ON INVERTING CIRCULANT MATRICES

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ABSTRACT

The elements of the inverse of a circulant matrix having only 3 nonzero elements in each row (located in cyclically adjacent columns) are derived analytically from the solution of a recurrence equation. Expressing any circulant as a product containing these 3-element type circulants then provides an algorithm for inverting circulants in general. Extension is also made to deriving generalized inverses of certain singular circulants.

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1. INTRODUCTION

A square matrix having the form

$$M = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} \\ m_{n-1} & m_0 & m_1 & \cdots & m_{n-2} \\ m_{n-2} & m_{n-1} & m_0 & \cdots & m_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & m_3 & \cdots & m_0 \end{bmatrix} \quad (1)$$

is a circulant. It is determined by its first row and will be denoted interchangeably as

$$M = C(m_0, m_1, m_2, \cdots, m_{n-1}) = C\{m_j\} \qquad \text{for } j = 0, 1, \cdots, n-1 . \quad (2)$$

On defining $P = C(0, 1, 0, 0, \cdots, 0)$ we express M as

$$M = \sum_{j=0}^{n-1} m_j P^j \quad (3)$$

and make considerable use of the following well known properties of circulants.

(i) $P^j = C(0, \cdots, 0, 1, 0, \cdots, 0)$ is a circulant. Its first row is null except for unity in the (j+1)'th position.

(ii) $P^n = P^j(P^{n-j}) = I = P^o$ and so $P^{n-j} = P^{-j} = (P^j)^{-1}$.

(iii) Circulants commute under multiplication.

(iv) The product of $C\{a_j\}$ and $C\{b_j\}$ is $C\left\{ \sum_{k=0}^{n-1} a_k b_{n+j-k} \right\}$ for $j = 0, 1, \cdots, n-1$.

(v) Multiplying a circulant by P^k shifts all elements in each row k columns to the right in a cyclic manner; and multiplication by P^{-k} similarly shifts them k columns to the left.

(vi) The inverse of a non-singular circulant is a circulant.

(vii) For $\underset{\sim}{1}$ denoting a vector of unities, $M\underset{\sim}{1} = s\underset{\sim}{1}$ where s is the row sum of M (the same for every row); and when M^{-1} exists, its row sum is 1/s. Also, for J a matrix of unities MJ = JM = sJ .

(viii) The characteristic roots of M are $\alpha_k = \sum_{j=0}^{n-1} m_j r_k^j$, where r_k is the k'th root of $r^n = 1$, for k = 1, 2, ···, n. (The corresponding characteristic vector is $\underset{\sim}{v}_k' = [1 \; r_k \; r_k^2 \cdots r_k^{n-1}]$) .

(ix) One characteristic root of M is s. [This is evident from (vii), and also from (viii) wherein r = 1 is always a root of $r^n = 1$.]

(x) For circulants of even order n, a characteristic root is $s' = \sum_{j=0}^{n-1} (-1)^j m_j$. [This arises from (viii) with r = -1 being a root of $r^n - 1$ for n even.]

Circulants and functions of them arise in a variety of applications. Their uses in solid state physics are discussed in some detail by Löwdin, Pauncz, and de Heer [6] and by Gilbert [4]. The first of these considers 3 methods of obtaining the inverse of certain special circulant (overlap) matrices using Chebyshev polynomials of the first and second kind and giving some asymptotic results. The second presents a method of transforming to diagonal form, inverting, and then transforming back to original form, using the characteristic roots of a circulant C and the matrix which diagonalizes it, which are known. Abraham and Weiss [1] and Calaib and Appel [3] also utilize this fact, although their expressions for inverting overlap matrices are rather complex, involving Chebyshev polynomials and integrals that have to be evaluated by numerical integration or infinite sums of complex numbers. Circulants also occur in statistical applications, for example in Anderson [2], Olkin and Press [7], and Wise [9].

