A PREHISTORY OF PROBABILITY THEORY

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Abstract

This paper is an attempt to deal with the development of the theory of probability from the beginning of recorded history until its official discovery in 1654 by Pascal and Fermat. I will attempt to analyze the actions of some of the figures associated with the development and attempt to draw inferences as to why certain events happened or didn't happen at specified time intervals. I will also look at some of the qualities that these men possessed that enabled these events to take place. The major conclusion is that basic probability theory came about as the merging of the idea of random phenomena (exemplified by gambling) with the exact science of mathematics. This apparent contradiction in terms is the reason why probability theory was so long in developing, and why it ultimately took men of great genius to master it.

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Introduction

This paper is an attempt to deal with the development of the theory of probability from the beginning of recorded history up until its official discovery in 1654 by Pascal and Fermat. I will attempt to analyze the actions of some of the figures associated with the development and attempt to draw inferences as to why certain events happened or didn't happen at specified time intervals. I will also look at some of the qualities that these men possessed that enabled these events to take place.

Probability in Antiquity

The playing of games of chance seems to be as old as civilization itself. Large number of heel bones of deer or sheep, called astragali or tali, and knucklebones have been found in many ancient civilizations, including those of Egypt, Babylon, Greece and Rome. There are many references in history to the use of these bones in game playing. F. N. David tells us that there are records of painting on Grecian urns of children playing with bones, much like marbles, and that according to Homer, Patrocles played with knucklebones as a child.[1] Cardano also tells us of another account in Homer of Palamedis, a Greek, inventing some games of chance in order to relieve the tedium during the ten-year siege of Troy.[2] David mentions an account of Herodotus, the Greek historian, writing on the famine at Lydia. Herodotus writes that the Lydians took up the playing of games of chance with great vigor so that they could forget about eating on every other day and last out the food shortage. Among the games he mentions are dice, knucklebones and backgammon.[3]

One of the typical games of this time was played with four astragali. Each of the four sides of each astragali was numbered 1, 3, 4 or 6. The one was called the "dog" by the Romans, and the most highly valued throw was called the Venus, occurring when each of the bones showed a different face. One wonders why this was the best, since under today's rules in games of chance, the best is the least probable.

The earliest dice were thought to have been made in the third millenium B.C. and have been found in both Tepe Gaivra in Northern Iraq and Mohenjo-Daro in India.[4] It is interesting to note that these fired pottery die were marked with pips, like our own.

Since the use of dice and astragali was so common in antiquity, particularly in ancient Greece, one wonders why the Greeks, for all of their other great scientific achievements, never came up with any sort of probability theory. Sambursky tells us that the concept of events being probable as opposed to being certain seemed to bother many of the great thinkers of ancient Greece.[5] The Greek equivalent of probable—elkos—meaning "to be expected with some degree of certainty"[6] was in use in Greek literature from the time of the pre-socratics through the Hellenistic period. In Plato's Phaedra, Socrates says that in courts of law, men "care nothing about truth, but only about conviction, and this is based on probability."[7] Sambursky gives us another good illustration of this:

> The awareness of the fact that empirical truth does not rest on the same safe foundations as does mathematical truth independent of the time factor, led to the coining of a very striking term for the natural sciences. Simplicius tells us: "Aptly did Plato call natural science the science of the probable; Aristotle was of the same opinion and postulated that exact evidence must spring from immediate and reliable principles and from exact and essentially primary causes.[8]

This characterization of the natural sciences as being based largely on the unpredictable again makes one wonder why the ancient Greeks never really developed a theory of probability as such. Certain ideas which are very much in harmony with it were expounded by the Stoics. These ideas are given by Sambursky:

> A definition of the Possible widely accepted in the post-Aristotelian era and down to the late Hellenistic period was that of Diodorus the Megarian who said: "The Possible is that which either is or will be true"[*]. This identification of the Possible with actual or potential happenings was challenged by the Stoics, who defined the Possible as "that which is not prevented by anything from happening even if it does not happen" [†]. At the same time the Stoics maintained that the Possible is part and parcel of "things happening according to fate", and they regarded the Possible as an integral part of the causal set-up of the world. According to the meager information available on the subject, mainly Alexander of Aphrodisias, de fato, the Stoics argued as follows: Let us suppose that there are two mutually exclusive possibilities, A and B; then the non-realization of A means the realization of B. The same causal nexus, therefore, which led to the happening of B was the reason which prevented A from happening, and both A and B have their place in the causal scheme. The very fact, however, that A could be assumed as possible, although it did not happen, is attributed to our ignorance of the future, i.e. of the complete causal nexus. It is this ignorance which gives meaning to the category of the Possible and which for a consistent determinist is the prerequisite of the existence of "equally possible cases", of which one alone is going to take place.
>
> * Plut., de Stoic. repugn., 1055E.
> † Plut., l. c. and Alex. Aphr., de Fato, ch. 10.

