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0. Summary. The concepts of defining contrast (DC), generalized defining relationship (GDR) and aliasing structure (AS) are now well established in the terminology of regression analysis and fractional factorial design theory. There is no complete agreement in the literature about the meaning of regular and irregular fractional factorial designs. This paper provides a workable definition of a regular fraction from a symmetrical prime powered factorial. It also characterizes the uniqueness of the GDR for fractions from the most general factorial. Finally, some results are presented on the uniqueness of the GDR for regular designs, on orthogonality aspects of regular and irregular designs, and on group-theoretic generation of the complete aliasing structure. Examples are provided to illustrate the developments.

1. Introduction. As the basic setting we take the orthogonal linear model for a complete replicate of the $s_1 \times s_2 \times \ldots \times s_n$ factorial, i.e.

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$$(1.1) \qquad \begin{cases} Y = X\beta + \varepsilon \ , \\ E[\varepsilon] = 0 \ , \quad E[\varepsilon\varepsilon'] = \sigma^2 I_N \ , \quad N = \prod_{i=1}^{n} s_i \ , \end{cases}$$

where X is an $N \times N$ columnwise orthogonal matrix with the first column consisting of 1's so that the sum of the elements in any other column is equal to zero. The $N \times 1$ parametric vector β has as first element the mean and the other elements are factorial effects (main effects and interactions). The $N \times 1$ vector ε is the usual random error vector. The structure of the matrix X will depend on the definition of factorial effects. Since in this paper both the general factorial and the special case of the symmetrical s^n prime powered factorial will be considered we will use both the product definition (i.e. the matrix X is the Kronecker product of basic columnwise orthogonal matrices with first column $\mathbf{1}$ associated with the factors) and the geometric definition (i.e. the matrix X is obtained from one basic columnwise orthogonal matrix with first column $\mathbf{1}$ associated with parallel pencils of flats of the finite Euclidean geometry $EG(n, s)$) of factorial effects. If no statement is made as to which definition (product or geometric) is being used then results in the prime powered case will hold for either ~~definitions~~ definition.

Identify the levels of the i-th factor with elements of the set $S_i = \{0, 1, 2, \ldots, s_i - 1\}$. For the symmetrical s^n prime powered factorial the levels of each factor can be taken as elements of the Galois field $GF(s) = F$. The set of treatment combinations is

$S = \mathop{\times}\limits_{i=1}^{n} S_i$, the Cartesian product of the sets $S_1, S_2, \ldots, S_n$,

which for the symmetrical prime powered factorial is written as F^n .

The set F^n is a vector space over F and it may be viewed as the finite Euclidean geometry $EG(n, s)$ over F. Denote a treatment combination by the n-tuple $(x_1, x_2, \ldots, x_n)$ with $x_i \in S_i$, $i = 1, 2, \ldots, n$, and an element of β by $A_1^{x_1} A_2^{x_2} \ldots A_n^{x_n}$. For convenience the mean $A_1^0 A_2^0 \ldots A_n^0$ will be indicated by μ and the other factorial effects by the Greek letter α with subscripts, i.e. the vector of effects β' is equal to $(\mu, \alpha_1, \alpha_2, \ldots, \alpha_{N-1})$. In the symmetrical prime powered case the elements of β other than the mean, i.e. the α's, are derived from $(s^n-1)/(s-1)$ components, which are associated with points of the finite projective geometry $PG(n-1, s)$, (e.g. see Kempthorne [1952]).

Let D be a factorial arrangement from the $s_1 \times s_2 \times \ldots \times s_n$ factorial consisting of k not necessarily distinct treatment combinations (for a precise combinatorial definition, see Hedayat, Raktoe and Federer [1974]). For the $k\times 1$ observation vector Y_D we then have the induced model

$$(1.2) \qquad \begin{cases} Y_D = X_{D,\beta}\beta + \varepsilon_D , \\ E[\varepsilon_D] = 0 , \quad E[\varepsilon_D\varepsilon_D'] = \sigma^2 I_k , \end{cases}$$

where the $k\times N$ matrix $X_{D,\beta}$ is read off from X of (1.1) taking repetitions into account. In the development below we will be especially interested in the partitioned vector $\beta' = (\beta_1' \vdots \beta_2')$, where β_1 is $N_1\times 1$, β_2 is $N_2\times 1$, $N_1 + N_2 = N$, such that: (i) the mean μ always belongs to β_1 and is its first entry, and (ii) for the design matrix of D, i.e.

$$(1.3) \qquad X_{D,\beta} = [X_{D,\beta_1} \vdots X_{D,\beta_2}] = [X_1 \vdots X_2] ,$$

we assume that $X_{D,\beta_1} = X_1$ is of full column rank. Note that assumptions (i) and (ii) imply that the number of distinct treatment combinations in D is at least equal to N_1. If β_2 is negligible then the BLUE of β_1 is equal to zero ?