This paper provides a method for obtaining the inverse of a circulant having real elements, using a method of recurrence equations applied to the circulant

$$M = C(a, b, 0, \cdots, 0, c) \quad . \tag{4}$$

When a, b and c are non-zero we call M a 3-element circulant. In section 2 we obtain closed form expressions for elements of its inverse. Extension to general circulants is described in Section 3. In section 4 the Moore-Penrose inverse is obtained for certain classes of singular circulants.

2. THE 3-ELEMENT CIRCULANTS

Any circulant having just 3 cyclically consecutive non-zero elements in the first row can, by property (v), be brought to the form (4) by multiplication by P^k for some k . Its inverse is then $P^{-k}M^{-1}$, representing just a cyclic shift of elements in the rows of M^{-1}. There is therefore no loss of generality in confining attention to M . We denote its inverse by

$$M^{-1} = C(a_0, a_1, a_2, \cdots, a_{n-1}) \tag{5}$$

and restrict ourselves to $n > 3$, since inverses of 2- and 3-order matrices are well known. Equations (3) and (4) and property (ii) then give

$$M = aP^0 + bP + cP^{n-1} = aI + bP + cP^{-1} \quad . \tag{6}$$

Furthermore, in terms of (viii), the characteristic roots of M are $\theta_i = a + br_i + cr_i^{n-1}$, which immediately gives us two conditions under which M of (4) is singular: (I) When $a + b + c = 0$, because $r = 1$ is always a root of $r^n = 1$ and then $\theta = a + b + c = 0$. (II) When $a = b + c$ and n is even, because then $r = -1$ is a root of $r^n = 1$ and $\theta = a - b - c = 0$. These cases are therefore exclusions in theorem 1, which otherwise provides the elements of the inverse of 3-element circulants.

Theorem 1 Except when $a + b + c = 0$ or when n is even and $a = b + c$, elements of

$$M^{-1} = [C(a,\ b,\ 0,\ 0,\ \cdots,\ 0,\ c)]^{-1} = C\{a_j\}$$

for $j = 0, 1, 2, \cdots, n-1$ with $n > 3$ are as follows.

(a) When $a^2 > 4bc$ and $b \neq 0$

$$a_j = \frac{z_1 z_2}{b(z_1 - z_2)} \left[\frac{z_1^j}{1-z_1^n} - \frac{z_2^j}{1-z_2^n} \right] \tag{7}$$

for $z_1, z_2 = \left[-a \pm (a^2 - 4bc)^{\frac{1}{2}}\right]/2c$.

(b) When $a^2 < 4bc$ and $b \neq 0$

$$a_j = \frac{t^{j+1}[\sin(j\theta) + t^n \sin(n-j)\theta]}{b \sin\theta (1 - 2t^n \cos n\theta + t^{2n})} \tag{8}$$

where $\theta = \sin^{-1}(1 - a^2/4bc)^{\frac{1}{2}} = \cos^{-1}[-a/2(bc)^{\frac{1}{2}}]$ and $t = (b/c)^{\frac{1}{2}}$, using positive square roots.

(c) When $a^2 = 4bc$ and $b \neq 0$, with $z = -a/2c$

$$a_j = \frac{z^{j+1}}{b(1-z^n)} \left[\frac{n}{1-z^n} - (n-j) \right] . \tag{9}$$

(d) When $b = 0$, with $z = -c/a$

$$a_o = \frac{1}{a(1-z^n)} \quad \text{and} \quad a_j = a_o j^{n-j} \quad \text{for } j = 1, 2, \cdots, n-1 . \tag{10}$$

In neither (9) nor (10) will z^n be unity because the exclusions of $a + b + c = 0$ and of $a = b + c$ for n even that are part of the theorem preclude this occurrence. Part (d) of the theorem is for 2-element circulants with $b = 0$ in M of (4). These are special cases of 3-element circulants and by applying property (v) the case

$b = 0$ also covers 2-element circulants obtained by having either $a = 0$ or $c = 0$.

