APPENDIX TO "PROOF"

BU-597-M

S. R. Searle

August, 1976

Biometrics Unit, Cornell University, Ithaca, New York

Abstract


_____________________________

Paper No. BU-597-M in the Biometrics Unit Mimeo Series, Department of Plant Breeding and Biometry, Cornell University, Ithaca, New York 14853.
APPENDIX TO "PROOF"

Simple Examples of Some Elementary Methods of Proof

Given here, with little comment, are some simple illustrative examples of
the twelve methods of proof listed in the last section of the paper. The familiar
abbreviations L.H.S. and R.H.S. refer to the two sides of an equality to be proven.

1. To prove \( x = y \) manipulate \( x \) until it becomes \( y \)

Prove: \( \frac{a}{1 - \frac{1}{a}} = (a - 1)(1 + \frac{1}{a - 1})^2 \), for \( a \neq 0 \) or 1.

Proof:

L.H.S. = \( \frac{a}{1 - \frac{1}{a}} = a \frac{a - 1}{a} = a^2/(a - 1) \)

= \( (a - 1)[a/(a - 1)]^2 = (a - 1)[1 + 1/(a - 1)]^2 \)

= R.H.S. Q.E.D.

2. To prove \( x = y \), manipulate \( x \) and \( y \) until each becomes \( t \)

Prove: \( \frac{a}{1 - \frac{1}{a}} = (a - 1)(1 + \frac{1}{a - 1})^2 \), for \( a \neq 0 \) or 1.

Proof:

L.H.S. = \( \frac{a}{1 - \frac{1}{a}} = a \frac{a - 1}{a} = a^2/(a - 1) \)

R.H.S. = \( (a - 1)[1 + 1/(a - 1)]^2 = (a - 1)[a/(a - 1)]^2 = a^2/(a - 1) \)

:. L.H.S. = R.H.S. Q.E.D.

Notice the last line of this proof. Without it we have only proved that
L.H.S. = \( a^2/(a - 1) \) and R.H.S. = \( a^2/(a - 1) \). It is the last line that draws
the conclusion from these two results that L.H.S. = R.H.S. This line was not
needed in method 1 because there we proved L.H.S. = R.H.D. directly, but here
it is needed, and it is also needed in other methods of proof such as 3 and 4.
It is a vital part of these proofs.
3. To prove \( x = y \), manipulate \( x - y \) until it becomes zero

**Prove:** \( \frac{a}{1 - \frac{1}{a}} = (a - 1)[1 + \frac{1}{(a - 1)}] \), for \( a \neq 0 \) or 1.

**Proof:**

\[
\text{L.H.S. - R.H.S.} = \frac{a}{1 - \frac{1}{a}} - (a - 1)[1 + \frac{1}{a - 1}]^2
\]

\[
= \frac{a^2}{a - 1} - (a - 1)[1 + \frac{1}{a - 1} + 2(a - 1)]
\]

\[
= \frac{a^2}{a - 1} - (a - 1) - \frac{1}{a - 1} - 2
\]

\[
= \frac{a^2 - 1}{a - 1} - a + 1 - 2
\]

\[
= a + 1 - a - 1
\]

\[
= 0.
\]

\[
\therefore \text{L.H.S.} = \text{R.H.S.} \tag{Q.E.D.}
\]

4. To prove \( x = y \), manipulate \( \frac{x}{y} \) until it becomes unity (for \( y \neq 0 \))

**Prove:** \( \frac{a}{1 - \frac{1}{a}} = (a - 1)[1 + \frac{1}{(a - 1)}] \), for \( a \neq 0 \) or 1.

**Proof:**

\[
\frac{\text{L.H.S.}}{\text{R.H.S.}} = \frac{a}{1 - \frac{1}{a}} \div (a - 1)[1 + \frac{1}{a - 1}]^2
\]

\[
= \frac{a^2}{a - 1} \div (a - 1)(\frac{a}{a - 1})^2 = \frac{a^2}{a - 1} \div \frac{a^2}{a - 1} = 1
\]

Therefore \( \text{L.H.S.} = \text{R.H.S.} \tag{Q.E.D.} \)