$$(1.4) \qquad \hat{\beta}_1 = [X_1'X_1]^{-1}X_1'Y_D .$$

If the negligibility assumption is false then

$$(1.5) \qquad E[\hat{\beta}_1] = \beta_1 + [X_1'X_1]^{-1}X_1'X_2\beta_2 = \beta_1 + A_{D,\beta}\beta_2 .$$

The matrix $A = [X_1'X_1]^{-1}X_1'X_2$ is known as the <u>alias matrix</u> of the design D relative to β_1 and β_2. The vector $\beta_1 + A\beta_2$ is known as the <u>aliasing structure</u> (AS) and its first element is called the <u>generalized defining relationship</u> (GDR) of D relative to β_1 and β_2. These concepts are due to Box and Wilson [1951] and may be found in a different context in Hedayat, Raktoe and Federer [1974] and Raktoe [1976]. We like to emphasize that in general both the AS and the GDR depend not only on the choice of the design D but also on the choice of β_1.

<u>2. Uniqueness of the GDR</u>. For a given factorial arrangement D the GDR is said to be unique if and only if the first entry of the AS is invariant under every partitioning $\beta' = (\beta_1' \vdots \beta_2')$ satisfying conditions (i) and (ii) above. The theorem below provides a characterization of the uniqueness of the GDR.

<u>Theorem 2.1</u>. Let D be any factorial arrangement of S. Then the GDR is unique if and only if every column of the design matrix $X_{D,\beta}$ is a multiple of $\mathbf{1}$ or is orthogonal to $\mathbf{1}$.

Proof. Suppose that each column of $X_{D,\beta}$ is either a multiple of $\mathbf{1}$ or is orthogonal to $\mathbf{1}$. In this latter case the sum of the entries in that column is equal to zero. Let $\beta' = (\beta_1' \vdots \beta_2')$ be any partition of β satisfying conditions (i) and (ii) above. Then an effect α will belong to β_2 if the column in $X_{D,\beta}$ determined by D and α is a multiple of $\mathbf{1}$. Since the first column of X_1 is $\mathbf{1}$ and X_1 is of full column rank it follows from the assumption that all the other columns of X_1 are orthogonal to $\mathbf{1}$. Hence if D has k treatment combinations, then

$$(2.1) \qquad [X_1'X_1]^{-1} = \left[\begin{array}{c|c} \frac{1}{k} & 0' \\ \hline 0 & C_1 \end{array}\right] \quad \text{and} \quad X_1'X_2 = \left[\begin{array}{c} c_1 \; c_2 \cdots c_{N_2} \\ \hline C_2 \end{array}\right]$$

for suitable matrices C_1 and C_2 and $c_j = 0$ or kd_j according as the j-th column of X_2 is orthogonal to $\mathbf{1}$ or equals $d_j\mathbf{1}$, $j = 1, 2, \ldots, N_2$. Thus if $\beta_2' = (\alpha_1\ \alpha_2 \ldots \alpha_{N_2})$ then the GDR corresponding to D and $\beta' = (\beta_1' \vdots \beta_2')$ is $\mu + d_1\alpha_1 + d_2\alpha_2 + \ldots + d_{N_2}\alpha_{N_2}$, and only those effects α_j in β_2 appear in this linear combination for which $d_j \neq 0$, that is precisely those α_j in β_2 appear for which the corresponding column in X_2 is a multiple of $\mathbf{1}$. Hence it follows that the GDR is unique.

Conversely, suppose that for the given design D, the GDR is unique. Further suppose that there exists a column $a' = (a_1\ a_2 \ldots a_k)$ in the design matrix $X_{D,\beta}$, which is not a multiple of $\mathbf{1}$. Then we must show that it is orthogonal to $\mathbf{1}$. Suppose that this column $a' = (a_1\ a_2 \ldots a_k)$ is determined in the design matrix $X_{D,\beta}$ by the design D and effect α_0. Let $\beta_1' = (\mu\ \alpha_0)$ and $\beta_2' = (\alpha_1\ \alpha_2 \ldots \alpha_{N_2})$ be the vector determined by placing in some order all the effects in β_2 besides μ and α_0, where $N_2 = N - 2$. Then clearly $\beta' = (\beta_1' \vdots \beta_2')$ is a partition of β satisfying (i) and (ii) above, and $X_1 = [\mathbf{1} \vdots a]$. The GDR corresponding to the partitioned β is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_{N_2}$, say $\mu + v_1\alpha_1 + v_2\alpha_2 + \ldots + v_{N_2}\alpha_{N_2}$ for suitable scalars $v_1, v_2, \ldots, v_{N_2}$. Next, consider the partition $\beta' = (\mu \vdots \alpha_0\ \alpha_1 \ldots \alpha_{N_2-1})$, where $N_2 = N - 1$. This partition also satisfies conditions (i) and (ii) above, and the GDR corresponding to this partition is easily shown to be of the form $\mu + \frac{1}{k}(\sum_{i=1}^{k} a_i)\alpha_0 + w_1\alpha_1 + w_2\alpha_2 + \ldots + w_{N_2-1}\alpha_{N_2-1}$. Since the GDR is unique it follows that $v_j = w_j$ for all j , and in particular $\sum_{i=1}^{k} a_i = 0$, that is, the column vector a is orthogonal to $\mathbf{1}$. This completes the proof.