The method of inverting certain patterned matrices from the use of recurrence equations given in Kounias [5] can be applied directly to (4). However*, as a result of (6) we have

$$MM^{-1} = (aI + bP + cP^{-1})[C(a_0, a_1, \cdots, a_{n-1})] = I ,$$

i.e.,

$$aC(a_0, a_1, \cdots, a_{n-1}) + bC(a_{n-1}, a_0, a_1, \cdots, a_{n-2}) + cC(a_1, a_2, \cdots, a_{n-1}, a_0) = I \quad .$$

Equating diagonal and off-diagonal elements gives

$$aa_{j+1} + ba_j + ca_{j+2} = 0 \text{ for } j = 0, 1, \cdots, n-2 \text{ and } a_n \equiv a_o \tag{11}$$

and

$$aa_o + ba_{n-1} + ca_1 = 1 \quad . \tag{12}$$

We find the solution to the recurrence equation (11) and show that it satisfies (12). The general solution to (11) is

$$a_j = pz_1^j + qz_2^j \tag{13}$$

where the quadratic equation

$$cz^2 + az + b = 0 \tag{14}$$

has solutions

$$z_1, z_2 = \left(-a \pm \sqrt{a^2-4bc}\right)/2c \quad . \tag{15}$$

To determine p and q for (13) use the initial condition $a_n = a_o$ given in (11), so that

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* As kindly noted by a referee.

$$pz_1^n + qz_2^n = p + q \quad . \tag{16}$$

A necessary second condition comes from applying property (vii) to (5) giving

$$\sum_{j=0}^{n-1} a_j = 1/(a + b + c) \tag{17}$$

so that from summing (13)

$$\frac{p(1-z_1^n)}{1-z_1} + \frac{q(1-z_2^n)}{1-z_2} = \frac{1}{a+b+c} \quad . \tag{18}$$

Solving (16) and (18) for p and q and substituting in (13) gives

$$a_j = \frac{(1-z_1)(1-z_2)}{(a+b+c)(z_1-z_2)} \left[\frac{z_1^j}{1-z_1^n} - \frac{z_2^j}{1-z_2^n} \right] \quad . \tag{19}$$

On making use of the identity

$$b(1-z_1)(1-z_2) \equiv (a+b+c)z_1 z_2 \ ,$$

arising from the definition of z_1 and z_2 in (14), it is easily shown that (19) satisfies (12). The same identity provides the alternative to (19) shown in (7) of Theorem 1. The condition $a^2 > 4bc$ is imposed to ensure that z_1 and z_2 are real but not equal.

Avoidance of complex numbers in calculating (7) when $a^2 < 4bc$ is achieved by use of elementary trigonometry. Define $t = (b/c)^{\frac{1}{2}}$ and θ by $\sin\theta = (1-a^2/4bc)^{\frac{1}{2}}$ and $\cos\theta = -a/2(bc)^{\frac{1}{2}}$, taking positive roots in all cases and noting that bc and b/c are always positive because $bc > a^2/4$. Then $z_1 = te^{i\theta}$ and $z_2 = te^{-i\theta}$, and substituting these values into (7) leads, after a little simplification, to (8).

It is clear that when $a^2 = 4bc$ neither (7) nor (8) can be used because then they both have zero denominators. However, for $y = x_2/x_1$ with $x_1 \neq 0$,

$$\frac{x_1^k - x_2^k}{x_1 - x_2} = x_1^{k-1}(1 + y + y^2 + \cdots + y^{k-1}) = kx^{k-1} \text{ for } x_1 = x_2 = x \quad . \tag{20}$$

Thus on observing from (15) that $a^2 = 4bc$ implies $z_1 = z_2 = z = -a/2c$, and by rewriting (7) as

$$a_j = \frac{z_1 z_2}{b(1-z_1^n)(1-z_2^n)} \left[\frac{z_1^j - z_2^j}{z_1 - z_2} + \frac{z_1^j z_2^j (z_1^{n-j} - z_2^{n-j})}{z_1 - z_2} \right] , \tag{21}$$

application of (20) to (21) gives (9).