5. To prove \( x = y \), manipulate \( x + m - m \) until it becomes \( y \)

**Prove:** That if \( x^2 + 8x - 20 = 0 \) then \( (x + 4)^2 = 36 \)

**Proof:**

\[
x^2 + 8x - 20 = 0
\]

\[
\therefore x^2 + 8x + 16 - 16 - 20 = 0;
\]

i.e., \( (x + 4)^2 - 36 = 0 \)

\[
\therefore (x + 4)^2 = 36 \tag{Q.E.D.}
\]
Although this is not a question of proving two things equal it illustrates what is called the 'add-and-subtract' procedure. A classical example of it is the solving of the quadratic equation

\[ ax^2 + bx + c = 0. \]

For \( a \neq 0 \) this is

\[ x^2 + (b/a)x + c/a = 0 \]

and so

\[ x^2 + (b/a)x + (b/2a)^2 - (b/2a)^2 + c/a = 0. \]

\[ \therefore (x + b/2a)^2 = (b^2 - 4ac)/4a^2 \]

\[ \therefore x = [-b \pm \sqrt{b^2 - 4ac}]/2a \]

6. To prove \( x = y \), manipulate \( xk/k \) until it is \( y \) (for \( k \neq 0 \))

**Proof:**

\[ \frac{3 - \sqrt{5}}{\sqrt{5} - 2} = 1 + \sqrt{5} \]

**Proof:**

\[ \frac{3 - \sqrt{5}}{\sqrt{5} - 2} = \frac{(3 - \sqrt{5})(\sqrt{5} + 2)}{(\sqrt{5} - 2)(\sqrt{5} + 2)} = \frac{3\sqrt{5} + 6 - 5 - 2\sqrt{5}}{5 - 4} = 1 + \sqrt{5} \]

Q.E.D.

7. To prove \( t/s = y \), manipulate \( sy \) until it is \( t \)

**Proof:**

\[ \frac{1 - x^n}{1 - x} = 1 + x + x^2 + \cdots + x^{n-1} \text{ for } n \text{ a positive integer and } x \neq 1. \]

**Proof:**

\[ (1 - x)(1 + x + x^2 + \cdots + x^{n-1}) = 1 + x + x^2 + \cdots + x^{n-1} - (x + x^2 + x^3 + \cdots + x^{n-1} + x^n) = 1 - x^n \]

\[ \therefore \frac{1 - x^n}{1 - x} = 1 + x + x^2 + \cdots + x^{n-1} \]

Q.E.D.

Once again notice how important is the last line of the proof.
8. Proof by "construction"

**Prove:** That if \( \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \) with \( b, d \) and \( f \) non-zero,

\[
\frac{a^3 b + 2c^2e - 3ae^2f}{b^4 + 2d^2f - 3bf^3} = \frac{ace}{bdf}
\]

**Proof:** Let \( \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \lambda \).

Then \( a = b\lambda, c = d\lambda \) and \( e = f\lambda \)

and so

\[
\frac{a^3 b + 2c^2e - 3ae^2f}{b^4 + 2d^2f - 3bf^3} = \frac{b^4 \lambda^3 - 2d^2f\lambda - 3bf^3\lambda^3}{b^4 - 2d^2f - 3bf^3} = \lambda^3 = \left(\frac{a}{b}\right)\left(\frac{c}{d}\right)\left(\frac{e}{f}\right) = \frac{ace}{bdf}
\]

Q.E.D.

9. Proof by substitution

**Prove:** That \( x = 3 \) is a solution of \( 2x^2 + 5x - 33 = 0 \).

**Proof:** When \( x = 3, 2x^2 + 5x - 33 = 2(9) + 5(3) - 33 = 18 + 15 - 33 = 0 \).

Therefore \( x = 3 \) is a solution. O.E.D.

Note the importance of the last line of this proof, so far as its logic is concerned.

