December, 1955 BU-60-M

COVARIANCE VERSUS THE METHOD OF FITTING CONSTANTS

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The purpose of this article is to illustrate that the adjusted treatment and error mean squares and the adjusted means from a multiple covariance analysis with the n-1 independent variates as per Federer and Schlottfeldt, Biometrics 10:282-290, 1954 and Outhwaite and Rutherford, Biometrics 11, December, 1955, and by the method of fitting constants, yields the same mean squares and means. The above cited authors do not illustrate this fact in their papers.

The simplest example that appears to be general enough to illustrate the stated purpose would be one with four rows or blocks, three columns, and three treatments. Also, it is desirable to have the treatments unbalanced with respect to columns. Therefore, the following example composed of three treatments, A, B, C, was selected:

	Column 1	Column 2	Column 3	Row Total
Block 1	A - X_{111}	C - X_{123}	B - X_{132}	$X_{1..}$
Block 2	C - X_{213}	B - X_{222}	A - X_{231}	$X_{2..}$
Block 3	B - X_{312}	C - X_{323}	A - X_{331}	$X_{3..}$
Block 4	A - X_{411}	C - X_{423}	B - X_{432}	$X_{4..}$
Column Totals	$X_{.1.}$	$X_{.2.}$	$X_{.3.}$	$X_{...}$ = grand total

The linear model is

$$X_{ijh} = \mu + \rho_i + \gamma_j + \tau_h + \epsilon_{ijh}, \tag{1}$$

which is the yield of the h'th treatment in the i'th block and in the j'th column. $i = 1,2,3,4, = r$; $j = 1,2,3 = k$; $h = 1,2,3 = k$. The normal equation for the mean is

$$X_{...} = 12\hat{\mu} + 3\Sigma\hat{\rho}_i + 4\Sigma\hat{\gamma}_j + 4\Sigma\hat{\tau}_i \tag{2}$$

The normal equations for the blocks are

$$X_{1\cdot\cdot} = 3\hat{\mu} + 3\hat{\rho}_1 + \Sigma\hat{\gamma}_j + \Sigma\hat{\tau}_h \tag{3}$$

$$X_{2\cdot\cdot} = 3\hat{\mu} + 3\hat{\rho}_2 + \Sigma\hat{\gamma}_j + \Sigma\hat{\tau}_h \tag{4}$$

$$X_{3\cdot\cdot} = 3\hat{\mu} + 3\hat{\rho}_3 + \Sigma\hat{\gamma}_j + \Sigma\hat{\tau}_h \tag{5}$$

$$X_{4\cdot\cdot} = 3\hat{\mu} + 3\hat{\rho}_4 + \Sigma\hat{\gamma}_j + \Sigma\hat{\tau}_h \tag{6}$$

The normal equations for the columns are:

$$X_{\cdot 1 \cdot} = 4(\hat{\mu} + \hat{\gamma}_1) + \Sigma\hat{\rho}_i + 2\hat{\tau}_1 + \hat{\tau}_2 + \hat{\tau}_3 \tag{7}$$

$$X_{\cdot 2 \cdot} = 4(\hat{\mu} + \hat{\gamma}_2) + \Sigma\hat{\rho}_i + \hat{\tau}_2 + 3\hat{\tau}_3 \tag{8}$$

$$X_{\cdot 3 \cdot} = 4(\hat{\mu} + \hat{\gamma}_3) + \Sigma\hat{\rho}_j + 2\hat{\tau}_1 + 2\hat{\tau}_2 \tag{9}$$

The normal equations for the treatments are:

$$X_{\cdot\cdot 1} = 4\hat{\mu} + \Sigma\hat{\rho}_i + 2\hat{\gamma}_1 + 2\hat{\gamma}_3 + 4\hat{\tau}_1 \tag{10}$$

$$X_{\cdot\cdot 2} = 4\hat{\mu} + \Sigma\hat{\rho}_i + \hat{\gamma}_1 + \hat{\gamma}_2 + 2\hat{\gamma}_3 + 4\hat{\tau}_2 \tag{11}$$

$$X_{\cdot\cdot 3} = 4\hat{\mu} + \Sigma\hat{\rho}_i + \hat{\gamma}_1 + 3\hat{\gamma}_2 + 4\hat{\tau}_3 \tag{12}$$