3. Regular fractional factorial designs. The set of treatment combinations F^n for the prime powered factorial forms a group under componentwise addition, and as stated earlier, it may be also viewed as a vector space or finite Euclidean geometry $EG(n, s)$ over the Galois field F . A fractional factorial design D in F^n will be called regular if and only if D is a subspace or coset of a subspace of the space F^n , and otherwise it is called irregular. Thus D will be regular if and only if it is the solution set to the consistent matrix equation $Ax = y$, where the $r \times n$ matrix A has r independent rows with elements in $GF(s) = F$, i.e. D is a subspace or coset of a subspace of F^n . Geometrically a regular fraction is an $(n-r)$-flat of $EG(n, s)$ with cardinality equal to s^{n-r} , $0 \le r \le n$. Note that when D is a regular design of F^n , then we may always write it as:

(3.1) $$D = z' + D_0 = \{z' + t';\ t' \in D_0\} ,$$

for a suitable sub-vector space D_0 of F^n and a suitable vector z' in F^n .

The definition of a regular fraction adopted in this paper depends only on the fraction and not on a specific parametric vector β_1 to be estimated under the assumption that β_2 is negligible. For other definitions, see Addelman [1961], Bose and Srivastava [1964] and Banerjee and Federer [1963]. The majority of authors appear implicitly to agree on the meaning of a regular fraction because the notation s^{n-r} for fractional factorials is reserved in the literature for regular fractions. Some authors specifically eliminate irregular s^{-r} fractions of s^n factorials from their discussions, because they prefer an s^{n-r} fraction. It is not clear how one would label an s^{-r} fraction which was not an s^{n-r} (regular) fraction. In regular s^{n-r} fractions certain factorial effects are often set equal to zero by statisticians, but it is not the statisticians' prerogative to do this for all experimenters. Indeed the experimenter must choose what he wants and this is the real art of experimental design as Youden [1961] has pointed out. This same author gives a detailed example of an irregular 2^{-3} fraction of a 2^7 factorial where the assumption on the nonexistence of two-factor interactions is in doubt. In this same example, each main effect is partially confounded with six two-factor interactions. The loss of information due to partial confounding is compensated for since the experimenter can identify the existence of interactions if they are present. Youden [1961] concludes that the choice of irregular s^{-r} fractions of an s^n factorial can lead to clarification of nonexistence assumptions of certain factorial effects.

Irregular fractions, not necessarily of cardinality s^{n-r}, are of course also celebrated in the literature, e.g. see Plackett and Burman [1946], Srivastava and Anderson [1970] and Srivastava and Chopra [1975].

The following theorem establishes the uniqueness of the GDR for regular designs under the geometric definition of factorial effects.

Theorem 3.1. Let D be a regular design in F^n. Then the GDR corresponding to D and $\beta' = (\beta_1' \vdots \beta_2')$ is unique if the design matrix $X_{D,\beta}$ is obtained using the geometric definition of factorial effects.

Proof. Let $F = \{0, 1, x, \ldots, x_{s-2}\}$ be the Galois field with s elements. In light of (3.1) we may without loss of generality assume D to be a subspace of the vector space F^n. If D is regular then by Theorem 2.1 we need to show that every column of $X_{D,\beta}$ is a multiple of $\mathbf{1}$ or is orthogonal to $\mathbf{1}$. Select an effect $\alpha = A_1^{a_1} A_2^{a_2} \ldots A_n^{a_n}$ with $a' = (a_1, a_2, \ldots, a_n) \in F^n$. Since we are considering the geometric definition for factorial effects, we may without loss assume $a_1 = 1$.

