CLASS NOTES ON EIGENVALUES OF ORTHOGONAL MATRICES

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Abstract

Established results on eigenvalues of orthogonal matrices are reviewed.

Orthogonal Matrices

A real matrix $\mathbf{P}$ is said to be row-wise orthonormal when $\mathbf{PP}' = \mathbf{I}$ (since its rows then constitute a set of orthonormal vectors); and it is column-wise orthonormal when $\mathbf{P}'\mathbf{P} = \mathbf{I}$.

Lemma 1. Any real matrix that is both row-wise and column-wise orthonormal is square, non-singular, and has determinant ± 1.

Proof: Suppose $\mathbf{P}_{r \times c}$ is row-wise orthonormal, with rank $\rho(\mathbf{P})$. Then $\mathbf{PP}' = \mathbf{I}_r$ and so $r = \rho(\mathbf{PP}') \leq \rho(\mathbf{P})$. But $\mathbf{P}$ has r rows; therefore $\rho(\mathbf{P}) = r$. Similarly if $\mathbf{P}$ is column-wise orthonormal, $\rho(\mathbf{P}) = c$; i.e., $r = c$, and so $\mathbf{P}$ is square. (This argument is identical to that of proving that $\mathbf{P}'$ can be both a right and a left inverse of $\mathbf{P}$ only if $\mathbf{P}$ is square.)

With $\mathbf{P}$ being square, taking the determinant of both sides of $\mathbf{PP}' = \mathbf{I}$ gives $|\mathbf{P}|^2 = 1$. Hence $|\mathbf{P}| \neq 0$ and so $\mathbf{P}$ is non-singular; and $|\mathbf{P}| = \pm 1$. Q.E.D.

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Corollary: For square $\underset{\sim}{P}$, either of $\underset{\sim}{P}\underset{\sim}{P}' = \underset{\sim}{I}$ or $\underset{\sim}{P}'\underset{\sim}{P} = \underset{\sim}{I}$ implies the other, and also implies $\underset{\sim}{P}$ non-singular with $|\underset{\sim}{P}| = \pm 1$.

Definition: A real, square matrix $\underset{\sim}{P}$ satisfying $\underset{\sim}{P}\underset{\sim}{P}' = \underset{\sim}{I} = \underset{\sim}{P}'\underset{\sim}{P}$ is called an orthogonal matrix. It is non-singular, with $|\underset{\sim}{P}| = \pm 1$.

The name "orthogonal matrix" is of long-standing; in view of the orthonormal property of its rows (and columns), orthonormal in contrast to just orthogonal, the name "orthonormal matrix" might seem more logical – but "orthogonal matrix" takes historic precedence.

Eigenvalues

We use $\underset{\sim}{P}$ to represent an orthogonal matrix and give some lemmas adapted from Mirsky [1955, pp. 222-226].

Lemma 2. Every eigenvalue λ of $\underset{\sim}{P}$ is non-zero; and $1/\lambda$ is also an eigenvalue.

Proof: Because $\underset{\sim}{P}$ is non-singular, $\lambda \neq 0$; and $1/\lambda$ is an eigenvalue of $\underset{\sim}{P}^{-1} = \underset{\sim}{P}'$. Therefore

$$0 = |\underset{\sim}{P}' - \lambda^{-1}\underset{\sim}{I}| = |(\underset{\sim}{P} - \lambda^{-1}\underset{\sim}{I})'| = |\underset{\sim}{P} - \lambda^{-1}\underset{\sim}{I}| .$$

Hence $1/\lambda$ is an eigenvalue of $\underset{\sim}{P}$. Q.E.D.

Some eigenvalues may be complex; e.g., not all orthogonal matrices are symmetric.

Lemma 3. If, for $i = \sqrt{-1}$, an eigenvalue of $\underset{\sim}{P}$ is $\lambda = \alpha + i\beta$ with corresponding eigenvector $\underset{\sim}{u} = \underset{\sim}{v} + i\underset{\sim}{w}$, then the complex conjugate of λ, namely $\bar{\lambda} = \alpha - i\beta$, is also an eigenvalue with the corresponding eigenvector being the complex conjugate of $\underset{\sim}{u}$, namely $\bar{\underset{\sim}{u}} = \underset{\sim}{v} - i\underset{\sim}{w}$.