For the special case $b = 0$ the recurrent equation (11) reduces to

$$a_{j+1}/a_{j+2} = -c/a \equiv z \text{ say, for } j = 0, 1, \cdots, n-2$$

with (22)

$$a_n = a_o;$$

and (12) and (17) become

$$aa_0 + ca_1 = 1 \tag{23}$$

and

$$\sum_{j=0}^{n-1} a_j = 1/(a + c) \quad . \tag{24}$$

From (22) we then have $a_j = a_o z^{n-j}$ for $j = 1, 2, \cdots, n-1$ which with (24) leads to results (10) of Theorem 1. It is easily shown that they satisfy (23).

3. k-ELEMENT CIRCULANTS

Consider a circulant having k cyclically consecutive non-zero elements, $H = \sum_{j=0}^{k-1} h_j P^j$. If δ_j for $j = 1, \cdots, k-1$ are the roots of the (k-1) order polynomial equation $\sum_{j=0}^{k-1} h_j x^j = 0$, then $H = h_{k-1} \prod_{j=1}^{k-1} (P - \delta_j I)$. When k-1 is even, the roots consist of $\frac{1}{2}(k-1)$ complex conjugate pairs, δ_{i1}, δ_{i2}, say, for $i = 1, \ldots, \frac{1}{2}(k-1)$. Then for $\mu_i = \delta_{i1} + \delta_{i2}$ and $\gamma_i = \delta_{i1}\delta_{i2}$

$$H = h_{k-1} \prod_{i=1}^{\frac{1}{2}(k-1)} (P^2 + \mu_i P + \gamma_i I) = h_{k-1} P^{-\frac{1}{2}(k-1)} \prod_{i=1}^{\frac{1}{2}(k-1)} (\mu_i I + P + \gamma_i P^{-1}) ,$$

so that its inverse is

$$H^{-1} = (1/h_{k-1}) P^{\frac{1}{2}(k-1)} \prod_{i=1}^{\frac{1}{2}(k-1)} (\mu_i I + P + \gamma_i P^{-1})^{-1} . \tag{25}$$

The inverses required in (25) are of 3-element circulants and hence can be obtained from Theorem 1. They are circulants, as are their products and so are readily calculated using properties (iii) and (iv). Thus (25) represents a method for deriving the inverse of any circulant based on the inverses of 3-element circulants of Theorem 1. When $k \ll n$ it can be a more economical way of calculating H^{-1} than direct numerical inversion. It will also be useful when the k-order polynomial that yields the μ_i's and γ_i's has easily obtained solutions. Furthermore, when inverses are required of circulants of different orders but having the same k non-zero consecutive elements, the method will be of particular use. The first step, of calculating the μ_i's and γ_i's, does not involve n and the second step, of applying Theorem 1, uses the same μ_i's and γ_i's (in the z's) for all values of n .

A particular form of n-element circulant based on a 2- or 3-element circulant can be noted. First, recall that for any non-singular matrix M and for J being a square matrix with every element unity, one form of the Householder result is

$$(M + \lambda J)^{-1} = M^{-1} - \frac{\lambda}{1 + \lambda \mathbf{1}' M^{-1} \mathbf{1}} M^{-1} J M^{-1} , \tag{26}$$

for any λ such that $1/\lambda \neq -\mathbf{1}' M^{-1} \mathbf{1}$. Verification is evident by direct multiplication. When M is a circulant of order n, with non-zero row sum s, then utilization of property (vii) in (26) gives

$$(M + \lambda J)^{-1} = M^{-1} - \frac{\lambda}{s(s + \lambda n)} J \tag{27}$$

for any $\lambda \neq -s/n$. Then, when M is a 2- or 3-element circulant, M^{-1} can be obtained from Theorem 1 and used in (27) to give the inverse of the corresponding n-element circulant $M + \lambda J$.