Let $D_0 = \{t' : t' = (t_1\ t_2 \ldots t_n) \in D,\ \sum_{i=1}^{n} a_i t_i = 0\}$, that is, the set of all vectors in D orthogonal to a'. Let $D_1 =$ $\{t' : t' \in D,\ \sum_{i=1}^{n} a_i t_i = 1\}$ and for $j = 2, \ldots, s-2$ let $D_{x_j} = \{t' : t' \in D,\ \sum_{i=1}^{n} t_i a_i = x_j\}$. If $D_0 = D$, then it follows from the geometric definition of factorial effects that the column of $X_{D,\beta}$ determined by D and α is a multiple of $\mathbf{1}$. If $D_0 \neq D$, then select a vector $y' = (y_1\ y_2 \ldots y_n) \in D - D_0$ and suppose $y'a = \sum_{i=1}^{n} a_i y_i = x_{k_0}$. Since D is regular $x_{k_0}^{-1} y' =$

$(x_{k_0}^{-1}y_1\ x_{k_0}^{-1}y_2 \ldots x_{k_0}^{-1}y_n) \in D$ and indeed also in D_1. Moreover for any j, $2 \le j \le s-2$, $x_j x_{k_0}^{-1} y' \in D_{x_j}$. We may thus write $D = D_0 \cup D_1 \cup D_{x_2} \cup \ldots \cup D_{x_{s-2}}$, a disjoint union, and further $D_1 = x_{k_0}^{-1}y' + D_0$, $D_{x_j} = (x_j x_{k_0}^{-1})y' + D_0$. That is, each component in the union is a coset of the subspace D_0. In particular $|D_0| = |D_1| = |D_{x_j}|$ for each j, $j = 2, 3, \ldots, s-2$. Since the basic coefficient matrix of order $s \times s$ used to obtain the design matrix $X_{D,\beta}$ is such that the sum of the entries in any column besides the first is zero, it follows that the column vector determined by D and α in $X_{D,\beta}$ is orthogonal to $\mathbf{1}$.

The reader should note that the definition of the GDR for regular *where defined* fractions leads to the well known DC (under the geometric definition of *except in summary* factorial effects), which has a specific and different meaning (e.g. see Cochran and Cox [1957]). *explain*

Example 3.1. Let $F = \{0, 1, 2\}$ be the three element Galois field under addition and multiplication modulo 3. Consider the 3^2 factorial and let $D = \{(00), (11), (22)\}$. Then D is a regular design of F^2. Using as our basic orthogonal matrix the matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

then under the geometric definition of factorial effects the design matrix is equal to:

$$X_{D,\beta} = \begin{bmatrix} 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 0 & 1 & -1 & -2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & -1 & 1 & 1 & -2 \end{bmatrix},$$

where $\beta' = (\mu, A_1^0A_2^1, A_1^0A_2^2, A_1^1A_2^0, A_1^1A_2^1, A_1^1A_2^2, A_1^2A_2^0, A_1^2A_2^1, A_1^2A_2^2)$. It may now be verified that the GDR is unique.

This same example can be used to show the falsity of Theorem 3.1 under the product definition of factorial effects. Using the same basic orthogonal matrix we obtain the following design matrix for D under the product definition of β :

$$X_{D,\beta} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The GDR is not unique since, for example, the final column of the design matrix is neither a multiple of $\mathbf{1}$ nor is orthogonal to $\mathbf{1}$.

The next example illustrates that the converse of Theorem 3.1 does not hold, that is, the uniqueness of the GDR need not ensure that the design D is regular, even if the geometric definition for factorial effects is used.

Example 3.2. Consider the 4^2 factorial with levels from the 4 element Galois field $F = \{0, 1, x, x+1\}$ with the usual modd. $(2, x^2 + x + 1)$ arithmetic. Let $D = \{(00), (10)\}$; then D is an irregular design of F^2 . Let the basic matrix used for the geometric definition of factorial effects be

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & 0 & 0 & -3 \end{bmatrix} .$$

We may then verify that the design matrix $X_{D,\beta}$ is equal to 2×16 matrix:

$$\begin{array}{cc}
 & \begin{array}{cccccccccccccccc} \mu & A_1^0A_2^1 & A_1^0A_2^x & A_1^0A_2^{x+1} & A_1^1A_2^0 & A_1^1A_2^1 & A_1^1A_2^x & A_1^1A_2^{x+1} & A_1^xA_2^0 & A_1^xA_2^1 & A_1^xA_2^x & A_1^xA_2^{x+1} & A_1^{x+1}A_2^0 & A_1^{x+1}A_2^1 & A_1^{x+1}A_2^x & A_1^{x+1}A_2^{x+1} \end{array} \\
\begin{array}{c} 00 \\ 10 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}
\end{array} .$$

Hence by Theorem 2.1 the GDR is unique, but D is not regular. However note that D consists of half the regular design $D^* = \{(00), (10), (x0), (x+10)\}$.

4. <u>Orthogonal fractional factorial designs</u>. Let D be a design in S . Then D is said to be <u>locally orthogonal</u> relative to a particular parametric sub-vector β_1 in the partitioning $\beta' = (\beta_1' \vdots \beta_2')$ under the assumption that β_2 is negligible if $X_1'X_1$ is a diagonal matrix, i.e. when the estimators of the elements of β_1 are uncorrelated. A design D in S is said to be <u>globally orthogonal</u> if $X_1'X_1$ is diagonal for

every choice of β_1 in the partitioning $\beta_1' = (\beta_1' \vdots \beta_2')$ under the assumption that each time the remaining parametric vector β_2 is negligible.