This idea of "equally probable cases", one of which must take place, puts one right into the heart of modern probability theory. All one would need to do is count up the number of ways each of the cases could occur and divide by the total overall to get the probability of occurrence. Certainly games of chance were popular enough at this time to give some impetus to such an idea, so why didn't it happen? Of course one can only speculate but one reason may be that for the most part the dice of this time, and certainly the astragali, were so asymmetrical as to give extremely inconsistent results from one set to another, so much so that it may never have occurred to anyone that certain faces come up a certain proportion of the time. A second reason is that, with the Greek number system, the calculations would have been intractable. An example of where the above reasoning may hold is that the Venus throw was considered the best with four astragali, when four of a kind is certainly harder to make, i.e. a Venus can be made in $4! = 24$ different ways and four of a kind in only 4 ways. A third reason is the superstition of gamblers, which still holds today, when the theory of probability is widely known. If the average gambler really didn't believe that there were outside forces which affected the dice, call it luck, the gods or whatever, he would hardly have any reason to play, since in a fair game his expected winnings are zero. Of course, gambling skill, such as knowing how much to bet at certain times, can make a difference, but the wide assortment of lucky charms used by gamblers today is something at which a rational man can only marvel.

A fourth reason, which Sambursky gives us, is a philosophical one. Aristotle's and Plato's idea of perfection in the heavens and imperfection in the sublunar region gives us a reason why probability theory nevery really caught on. Recurrent, perfect motion exists only in the outer heavens. Recurrent phenomena may also exist in the lower world, but it is due only to a direct affect from the heavens, such as day and night or the change of seasons. The idea of man producing regular sequences of anything goes against this principle and following from this, the idea of laws governing as chaotic an event as the throws of a die, i.e. that certain faces turn up a regular proportion of the time, is absurd.[10]

A fifth reason, which will show itself to be most important when we actually get to the actual discoveries of probability theory, is that the Greek intellectuals and scientists themselves seem to have had very little amicable contact with gambling and gamblers, as Cardano hints at in his chapter entitled "Why Gambling Was Condemned by Aristotle." As we will see, it will take a combination of gamblers' intuition and intellectual thought to actually come up with the theory of probability. In order for a great thinker to make use of his abilities, he will have to actually "get his hands dirty" with the real data to get an idea of what is truly going on.

Probability in the Middle Ages

As with most other sciences in the Middle Ages, probability made almost no progress in Europe. We also know of very little progress in the Arab world, aside from the great improvements in mathematics which would help in development at a much later time. One of these indirect accomplishments was the discovery of binomial coefficients by Omar Khayyam (died 1214), a Persian.

With the coming to power of the Christian Church, the idea of a random event could not exist, as God was responsible for absolutely everying, even the throw of your die. Dark Ages man, however, was quick to take advantage of this and the practice of casting die to determine God's will became very popular. All one needed to do was to tell God how you would interpret each throw, i.e. say a prayer or do a good deed and throw the dice. He, of course, would control the result and in doing so would tell you what to do. I am surprised that this "heavenly hotline" didn't become more popular than it did.

The idea of randomness to discover God's will was not new even at this time. The drawing of lots to determine important information is prevalent throughout the Bible and is still in use today. The game of odds and evens, played by two people holding out one or two fingers simultaneously, has also been used for thousands of years including the present day. The religious Greeks used five astragali thrown simultaneously to predict fortunes. David gives us an example from Frazier of tables of these:

> 1.3.3.4.4. = 15. <u>The throw of Saviour Zeus</u>.
> One one, two threes, two fours,
> The deed which thou meditatest, go do it boldly.
> Put thy hand to it. The gods have given these favourable omens.
> Shrink not from it in thy mind, for no evil shall befall thee.
>
> 6.3.3.3.3. = 18. <u>The throw of Good Cronos</u>.
> A six and four threes.
> Haste not, for a divinity opposes. Bide thy time.
> Not like a bitch that has brought forth a litter of blind puppies.
> Lay thy plans quietly, and they shall be brought to a fair completion.
>
> 6.4.4.4.4. = 22. <u>The throw of Poseidon</u>.
> One six and all the rest are fours.
> To throw a seed into the sea and to write letters,
> Both these things are empty toil and a mean act.
> Mortal as thou art, do no violence to a god, who will injure thee.
>
> 4.4.4.6.6. = 24. <u>The throw of child-eating Cronos</u>.
> Three fours and two sixes. God speaks as follows:
> Abide in thy house, nor go elsewhere,
> Lest a ravening and destroying beast come nigh thee.
> For I see not that this business is safe. But bide thy time.[11]

There are many more examples of this sort of thing throughout antiquity and up to the present day, including the modern practice of divination with playing cards.

From this use of dice throwing for divination, we come across one of the earliest enumerations of the throws of dice. In about 960 A.D. a Bishop Wiebold of Cambray incorporated a device to let God decide what prayers his monks should say. Wiebold enumerated the 56 possible outcomes of the three die, irrespective of order, and set to each a virtue. The monks would roll the dice each day and see which virtue God wanted them to practice.

Not all of the dicing at this time was for holy purposes. There are many records of church edicts forbidding gambling. Kendall gives us one from Louis IX in 1255: "They shall abstain from dice and chess, from fornication and frequenting taverns. Gaming houses and manufacture of dice are prohibited throughout the realm."[12] However, it appears that the church wasn't really against dice as such, but against the evil behavior that went with them. Kendall gives us this example of a gambler's blasphemy from Chaucer's "The Pardoner's Tale":

> By Goddës precious heart and by his nails
> And by the blood of Christ that is in Hayles
> Seven is my chance, and thine is cinq and trey.
> By Goddës armës, if thou falsely play
> This dagger shall throughout thine hertë go!--
> This fruit cometh of the bitchëd bonës two:
> Forswearing, irë, falseness, homicide.[13]

The first known work dealing with the counting of the number of ways in which three dice can fall, including permutations, is found in a Latin poem called De Vetula, which deals with sports and games. It has been attributed to several authors, but the general consensus is that it was written by one Richard de Fourneval (1200-1250), the chancellor of the Cathedral of Amiens. Kendall gives the relevant passage:

> If all three numbers are alike there are six possibilities; if two are alike and the other different there are 30 cases, because the pair can be chosen in six ways and the other in five; and if all three are different there are 20 ways, because 30 times 4 is 120 but each possibility arises in 6 ways. There are 56 possibilities.
>
> But if all three are alike there is only one way for each number; if two are alike and one different there are three ways; and if all are different there are six ways. The accompanying figure shows the various ways.
>
> It follows, but is not stated, that the total number of ways is $(6 \times 1) + (30 \times 3) + (20 \times 6) = 216$.[14]

Thus we have a start for the probability of dice.

The first known attempt at solving a probability problem is that of Fra Luca de Borga, also called Paccioli. In his book <u>Summa di Arithmetica, Geometria, Proportioni et Proportionalita</u> (1494) he considers a problem which later became known as the problem of points and inspired Fermat and Pascal. In this problem we have two gamblers, A and B, who make a bet and agree that the winner will be the first to take six rounds of some game, but have to quit when A has won five and B three. How should they divide the stakes? The correct answer is that A should get seven parts and B one, since the game would end in at most three more rounds, with $2^3 = 8$ possible outcomes, 7 of these favoring A. Paccioli decides A should get five, B three. Approximately sixty years later, another famous mathematician of the time, Tartaglia in his General Trattalo published in 1556 decides that this is wrong, based on the reasoning that if A has one game, B none, then A would take the entire stakes, which would not be fair. He decides instead that if A has won X, B has won Y, where $X + Y = Z$. $X > Y$, then the formula should be that A gets $\left(\frac{1}{2} + \frac{X-Y}{2Z}\right)$, B getting the rest.[15]

Two years later, G. F. Peverone in his work Breve e Facili Trattali, il Primo d'Arithmetica, l'Altra di Geometria, considers a similar problem, without reference to his predecessors, and almost gets the correct answer. In his example, A has taken seven games, B has nine games, with ten needed to win. Kendall gives his argument:

> A should put 2 crowns and B 12 crowns [or, equivalently, the stake should be divided in the proportion 1:6]. For if A, like B, had one game to go each would put two crowns [or divide the stakes in equal proportions]. If A had two games to go against B's one, he should put 6 crowns against B's two, because, by winning two games he would have won four crowns, but with the risk of losing the second after winning the first; and with three games to go he should put 12 crowns because the difficulty and risk are doubled.[16]

Kendall calls this one of the near misses of mathematics. The arguments for the first and second games are correct. If B has one game to go and is betting two crowns, then for A:

> With one game to go he stakes 2 crowns. With two games to go he stakes 2 + 4 crowns. With three games to go he stakes 2 + 4 + 8 crowns.[17]

Had Peverone been consistent and followed his own rules, he would have solved the problem.

