GENERAL N-ARY BALANCED BLOCK DESIGN

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SUMMARY

The concept of N-ary balanced incomplete block design where the incidence matrix n takes the N-values, $0,1,2,\cdots,N-1$, is extended to general N-ary balanced block designs, where the incidence matrix n^* takes on the N values m_a : $a = 0,1,2,\cdots N-1$ and $m_a = am_1 - (a-1)m_0$ and $0 \leq m_0 < m_1$. For ternary designs $m_2 = 2m_1 - m_0$ and thus $0 \leq m_0 < m_1 < m_2$.

The parameters and necessary conditions for n^* are evaluated. Given a fixed number of units N^* (say) and fixed number of treatments v (say), more than one general N-ary balanced block design for different values of m_a : $a = 0,1,\cdots,N-1$, is possible A criterion for selecting an optimal design from its class is derived.

1. INTRODUCTION

N-ary balanced incomplete block designs were introduced by Tocher [1952]. The number of occurances of treatments in block were $0,1,\cdots,N-1$ with all occurrences being represented. Statistical literature on these designs since their introduction has been confined to designs with these occurrences. We shall generalize these N-ary designs such that the occurrences of a treatment in the block is some non-negative integer $m_0,m_1,\cdots,m_{N-1}$. The generalizations represent a sequel to those given by Shefiq and Federer [1977] for generalized binary balanced block design (GBBBD) and they provide a generalization for the present experimental

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design theory such that the experimenter is provided with many new N-ary balanced block designs. This allows flexibility in the statistical designs, and in the use of all homogeneous material for given block sizes. We shall confine our attention to block designs which are equireplicated and equal sized blocks.

In the next section a presentation of parameters for basic ternary balanced incomplete block design (BTBIBD) and general ternary balanced block design (GTBBD) is made, and some definitions are presented. Some results on the existence of GTBBD and on their optimality are presented in section three. An example, illustrating the results, is presented in the fourth section. In the fifth section, all the previous results are extended to general N-ary balanced block design (GNBBD).

2. PARAMETERS OF BTBIBD AND GTBBD AND SOME DEFINITIONS

Let $(v,b,r,k,\lambda;0,1,2)$ be the parameters of a basic ternary balanced incomplete block design, (BTBIBD), where $2 \leq k$ and where the incidence matrix $\underline{n} = (n_{ij})$ contains only three values for n_{ij}, i.e., 0, 1, and 2. n_{ij} denotes the frequency of the i^{th} treatment in the j^{th} block, $j=1,2,\cdots,b$. Further, let r_a, $a=0,1,2$, denote the number of times an element a appears in the i^{th} row of $\underline{n}$; the occurrences are assumed independent of i. Then the following relations hold:

$$b = r_0 + r_1 + r_2 \tag{2.1}$$

$$r = r_1 + 2r_2 \tag{2.2}$$

$$\sum_{j=1}^{b} n_{ij}n_{\ell j} = r_1 + 4r_2 \quad \text{if} \quad \ell = i \tag{2.3}$$

$$= \lambda \qquad \text{if} \quad \ell \neq i \tag{2.4}$$

$$vr = bk \tag{2.5}$$

$$\lambda(v-1) = r(k-1) - 2r_2 = r(k-2) + r_1 \; . \tag{2.6}$$

To obtain (2.4) note that

$$\sum_{\ell=1}^{v}\left(\sum_{j=1}^{b} n_{ij}n_{\ell j}\right) = \sum_{j=1}^{b} n_{ij} \sum_{\ell=1}^{v} n_{\ell j} = rk$$

and that

$$\sum_{\ell=1}^{v}\left(\sum_{j=1}^{b} n_{ij}n_{\ell j}\right) = \sum_{j=1}^{b}\left(n_{ij}^2 + \sum_{i\neq\ell=1}^{v} n_{ij}n_{\ell j}\right)$$

$$= r_1 + 4r_2 + (v-1)\lambda$$

hence

$$\lambda(v-1) = rk - \sum_{j=1}^{b} n_{ij}^2 = rk - r_1 - 4r_2 = r(k-1) - 2r_2 = r(k-2) + r_1 .$$

In order to fix λ uniquely, note that $r(k-1) - 2r_2$ must be a positive multiple of $v-1$. For example, if $v=5$, $b=15$, $k=4$, $r=12$, consider values of $r_2 = 1,2,3,4$, or 5. If $r_2 = 1,3$ or 5, λ is not an integer. If $r_2=2$, $\lambda=8$ and if $r_2=4$, $\lambda=7$.