As an illustration of these concepts consider Examples 3.1 and 3.2. In Example 3.1 the given regular design $D = \{(00), (11), (22)\}$ is locally orthogonal relative to $\beta_1' = (\mu, A_1^0A_2^1, A_1^0A_2^2)$ under both the geometric and product definitions of factorial effects. However it is not locally orthogonal relative to $\beta_1' = (\mu, A_1^2A_2^0, A_1^2A_2^2)$ under both definitions. It follows that D is not globally orthogonal. This example demonstrates that in general a regular fraction need not be locally orthogonal nor globally orthogonal. However, as will be seen shortly, regular fractions of the 2^n factorial are globally orthogonal. In Example 3.2 the design $D = \{(00), (10)\}$ is a simple example of a globally orthogonal design.

Example 4.1. Consider the 2^3 factorial with the Galois field $F = \{0, 1\}$ and the usual modulo 2 arithmetic. The fraction $D = \{(000), (100), (010), (001)\}$ of F^3 is not a 2-flat of $EG(3, 2)$ and hence it is irregular. Let $\beta_1' = (\mu, A_1^1A_2^1A_3^0, A_1^1A_2^0A_3^1, A_1^0A_2^1A_3^1)$ and $\beta_2' = (A_1^1A_2^0A_3^0, A_1^0A_2^1A_3^0, A_1^0A_2^0A_3^1, A_1^1A_2^1A_3^1)$, then under the assumption that β_2 is negligible and the basic matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ under the product definition of factorial effects, the BLUE of β_1 is equal to:

$$\hat{\beta}_1 = [X_1'X_1]^{-1}X_1'Y_D = \begin{bmatrix} \hat{\mu} \\ \widehat{A_1^1A_2^1A_3^0} \\ \widehat{A_1^1A_2^0A_3^1} \\ \widehat{A_1^0A_1^1A_3^1} \end{bmatrix} = \begin{bmatrix} 1/4 & & & \\ & 1/4 & & \\ & & 1/4 & \\ & & & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} Y_D .$$

Hence D is locally orthogonal relative to β_1. It may be verified that D is not a globally orthogonal design.

The following theorem provides a characterization of globally orthogonal designs.

Theorem 4.1. A design D of S is globally orthogonal if and only if any two columns of the design matrix $X_{D,\beta}$ are multiples of or orthogonal to each other. i.e. complete confounding of effects

Proof. Suppose that any two columns of the design matrix $X_{D,\beta}$ are multiples of or orthogonal to each other. Then the design matrix $X_{D,\beta}$ can be written as:

$$X_{D,\beta} = [W_1 \vdots W_2] ,$$

where W_1 is a $k \times m$ columnwise orthogonal matrix with the first column consisting of 1's, m being equal to the number of mutually orthogonal vectors in $X_{D,B}$, and W_2 consists of multiples of vectors of W_1. Since β_1 is selected in such a way that X_1 must be of full column rank it follows that every feasible choice of β_1 leads to a design matrix of the type

$$X_1 = [1 \vdots \tilde{X}_1],$$

where $\tilde{X}_1$ consists of N_1-1 columns of W_1 excepting the first column or of substitutions of corresponding multiples in W_2. It follows from the assumption that $X_1'X_1$ is diagonal. Hence D is globally orthogonal. The converse of the theorem follows directly from the assumption that D is globally orthogonal.

We now turn to the special case of the 2^n factorial. Let $F = \{0, 1\}$ be the two element Galois field and let D be a design of F^n. Let $X_{D,\beta}$ and $X^*_{D,\beta}$ be the design matrices obtained by the geometric and product definitions of factorial effects respectively, with the coefficient matrix being the 2×2 matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then one may show that there exists a diagonal matrix V of order $2^n \times 2^n$ such that $X_{D,\beta}V = X^*_{D,\beta}$. Thus in the special case of the 2^n factorial the two design matrices are in this sense equivalent. The following result is valid no matter which of the two definitions of factorial effects is used to obtain the design matrix $X_{D,\beta}$.

<u>Theorem 4.2</u>. Let D be a regular design in F^n, where $F = \{0, 1\}$ is the two element Galois field. Then D is a globally orthogonal design.

