, are as efficient or more so than the α designs given by Patterson and Williams (1976). Efficiency factors for a (0,1) concurrence design constructed for $v \leq 25$ treatments in blocks of size k and $k+1$ in $r=4$ replicates are given in Table 5.1.

Table 5.1. Efficiency factors ($r=4$)

v	Block size	Efficiency factor
18	3,4	.7399
19	3,4	.7551
20	4	.7686
21	4,5	.7804
22	4,5	.7911
23	4,5	.8009
24	4,5	.8099
25	5	.8182

For 3 replicate designs for any $v=n^2$, n an even integer, our method produces zero-one designs whereas that of Patterson and Williams (1976) produces a zero-one-two (i.e., $\alpha(0,1,2)$) design. Efficiencies for some of the constructed designs are given in Table 5.2. Our designs are as, or more, efficient as others in the literature.

Table 5.2. Efficiency factors for some designs ($r=3$)

		Efficiency factors	
v	k	Our designs*	$\alpha(0,1,2)$ designs
16	4	.7692	.7538
36	6	.8235	.8186
64	8	.8571	.8549
100	10	.8800	.8788

*The efficiencies are the same as for lattice designs.

For pairwise balanced designs for $v=pk$ treatments with blocks of size p and k in $p+1$ replicates, the efficiencies are given in Table 5.3, for some of the constructed designs.

Table 5.3. Efficiency factors ($r = 6$)

v	Block size	Efficiency factor
10	5,2	.6034
15	5,3	.7362
20	5,4	.7979
25	5	.8383

The BIB α-designs do not exist, whereas our procedure starts with the construction of BIB designs. It is suspected that (0,1) concurrence designs are always better than (0,1,2) concurrence designs. We know this is true for variance balanced designs (Shafiq and Federer (1979)). The relative efficiency of basic balanced ternary to binary designs is given by Shafiq and Federer (1979) as $E_3/E_2 = 1 - 2r_2/r(k-1)$, where r_2 is the number of occurrences of the number 2 in each row of NN'. They also present results demonstrating that balanced incomplete block designs with a zero-one concurrence will be more efficient than zero-two concurrence designs.

6. INTRA-BLOCK AND INTER-BLOCK ANALYSES FOR INCOMPLETE BLOCK DESIGNS

For completeness we give the normal equations for intra-block and inter-block analyses. A response equation for a resolvable incomplete block design is

$$Y_{hij} = (\mu + \rho_h + \beta_{hi} + \tau_j + \epsilon_{hij})n_{hij} \quad , \tag{6.1}$$

where $n_{hij} = 1$ if treatment j occurs in $\underline{hi}$th incomplete block and 0 otherwise, the ϵ_{hij} are $NIID(0,\sigma_\epsilon^2)$, the least squares equations in matrix form are

$$\begin{bmatrix} K & N'_{b\times v} \\ N_{v\times b} & rI_{v\times v} \end{bmatrix} \begin{bmatrix} \underline{\mu + \rho + \beta}_{b\times 1} \\ \underset{\sim}{\tau}_{v\times 1} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{Y}_b \\ \underset{\sim}{Y}_t \end{bmatrix} \quad , \tag{6.2}$$

where K is a diagonal matrix of block sizes, $N_{v\times b} = (n_{hij})$ is the $v \times b$ incidence matrix for the design, $\underset{\sim}{Y}_b$ is the $b \times 1$ vector of incomplete block totals, and

$\underset{-}{Y}_t$ is the $v \times 1$ vector of treatment totals. The least squares solutions with the usual restraints for μ is $\bar{y}_{...}$, the overall mean, and of $\mu + \rho_h$ is $\bar{y}_{h..}$, the mean of the $\underline{h}$th complete block. The reduced intra-block normal equations for the treatment effects τ are

$$(rI_{v\times v} - NK^{-1}N')\underset{-}{\tau} = (\underset{-}{Y}_t - NK^{-1}\underset{-}{Y}_b) = \hat{\underset{\sim}{Q}}_t \quad \text{(say)} \quad . \tag{6.3}$$

Adding $\frac{1}{k} J_{v\times v}$, where $J_{v\times v}$ is a matrix of ones, to the above equations to make them full rank matrices and assuming that $\Sigma_j \hat{\tau}_j = 0$, we obtain least squares solutions as:

$$\hat{\underset{-}{\tau}} = (rI_{v\times v} - N_{v\times b}K^{-1}_{b\times b}N'_{b\times v} + \frac{1}{k} J_{v\times v})^{-1}\underset{\sim}{Q}_t \quad . \tag{6.4}$$

Then, $\sigma^2_\epsilon(rI - NK^{-1}N' + \frac{1}{k} J)^{-1}$ is used to obtain variances of linear contrasts among $\hat{\tau}_j$'s .

For the inter-block analysis, substitute $(K + I\sigma^2_\epsilon/\sigma^2_\beta)^{-1}$ for K^{-1} in (6.2-3) to obtain inter-block solutions $\tilde{\tau}_j$ for the τ_j, and then obtain the estimated variance of linear contrasts of $\underset{-}{\tilde{\tau}}_j$ from $(rI - N(K + I\sigma^2_\epsilon/\sigma^2_\beta)^{-1}N' + (k + \sigma^2_\epsilon/\sigma^2_\beta)^{-1}J)^{-1}$, where $\hat{\sigma}^2_\epsilon$ and $\hat{\sigma}^2_\beta$ are substituted for σ^2_ϵ and σ^2_β, respectively. Here, we consider the β_{hi} to be NIID$(0,\sigma^2_\beta)$.

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