Given that $\underline{n}$ is the incidence matrix of BTBIBD with parameters $(v,b,r,k,\lambda;0,1,2)$, the incidence matrix of a GTBBD is defined to be:

$$\underline{n}^* = \underline{n}(m_1-m_0) + \underline{J}m_0 \tag{2.7}$$

where $\underline{J}$ is a $v \times b$ matrix whose elements are all ones and where $0 \leq m_0 < m_1$. The parameters of the GTBBD are $(v,b,r^*,k^*,\lambda^*;m_0,m_1,m_2=2m_1-m_0)$ where

$$r^* = rm_1 + (b-r)m_0 \tag{2.8}$$

$$k^* = km_1 + (v-k)m_0 \tag{2.9}$$

$$(v-1)\lambda^* = r^*(k^*-m_1-m_0) + bm_1m_0 - 2r_2(m_1-m_0)^2 \tag{2.10}$$

$$vr^* = bk^* \tag{2.11}$$

$$v \leq b \tag{2.12}$$

Definition 2.1. A GTBBD is said to be incomplete if $m_0 = 0$; otherwise, it is said to be complete.

To illustrate this definition consider the following two designs:

Design 2.1

$m_0 = 0$, $m_1 = 2$, $m_2 = 4$

$v = b = 3$; $k^* = r^* = 6$

blocks		
1	2	3
A	B	C
A	B	C
A	B	C
A	B	C
B	C	A
B	C	A

Design 2.2

$m_0 = 1$, $m_1 = 2$, $m_2 = 3$

$v = b = 3$; $k^* = r^* = 6$

blocks		
1	2	3
A	A	A
B	B	B
C	C	C
A	B	C
A	B	C
B	C	A

Design 2.1 is incomplete, whereas design 2.2 is complete.

Definition 2.2. A complete GTBBD is said to be orthogonal if $n^*_{ij} = r^*_i k^*_j / N^*$, where N^* is the total number of observations, r^*_i is the number of replications of the i^{th} treatment, k^*_j is the number of entries in the j^{th} block, and n^*_{ij} is the ij^{th} element of $\underline{n}^*$.

Design 2.1 above is incomplete and nonorthogonal, and design 2.2 is complete and nonorthogonal. The following design is both complete and orthogonal.

Design 2.3. $v = 3 = b$, $r^*_1 = 12$, $r^*_2 = 6 = r^*_3$, $N^* = 24$

Blocks		
1	AAAABBCC	$8 = k^*_1$
2	AABC	$4 = k^*_2$
3	AAAAAABBBCCC	$12 = k^*_3$

$$\underline{n}^* = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

$n^*_{11} = 8(12)/24 = 4 \qquad n^*_{12} = 4(12)/24 = 2, \qquad$ etc.

Definition 2.3. A GTBBD is variance balanced if the coefficient matrix $\underset{\sim}{C}^*_{v\times v} = c_1^* \underset{\sim}{I}_{v\times v} + c_2^* \underset{\sim}{J}_{v\times v}$ where c_1^* is the non-zero eigen value of $\underset{\sim}{C}^*$, $c_2^* = c_1^*/v$, $\underset{\sim}{I}$ is the identity matrix, and $\underset{\sim}{C}^* = \mathrm{diag}(r_1^*,\cdots,r_v^*) - \underset{\sim}{n}^* \,\mathrm{diag}\left(\frac{1}{k_1^*},\cdots,\frac{1}{k_b^*}\right) \underset{\sim}{n}^{*\prime}$.