Imposing the restrictions

$$\Sigma\hat{\rho}_i = 0, \; \Sigma\hat{\gamma}_j = 0, \text{ and } \Sigma\hat{\tau}_i = 0, \tag{13}$$

we obtain

$$\hat{\mu} = X_{\cdot\cdot\cdot}/12 = \bar{x} \tag{14}$$

$$\hat{\rho}_1 + \hat{\mu} = X_{1\cdot\cdot}/3 = \bar{x}_{1\cdot\cdot} \tag{15}$$

$$\hat{\rho}_2 + \hat{\mu} = X_{2\cdot\cdot}/3 = \bar{x}_{2\cdot\cdot} \tag{16}$$

$$\hat{\rho}_3 + \hat{\mu} = X_{3\cdot\cdot}/3 = \bar{x}_{3\cdot\cdot} \tag{17}$$

$$\hat{\rho}_4 + \hat{\mu} = X_{4\cdot\cdot}/3 = \bar{x}_{4\cdot\cdot} \tag{18}$$

The remaining equations cannot be solved so easily. From the $\hat{\gamma}_j$ equations (7, 8, and 9) we obtain values of $\hat{\gamma}_j$ in terms of the observations and $\hat{\tau}_h$. These values for $\hat{\gamma}_j$ are substituted in the $\hat{\tau}_h$ equations (10, 11, and 12) to obtain equations involving only the $\hat{\tau}_h$. Thus, the $\hat{\gamma}_j$ equations may be written as

$$\hat{\gamma}_1 = \bar{x}_{\cdot 1 \cdot} - \bar{x} - \hat{\tau}_1/4 \qquad (19)$$

$$\hat{\gamma}_2 = \bar{x}_{\cdot 2 \cdot} - \bar{x} - \hat{\tau}_2/4 - 3\hat{\tau}_3/4 \qquad (20)$$

$$\hat{\gamma}_3 = \bar{x}_{\cdot 3 \cdot} - \bar{x} + \hat{\tau}_3/2 \qquad (21)$$

Therefore,

$$(\bar{x}_{\cdot\cdot 1} - \bar{x}) + \frac{2}{4}(\bar{x}_{\cdot 2 \cdot} - \bar{x}) = \hat{\tau}_1 + \frac{\hat{\tau}_2}{2(4)} + \frac{3\hat{\tau}_3}{2(4)}, \qquad (22)$$

$$(\bar{x}_{\cdot\cdot 2} - \bar{x}) - \frac{1}{4}(\bar{x}_{\cdot 3 \cdot} - \bar{x}) = \hat{\tau}_2 + \hat{\tau}_3/8, \qquad (23)$$

and

$$(\bar{x}_{\cdot\cdot 3} - \bar{x}) - \frac{1}{4}(\bar{x}_{\cdot 1 \cdot} - \bar{x}) - \frac{3}{4}(\bar{x}_{\cdot 2 \cdot} - \bar{x}) = \hat{\tau}_3 - \frac{\hat{\tau}_1}{16} - \frac{3\hat{\tau}_2}{16} - \frac{9\hat{\tau}_3}{16}$$

$$= \frac{7\hat{\tau}_3}{16} - \frac{\hat{\tau}_1}{16} - \frac{3\hat{\tau}_2}{16} \qquad (24)$$

In (22) substitute $-\hat{\tau}_2 - \hat{\tau}_3$ for $\hat{\tau}_1$.

Then (22) becomes

$$(\bar{x}_{\cdot\cdot 1} - \bar{x}) + \frac{1}{2}(\bar{x}_{\cdot 2 \cdot} - \bar{x}) = -\frac{7}{8}\hat{\tau}_2 - \frac{5}{8}\hat{\tau}_3. \qquad (25)$$

Multiply (23) by 5 to obtain

$$5\left\{\bar{x}_{\cdot\cdot 2} - \bar{x} - \frac{1}{4}(\bar{x}_{\cdot 3 \cdot} - \bar{x}) = \hat{\tau}_2 + \frac{\hat{\tau}_3}{8}\right\}. \qquad (26)$$

Add equation (25) to equation (26) to obtain

$$\hat{\tau}_2 = \frac{8}{33}[\,\bar{x}_{\cdot\cdot 1} - \bar{x} + 5(\bar{x}_{\cdot\cdot 2} - \bar{x}) + \frac{1}{2}(\bar{x}_{\cdot 2 \cdot} - \bar{x}) - \frac{5}{4}(\bar{x}_{\cdot 3 \cdot} - \bar{x})\,]. \qquad (27)$$