Proof: The equation defining λ and u is $Pu = \lambda u$; i.e.,

$$P(v + iw) = (\alpha + i\beta)(v + iw) = \alpha v - \beta w + i(\beta v + \alpha w) .$$

Therefore $Pv = \alpha v - \beta w$ and $Pw = \beta v + \alpha w$. Hence

$$P(v - iw) = \alpha v - \beta w - i(\beta v + \alpha w) = (\alpha - i\beta)(v - iw) . \qquad \text{Q.E.D.}$$

This lemma is, of course, true for all matrices P, not just orthogonal matrices.

Lemma 4. Every eigenvalue of P has unit modulus: $|\lambda| = 1$.

Proof: $$Pu = \lambda u \quad \text{and} \quad P\bar{u} = \bar{\lambda}\bar{u} .$$

Therefore $\bar{u}'P' = \bar{u}'\bar{\lambda}$ and so

$$\bar{u}'P'Pu = \bar{u}'\bar{\lambda}\lambda u; \text{ i.e., } \bar{u}'u = \bar{\lambda}\lambda\bar{u}'u .$$

But if $\alpha_k + i\beta_k$ is a typical element of u, then $\bar{u}'u = \sum_k(\alpha_k^2 + \beta_k^2) \neq 0$ because α_k and β_k are real. Therefore $\bar{u}'u = \bar{\lambda}\lambda\bar{u}'u$ implies $\bar{\lambda}\lambda = 1$; i.e., $|\lambda|^2 = 1$ and so $|\lambda| = 1$. Q.E.D.

Corollary 4.1: All real eigenvalues of P are +1 or -1.

Corollary 4.2: Symmetric orthogonal matrices have eigenvalues that are all ± 1.

Note that when $\lambda = \pm 1$ then $1/\lambda = \pm 1$ also, and $1/\lambda$ is, of course, an eigenvalue. But this does not mean that if P has an eigenvalue of 1 then it necessarily has another eigenvalue of $1/1 = 1$. Lemma 2 states only that if λ is an eigenvalue then so is $1/\lambda$; it does not state that if λ is an eigenvalue then $1/\lambda$ is another eigenvalue. If it did and there was one eigenvalue of 1 then there would always be two such eigenvalues. Such is not the case (and the same is true for $\lambda = -1$). For example,

$$\underset{\sim}{P} = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix} \quad \text{has} \quad |\underset{\sim}{P}| = -1$$

and the characteristic equation of $\underset{\sim}{P}$ is $(.6 - \lambda)(-.6 - \lambda) - .64 = 0$; i.e., $\lambda^2 = 1$ and so $\lambda = \pm 1$. The equations $\underset{\sim}{P}\underset{\sim}{u} = \lambda \underset{\sim}{u}$ are

$$\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (1) \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ -2 \end{bmatrix} .$$

This exemplifies Corollary 4.2.

Lemma 5. Each complex conjugate pair of eigenvalues of $\underset{\sim}{P}$, satisfy Lemmas 3 and 5; i.e., if λ and $\bar{\lambda}$ are the eigenvalues then $|\lambda| = |\bar{\lambda}| = 1$ and $1/\lambda = \bar{\lambda}$.

Proof: Let $\lambda = \alpha + i\beta$ and $\bar{\lambda} = \alpha - i\beta$. Then $|\lambda|^2 = |\bar{\lambda}|^2 = \alpha^2 + \beta^2 = 1$. Therefore

$$\frac{1}{\lambda} = \frac{1}{\alpha + i\beta} = \frac{\alpha - i\beta}{(\alpha + i\beta)(\alpha - i\beta)} = \frac{\alpha - i\beta}{\alpha^2 + \beta^2} = \alpha - i\beta = \bar{\lambda} . \qquad \text{Q.E.D.}$$

Reference

Mirsky, L. [1955]. An Introduction to Linear Algebra. Oxford University Press.