In design 2.1, $\underset{\sim}{C}^* = 4\underset{\sim}{I} - 4\underset{\sim}{J}/3$, and in design 2.2, $\underset{\sim}{C}^* = (33\underset{\sim}{I} - 11\underset{\sim}{J})/6$. Thus, both are variance balanced. However, in design 2.3

$$\underset{\sim}{C}^* = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{8} & & \\ & \frac{1}{4} & \\ & & \frac{1}{12} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 6 & 3 & 3 \end{bmatrix}$$

$$= \frac{3}{2} \begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ -2 & -1 & 3 \end{bmatrix} \neq c_1^* \underset{\sim}{I} + c_2^* \underset{\sim}{J} \; .$$

The design 2.3 is not variance balanced, but it is orthogonal.

3. EXISTENCE AND VARIANCE OPTIMALITY OF GTBBD

Theorem 3.1. The existence of a balanced ternary incomplete block design with parameters $(v,b,r,k,\lambda;\ n_{ij}=0,1,2)$ implies the existence of a GTBBD with parameters $(v,b,r^*,k^*,\lambda^*;\ n_{ij}^* = m_0,m_1,m_2)$.

Proof: From the definition of a GTBBD, note that $\underset{\sim}{n}^* = \underset{\sim}{n}(m_1-m_0) + \underset{\sim}{J}m_0$. The ij^{th} entry of n^* is $n^*_{ij} = n_{ij}(m_1-m_0) + m_0$. Then,

$$\begin{aligned} n_{ij}^* &= m_0 && \text{if } n_{ij} = 0 \\ &= m_1 && \text{if } n_{ij} = 1 \\ &= m_2 = 2m_1 - m_0 && \text{if } n_{ij} = 2 \; . \end{aligned}$$

Starting with a BTBIBD with incidence matrix $\underline{n}$, a GTBBD may be easily constructed by replacing all zeros in the BTBIBD with m_0, all the ones with m_1, and all the twos with m_2. The resulting GTBBD has parameters $(v,b,r^*,k^*,\lambda^*; m_0,m_1,m_2)$ where r^*, k^*, λ^* satisfy equations (2.8) to (2.11). We shall now derive equations (2.8) to (2.11) formally. Let $\underline{1}_b$ and $\underline{1}_v$ denote column vectors whose elements are all ones and whose orders are b and v respectively; now,

$$\underline{n}^*\underline{1}_b = [\underline{n}(m_1-m_0) + \underline{J}m_0]\underline{1}_b$$

$$= [r(m_1-m_0) + bm_0]\underline{1}_v$$

$$= [rm_1 + (b-r)m_0]\underline{1}_v = r^*\underline{1}_v$$

and

$$\underline{n}^{*\prime}\underline{1}_v = [\underline{n}'(m_1-m_0) + \underline{J}'m_0]\underline{1}_v$$

$$= [k(m_1-m_0) + vm_0]\underline{1}_b$$

$$= [km_1 + (v-k)m_0]\underline{1}_b = k^*\underline{1}_b \quad .$$

Also,

$$\underline{n}^*\underline{n}^{*\prime} = [\underline{n}(m_1-m_0) + \underline{J}m_0][\underline{n}'(m_1-m_0) + \underline{J}'m_0]$$

$$= \underline{n}\underline{n}'(m_1-m_0)^2 + [2r(m_1-m_0)m_0 + bm_0^2]\underline{J} \quad .$$

The $(i\ell)^{th}$ entry of $\underline{n}^*\underline{n}^{*\prime}$ for $\ell \neq i$, is denoted by λ^* and is written as $\lambda^* = \lambda(m_1-m_0)^2 + 2r(m_1-m_0)m_0 + bm_0^2$, where $\lambda = [r(k-1) - 2r_2]/(v-1)$. Then,