Therefore,

$$\hat{\tau}_3 = -\frac{56}{33}(\bar{x}_{\cdot\cdot 2} - \bar{x}) - \frac{64}{33}(\bar{x}_{\cdot\cdot 1} - \bar{x})$$

$$- \frac{32}{33}(\bar{x}_{\cdot 2 \cdot} - \bar{x}) + \frac{14}{33}(\bar{x}_{\cdot 3 \cdot} - \bar{x}). \qquad (28)$$

Also, $\hat{\tau}_1 = -\hat{\tau}_2 - \hat{\tau}_3 = \frac{-4}{33}\left\{-14(\bar{x}_{\cdot\cdot 1}-\bar{x}) - 4(\bar{x}_{\cdot\cdot 2}-\bar{x}) - 7(\bar{x}_{\cdot 2 \cdot}-\bar{x}) + (\bar{x}_{\cdot 3 \cdot}-\bar{x})\right\}$ (29)

Substitution of the values for $\hat{\tau}_h$ in equations (19) to (21) results in the following equations:

$$\hat{\gamma}_1 = \bar{x}_{.1.} - \bar{x} + \frac{1}{33}\left\{-14(\bar{x}_{..1}-\bar{x}) - 4(\bar{x}_{..2}-\bar{x}) - 7(\bar{x}_{.2.}-\bar{x}) + (\bar{x}_{.3.}-\bar{x})\right\}. \qquad (30)$$

$$\hat{\gamma}_2 = \bar{x}_{.2.} - \bar{x} - \frac{1}{33}\left\{-46(\bar{x}_{..1}-\bar{x}) - 32(\bar{x}_{..2}-\bar{x}) - 23(\bar{x}_{.2.}-\bar{x}) + 8(\bar{x}_{.3.}-\bar{x})\right\}$$

$$= \frac{56}{33}(\bar{x}_{.2.}-\bar{x}) + \frac{1}{33}\left\{46(\bar{x}_{..1}-\bar{x}) + 32(\bar{x}_{..2}-\bar{x}) - 8(\bar{x}_{.3.}-\bar{x})\right\}. \qquad (31)$$

$$\hat{\gamma}_3 = \bar{x}_{.3.} - \bar{x} + \frac{1}{2(33)}\left\{-56(\bar{x}_{..2}-\bar{x}) - 64(\bar{x}_{..1}-\bar{x}) - 32(\bar{x}_{.2.}-\bar{x}) + 14(\bar{x}_{.3.}-\bar{x})\right\}$$

$$= \frac{40}{33}(\bar{x}_{.3.}-\bar{x}) - \frac{1}{33}\left\{28(\bar{x}_{..2}-\bar{x}) + 32(\bar{x}_{..1}-\bar{x}) + 16(\bar{x}_{.2.}-\bar{x})\right\}. \qquad (32)$$

The problem now is to show algebraically that the quantities $\hat{\mu} + \hat{\tau}_h$ are identically equal to the adjusted treatment means from a multiple covariance analysis with covariates Y and Z, where Y has the value -1 for every cell in column 1, the value 0 for every cell in column 2, and the value +1 for every cell in column 3, and where Z has the value of +1 in every cell in columns 1 and 3 and the value -2 in every cell in column 2. However, a numerical example will be used to illustrate the equality first and then the algebraic solution will be carried through.

The constructed example consists of setting $\hat{\mu} = 10$, $\hat{\rho}_1 = -1$, $\hat{\rho}_2 = 0$, $\hat{\rho}_3 = 1$, $\hat{\rho}_4 = 0$, $\hat{\gamma}_1 = -1$, $\hat{\gamma}_2 = -2$, $\hat{\gamma}_3 = -3$, $\hat{\tau}_1 = -2$, $\hat{\tau}_2 = -2$, and $\hat{\tau}_3 = 4$. With these values, the following results were obtained:

Column number and treatment (in parentheses) yields

	1			2			3			Totals			Mean
Row	X	Y	Z	X	Y	Z	X	Y	Z	X	Y	Z	X
1	(A) 6	-1	1	(C) 11	0	-2	(B) 10	1	1	27	0	0	9
2	(C) 13	-1	1	(B) 6	0	-2	(A) 11	1	1	30	0	0	10
3	(B) 8	-1	1	(C) 13	0	-2	(A) 12	1	1	33	0	0	11
4	(A) 7	-1	1	(C) 12	0	-2	(B) 11	1	1	30	0	0	10
Total	34	-4	4	42	0	-8	44	4	4	120	0	0	--
Mean	8.5	-1	1	10.5	0	-2	11.0	1	1	10.0	0	0	--

$\bar{x}_{..1} = 36/4 = 9.00$; $\bar{x}_{..2} = 35/4 = 8.75$; $\bar{x}_{..3} = 49/4 = 12.25$, where A = 1, B = 2, and C = 3.