<u>Proof</u>. We must show that any two columns of the design matrix $X_{D,\beta}$ are multiples of each other or are orthogonal to each other. Let $\alpha_1 = A_1^{a_1}A_2^{a_2} \dots A_n^{a_n}$ and $\alpha_2 = A_1^{b_1}A_2^{b_2} \dots A_n^{b_n}$ be any two effects with $a' = (a_1\ a_2 \dots a_n)$, $b' = (b_1\ b_2 \dots b_n)$ in F^n, where $F = \{0, 1\}$ is the two element Galois field. Suppose that the columns of the design matrix $X_{D,\beta}$ determined by the regular design D and the effects α_1 and α_2 are not multiples of each other. Then we must show that these columns are orthogonal to each other. Let $C_0 = \{t' : (t_1\ t_2 \dots t_n)$

ϵ D such that $\sum_{i=1}^{n} t_i a_i = 0\}$, $C_1 = \{t' : t' = (t_1 \, t_2 \, \ldots \, t_n) \in D$ such that $\sum_{i=1}^{n} t_i a_i = 1\}$, $D_0 = \{t' : t' = (t_1 \, t_2 \, \ldots \, t_n) \in D$ such that $\sum_{i=1}^{n} t_i b_i = 0\}$, and $D_1 = \{t' : t' = (t_1 \, t_2 \, \ldots \, t_n) \in D$ such that $\sum_{i=1}^{n} t_i b_i = 1\}$. Since D is a subspace of F^n in view of (3.1) it follows that C_0 is a subspace of D , C_1 is a coset of C_0 and $D = C_0 \cup C_1$. Similarly, D_0 is a subspace of D , D_1 is a coset of D_0 and $D = D_0 \cup D_1$. Hence D is a pairwise disjoint union: $D = (C_0 \cap D_0) \cup (C_0 \cap D_1) \cup (C_1 \cap D_0) \cup (C_1 \cap D_1)$, and it may be verified that $C_0 \cap D_1$, $C_1 \cap D_0$, $C_1 \cap D_1$ are pairwise distinct cosets of the subspace $C_0 \cap D_0$. In particular we have:

$$(4.1) \qquad |C_0 \cap D_0| = |C_i \cap D_j| \, , \quad \text{for all } i, j \in \{0, 1\} .$$

Let $h = |C_0 \cap D_0|$, the cardinality of $C_0 \cap D_0$. Then since the basic matrix is $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, the geometric definition of factorial effects implies using (4.1) that there are h pairs each of (-1,-1), (-1, 1), (1, -1) and (1, 1) between the two columns of $X_{D,\beta}$ determined by D and α_1 and, D and α_2 . It follows that these columns are orthogonal, which proves the theorem.

To illustrate this theorem we provide the following example.

Example 4.2. Consider the 2^3 factorial with the Galois field $F = \{0, 1\}$ and modulo 2 arithmetic, and take the regular design $D = \{(000), (110), (101), (011)\}$. The design matrix $X_{D,\beta}$, using the basic matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, is equal to:

$$
\begin{array}{c}
 \\
(000) \\
(110) \\
(101) \\
(011)
\end{array}
\begin{array}{c}
\begin{array}{cccccccc}
\mu & A_1^1A_2^0A_3^0 & A_1^0A_2^1A_3^0 & A_1^0A_2^0A_3^1 & A_1^1A_2^1A_3^0 & A_1^1A_2^0A_3^1 & A_1^0A_2^1A_3^1 & A_1^1A_2^1A_3^1
\end{array} \\
\begin{bmatrix}
1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & -1 & 1 & -1
\end{bmatrix}
\end{array} .
$$

The reader may verify that D is globally orthogonal.

5. <u>Group theoretic generation of the aliasing structure</u>. Let $\alpha_1 = A_1^{x_1} A_2^{x_2} \ldots A_n^{x_n}$ and $\alpha_2 = A_1^{y_1} A_2^{y_2} \ldots A_n^{y_n}$ be factorial effects of β, x_i, y_i in S_i for each i or x_i, y_i in the Galois field F for each i. Then by $\alpha_1 \cdot \alpha_2$ we mean the effect $A_1^{z_1} A_2^{z_2} \ldots A_n^{z_n}$, where for each i : $z_i = x_i + y_i$ with the addition modulo s_i if x_i, y_i are in S_i and the addition in F if x_i, y_i are in F. With multiplication on effects, the vector β as a set of effects forms an Abelian group and the identity element of this group is μ.

For a given design D of S and a particular partition $\beta' = (\beta_1' \vdots \beta_2')$ of the parametric vector suppose that the GDR is explicitly given by

$$
E[\hat{\mu}] = \mu + \sum_{i=1}^{q} c_i A_1^{v_{i1}} A_2^{v_{i2}} \ldots A_n^{v_{in}} , \tag{5.1}
$$

where $\{(v_{i1}\ v_{i2} \ldots v_{in})$, $i = 1, 2, \ldots, q\}$ is a q-subset of S or F^n under the appropriate arithmetics as indicated above. Let $A_1^{w_1} A_2^{w_2} \ldots \ldots A_n^{w_n}$ be an arbitrary effect of β_1. If the element of the aliasing structure $\beta_1 + A_{D,\beta}\beta_2$ corresponding to the effect $A_1^{w_1} A^{w_2} \ldots A_n^{w_n}$ is given by

$$(5.2) \qquad E[\widehat{A_1^{w_1} A_2^{w_2} \ldots A_n^{w_n}}] = A_1^{w_1} A_2^{w_2} \ldots A_n^{w_n}$$

$$+ \sum_{i=1}^{n} c_i \, A_1^{v_{i1}+w_1} A_2^{v_{i2}+w_2} \ldots A_n^{v_{in}+w_n} \; ,$$

then the aliasing structure of D is said to be locally group-theoretically generated. It is globally group-theoretically generated if (5.2) is true for every partitioning $\beta' = (\beta_1' \vdots \beta_2')$.