Gerolamo Cardano (1501-1576)

We now come to the original pioneer in the theory of probability, Gerolamo Cardano. His treatise, the Liber de Ludo Aldae (The Book on Games of Chance)[18] is really the first text of any sort on probability theory. The fact that it has been maligned by many scholars up to the present day, including David and Todhunter, seems to be because of the wretched style of the book (which Goulds seems to have miraculously cleared up in his translation) and its inconsistencies, which I think only add to the educational value of the text. It was never published by Cardano and only appeared in 1663 in a ten-volume set of Cardano's works, 138 years after 1525, when Cardano claims to have written it. The principal inconsistency involved Cardano's derivation of the so-called "power law" of probability, an example of which is: if the probability of flipping a coin and getting a head on one toss is $\frac{1}{2}$, then the probability of flipping the coin n times and getting n heads is $(\frac{1}{2})^n$. Cardano states this relationship incorrectly several times before getting it right, which he ultimately does. However, his attempts to derive this law, although not mathematically very elegant, greatly enhance the insight one gets into understanding just what it took to initiate the theory of probability.

Cardano was born in Milan, Italy, on September 24, 1501. His father was a well-known lawyer and intellectual, and personal friend of Leonardo da Vinci. However, his mother and father did not marry until many years later, a fact which was to plague Cardano for much of his later life. He attended medical schools in Pavia and Padua, where his remarkable intelligence made him many friends and enemies. He was soon elected to the rectorship of the university, a post apparently equivalent to today's president of the student body. As the rector he was expected to do a lot of entertaining and, as he wasn't wealthy, he started to gamble to support himself, implying that he must have been quite good. Upon graduation, he applied for a position in the College of Physicians at Milan but was officially turned down owing to his bastard status. Because of this, he spent many lean years in poor towns as town physician and always needed to gamble to support himself and his family. Finally, due to his proficiency as a physician and his father's influential friends, he was appointed to the College of Physicians and also to a position as a public lecturer, a job which won him great acclaim. It was at this time that he began publishing books on a wide variety of subjects, inspired by his lectures. He was especially noted for his works on mathematics which were widely read. He was also acclaimed one of the greatest physicians of his time, and turned down appointments as court physician to crowned heads all over Europe. However, his good fortune didn't last, and after several family tragedies, he was arrested for heresy. Due to his age at the time (70) and his previous good behavior, he was allowed to live at home and died there at the age of 76.

Cardano's Book on Games of Chance was apparently written as a gamblers' manual, and could very well have been called "The Compleat Gambler." Along with probability theory, he also includes such topics as how to recognize cheaters, the psychology of gambling and gamblers, why Aristotle condemned gambling, some history of games of chance and, of course, since he was a physician first and foremost, the therapeutic value of gambling. In his discussions on probability theory, he writes of equiprobability, mathematical expectation, frequency tables for the enumeration of casts of dice in several different games, on the additive property of probabilities, and on the power law of probabilities. He even seems to possess a vague understanding of the laws of large numbers. In his chapters on the casting of one, two, or three dice, he calculates the proportion of favorable cases to the proportion of unfavorable cases for certain throws, just as it is done today. For two dice, he states:

> The number of throws containing at least one ace is eleven out of the circuit of thirty-six; or somewhat more than half of equality; and in two casts of two dice the number of ways of getting at least one ace twice is more than 1/6 but less than 1/4 of equality.[19]

When he says "the circuit" he means the number of ways that the dice can fall, and "equality" is one-half the circuit. Ore implies that the reason for the term equality is because, in a two-man game, each would have a probability of winning equal to $\frac{1}{2}$, so the expected number of winning cases would be $\frac{1}{2}$(circuit) = equality, which would therefore be each player's possible number of favorable occurrences. He then goes on to say:

> In three casts of two dice the number of times that at least one ace will turn up three times in a row falls far short of the whole circuit, but its turning up twice differs from equality by about 1/12. The argument is based upon the fact that such a succession is in conformity with a series of trials and would be inaccurate apart from such a series.[20]

This is what Ore calls "reasoning on the mean."[21] It is based on the fact that the probability of making a 6 with one die on one toss is 1/6, on two throws is 2/6, and so on until we get to the probability of a 6 on 6 throws to be one. His error here is in confusing the probability with the expected value of the toss, something still common today in basic probability classes. Cardano realizes that this doesn't always happen in a small number of tosses but on the whole bears itself out, thus demonstrating an understanding of the law of large numbers. He makes what is today known as a large sample approximation.