From formula (14) we note that $\hat{\mu} = \bar{x} = 10$ which corresponds to the value used in setting up the example. From formulae (15) to (18), we find that $\hat{\rho}_1 = -1$, $\hat{\rho}_2 = 0$, $\hat{\rho}_3 = 1$, and $\hat{\rho}_4 = 0$. From formulae (27) to (29), the values for the $\hat{\tau}_h$ are obtained as

$$\hat{\tau}_1 = -2,$$

$$\hat{\tau}_2 = -2,$$

$$\text{and} \quad \hat{\tau}_3 = 4.$$

Likewise, from formulae (30) to (32), the values for $\hat{\gamma}_j$ are obtained as:

$$\hat{\gamma}_1 = -1,$$

$$\hat{\gamma}_2 = -2,$$

$$\text{and} \quad \hat{\gamma}_3 = 3.$$

The various sums of squares for the analysis of variance are obtained as follows:

Correction for the mean

$$\frac{X_{\cdot\cdot\cdot}^2}{bk} = \frac{120^2}{12} = 1200, \text{ with 1 degree of freedom}$$

Total sum of squares corrected for the mean

$$6^2 + 13^2 + \dots + 12^2 + 11^2 - 1200 = 74, \text{ with 11 d.f.}$$

Row sum of squares

$$\frac{27^2 + 30^2 + 33^2 + 30^2}{3} - 1200 = 6, \text{ with 3 d.f.}$$

Column (ignoring treatment effect) sum of squares

$$\frac{34^2 + 42^2 + 44^2}{4} - 1200 = 14, \text{ with 2 d.f.}$$

Treatment (ignoring column effect) sum of squares

$$\frac{36^2 + 35^2 + 49^2}{4} - 1200 = 30.5, \text{ with 2 d.f.}$$

Treatment (eliminating column effect) sum of squares

$$\hat{\mu}X_{\cdot\cdot\cdot} + \Sigma\hat{\rho}_i X_{i\cdot\cdot} + \Sigma\hat{\gamma}_j X_{\cdot j\cdot} + \Sigma\hat{\tau}_h X_{\cdot\cdot h} - (\hat{\mu}' X_{\cdot\cdot\cdot} + \Sigma\hat{\rho}_i' X_{i\cdot\cdot} + \Sigma\,\hat{\gamma}' X_{\cdot j\cdot}) \quad (33)$$

$$= 10(120) + [-1(27) + 0(30) + 1(33) + 0(30)]$$

$$+ [-1(34) - 2(42) + 3(44)] \div [-2(36) - 2(35) + 4(49)]$$

$$- \Big\{ 10(120) + [-1(27) + 0(30) + 1(33) + 0(30)]$$

$$+ [(8.5 - 10)(34) + (10.5 - 10)(42) + (11.0 - 10)(44)] \Big\}$$

$$= 1200 + 6 + 14 + 54 - (1200 + 6 + 14) = 54, \text{ with 2 d.f.}$$

In the above, $\hat{\mu}' = \hat{\mu}$, $\hat{\rho}_i' = \hat{\rho}_i$, and $\hat{\gamma}_j' = \bar{x}_{.j.} - \bar{\bar{x}}$ (34)

It so happens that the sum of squares $\Sigma\hat{\gamma}_j X_{.j.}$ is equal to the sum of squares $\Sigma(\bar{x}_{.j.} - \bar{\bar{x}})X_{.j.}$. This happenstance does not hold for all examples with confounding of column and treatment effects.

The above results are summarized in the following analysis of variance table:

Source of variation	d.f.	Sum of squares
Total (corrected for mean)	11	74
Row	3	6
Column (ignoring tr.)	2	14
Treatment (eliminating col.)	2	54
Residual	4	0

The residual sum of squares is zero since no allowance was made for error deviations in the constructed example.