One might expect that when D is a regular design in F^n , and the design matrix is obtained by using the geometric definition for factorial effects, then since the GDR is unique the aliasing structure is globally group-theoretically generated. This, however, is not the case in general and an example is given below to illustrate this fact. We then show that in the special setting of the 2^n factorial the aliasing structure corresponding to a regular design is globally group-theoretically generated.

Example 5.1. Consider the example which was introduced in Example 3.1, namely, the 3^2 factorial with $F = \{0, 1, 2\}$ and modulo 3 arithmetic. The regular design introduced there was $D = \{(00), (11), (22)\}$. Using the geometric definition of factorial effects and the same basic matrix one may verify that for the partitioning $\beta' = (\beta_1' \vdots \beta_2')$ with $\beta_1' = (\mu, A_1^0A_2^1, A_1^0A_2^2)$, and $\beta_2' = (A_1^1A_2^2, A_1^2A_2^1, A_1^1A_2^0, A_1^2A_2^0, A_1^1A_2^1, A_1^2A_2^2)$ the GDR is $\mu + A_1^2A_2^1 - A_1^1A_2^2$ and the aliasing structure is

$$\beta_1 + A_{D,\beta}\beta_2 = \begin{bmatrix} \mu + A_1^2A_2^1 - A_1^1A_2^2 \\ A_1^0A_2^1 + A_1^1A_2^0 + \frac{1}{2} A_1^1A_2^1 - \frac{3}{2} A_1^2A_2^2 \\ A_1^0A_2^2 + A_1^2A_2^0 - \frac{1}{2} A_1^1A_2^1 - \frac{1}{2} A_1^2A_2^2 \end{bmatrix} .$$

It is clear that the second and third entries in the aliasing structure do not arise from group-theoretic generation from the GDR. Hence D is not locally nor globally group-theoretically generated.

Theorem 5.1. If D is a regular design of F^n, where $F = \{0, 1\}$ is a two element Galois field, then the aliasing structure of D relative to $\beta' = (\beta_1' \vdots \beta_2')$ is globally group-theoretically generated.

Proof. Let $\gamma_0 = \{\alpha : \alpha = A_1^{a_1} A_2^{a_2} \ldots A_n^{a_n} \ni \sum_{i=1}^{n} a_i y_i = 0$ for each $y' = (y_1\ y_2 \ldots y_n) \in D\}$, where D in view of (3.1) is taken as a subspace of F^n. Then γ_0 is a subgroup of the Abelian group β of all factorial effects under the multiplication of effects as introduced above. Write $\beta = \gamma_0 \cup \gamma_1 \cup \ldots \cup \gamma_m$ as a pairwise disjoint union, where $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ are the distinct cosets of γ_0 in β. Then for each j, $1 \le j \le m$, there exist effects α_j such that $\gamma_j = \alpha_j \gamma_0 = \{\alpha_j \alpha : \alpha \in \gamma_0\}$. We now have the following: (i) it can be easily verified that for each j the columns of $X_{D,\beta}$ corresponding to the effects in γ_j are precisely the columns which are multiples of each other, (ii) by Theorem 4.2 any two columns of $X_{D,\beta}$ corresponding to effects α_1, α_2 chosen from two distinct cosets among the cosets $\gamma_0, \gamma_1, \ldots, \gamma_m$ are orthogonal, (iii) the columns of $X_{D,\beta}$ corresponding to the effects in γ_0 are multiples of $\mathbb{1}$, and since D is regular it follows from Theorem 3.1 that each of the columns corresponding to the effects in $\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_m$ is orthogonal to $\mathbb{1}$. Hence the GDR corresponding to D and any partition of $\beta' = (\beta_1' \vdots \beta_2')$ is a linear combination of the effects in γ_0. Furthermore, it follows from (i), (ii), and (iii) above that any entry in the aliasing structure $\beta_1 + A_{D,\beta}\beta_2$ is a linear combination of the effects in exactly one of the γ_j . Since $\gamma_j = \alpha_j \gamma_0$, it follows that the aliasing structure is globally group-theoretically generated.