4. SOME SINGULAR CIRCULANTS

Experimental designs discussed by Anderson [2] give rise to information matrices that are singular circulants of order n and rank n - 1, having row sums equal to zero. Although such a circulant M has no regular inverse its Moore-Penrose (generalized) inverse can be derived from $(M + J)^{-1}$. This and some allied results are now obtained.

Consideration is given first to the existence of $(M + \lambda J)^{-1}$ for singular M . Characteristic roots of $M + \lambda J$ are, from property (viii),

$$\alpha'_k = \sum_{j=0}^{n-1} (m_j + \lambda) r_k^j = \alpha_k + \lambda \sum_{j=0}^{n-1} r_k^j \quad . \tag{28}$$

Hence for $r_1 = 1$, $\alpha_1 = s$ as in property (ix) and $\alpha'_1 = s + n\lambda$. Furthermore, in (28)

$$\alpha_k' - \alpha_k = \lambda \sum_{j=0}^{n-1} r_k^j = \frac{\lambda\left(r_k^n - 1\right)}{r_k - 1} = 0 \quad \text{for } r_k \neq 1 \; ;$$

i.e., the characteristic roots of M and of $M + \lambda J$ are the same except for $\alpha_1 = s$ and $\alpha_1' = s + n\lambda$. Hence for M singular, $M + \lambda J$ is non-singular if and only if the sole characteristic root of M that is zero is $\alpha_1 = s = 0$. Post-multiplication of the identity $(M + \lambda J)^{-1}(M + \lambda J) \equiv I$ by J then leads to

$$(M + \lambda J)^{-1}M = I - (1/n)J = M(M + \lambda J)^{-1} \tag{29}$$

the second equality arising from the commutative property (iii). From (29) it is then easy to show that

$$M^{*} = (M + \lambda J)^{-1} - (\lambda/n^2)J$$

is the Moore-Penrose inverse of M . Since this is unique for given M we use $\lambda = 1$ as the easiest value for λ and get

$$M^{*} = (M + J)^{-1} - (1/n^2)J \quad . \tag{30}$$

Additional results, similar to the preceding, can be obtained for n even, when $r_2 = -1$ is then also a root of $r^n = 1$. In that case property (x) gives $\alpha_2 = s'$ as a characteristic root. Now consider $M + \mu T$ for T defined as $T = (-1)^{i+j}$ for $i, j = 0, 1, \ldots, n-1$. Then, similar to (28), characteristic roots of $M + \mu T$ are

$$\alpha_k'' = \sum_{j=0}^{n-1}\left[m_j + \mu(-1)^j\right]r_k^j = \alpha_k + \mu \sum_{j=0}^{n-1}(-r_k)^j \quad . \tag{31}$$

For $r_1 = 1$ and $r_2 = -1$ this gives $\alpha_1'' = \alpha_1 = s$ and $\alpha_2'' = \alpha_2 + n\mu = s' + n\mu$. Hence, for M singular of even order, $M + \mu T$ is non-singular if and only if the sole characteristic root of M that is zero is $\alpha_2 = s' = 0$. An exactly similar argument shows that if and only if $s = 0$ and $s' = 0$ are the two zero characteristic roots of M for n even, then $M + \lambda J + \mu T$ has an inverse although M does not. Extensions of (30) using T and $J + T$ in place of J can then be established for these two cases of n even whereupon we have the following theorem.