$$(v-1)\lambda^* = (rk-r-2r_2)(m_1-m_0)^2 + (v-1)(2r_2(m_1-m_0)m_0 + bm_0^2)$$

$$= (r^*-bm_0)(k^*-vm_0) - (r^*-bm_0)(m_1-m_0) - 2r_2(m_1-m_0)^2$$

$$+ 2(v-1)(r^*-bm_0)m_0 + (v-1)bm_0^2$$

$$= r^*(k^*-m_1-m_0) + bm_1m_0 - 2r_2(m_1-m_0)^2 \quad .$$

When $m_0 = 0$ and $m_1 = 1$, equation (2.12) is known as Fisher's inequality. We generalize his inequality here. To prove (2.12) for $0 \le m_0 < m_1 < m_2 = 2m_1 - m_0$, note that $\lambda^*_{ii} + (v-1)\lambda^* = r^*k^* = \Sigma_{\ell=1}^{v} \Sigma_{j=1}^{b} n^*_{ij}n^*_{i\ell}$, where λ^*_{ii} is the i^{th} diagonal entry of $\underline{n}^*\underline{n}^{*\prime}$ and is the same for all i. Thus, $\underline{n}^*\underline{n}^{*\prime} = (r^*k^*-\lambda^*v)\underline{I} + \lambda^*\underline{J}$. The determinant of $\underline{n}^*\underline{n}^{*\prime}$ is $|\underline{n}^*\underline{n}^{*\prime}| = r^*k^*(r^*k^*-\lambda^*v)^{v-1} = r^*k^*(r-\lambda+2r_2)^{v-1}(m_1-m_0)^{2(v-1)}$, since $(r^*k^*-\lambda^*v) = (rk-\lambda v)(m_1-m_0)^2$ and since $rk - \lambda v = r - \lambda + 2r_2 = \lambda_{ii} - \lambda$; where $\lambda_{ii} = \Sigma_{j=1}^{b} n_{ij}^2 = r_1 + 4r_2$ from (2.3) we know that $r - \lambda + 2r_2 > 0$. This is because λ_{ii} must be greater than or equal to $\Sigma_{j=1}^{b} n_{ij}n_{\ell j}$, $\ell \ne i$, because the numbers r_0, r_1 and r_2 of zeros, ones and twos, respectively, are independent of the i^{th} treatment, and correlation can only be one if the symbols in rows i and ℓ are identical. But, this would mean that rank of $\underline{n}^*\underline{n}^{*\prime}$ is less than v, since two rows would be identical. This is impossible, since the BTBIBD we started with was connected and had no two rows of $\underline{n}$ identical. Thus, the rank of $\underline{n}^*\underline{n}^{*\prime}$ is v. Now, $\underline{n}^*$ is v x b and has rank less than or equal to the minimum of v and b. Also, the rank of a product of two matrices is less than or equal to the minimum of the rank of the two matrices. Hence, since the rank of $\underline{n}^*\underline{n}^{*\prime}$ is v, the $v \le b$ and the Fisher's inequality is proved for GTBBD.

Under the assumptions of homoscedasticity and usual linear model theory, the coefficient matrix for obtaining solution for the treatment effect of a GTBBD is

$$\underline{C}^* = r^*\underline{I} - \underline{n}^*\underline{n}^{*\prime}/k^* = r^*\underline{I} - (r^*k^*-\lambda^*v)\underline{I}/k^* - \lambda^*\underline{J}/k^* = \lambda^*(v\underline{I}-\underline{J})/k^* \; . \quad (3.1)$$

This form is identical to the coefficient matrix $\underline{C}$ of the BTBIBD when * is dropped. The rank of $\underline{C}^*$ is (v-1) and the covariance matrix (intrablock) of treatment effects is $\sigma^2 k^* \underset{\sim}{I} / \lambda^* v$, when the restraint that the sum of the treatment effects equal zero is utilized.

In the class of all equireplicated and equi-sized block GTBBD the question arises as to which one(s) of these balanced designs has(have) the smallest variance. This problem is not encountered in the case of the BTBIBD, since there is only one variance. The same situation arises for the binary designs discussed by Shafiq and Federer [1977]. Now, as may be noted from the definition of the GTBBD, there are many possible values for m_0 and m_1. In the search of an optimal design in the class, note that maximizing the quantity $\lambda^* v/k^*$ will minimize the variance of estimable treatment effects. Since v is constant in the class, we need only confine our attention to λ^*/k^*. Of course, the comparison is made among designs having fixed N^* or r^* as $r^* v = N^*$. The following theorem is in this spirit.

Theorem 3.2. In the class of all equireplicated and equi-sized blocks GTBBD with parameters $(v, b_d, r^*, k_d^*, \lambda_d^*; m_{0d}, m_{1d}, m_{2d})$, the design(s) having the minimal value of $[r_d(b_d - r_d) + 2b_d r_{2d}](m_{1d} - m_{0d})^2$ is(are) optimal in the sense of A-, D-, and E-optimality.