> Moreover, a repeated succession, such as favorable points occurring twice, arises from circuits performed in turn; for example, in 3,600 casts, the equality is $\frac{1}{2}$ of that number, namely, 1,800 casts; for in such a number of casts the desired result may or may not happen with equal probability. So the whole set of circuits is not inaccurate, except insofar as there can be repetition, even twice or three times, in one of them. Accordingly, this knowledge is based on conjecture which yields only an approximation, and the reckoning is not exact in these details; yet it happens in the case of many circuits that the matter falls out very close to conjecture.[22]

In the next few chapters, he correctly enumerates the number of ways of making certain points with dice in several different games.

In his Chapter 14, he gives a general rule for gambling, since "mathematicians may be deceived." He states:

> I have wishes this matter not to lie hidden because many people, not understanding Aristotle, have been deceived, and with loss. So there is one general rule, namely, that we should consider the whole circuit, and the number of those casts which represents in how many ways the favorable result can occur, and compare that number to the remainder of the circuit, and according to that proportion should the mutual wagers be laid so that one may contend on equal terms.[23]

This is exactly the concept of a fair game and represents a hitherto unknown abstraction from empirical observation to theoretical concept. But he then uses this rule incorrectly to derive the power law, coming closer but not quite getting it. He again brings up the problem of throwing an ace twice, once on each of two tosses of three die. The actual probability of one ace is $\frac{91}{216}$, two aces is $\left(\frac{91}{216}\right)^2$. Cardano realizes that it is a power rule but trips over his own notation. He states:

> But if two casts are necessary, we shall multiply them in turn, and the remainders for those numbers in turn, and if three are necessary, or four, we shall do the same, and then we shall have to make the comparison in accordance with the numbers thus obtained. Thus, if it is necessary for someone that he should throw an ace twice, then you know that the throws favorable for it are 91 in number, and the remainder is 125; so we multiply each of these numbers by itself and get 8,281 and 15,625, and the odds are about 2 to 1. Thus, if he should wager double, he will contend under an unfair condition, although in the opinion of some the condition of the one offering double stakes would be better. In three successive casts, therefore, if an ace is necessary, the odds will be 753,571 to 1,953,125, or very nearly 5 to 2, but somewhat greater.[24]

In Chapter 15 he again decides that this can't be true by working through some simple examples, so he again sets out to find the proper formula. Using the example of an even or odd roll, he correctly gets the formula for an even roll on two tries to be $\left(\frac{1}{2}\right)^2$, but errs again when he generalizes this to $\left(\frac{1}{n}\right)^2$ for an even roll on n tries. He works through some more simple examples and again realizes he has come up with the wrong answer. So now he goes to the case of a four-sided talus with three odd faces and one even, so the probability of getting an odd face on one toss is $\frac{3}{4}$. Using this example, he finally comes up with the correct rule, i.e. if p is the probability of making a certain point on 1 toss, then $(p)^n$ is the probability of making it n times in n tosses. He works through a few more examples and ultimately convinces himself that his is right.

In reading through these passages, we begin to realize exactly what combination of circumstances it took for Cardano to develop the theory of probability. First of all, Cardano was a physician, trained in the observational methods of Galen and Hippocrates. Secondly, he was a mathematician, well versed in all of the latest methods, and last of all, he was a gambler. The claim is that all three of these sides of Cardano contributed to his ability to do what he did. As a mathematician, he was able to make conjectures about the laws of probability, as had Tartaglia and Paccioli before him. But as a physician he was trained in the method of actually taking careful observation of the hypothesized solutions to his medical problems, i.e. so he could reformulate if his patient got worse, and this put him in the frame of mind where he would check his mathematical solutions, unlike his predecessors, against actual observation. And finally, his gambling experience gave him the intuitive feel for the observations that he needed to tie all of this together. Cardano's methods also remind one a bit of those of Archimedes, in the famous letter in which Archimedes explains how he came up with the formulas for the centers of mass of certain plane figures by first cutting them out of sheepskin and balancing them on a point to see where the center was, and then finding a formula to fit his observation. Perhaps if he had included this in his works, as Cardano had, he also would have been maligned by modern mathematicians as being "inelegant," one of the most insulting terms used in present day mathematical literature.