Whether the converse of Theorem 5.1 is true or false has not been settled yet. The following example shows the abundance of local group-theoretic generation of the AS.

Example 5.2. Consider the 2^2 factorial with underlying Galois field $F = \{0, 1\}$ and basic matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Let $D = \{(00), (10), (01)\}$, which is an irregular design. Among the four parameters in $\beta' = (\mu\ A_1^1A_2^0\ \ A_1^0A_2^1,\ A_1^1A_2^1)$ there are six partitions to be considered, namely:

1. $\beta' = (\mu\ A_1^1A_2^0 : A_1^0A_2^1\ A_1^1A_2^1)$,

2. $\beta' = (\mu\ A_1^0A_2^1 : A_1^1A_2^0\ A_1^1A_2^1)$,

3. $\beta' = (\mu\ A_1^1A_2^1 : A_1^1A_2^0\ A_1^0A_2^1)$,

4. $\beta' = (\mu\ A_1^1A_2^0\ A_1^0A_2^1 : A_1^1A_2^1)$,

5. $\beta' = (\mu\ A_1^1A_2^0\ A_1^1A_2^1 : A_1^0A_2^1)$,

and 6. $\beta' = (\mu\ A_1^0A_2^1\ A_1^1A_2^1 : A_1^1A_2^0)$.

The calculation of the corresponding GDR's are summarized below:

Partition	GDR
1	$\left\{\begin{array}{r} \mu - \frac{1}{2} A_1^0A_2^1 - \frac{1}{2} A_1^1A_2^1 \\ A_1^1A_2^0 - \frac{1}{2} A_1^0A_2^1 - \frac{1}{2} A_1^1A_2^1 \end{array}\right\}$
2	$\left\{\begin{array}{r} \mu - \frac{1}{2} A_1^1A_2^0 - \frac{1}{2} A_1^1A_2^1 \\ A_1^0A_2^1 - \frac{1}{2} A_1^1A_2^0 - \frac{1}{2} A_1^1A_2^1 \end{array}\right\}$
3	$\left\{\begin{array}{r} \mu - \frac{1}{2} A_1^1A_2^0 - \frac{1}{2} A_1^0A_2^1 \\ A_1^1A_2^1 - \frac{1}{2} A_1^1A_2^0 - \frac{1}{2} A_1^0A_2^1 \end{array}\right\}$
4	$\left\{\begin{array}{r} \mu - A_1^1A_2^1 \\ A_1^1A_2^0 - A_1^1A_2^1 \\ A_1^0A_2^1 - A_1^1A_2^1 \end{array}\right\}$
5	$\left\{\begin{array}{r} \mu - A_1^0A_2^1 \\ A_1^1A_2^0 - A_1^0A_2^1 \\ A_1^1A_2^1 - A_1^0A_2^1 \end{array}\right\}$
6	$\left\{\begin{array}{r} \mu - A_1^1A_2^0 \\ A_1^0A_2^1 - A_1^1A_2^0 \\ A_1^1A_2^1 - A_1^1A_2^0 \end{array}\right\}$

The GDR is locally group-theoretically generated for the first three partitions, i.e. in 50% of the cases.

A natural question to ask is whether a globally orthogonal design implies globally group-theoretic generation of the AS. This conjecture is false as the following example demonstrates.

Example 5.3. Consider the globally orthogonal design introduced in Example 3.2 in the case of the 4^2 factorial, i.e. $D = \{(00), (10)\}$. On inspection of the design matrix $X_{D,\beta}$ we observe that there are four possible partitions. Taking $\beta_1' = (\mu, A_1^1A_2^0)$ in the first case we see that the GDR contains exactly eleven effects besides the mean, while the second element of the AS has three effects besides $A_1^1A_2^0$. Hence the AS cannot be locally grouped-theoretically generated.

6. Computation of the GDR and the AS. The AS is given by the expression $\beta_1 + [X_1'X_1]^{-1}X_1'X_2\beta_2$, where the first element is the GDR. From the examples given one might obtain the impression that when n is large and k small, the problem of calculating the GDR or the AS is quite formidable. However Margolin [1967] has given an algorithm which allows us to compute the elements of the GDR in one operation. This algorithm in view of Theorem 5.1 will provide us the AS of any regular fraction of the 2^n factorial.

In the case of orthogonal designs the GDR is obtained in the following way:

$$\begin{aligned} \text{GDR} &= \mu + \frac{1}{k}(1\ 1\ \ldots\ 1)\ X_2\beta_2 \\ &= \mu + \frac{1}{k}\left(\sum_{j=1}^{k} X_{2j}\right)\beta_2 , \end{aligned}$$

where X_{2j} is the j-th row of X_2. Hence the GDR of Plackett-Burman [1946] patterns can be calculated in this way. In the case of globally orthogonal designs one may ignore those effects in β_2 whose columns are orthogonal to $\mathbf{1}$.

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