And note also that not even here do we use the "if \( p \Rightarrow q \) then \( q \Rightarrow p \)" fallacy; i.e., we do not "prove" that \( x = 3 \) is a solution by writing

"**Proof**" \( 2x^2 + 5x - 33 = 0 \)

For \( x = 3 \) \( 2(9) + 5(3) - 33 = 0 \)

\[18 + 15 - 33 = 0 \]

\[0 = 0 \]

We know that \( 0 = 0 \) and this is no proof. It contains appropriate arithmetic but it is not always correct logic.
10. Proof by induction

Prove: That for $n$ a positive integer

$$1 + 2 + 3 + \cdots + (n - 1) + n = \frac{1}{2}n(n + 1). \quad \text{(A)}$$

Proof: (in 3 parts)

(i) When $n = 1$, L.H.S. = 1 and R.H.S. = $\frac{1}{3}(1)^2 = 1 = \text{L.H.S.}$ \quad \text{(B)}

Thus for $n = 1$, (A) is proven.

(ii) Assume that (A) is true for some particular positive integer $k$; i.e., that

$$1 + 2 + 3 + \cdots + (k - 1) + k = \frac{1}{2}k(k + 1) \quad \text{(C)}$$

Then, adding $k + 1$ to both sides,

$$1 + 2 + 3 + \cdots + (k - 1) + k + (k + 1) = \frac{1}{2}k(k + 1) + (k + 1)$$

$$= (k + 1)(\frac{1}{2}k + 1)$$

Hence

$$1 + 2 + 3 + \cdots + k + k + 1 = \frac{1}{2}(k + 1)(k + 2)$$

i.e.,

$$1 + 2 + 3 + \cdots + k + k + 1 = \frac{1}{2}(k + 1)[(k + 1) + 1]. \quad \text{(D)}$$

(iii) Now (D) is exactly the same as (C) but with $k$ replaced by $k + 1$; i.e.,

(A) holds for $n = k + 1$. Thus on assuming as we did in (C) that (A) holds for $n = k$, we have shown in (D) that (A) also holds for $n = k + 1$.

But in (B) we first showed that (A) does hold for $n = 1$. Therefore (D) shows that it also holds for $n = 2$. This being so, further use of (D) shows that (A) also holds for $n = 3$; and so on, for $n = 4, 5, 6, \ldots$.

Thus (A) holds in general, for all positive integers $n$.

Note the 3 parts to this form of proof. The first consists of showing that the formula to be proven does hold for $n = 1$. The second part consists of assuming that the formula holds for $n = k$ and showing, as a consequence, that it holds for $n = k + 1$. And the third part is the logic of tying the first two parts together.
11. Proof by exhaustion

Prove: That the roots of
\[ x^2 - 5x + 6 = 0 \]
are \( x = 2 \) and \( x = 3 \).

Proof: For \( x = 2 \),
\[ x^2 - 5x + 6 = 4 - 10 + 6 = 0 \]
For \( x = 3 \),
\[ x^2 - 5x + 6 = 9 - 15 + 6 = 0 \]
\[ \therefore x = 2 \text{ and } x = 3 \text{ are roots}. \]

But the equation is a quadratic and therefore has only two roots, and so \( x = 2 \) and \( x = 3 \) are the roots. Q.E.D.

12. Proof by contradiction

Prove: That 7 is not a root of \( x^2 - 3x - 40 = 0 \).

Proof: When \( x = 7 \),
\[ x^2 - 3x + 40 = 49 - 21 + 40 = 68 \neq 0 \]
\[ \therefore 7 \text{ is not a root}. \] Q.E.D.

The reductio ad absurdum approach is as follows:
 Assume that 7 is a root.
\[ \therefore 49 - 21 + 40 = 0 \]
\[ \therefore 68 = 0 \]

The absurdity of this conclusion leads to rejection of the assumption that 7 is a root; i.e., 7 is not a root.

Combining proof by contradiction with proof by substitution is often useful in obtaining roots of equations by trial and error. For example, consider
\[ 6x^5 + 18x^4 - 121x^3 - 39x^2 + 77x = 0 , \]
and try \( x = 1 \) and \( x = -1 \) as possible roots:
\[ x = 1: \quad 8 + 18 - 121 - 39 + 77 - 15 = 103 - 175 = -72 \neq 0 \]
\[ x = -1: \quad -8 + 18 + 121 - 39 - 77 - 15 = 139 - 139 = 0 . \]

Hence \( x = -1 \) is a root but \( x = 1 \) is not.