The adjusted treatment means are:

$$\hat{\mu} + \hat{\tau}_1 = 10 + (-2) = 8,$$

$$\hat{\mu} + \hat{\tau}_2 = 10 + (-2) = 8,$$

$$\text{and} \quad \hat{\mu} + \hat{\tau}_3 = 10 + 4 = 14.$$

Now consider the alternative analysis, i.e., the multiple covariance analysis with the covariates corresponding to an orthogonal set of comparisons among the columns. The orthogonal set used in the example corresponds to the linear (Y) and quadratic (Z) effects for equally spaced values of the covariate. With the values for Y(linear effect) and Z(quadratic effect) and the yields as given above in the constructed example, the following sums of squares and products were obtained:

Source of Variation	Sum of products						
	d.f.	y^2	z^2	yz	xy	xz	x^2
Total (uncorr.)	12	8	24	0	10	-6	1274
Correction term	1	0	0	0	0	0	1200
Total (corr.)	11	8	24	0	10	-6	74
Row	3	0	0	0	0	0	6
Treatment	2	.5	10.5	1.5	-3.5	-16.5	30.5
Error	6	7.5	13.5	-1.5	13.5	10.5	37.5
Tr. + Error	8	8	24.0	0	10	-6	68

The various correlations as computed on the error (or residual)line are:

$$r_{xy(E)} = \frac{13.5}{\sqrt{7.5(37.5)}} = .804984$$

$$r_{xz(E)} = \frac{10.5}{\sqrt{13.5(37.5)}} = .466667$$

$$r_{yz(E)} = \frac{-1.5}{\sqrt{7.5(13.5)}} = .149071$$

$$R^2_{xyz(E)} = \frac{r^2_{xy(E)} + r^2_{xz(E)} - 2r_{yz(E)}r_{xy(E)}r_{xz(E)}}{1 - r^2_{yz(E)}} \quad (35)$$

$= \frac{.977777}{.977778}$ which equals unity within rounding errors. This value must be unity for this example since no allowance was made for any error deviations in constructing the example. Therefore,

$$(1 - R^2_{xyz(E)})E_{xx} = (1-1)(37.5) = \text{zero.}$$

The next correlations required are those from the treatment + error line in the analysis of variance table; thus,

$$r_{xy(T+E)} = \frac{10}{\sqrt{8(68)}} = .428746,$$

$$r_{xz(T+E)} = \frac{-6}{\sqrt{24(68)}} = -.148847,$$

and $r_{yz(T+E)} = 0$.

Therefore, $R^2_{xyz(T+E)} = .205979$, $(1-R^2_{xyz(T+E)}) = .794021$, and $.794021(68) = 53.9934$, which equals the treatment (eliminating column effect) sum of squares, 54, obtained previously. Likewise, the column (ignoring treatment) sum of squares equals $R^2_{xyz(T+E)}$ (treatment + error ss) = .205979(68) = 14.0066. Thus, the sums of squares from the covariance analysis agree, within rounding errors, with the corresponding ones from the method of fitting constants analysis.

The adjusted treatment means from the covariance analysis are obtained from the formula,

$$\bar{x}_{..h}' = \bar{x}_{..h} - b_{xy\cdot z}(\bar{y}_{..h} - \bar{y}) - b_{xz\cdot y}(\bar{z}_{..h} - \bar{z}), \tag{36}$$

where

$$b_{xy\cdot z} = \frac{13.5(13.5) - (-1.5)(10.5)}{7.5(13.5) - (-1.5)^2} = 2$$

and

$$b_{xz\cdot y} = \frac{7.5(10.5) - (-1.5)(13.5)}{7.5(13.5) - (-1.5)^2} = 1;$$

$$\bar{x}_{..1}' = 9.00 - 2(0 - 0) - 1(1 - 0) = 8,$$

$$\bar{x}_{..2}' = 8.75 - 2(.25 - 0) - 1(.25 - 0) = 8,$$

and

$$\bar{x}_{..3}' = 12.25 - 2(-.25) - 1(-1.25) = 14.$$

The adjusted means from the covariance analysis are identical with the $\hat{\mu} + \hat{\tau}_h$ values obtained from the method of fitting constants analysis.