As for the philosophical questions of random phenomena, Cardano seems to feel that for the most part, what appears to one man to be luck is usually skillful gambling, i.e. betting high on good cards and low on bad, intelligent handling of circumstances, or to cheating, i.e. playing with marked cards. The final passage from his chapter on luck gives a good illustration of how he feels on the subject:

> Now I think it worthy of consideration that this fortune of mine seems to have been something greater than mere chance, since we see in it a beginning, an increase, and a certain continuance so that certain remarkable things happen, as for instance that two aces occurred twice when defeat could not otherwise be brought about, and other things of this sort. We would also see a decline and then very often a change, and then great calamity or great good fortune, and other things in the same way. In view of all this I should think we ought to decide that there is something in this, although we do not know the law which connects the parts. It is as though you were fated in advance to be enriched or despoiled; especially seeing that from this there can follow something more important, as it happened to the man who, on leaving a game after losing all his money, injured the image of the Blessed Virgin with his fist. He was arrested and condemned to be hanged.
>
> But whether the cause of that luck, be it in the conjunction of the stars or in the construction of a certain order of the universe, can affect the cards, which are considered bad or good only according to the conventions of men (since they signify nothing of themselves), is so worthy of doubt that it is easier to find a cause of this fact without that purpose than with it; without it the matter can well be reduced to chance, as in the constitution of the clouds, the scattering of beans, and the like.[25]

It is hard to say how much effect philosophy had on his calculations but one can probably say that without a visceral belief in some sort of laws of recurrent phenomena, he would have had no reason to worry about probability theory.

The next person to publish any results on probability theory was Galileo. He wrote a short treatise Guiocco Dei Dadi (loosely: Thoughts on Dice Games[26]), which consists merely of a table of the enumeration of the possible tosses of three dice, and a short passage on how he derived these results. David informs us that it was written at the bidding of Galileo's benefactor, the Duke of Tuscany, and apparently for no other reason, certainly not his own interest in the problem. He states:

> Now I, to oblige him who has ordered me to produce whatever occurs to me about such a problem, will expound my ideas.[27]

This passage is included only because historically it is recognized as a big step, since Cardano's work wasn't really appreciated until Ore's book in 1953.

Pascal (1623-1662) and Fermat (1601-1665)

We now come to what is officially considered to be the birth of probability theory. Our story begins in 1654, when Antoince Gombaud Chevalier de Méré came to his friend Blaise Pascal with two gambling problems. In many texts, de Méré is listed as a gambler, but Ore tells us that he was in fact a gentleman who dabbled slightly in mathematics and was confused about the solution to the problem of points stated and unsolved in many textbooks of the time. He actually shared Pascal's disdain of professional gamblers, but together they did do some friendly gambling. Blaise Pascal was born in Clermont-Ferrand, France, in 1623. His father, like Cardano's, was also a well-known jurist and intellect. As a child, Pascal was reported to have been of wondrous intelligence, but poor health. His life was torn between his great scientific achievements in physics and mathematics and his fervent devotion to Jansenism, an anti-science sect of Christianity, which ultimately won out in his later years.

De Méré proposed two problems to Pascal, the first concerning how many times one must be allowed to toss two dice in order to have a better than even chance of obtaining two sixes at least once, and another which was a version of the problem of points.

The proper solution to the first problem is found by the following method. Let $p_n = p\{\text{one gets at least one double six in n tosses}\} = 1 - (q)^n$, where $q = \frac{35}{36}$.

Solving, we get:

$$\frac{1}{2} = 1 - \left(\frac{35}{36}\right)^n$$

$$\Rightarrow \quad \frac{1}{2} = \left(\frac{35}{36}\right)^n$$

$$\Rightarrow \quad \log\left(\frac{1}{2}\right) = n \log\left(\frac{35}{36}\right)$$

$$\Rightarrow \quad n = 24.605 \quad .$$

Therefore, for $p_n \geq \frac{1}{2}$, we want $n \geq 25$.