Proof: The three criteria of variance optimality known as A-optimality, D-optimality, and E-optimality involve functions of the non-zero eigen values of the coefficient matrix $\underset{\sim}{C}^*$ for treatment effects. Let γ_g, $g=1,2,\cdots,v-1$, be the set of non-zero eigen values of $\underset{\sim}{C}^*$. Then, the various optimalities in terms of γ_g are:

i) A-optimality: $f_A(\underset{\sim}{C}^*) = \sum_{g=1}^{v-1} \gamma_g^{-1}$

ii) D-optimality: $f_D(\underset{-}{C}^*) = \prod_{g=1}^{v-1} \gamma_g^{-1}$

iii) E-optimality: $f_E(\underset{-}{C}^*) = \max_{1 \le g \le v-1} \gamma_g^{-1}$.

Kiefer [1958, 1959] and others have presented discussions on these and other various optimality criteria.

In the case of GTBBD the v-1 non-zero eigen values of $\underset{-}{C}_d^*$ are all equal to $v\lambda_d^*/k_d^* = \gamma_d$ for each g. Therefore, by minimizing λ_d^*/k_d^* all the three criteria will be achieved. Thus

$$\max_d (\lambda_d^*/k_d^*) \equiv \max_d \left[r^* - \frac{r^*(m_{1d}+m_{0d}) - b_d m_{1d} m_{0d} + 2r_{2d}(m_{1d}-m_{0d})^2}{k_d^*} \right]$$

$$\equiv \min_d [r^* b_d(m_{1d}+m_{0d}) - b_d^2 m_{1d} m_{0d} + 2b_d r_d (m_{1d}-m_{0d})^2]$$

$$\equiv \min_d [r^{*2} - (r^*-m_{1d}b_d)(r^*-m_{0d}b_d) + 2b_d r_d (m_{1d}-m_{0d})^2]$$

$$\equiv \min_d [r^{*2} + r_d(b_d-r_d)(m_{1d}-m_{0d})^2 + 2b_d r_d (m_{1d}-m_{0d})^2]$$

since $(r^*-m_{0d}b_d) = r_d(m_{1d}-m_{0d})$ and $(m_{1d}b_d-r^*) = (b_d-r_d)(m_{1d}-m_{0d})$. Therefore,

$$\max_d (\lambda_d^*/k_d^*) \equiv \min_d [r_d(b_d-r_d) + 2b_d r_d)(m_{1d}-m_{0d})^2] .$$

The following two corollaries follow directly from theorem 3.2.

Corollary 3.1. In a subclass of GTBBD with parameters $(v,b_d,r^*,k_d^*,\lambda_d^*;m_{0d},m_{1d},m_{2d})$ and derived from BTBIBD's with parameters $(v,b_d,r_d,k_d,\lambda_d;0,1,2)$ for the d^{th} design, in which the difference $(m_{1d}-m_{0d})$ is a constant, the one(s) having a minimal value of $r_d(b_d-r_d) + 2b_d r_{2d}$ is(are) optimal.

Corollary 3.2. In a subclass of GTBBD with parameters $(v,b_d,r^*,k_d^*,\lambda_d^*;m_{0d},m_{1d},m_{2d})$ and derived from BTBIBD's with parameters $(v,b_d,r_d,k_d,\lambda_d;0,1,2)$, in which the quantity $r_d(b_d-r_d) + 2b_d r_{2d}$ is constant, the design(s) having minimal value of $(m_{1d}-m_{0d})$ is(are) optimal.

4. EXAMPLE

The following BTBIBD's are used to construct GTBBD's with v = 4 and $r^* = 45$.