Intuitively one would think that the results from the covariance analysis and from the method of fitting constants analysis should be identical. The example given above confirms what one would expect. For additional evidence it is required to show that the adjusted means from the covariance analysis $\bar{x}_{..h}'$ are algebraically equal to the $\hat{\mu} + \hat{\tau}_h$ values. The algebraic solution is given for $\bar{x}_{..1}'$, which is equal to

$$\bar{x}_{..1} - b_{xy\cdot z}(\bar{y}_{..1} - \bar{y}) - b_{xz\cdot y}(\bar{z}_{..1} - \bar{z})$$

$$= \bar{x}_{..1} - b_{xy\cdot z}(0 - 0) - b_{xz\cdot y}(1 - 0)$$

$$= \bar{x}_{..1} - b_{xz\cdot y}$$

$$= \bar{x}_{\cdot\cdot 1} - \frac{E_{yy}E_{xy} - E_{yz}E_{xz}}{E_{yy}E_{zz} - E_{zz}^2}$$

$$= \bar{x}_{\cdot\cdot 1} - [7.5(X_{\cdot 1 \cdot} + X_{\cdot 3 \cdot} - 2X_{\cdot 2 \cdot} - X_{\cdot\cdot 1} - \frac{X_{\cdot\cdot 2}}{4} + \frac{5}{4}X_{\cdot\cdot 3})$$

$$- (-1.5)(-X_{\cdot 1 \cdot} + X_{\cdot 3 \cdot} - \frac{X_{\cdot\cdot 2}}{4} + \frac{X_{\cdot\cdot 3}}{4})] /99$$

$$= \bar{x}_{\cdot\cdot 1} - [24\bar{x}_{\cdot 1 \cdot} + 36\bar{x}_{\cdot 3 \cdot} - 60\bar{x}_{\cdot 2 \cdot} - 30\bar{x}_{\cdot\cdot 1} - 9\bar{x}_{\cdot\cdot 2} + 39\bar{x}_{\cdot\cdot 3}]/99$$

$$= x_{\cdot\cdot 1} - [-84(\bar{x}_{\cdot 2 \cdot} - \bar{x}) + 12(\bar{x}_{\cdot 3 \cdot} - \bar{x}) - 69(\bar{x}_{\cdot\cdot 1} - \bar{x}) - 48(\bar{x}_{\cdot\cdot 2} - \bar{x})]/99$$

$$= \bar{x}_{\cdot\cdot 1} - \frac{12}{99}[-7(\bar{x}_{\cdot 2 \cdot} - \bar{x}) + (\bar{x}_{\cdot 3 \cdot} - \bar{x}) - 14(\bar{x}_{\cdot\cdot 1} - \bar{x}) - 4(\bar{x}_{\cdot\cdot 2} - \bar{x})$$

$$+ \frac{99}{12}(\bar{x}_{\cdot\cdot 1} - \bar{x})]$$

$$= \bar{x}_{\cdot\cdot 1} - [-\hat{\tau}_1 + \bar{x}_{\cdot\cdot 1} - \bar{x}]$$

$$= \bar{x} + \hat{\tau}_1$$

$$= \hat{\mu} + \hat{\tau}_1.$$

In the above the E_{yy}, E_{zy}, etc., values refer to the sum of products in the analysis of covariance table. Therefore, the algebraic solution confirms the arithmetic solution.

The one remaining problem is to show that the variance of $\hat{\mu} + \hat{\tau}_h$ is equal to that for $\bar{x}_{\cdot\cdot h}$. The solution follows for h = 1.

$$V(\hat{\mu} + \hat{\tau}_1) = E[\hat{\mu} + \hat{\tau}_1 - E(\hat{\mu} + \hat{\tau}_1)]^2$$

$$= E[\mu + \Sigma\Sigma\Sigma\epsilon_{ijh}/12 - \frac{4}{33}\left\{-14\bar{x}_{\cdot\cdot 1} - 4\bar{x}_{\cdot\cdot 2} - 7\bar{x}_{\cdot 2 \cdot}\right.$$

$$\left. + \bar{x}_{\cdot 3 \cdot} + 24\bar{x}\right\} - E(\hat{\mu} + \hat{\tau}_1)]^2$$

$$= E[\mu + \frac{1}{12}\Sigma\Sigma\Sigma\epsilon_{ijh} - \frac{4}{33}\left\{-14(\mu + \tau_1 + \frac{1}{2}(\gamma_1 + \gamma_3) + \frac{1}{4}\Sigma\Sigma\epsilon_{ij1})\right.$$

$$- 4(\mu + \tau_2 + \frac{\gamma_3}{4} + \frac{1}{4}\Sigma\Sigma\epsilon_{ij2}) - 7(\mu + \gamma_2 + \frac{1}{4}(3\tau_3 + \tau_2) + \frac{1}{4}\Sigma\Sigma\epsilon_{ijh})$$