This answer apparently upset de Méré, as Pascal writes to Fermat:

> He told me that he had found a fallacy in the theory of numbers, for this reason:
>
> If one undertakes to get a six with one die, the advantage in getting it in 4 throws is as 671 is to 625.
>
> If one undertakes to throw 2 sixes with two dice, there is a disadvantage in undertaking it in 24 throws.
>
> And nevertheless 24 is to 36 (which is the number of pairings of the faces of two dice) as 4 is to 6 (which is the number of faces of one die).
>
> This is what made him so indignant and which made him say to one and all that the propositions were not consistent and that Arithmetic was self-contradictory: but you will very easily see that what I say is correct, understanding the principles as you do.[28]

In the second problem, Pascal gives de Méré an almost correct answer, but another well-known mathematician, Roberval, hears of the solution and ridicules it, obviously because he didn't understand it. So apparently because he is still a little unsure of himself, he writes to his father's old friend, Pierre de Fermat, the man considered to be the greatest mathematician in France.

Fermat was born in Beaumont-de-Lomagne near Gascony in 1601. He was the son of a merchant. He studied law at the University of Toulouse and finished in 1631. He was extremely well-read and famous in his time, and several theorems in algebra bear his name and are still studied today.

The first letter from Pascal to Fermat has been lost, but we have Fermat's reply. In it, he reviews Pascal's solution to the problem of points and finds him correct on all but one facet, which he then corrects. The problem concerns a game in which eight attempts are made to make a six with a single die, and the way the stakes should be divided in the event that the game is halted at certain stages of play with the point not having been made. They agree that the thrower should get $\frac{1}{6}$ of the pot if he doesn't take his first throw, $\frac{1}{6}$ of the remaining $\frac{5}{6}$ if he doesn't take his second, and so on. This is because his expected winnings are $p\{\text{winning}\} \cdot \text{pot} = \frac{1}{6}$ pot. This generalizes to $\sum_{i=1}^{n}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{i-1}$ portion of the pot for n throws not taken. Pascal then trips over his notation, and says that if he takes three throws and not his fourth, he should get $\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{3}$ for the fourth throw, but Fermat corrects him, explaining that the expected value of a throw does not depend on previous throws, i.e. the fourth throw is still the first not taken, so its value is $\frac{1}{6}$ of the pot.

Pascal's reply to Fermat is joyous in tone, with Pascal in complete agreement with Fermat. He then proposes another problem to Fermat and presents his solution for Fermat to check. The problem again involves stopping before certain points are made. He considers three cases in a game in which the first to make 3 wins. In the first case, he writes:

> Here, more or less, is what I do to show the fair value of each game, when two opponents play, for example, in three games, and each person has staked 32 pistoles.
>
> Let us say that the first man had won twice and the other once; now they play another game, in which the conditions are that, if the first wins, he takes all the stakes, that is 64 pistoles, if the other wins it, then they have each won two games, and therefore, if they wish to stop playing, they must each take back their own stake, that is, 32 pistoles each.
>
> Then consider, Sir, if the first man wins, he gets 64 pistoles, if he loses he gets 32. Thus if they do not wish to risk this last game, but wish to separate without playing it, the first man must say: "I am certain to get 32 pistoles, even if I lose I still get them; but as for the other 32, perhaps I will get them, perhaps you

> will get them, the chances are equal. Let us then divide these 32 pistoles in half and give one half to me as well as my 32 which are mine for sure." He will then have 48 pistoles and the other 16.[29]

From here he goes to the case where A has two, B has none. In the next game, a win for A gives him the total, a win for B puts them back in case one, so A gets the average of the two, i.e. $\frac{1}{2}(64+48) = 32 + 24 = 56$. The third case is A having one, B zero. A win for A puts them in case two, a win for B evens them, so A gets $\frac{1}{2}(56+32) = 44 = 32 + 12$. Pascal then generalizes:

> Now, in this way, you see, by simple subtraction, that for the first game, 12 pistoles of the other man's money are due to him, for the second another 12; and for the last, 8.
>
> Now, to make no mystery of it, since you understand it so well, and I only wish to see that I have made no mistake, the value (by which I mean only the value of the opponent's money) of the last game of two is double that of the last game of three and four times the last game of four and eight times the last game of five, etc.[30]

i.e. for the case of a game to n where A has n-1 wins and B zero, A should get $1 - (\frac{1}{2})^{n-1}$ proportion of B's ante.