BTBIBD-1

Treatment	Blocks 1	2	3	4	5	6	7	8	9	10
A	2	0	0	0	1	1	1	0	0	0
B	0	2	0	0	1	0	0	1	1	0
C	0	0	2	0	0	1	0	1	0	1
D	0	0	0	2	0	0	1	0	1	1

$v = 4$ $b = 10$

$r = 5$ $k = 2$

$\lambda = 1$ $r_2 = 1$

BTBIBD-2

Treatment	Blocks 1	2	3	4	5	6	7	8	9	10	11	12
A	2	2	2	1	1	1	0	0	0	0	0	0
B	1	0	0	2	0	0	2	2	1	1	0	0
C	0	1	0	0	2	0	1	0	2	0	2	1
D	0	0	1	0	0	2	0	1	0	2	1	2

$v = 4$ $b = 12$

$r = 9$ $k = 3$

$\lambda = 4$ $r_2 = 3$

BTBIBD-3

Treatment	Blocks 1	2	3	4	5	6	7	8	9
A	2	2	2	1	1	1	0	0	0
B	2	0	0	1	1	1	2	2	0
C	0	2	0	1	1	1	2	0	2
D	0	0	2	1	1	1	0	2	2

$v = 4$ $b = 9$

$r = 9$ $k = 4$

$\lambda = 7$ $r_2 = 3$

TABLE 4.1: GTBBD for $v = 4$ and $r^* = 45$.

BTBIBD	Parameters of BTBIBD					Parameters of GTBBD					Optimality Measures		
	b_d	r_d	k_d	λ_d	r_{2d}	k_d^*	λ_d^*	m_{0d}	m_{1d}	m_{2d}	$m_{1d}-m_{0d}$	I_d^*	II_d^*
1	10	5	2	1	1	18	201	4	5	6	1	45	1
1	10	5	2	1	1	18	189	3	6	9	3	45	9
1	10	5	2	1	1	18	165	2	7	12	5	45	25
1	10	5	2	1	1	18	129	1	8	15	7	45	49
1	10	5	2	1	1	18	81	0	9	18	9	45	81
2	12	9	3	4	3	15	166	3	4	5	1	99	2.2
2	12	9	3	4	3	15	100	0	5	10	5	99	55
3	9	9	4	7	3	20	223	4	5	6	1	54	1.8
3	9	9	4	7	3	20	217	3	5	7	2	54	4.8
3	9	9	4	7	3	20	207	2	5	8	3	54	10.8
3	9	9	4	7	3	20	193	1	5	9	4	54	19.2
3	9	9	4	7	3	20	175	0	5	10	5	54	30.0

$$I_d^* = r_d(b_d - r_d) + 2b_d r_{2d}$$

$$II_d^* = \left(r_d(b_d - r_d) + 2b_d r_{2d}\right)(m_{1d} - m_{0d})^2/r^*.$$

5. PARAMETERS OF BASIC N-ARY AND GENERAL N-ARY DESIGN

Let $(v,b,r,k,\lambda;n_{ij}=0,1,2,\cdots,N-1)$ be the parameters of BNBIBD, where $N-1 \leq k$ and whose incidence matrix $\underline{n}$ takes only N values namely, $0,1,2,\cdots,N-1$. Let n_{ij} denote the frequency of the i^{th} treatment in the j^{th} block. Let r_{ai} denote the frequency of an element a in the i^{th} row of $\underline{n}$. Similarly, let k_{aj} denote the frequency of element a in the j^{th} column of $\underline{n}$. We assume that $r_{ai} = r_a$ for all i. Thus,

$$b = \sum_{a=0}^{N-1} r_a \tag{5.1}$$

$$v = \sum_{a=0}^{N-1} k_{aj} \tag{5.2}$$

$$r = \sum_{a=0}^{N-1} a r_a \tag{5.3}$$

$$k = \sum_{a=0}^{N-1} a k_{aj} \tag{5.4}$$

$$\sum_{j=1}^{b} n_{ij} n_{\ell j} = \sum_{a=0}^{N-1} a^2 r_a \quad \text{if} \quad \ell = i \tag{5.5}$$

$$= \lambda \qquad\qquad \text{if} \quad \ell \neq i \,. \tag{5.6}$$

Necessary conditions:

$$vr = bk \tag{5.7}$$

$$(v-1)\lambda = r(k-1) - \sum_{a=0}^{N-1} a(a-1) r_a = \sum_{a=0}^{N-1} a(k-a) r_a \,. \tag{5.8}$$