<u>Theorem</u> 2 For square matrices of order $n > 3$, for J having every element unity and for $T = (-1)^{i+j}$ for $i, j = 0, 1, \cdots, n-1$ (a "checkerboard" of +1's and -1's, with +1 as the leading element), the following classes of singular circulants $M = C\{m_j\}$ for $j = 0, 1, \cdots, n-1$ with $s = \sum_{j=1}^{n-1} m_j$ and $s' = \sum_{j=1}^{n-1} (-1)^j m_j$ have the Moore-Penrose inverses $M^{\#}$ as shown:

(a) M of rank $n-1$ with $s = 0$ and, for n even, $s' \neq 0$:

$$M^{\#} = (M + J)^{-1} - (1/n^2)J \quad . \tag{32}$$

(b) M of rank $n-1$ with n even, $s \neq 0$, $s' = 0$:

$$M^{\#} = (M + T)^{-1} - (1'n^2)T \quad . \tag{33}$$

(c) M of rank $n-2$ with n even, $s = 0 = s'$:

$$M^{\#} = (M + J + T)^{-1} - 1/n^2(J + T) \quad . \tag{34}$$

<u>Corollary</u>

The first term in (32), (33) and (34) is a generalized inverse G of the respective circulants M satisfying just the first of the four Penrose [8] conditions, namely $MGM = M$. Such a matrix G is sometimes called a g_1-generalized inverse.

Part (a) of the theorem covers the form of information matrix used by Anderson [2].

In passing it might be noted that a slight generalization of (29) is that for any matrix A having non-zero row sums s and hence $AJ = sJ$,

$$(A + \lambda J)^{-1}A = I - \frac{\lambda}{s + n\lambda} J \tag{35}$$

provided $(A + \lambda J)^{-1}$ exists. However, this condition is not met if A is singular and so (35) does not lead to the Moore-Penrose inverse of A .

Closed form expressions for the elements of $M^{\#}$ of Theorem 2, for M a 3-element circulant, are given in Theorem 3.

Theorem 3 For $M = C(a, b, 0, 0, \cdots, 0, c)$ of order $n > 3$, the Moore-Penrose inverses $M^{\#} = C\{m_j\}$ for $j = 0, 1, \cdots, n-1$ of Theorem 2 have elements as follows.

(a) M of rank $n - 1$ with $s = a + b + c = 0$, and for n even, $s' = a - b - c \neq 0$:

(a_1), for $z = b/c \neq 1$

$$m_j^{\#} = \frac{1}{nc(1-z)}\left[\frac{n-1}{2} - j + \frac{1}{1-z} - \frac{nz^j}{1-z^n}\right], \tag{36}$$

(a_2), for $b = c$

$$m_j^{\#} = \frac{6j(n-j) - (n^2 - 1)}{12nc} \quad . \tag{37}$$

(b) M of rank $n - 1$ with n even, $s = a + b + c \neq 0$ but $s' = a - b - c = 0$:

(b_1), for $z = b/c \neq 1$

$$m_j^{\#} = \frac{(-1)^{j+1}}{nc(1-z)}\left[\frac{n-1}{2} - j + \frac{1}{1-z} - \frac{nz^j}{1-z^n}\right], \tag{38}$$

(b_2), for $b = c$

$$m_j^{\#} = (-1)^{j+1}\left[\frac{6j(n-j) - (n^2-1)}{12nc}\right] \quad . \tag{39}$$

(c) M, of rank $n-2$ with n even and $s = a + b + c = 0 = s' = a - b - c$ so that $a = 0$ and $b = -c$:

$$\begin{aligned} m_j^{\#} &= 0 \quad \text{for} \quad j = 0, 2, 4, \cdots, n-2 \\ &= (n-2j)/2nc \quad \text{for} \quad j = 1, 3, 5, \cdots, n-1 \quad . \end{aligned} \tag{40}$$