The other case of A having n-1, B having 1 is not so easy. In the example of A having 2, B having 1, A is owed $\frac{3}{8}=\frac{1.3}{2.4}$ of B's share. Pascal then generalizes this to the case of A having n-1, B having 1, saying that A is owed $\frac{1.3.5\cdots}{1.3.5\cdots}\frac{2n-1}{2n}$ of B's ante. His reasoning is:

> In the first place, it must be said that if one has gained 1 out of 5 games, for example, and if one needs 4, the match will be decided for certain in 8 which is double 4.
>
> The value in terms of the opponent's stakes, of the first game in a set of 5, is a fraction whose numerator is half the combinations of 4 out of 8 (I take 4 because it is equal to the number of games required and 8 because it is double 4) and whose denominator is this same numerator plus all the combinations of higher numbers.
>
> Thus, if I have won the first game out of 5, 35/128 of my opponent's stake is due to me: that is to say, if he has staked 128 pistoles, I take 35 and leave him the remainder, 93.
>
> Now this fraction 35/128 is the same as 105/384 which is made by taking the product of even numbers as denominator and the product of odd numbers as numerator.
>
> You will undoubtedly understand all this well, if you take a little trouble: that is why I think it unnecessary to go on with it any longer.[31]

I.e. the fraction owed = $\dfrac{\frac{1}{2}\binom{8}{4}}{\frac{1}{2}\binom{8}{4} + \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8}}$

and since

$$(1+1)^8 = \binom{8}{0} + \binom{8}{1} + \cdots + \binom{8}{8} \text{ and } \binom{n}{k} = \binom{n}{n-k}$$

we have

$$\text{fraction owed: } \frac{\frac{1}{2}\binom{8}{4}}{\frac{1}{2}(1+1)^8} = \frac{8.7.6.5}{4.3.2.1}\,\frac{1}{2^8} = \frac{7.5.3.1}{8.6.4.2}\ .$$

He gives no particular reasoning for the combinations used in the derivation, assuming that Fermat would understand. Mahoney points out that the solution has been written down incorrectly and should actually be that A is due $\frac{1.3.5\cdots(2n-3)}{2.4.6\cdots(2n-2)}$ of B's ante.[32] Even with this slight error, this solution represents a major breakthrough in the theory of probability. This is the first time that anyone has ever solved the problem of points and in some generality.

The reply of Fermat to Pascal has been lost. However, in Pascal's reply, he writes of the inapplicability of Fermat's combinatorial method to the problem involving three gamblers. From this we assume that in the last letter Fermat offered a combinatoric solution to the problem. Pascal incorrectly interprets the extension of Fermat's solution and shows how his own method gives the correct answer. In Fermat's reply he shows again that Pascal is in error and offers yet another method of solution, in which he gives the probability of A winning on each of one, two or three tosses, sums them up, and has the probability of A winning, once again demonstrating the additivity of probabilities which Cardano had shown 100 years before. Pascal in his return to Fermat says that he is glad that they are in complete agreement, but he doesn't wish to correspond on the subject any more. They had only one more exchange of letters in 1660 on unrelated subjects. In reading these last letters, and in some parts of the earlier letters, one gets the idea that Pascal may have been slightly intimidated by the great genius of Fermat, and how easily he could come up with several methods of solution when Pascal spent so much effort on one, which would often have an error in it. Perhaps this was the reason he wished to break off the correspondence, or perhaps it was because of his religious feelings against science; but for whatever reason, here ends the recognized story of the invention of probability theory.

Now making my own subjective analysis of why these men were able to deduce what people had wondered about for two centuries, it seems that the necessary ingredients consisted of a great genius who had done a little gambling, Pascal; a nobleman who had wondered about gambling, de Méré, to motivate the problem; and a mathematician, perhaps the greatest who ever lived, Fermat, to get interested. Where most men would need a feel for gambling to know what the probabilities ought to be and work from there, Fermat's genius was so great that he only needed to be motivated to look at the problem to solve it. One gets the idea that Pascal also felt this way from the tone of the letters.

Conclusion

We have thus traced the development of the theory of probability from antiquity until its official invention in 1654. Many reasons have been given as to why certain events happened and didn't happen when they did. The overriding conclusion is that basic probability theory came about as the merging of the idea of random phenomena (exemplified by gambling) with the exact science of mathematics. This apparent contradiction in terms is the reason why probability was so long in developing, and why it ultimately took men of such great genius to master it.

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