Equations (5.1) to (5.7) are obvious and (5.8) is derived as follows:

$$\lambda(v-1) = rk - \sum_{j=1}^{N-1} n_{ij}^2 = rk - \sum_{a=0}^{N-1} a^2 r_a$$

$$= r(k-1) - \sum_{a=0}^{N-1} a(a-1) r_a$$

or we can express it as

$$\lambda(v-1) = rk - \sum_{a=0}^{N-1} a^2 r_a = \sum_{a=0}^{N-1} a r_a k - \sum_{a=0}^{N-1} a^2 r_a$$

$$= \sum_{a=0}^{N-1} a(k-a) r_a \ .$$

Some restrictions on r_a could easily be imposed. For example,

$$r_a < \left[r - \sum_{i=1}^{N-a-1} (a+i) r_{a+i}\right]/a \quad \text{for } a=2,\cdots,N-1, \tag{5.9}$$

and only those values of r_a for which λ is integer are algebraically possible.

Given $\underline{n}$, we define the incidence matrix of GNBBD to be

$$\underline{n}^* = \underline{n}(m_1 - m_0) + \underline{J} m_0 \tag{5.10}$$

where m_0 and m_1 are non-negative integers such that $0 \leq m_0 < m_1$ and $\underline{J}$ is a $v \times b$ matrix whose elements are all ones. The parameters of GNBBD are $(v, b, r^*, k^*, \lambda^*;$ $m_a : a = 0, 1, \cdots, N-1)$ where $m_a = am_1 - (a-1)m_0$ and

$$r^* = rm_1 + (b-r)m_0 \tag{5.11}$$

$$k^* = km_1 + (v-k)m_0 \tag{5.12}$$

$$(v-1)\lambda^* = r^*(k^*-m_1-m_0) + bm_1m_0 - \sum_{a=0}^{N-1} a(a-1)r_a(m_1-m_0)^2 \tag{5.13}$$

where

$$\lambda^* = \sum_{j=1}^{b} n^*_{ij}n^*_{\ell j} \quad \text{for all } \ell \neq i$$

$$vr^* = bk^* = N^* \tag{5.14}$$

$$v \leq b \ . \tag{5.15}$$

The definitions 2.1, 2.2, and 2.3 also hold true for GNBBD.

6. EXISTENCE AND VARIANCE OPTIMALITY OF GNBBD

Theorem 6.1. The existence of a balanced N-ary incomplete block design with parameters $(v,b,r,k,\lambda;0,1,\cdots,N-1)$ implies the existence of a GNBBD with parameters $(v,b,r^*,k^*,\lambda^*;n^*_{ij}=m_0,m_1,\cdots,m_{N-1})$.

Proof: From the definition of a GNBBD, note that $\underline{n}^* = \underline{n}(m_1-m_0) + \underline{J}m_0$, the ij^{th} entry of n^* denoted by n^*_{ij} is

$$\begin{aligned} n^*_{ij} &= n_{ij}(m_1-m_0) + m_0 \\ &= a(m_1-m_0) + m_0 \quad \text{if} \quad n_{ij} = a \text{ and} \\ &\qquad\qquad\qquad\qquad\quad a = 0,1,2,\cdots,N-1. \end{aligned}$$

Let us define $m_a = am_1 - (a-1)m_0$. Starting with a BNBIBD with incidence matrix $\underline{n}$, a GNBIBD may be easily constructed by replacing all a's by m_a's. The resulting GNBBD has parameters $(v,b,r^*,k^*,\lambda^*,m_a : a = 0,1,\cdots,N-1)$. r^*, k^*, and λ^* satisfy equations (5.11) to (5.13). We shall now derive equations (5.11) to (5.13) formally.