Derivation of (36) and (37) comes from writing $(M + J)^{-1}$ as $C\{a_j\}$ as in (5) and using (29) expressed as

$$M(M + J)^{-1} + (1/n)J = I$$

for M of (4). This gives rises to equations like (12) and (13) only with $1/n$ added to each left-hand side. Also, similar to (17) we have $\sum_{j=0}^{n-1} = 1/(a+b+c+n)$ $= 1/n$ because $a + b + c = 0$, and on defining $z = b/c$ we find that the recurrent equations whose general solution in this case is $a_j = pz^j + q + j\theta$ for some constant θ, turns out to be

$$a_j = \frac{1}{n^2} + \frac{1}{nc(1-z)}\left[\frac{n-1}{2} - j + \frac{1}{1-z} - \frac{nz^j}{1-z^n}\right] \quad . \tag{41}$$

This is the element of $(M + J)^{-1}$ in (32). Subtracting $1/n^2$ in accord with (32) gives $m_j^{\#}$ shown in (36); and using L'Hôpital's rule to find the limit of (36) as $z \to 1$ gives (37).

Derivation of (38) and (39) proceeds similarly. Writing $(M + T)^{-1}$ as $C\{a_j\}$ and using

$$M(M + T)^{-1} + (1/n)T = T$$

gives recurrent equations like (12) and (13) only with $(-1)^{j+1}/n$ and $1/n$ on their left-hand sides respectively. Similar to (17) we now have $\sum_{j=0}^{n-1} a_j = 1/a+b+c) = 1/(2b+2c)$ because $a = b + c$ and n is even, and also using $\sum_{j=0}^{n-1}(-1)^j a_j = 1/n$ [because $T(M + T)^{-1} = T/n$, as may be verified], we find the the recurrent equations in this case have solution

$$a_j = (-1)^j\left\{\frac{1}{n^2} - \frac{1}{nc(1-z)}\left[\frac{n-1}{2} - j + \frac{1}{1-z} - \frac{nz^j}{1-z^n}\right]\right\} . \tag{42}$$

Subtracting $(-1)^j/n^2$ from this, in accord with (33) gives m_j^* of (38) and deriving the limit as $z \to 1$ gives (39). The term in the square brackets in (42) appears to be formally the same as that in (41); and the elements (38) and (39), apart from the factor $(-1)^{j+1}$, appear to be the same as (36) and (37) respectively. However, it must be remembered that $a + b + c = 0$ with n odd or even for (36) and (37), whereas $a - b - c = 0$ with n even for (38) and (39). Hence, apart from the factor $(-1)^{j+1}$ these results are equal only when $c = -(a+b)$ in (36) and (37) is the same as $c = a - b$ in (38) and (39). This implies $a = 0$ and $b = -c$ and then $M = C(0, -c, 0, 0, \cdots, 0, c)$ in both places. This is exactly part (c) of Theorems 2 and 3 (where we have $a + b + c = 0 = a - b - c$) which is now considered.

Define, for part (c), $(M + J + T) = C\{a_j\}$ and use

$$M(M + J + T)^{-1} + (1/n)(J + T) = I ,$$

as may be verified. Arising from this, with $a = 0$ and $b = -c$ the equations like (11) and (12) are

$$a_j - a_{j+2} = \left[(-1)^{j+1} + 1\right] \Big/ nc \quad \text{and} \quad a_{n-1} - a_1 = -(1-2/n)/c \quad .$$

Repetitive use of the first of these shows that $a_0 = a_2 = a_4 \cdots$, and that for j odd $a_j = a_1 - (j-1)/nc$. Then, using $\sum_{j=0}^{n-1} (-1)^j a_j = 1/n$ immediately gives, with n being even, $a_j = 2/n^2$ for j even and then

$$a_1 + a_3 + \cdots + a_{n-1} = \tfrac{1}{2}na_1 - (0+2+4+\cdots+n-2)/nc = 0 \quad .$$

Hence $a_1 = (n-2)/2nc$ which leads at once to $a_j = (n-2j)/2nc$ for j odd. Finally, subtraction of $2/n^2$ from a_0, a_2, a_4, $\cdots$, in accord with (34), gives (40).

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