$$\left. + (\mu + \gamma_3 + \frac{1}{2}(\tau_1 + \tau_2) + \frac{1}{4}\Sigma\Sigma\epsilon_{i3h}) + 24\mu + 2\Sigma\Sigma\Sigma\epsilon_{ijh}\right\}$$

$$- E(\hat{\mu} + \hat{\tau}_1)]^2$$

$$= E[\epsilon_{111}\left\{\frac{-63}{12(33)} + \frac{14}{33}\right\} + \epsilon_{231}\left\{\frac{-63}{12(33)} + \frac{14}{33} - \frac{1}{33}\right\}$$

$$+ \epsilon_{331}\left\{\frac{-63}{12(33)} + \frac{14}{33} - \frac{1}{33}\right\} + \epsilon_{411}\left\{\frac{-63}{12(33)} + \frac{14}{33}\right\}$$

$$+ \epsilon_{132}\left\{\frac{-63}{12(33)} + \frac{4}{33} - \frac{1}{33}\right\} + \epsilon_{222}\left\{\frac{-63}{12(33)} + \frac{4}{33} + \frac{7}{33}\right\}$$

$$+ \epsilon_{312}\left\{\frac{-63}{12(33)} + \frac{4}{33}\right\} + \epsilon_{432}\left\{\frac{-63}{12(33)} + \frac{4}{33} - \frac{1}{33}\right\}$$

$$+ \epsilon_{123}\left\{\frac{-63}{12(33)} + \frac{7}{33}\right\} + \epsilon_{323}\left\{\frac{-63}{12(33)} + \frac{7}{33}\right\}$$

$$+ \epsilon_{423}\left\{\frac{-63}{12(33)} + \frac{7}{33}\right\} + \epsilon_{213}\left\{\frac{-63}{12(33)}\right\}$$

$$+ \gamma_1\left(\frac{28}{33}\right) + \gamma_2\left(\frac{28}{33}\right) + \gamma_3\left(\frac{28}{33} + \frac{4}{33} - \frac{4}{33}\right)$$

$$+ \frac{(56-2)}{33}\tau_1 + \frac{(16+7-2)}{33}\tau_2 + \frac{21}{33}\tau_3 - E(\hat{\mu} + \hat{\tau}_1)]^2$$

$$= E[\frac{1}{121}(35\epsilon_{111} + 31\epsilon_{231} + 31\epsilon_{331} + 35\epsilon_{411} - 9\epsilon_{132}$$

$$+ 23\epsilon_{222} - 5\epsilon_{312} + 9\epsilon_{432} + 7\epsilon_{123} + 7\epsilon_{323} + 7\epsilon_{423} - 21\epsilon_{213}]^2$$

$$= .32575758\,\sigma_\epsilon^2.$$

The variance of $\bar{\bar{x}}_{\cdot\cdot 1}'$ is equal to

$$V(\bar{\bar{x}}_{\cdot\cdot 1}') = V(\hat{\bar{\bar{x}}}_{\cdot\cdot 1}) + V(b_{xy\cdot z})(\bar{\bar{y}}_{\cdot\cdot 1} - \bar{\bar{y}})^2 + V(b_{xz\cdot y})(\bar{\bar{z}}_{\cdot\cdot 1} - \bar{\bar{z}})^2$$

$$= V(\bar{\bar{x}}_{\cdot\cdot 1}) + V(b_{xz\cdot y})$$

$$= \sigma_\epsilon^2\left\{\frac{1}{r} + \frac{E_{yy}}{E_{yy}E_{zz} - E_{yz}^2}\right\}$$

$$= \sigma_\epsilon^2\left\{.25 + \frac{7.5}{99}\right\}$$

$$= .32575758\,\sigma_\epsilon^2,$$

and the variances are identical.

The same procedures could be used to show that the following relationships hold.

$$\bar{\bar{x}}_{..2}' = \hat{\mu} + \hat{\tau}_2,$$

$$\bar{\bar{x}}_{..3}' = \hat{\mu} + \hat{\tau}_3,$$

$$V(\bar{\bar{x}}_{..2}') = V(\hat{\mu} + \hat{\tau}_2),$$

and $$V(\bar{\bar{x}}_{..3}') = V(\hat{\mu} + \hat{\tau}_3).$$