Let $\underline{1}_b$ and $\underline{1}_v$ denote the column vectors whose elements are all ones and whose orders are b and v, respectively. Now

$$\begin{aligned} \underline{n}^*\underline{1}_b &= [\underline{n}(m_1-m_0) + \underline{J}m_0]\underline{1}_b \\ &= [r(m_1-m_0) + bm_0]\underline{1}_v \\ &= [rm_1 + (b-r)m_0]\underline{1}_v = r^*\underline{1}_v, \end{aligned}$$

and

$$\begin{aligned} \underline{n}^{*\prime}\underline{1}_v &= [\underline{n}'(m_1-m_0) + \underline{J}'m_0]\underline{1}_v \\ &= [k(m_1-m_0) + vm_0]\underline{1}_b \\ &= [km_1 + (v-k)m_0]\underline{1}_b, \end{aligned}$$

also

$$\begin{aligned} \underline{n}^*\underline{n}^{*\prime} &= [\underline{n}(m_1-m_0) + \underline{J}m_0][\underline{n}(m_1-m_0) + \underline{J}m_0]' \\ &= \underline{n}\underline{n}'(m_1-m_0)^2 + [2r(m_1-m_0)m_0 + bm_0^2]\underline{J} \end{aligned}$$

where $\underline{J}$ is a v × v matrix of ones. Thus the $(i\ell)^{th}$ entry of $\underline{n}^*\underline{n}^{*\prime}$ for $\ell \neq i$, denoted by λ^*, is written as $\lambda^* = \lambda(m_1-m_0)^2 + 2r(m_1-m_0)m_0 + bm_0^2$, where

$$\lambda = \left[r(k-1) - \sum_{a=0}^{N-1} a(a-1)r_a\right]/(v-1).$$

Thus,

$$\lambda^*(v-1) = \left(rk-r - \sum_{a=0}^{N-1} a(a-1)r_a\right)(m_1-m_0)^2 + 2(v-1)r$$

$$\cdot\ (m_1-m_0)m_0 + b(v-1)m_0^2$$

$$= (r^*-bm_0)(k^*-vm_0) - (r^*-bm_0)(m_1-m_0)$$

$$- \left(\sum_{a=0}^{N-1} a(a-1)r_a\right)(m_1-m_0)^2 + 2(v-1)(r^*-bm_0)m_0$$

$$+ b(v-1)m_0^2$$

$$= r^*(k^*-m_1-m_0) + bm_1m_0 - \sum_{a=0}^{N-1} a(a-1)r_a(m_1-m_0)^2 .$$

To prove (5.15) for $0 \le m_0 < m_1 < \cdots < m_{N-1}$, imitate the steps used to prove this under GTBBD except $r-\lambda + 2r_2$ is replaced by $r-\lambda + \sum_{a=0}^{N-1} a(a-1)r_a$.

The coefficient matrix $\underline{C}^*$ assumes the same relation as in (3.1).

Theorem 6.2. In the class of all equireplicated and equisized blocks GNBBD with parameters $(v,b_d,r^*,k_d^*,\lambda_d^*;m_{0d},m_{1d},\cdots,m_{N-1d})$, the design(s) having the minimal value of

$$\left[r_d(b_d-r_d) + \sum_{a=2}^{N-1} a(a-1)r_{ad}\right](m_{1d}-m_{0d})^2$$

is(are) optimal in the sense of A- D- E-optimality.

The proof is a straightforward extension of Theorem 3.2. Corollaries 3.1 and 3.2 are similarly extended and are restated as:

Corollary 6.1. *In a subclass of GNBBD with parameters* $(v,b_d,r^*,k_d^*,\lambda_d^*;$ $m_{ad}: a = 0,1,\cdots,N-1)$ *and derived from* BNBIBD *with parameters* $(v,b_d,r_d,k_d,\lambda_d;$ $a: a = 0,1,\cdots,N-1)$ *for the* d^{th} *design, in which the difference* $(m_{1d}-m_{0d})$ *is constant, the one(s) having the minimal value of* $r_d(b_d-r_d) + \sum_{a=0}^{N-1} a(a-1)r_{ad}$ *is(are) optimal.*

Corollary 6.2. *In a subclass of GNBBD with parameters* $(v,b_d,r^*,k_d^*,\lambda_d^*;$ $m_{ad}: a = 0,1,\cdots,N-1)$ *and derived from* BNBIBD *with parameters* $(v,b_d,r_d,k_d,\lambda_d;$ $a: a = 0,1,\cdots,N-1)$ *in which the quantity* $r_d(b_d-r_d) + \sum_{a=0}^{N-1} a(a-1)r_{ad}$ *is constant, the design(s) having the minimal value of* $(m_{1d}-m_{0d})$ *is(are) optimal.*

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